Mean Square Convergence

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- Learn mean-square convergence
- Formulate necessary and sufficient condition for invertibility of MA(1) process

Mean-square convergence

Let

$$X_1, X_2, X_3, \dots$$

be a sequence of random variables (i.e. a stochastic process).

We say X_n converge to a random variable X in the mean-square sense

if

$$E[(X_n - X)^2] \to 0 \; as \; n \to \infty$$

MA(1) model

We inverted MA(1) model

$$X_t = Z_t + \beta Z_{t-1}$$

as

$$Z_t = \sum_{k=0}^{\infty} (-\beta)^k X_{t-k}$$

Infinite sum above is convergent in mean-square sense under some condition on β.

Auto covariance function

$$\gamma(k) = \begin{cases} 0, & k > 1 \\ \beta\sigma_Z^2, & k = 1 \\ (1+\beta^2)\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}$$

Series convergence

Lets find $\beta' s$ that partial sum

$$\sum_{k=0}^{n} (-\beta)^k X_{t-k}$$

converges to Z_t in mean-square sense.

$$E\left[\left(\sum_{k=0}^{n}(-\beta)^k X_{t-k} - Z_t\right)^2\right] = E\left[\left(\sum_{k=0}^{n}(-\beta)^k X_{t-k}\right)^2\right] - 2E\left[\sum_{k=0}^{n}(-\beta)^k X_{t-k}\, Z_t\right] + E[Z_t^2]$$

$$= E\left[\sum_{k=0}^{n} \beta^{2k} X_{t-k}^2\right] + 2E\left[\sum_{k=0}^{n-1}(-\beta)^{2k+1} X_{t-k} X_{t-k+1}\right] - 2E[X_t Z_t] + \sigma_Z^2$$

$$= \sum_{k=0}^{n} \beta^{2k} E\left[X_{t-k}^2\right] - 2\sum_{k=0}^{n-1} \beta^{2k+1}\, E[X_{t-k} X_{t-k+1}] - 2E[Z_t^2$$

To get

$$E\left[\left(\sum_{k=0}^{n}(-\beta)^k X_{t-k} - Z_t\right)^2\right] \to 0 \; as \; n \to \infty$$

We need

$$\sigma_Z^2 \beta^{2n+2} \to 0 \; as \; n \to \infty$$

Thus, $|\beta| < 1$.

i.e.,

$$\left|-\frac{1}{\beta}\right| > 1$$

i.e., zero of the polynomial

$$\beta(B) = 1 + \beta B$$

Lies outside of the unit circle.

What We've Learned

- Definition of the mean square convergence
- Necessary and sufficient condition for invertibility of MA(1) process