Electrodynamics

SECTION

Time-dependence

LECTURE

Maxwell’s equations

BÖRGE GÖBEL

Maxwell‘s equations

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}$$

- Charges are sources of the electric field
- The magnetic field as no sources (no monopoles)
- Time-dependent magnetic fields generate electric fields
- Time-dependent electric fields and currents generate magnetic fields

Vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{N}}{\mathrm{A}^2}$

Vacuum permittivity $\epsilon_0 = 8.854... \cdot 10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}}$

Maxwell's equations

$$\nabla \cdot \boldsymbol{B} = 0$$ ... Like in magnetostatics

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ ... different

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r})$$

Vectorpotential $\boldsymbol{A}$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) = 0$$

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A}\right) = 0$$

Electrostatic potential φ

$$\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A} = -\nabla \varphi$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla \varphi$$

Four-potential

Maxwell's equations

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \left(-\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi\right) = \frac{1}{\epsilon_0}\rho$$

$$-\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} - \Delta\varphi = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi\right) + \mu_0\boldsymbol{j}$$

Identity $\nabla \times (\nabla \times \boldsymbol{A}) = -\Delta\boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A})$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A} - [\Delta\boldsymbol{A} - \nabla(\nabla \cdot \boldsymbol{A})] + \epsilon_0\nabla\frac{\partial\varphi}{\partial t} = \mu_0\boldsymbol{j}$$

New operator $\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$

$$\square\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\square\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}$$

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

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Lorenz gauge

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$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}$$

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

Lorentz gauge

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

- Both potentials are no observables (can be gauged)

$$\nabla\cdot\boldsymbol{B} = 0$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$ not unique

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A} = 0$$ Lorenz gauge

Lorentz gauge

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

- Both potentials are no observables (can be gauged)

$$\boldsymbol{A} \rightarrow \boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$$

$$\varphi \rightarrow \varphi' = \varphi - \frac{\partial}{\partial t}\chi$$

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A} = 0$$

Lorenz gauge

$$\frac{1}{c^2}\frac{\partial\varphi'}{\partial t} + \nabla\times\boldsymbol{A}' = \left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\times\boldsymbol{A}\right) + \left(\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\chi + \nabla\times\nabla\chi\right) = 0$$

if χ chosen accordingly

$$\begin{aligned}\boldsymbol{E}' &= -\frac{\partial}{\partial t}\boldsymbol{A}' - \nabla\varphi' \\ &= -\frac{\partial}{\partial t}\boldsymbol{A} - \frac{\partial}{\partial t}\nabla\chi - \nabla\varphi + \nabla\frac{\partial}{\partial t}\chi \\ &= -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi \\ \boldsymbol{E}' &= \boldsymbol{E}\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}' &= \nabla\times\boldsymbol{A}' \\ &= \nabla\times\boldsymbol{A} + \nabla\times\nabla\chi \\ \boldsymbol{B}' &= \boldsymbol{B}\end{aligned}$$

Maxwell's equations

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j}$$

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \mu_0 \boldsymbol{j}$$

Lorenz gauge

$$\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0$$

$$\Box\varphi = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} = \mu_0 \boldsymbol{j}$$

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Retarded potentials

Maxwell's equations

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}$$

→

$$\Box\varphi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} + \nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \mu_0\boldsymbol{j}$$

→ Lorenz gauge

$$\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0$$

$$\Box\varphi = \frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Box\boldsymbol{A} = \mu_0\boldsymbol{j}$$

Retarded potentials

$$\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$$

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi$$

$$\Delta\boldsymbol{A} = -\mu_0\boldsymbol{j} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\Delta\varphi = -\frac{1}{\epsilon_0}\rho$$

$$\boldsymbol{E} = -\nabla\varphi$$

Electrostatics

$$\Delta\boldsymbol{A} = -\mu_0\boldsymbol{j}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$(\nabla\cdot\boldsymbol{A} = 0)$$

Magnetostatics

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

Retarded potentials

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\boldsymbol{r}',t\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}\left(\boldsymbol{r}',t\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

Retarded potentials

Verify $$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$ solves $$\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' \qquad \Delta\varphi = \nabla\cdot\nabla\varphi$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\int \nabla\left(\frac{\rho(\boldsymbol{r}', t - R/c)}{R}\right)\mathrm{d}V'$$

$$= \frac{1}{4\pi\epsilon_0}\int\left(-\frac{\rho(\boldsymbol{r}', t-R/c)\nabla R}{R^2} + \frac{\nabla\rho(\boldsymbol{r}', t-R/c)}{R}\right)\mathrm{d}V'$$

$$\nabla\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{\dot\rho}{c}\frac{\boldsymbol{R}}{R^2} + \rho\frac{\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\nabla R = \nabla\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$$

$$= \frac{1}{2}\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}\cdot 2\begin{pmatrix} x-x' \\ y-y' \\ z-z'\end{pmatrix}$$

$$\nabla R = \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\boldsymbol{R}}{R}$$

$$\nabla\rho(\boldsymbol{r}', t_{\mathrm{ret}}) = \frac{\partial\rho}{\partial t_{\mathrm{ret}}}\nabla t_{\mathrm{ret}} = \dot\rho\cdot\nabla(t - R/c) = \dot\rho\frac{(-1)}{c}\nabla R = -\frac{\dot\rho}{c}\frac{\boldsymbol{R}}{R}$$

Retarded potentials

Verify $\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ solves $\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$

$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ $\Delta\varphi = \nabla\cdot\nabla\varphi$

$$\nabla\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{\dot\rho}{c}\frac{\boldsymbol{R}}{R^2} + \rho\frac{\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = \frac{1}{c}\dot\rho\nabla\cdot\frac{\boldsymbol{R}}{R^2} + \frac{1}{c}\frac{\boldsymbol{R}}{R^2}\cdot\nabla\dot\rho + \rho\nabla\cdot\frac{\boldsymbol{R}}{R^3} + \frac{\boldsymbol{R}}{R^3}\cdot\nabla\rho$$

$$= \frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} + \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} + \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho + \frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} + \boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} + \boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho$$

Retarded potentials

Verify $\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ solves $\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$

$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ $\qquad \Delta\varphi = \nabla\cdot\nabla\varphi$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = \frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} + \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} + \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho + \frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} + \boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} + \boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho$$

$$\frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} = 3\frac{\dot\rho}{cR^2}$$

$$\frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} = \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\frac{(-2)}{R^3}\nabla R = -2\frac{1}{c}\boldsymbol{R}\dot\rho\cdot\frac{1}{R^3}\frac{\boldsymbol{R}}{R} = -2\frac{\dot\rho}{cR^2}$$

$$\frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho = \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\frac{-\ddot\rho}{c}\frac{\boldsymbol{R}}{R} = -\frac{\ddot\rho}{c^2R}$$

$$\frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} = 3\frac{\rho}{R^3}$$

$$\boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} = \boldsymbol{R}\rho\cdot\frac{-3}{R^4}\nabla R = \boldsymbol{R}\rho\cdot\frac{-3}{R^4}\frac{\boldsymbol{R}}{R} = -3\frac{\rho}{R^3}$$

$$\boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho = \boldsymbol{R}\frac{1}{R^3}\cdot\frac{-\dot\rho}{c}\frac{\boldsymbol{R}}{R} = -\frac{\dot\rho}{cR^2}$$

Retarded potentials

Verify $$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$ solves $$\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' \qquad \Delta\varphi = \nabla\cdot\nabla\varphi$$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = -\frac{\ddot\rho}{c^2R}$$ if $R \neq 0$

$$\Delta\varphi = \frac{1}{4\pi\epsilon_0}\int\frac{\ddot\rho}{c^2R}\,\mathrm{d}V' + \ldots$$

$$\Delta\varphi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r},t-R/c)}{R}\,\mathrm{d}V\right) + \ldots$$

$$\Delta\varphi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(\boldsymbol{r},t) + \ldots$$

... Good. But where is $-\frac{1}{\epsilon_0}\rho$?

Our derivation is only valid for

$R \neq 0 \qquad \boldsymbol{r} \neq \boldsymbol{r}'$

Retarded potentials

Verify $\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ solves $\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$

$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ $\qquad \Delta\varphi = \nabla\cdot\nabla\varphi$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = \frac{1}{c}\dot\rho\nabla\cdot\frac{\boldsymbol{R}}{R^2} + \frac{1}{c}\frac{\boldsymbol{R}}{R^2}\cdot\nabla\dot\rho + \boxed{\rho\nabla\cdot\frac{\boldsymbol{R}}{R^3}} + \frac{\boldsymbol{R}}{R^3}\cdot\nabla\rho$$

$$= \frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} + \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} + \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho + \frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} + \boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} + \boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho \quad + \ldots$$

Gauss's theorem $\int\nabla\cdot\frac{\boldsymbol{R}}{R^3}\,\mathrm{d}V' = \oint_{\partial V}\frac{\boldsymbol{R}}{R^3}\cdot\mathrm{d}\boldsymbol{S}$

$$= \lim_{R\to\infty}\oint_{\partial(\text{sphere})_R}\frac{\boldsymbol{R}}{R^3}\cdot\boldsymbol{e}_R R^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi = \lim_{R\to\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi = 2\pi[-\cos\theta]_0^{\pi} = 4\pi$$

Retarded potentials

Verify $\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ solves $\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$

$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ $\qquad \Delta\varphi = \nabla\cdot\nabla\varphi$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = \frac{1}{c}\dot\rho\nabla\cdot\frac{\boldsymbol{R}}{R^2} + \frac{1}{c}\frac{\boldsymbol{R}}{R^2}\cdot\nabla\dot\rho + \boxed{\rho\nabla\cdot\frac{\boldsymbol{R}}{R^3}} + \frac{\boldsymbol{R}}{R^3}\cdot\nabla\rho$$

$$= \frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} + \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} + \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho + \frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} + \boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} + \boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho \quad + \ldots$$

$$\int\nabla\cdot\frac{\boldsymbol{R}}{R^3}\,\mathrm{d}V' = \int 4\pi\delta(\boldsymbol{R})\mathrm{d}V = 4\pi$$

$$\nabla\cdot\frac{\boldsymbol{R}}{R^3} = 4\pi\delta(\boldsymbol{R}) = \boxed{4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')}$$

Retarded potentials

Verify $\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ solves $\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$

$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ $\Delta\varphi = \nabla\cdot\nabla\varphi$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = \frac{1}{c}\dot\rho\nabla\cdot\frac{\boldsymbol{R}}{R^2} + \frac{1}{c}\frac{\boldsymbol{R}}{R^2}\cdot\nabla\dot\rho + \rho\nabla\cdot\frac{\boldsymbol{R}}{R^3} + \frac{\boldsymbol{R}}{R^3}\cdot\nabla\rho$$

$$= \frac{1}{c}\frac{\dot\rho}{R^2}\nabla\cdot\boldsymbol{R} + \frac{1}{c}\boldsymbol{R}\dot\rho\cdot\nabla\frac{1}{R^2} + \frac{1}{c}\boldsymbol{R}\frac{1}{R^2}\cdot\nabla\dot\rho + \frac{\rho}{R^3}\nabla\cdot\boldsymbol{R} + \boldsymbol{R}\rho\cdot\nabla\frac{1}{R^3} + \boldsymbol{R}\frac{1}{R^3}\cdot\nabla\rho + \boxed{4\pi\delta(\boldsymbol{R})\rho}$$

$$\int\nabla\cdot\frac{\boldsymbol{R}}{R^3}\,\mathrm{d}V' = \int 4\pi\delta(\boldsymbol{R})\mathrm{d}V = 4\pi$$

$$\nabla\cdot\frac{\boldsymbol{R}}{R^3} = 4\pi\delta(\boldsymbol{R}) = \boxed{4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')}$$

Delta distribution

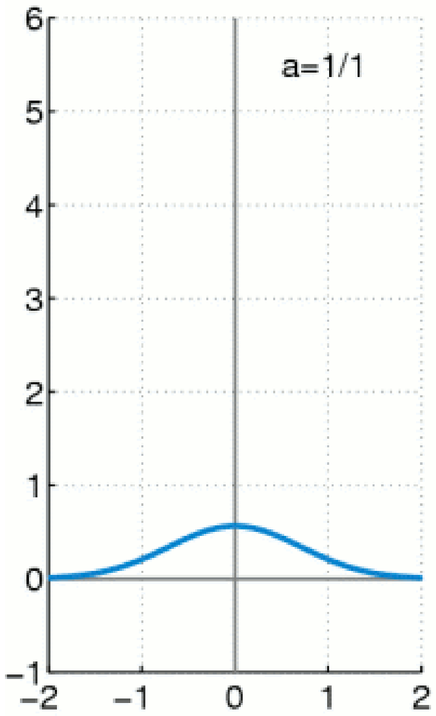
- Continuous version $$\delta(x-x_0)=\lim_{a\to 0}\frac{1}{|a|\sqrt{\pi}}e^{-\left(\frac{x-x_0}{a}\right)^2}$$

... not the strict definition

- Acts in integrals $$\int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\delta(x-x_0)=f(x_0)$$

Here $$\int \nabla\cdot\frac{\boldsymbol{R}}{R^3}\mathrm{d}V'=\int 4\pi\delta(\boldsymbol{R})\mathrm{d}V=4\pi$$

$$\nabla\cdot\frac{\boldsymbol{R}}{R^3}=4\pi\delta(\boldsymbol{R})=4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$$



https://upload.wikimedia.org/wikipedia/commons/b/b4/Dirac_function_approximation.gif

Retarded potentials

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

solves

$$\Delta\varphi = -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$$

✓

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' \qquad \Delta\varphi = \nabla\cdot\nabla\varphi$$

$$\Delta\varphi = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3}\right)\mathrm{d}V'$$

$$\frac{1}{c}\nabla\cdot\frac{\dot\rho\boldsymbol{R}}{R^2} + \nabla\cdot\frac{\rho\boldsymbol{R}}{R^3} = -\frac{\ddot\rho}{c^2R} + 4\pi\delta(\boldsymbol{R})\rho$$

$$\Delta\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\ddot\rho(\boldsymbol{r}',t-R/c)}{c^2R}\,\mathrm{d}V' - \frac{1}{4\pi\epsilon_0}\int 4\pi\rho\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\delta(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}V'$$

$$\Delta\varphi(\boldsymbol{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(\boldsymbol{r},t) - \frac{1}{\epsilon_0}\rho(\boldsymbol{r},t)$$

… vector potential $\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$ very similar

Electrodynamics

SECTION

Time-dependence

LECTURE

Hertzian dipole

Electric and magnetic dipole

magnetic

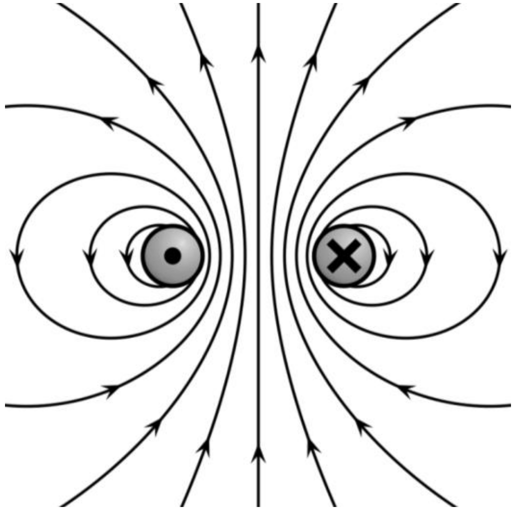
$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{m}r^2}{r^5}$$

$$\boldsymbol{m} = \frac{1}{2} I \oint_{\text{loop}} (\boldsymbol{r}' \times \mathrm{d}\boldsymbol{r}') = I \cdot A\boldsymbol{n}$$

electric

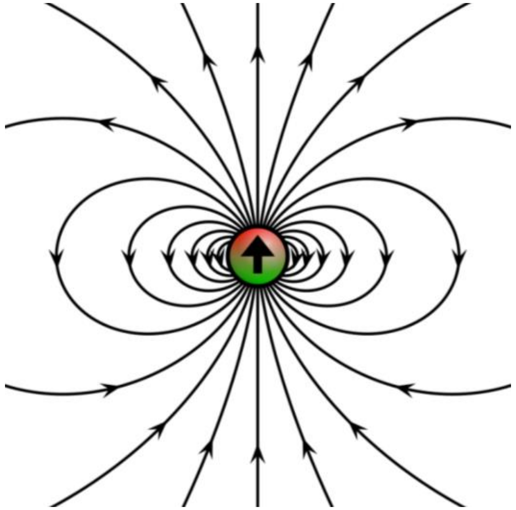
$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}$$

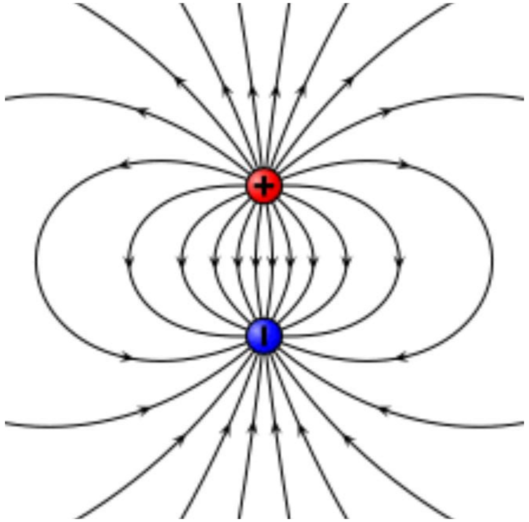
$$\boldsymbol{p} = Q\boldsymbol{a}$$



zoom out

$|\boldsymbol{r} - \boldsymbol{r}'| \gg a$

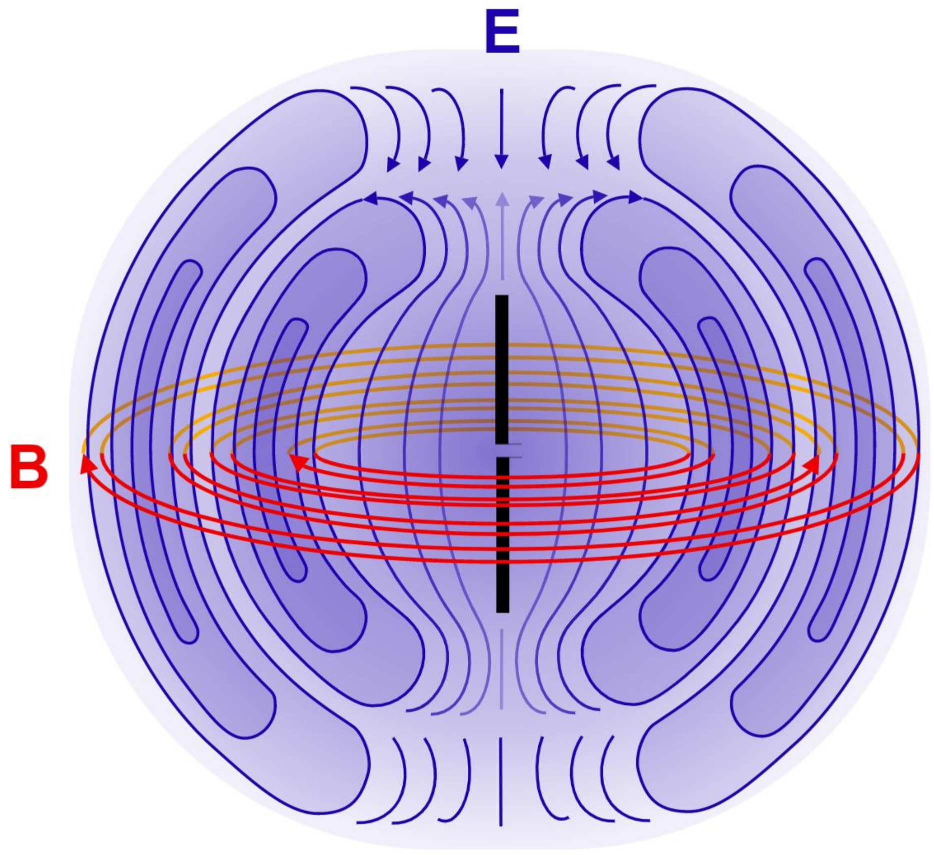




https://upload.wikimedia.org/wikipedia/commons/7/76/VFPt_dipoles_magnetic.svg

https://upload.wikimedia.org/wikipedia/commons/a/aa/VFPt_dipole_animation_electric.gif

Hertzian dipole



https://upload.wikimedia.org/wikipedia
/commons/c/c5/Felder_um_Dipol.svg

Tiny segment of an antenna

- Electric dipole moment & current density

$$\boldsymbol{j}(\boldsymbol{r},t) = Q\dot{\boldsymbol{a}}\delta(\boldsymbol{r})$$
$$\boldsymbol{j}(\boldsymbol{r},t) = \dot{\boldsymbol{p}}\delta(\boldsymbol{r})$$

$$\int \dot{\boldsymbol{p}}\delta(\boldsymbol{r})\,\mathrm{d}V = \int \boldsymbol{j}(\boldsymbol{r},t)\,\mathrm{d}V$$
$$\dot{\boldsymbol{p}} = \boldsymbol{J}(t)$$

Hertzian dipole

- current density $\boldsymbol{j}(\boldsymbol{r},t) = \dot{\boldsymbol{p}}\delta(\boldsymbol{r})$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\dot{\boldsymbol{p}}\left(t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\delta(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}$$

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A} = 0$$

Lorenz gauge

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= -c^2\frac{\mu_0}{4\pi}\nabla\cdot\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}\\ &= -c^2\frac{\mu_0}{4\pi}\left(\frac{1}{r}\nabla\cdot\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right) + \dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\nabla\cdot\frac{1}{r}\right)\\ &= -\frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\ddot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\cdot\nabla\left[t-\frac{r}{c}\right] - \dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\cdot\frac{\boldsymbol{r}}{r^3}\right)\\ &= -\frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\ddot{\boldsymbol{p}}\cdot\frac{(-1)}{c}\frac{\boldsymbol{r}}{r} - \dot{\boldsymbol{p}}\cdot\frac{\boldsymbol{r}}{r^3}\right)_{t'=t-\frac{r}{c}}\end{aligned}$$

$$\frac{\partial\varphi}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r}\cdot\left(\frac{1}{cr}\ddot{\boldsymbol{p}} + \frac{1}{r^2}\dot{\boldsymbol{p}}\right)_{t'=t-\frac{r}{c}}$$

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r}\cdot\left[\frac{1}{cr}\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right) + \frac{1}{r^2}\boldsymbol{p}\left(t-\frac{r}{c}\right)\right]$$

Hertzian dipole

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r}\cdot\left[\frac{1}{cr}\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r^2}\boldsymbol{p}\left(t-\frac{r}{c}\right)\right]$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}$$

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\varphi$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\nabla\times\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\nabla\times\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)-\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\times\nabla\frac{1}{r}\right]$$

$$\nabla\times\phi\boldsymbol{a} = \phi\nabla\times\boldsymbol{a} - \boldsymbol{a}\times\nabla\phi$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\ddot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\times\frac{(-1)}{c}\nabla r-\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)\times\frac{-\boldsymbol{r}}{r^3}\right]$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\ddot{\boldsymbol{p}}\times\frac{(-1)}{c}\frac{\boldsymbol{r}}{r}-\dot{\boldsymbol{p}}\times\frac{-\boldsymbol{r}}{r^3}\right]_{t-\frac{r}{c}}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r}\times\left[\frac{\ddot{\boldsymbol{p}}}{c}-\frac{\dot{\boldsymbol{p}}}{r}\right]_{t-\frac{r}{c}}$$

Hertzian dipole

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r}\cdot\left[\frac{1}{cr}\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r^2}\boldsymbol{p}\left(t-\frac{r}{c}\right)\right]$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\varphi$$

$$\dot{\boldsymbol{A}} = \frac{\mu_0}{4\pi}\frac{\ddot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r} = \frac{1}{4\pi\epsilon_0 r}\left[\frac{\ddot{\boldsymbol{p}}}{c^2}\right]_{t-\frac{r}{c}}$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\nabla\left[\frac{\boldsymbol{r}}{r}\cdot\left(\frac{1}{cr}\dot{\boldsymbol{p}}+\frac{1}{r^2}\boldsymbol{p}\right)\right]_{t-\frac{r}{c}}$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{c}\nabla\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{p}}}{r^2}+\nabla\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{r^3}\right]_{t-\frac{r}{c}}$$

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}+(\boldsymbol{b}\cdot\nabla)\boldsymbol{a}+\boldsymbol{a}\times(\nabla\times\boldsymbol{b})+\boldsymbol{b}\times(\nabla\times\boldsymbol{a})$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0 c}\left[(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2}+\left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}}+\frac{\boldsymbol{r}}{r^2}\times(\nabla\times\dot{\boldsymbol{p}})+\dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right)\right]_{t-\frac{r}{c}}$$
$$+\frac{1}{4\pi\epsilon_0}\left[(\boldsymbol{p}\cdot\nabla)\frac{\boldsymbol{r}}{r^3}+\left(\frac{\boldsymbol{r}}{r^3}\cdot\nabla\right)\boldsymbol{p}+\frac{\boldsymbol{r}}{r^3}\times(\nabla\times\boldsymbol{p})+\boldsymbol{p}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^3}\right)\right]_{t-\frac{r}{c}}$$

Let us calculate these terms

Hertzian dipole

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0 c}\left[(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} + \left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}} + \frac{\boldsymbol{r}}{r^2}\times(\nabla\times\dot{\boldsymbol{p}}) + \dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right)\right]_{t-\frac{r}{c}}$$

$$+\frac{1}{4\pi\epsilon_0}\left[(\boldsymbol{p}\cdot\nabla)\frac{\boldsymbol{r}}{r^3} + \left(\frac{\boldsymbol{r}}{r^3}\cdot\nabla\right)\boldsymbol{p} + \frac{\boldsymbol{r}}{r^3}\times(\nabla\times\boldsymbol{p}) + \boldsymbol{p}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^3}\right)\right]_{t-\frac{r}{c}}$$

Let us calculate these terms

$$(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} = \left(\dot{p}_x\frac{\partial}{\partial x} + \dot{p}_y\frac{\partial}{\partial y} + \dot{p}_z\frac{\partial}{\partial z}\right)\begin{pmatrix} x/r^2 \\ y/r^2 \\ z/r^2 \end{pmatrix}$$

$$(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} = \begin{pmatrix} \frac{1}{r^2}\dot{p}_x + \frac{x\cdot(-1)\cdot 2x}{r^4}\dot{p}_x + \frac{x\cdot(-1)\cdot 2y}{r^4}\dot{p}_y + \frac{x\cdot(-1)\cdot 2z}{r^4}\dot{p}_z \\ \frac{1}{r^2}\dot{p}_y + \frac{y\cdot(-1)\cdot 2x}{r^4}\dot{p}_x + \frac{y\cdot(-1)\cdot 2y}{r^4}\dot{p}_y + \frac{y\cdot(-1)\cdot 2z}{r^4}\dot{p}_z \\ \frac{1}{r^2}\dot{p}_z + \frac{z\cdot(-1)\cdot 2x}{r^4}\dot{p}_x + \frac{z\cdot(-1)\cdot 2y}{r^4}\dot{p}_y + \frac{z\cdot(-1)\cdot 2z}{r^4}\dot{p}_z \end{pmatrix}$$

$$(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} = \frac{\dot{\boldsymbol{p}}}{r^2} - 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{p}}}{r^4} = \frac{\dot{\boldsymbol{p}}}{r^2} - 2(\boldsymbol{r}\cdot\dot{\boldsymbol{p}})\frac{\boldsymbol{r}}{r^4}$$

Hertzian dipole

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0 c}\left[(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} + \left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}} + \frac{\boldsymbol{r}}{r^2}\times(\nabla\times\dot{\boldsymbol{p}}) + \dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right)\right]_{t-\frac{r}{c}}$$

$$+\frac{1}{4\pi\epsilon_0}\left[(\boldsymbol{p}\cdot\nabla)\frac{\boldsymbol{r}}{r^3} + \left(\frac{\boldsymbol{r}}{r^3}\cdot\nabla\right)\boldsymbol{p} + \frac{\boldsymbol{r}}{r^3}\times(\nabla\times\boldsymbol{p}) + \boldsymbol{p}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^3}\right)\right]_{t-\frac{r}{c}}$$

Let us calculate these terms

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\boldsymbol{e}_\varphi\frac{\partial}{\partial\varphi}$$

$$\frac{\boldsymbol{r}}{r}\frac{\partial}{\partial r}$$

... spherical coordinates

$$(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} = \frac{\dot{\boldsymbol{p}}}{r^2} - 2(\boldsymbol{r}\cdot\dot{\boldsymbol{p}})\frac{\boldsymbol{r}}{r^4}$$

$$\left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)} = \frac{1}{r}\frac{\partial}{\partial r}\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)} = -\frac{1}{cr}\ddot{\boldsymbol{p}}$$

$$\frac{\boldsymbol{r}}{r^2}\times\left[\nabla\times\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)}\right] = \frac{1}{r^3}\boldsymbol{r}\times\left[\boldsymbol{r}\times\frac{\partial}{\partial r}\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)}\right] = -\frac{1}{cr^3}\boldsymbol{r}\times[\boldsymbol{r}\times\ddot{\boldsymbol{p}}] = -\frac{1}{cr^3}\boldsymbol{r}(\boldsymbol{r}\cdot\ddot{\boldsymbol{p}}) + \frac{1}{cr}\ddot{\boldsymbol{p}}$$

$$\dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right) = \dot{\boldsymbol{p}}\times\left(\frac{\boldsymbol{r}}{r}\frac{\partial}{\partial r}\times\frac{\boldsymbol{r}}{r^2}\right) = 0$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

$$\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c}) = \boldsymbol{a}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}a^2$$

Hertzian dipole

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0 c}\left[(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} + \left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}} + \frac{\boldsymbol{r}}{r^2}\times(\nabla\times\dot{\boldsymbol{p}}) + \dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right)\right]_{t-\frac{r}{c}}$$
$$+\frac{1}{4\pi\epsilon_0}\left[(\boldsymbol{p}\cdot\nabla)\frac{\boldsymbol{r}}{r^3} + \left(\frac{\boldsymbol{r}}{r^3}\cdot\nabla\right)\boldsymbol{p} + \frac{\boldsymbol{r}}{r^3}\times(\nabla\times\boldsymbol{p}) + \boldsymbol{p}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^3}\right)\right]_{t-\frac{r}{c}}$$

Let us calculate these terms

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\boldsymbol{e}_\varphi\frac{\partial}{\partial\varphi}$$

$$(\dot{\boldsymbol{p}}\cdot\nabla)\frac{\boldsymbol{r}}{r^2} = \frac{\dot{\boldsymbol{p}}}{r^2} - 2(\boldsymbol{r}\cdot\dot{\boldsymbol{p}})\frac{\boldsymbol{r}}{r^4}$$

$$\left(\frac{\boldsymbol{r}}{r^2}\cdot\nabla\right)\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)} = -\frac{1}{cr}\ddot{\boldsymbol{p}}$$

$$\frac{\boldsymbol{r}}{r^2}\times\left[\nabla\times\dot{\boldsymbol{p}}_{\left(t-\frac{r}{c}\right)}\right] = -\frac{1}{cr^3}\boldsymbol{r}(\boldsymbol{r}\cdot\ddot{\boldsymbol{p}}) + \frac{1}{cr}\ddot{\boldsymbol{p}}$$

$$\dot{\boldsymbol{p}}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^2}\right) = 0$$

$$(\boldsymbol{p}\cdot\nabla)\frac{\boldsymbol{r}}{r^3} = \frac{\boldsymbol{p}}{r^3} - 3\frac{1}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}$$

$$\left(\frac{\boldsymbol{r}}{r^3}\cdot\nabla\right)\boldsymbol{p}_{\left(t-\frac{r}{c}\right)} = -\frac{1}{cr^2}\dot{\boldsymbol{p}}$$

$$\frac{\boldsymbol{r}}{r^3}\times\left[\nabla\times\boldsymbol{p}_{\left(t-\frac{r}{c}\right)}\right] = -\frac{1}{cr^4}\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{p}}) + \frac{1}{cr^2}\dot{\boldsymbol{p}}$$

$$\boldsymbol{p}\times\left(\nabla\times\frac{\boldsymbol{r}}{r^3}\right) = 0$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\left[-\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} + \frac{\boldsymbol{p}}{r^3} + \frac{\dot{\boldsymbol{p}}}{cr^2}\right]$$

Hertzian dipole

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r}\cdot\left[\frac{1}{cr}\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r^2}\boldsymbol{p}\left(t-\frac{r}{c}\right)\right]$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r}$$

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\left[-\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}-\frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}-\frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}+\frac{\boldsymbol{p}}{r^3}+\frac{\dot{\boldsymbol{p}}}{cr^2}\right]_{t-\frac{r}{c}}$$

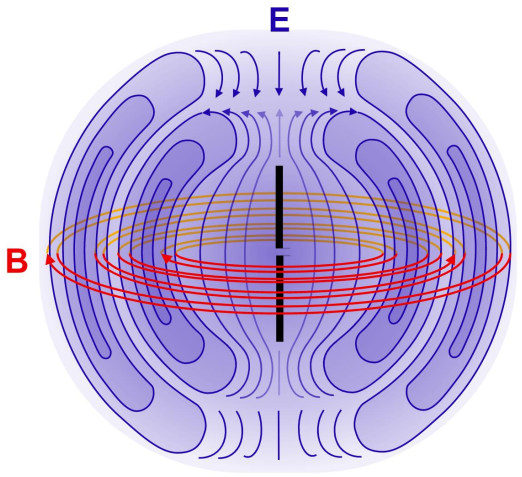
$$\dot{\boldsymbol{A}} = \frac{\mu_0}{4\pi}\frac{\ddot{\boldsymbol{p}}\left(t-\frac{r}{c}\right)}{r} = \frac{1}{4\pi\epsilon_0 r}\left[\frac{\ddot{\boldsymbol{p}}}{c^2}\right]_{t-\frac{r}{c}}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r}\times\left[\frac{\ddot{\boldsymbol{p}}}{c}-\frac{\dot{\boldsymbol{p}}}{r}\right]_{t-\frac{r}{c}}$$

$$\boldsymbol{E} = -\dot{\boldsymbol{A}}-\nabla\varphi$$

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}-\frac{\boldsymbol{p}}{r^3}+\frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}-\frac{\dot{\boldsymbol{p}}}{cr^2}+\frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}-\frac{\ddot{\boldsymbol{p}}}{c^2r}\right]_{t-\frac{r}{c}}$$



https://upload.wikimedia.org/wikipedia/commons/c/c5/Felder_um_Dipol.svg

Electrodynamics

SECTION

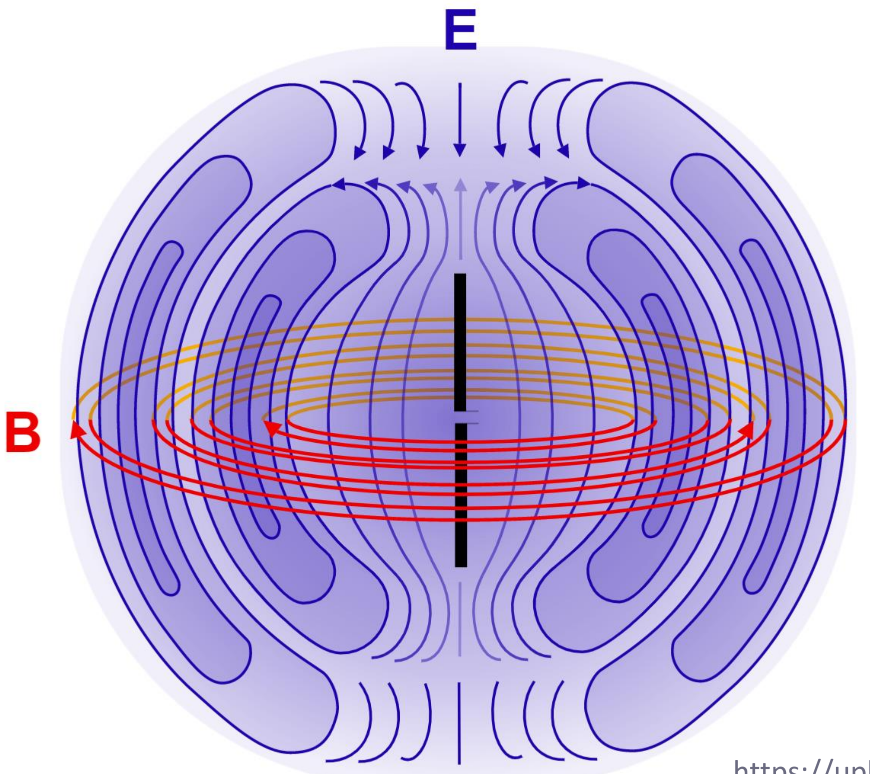
Time-dependence

LECTURE

Discussion of Hertzian dipole

BÖRGE GÖBEL

Hertzian dipole



https://upload.wikimedia.org/wikipedia/commons/c/c5/Felder_um_Dipol.svg

Tiny segment of an antenna

- Electric dipole moment & current density

$$\boldsymbol{j}(\boldsymbol{r},t) = Q\dot{\boldsymbol{a}}\delta(\boldsymbol{r})$$
$$\boldsymbol{j}(\boldsymbol{r},t) = \dot{\boldsymbol{p}}\delta(\boldsymbol{r})$$
$$\int \dot{\boldsymbol{p}}\delta(\boldsymbol{r})\,\mathrm{d}V = \int \boldsymbol{j}(\boldsymbol{r},t)\,\mathrm{d}V$$
$$\dot{\boldsymbol{p}} = \boldsymbol{J}(t)$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r} \times \left[\frac{\ddot{\boldsymbol{p}}}{c} - \frac{\dot{\boldsymbol{p}}}{r}\right]_{t-\frac{r}{c}}$$

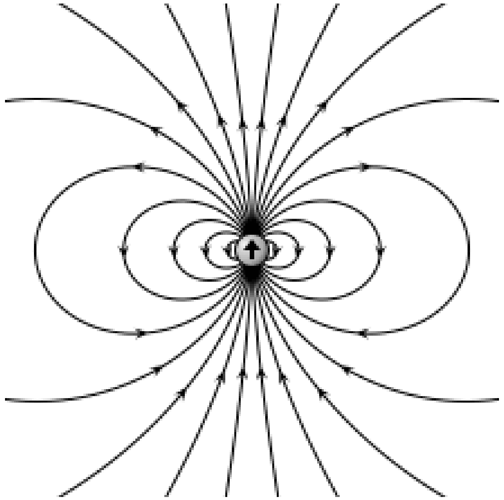
$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3} + \frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\dot{\boldsymbol{p}}}{cr^2} + \frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\ddot{\boldsymbol{p}}}{c^2r}\right]_{t-\frac{r}{c}}$$

Hertzian dipole

- For time-independent dipole moments $\dot{\boldsymbol{p}} = 0$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r} \times \left[\cancel{\frac{\ddot{\boldsymbol{p}}}{c}} - \cancel{\frac{\dot{\boldsymbol{p}}}{r}}\right]_{t-\frac{r}{c}}$$

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3} + \cancel{\frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}} - \cancel{\frac{\dot{\boldsymbol{p}}}{cr^2}} + \cancel{\frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r}} - \cancel{\frac{\ddot{\boldsymbol{p}}}{c^2r}}\right]_{t-\frac{r}{c}}$$



$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}$$

https://upload.wikimedia.org/wikipedia/commons/a/aa/VFPt_dipole_animation_electric.gif

Hertzian dipole

- Harmonically oscillating dipole

$$\boldsymbol{p} = \boldsymbol{p}_0\, e^{-i\omega t}$$
$$\dot{\boldsymbol{p}} = -i\omega \boldsymbol{p}_0\, e^{-i\omega t} = -i\omega \boldsymbol{p}$$
$$\ddot{\boldsymbol{p}} = -\omega^2 \boldsymbol{p}_0\, e^{-i\omega t} = -\omega^2 \boldsymbol{p}$$

$$k = \frac{\omega}{c} \qquad k = \frac{2\pi}{\lambda}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r} \times \left[\frac{\ddot{\boldsymbol{p}}}{c} - \frac{\dot{\boldsymbol{p}}}{r}\right]_{t-\frac{r}{c}}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r} \times \left(-\frac{\omega^2 \boldsymbol{p}}{c} + i\omega\frac{\boldsymbol{p}}{r}\right)_{t-\frac{r}{c}}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\frac{\boldsymbol{r}}{r} \times \left(-k + \frac{i}{r}\right) \boldsymbol{p}_0\, e^{i(kr-\omega t)}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\boldsymbol{e}_r \times \left(\frac{i}{r^2} - \frac{2\pi}{\lambda r}\right) \boldsymbol{p}_0\, e^{i(kr-\omega t)}$$

Hertzian dipole

- Harmonically oscillating dipole

$$\boldsymbol{p} = \boldsymbol{p}_0\, e^{-i\omega t}$$
$$\dot{\boldsymbol{p}} = -i\omega \boldsymbol{p}_0\, e^{-i\omega t} = -i\omega \boldsymbol{p}$$
$$\ddot{\boldsymbol{p}} = -\omega^2 \boldsymbol{p}_0\, e^{-i\omega t} = -\omega^2 \boldsymbol{p}$$

$$k = \frac{\omega}{c} \qquad k = \frac{2\pi}{\lambda}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\boldsymbol{e}_r \times \left(\frac{i}{r^2} - \frac{2\pi}{\lambda r}\right) \boldsymbol{p}_0\, e^{i(kr-\omega t)}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3} + \frac{3}{cr^4}(\dot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\dot{\boldsymbol{p}}}{cr^2} + \frac{1}{c^2r^3}(\ddot{\boldsymbol{p}}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\ddot{\boldsymbol{p}}}{c^2 r}\right]$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5} - ik\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5} - k^2\frac{(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}\right)_{t-\frac{r}{c}}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}(1-ikr) - k^2r^2\frac{(\boldsymbol{p}_0\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}\right] e^{i(kr-\omega t)}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}_0 r^2}{r^5}(1-ikr) + k^2r^2\frac{\boldsymbol{r}\times(\boldsymbol{p}_0\times\boldsymbol{r})}{r^5}\right] e^{i(kr-\omega t)}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}_0}{r^3} - 2\pi i\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}_0}{r^2\lambda} + 4\pi^2\frac{\boldsymbol{e}_r\times(\boldsymbol{p}_0\times\boldsymbol{e}_r)}{r\lambda^2}\right] e^{i(kr-\omega t)}$$

Hertzian dipole

- Harmonically oscillating dipole

$$\boldsymbol{p} = \boldsymbol{p}_0\, e^{-i\omega t}$$
$$\dot{\boldsymbol{p}} = -i\omega \boldsymbol{p}_0\, e^{-i\omega t} = -i\omega \boldsymbol{p}$$
$$\ddot{\boldsymbol{p}} = -\omega^2 \boldsymbol{p}_0\, e^{-i\omega t} = -\omega^2 \boldsymbol{p}$$

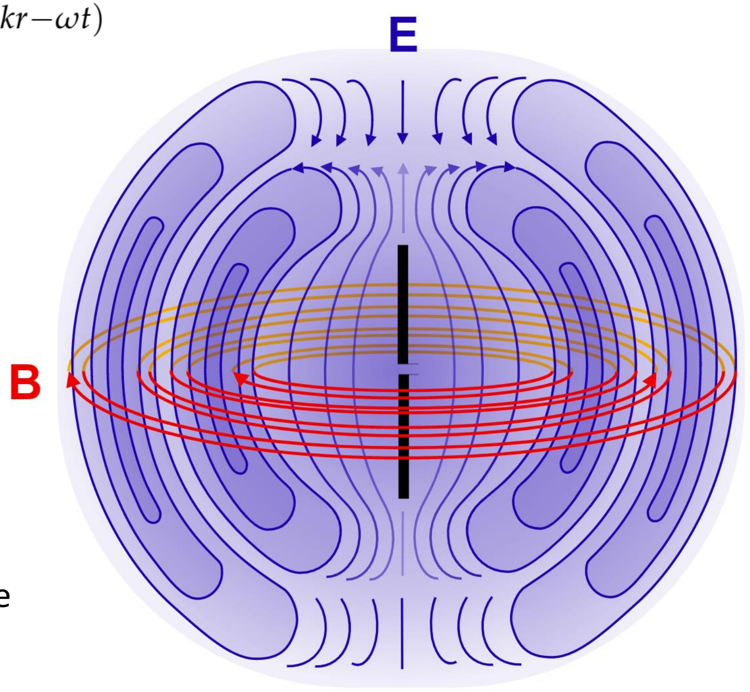
$$k = \frac{\omega}{c}$$

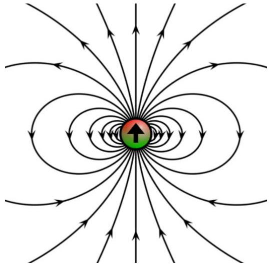
$$k = \frac{2\pi}{\lambda}$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}_0}{r^3} - 2\pi i \frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}_0}{r^2\lambda} + 4\pi^2 \frac{\boldsymbol{e}_r \times (\boldsymbol{p}_0 \times \boldsymbol{e}_r)}{r\lambda^2}\right] e^{i(kr-\omega t)}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\boldsymbol{e}_r \times \left(\frac{i}{r^2} - \frac{2\pi}{\lambda r}\right) \boldsymbol{p}_0\, e^{i(kr-\omega t)}$$

https://upload.wikimedia.org/wikipedia/commons/c/c5/Felder_um_Dipol.svg





https://upload.wikimedia.org/wikipedia/commons/7/76/VFPt_dipoles_magnetic.svg

Near field

- Electric field dominates
- Similar to static dipole
  - along $\boldsymbol{e}_r$ near z axis
  - along $\boldsymbol{e}_\theta$ near xy plane

Far field

- $\boldsymbol{B} \perp \boldsymbol{E}$
- Electric field largest near xy plane (along $\boldsymbol{e}_\theta$)
- Magnetic field largest near xy plane (along $\boldsymbol{e}_\phi$)