Solutions

Modeling Flight Dynamics with Tensors

Lecture 11

Problem 1 Pitching Moment Taylor Series Expansion

Slide 6 gives the compact form of the aerodynamic Taylor series development $y_i = d_i(z_j)$ with $y_i = \{X, Y, Z, L, M, N\};\ i = 1,2,\ldots,6$ and $z_j = \{u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \delta p, \delta q, \delta r\}$, $j = 1,2,\ldots,12$.
Expand the aerodynamic pitching moment *M* up to second order of the Taylor series and present it in matrix form.

Solution

$M = M(z_j)$, $z_j = \{u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \delta p, \delta q, \delta r\}$, $j = 1,2,\ldots,12$

$$M = M(z_j) + \frac{\partial M}{\partial z_{j_1}} \Delta z_{j_1} + \frac{1}{2} \frac{\partial^2 M}{\partial z_{j_1} \partial z_{j_2}} \Delta z_{j_1} \Delta z_{j_2}$$

First Term: $M(z_j) = M(u, v, .., \delta r)$, **Second Term**: $\frac{\partial M}{\partial z_{j_1}} \Delta z_{j_1} = \begin{bmatrix} \frac{\partial M}{\partial u} & \frac{\partial M}{\partial v} & . & . & \frac{\partial M}{\partial \delta r} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ . \\ . \\ \Delta \delta r \end{bmatrix}$

Third Term:

$$\frac{1}{2} \frac{\partial^2 M}{\partial z_{j_1} \partial z_{j_2}} \Delta z_{j_1} \Delta z_{j_2} = \frac{1}{2} \begin{bmatrix} \Delta u & \Delta v & . & . & \Delta \delta r \end{bmatrix} \begin{bmatrix} \frac{\partial^2 M}{\partial u^2} & \frac{\partial^2 M}{\partial u \partial v} & . & . & \frac{\partial^2 M}{\partial u \partial \delta r} \\ \frac{\partial^2 M}{\partial v \partial u} & \frac{\partial^2 M}{\partial v^2} & . & . & \frac{\partial^2 M}{\partial v \partial \delta r} \\ . & . & . & & . \\ . & . & & . & . \\ \frac{\partial^2 M}{\partial r \partial u} & \frac{\partial^2 M}{\partial r \partial v} & & & \frac{\partial^2 M}{\partial \delta r^2} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ . \\ . \\ \Delta \delta r \end{bmatrix}$$

Comment: The 1x12 matrix of the second term is called a *gradient*, and the 12x12 matrix of the third term is called a *Hessian* matrix. Note there are 144 second order derivatives just for the aerodynamic pitching moment. Can they all be neglected? Stay tuned.

Problem 2 Replacing Tabular Dependency by Polynomial

Slide 4 depicts the aerodynamic model for the F16. The pitching moment is expanded into a mixture of tables and derivatives:

$$C_m = C_{m_o}(M,\alpha,\delta e) + \underline{\frac{c}{2V} C_{m_q}(M,\alpha)q} + \frac{C_Z}{c}(x_{cg\,R} - x_{cg})$$

The pitch damping derivative is itself a tabular function of Mach and angle-of-attack. You want to reduce that two-dimensional table to a one-dimensional table of Mach only, while expanding the angle-of-attack dependency into a Taylor series of up to third order. Show the process and determine whether all these derivatives exist based on the mirror symmetry of the aircraft.

Solution

$$C_{m_q}(M,\alpha)q = C_{m_q}(M)\alpha_0 q + \frac{\partial C_{m_q}}{\partial \alpha}(M)\Delta\alpha q + \frac{1}{2!}\frac{\partial^2 C_{m_q}}{\partial \alpha^2}(M)\Delta\alpha^2 q + \frac{1}{3!}\frac{\partial^3 C_{m_q}}{\partial \alpha^3}(M)\Delta\alpha^3 q$$

Checking existence

First Term $C_{m_q}(M)\alpha_0 q:\ D_5^5 = 5+1+5+1 = 12\ even \rightarrow exists$

Second Term $\frac{\partial C_{m_q}}{\partial \alpha}(M)\Delta\alpha q:\ D_5^{5\,3} = 8+2+5+1 = 16\ even \rightarrow exists$

Third Term $\frac{1}{2!}\frac{\partial^2 C_{m_q}}{\partial \alpha^2}(M)\Delta\alpha^2 q:\ D_5^{5\,3\,3} = 11+3+5+1 = 20\ even \rightarrow exists$

Fourth Term $\frac{1}{3!}\frac{\partial^3 C_{m_q}}{\partial \alpha^3}(M)\Delta\alpha^3 q:\ D_5^{5\,3\,3\,3} = 14+4+5+1 = 24\ even \rightarrow exists$

Comment: The last aero-derivative is of fourth order. Who can fathom that?

Problem 3 Aero Derivative Table for Missiles

You are to build a derivative map up to second order for a missile with tetragonal symmetry, with particular emphasis on roll rate and roll control coupling, using the chart on Slide 11. Forces and moments are in body axes: Y, Z, M, N, L. Eliminate all those derivatives that do not exist because of tetragonal symmetry.

Solution

	Linear Derivatives	Roll Rate Coupling	Roll Control Coupling	
Y	$Y_v, Y_r, Y_{\dot{v}}, Y_{\delta r}$	$Y_{wp}, Y_{\dot{w}p}, Y_{qp}, Y_{\delta qp}$	$Y_{w\delta p}, Y_{\dot{w}\delta p}, Y_{q\delta p}, Y_{\delta q\delta p}$	
Z	$Z_w, Z_q, Z_{\dot{w}}, Z_{\delta q}$	$Z_{vp}, Z_{\dot{v}p}, Z_{rp}, Z_{\delta rp}$	$Z_{v\delta p}, Z_{\dot{v}\delta p}, Z_{r\delta p}, Z_{\delta r\delta p}$	
M	$M_w, M_q, M_{\dot{w}}, M_{\delta q}$	$M_{vp}, M_{\dot{v}p}, M_{rp}, M_{\delta rp}$	$M_{v\delta p}, M_{\dot{v}\delta p}, M_{r\delta p}, M_{\delta r\delta p}$	
N	$N_v, N_r, N_{\dot{v}}, N_{\delta r}$	$N_{wp}, N_{\dot{w}p}, N_{qp}, N_{\delta qp}$	$N_{w\delta p}, N_{\dot{w}\delta p}, N_{q\delta p}, N_{\delta q\delta p}$	
	Linear Derivatives	Incidence Coupling	Rate Coupling	Control Coupling
L	$L_p, L_{\delta p}$	$L_{wv}, L_{\dot{w}\dot{v}}, L_{\dot{w}v}, L_{w\dot{v}},$	$L_{vq}, L_{\dot{v}q}, L_{wr}, L_{\dot{w}r}, L_{rq}$	$L_{v\delta q}, L_{\dot{v}\delta q}, L_{r\delta q},$ $L_{w\delta r}, L_{\dot{w}\delta r}, L_{q\delta r}, L_{\delta q\delta r}$

Comment: The L_{wv} derivative is of particular interest in Lesson 12