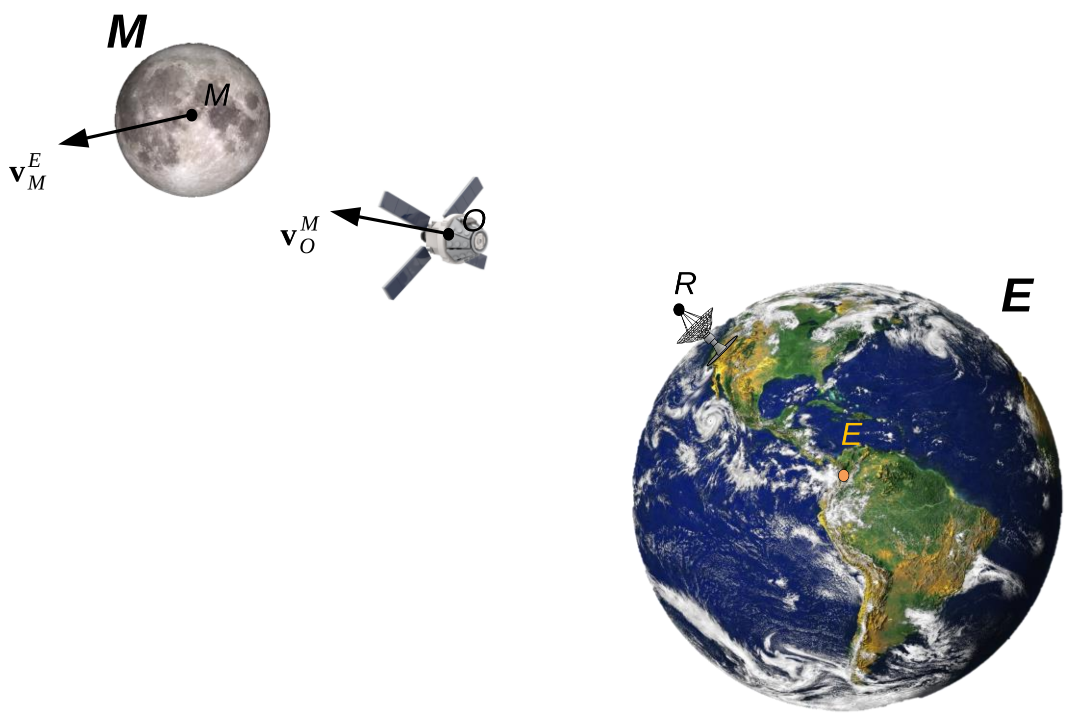
Solutions

Modeling Flight Dynamics with Tensors

Lecture 5

Problem 1 Orion Velocity

Derive the relative velocity $\mathbf{v}_O^M$ of the Orion wrt the Moon, given the radar measured velocity $\mathbf{v}_O^R$ and the displacement vector $\mathbf{s}_{OE}$ of the Orion. Velocities and positions of Moon and Earth are given by the Celestial Almanac.



Solution

By definition $\mathbf{v}_O^M = D^M \mathbf{s}_{OM}$. I break up the position vector into three parts

$$\mathbf{s}_{OM} = \mathbf{s}_{OR} + \mathbf{s}_{RE} + \mathbf{s}_{EM}$$

And subject them to the rotational time derivative

$$D^M \mathbf{s}_{OM} = D^M \mathbf{s}_{OR} + D^M \mathbf{s}_{RE} + D^M \mathbf{s}_{EM}$$

Then shift each derivative to the appropriate reference frame one term at-a-time, using the Euler Transformation

Term1 $D^M \mathbf{s}_{OR} = D^R \mathbf{s}_{OR} + \mathbf{\Omega}^{RM} \mathbf{s}_{OR} = D^R \mathbf{s}_{OR} + \mathbf{\Omega}^{EM} \mathbf{s}_{OR} = \mathbf{v}_O^R + \mathbf{\Omega}^{EM} \mathbf{s}_{OR}$ -> radar is part of the Earth frame ***R***=***E***

Term 2 $D^M \mathbf{s}_{RE} = D^E \mathbf{s}_{RE} + \mathbf{\Omega}^{EM} \mathbf{s}_{RE} = \mathbf{v}_R^E + \mathbf{\Omega}^{EM} \mathbf{s}_{RE} = \mathbf{\Omega}^{EM} \mathbf{s}_{RE}$ -> radar is fixed in the Earth frame

Term 3 $D^M \mathbf{s}_{EM} = \mathbf{v}_E^M$

Collecting the terms

$$\mathbf{v}_O^M = \mathbf{v}_O^R + \mathbf{\Omega}^{EM} \mathbf{s}_{OR} + \mathbf{\Omega}^{EM} \mathbf{s}_{RE} + \mathbf{v}_E^M$$

And the result

$$\boxed{\mathbf{v}_O^M = \mathbf{v}_O^R + \mathbf{v}_E^M + \mathbf{\Omega}^{EM} \mathbf{s}_{OE}}$$

If you want to express the result in terms of Moon wrt Earth-like terms $\mathbf{v}_M^E$, $\mathbf{\Omega}^{ME}$, reformulate

$$\mathbf{v}_E^M = D^M \mathbf{s}_{EM} = -D^M \mathbf{s}_{ME} = -D^E \mathbf{s}_{ME} - \mathbf{\Omega}^{EM} \mathbf{s}_{ME} = -\mathbf{v}_M^E + \mathbf{\Omega}^{ME} \mathbf{s}_{ME}$$

And substitute

$$\begin{aligned}\mathbf{v}_O^M &= \mathbf{v}_O^R - \mathbf{v}_M^E - \mathbf{\Omega}^{EM} \mathbf{s}_{ME} + \mathbf{\Omega}^{EM} \mathbf{s}_{OE} \\ &= \mathbf{v}_O^R - \mathbf{v}_M^E + \mathbf{\Omega}^{EM} \mathbf{s}_{OE} + \mathbf{\Omega}^{EM} \mathbf{s}_{EM}\end{aligned}$$

With the result now Earth related

$$\mathbf{v}_O^M = \mathbf{v}_O^R - \mathbf{v}_M^E - \mathbf{\Omega}^{ME} \mathbf{s}_{MO}$$

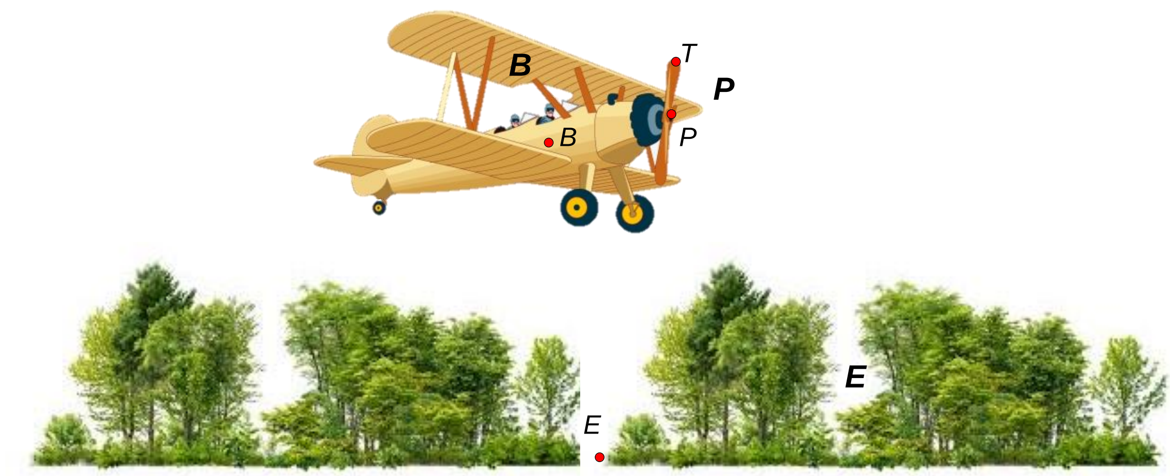
But now we need to know $\mathbf{s}_{MO}$

Problem 2 Propeller Tip-Speed

Calculate the speed $\mathbf{v}_T^E$ at the tip of the propeller to verify that it does not exceed sonic speed. The three frames ***E, B, P***, the four points *E*, c.m. *B, P, T*, and the three coordinate systems $]^E$, $]^B$, $]^P$ are used in the derivation.

First derive the tensor relationship for $\mathbf{v}_T^E$ given $\mathbf{v}_B^E, \boldsymbol{\omega}^{BE}, \boldsymbol{\omega}^{PB}, \mathbf{s}_{TP}, \mathbf{s}_{PB}$

Then compute the tip-speed for the aircraft making a barrel roll with speed *V*=100 *m/s*, roll rate $\dot{\phi}=1$ *rad/sec*, pitch-up rate $\dot{\theta}=0.1$ *rad/sec* (starting at horizontal and level) and the propeller's max angular velocity $\omega = 3000\ RPM$. The geometrical distances are $[\overline{s_{TP}}]^P = [0 \quad 1 \quad 0]\, m$; $[\overline{s_{PB}}]^B = [3 \quad 0 \quad 0]\, m$. Fly the aircraft for 10 seconds.



Solution

Tensor Formulation

For most kinematic problem I start with a displacement vector sequence of the points involved. The tip-speed is related to the displacement vector by $\mathbf{v}_T^E = D^E \mathbf{s}_{TE}$ Dividing up the displacement vector between the points $\mathbf{s}_{TE} = \mathbf{s}_{TP} + \mathbf{s}_{PB} + \mathbf{s}_{BE}$ and using it in the velocity expression $\mathbf{v}_T^E = D^E \mathbf{s}_{TE} = D^E \mathbf{s}_{TP} + D^E \mathbf{s}_{PB} + D^E \mathbf{s}_{BE}$

First term: $D^E \mathbf{s}_{TP} = D^B \mathbf{s}_{TP} + \boldsymbol{\Omega}^{BE} \mathbf{s}_{TP} = \underbrace{D^P \mathbf{s}_{TP}}_{\mathbf{v}_T^P = 0} + \boldsymbol{\Omega}^{PB} \mathbf{s}_{TP} + \boldsymbol{\Omega}^{BE} \mathbf{s}_{TP} = \boldsymbol{\Omega}^{PE} \mathbf{s}_{TP}$ -> $\mathbf{v}_T^P = \mathbf{0}$ because the propeller tip *T* is part of the propeller frame.

Second term: $D^E \mathbf{s}_{PB} = \underbrace{D^B \mathbf{s}_{PB}}_{\mathbf{v}_P^B = 0} + \boldsymbol{\Omega}^{BE} \mathbf{s}_{PB}$ -> $\mathbf{v}_P^B = \mathbf{0}$ because the propeller hub *P* is also part of the body frame ***B***.

Third term: $D^E \mathbf{s}_{BE} = \mathbf{v}_B^E$

Collecting terms $\boxed{\mathbf{v}_T^E = \mathbf{\Omega}^{PE}\mathbf{s}_{TP} + \mathbf{\Omega}^{BE}\mathbf{s}_{PB} + \mathbf{v}_B^E}$ where $\mathbf{\Omega}^{PE}\mathbf{s}_{TP} = \mathbf{\Omega}^{PB}\mathbf{s}_{TP} + \mathbf{\Omega}^{BE}\mathbf{s}_{TP}$

Matrix Formulation

First express the tensor relationship in Earth coordinates

$$[v_T^E]^E = [\Omega^{PE}]^E[s_{TP}]^E + [\Omega^{BE}]^E[s_{PB}]^E + [v_B^E]^E$$

Then convert to the appropriate coordinate systems

First Term: $[\Omega^{PE}]^E[s_{TP}]^E = \left([T]^{EB}[\Omega^{PB}]^B[\bar{T}]^{EB} + [\Omega^{BE}]^E\right)[T]^{EP}[s_{TP}]^P$

Second Term: $[\Omega^{BE}]^E[s_{PB}]^E = [\Omega^{BE}]^E[T]^{EB}[s_{PB}]^B$

Third Term: $[v_B^E]^E$ no change

Computation:

$$[v_T^E]^E = \left([T]^{EB}[\Omega^{PB}]^B[\bar{T}]^{EB} + [\Omega^{BE}]^E\right)[T]^{EP}[s_{TP}]^P + [\Omega^{BE}]^E[T]^{EB}[s_{PB}]^B + [v_B^E]^E$$

$$M_T = \frac{\left|[v_T^E]^E\right|}{a}$$

Details:

$$[T]^{BE} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta \\ \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}; \quad \begin{matrix} \theta = 10^o \\ \phi = \dot{\phi}\cdot t \end{matrix}$$

$$[T]^{PE} = [T]^{PB}[T]^{BE} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\sigma & \sin\sigma \\ 0 & -\sin\sigma & \cos\sigma \end{bmatrix}[T]^{BE}, \ \sigma = \omega\cdot t$$

$$[\Omega^{PB}]^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

$$[\omega]^B = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{bmatrix}; [\omega]^E = \overline{[T]}^{BE}[\omega]^B = \begin{bmatrix} \omega_1^E \\ \omega_2^E \\ \omega_3^E \end{bmatrix} \to [\Omega^{BE}]^E = \begin{bmatrix} 0 & -\omega_3^E & \omega_2^E \\ \omega_3^E & 0 & -\omega_1^E \\ -\omega_2^E & \omega_1^E & 0 \end{bmatrix}$$

$$[s_{TP}]^P = \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}; [s_{PB}]^B = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}; [v_B^E]^E = V \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix}$$

$$\omega = 3000 \ \ RPM = 314.1 rad/\sec$$

Now I go to SCILAB to program the matrices and run for 5 sec with the result:

Time - sec	Tip Mach
0	0.9706184
1	0.9706146
2	0.9706138
3	0.9706159
4	0.9706210
5	0.9706291

Comment: This is another example of my motto: "From tensor modeling to matrix coding". The tip-speed is near sonic with local shock waves. So, the pilot should lower the RPM of the propeller (3000 *RPM* anyway is extreme). Note, there is a small dependency on pitch rate.

Problem 3 Matrix Manipulation

Slide 14 uses the result $\left[\frac{dT}{dt}\right]^{BA} = [\overline{\Omega^{BA}}]^B [T]^{BA}$ without details. As an exercise you are to derive it from $[T]^{BA}\overline{\left[\frac{dT}{dt}\right]}^{BA} = [\Omega^{BA}]^B$.

Solution

Pre-multiply by $[\bar{T}]^{BA}$: $\underbrace{[\bar{T}]^{BA}[T]^{BA}}_{[E]}\overline{\left[\frac{dT}{dt}\right]}^{BA} = [\bar{T}]^{BA}[\Omega^{BA}]^B$

Apply the transposed to both sides $\left[\frac{dT}{dt}\right]^{BA} = \overline{[\Omega^{BA}]}^B [T]^{BA}$ *QED*