Series and series representation

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- Recall infinite series and their convergence
- Examine geometric series
- Represent rational functions as a geometric series

Sequence and series

- Sequence $\{a_n\}$ is list of numbers in definite order

$$a_1, a_2, a_3, \dots a_n, \dots$$

- If the limit of the sequence exists, i..e,

$$\lim_{n\to\infty} a_n = a$$

then we say the sequence is convergent.

Examples

- $a_n = \frac{n}{n+1}$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \to 1$$

- $a_n = 3^n$

$$3, 9, 27, \dots, 3^n, \dots$$

- $a_n = \sqrt{n}$

$$1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots$$

- $a_n = \frac{1}{n^2}$

$$1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots \to 0$$

Partial sums

- Partial sums of a sequence $\{a_n\}$ are defined as

$$s_n = a_1 + a_2 + \cdots + a_n$$

- $s_1 = a_1$
- $s_2 = a_1 + a_2$
- $s_3 = a_1 + a_2 + a_3$

$\vdots$

Series

- If the partial sums $\{s_n\}$ is convergent to a number s, then we say

  the infinite series $\sum_{k=1}^{\infty} a_k$ is convergent, and is equal to s.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = s$$

- Otherwise, we say $\sum_{k=1}^{\infty} a_k$ is divergent.

Some convergent series

- $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$
- $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2)$

Some divergent series

- $\sum_{k=1}^{\infty} 3^k$
- $\sum_{k=1}^{\infty}(2k+1)$
- $\sum_{k=1}^{\infty} \frac{1}{k}$

Absolute convergence

- Series is absolutely convergent if

$$\sum_{k=1}^{\infty} |a_k|$$

is convergent.

- Absolute convergence implies convergence.

Convergence tests

- Integral test
- Comparison test
- Limit comparison test
- Alternating series test
- Ratio test
- Root test

Geometric series

- Geometric sequence

$$\{ar^{n-1}\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \dots\}$$

- Geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

- $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

  since $a = \frac{1}{2}, r = \frac{1}{2}$.

Series representation

- Series representation for $\frac{1}{1-x}$ $where\, a = 1, r = x$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

if $|x| < 1$.

Series representation cont.

- Series representation for $\frac{1}{(1-x)\left(1-\frac{x}{2}\right)}$

$$\frac{1}{(1-x)\left(1-\frac{x}{2}\right)} = \frac{2}{1-x} + \frac{-1}{1-\frac{x}{2}} = \sum_{k=0}^{\infty}\left(2-\frac{1}{2^k}\right)x^k$$

If $|x| < 1$ and $\left|\frac{x}{2}\right| < 1$, i.e., if $|x| < 1$.

Complex functions

Assume z is a complex number

$$\frac{a}{1-z} = a + az + az^2 + \cdots = \sum_{k=1}^{\infty} az^{k-1}$$

if $|z| < 1$.

What We've Learned

- The definition of infinite series and their convergence
- Geometric series is convergent if the multiplier has norm less than 1
- How to represent some rational functions as a geometric series