Introduction to Invertibility

PRACTICAL TIME SERIES ANALYSIS

THISTLETON AND SADIGOV

Objectives

- Learn invertibility of a stochastic process

Two MA(1) models

- Model 1

$$X_t = Z_t + 2Z_{t-1}$$

- Model 2

$$X_t = Z_t + \frac{1}{2}Z_{t-1}$$

Theoretical Auto Covariance Function of Model 1

$$\gamma(k) = Cov\,[X_{t+k}, X_t] = Cov\,[Z_{t+k} + 2Z_{t+k-1}, Z_t + 2Z_{t-1}]$$

If $k > 1$, then $t + k - 1 > t$, so all Z's are uncorrelated, thus $\gamma(k) = 0$.

If $k = 0$, then

$$\gamma(0) = Cov\,[Z_t + 2Z_{t-1}, Z_t + 2Z_{t-1}] =$$

$$Cov[Z_t, Z_t] + 4Cov[Z_{t-1}, Z_{t-1}] = \sigma_Z^2 + 4\sigma_Z^2 = 5\sigma_Z^2.$$

If $k = 1$, then

$$\gamma(1) = Cov\,[Z_{t+1} + 2Z_t, Z_t + 2Z_{t-1}] = Cov\,[2Z_t, Z_t] = 2\sigma_Z^2$$

If $k < 0$, then

$$\gamma(k) = \gamma(-k)$$

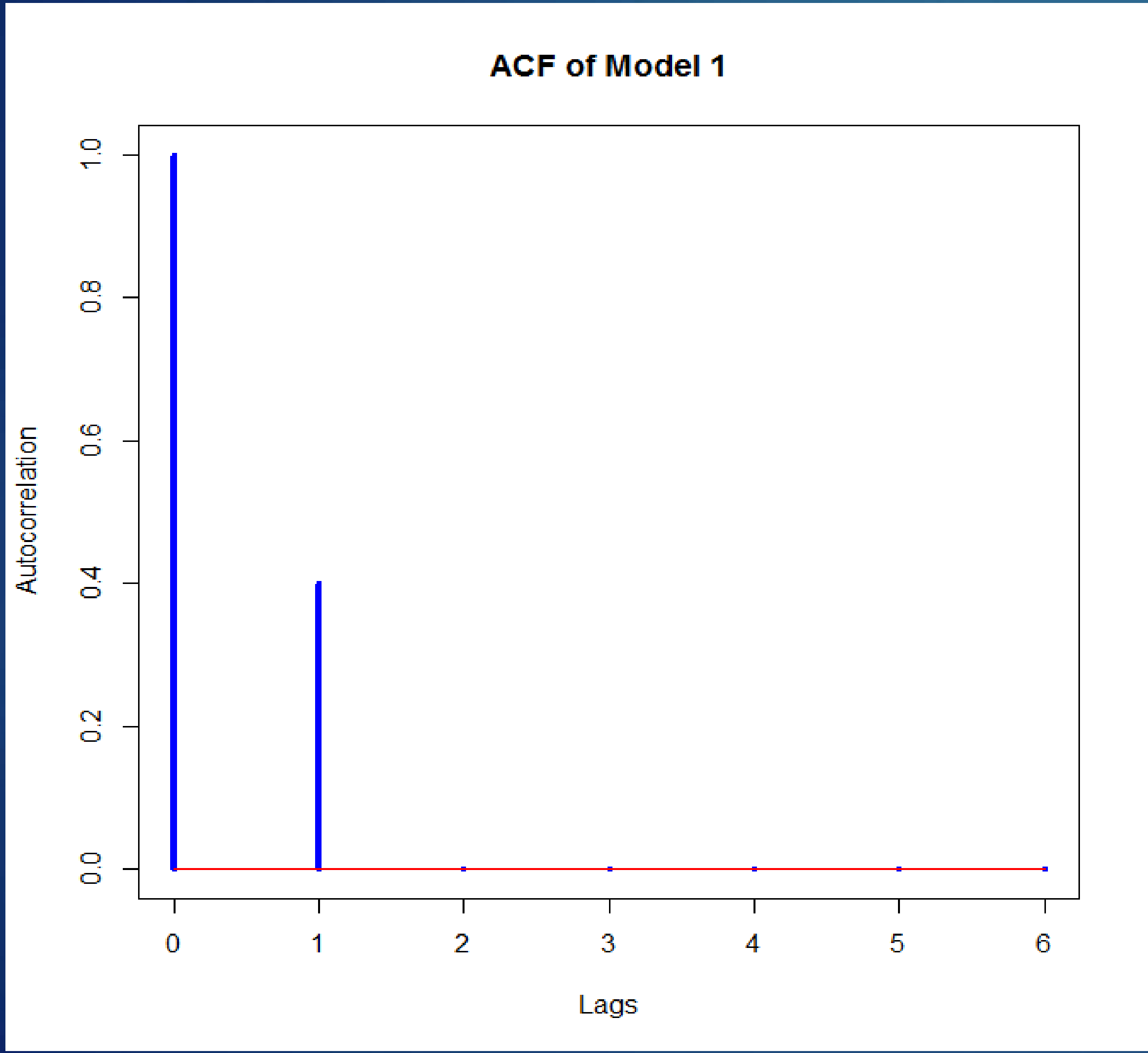
Auto Covariance Function and ACF of Model 1

$$\gamma(k) = \begin{cases} 0, & k > 1 \\ 2\sigma_Z^2, & k = 1 \\ 5\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}$$

Then, since $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$,

$$\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}$$

ACF



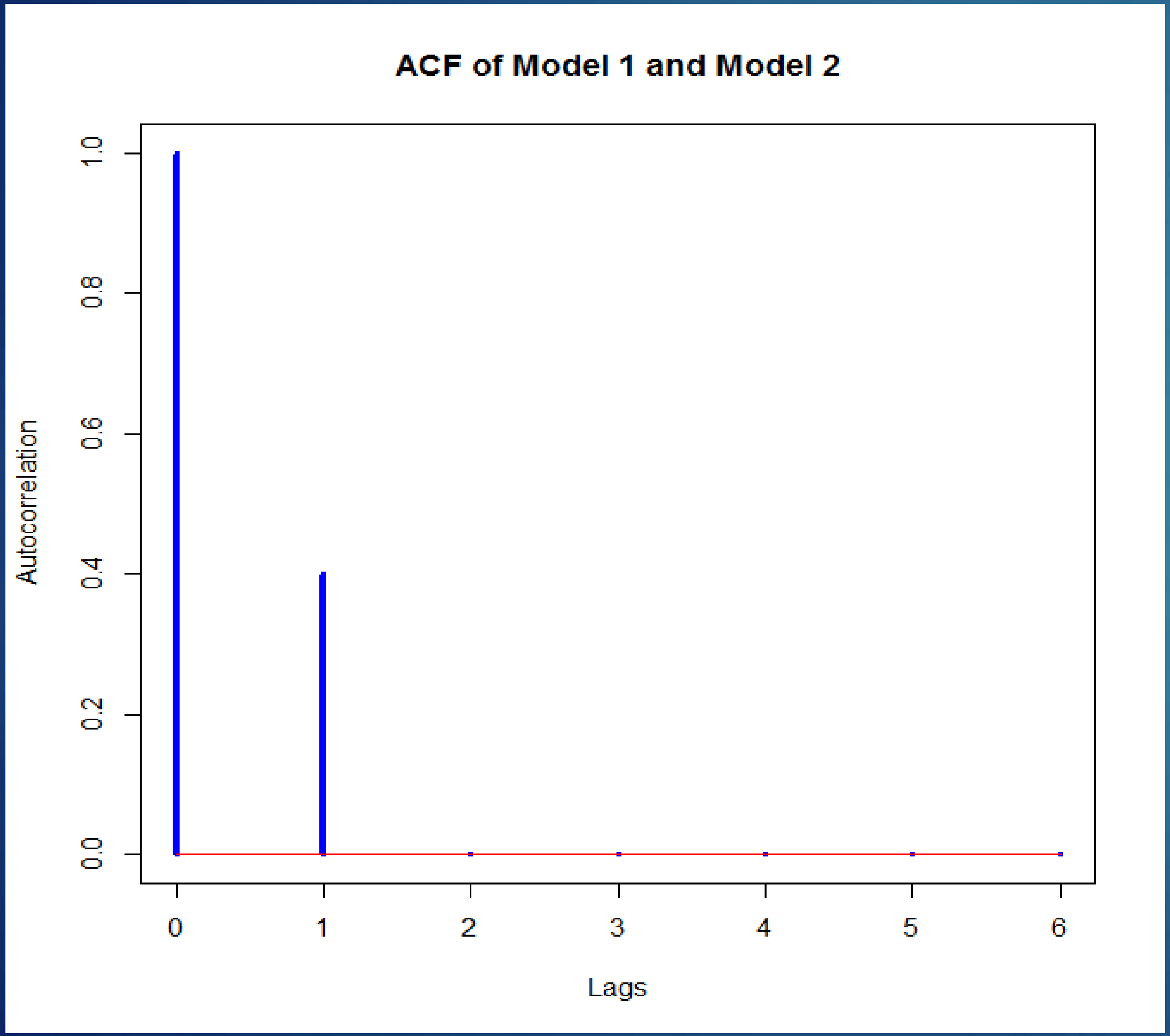
ACF of Model 2

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{Cov\left[Z_{t+1} + \frac{1}{2}Z_t,\, Z_t + \frac{1}{2}Z_{t-1}\right]}{Cov[Z_t + \frac{1}{2}Z_{t-1},\ \ Z_t + \frac{1}{2}Z_{t-1}]} = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = \frac{2}{5}.$$

Thus we obtain the same ACF:

$$\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}$$

ACFs are same!



Inverting through backward substitution

MA(1) process

$$X_t = Z_t + \beta Z_{t-1},$$

$$Z_t = X_t - \beta Z_{t-1} = X_t - \beta(X_{t-1} - \beta Z_{t-2}) = X_t - \beta X_{t-1} + \beta^2 Z_{t-2}$$

In this manner,

$$Z_t = X_t - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \cdots$$

i.e.,

$$X_t = Z_t + \beta X_{t-1} - \beta^2 X_{t-2} + \beta^3 X_{t-3} - \cdots$$

We 'inverted' MA(1) process to AR(∞).

Inverting using Backward shift operator

$$X_t = \beta(B)Z_t$$

where

$$\beta(B) = 1 + \beta B$$

Then, we find Z_t by inverting the polynomial operator $\beta(B)$:

$$\beta(B)^{-1}X_t = Z_t$$

Inverse of $\beta(B)$

$$\beta(B)^{-1} = \frac{1}{1+\beta B} = 1 - \beta B + \beta^2 B^2 - \beta^3 B^3 + \cdots$$

Here we expand the inverse of the polynomial operator as a 'rational function where βB is a complex number'.

Thus we obtain,

$$\beta(B)^{-1} X_t = 1 - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \cdots$$

$$Z_t = \sum_{n=0}^{\infty} (-\beta)^n X_{t-n}$$

In order to make sure that the sum on the right is convergent (in the mean-square sense), we need $|\beta| < 1$.

There is an optional reading titled “Mean-square convergence” where we explain this result.

Invertibility - Definition

Definition:

$\{X_t\}$ is a stochastic process.

$\{Z_t\}$ is innovations, i.e., random disturbances or white noise.

$\{X_t\}$ is called **invertible**, if $Z_t = \sum_{k=0}^{\infty} \pi_k X_{t-k}$ where $\sum_{k=0}^{\infty} |\pi_k|$ is convergent.

Model 1 vs Model 2

- Model 1 is **not invertible** since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} 2^k, \qquad \mathit{Divergent}$$

- Model 2 is **invertible** since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} \frac{1}{2^k}, \qquad \mathit{Geometric\ Series}, \qquad \mathit{Convergent}$$

Model choice

- For 'invertibility' to hold, we choose Model 2, since $\left|\frac{1}{2}\right| < 1$.
- This way, ACF uniquely determines the MA process.

What We've Learned

- Definition of invertibility of a stochastic process
- Invertibility condition guarantees unique MA process corresponding to observed ACF