# Electrodynamics

SECTION

## Electrostatics

LECTURE

## Maxwell's equations

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# Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

- Charges are sources of the electric field

$$\nabla \cdot \boldsymbol{B} = 0$$

- The magnetic field as no sources (no monopoles)

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

- Time-dependent magnetic fields generate electric fields

$$\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j}$$

- Time-dependent electric fields and currents generate magnetic fields

Vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{N}}{\mathrm{A}^2}$

Vacuum permittivity $\epsilon_0 = 8.854... \cdot 10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}}$

# Maxwell's equations

Differential formulation

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Integral formulation

$$\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \, \mathrm{d}V$$

$$\oint_{\partial A} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_A \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$$

# Electrodynamics

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## Electrostatics

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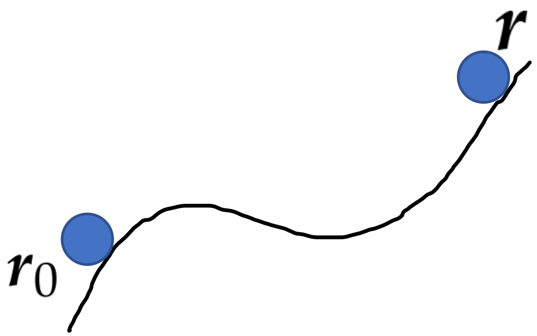
## Poisson equation

# Poisson equation

- Maxwell $\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$

- We can introduce the electrostatic potential $\boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r})$

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \mathbf{E}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'$$




- Like in classical mechanics $\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$

$$V(\boldsymbol{r}) - V(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'$$

- Maxwell $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r})$

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

Poisson equation

# Electrodynamics

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## Electrostatics

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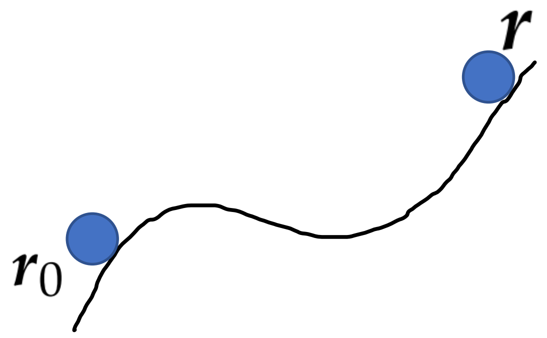
## Rotational-symmetric charge distributions

# Poisson equation

- Maxwell $\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$

- We can introduce the electrostatic potential $\boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r})$

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \mathbf{E}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'$$




- Like in classical mechanics $\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$

$$V(\boldsymbol{r}) - V(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'$$

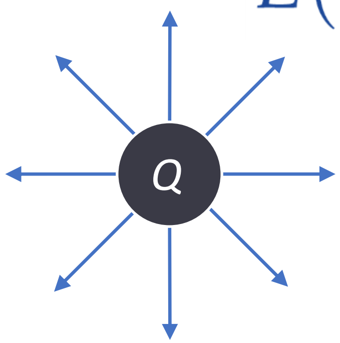
- Maxwell $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r})$

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

Poisson equation

# Coulomb's law

- Maxwell $\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \mathrm{d}V = Q_v$



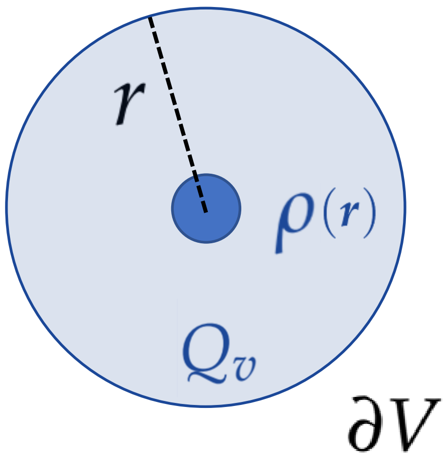

$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r \qquad \boldsymbol{E} \parallel \mathrm{d}\boldsymbol{S}$$

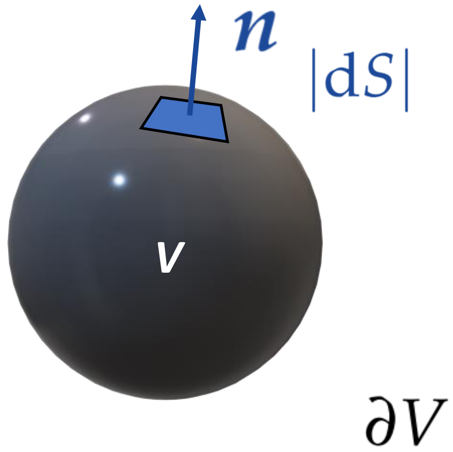
$$\epsilon_0 E(r) \oint_{\partial V} \mathrm{d}S = Q_v$$

$$\epsilon_0 E(r)\, 4\pi r^2 = Q_v$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q_v}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_v}{r^2} \boldsymbol{e}_r$$



# Electrostatic potential

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

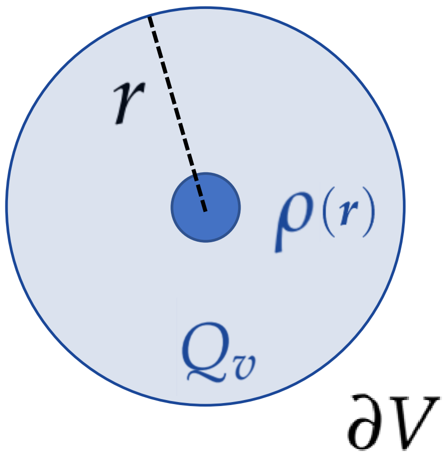
$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}\boldsymbol{e}_r$$

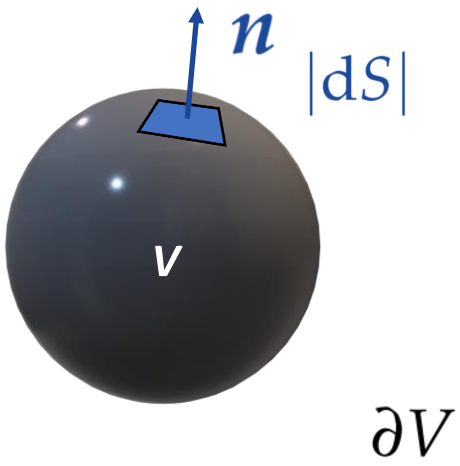
If we have ***E***(***r***) it is easy

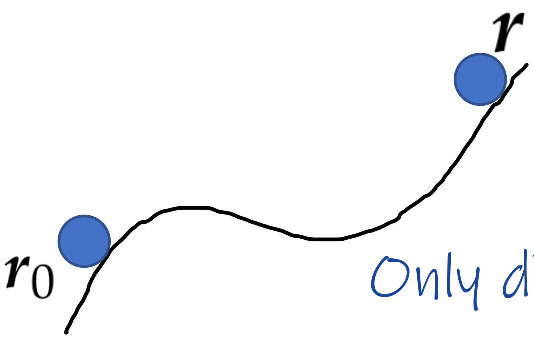
$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \mathrm{E}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'$$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r} - \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r_0}$$

Solve Poisson equation?




Only differences matter!

# Electrodynamics

SECTION

## Electrostatics

LECTURE

## General charge distributions

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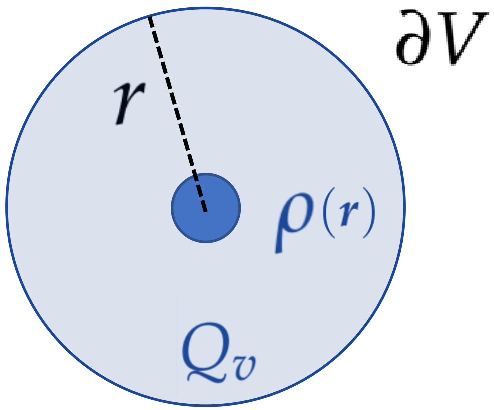
# Electrostatic potential

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

Poisson equation




- Point charge

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}$$

- Multiple point charges

$$\varphi(\boldsymbol{r}) = \sum_i \frac{1}{4\pi\epsilon_0}\frac{Q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}$$

- General charge distribution

$$\varphi(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0}\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}'$$

$$Q = \int \rho(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'$$

# Electrodynamics

SECTION

## Electrostatics

LECTURE

## Exercise: charged sphere

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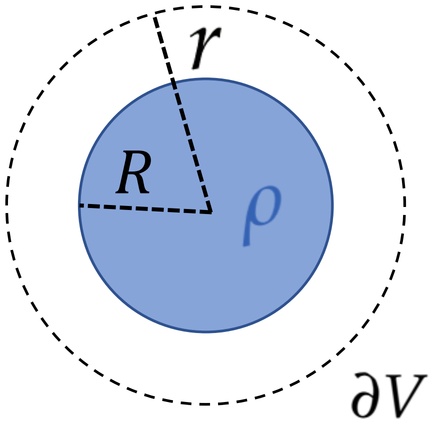
# Exercise

1. Calculate the electric field $\boldsymbol{E}(\boldsymbol{r})$ of a homogeneously charged sphere (distinguish inside and outside).

   *Hint: Use rotational symmetry to simplify the problem.*

   *Hint: "Homogeneously charged" means that $\rho$ = const.*

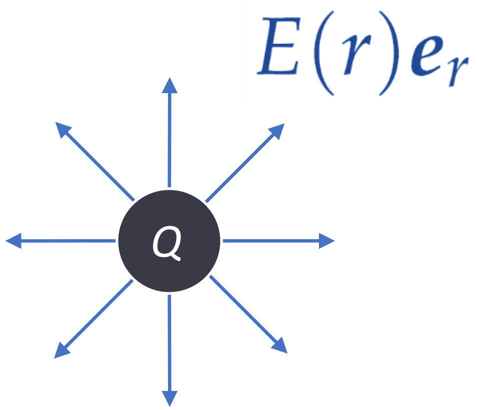
2. Calculate the electrostatic potential $\varphi(\boldsymbol{r})$.



# Coulomb's law

- Maxwell $\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \mathrm{d}V = Q_v$

$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r \qquad \boldsymbol{E} \parallel \mathrm{d}\boldsymbol{S}$$



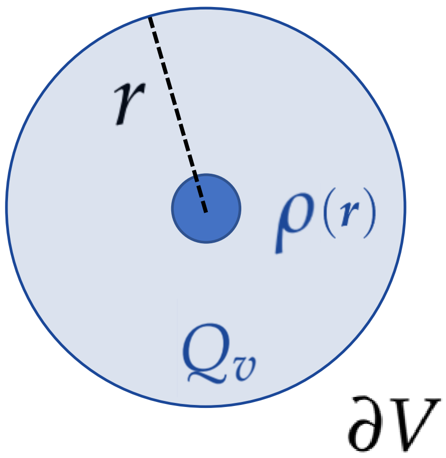

$$\epsilon_0 E(r) \oint_{\partial V} \mathrm{d}S = Q_v$$

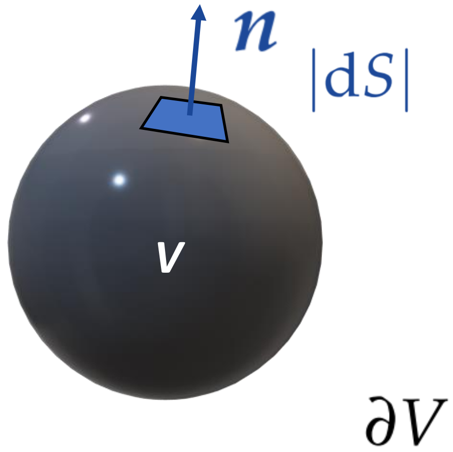
$$\epsilon_0 E(r)\, 4\pi r^2 = Q_v$$

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}$$

What about inside of the sphere?

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}\boldsymbol{e}_r$$



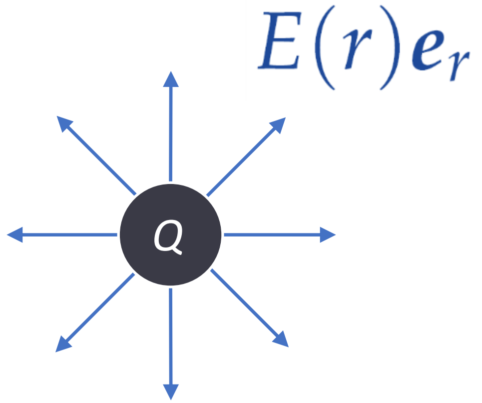
# Charged sphere

- Maxwell

$$\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \mathrm{d}V$$

$$\boxed{\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r}$$

$$\boldsymbol{E} \parallel \mathrm{d}\boldsymbol{S}$$

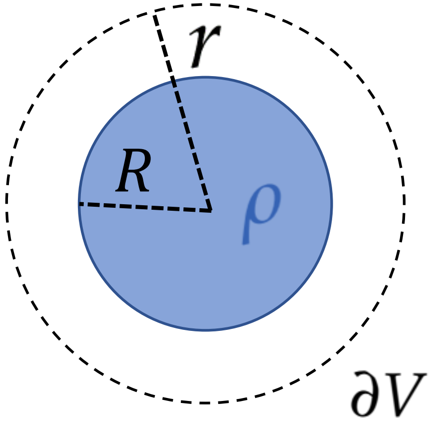



- Outside $r > R$

$$\epsilon_0 E(r) \oint_{\partial V} \mathrm{d}S = \frac{4}{3}\pi R^3 \rho = Q_v$$

$$\epsilon_0 E(r)\, 4\pi r^2 = \frac{4}{3}\pi R^3 \rho = Q_v$$

$$\boxed{E(r) = \frac{R^3}{3\epsilon_0}\frac{\rho}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}}$$




- Inside $r < R$

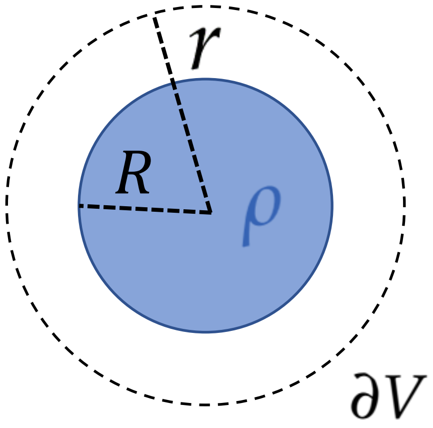
$$\epsilon_0 E(r) \oint_{\partial V} \mathrm{d}S = \rho \int_V \mathrm{d}V$$

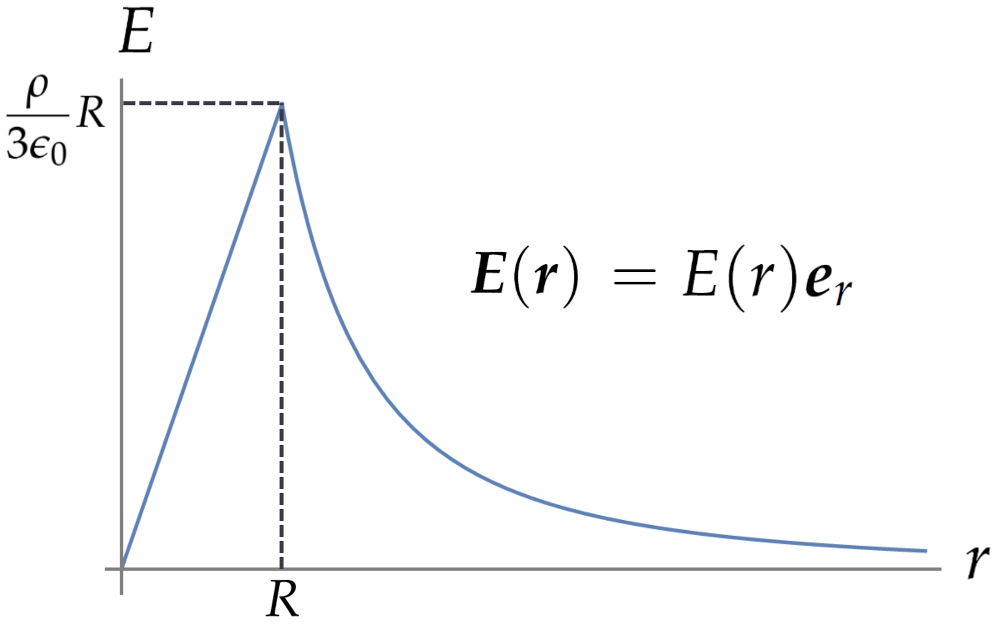
$$\epsilon_0 E(r)\, 4\pi r^2 = \frac{4}{3}\pi r^3 \rho$$

$$\boxed{E(r) = \frac{\rho}{3\epsilon_0} r}$$

# Charged sphere

1. Calculate the electric field ***E***(***r***) of a homogeneously charged sphere (distinguish inside and outside).




$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$

- Inside $r < R$

$$E(r) = \frac{\rho}{3\epsilon_0}r$$

- Outside $r > R$

$$E(r) = \frac{R^3}{3\epsilon_0}\frac{\rho}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}$$

# Charged sphere

2. Calculate the electrostatic potential $\varphi(\boldsymbol{r})$.

- electrostatic potential

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'$$

- Inside $r < R$

$$E(r) = \frac{\rho}{3\epsilon_0} r$$

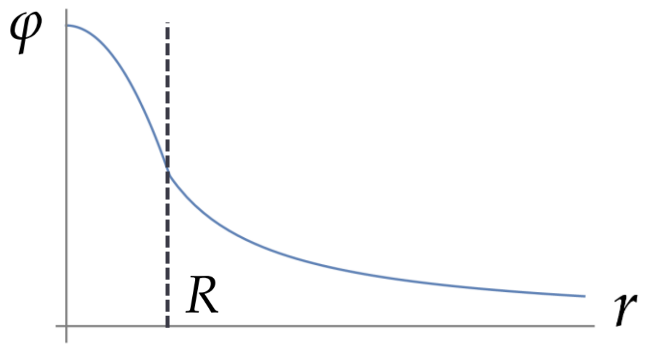
$$\varphi(\boldsymbol{r}) = -\frac{\rho}{6\epsilon_0} r^2 + \frac{1\rho}{2\epsilon_0} R^2$$

$$\varphi(\boldsymbol{r}) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2)$$

- Outside $r > R$

$$E(r) = \frac{R^3}{3\epsilon_0}\frac{\rho}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}$$

$$\varphi(\boldsymbol{r}) = \frac{R^3}{3\epsilon_0}\frac{\rho}{r} = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r}$$



# Electrodynamics

SECTION

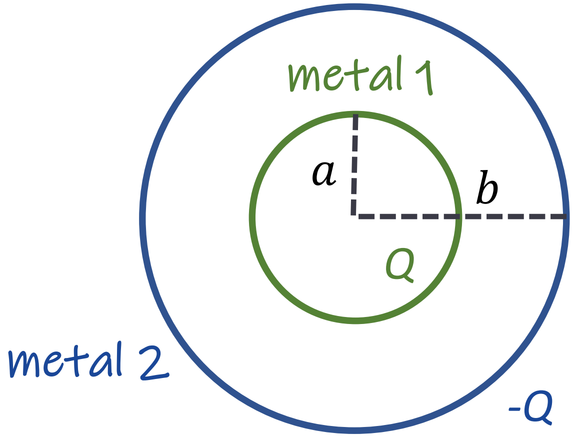
## Electrostatics

LECTURE

## Exercise: spherical capacitor

# Spherical capacitor

1. Calculate the electric field $\boldsymbol{E}(\boldsymbol{r})$ of the following capacitor for all regions.
2. Calculate the electrostatic potential $\varphi(\boldsymbol{r})$ for all regions.
3. Calculate the voltage between the two metals $U = \varphi(r = a) - \varphi(r = b)$.
4. Calculate the capacitance $C = Q/U$.



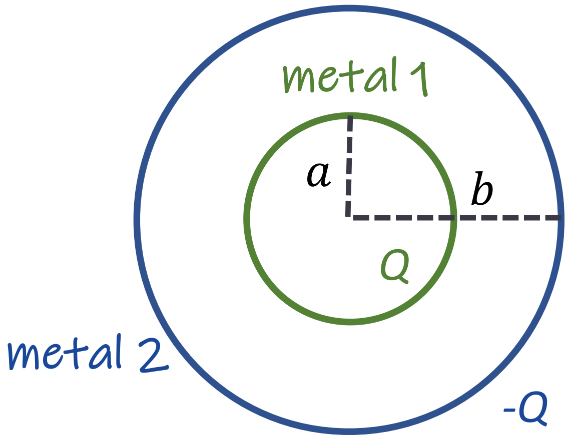

- Maxwell $$\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \mathrm{d}V$$

$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$

$$\boldsymbol{E} \parallel \mathrm{d}\boldsymbol{S}$$

# Spherical capacitor

1. Calculate the electric field ***E***(***r***) of the following capacitor for all regions.




- Inside *r* < *a*: $E(\boldsymbol{r}) = 0$

  No enclosed charge $\rho(\boldsymbol{r}) = 0$

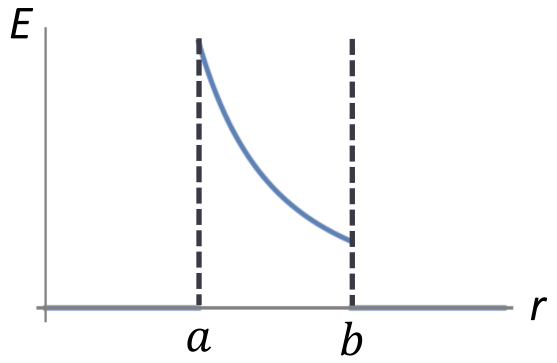
$$\epsilon_0 \oint_{\partial V} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \mathrm{d}V$$

- Between *a* < *r* < *b*:

$$\epsilon_0 \oint_{\partial V} E\,\mathrm{d}S = Q$$

$$\epsilon_0 4\pi r^2 E = Q$$

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \qquad \boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$




- Outside *r* > *b*: $E(\boldsymbol{r}) = 0$

  No enclosed charge $Q_v = Q - Q = 0$

# Spherical capacitor

2. Calculate the electrostatic potential $\varphi(\boldsymbol{r})$.

   - Inside $r < a$: $E(\boldsymbol{r}) = 0$

     $$\varphi(\boldsymbol{r}) = const.$$

   - Between $a < r < b$: $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$

     $$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + const.$$

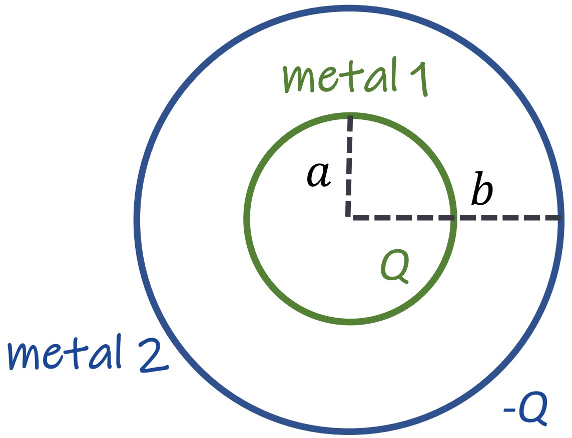
   - Outside $r > b$: $E(\boldsymbol{r}) = 0$

     $$\varphi(\boldsymbol{r}) = const.$$

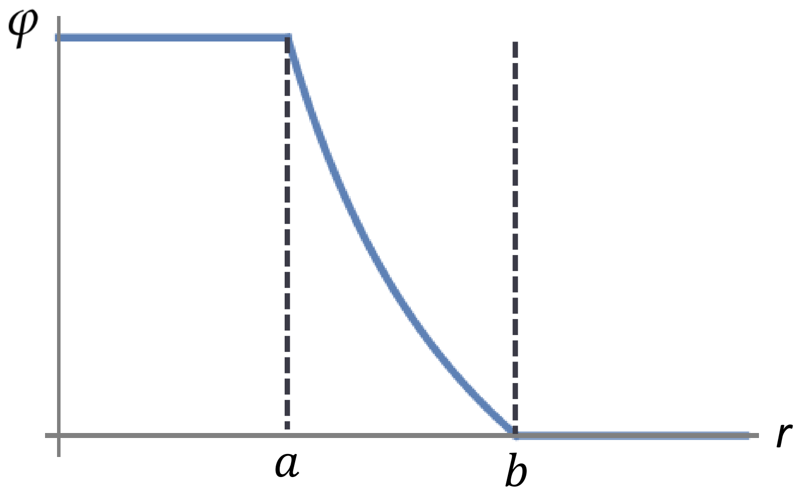
electrostatic potential

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \mathbf{E}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'$$




$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$



# Spherical capacitor

2. Calculate the electrostatic potential $\varphi(\boldsymbol{r})$.

- Inside $r < a$: $E(\boldsymbol{r}) = 0$

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

- Between $a < r < b$: $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

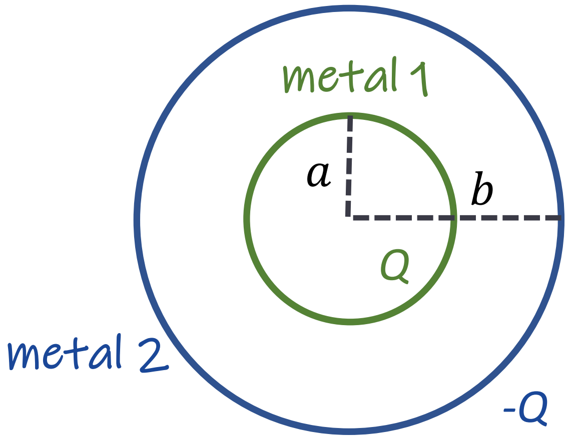
- Outside $r > b$: $E(\boldsymbol{r}) = 0$

$$\varphi(\boldsymbol{r}) = 0$$

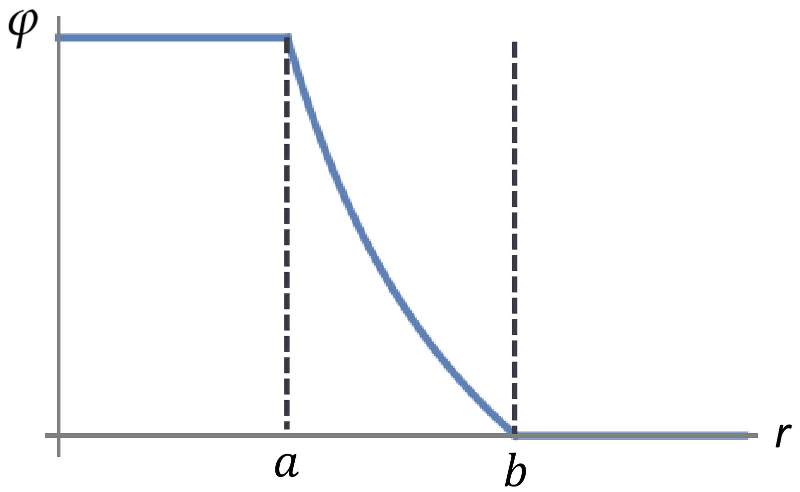
electrostatic potential

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'$$




$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$



# Spherical capacitor

3. Calculate the voltage between the two metals $U = \varphi(r = a) - \varphi(r = b)$.

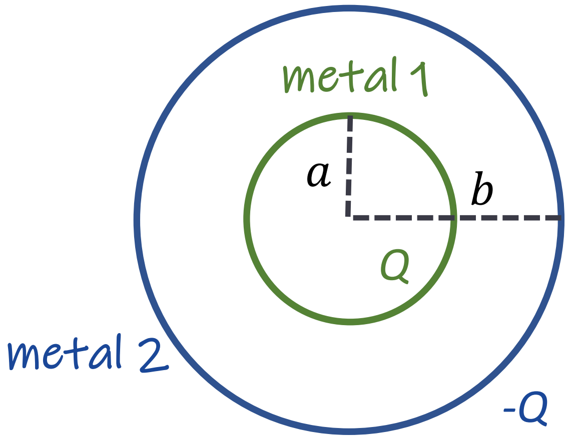
- Inside $r < a$: $E(\mathbf{r}) = 0$

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

- Outside $r > b$: $E(\mathbf{r}) = 0$

$$\varphi(\mathbf{r}) = 0$$

$$U = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$




4. Calculate the capacitance C = Q/U.

$$C = Q/U = \frac{Q}{\frac{1}{4\pi\epsilon_0} Q\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$C = 4\pi\epsilon_0 \frac{ab}{b - a}$$

# Electrodynamics

SECTION

## Electrostatics

LECTURE

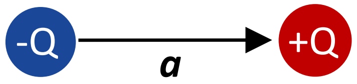
## Electric dipole

# Electric dipole

- For multiple point charges

$$\varphi(\boldsymbol{r}) = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{|\boldsymbol{r} - \boldsymbol{r}_1|}$$

$$\boldsymbol{p} = Q\boldsymbol{a}$$




- For electric dipole

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{|\boldsymbol{r} + \boldsymbol{a}|}\right)$$

Taylor expansion

$$\frac{1}{|\boldsymbol{r} + \boldsymbol{a}|} = \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{a} \cdot \nabla)^n \frac{1}{r}$$

For $r \gg a$

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r} - \boldsymbol{a} \cdot \nabla \frac{1}{r}\right)$$

$$\nabla \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$$

$$\nabla \frac{1}{r} = \nabla (x^2 + y^2 + z^2)^{-1/2} = \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

- Electrostatic potential

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\boldsymbol{a} \cdot \boldsymbol{r}}{r^3}$$

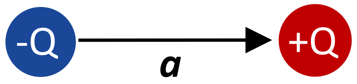
$$\boxed{\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3}}$$

# Electric dipole

- Electrostatic potential

$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{|\boldsymbol{r}+\boldsymbol{a}|}\right)$$

$$\boldsymbol{p} = Q\boldsymbol{a}$$




For $r \gg a$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$

- Electric field

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0}\nabla\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$

$$\nabla\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} = \nabla\frac{p_x x + p_y y + p_z z}{r^3} = \begin{pmatrix} p_x/r^3 \\ p_y/r^3 \\ p_z/r^3 \end{pmatrix} + (p_x x + p_y y + p_z z)\nabla\frac{1}{r^3}$$

$$\nabla\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} = \frac{\boldsymbol{p}}{r^3} + \boldsymbol{p}\cdot\boldsymbol{r}\nabla\frac{1}{r^3}$$

$$\nabla\frac{1}{r^3} = \nabla(x^2+y^2+z^2)^{-3/2} = \left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\nabla\frac{1}{r^3} = -3\frac{\boldsymbol{r}}{r^5}$$

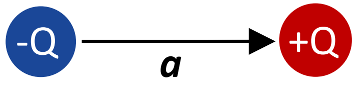
$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0}\nabla\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\boldsymbol{p}}{r^3} - 3\boldsymbol{p}\cdot\boldsymbol{r}\frac{\boldsymbol{r}}{r^5}\right) = -\frac{1}{4\pi\epsilon_0}\frac{r^2\boldsymbol{p} - 3\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{r})}{r^5}$$

# Electric dipole

- Electrostatic potential

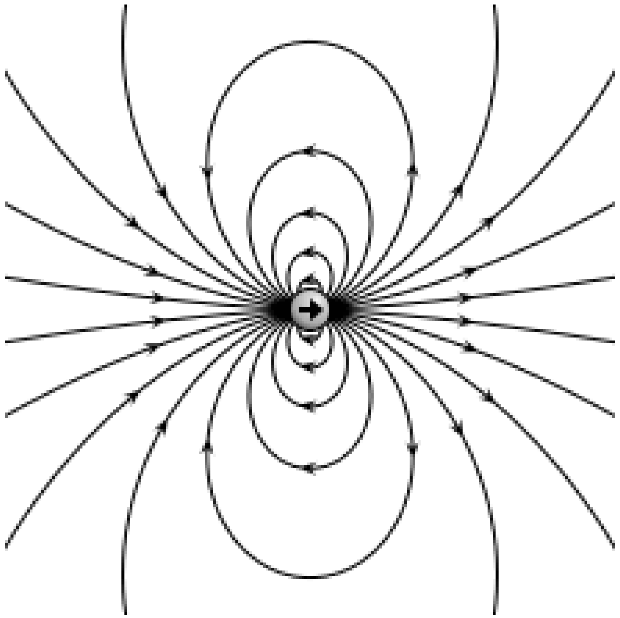
$$\varphi(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{|\boldsymbol{r}+\boldsymbol{a}|}\right)$$

$$\boldsymbol{p} = Q\boldsymbol{a}$$




For $r \gg a$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$



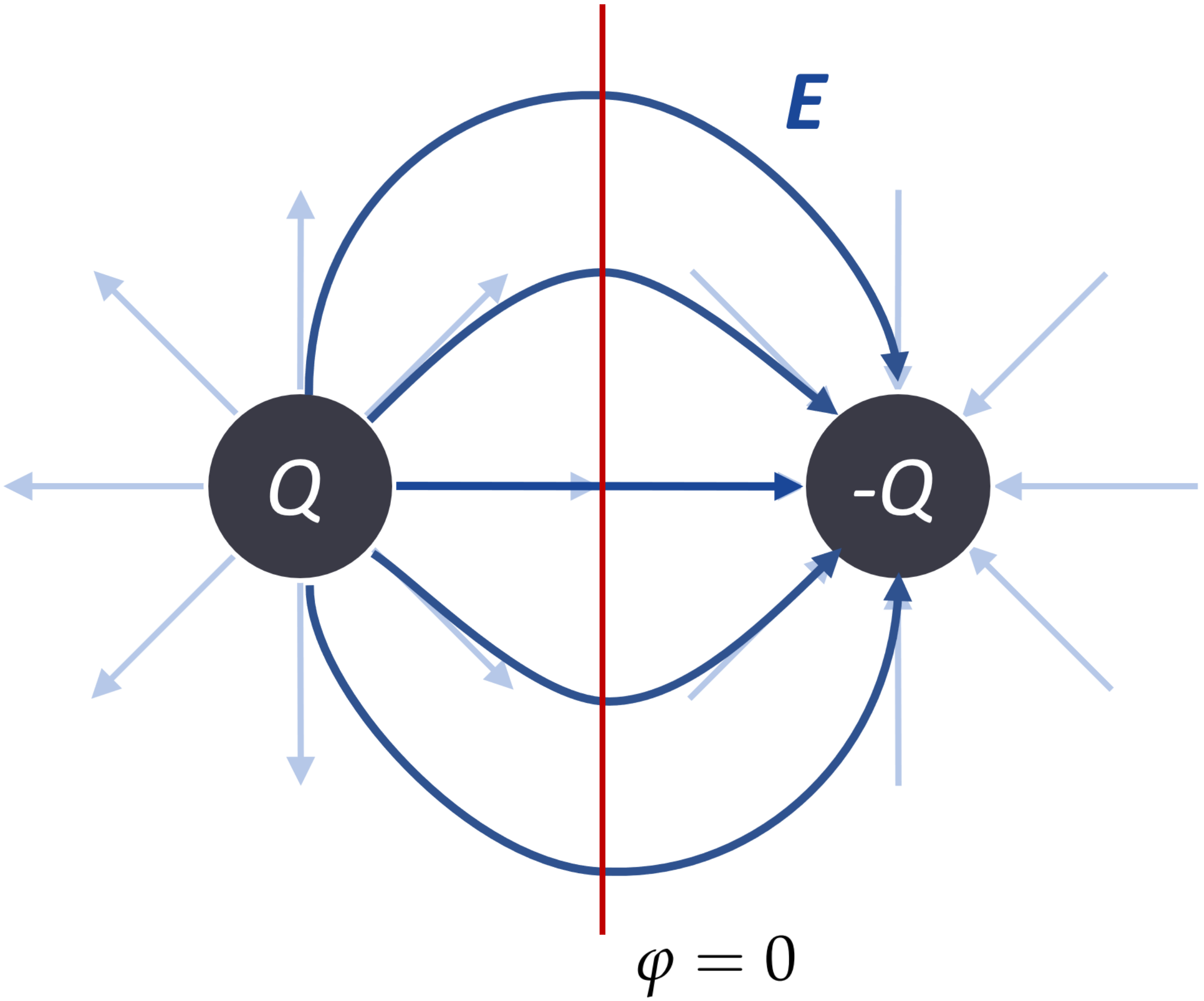
- Electric field

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0}\frac{r^2\boldsymbol{p} - 3\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{r})}{r^5}$$

$$\boldsymbol{E}(\boldsymbol{r}) \sim \frac{1}{r^3}$$

https://upload.wikimedia.org/wikipedia/commons/a/aa/VFPt_dipole_animation_electric.gif

# Electric dipole



# Electrodynamics

SECTION

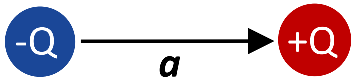
## Electrostatics

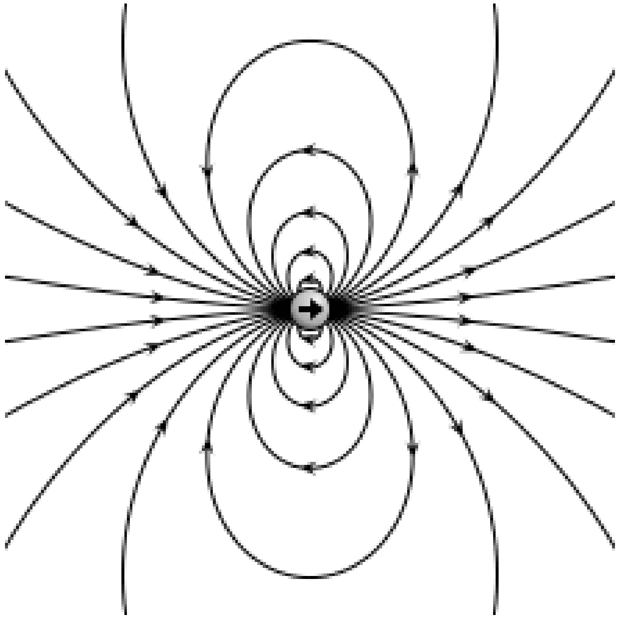
LECTURE

## Force and torque acting on an electric dipole

# Electric dipole

$$\boldsymbol{p} = Q\boldsymbol{a}$$



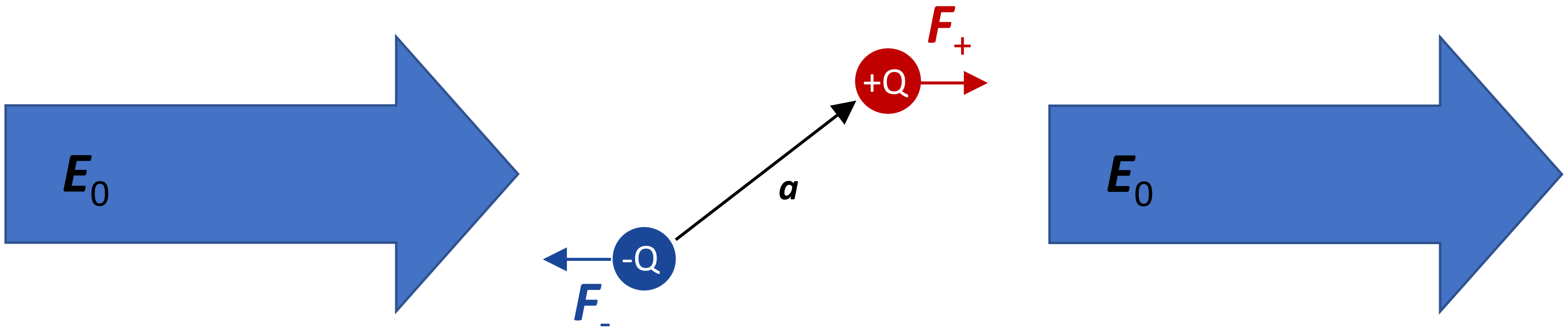
https://upload.wikimedia.org/wikipedia/commons/a/aa/VFPt_dipole_animation_electric.gif

- Electric field

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0}\frac{r^2\boldsymbol{p} - 3\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{r})}{r^5}$$

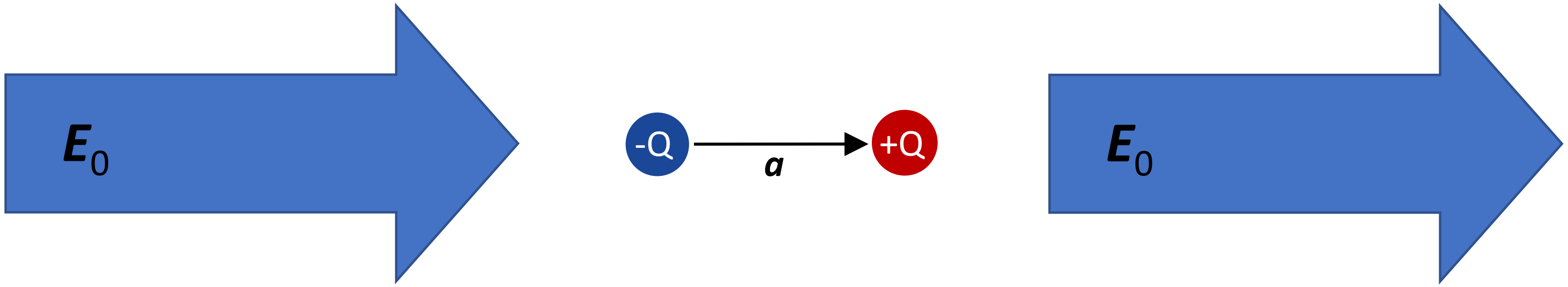
# Electric dipole in a homogeneous *E* field

$$\boldsymbol{p} = Q\boldsymbol{a}$$



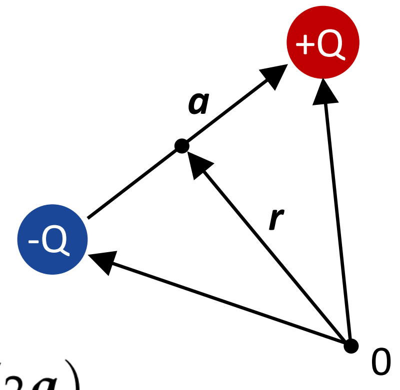
# Electric dipole in a homogeneous $E$ field

$$\boldsymbol{p} = Q\boldsymbol{a}$$




Reorientation
No motion

# Electric dipole in a homogeneous $E$ field




**Force** No motion

$$F = QE(r + 1/2a) - QE(r - 1/2a)$$

$$= Q\left[E(r) + (1/2a \cdot \nabla)\, E(r)\right] + \dots$$
$$- Q\left[E(r) - (1/2a \cdot \nabla)\, E(r)\right] + \dots$$

$$= Q\,(a \cdot \nabla)\, E(r) + \dots$$

$$\boxed{F = (p \cdot \nabla)E(r)}$$

$$F = 0 \quad \text{for } E(r) = E_0 = const.$$

**Torque** Reorientation

$$M = r \times F$$

$$M = 1/2a \times QE\,(r + 1/2a) + 1/2a \times QE\,(r - 1/2a)$$

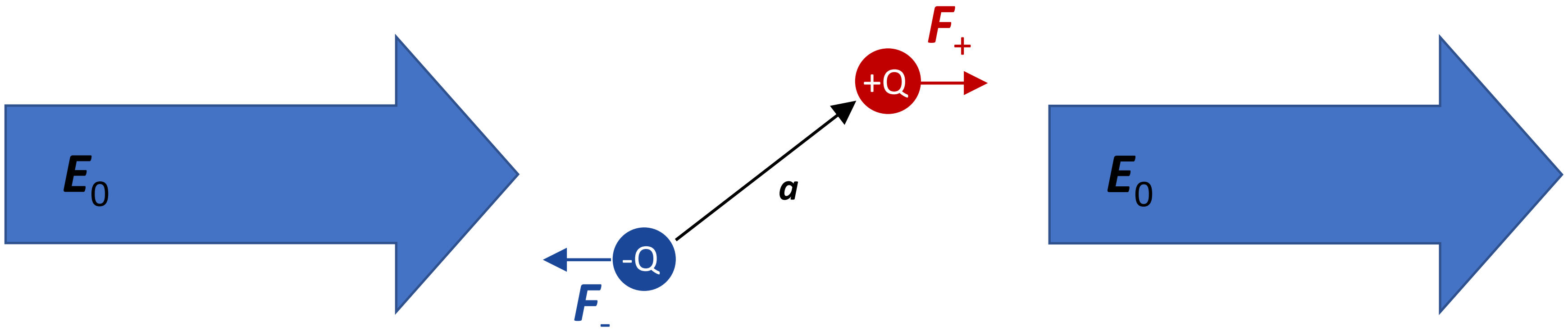
$$= 1/2a \times Q\{\left[E(r) + (1/2a \cdot \nabla)\, E(r)\right] + \cdots$$
$$+ \left[E(r) - (1/2a \cdot \nabla)\, E(r)\right] + \dots\}$$

$$= Qa \times E(r) + \dots$$

$$\boxed{M = p \times E(r)}$$

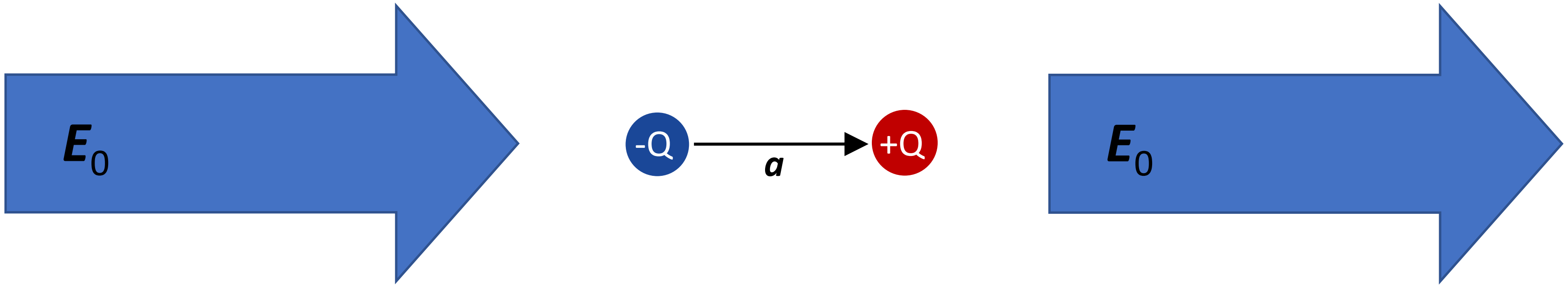
# Electric dipole in a homogeneous *E* field

$$\boldsymbol{p} = Q\boldsymbol{a}$$



# Electric dipole in a homogeneous *E* field

$$\boldsymbol{p} = Q\boldsymbol{a}$$




Reorientation $\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{E}(\boldsymbol{r})$

No motion $\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}(\boldsymbol{r})$

# Electrodynamics

SECTION

## Electrostatics

LECTURE

## Boundary conditions & mirror charges

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# Boundary conditions

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

Poisson equation

- Special solution

$$\varphi_s(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}'$$

- General solution

$$\varphi(\boldsymbol{r}) = \varphi_s(\boldsymbol{r}) + \varphi_h(\boldsymbol{r})$$

Solution of the homogeneous equation

$$\Delta\varphi_h(\boldsymbol{r}) = 0$$

... has to fulfill boundary conditions

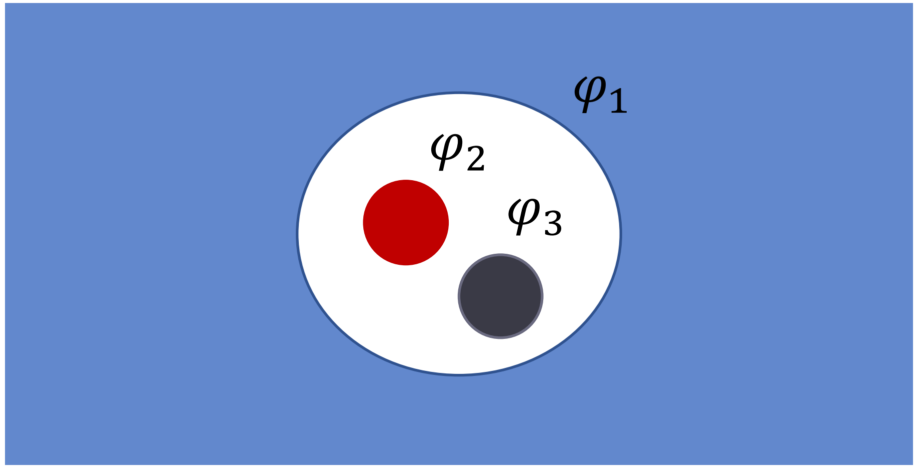
so far $\varphi = 0$ for $\boldsymbol{r} \to \infty$ $\quad \varphi_h(\boldsymbol{r}) = 0$

# Boundary conditions

$$\varphi(\boldsymbol{r}) = \varphi_s(\boldsymbol{r}) + \boxed{\varphi_h(\boldsymbol{r})}$$ ... has to fulfill boundary conditions

so far $\varphi = 0$ for $\boldsymbol{r} \to \infty$ $\varphi_h(\boldsymbol{r}) = 0$

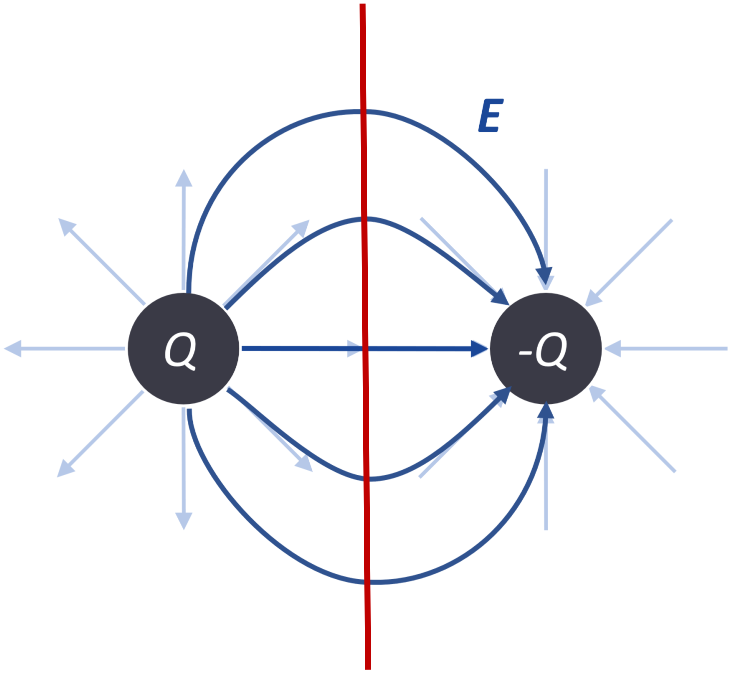
- Charged wire or metal surface: constant potential



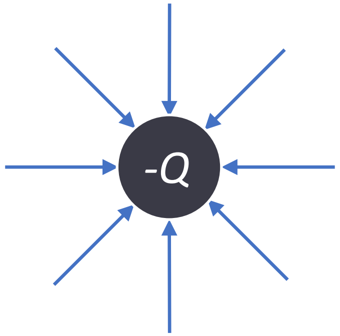
# Boundary conditions

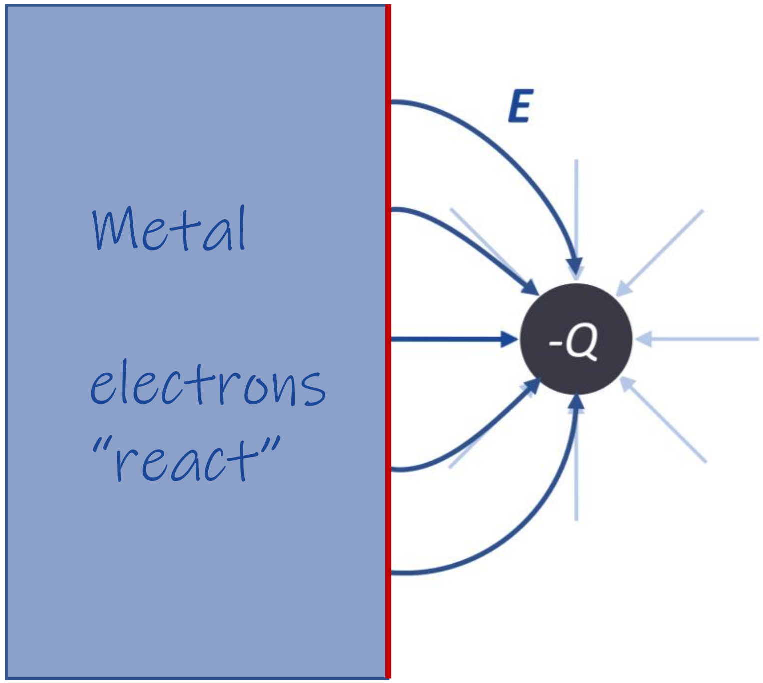
$$\varphi(\boldsymbol{r}) = \varphi_s(\boldsymbol{r}) + \boxed{\varphi_h(\boldsymbol{r})}$$

... has to fulfill boundary conditions




$\varphi = 0$



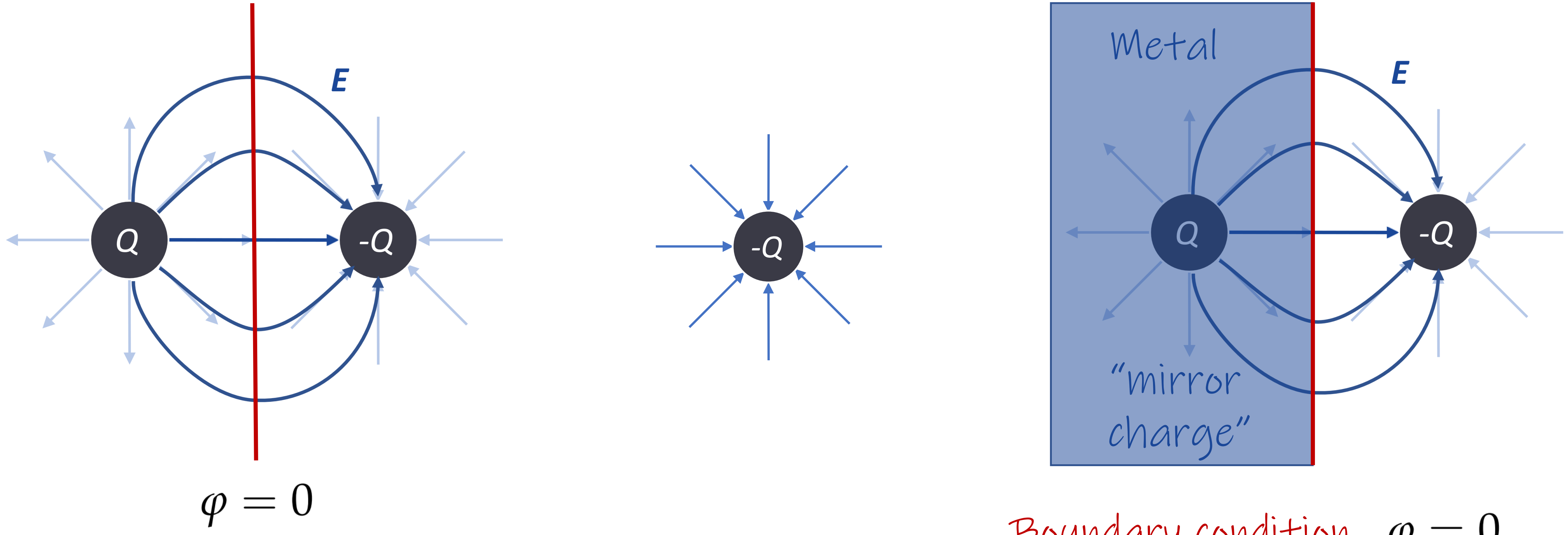




Boundary condition $\varphi = 0$

# Boundary conditions

$$\varphi(\boldsymbol{r}) = \varphi_s(\boldsymbol{r}) + \boxed{\varphi_h(\boldsymbol{r})}$$

... has to fulfill boundary conditions




Boundary condition $\varphi = 0$

# Electrodynamics

SECTION

## Electrostatics

LECTURE

## Energy in electrostatics

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# Field energy

$$-\boldsymbol{j} \cdot \boldsymbol{E} = \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\boldsymbol{E}^2}{2} + \frac{1}{2\mu_0}\boldsymbol{B}^2\right) + \nabla \cdot \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$$

$$\nu = \frac{\partial}{\partial t} w + \nabla \cdot \boldsymbol{S}$$

Continuity equation for energy density

- Power density (energy generation density) $\qquad \nu = -\boldsymbol{j} \cdot \boldsymbol{E}$
- Energy density $\qquad w = \epsilon_0 \frac{\boldsymbol{E}^2}{2} + \frac{1}{2\mu_0}\boldsymbol{B}^2$
- Poynting vector (energy-current density) $\qquad \boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$

# Electrostatic energy

- Energy density $$w = \epsilon_0 \frac{\boldsymbol{E}^2}{2} + \frac{1}{2\mu_0}\boldsymbol{B}^2$$
- Energy $$W_e = \int w_e \,\mathrm{d}V = \int \frac{\epsilon_0}{2}\boldsymbol{E}^2 \,\mathrm{d}V$$

$$W_e = -\int \frac{\epsilon_0}{2}\boldsymbol{E}\cdot\nabla\varphi \,\mathrm{d}V$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$W_e = -\int \frac{\epsilon_0}{2}\nabla\cdot(\boldsymbol{E}\varphi)\,\mathrm{d}V + \int \frac{\epsilon_0}{2}\varphi\nabla\cdot\boldsymbol{E}\,\mathrm{d}V$$

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

$$W_e = -\oint \frac{\epsilon_0}{2}(\boldsymbol{E}\varphi)\cdot\mathrm{d}\boldsymbol{S} + \frac{1}{2}\int \rho\varphi\,\mathrm{d}V$$

$$W_e = \frac{1}{2}\int \rho(\boldsymbol{r})\varphi(\boldsymbol{r})\,\mathrm{d}V$$

If no surface charges

# Electrostatic energy

$$W_e = \int w_e \, \mathrm{d}V = \int \frac{\epsilon_0}{2} \boldsymbol{E}^2 \, \mathrm{d}V$$

$$W_e = \frac{1}{2} \int \rho(\boldsymbol{r}) \varphi(\boldsymbol{r}) \, \mathrm{d}V$$

If no surface charges

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'$$

$$W_e = \frac{1}{8\pi\epsilon_0} \int \int \frac{\rho(\boldsymbol{r}')\rho(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V \mathrm{d}V'$$

general

$$W_e = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{Q_i Q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

point charges

$$W_e = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$$

2 charges

# Electrodynamics

SECTION

## Electrostatics

LECTURE

## Summary

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# Electrostatics

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

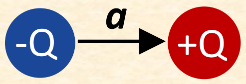
$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$\Delta\varphi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r})$$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}V'$$

$$\boldsymbol{p} = Q\boldsymbol{a}$$

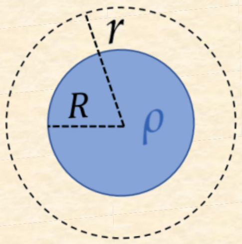



$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p}r^2}{r^5}$$

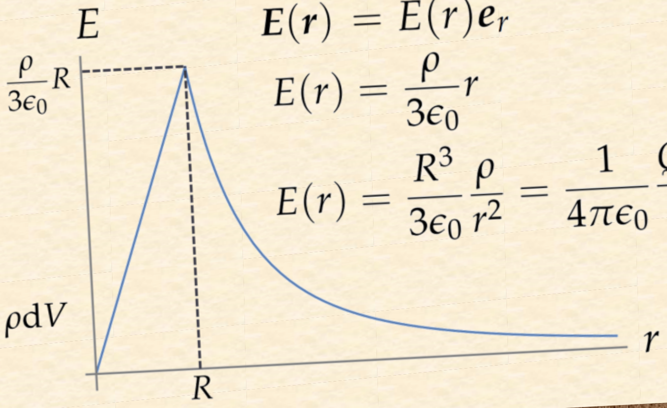
$$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}(\boldsymbol{r}) \qquad \boldsymbol{M} = \boldsymbol{p}\times\boldsymbol{E}(\boldsymbol{r})$$

$$W_e = \int w_e\,\mathrm{d}V = \int \frac{\epsilon_0}{2}E^2\,\mathrm{d}V$$

$$W_e = \frac{1}{2}\int \rho(\boldsymbol{r})\varphi(\boldsymbol{r})\,\mathrm{d}V$$



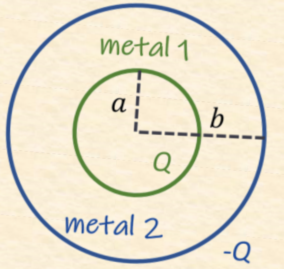

$$\epsilon_0 \oint_{\partial V} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = \int_V \rho\,\mathrm{d}V$$




$$\boldsymbol{E}(\boldsymbol{r}) = E(r)\boldsymbol{e}_r$$

$$E(r) = \frac{\rho}{3\epsilon_0}r$$

$$E(r) = \frac{R^3}{3\epsilon_0}\frac{\rho}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_v}{r^2}$$




$$U = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$