SARIMA processes

PRACTICAL TIME SERIES ANALYSIS

THISTLETON AND SADIGOV

Objectives

- Describe Seasonal ARIMA models
- Rewrite Seasonal ARIMA models using backshift and difference operators

ARIMA processes $\{X_t\}$

Let

$$Y_t = \nabla^d X_t$$

then

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q}$$

can be written as

$$\phi(B) Y_t = \theta(B) Z_t$$

where

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

$$\phi\,(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

Box-Jenkins Seasonal ARIMA model

- Data might contain seasonal periodic component in addition to correlation with recent lags
- It repeats every s observations
- For a time series of monthly observations, X_t might depend on annual lags
- $X_{t-12}, X_{t-24}, \ldots$
- Quarterly data might have period of $s = 4$
- Seasonal ARIMA model

Pure Seasonal ARMA process

$ARMA(P,Q)_s$ has the form

$$\Phi_{\mathrm{P}}(B^s)X_t = \Theta_{\mathrm{Q}}(B^s)Z_t$$

where

$$\Phi_{\mathrm{P}}(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps}$$

and

$$\Theta_{\mathrm{Q}}(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \cdots + \Theta_Q B^{Qs}$$

Stationarity and invertibility

Just like pure ARMA processes, for Seasonal ARMA process to be stationary and invertible, we need that the complex roots of the polynomials

$$\Phi_{\mathrm{P}}(z^s)$$

and

$$\Theta_{\mathrm{Q}}(z^s)$$

are outside of the unit circle.

Example 1

$Seasonal\ ARMA(1,0)_{12}$ has the form

$$(1-\Phi_1 B^{12})X_t = Z_t$$

i.e.,

$$X_t = \Phi_1 \mathrm{X}_{\mathrm{t}-12} + \mathrm{Z}_{\mathrm{t}}$$

Example 2

$Seasonal\ ARMA(1,1)_{12}$ has the form

$$(1-\Phi_1 B^{12})X_t = (1+\Theta_1 B^{12})Z_t$$

i.e.,

$$X_t = \Phi_1 \mathrm{X}_{\mathrm{t}-12} + \mathrm{Z}_{\mathrm{t}} + \Theta_1 Z_{t-12}$$

Seasonal ARIMA process (SARIMA)

$SARIMA(p, d, q, P, D, Q)_s$ has the form

$$\Phi_{\mathrm{P}}(B^s)\phi_p(B)(1-B^S)^D(1-B)^d X_t = \Theta_{\mathrm{Q}}(B^s)\theta_q(B)Z_t$$

where

$$\theta_q(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

$$\Theta_{\mathrm{Q}}(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \cdots + \Theta_Q B^{Qs}$$

$$\phi_p\ (B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

$$\Phi_{\mathrm{P}}(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps}$$

SARIMA models

$SARIMA(p, d, q, P, D, Q)_s$ has two parts:

Non-seasonal part (p, d, q) and seasonal parts $(P, D, Q)_S$.

1. p – order of non-seasonal AR terms
2. d – order of non-seasonal differencing
3. q – order of non-seasonal MA terms
4. P – order of seasonal AR (i.e., SAR) terms
5. D – order of seasonal differencing (i.e., power of $(1 - B^S)$)
6. Q – order of seasonal MA (i.e., SMA) terms

Seasonal Differencing

- $D = 1$

$$\nabla_S X_t = (1 - B^S)X_t = X_t - X_{t-S}$$

- $D = 2$

$$\nabla_S^2 X_t = (1 - B^S)^2 X_t = (1 - 2B^S + B^{2S})X_t = X_t - 2X_{t-S} + X_{t-2S}$$

Example 3- $SARIMA(1,0,0,1,0,1)_{12}$

$$(1-\phi_1 B)(1-\Phi_1 B^{12})X_t = (1+\Theta_1 B^{12})Z_t$$

$$(1-\phi_1 B-\Phi_1 B^{12}+\phi_1\Phi_1 B^{13})X_t = Z_t+\Phi_1 Z_{t-12}$$

Thus

$$X_t = \phi_1 X_{t-1}+\Phi_1 X_{t-12}-\phi_1\Phi_1 X_{t-13}+Z_t+\Phi_1 Z_{t-12}$$

Example 4 - $SARIMA(0,1,1,0,0,1)_4$

$$(1-B)X_t = (1+\Theta_1 B^4)(1+\theta_1 B)Z_t$$

Then,

$$X_t - X_{t-1} = (1+\theta_1 B + \Theta_1 B^4 + \theta_1\Theta_1 B^5)Z_t$$

Thus

$$X_t = X_{t-1} + Z_t + \theta_1 Z_{t-1} + \Theta_1 Z_{t-4} + \theta_1\Theta_1 Z_{t-5}$$

What We've Learned

- Describe seasonal, autoregressive, integrated, moving average models
- Rewrite seasonal, autoregressive, integrated, moving average models using backshift and difference operators