Solutions

Modeling Flight Dynamics with Tensors

Lecture 7

Problem1 Tensor Identity

On Slide 6 I use the tensor identity $\bar{\mathbf{s}}_{iR}\mathbf{s}_{iR}\mathbf{E}-\mathbf{s}_{iR}\bar{\mathbf{s}}_{iR}=\bar{\mathbf{S}}_{iR}\mathbf{S}_{iR}$ for the moment of inertia tensor. Though the proof using tensor calculus is quite advanced, you should confirm the validity of this identity by using the 3x1 matrix $[\bar{s}]=\begin{bmatrix}a & b & c\end{bmatrix}$.

Solution

$$[\bar{s}][s][E]-[s][\bar{s}]=[\bar{S}][S]$$

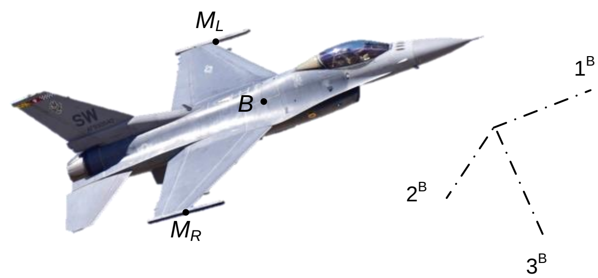
$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}a\\ b\\ c\end{bmatrix}\begin{bmatrix}a & b & c\end{bmatrix}=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}\begin{bmatrix}0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{bmatrix}$$

$$\begin{bmatrix}b^2+c^2 & -ab & -ac\\ -ab & a^2+c^2 & -bc\\ -ac & -bc & a^2+b^2\end{bmatrix}=\begin{bmatrix}b^2+c^2 & -ab & -ac\\ -ab & a^2+c^2 & -bc\\ -ac & -bc & a^2+b^2\end{bmatrix}\text{ QED}$$

Comment: These are symmetrical matrices, just like the MOI matrices

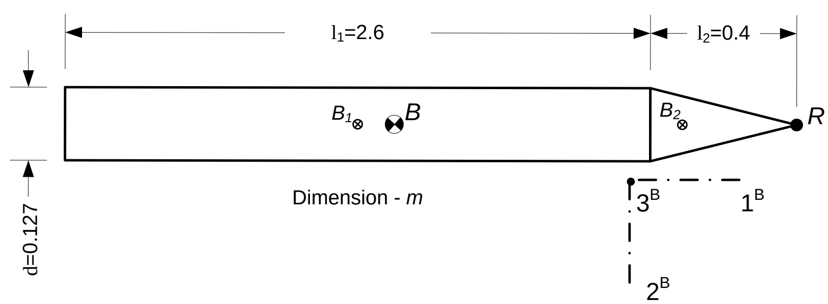
Problem 2 Moment of Inertia

Before take-off, the flight-line crew uploads two sidewinders at the wing tips of the F16 aircraft. The shape of the sidewinder is approximated by a cylinder of 2.6 *m* length and 0.127 *m* diameter, topped by a cone of 0.4 *m*, having the total mass of 85 kg. What is the contribution of the two sidewinders to the F16 moment of inertia, if the displacement of the c.m. M_R of the right sidewinder from the center of mass *B* of the F16 is in aircraft body coordinates $[\overline{s_{M_rB}}]^B = \begin{bmatrix} -1 & 5 & 0 \end{bmatrix}$ *m*? (Both sidewinders are aligned with the 1^B axis.)



Solution

<u>FIRST</u> MOI of Sidewinder. Here the *B*'s refer to the Sidewinder



Volume of Sidewinder

Cylinder: $V_1 = l_1\pi(d/2)^2 = 2.6*\pi*(0.127/2)^2 = 0.0329\,m^3$

Cone: $V_2 = l_2\pi(d/2)^2/3 = 0.4*\pi*(0.127/2)^2/3 = 0.00169 m^3$

Mean density of sidewinder: $\rho = \dfrac{m^M}{V_1 + V_2} = \dfrac{85}{0.0329 + 0.00169} = 2457\ kg/m^3$

Mass of Sidewinder

Cylinder: $m_1 = V_1 \cdot \rho = 0.0329 * 2457 = 80.8kg$

Cone: $m_2 = V_2 \cdot \rho = 0.00169 * 2457 = 4.15kg$

Center-of-mass of Sidewinder (missile convention uses the tip *R* as geometrical reference)

Cylinder: $[\overline{s_{B_1R}}]^B = \begin{bmatrix} -((l_1/2) + l_2) & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1.7 & 0 & 0 \end{bmatrix} m$

Cone: $[\overline{s_{B_2R}}]^B = \begin{bmatrix} -(3/4)l_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.3 & 0 & 0 \end{bmatrix} m$

Total: in Sidewinder body coordinates

$$\mathbf{s}_{BR} = \frac{m_1 \mathbf{s}_{B_1R} + m_2 \mathbf{s}_{B_2R}}{m} \quad \Rightarrow \quad [s_{BR}]^B = \frac{m_1 [s_{B_1R}]^B + m_2 [s_{B_2R}]^B}{m}$$

$$[s_{BR}]^B = \left\{ 80.8 \begin{bmatrix} -1.7 \\ 0 \\ 0 \end{bmatrix} + 4.15 \begin{bmatrix} -0.3 \\ 0 \\ 0 \end{bmatrix} \right\} \frac{1}{85} = \begin{bmatrix} -1.63 \\ 0 \\ 0 \end{bmatrix} m$$

Moment of Inertia of Sidewinder (introducing $r = d/2$)

$$\text{Cylinder: } [I_{B_1}^{B_1}]^B = \begin{bmatrix} m_1 r^2/2 & 0 & 0 \\ 0 & m_1(3r^2 + l_1^2)/12 & 0 \\ 0 & 0 & m_1(3r^2 + l_1^2)/12 \end{bmatrix}$$

$$\text{Cone: } [I_{B_2}^{B_2}]^B = \begin{bmatrix} 3m_2 r^2/10 & 0 & 0 \\ 0 & 3m_2(4r^2 + l_2^2)/80 & 0 \\ 0 & 0 & 3m_2(4r^2 + l_2^2)/80 \end{bmatrix}$$

Total: **(**moment of inertia referred to the common center of mass *B* of the sidewinder.)

$$\mathbf{I}_B^B = \mathbf{I}_{B_1}^{B_1} + m^{B_1} \overline{\mathbf{S}}_{B_1B} \mathbf{S}_{B_1B} + \mathbf{I}_{B_2}^{B_2} + m^{B_2} \overline{\mathbf{S}}_{B_2B} \mathbf{S}_{B_2B}$$

$$[I_B^B]^B = [I_{B_1}^{B_1}]^B + m_1 [\overline{S}_{B_1B}]^B [S_{B_1B}]^B + [I_{B_2}^{B_2}]^B + m_2 [\overline{S}_{B_2B}]^B [S_{B_2B}]^B$$

$$[s_{B_1B}]^B = [s_{B_1R}]^B - [s_{BR}]^B = \begin{bmatrix} -1.7 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1.63 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.07 \\ 0 \\ 0 \end{bmatrix} m$$

where:

$$[s_{B_2B}]^B = [s_{B_2R}]^B - [s_{BR}]^B = \begin{bmatrix} -0.3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1.63 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.33 \\ 0 \\ 0 \end{bmatrix} m$$

<u>SECOND</u> Contribution of the two sidewinders to the F16 MOI (Now we give the sidewinders the designation *M* to distinguish from the F16 *B*)

Moment of inertia of the two sidewinders referred to the c.m. of the F16

$$\mathbf{I}_B^{\Sigma M} = \mathbf{I}_M^M + m^M \overline{\mathbf{S}}_{M_rB}\mathbf{S}_{M_rB} + \mathbf{I}_M^M + m^M \overline{\mathbf{S}}_{M_lB}\mathbf{S}_{M_lB}$$
$$[I_B^{\Sigma M}]^B = [I_M^M]^B + m^M [\overline{S}_{M_rB}]^B [S_{M_rB}]^B + [I_M^M]^B + m^M [\overline{S}_{M_lB}]^B [S_{M_lB}]^B$$

where $[\overline{s_{M_rB}}]^B = \begin{bmatrix} -1 & 5 & 0 \end{bmatrix}$, $[\overline{s_{M_lB}}]^B = \begin{bmatrix} -1 & -5 & 0 \end{bmatrix}$ *m*

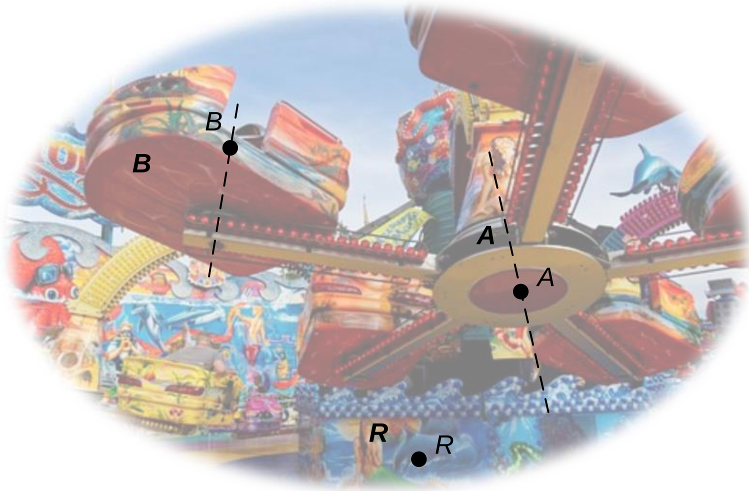
My SCILAB results

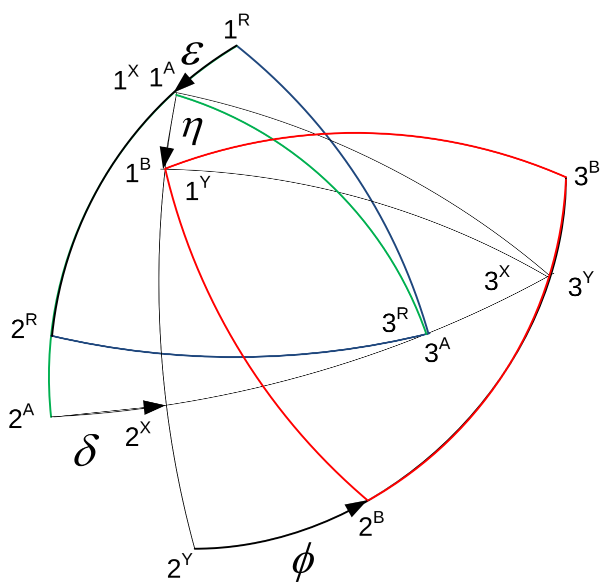
```
"Sidewinder MOI - kg*m^2"
  0.1679231   0.        0.
  0.          53.36305  0.
  0.          0.        53.36305
 "Both sidewinder contribution to the F16 MOI - kg*m^2"
  4250.3358   0.        0.
  0.          276.7261  0.
  0.          0.        4526.7261
```

Comment: Comparing with a typical value of the F16 MOI. The sidewinders make a noticeable contribution to the roll and yaw MOI of the F16

$$MOI\_F16 = \begin{bmatrix} 12875 & 0 & -1331 \\ & 75673 & \\ -1331 & & 85551 \end{bmatrix} \text{kg*m^2}$$

Problem 3 Rotary and Point-Mass Angular Momenta





Let's go to the amusement park and whirl around in the chair ***B*** *w*hich in turn is orbiting with the arms assembly ***A*** relative to the ground ***R***. After all that fun you are to calculate your angular momentum caused by the rotating chair and the orbiting arm, using the equation of Slide 9.

The coordinate system arrangement is quite complex, so I have given you my orange-peel diagram to help you along. Study it carefully and associate it with the picture above. Then you are ready to calculate your angular momentum in your body coordinates $[l_R^{BR}]^B$ and satisfy your curiosity of the contribution of the rotary vs the point-mass part to your total angular momentum.

Here are some details. The arm assembly ***A*** with its coordinate system $]^A$ is tilted wrt the ground ***R*** and its coordinate system $]^R$ by the angle $\varepsilon = 20$ *deg*, and orbits about its 1^A axis with the angle $\delta = \dot{\delta} t, \dot{\delta} = 6\, RPM$. The displacement vector of the c.m. *B* of the chair wrt to the center of the arm assembly *A* is $[\overline{s_{BA}}]^A = \begin{bmatrix} 0.8 & 3 & 0 \end{bmatrix} m$, and *A* wrt *R* $[\overline{s_{AR}}]^R = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} m$. In turn the chair ***B*** is also tilted wrt ***A*** by the angle $\eta = 10$ deg and rotates about its 1^B axis with the angle $\phi = \dot{\phi} t, \dot{\phi} = 120\ RPM$. Finally, the MOI of the chair with you sitting in it is approximated by a cylinder of *r*= 0.75 *m* and *h*=1 *m* at uniform density *d*=1500 *kg/m^3*.

Now it is up to you, sitting in the chair, to determine your *rotary* and *point-mass* angular momenta in the chair's body coordinates. Display the result for one orbit of the chair.

Solution

Tensor Solution

From Slide 9 $\mathbf{l}_R^{BR} = \mathbf{I}_B^B \boldsymbol{\omega}^{BR} + m^B \mathbf{S}_{BR} D^R \mathbf{s}_{BR}$ I divide the angular velocity and the displacement vector into their individual parts $\mathbf{l}_R^{BR} = \mathbf{I}_B^B \left(\boldsymbol{\omega}^{BA} + \boldsymbol{\omega}^{AR} \right) + m^B \left(\mathbf{S}_{BA} + \mathbf{S}_{AR} \right) D^R \left(\mathbf{s}_{BA} + \mathbf{s}_{AR} \right)$.

Let's work with the rotational time derivative term

$$D^R(\mathbf{s}_{BA} + \mathbf{s}_{AR}) = D^R\mathbf{s}_{BA} + D^R\mathbf{s}_{AR} = \underbrace{D^A\mathbf{s}_{BA}}_{0} + \mathbf{\Omega}^{AR}\mathbf{s}_{BA} + \underbrace{D^R\mathbf{s}_{AR}}_{0} = \mathbf{\Omega}^{AR}\mathbf{s}_{BA}$$

Our solution is in tensor form

$$\mathbf{l}_R^{BR} = \mathbf{I}_B^B \left(\boldsymbol{\omega}^{BA} + \boldsymbol{\omega}^{AR} \right) + m^B \left(\mathbf{S}_{BA} + \mathbf{S}_{AR} \right) \mathbf{\Omega}^{AR} \mathbf{s}_{BA}$$

Numerical Solution

The angular momentum is to be calculated in body coordinates:

$$\begin{aligned} &[l_R^{BR}]^B = \\ &[I_B^B]^B \left([T]^{BA}[\omega^{BA}]^A + [T]^{BR}[\omega^{AR}]^R \right) \quad \text{rotary part} \\ &+ m^B \left([T]^{BA}[S_{BA}]^A[\bar{T}]^{BA} + [T]^{BR}[S_{AR}]^R[\bar{T}]^{BR} \right) [T]^{BR}[\Omega^{AR}]^R[\bar{T}]^{BR}[T]^{BA}[s_{BA}]^A \quad \text{point - mass part} \end{aligned}$$

Coordinate systems derived from the orange-peel diagram

$$[T]^{AR} = \begin{bmatrix} \cos\varepsilon & \sin\varepsilon & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, [T]^{XA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{bmatrix}, [T]^{YX} = \begin{bmatrix} \cos\eta & \sin\eta & 0 \\ -\sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]^{BY} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

The MOI and mass

$$[I_B^B]^B = \begin{bmatrix} m\,r^2/2 & 0 & 0 \\ 0 & m\,(h^2+3r^2) & 0 \\ 0 & 0 & m\,(h^2+3r^2) \end{bmatrix} \;,\; m = \pi r^2 h d$$

Angular velocities

$$[\overline{\omega^{BA}}]^A = \begin{bmatrix} \dot{\phi} & 0 & 0 \end{bmatrix}, [\overline{\omega^{AR}}]^R = \begin{bmatrix} \dot{\delta} & 0 & 0 \end{bmatrix}$$

My SCILAB results of the angular momenta while the chair is making one orbit

```
"ONE REVOLUTION (data entry every second)”
 " Magnitude of rotary angular momentum - kgm^2/s"
  19979.142
  19364.432
  18575.117
  17909.820
  17648.468
  17908.875
  18573.645
  19363.022
  19978.299
  20208.766
  19979.984
 "Magnitude of point-mass angular momentum - kgm^2/s"
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
  18229.788
 "Angular momentum of spinning seat only - kgm^2/s"
  9361.8977
```

Comments: The rotary and point-mass contributions to the total angular momentum are almost of equal value and twice the size of the chair’s only angular momentum. As expected, there is no change in the point-mass angular momentum during the orbit.