Invertibility and stationarity conditions

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- Articulate invertibility condition for MA(q) processes
- Discover stationarity condition for AR(p) processes
- Relate MA and AR processes through duality

MA(q) process

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots \beta_q Z_{t-q}$$

Using Backward shift operator,

$$X_t = \left(\beta_0 + \beta_1 B + \cdots + \beta_q B^q\right) Z_t = \beta(B) Z_t$$

We obtain innovations Z_t in terms of present and past values of X_t,

$$Z_t = \beta(B)^{-1} X_t = (\alpha_0 + \alpha_1 B + \alpha_2 B^2 + \cdots) X_t$$

For this to hold, "complex roots of the polynomial $\beta(B)$ must lie outside of the unit circle where B is regarded as complex variable".

Invertibility condition for MA(q)

MA(q) process is invertible if the roots of the polynomial

$$\beta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

(Proof is done using mean-square convergence, see optional reading)

EX: MA(1) process

- $X_t = Z_t + \beta Z_{t-1}$

- $\beta(B) = 1 + \beta B$

- In this case only one (real) root $B = -\frac{1}{\beta}$
- $\left|-\frac{1}{\beta}\right| > 1 \Rightarrow |\beta| < 1.$

- Then, $Z_t = \sum_{k=0}^{\infty}(-\beta)^k B^k X_t = \sum_{k=0}^{\infty}(-\beta)^k X_{t-k}$

Example – MA(2) process

$$X_t = Z_t + \frac{5}{6}Z_{t-1} + \frac{1}{6}Z_{t-2}$$

Then,

$$X_t = \beta(B)Z_t$$

Where

$$\beta(B) = 1 + \frac{5}{6}B + \frac{1}{6}B^2$$

Example cont.

$$1 + \frac{5}{6}z + \frac{1}{6}z^2 = 0$$

$$z_1 = 2,\ z_2 = 3$$

Example cont.

$$\beta(B)^{-1} = \frac{1}{1+\frac{5}{6}B+\frac{1}{6}B^2} = \frac{3}{1+\frac{1}{2}B} - \frac{2}{1+\frac{1}{3}B}$$

$$\beta(B)^{-1} = \sum_{k=0}^{\infty}\left[3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k\right]B^k$$

$$Z_t = \sum_{k=0}^{\infty} \left[3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k \right] B^k X_t$$

$$Z_t = \sum_{k=1}^{\infty} \pi_k B^k X_t = \sum_{k=1}^{\infty} \pi_k X_{t-k}$$

Where

$$\pi_k = 3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k$$

MA(2) process $\Longrightarrow$ AR(∞) process

Stationarity condition for AR(p)

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

is (weakly) stationary if the roots of the polynomial

$$\phi\,(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

AR(1) process

$$X_t = \phi_1 X_{t-1} + Z_t \;\; \Longrightarrow \;\; (1 - \phi_1 B) X_t = Z_t$$

$$\phi(B) = 1 - \phi_1 B$$

$$\phi(z) = 1 - \phi_1 z = 0 \;\; \Longrightarrow \;\; z = \frac{1}{\phi_1}$$

$$|z| = \left|\frac{1}{\phi_1}\right| > 1 \;\Rightarrow |\phi_1| < 1$$

Thus, when $|\phi_1| < 1$, the AR(1) process is stationary.

$$X_t = \frac{1}{1 - \phi_1 B} Z_t = (1 + \phi_1 B + \phi_1 B^2 - \cdots) Z_t = \sum_{k=0}^{\infty} \phi_1^k Z_{t-k}$$

Another look at ϕ_1

Take Variance from both side,

$$Var[X_t] = Var\left[\sum_{k=0}^{\infty} \phi_1^k Z_{t-k}\right] = \sum_{k=0}^{\infty} \phi_1^{2k} \sigma_Z^2 = \sigma_Z^2 \sum_{k=0}^{\infty} \phi_1^{2k}$$

which is a convergent geometric series if $|\phi_1^2| < 1$, i.e.,

$$|\phi_1| < 1.$$

AR(p) process $\Longrightarrow$ MA(∞) process

Duality between AR and MA processes

Under invertibility condition of MA(q),

$$MA(q) \Longrightarrow AR(\infty)$$

Under stationarity condition of AR(p)

$$AR(p) \Longrightarrow MA(\infty)$$

What We’ve Learned

- Invertibility condition for MA(q) processes
- Stationarity condition for AR(p) processes
- Duality MA and AR processes