Difference equations

PRACTICAL TIME SERIES ANALYSIS

THISTLETON AND SADIGOV

Objectives

- Recall and solve difference equations

Difference equation

- General term of a sequence is given, ex: $a_n = 2n + 1$. So,

$$3, 5, 7, \ldots$$

- General term not given, but a relation is given, ex:

$$a_n = 5a_{n-1} - 6a_{n-2}$$

- This is a difference equation (recursive relation)

How to solve difference equations?

- We look for a solution in the format

$$a_n = \lambda^n$$

- For the previous problem,

$$\lambda^n = 5\lambda^{n-1} - 6\lambda^{n-2}$$

We simplify

$$\lambda^2 - 5\lambda + 6 = 0$$

- Auxiliary equation or characteristic equation.

- $\lambda = 2, \lambda = 3$
- $a_n = c_1 2^n + c_2 3^n$
- With some initial conditions, say $a_0 = 3, a_1 = 8$.

We get

$$\begin{cases} c_1 + c_2 = 3 \\ 2c_1 + 3c_2 = 8 \end{cases}$$

Thus,

$$c_1 = 1, c_2 = 2.$$

Solution

$$a_n = 2^n + 2 \cdot 3^n$$

Is the solution of 2nd order difference equation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

k-th order difference equation

$$a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \cdots + \beta_k a_{n-k}$$

Its characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1}\lambda - \beta_k = 0$$

Then we look for the solutions of the characteristic equation. Say, all k solutions are distinct real numbers, $\lambda_1, \lambda_2, \ldots, \lambda_k$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_n \lambda_k^n$$

Coefficients $c_j's$ are determined using initial values.

Example - Fibonacci sequence

Fibonacci sequence is defined as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

i.e., every term starting from the 3rd term is addition of the previous two terms.

Question: What is the general term, a_n, of the Fibonacci sequence?

Formulation

We are looking for a sequence $\{a_n\}_{n=0}^{\infty}$, such that

$$a_n = a_{n-1} + a_{n-2}$$

where $a_0 = 1, a_1 = 1$.

Characteristic equation becomes

$$\lambda^2 - \lambda - 1 = 0$$

Then $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

Thus

$$a_n = c_1\left(\frac{1-\sqrt{5}}{2}\right)^n + c_2\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Use initial data

$$\begin{cases} c_1 + c_2 = 1 \\ c_1\left(\frac{1-\sqrt{5}}{2}\right) + c_2\left(\frac{1+\sqrt{5}}{2}\right) = 1 \end{cases}$$

General term of Fibonacci sequence

We obtain

$$c_1 = \frac{5-\sqrt{5}}{10} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_2 = \frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$a_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Relation to differential equations

k –th order linear ordinary equation

$$y^{(k)} = \beta_1 y^{(k-1)} + \cdots \beta_{k-1} y + \beta_k$$

Solution format $y = e^{\lambda t}$ gives characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1}\lambda - \beta_k = 0$$

Then we solve the characteristic equation.

What We’ve Learned

- Definition of difference equations and how to solve them