

CHAPTER 1

P.E. 1.1

$$(a) \vec{A} + \vec{B} = (1, 0, 3) + (5, 2, -6) = (6, 2, -3)$$

$$|\vec{A} + \vec{B}| = \sqrt{36 + 4 + 9} = \underline{7}$$

$$(b) 5\vec{A} - \vec{B} = (5, 0, 15) - (5, 2, -6) = \underline{(0, -2, 21)}$$

(c) The component of $\vec{A}$ along $\vec{a}_y$ is

$$A_y = \underline{0}$$

$$(d) 3\vec{A} + \vec{B} = (3, 0, 9) + (5, 2, -6) = (8, 2, 3)$$

A unit vector parallel to this vector

$$\text{is } \vec{a}_{||} = \frac{(8, 2, 3)}{\sqrt{64 + 4 + 9}}$$

$$= \underline{\underline{\pm (0.9117 \vec{a}_x + 0.2279 \vec{a}_y + 0.3419 \vec{a}_z)}}$$

P.E. 1.2

(a) The distance vector

$$\vec{r}_{QR} = \vec{r}_R - \vec{r}_Q = (0, 3, 2) - (2, 4, 6)$$

$$= \underline{\underline{-2\vec{a}_x - \vec{a}_y + 2\vec{a}_z}}$$

(b) The distance between Q and R is

$$|\vec{r}_{QR}| = \sqrt{4 + 1 + 4} = \underline{3}$$

$$(c) \text{Vector } \vec{r}_{QP} = \vec{r}_P - \vec{r}_Q = (1, -3, 5) - (2, 4, 6) = (-1, -7, -1)$$

$$\cos \theta_{PQR} = \frac{\vec{r}_{QR} \cdot \vec{r}_{QP}}{|\vec{r}_{QR}| |\vec{r}_{QP}|} = \frac{7}{3\sqrt{51}}$$

$$\theta_{PQR} = \underline{\underline{70.93^\circ}}$$

$$(d) \text{ Area} = \frac{1}{2} |\vec{r}_{QR} \times \vec{r}_{QP}| = \frac{1}{2} |(15, -4, 13)|$$

$$= \underline{\underline{10.12}}$$

P.E. 1.3

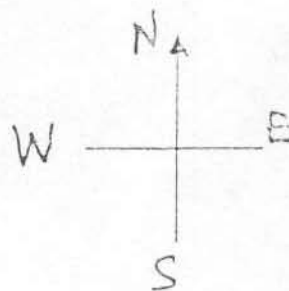
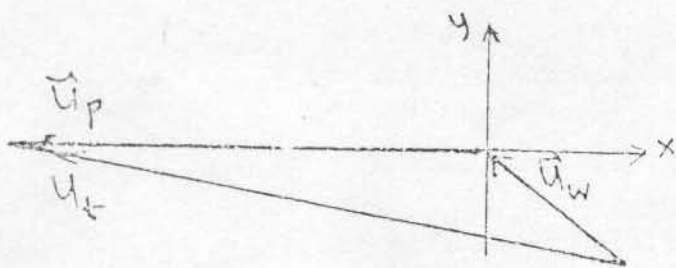
Consider the figure shown below.

$$\vec{U}_t = \vec{U}_p + \vec{U}_w = -350 \vec{a}_x + \frac{40}{\sqrt{2}} (-\vec{a}_x + \vec{a}_y)$$

$$= -278 \vec{a}_x + 28.28 \vec{a}_y$$

$$\text{or } U = 379.3 \angle -4.275^\circ$$

i.e. 379.3 km/hr at 4.275° north of west.



P.E. 1.4

At point $(1, 0)$, $\vec{G} = \vec{a}_y$;

at point $(0, 1)$, $\vec{G} = -\vec{a}_x$;

at point $(2, 0)$, $\vec{G} = \vec{a}_y$;

at point $(1, 1)$, $\vec{G} = \frac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}}$; and so on.

It is evident that $\vec{G}$ is a unit vector at each point. Thus the vector field $\vec{G}$ is as sketched in fig. 1.8.

P.E. 1.5

Using dot product,

$$\cos \theta_{AB} = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

or using cross product,

$$\sin \theta_{AB} = \frac{|\vec{A} \times \vec{B}|}{AB} = \sqrt{\frac{481}{650}}$$

Either way,

$$\theta_{AB} = \underline{\underline{120.56^\circ}}$$

P.E. 1.6

$$\begin{aligned} \text{(a)} \quad \vec{E}_F &= (\vec{E} \cdot \vec{a}_F) \vec{a}_F = \frac{(\vec{E} \cdot \vec{F}) \vec{F}}{|\vec{F}|^2} = \frac{-10(4, -10, 5)}{141} \\ &= \underline{\underline{-0.2837 \vec{a}_x + 0.7092 \vec{a}_y - 0.3546 \vec{a}_z}} \end{aligned}$$

$$(b) \vec{E} \times \vec{F} = \begin{vmatrix} 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55, 16, -12)$$

$$\begin{aligned} \vec{a}_{E \times F} &= \pm \frac{(55, 16, -12)}{\sqrt{55^2 + 16^2 + 12^2}} \\ &= \pm (0.9398, 0.2734, -0.205) \end{aligned}$$

P.E. 1.7

$$\vec{a} + \vec{b} + \vec{c} = 0$$

showing that $\vec{a}$, $\vec{b}$, and $\vec{c}$ form the sides of a triangle.

$$\vec{a} \cdot \vec{b} = 0,$$

hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\vec{b} \times \vec{c}| = \frac{1}{2} |\vec{c} \times \vec{a}|$$

$$\frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} \begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2} |(3, -17, 12)|$$

$$\text{Area} = \frac{1}{2} \sqrt{9 + 289 + 144} = \underline{\underline{10.5}}$$

P.E. 1.8

$$\begin{aligned} (a) P_1 P_2 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{25 + 4 + 64} = \underline{\underline{9.644}} \end{aligned}$$

$$\begin{aligned} (b) \vec{r}_p &= \vec{r}_{p_1} + \lambda (\vec{r}_{p_2} - \vec{r}_{p_1}) \\ &= (1, 2, -3) + \lambda (-5, -2, 8) \end{aligned}$$

$$= \underline{(1-5\lambda, 2-2\lambda, -3+8\lambda)}.$$

(e) The shortest distance is

$$d = P_1 P_3 \sin \theta = \left| \vec{P_1 P_3} \times \vec{a}_{P_1 P_2} \right|$$

$$= \frac{1}{\sqrt{93}} \begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{\sqrt{93}} \left| (-14, -73, -27) \right| = \underline{8.2}$$

Prob. 1.1

(a) $\vec{A} + 2\vec{B} = (2, 5, -3) + (6, -8, 0) = \underline{8\vec{a}_x - 3\vec{a}_y - 3\vec{a}_z}$

(b) $\vec{A} - 5\vec{C} = (2, 5, -3) - (5, 5, 5) = (-3, 0, -8)$

$$|\vec{A} - 5\vec{C}| = \sqrt{9 + 0 + 64} = \underline{8.544}$$

(c) $k\vec{B} = 3k\vec{a}_x - 4k\vec{a}_y$

$$|k\vec{B}| = \sqrt{9k^2 + 16k^2} = \pm 5k = 2$$

i.e. $\underline{k = \pm 0.4}$

(d) $\vec{A} \cdot \vec{B} = (2, 5, -3) \cdot (3, -4, 0) = 6 - 20 + 0 = 14$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 2 & 5 & -3 \\ 3 & -4 & 0 \end{vmatrix} = (-12, -9, -23)$$

$$\frac{\vec{A} \times \vec{B}}{\vec{A} \cdot \vec{B}} = \left(\frac{12}{14}, \frac{9}{14}, \frac{23}{14} \right) = \underline{0.857\vec{a}_x + 0.6428\vec{a}_y + 1.642\vec{a}_z}$$

Prob. 1.2

$$(a) \vec{A} \cdot \vec{B} = 6 + 5 + 20 = \underline{\underline{31}}$$

$$(b) \vec{A} \times \vec{B} = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 5 \end{vmatrix} = \underline{\underline{2\vec{a}_x + 2\vec{a}_y - 13\vec{a}_z}}$$

$$(c) \vec{A} \cdot (\vec{B} - \vec{C}) = (2, 5, 4) \cdot (2, 1, 11) = \underline{\underline{53}}$$

$$(d) \vec{A} \times (\vec{B} + \vec{C}) = \begin{vmatrix} 2 & 5 & 4 \\ 4 & 1 & -1 \end{vmatrix} = \underline{\underline{-9\vec{a}_x + 18\vec{a}_y - 18\vec{a}_z}}$$

Prob. 1.3

$$(a) \vec{P} \cdot \vec{Q} = 3 - 10 + 12 = \underline{\underline{5}}$$

$$(b) \cos \theta_{PQ} = \frac{\vec{P} \cdot \vec{Q}}{PQ} = \frac{5}{\sqrt{35}\sqrt{29}} = \underline{\underline{0.1569}}$$

$$(c) \vec{Q} \times \vec{R} = \begin{vmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \end{vmatrix} = \underline{\underline{4\vec{a}_x + 4\vec{a}_y - 5\vec{a}_z}}$$

$$(d) \sin \theta_{PQ} = \frac{|\vec{Q} \times \vec{R}|}{QR} = \frac{\sqrt{57}}{\sqrt{2}\sqrt{29}} = \underline{\underline{0.9913}}$$

$$(e) \vec{P} \cdot (\vec{Q} \times \vec{R}) = (1, -5, 3) \cdot (4, 4, -5) = 4 - 20 - 15 = \underline{\underline{-31}}$$

$$(f) \vec{Q} \cdot (\vec{R} \times \vec{P}) = \vec{P} \cdot (\vec{Q} \times \vec{R}) = \underline{\underline{-31}}$$

$$(g) \vec{P} \times (\vec{Q} \times \vec{R}) = \begin{vmatrix} 1 & -5 & 3 \\ 4 & 4 & -5 \end{vmatrix} = \underline{\underline{13\vec{a}_x + 17\vec{a}_y + 24\vec{a}_z}}$$

$$(h) \vec{P} \times \vec{Q} = \begin{vmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \end{vmatrix} = (-26, 5, 17)$$

$$(\vec{P} \times \vec{Q}) \times \vec{R} = \begin{vmatrix} -26 & 5 & 17 \\ 1 & -1 & 0 \end{vmatrix} = \underline{\underline{17\vec{a}_x + 17\vec{a}_y + 21\vec{a}_z}}$$

Prob. 1.4

$$\vec{A} \cdot \vec{B} = 0 \rightarrow 4\alpha^2 + 16 - 20\alpha = 0$$

$$\alpha^2 - 5\alpha + 4 = 0$$

$$\alpha = \frac{5 \pm \sqrt{25 - 16}}{2} = \underline{\underline{1, 4}}$$

Prob. 1.5

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal,

$$\vec{u} \cdot \vec{v} = 0 \rightarrow 2u_x - 5v_y - 3 = 0 \quad (1)$$

$$\vec{u} \cdot \vec{w} = 0 \rightarrow 6u_x + 5 - w_z = 0 \quad (2)$$

$$\vec{v} \cdot \vec{w} = 0 \rightarrow 12 - v_y + 3w_z = 0 \quad (3)$$

Solving (1) to (3) gives

$$\underline{\underline{u_x = -\frac{69}{44}, \quad v_y = -\frac{27}{22}, \quad w_z = -\frac{97}{22}}}$$

Prob. 1.6

$$\begin{aligned} (a) \quad \bar{T}_s &= \frac{\vec{T} \cdot \vec{a}_s}{s} = \frac{\vec{T} \cdot \vec{s}}{s} = \frac{(2, -6, -3) \cdot (1, 2, 1)}{\sqrt{6}} \\ &= \frac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}} \end{aligned}$$

$$\begin{aligned} (b) \quad \bar{S}_T &= (\vec{S} \cdot \vec{a}_T) \vec{a}_T = \frac{(\vec{S} \cdot \vec{T})}{T^2} \vec{T} = \frac{-7(2, -6, 3)}{(7)^2} \\ &= \underline{\underline{-0.2857 \vec{a}_x + 0.8571 \vec{a}_y - 0.4286 \vec{a}_z}} \end{aligned}$$

$$\begin{aligned} \text{(c) } \sin \theta_{TS} &= \frac{|\vec{T} \times \vec{S}|}{TS} = \frac{\begin{vmatrix} 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix}}{7\sqrt{6}} = \frac{|(-12, 1, 10)|}{7\sqrt{6}} \\ &= \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129 \\ \theta_{TS} &= \underline{\underline{65.91^\circ}} \end{aligned}$$

Prob. 1.7

$$\text{Let } \vec{P} = \vec{A} \times \vec{B} = \begin{vmatrix} 2 & 3 & -4 \\ -6 & -4 & 1 \end{vmatrix} = (-13, 22, 10)$$

$$\begin{aligned} \text{Scalar component} &= \vec{P} \cdot \vec{a}_c = \frac{\vec{P} \cdot \vec{C}}{C} = \frac{-25}{\sqrt{3}} \\ &= \underline{\underline{-14.43}} \end{aligned}$$

$$\text{Vector component} = (\vec{P} \cdot \vec{a}_c) \vec{a}_c = \frac{-25}{\sqrt{3}} \frac{(1, -1, 1)}{\sqrt{3}}$$

$$= \underline{\underline{-8.33 \vec{a}_x + 8.33 \vec{a}_y - 8.33 \vec{a}_z}}$$

Prob. 1.8

$$\cos \theta_x = \frac{\vec{H} \cdot \vec{a}_x}{H} = \frac{3}{\sqrt{9+25+64}} = \frac{3}{\sqrt{98}}$$

$$\theta_x = \underline{\underline{72.36^\circ}}$$

$$\cos \theta_y = \frac{\vec{H} \cdot \vec{a}_y}{H} = \frac{5}{\sqrt{98}}$$

$$\theta_y = \underline{\underline{59.66^\circ}}$$

$$\cos \theta_z = \frac{\vec{H} \cdot \vec{a}_z}{H} = \frac{-8}{\sqrt{98}} \rightarrow \theta_z = \underline{\underline{143.91^\circ}}$$

Prob. 1.9

$$\vec{Q} \times \vec{R} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = (3, -1, -2)$$

$$\vec{P} \cdot (\vec{Q} \times \vec{R}) = (2, -1, 1) \cdot (3, -1, 2) = 6 + 1 - 2 = \underline{\underline{5}}$$

Prob. 1.10

(a) Method 1: $\vec{A} \cdot (\vec{A} \times \vec{B}) = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$

Since two rows are repeated. for the same reason, $\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$.

Method 2:

$$\begin{aligned} \vec{A} \cdot (\vec{A} \times \vec{B}) &= A_x (\cancel{A_y B_z} - \cancel{A_z B_y}) \\ &\quad + A_y (\cancel{A_z B_x} - \cancel{A_x B_z}) \\ &\quad + A_z (\cancel{A_x B_y} - \cancel{A_y B_x}) = 0 \end{aligned}$$

The same idea applies for $\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$.

Method 3:

$$\begin{aligned} \text{Since } \vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{C} \cdot (\vec{A} \times \vec{B}), \\ \vec{A} \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\vec{A} \times \vec{A}) = 0 \\ \vec{B} \cdot (\vec{A} \times \vec{B}) &= \vec{A} \cdot (\vec{B} \times \vec{B}) = 0 \end{aligned}$$

(b) $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$, $|\vec{A} \times \vec{B}| = AB \sin \theta_{AB}$

$$(\vec{A} \cdot \vec{B})^2 + |\vec{A} \times \vec{B}|^2 = (AB)^2 (\cos^2 \theta_{AB} + \sin^2 \theta_{AB}) \\ = (AB)^2$$

(c) Since $A_x = \vec{A} \cdot \vec{a}_x$, $A_y = \vec{A} \cdot \vec{a}_y$ and $A_z = \vec{A} \cdot \vec{a}_z$,

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z \\ = (A_x \cdot \vec{a}_x) \vec{a}_x + (\vec{A} \cdot \vec{a}_y) \vec{a}_y + (\vec{A} \cdot \vec{a}_z) \vec{a}_z$$

Prob. 1.11

$$\vec{P}_1 P_2 = \vec{r}_{P_2} - \vec{r}_{P_1} = (-6, 0, -3)$$

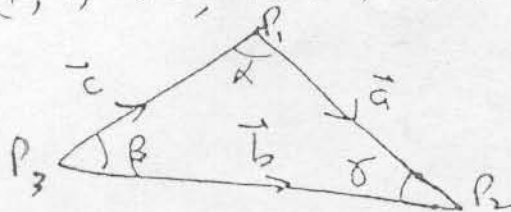
$$\vec{P}_1 P_3 = \vec{r}_{P_3} - \vec{r}_{P_1} = (1, 5, -6)$$

$$\vec{P}_1 P_2 \times \vec{P}_1 P_3 = \begin{vmatrix} -6 & 0 & -3 \\ 1 & 5 & -6 \end{vmatrix} = (15, 39, -30)$$

$$\text{Area of the triangle} = \frac{1}{2} |\vec{P}_1 P_2 \times \vec{P}_1 P_3| \\ = \frac{1}{2} \sqrt{15^2 + 39^2 + 30^2} \\ = \underline{\underline{25.72}}$$

Prob. 1.12

Let $P_1 = (4, 1, -3)$, $P_2 = (-2, 5, 4)$, and $P_3 = (0, 1, 6)$



$$\vec{a} = \vec{r}_{p_2} - \vec{r}_{p_1} = (-2, 5, 4) - (4, 1, -3) = (-6, 4, 7)$$

$$\vec{b} = \vec{r}_{p_3} - \vec{r}_{p_2} = (0, 1, 6) - (-2, 5, 4) = (2, -4, 2)$$

$$\vec{c} = \vec{r}_{p_1} - \vec{r}_{p_3} = (4, 1, -3) - (0, 1, 6) = (4, 0, -9)$$

Note that $\vec{a} + \vec{b} + \vec{c} = 0$.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= ab \cos(180 - \theta) \rightarrow -\cos \theta = \frac{\vec{a} \cdot \vec{b}}{ab} \\ &= \frac{-12 - 16 + 14}{\sqrt{101} \sqrt{24}} \end{aligned}$$

$$\theta = \cos^{-1} \frac{14}{\sqrt{101} \sqrt{24}} = \underline{\underline{73.47^\circ}}$$

$$\begin{aligned} \vec{b} \cdot \vec{c} &= bc \cos(180 - \beta) \rightarrow -\cos \beta = \frac{\vec{b} \cdot \vec{c}}{bc} \\ &= \frac{8 + 0 - 18}{\sqrt{24} \sqrt{97}} \end{aligned}$$

$$\beta = \cos^{-1} \frac{10}{\sqrt{24} \sqrt{97}} = \underline{\underline{78.04^\circ}}$$

$$\begin{aligned} \vec{a} \cdot \vec{c} &= ac \cos(180 - \alpha) \rightarrow -\cos \alpha = \frac{\vec{a} \cdot \vec{c}}{ac} \\ &= \frac{-24 + 0 - 63}{\sqrt{101} \sqrt{97}} \end{aligned}$$

$$\alpha = \cos^{-1} \frac{87}{\sqrt{101} \sqrt{97}} = \underline{\underline{28.48^\circ}}$$

Prob. 1.13

(a) let $\vec{A} = (A, B, C)$ and $\vec{r} = (x, y, z)$

$$(\vec{r} - \vec{A}) \cdot \vec{A} = (x - A)A + (y - B)B + (z - C)C$$

$$= Ax + By + Cz + D$$

where $D = -A^2 - B^2 - C^2$. Hence

$$(\vec{r} - \vec{A}) \cdot \vec{A} = 0 \longrightarrow Ax + By + Cz + D = 0$$

which is the equation of a plane.

$$(b) (\vec{r} - \vec{A}) \cdot \vec{r} = (x - A)x + (y - B)y + (z - C)z$$

$$= x^2 + y^2 + z^2 - Ax - By - Cz$$

If $(\vec{r} - \vec{A}) \cdot \vec{r} = 0$, then

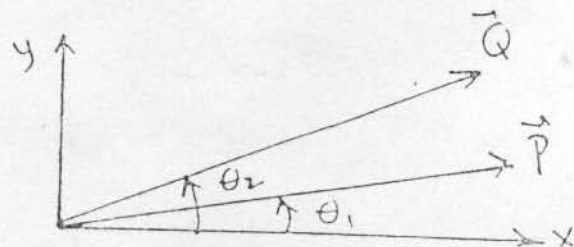
$$x^2 + y^2 + z^2 - Ax - By - Cz = 0$$

which is the equation of a sphere whose surface touches the origin.

(c) See parts (a) and (b).

Prob. 1.14

(a) Let $\vec{P}$ and $\vec{Q}$ be as shown below.



$$|\vec{P}| = \cos^2 \theta_1 + \sin^2 \theta_1 = 1, \quad |\vec{Q}| = \cos^2 \theta_2 + \sin^2 \theta_2 = 1$$

Hence $\vec{P}$ and $\vec{Q}$ are unit vectors.

$$(b) \vec{P} \cdot \vec{Q} = (1)(1) \cos(\theta_2 - \theta_1)$$

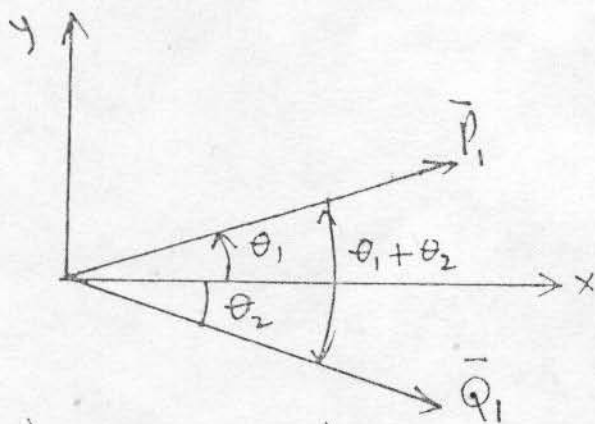
But $\vec{P} \cdot \vec{Q} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$. Thus

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}}$$

Let $\vec{P}_1 = \vec{P} = \cos \theta_1 \vec{a}_x + \sin \theta_1 \vec{a}_y$ and

$$\vec{Q}_1 = \cos \theta_2 \vec{a}_x - \sin \theta_2 \vec{a}_y,$$

$\vec{P}_1$ and $\vec{Q}_1$ are unit vectors shown below



$$\vec{P}_1 \cdot \vec{Q}_1 = (1)(1) \cos(\theta_1 + \theta_2)$$

But $\vec{P}_1 \cdot \vec{Q}_1 = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$, thus

$$\underline{\underline{\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in $\vec{Q}$.

$$\begin{aligned}
 \text{c) } \frac{1}{2} |\vec{P} - \vec{Q}| &= \frac{1}{2} |(\cos \theta_1 - \cos \theta_2) \vec{a}_x \\
 &\quad + (\sin \theta_1 - \sin \theta_2) \vec{a}_y| \\
 &= \frac{1}{2} \sqrt{\cos^2 \theta_1 + \sin^2 \theta_1 + \cos^2 \theta_2 + \sin^2 \theta_2 \\
 &\quad - 2 \cos \theta_1 \cos \theta_2 - 2 \sin \theta_1 \sin \theta_2} \\
 &= \frac{1}{2} \sqrt{2 - 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)} \\
 &= \frac{1}{2} \sqrt{2 - 2 \cos(\theta_2 - \theta_1)}
 \end{aligned}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between $\vec{P}$ and $\vec{Q}$.

$$\frac{1}{2} |\vec{P} - \vec{Q}| = \frac{1}{2} \sqrt{2 - 2 \cos \theta}$$

$$\text{But } \cos 2A = 1 - 2 \sin^2 A$$

$$\frac{1}{2} |\vec{P} - \vec{Q}| = \frac{1}{2} \sqrt{2 - 2 + 4 \sin^2 \theta/2} = \sin \theta/2$$

$$\therefore \frac{1}{2} |\vec{P} - \vec{Q}| = \left| \sin \frac{\theta_2 - \theta_1}{2} \right|$$

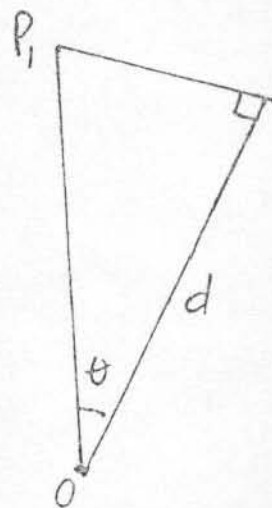
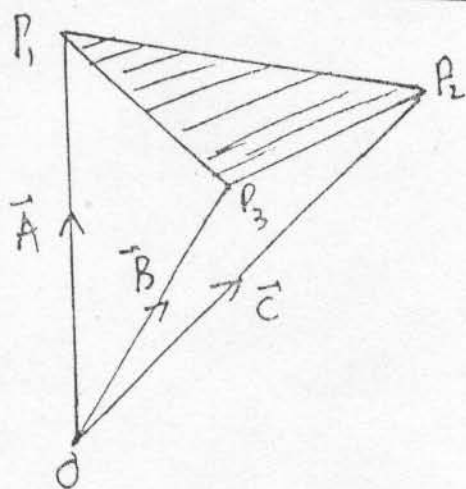
Prob. 1.15

a) Consider the figure on the next page.

$$\vec{P_2 P_1} = \vec{A} - \vec{B}, \quad \vec{P_3 P_1} = \vec{A} - \vec{C}$$

A unit normal to the plane is

$$\vec{\hat{n}} = \frac{\vec{P_2 P_1} \times \vec{P_3 P_1}}{|\vec{P_2 P_1} \times \vec{P_3 P_1}|} = \frac{(\vec{A} - \vec{B}) \times (\vec{A} - \vec{C})}{|(\vec{A} - \vec{B}) \times (\vec{A} - \vec{C})|}$$



The shortest distance from plane $P_1P_2P_3$ is the perpendicular distance from the plane to the origin, i.e.

$$d = OP_1 \cos \theta = \vec{A} \cdot \vec{a}_n = \vec{B} \cdot \vec{a}_n = \vec{C} \cdot \vec{a}_n.$$

Hence

$$d = \frac{\vec{A} \cdot [(\vec{A} - \vec{B}) \times (\vec{A} - \vec{C})]}{|(\vec{A} - \vec{B}) \times (\vec{A} - \vec{C})|}$$

as required.

(b) In (a), it is shown that a vector normal to the plane is

$$\begin{aligned} \vec{N} &= (\vec{A} - \vec{B}) \times (\vec{A} - \vec{C}) \\ &= \vec{A} \times \vec{A} - \vec{A} \times \vec{C} - \vec{B} \times \vec{A} + \vec{B} \times \vec{C} \\ &= \vec{A} \times \vec{B} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} \end{aligned}$$

(c) for this case,

$$\vec{A} - \vec{B} = (3, -3, -3), \quad \vec{A} - \vec{C} = (0, -2, 5)$$

$$(\vec{A} - \vec{B}) \times (\vec{A} - \vec{C}) = \begin{vmatrix} 3 & -3 & -3 \\ 0 & -2 & 5 \end{vmatrix} = -3(7, 5, 2)$$

$$d = \frac{(2, 1, 0) \cdot (-21, -15, -6)}{|21^2 + 15^2 + 6^2|^{1/2}} = \frac{-57}{\sqrt{702}} \\ = \underline{\underline{-2.151}}$$

Prob. 1.16

$$\vec{w} = \frac{w(1, -2, 3)}{3} = (1, -2, 2)$$

$$\vec{r} = \vec{r}_p - \vec{r}_0 = (1, 3, 4) - (2, -3, 1) = (-1, 6, 3)$$

$$\vec{u} = \vec{w} \times \vec{r} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18, -5, 4)$$

$$\underline{\underline{\vec{u} = -18\vec{a}_x - 5\vec{a}_y + 4\vec{a}_z}}$$

Prob. 1.17

$$\text{Let } \vec{A} = (10, -5, 4), \quad \vec{B} = (3, 1, 0)$$

$$\vec{A}_B = A \cos \theta_{AB} \vec{a}_B = \vec{A} \cdot \frac{\vec{B}}{B} \frac{\vec{B}}{B} = \frac{(\vec{A} \cdot \vec{B}) \vec{B}}{B^2}$$

$$= \frac{(25)(3, 1, 0)}{10} = \underline{\underline{7.5\vec{a}_x + 2.5\vec{a}_y}}$$

$$A_{\perp} = \vec{A} - \vec{A}_B = \underline{\underline{2.5\vec{a}_x - 7.5\vec{a}_y + 4\vec{a}_z}}$$

Prob. 1.18

(a) At T, $\vec{A} = (-4, 3, -9)$

$$|\vec{A}| = \sqrt{16 + 9 + 81} = \sqrt{106} = \underline{\underline{10.3}}$$

(b) Let $\vec{r}_{Ts} = \vec{B} = \vec{B} \vec{a}_B$

$$B = 5.6, \quad \vec{a}_B = \vec{a}_A = \frac{(-4, 3, -9)}{10.3}$$

$$\begin{aligned} \vec{r}_{Ts} = \vec{B} &= \frac{5.6(-4, 3, -9)}{10.3} \\ &= \underline{\underline{-2.175\vec{a}_x + 1.631\vec{a}_y - 4.893\vec{a}_z}} \end{aligned}$$

(c) $\vec{r}_{Ts} = \vec{r}_s - \vec{r}_T \longrightarrow \vec{r}_s = \vec{r}_T + \vec{r}_{Ts}$

$$\therefore \vec{r}_s = \underline{\underline{-0.175\vec{a}_x + 0.631\vec{a}_y - 1.893\vec{a}_z}}$$

Prob. 1.19

(a) At P(1, 2, 1), $\vec{Q} = (0, 9, 2)$ so that

$$\vec{a}_Q = \frac{\vec{Q}}{Q} = \frac{(0, 9, 2)}{\sqrt{85}} = \underline{\underline{(0, 0.9762, 0.2169)}}$$

(b) The component of $\vec{Q}$ along PT is

$$\vec{Q}_{PT} = |\vec{Q}| \cos \theta_{PT} \vec{a}_{PT} = (\vec{Q} \cdot \vec{a}_{PT}) \vec{a}_{PT}$$

$$\begin{aligned} \text{But } \vec{PT} &= \vec{r}_T - \vec{r}_P = (5, 3, 4) - (1, 2, 1) \\ &= (4, 1, 5) \end{aligned}$$

$$\vec{a}_{PT} = \frac{(4, 1, -5)}{\sqrt{42}}$$

$$\begin{aligned} \text{Hence } \vec{Q}_{PT} &= \frac{[(0, 9, 2) \cdot (4, 1, -5)]}{\sqrt{42}} \frac{(4, 1, -5)}{\sqrt{42}} \\ &= \frac{(-1)(4, 1, -5)}{42} \\ &= \underline{\underline{-0.09523 \vec{a}_x - 0.02381 \vec{a}_y + 0.119 \vec{a}_z}} \end{aligned}$$

c) When $\vec{Q} = (1, 11, 10)$.

$$2x - y = 1 \quad \text{--- (1)}$$

$$4y + z = 11 \quad \text{--- (2)}$$

$$4x - 2z = 10 \quad \text{--- (3)}$$

Solving for x , y , and z in (1) to (3) gives

$$(x, y, z) = \underline{\underline{(2, 3, -1)}}.$$

Prob 1.20

(a) At $(1, 2, 3)$, $\vec{E} = (2, 1, 6)$

$$|\vec{E}| = \sqrt{4 + 1 + 36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At $(1, 2, 3)$, $\vec{F} = (2, -4, 6)$

$$\begin{aligned} \hat{E}_F &= (\vec{E} \cdot \vec{a}_F) \vec{a}_F = \frac{(\vec{E} \cdot \vec{F})}{F^2} \vec{F} = \frac{36}{56} (2, -4, 6) \\ &= \underline{\underline{1.286 \vec{a}_x - 2.571 \vec{a}_y + 3.857 \vec{a}_z}} \end{aligned}$$

(c) At $(0, 1, -3)$, $\vec{E} = (0, 1, -3)$, $\vec{F} = (0, -1, 0)$

$$\vec{E} \times \vec{F} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3, 0, 0)$$

$$\hat{a}_{E \times F} = \pm \frac{\vec{E} \times \vec{F}}{|\vec{E} \times \vec{F}|} = \pm \underline{\underline{\hat{a}_x}}$$

Prob. 1.21

In the bac-cab rule,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}),$$

let $\vec{A} = \vec{C}$ be replaced by $\vec{a}_B$ and $\vec{B}$ by $\vec{A}$. We obtain

$$\vec{a}_B \times (\vec{A} \times \vec{a}_B) = \vec{A} - \vec{a}_B(\vec{a}_B \cdot \vec{A})$$

$$\therefore \vec{A} = (\vec{A} \cdot \vec{a}_B) \vec{a}_B + \vec{a}_B \times (\vec{A} \times \vec{a}_B).$$

CHAPTER 2

P.E. 2.1

(a) At $P(1, 3, 5)$, $x=1$, $y=3$, $z=5$,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{10}, \quad z=5, \quad \phi = \tan^{-1} \frac{y}{x} = 3$$

$$P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^\circ, 5)}}.$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916,$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\sqrt{10}}{5} = \tan^{-1} 0.6325 = 32.31^\circ$$

$$P(r, \theta, \phi) = \underline{\underline{P(5.916, 32.31^\circ, 71.56^\circ)}}$$

At $T(0, -4, 3)$, $x=0$, $y=-4$, $z=3$,

$$\rho = \sqrt{x^2 + y^2} = 4, \quad z=3, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-4}{0} = 270^\circ$$

$$T(\rho, \phi, z) = \underline{\underline{T(4, 270^\circ, 3)}}.$$

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \quad \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{4}{3} = 53.13^\circ$$

$$T(r, \theta, \phi) = \underline{\underline{T(5, 53.13^\circ, 270^\circ)}}.$$

At $S(-3, -4, -10)$, $x=-3$, $y=-4$, $z=-10$,

$$\rho = \sqrt{x^2 + y^2} = 5, \quad \phi = \tan^{-1} \frac{-4}{-3} = 233.1^\circ$$

$$S(\rho, \phi, z) = \underline{\underline{S(5, 233.1^\circ, -10)}}.$$

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18,$$

$$\theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{5}{-10} = 153.43^\circ$$

$$S(r, \theta, \phi) = \underline{\underline{S(11.18, 153.43^\circ, 233.1^\circ)}}.$$

(b) In cylindrical system,

$$\sqrt{x^2 + y^2} = \rho, \quad yz = z \rho \sin \phi,$$

$$Q_x = \frac{r}{\sqrt{r^2 + z^2}}, \quad Q_y = 0, \quad Q_z = -\frac{z r \sin \phi}{\sqrt{r^2 + z^2}}$$

$$\begin{bmatrix} Q_r \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix}$$

$$Q_r = Q_x \cos \phi = \frac{r \cos \phi}{\sqrt{r^2 + z^2}}, \quad Q_\phi = -Q_x \sin \phi = \frac{-r \sin \phi}{\sqrt{r^2 + z^2}}$$

Hence

$$\vec{Q} = \frac{r}{\sqrt{r^2 + z^2}} (\cos \phi \vec{a}_r - \sin \phi \vec{a}_\phi - z \sin \phi \vec{a}_z).$$

In spherical coordinates,

$$Q_x = \frac{r \sin \phi}{r} = \sin \phi, \quad Q_z = \frac{-r \sin \phi \sin \theta}{r \cos \theta} = -\sin \theta \cos \theta \sin \phi$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \phi \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix}$$

$$Q_r = Q_x \sin \theta \cos \phi + Q_z \cos \theta \\ = \sin^2 \theta \cos \phi - r \sin \theta \cos^2 \theta \sin \phi$$

$$Q_\theta = Q_x \cos \theta \cos \phi - Q_z \sin \theta \\ = \sin \theta \cos \theta \cos \phi + r \sin^2 \theta \cos \theta \sin \phi$$

$$\ddot{Q}_\phi = -Q_x \sin \phi = -\sin \theta \sin \phi$$

$$\begin{aligned} \ddot{\vec{Q}} = & \sin \theta (\sin \theta \cos \phi - r \cos^2 \theta \sin \phi) \ddot{a}_r \\ & + \sin \theta \cos \phi (\cos \phi + r \sin \theta \sin \phi) \ddot{a}_\theta \\ & - \sin \theta \sin \phi \ddot{a}_\phi \end{aligned}$$

At T,

$$\ddot{\vec{Q}}(x, y, z) = \frac{4}{5} \ddot{a}_x + \frac{12}{5} \ddot{a}_z = \underline{\underline{0.8 \ddot{a}_x + 2.4 \ddot{a}_z}}$$

$$\begin{aligned} \ddot{\vec{Q}}(\rho, \phi, z) = & \frac{4}{5} (\cos 270^\circ \ddot{a}_\rho - \sin 270^\circ \ddot{a}_\phi - 3 \sin 270^\circ \ddot{a}_z) \\ = & \underline{\underline{0.8 \ddot{a}_\phi + 2.4 \ddot{a}_z}}, \end{aligned}$$

$$\begin{aligned} \ddot{\vec{Q}}(r, \theta, \phi) = & \frac{4}{5} \left(0 - \frac{45}{25}(-1)\right) \ddot{a}_r + \frac{4}{5} \left(\frac{3}{5}\right) \left(0 - \frac{20}{5}(-1)\right) \ddot{a}_\theta \\ & - \frac{4}{5}(-1) \ddot{a}_\phi = \frac{36}{25} \ddot{a}_r + \frac{48}{25} \ddot{a}_\theta + \frac{4}{5} \ddot{a}_\phi \\ = & \underline{\underline{1.44 \ddot{a}_r + 1.92 \ddot{a}_\theta + 0.8 \ddot{a}_\phi}}. \end{aligned}$$

Note that $|\ddot{\vec{Q}}| = 2.53$ in the three cases.

P.E. 2.2

$$(a) \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z \sin \phi \\ 3 \rho \cos \phi \\ \rho \cos \phi \sin \phi \end{bmatrix}$$

$$\begin{aligned} \vec{A} = & (\rho z \cos \phi \sin \phi - 3 \rho \cos \phi \sin \phi) \ddot{a}_x \\ & + (\rho z \sin^2 \phi + 3 \rho \cos^2 \phi) \ddot{a}_y + \rho \cos \phi \sin \phi \ddot{a}_z. \end{aligned}$$

But $\rho = \sqrt{x^2 + y^2}$, $\tan \phi = \frac{y}{x} \rightarrow \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$,
 $\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}$. Substituting all this yields

$$\vec{A} = \frac{1}{\sqrt{x^2 + y^2}} \left[(xyz - 3xy) \vec{a}_x + (zy^2 + 3x^2) \vec{a}_y + xy \vec{a}_z \right].$$

$$(b) \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \theta \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \theta \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin \theta \end{bmatrix}$$

Since $r = \sqrt{x^2 + y^2 + z^2}$, $\tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$, $\tan \phi = \frac{y}{x}$,

$$\sin \theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_x = r^2 \sin \theta \cos \phi - \sin \theta \sin \phi = rx - \frac{y}{r} = \frac{1}{r} (r^2 x - y)$$

$$B_y = r^2 \sin \theta \sin \phi + \cos \phi \sin \theta = ry + \frac{x}{r} = \frac{1}{r} (r^2 y + x)$$

$$B_z = r^2 \cos \theta = rz = \frac{1}{r} (zr^2). \text{ Hence}$$

$$\vec{B} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[\{ x(x^2 + y^2 + z^2) - y \} \vec{a}_x + \{ y(x^2 + y^2 + z^2) + x \} \vec{a}_y + z(x^2 + y^2 + z^2) \vec{a}_z \right].$$

P.E.2.3

(a) At $(1, \pi/3, 0)$, $\vec{H} = (0, 0.5, 1)$

$$\vec{a}_x = \cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi = \frac{1}{2} (\vec{a}_\rho - \sqrt{3} \vec{a}_\phi)$$

$$\vec{H} \cdot \vec{a}_x = -\frac{\sqrt{3}}{4} = \underline{\underline{-0.433}}$$

(b) At $(1, \pi/3, 0)$, $\vec{a}_\theta = \cos \theta \vec{a}_1 - \sin \theta \vec{a}_2 = -\vec{a}_2$

$$\vec{H} \times \vec{a}_\theta = \begin{vmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & -1 \end{vmatrix} = \underline{\underline{-0.5 \vec{a}_\rho}}$$

(c) $(\vec{H} \cdot \vec{a}_\rho) \vec{a}_\rho = \underline{\underline{0 \vec{a}_\rho}}$

(d) $\vec{H} \times \vec{a}_2 = \begin{vmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.5 \vec{a}_\rho$

$$|\vec{H} \times \vec{a}_2| = \underline{\underline{0.5}}$$

P.E.2.4

(a) $\vec{A} \cdot \vec{B} = (3, 2, -6) \cdot (4, 0, 3) = \underline{\underline{-6}}$

(b) $|\vec{A} \times \vec{B}| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6 \vec{a}_r - 33 \vec{a}_\theta - 8 \vec{a}_\phi|$
 $= \underline{\underline{34.48}}$

(c) At $(1, \pi/3, 5\pi/4)$, $\theta = \pi/3$,

$$\vec{a}_r = \cos \theta \vec{a}_1 - \sin \theta \vec{a}_2 = \frac{1}{2} \vec{a}_r - \frac{\sqrt{3}}{2} \vec{a}_\theta$$

$$(\vec{A} \cdot \vec{a}_r) \vec{a}_r = \left(\frac{3}{2} - \sqrt{3}\right) \left(\frac{1}{2} \vec{a}_r - \frac{\sqrt{3}}{2} \vec{a}_\theta\right)$$

$$= \underline{\underline{-0.116 \vec{a}_r + 0.201 \vec{a}_\theta}}$$

Prob. 2.1

$$(a) \begin{aligned} x &= \rho \cos \phi = 5 \cos 120^\circ = -2.5 \\ y &= \rho \sin \phi = 5 \sin 120^\circ = 4.33, \quad z = 0 \\ P_1(x, y, z) &= \underline{\underline{P_1(-2.5, 4.33, 0)}}. \end{aligned}$$

$$(b) \begin{aligned} x &= 1 \cos 30^\circ = 0.866, \quad y = 1 \sin 30^\circ = 0.5, \quad z = -10 \\ P_2(x, y, z) &= \underline{\underline{P_2(0.866, 0.5, -10)}}. \end{aligned}$$

$$(c) \begin{aligned} x &= r \sin \theta \cos \phi = 10 \sin 135^\circ \cos 90^\circ = 0 \\ y &= r \sin \theta \sin \phi = 10 \sin 135^\circ \sin 90^\circ = 7.071 \\ z &= r \cos \theta = 10 \cos 135^\circ = -7.071 \\ P_3(x, y, z) &= \underline{\underline{P_3(0, 7.071, -7.071)}} \end{aligned}$$

$$(d) \begin{aligned} x &= 3 \sin 30^\circ \cos 240^\circ = -0.75 \\ y &= 3 \sin 30^\circ \sin 240^\circ = -1.299 \\ z &= r \cos \theta = 3 \cos 30^\circ = 2.598 \\ P_4(x, y, z) &= \underline{\underline{P_4(-0.75, -1.299, 2.598)}}. \end{aligned}$$

Prob. 2.2

$$(a) \begin{aligned} \rho &= \sqrt{x^2 + y^2} = \sqrt{17} = 4.123, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-4}{1} \\ &= 284.04^\circ \end{aligned}$$

$$P(\rho, \phi, z) = \underline{\underline{P(4.123, 284.04^\circ, -3)}}.$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 9} = 5.099, \quad \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{\sqrt{17}}{-3} = 126.04^\circ$$

$$P(r, \theta, \phi) = \underline{\underline{P(5.099, 126.04^\circ, 284.04^\circ)}}.$$

$$(b) \rho = 3, \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{0}{3} = 0^\circ$$

$$Q(\rho, \phi, z) = \underline{\underline{Q(3, 0^\circ, 5)}}.$$

$$r = \sqrt{9 + 0 + 25} = 5.831, \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{3}{5} = 30.96^\circ$$

$$Q(r, \theta, \phi) = \underline{\underline{Q(5.831, 30.96^\circ, 0^\circ)}}.$$

$$(c) \rho = \sqrt{4 + 36} = 6.325, \phi = \tan^{-1} \frac{6}{-2} = 108.4^\circ$$

$$R(\rho, \phi, z) = \underline{\underline{R(6.325, 108.4^\circ, 0)}}.$$

$$r = \rho = 6.325, \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{6.325}{0} = 90^\circ$$

$$R(r, \theta, \phi) = \underline{\underline{R(6.325, 90^\circ, 108.4^\circ)}}.$$

Prob. 2.3

$$(a) \begin{bmatrix} p_\rho \\ p_\phi \\ p_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{p} = (y+z) (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi)$$

But $y = \rho \sin \phi, z = z$

$$\vec{p} = \underline{\underline{(\rho \sin \phi + z) (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi)}}.$$

$$\begin{bmatrix} P_r \\ P_\theta \\ P_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{P} = (y+z) (\sin\theta \cos\phi \vec{a}_r + \cos\theta \cos\phi \vec{a}_\theta - \sin\phi \vec{a}_\phi)$$

But $y = r \sin\theta \sin\phi$, $z = r \cos\theta$,

$$\vec{P} = r (\sin\theta \sin\phi + \cos\theta) (\sin\theta \cos\phi \vec{a}_r + \cos\theta \cos\phi \vec{a}_\theta - \sin\phi \vec{a}_\phi).$$

$$(b) \vec{Q}(r, \phi, z) = (y \cos\phi + xz \sin\phi) \vec{a}_r \\ + (-y \sin\phi + xz \cos\phi) \vec{a}_\theta + (x+y) \vec{a}_z.$$

But $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$,

$$\vec{Q} = \frac{1}{2} \rho \sin 2\phi (1+z) \vec{a}_r + \rho (z \cos^2\phi - \sin^2\phi) \vec{a}_\theta \\ + \rho (\cos\phi + \sin\phi) \vec{a}_z.$$

$$\vec{Q}(r, \theta, \phi) = [y \sin\theta \cos\phi + xz \sin\theta \sin\phi + (x+y) \cos\theta] \vec{a}_r \\ + [y \cos\theta \cos\phi + xz \cos\theta \sin\phi - (x+y) \sin\theta] \vec{a}_\theta \\ + [-y \sin\phi + xz \cos\phi] \vec{a}_\phi$$

But $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$,

$$\vec{Q} = \left[\frac{1}{2} r \sin 2\phi \sin^2\theta + \frac{1}{2} r^2 \sin^2\theta \cos\theta \sin 2\phi \right. \\ \left. + \frac{1}{2} r (\cos\phi + \sin\phi) \cos 2\theta \right] \vec{a}_r$$

$$+ \left[\frac{1}{4} r \sin 2\theta \sin 2\phi + \frac{1}{2} r^2 \cos^2 \theta \sin \theta \sin 2\phi - \frac{r}{2} (\cos \phi + \sin \phi) \sin^2 \theta \right] \bar{a}_\theta \\ + \left(-r \sin \theta \sin^2 \phi + \frac{1}{2} r^2 \sin 2\theta \cos^2 \phi \right) \bar{a}_\phi.$$

$$\textcircled{c} \vec{T}(r, \phi, z) = \left[\left(\frac{x^2}{x^2+y^2} - y^2 \right) \cos \phi + \left(\frac{xy}{x^2+y^2} + xy \right) \sin \phi \right] \bar{a}_r \\ + \left[\left(\frac{-x^2}{x^2+y^2} + y^2 \right) \sin \phi + \left(\frac{xy}{x^2+y^2} + xy \right) \cos \phi \right] \bar{a}_\phi + \bar{a}_z$$

But $x = r \cos \phi$, $y = r \sin \phi$, $z = z$,

$$\vec{T} = (\cos^3 \phi - r^2 \sin^2 \phi \cos \phi + \sin^2 \phi \cos \phi + r^2 \sin^2 \phi \cos \phi) \bar{a}_r \\ + (-\sin \phi \cos^2 \phi + r^2 \sin^3 \phi + \sin \phi \cos^2 \phi + r^2 \sin \phi \cos^2 \phi) \bar{a}_\phi + \bar{a}_z$$

$$\vec{T}(r, \phi, z) = \underline{\underline{\cos \phi \bar{a}_r + r^2 \sin \phi \bar{a}_\phi + \bar{a}_z}}$$

$$\begin{bmatrix} T_r \\ T_\theta \\ T_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \sin \phi & \cos \theta \cos \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x^2}{x^2+y^2} - y^2 \\ \frac{xy}{x^2+y^2} + xy \\ 1 \end{bmatrix}$$

Since $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$;

$$T_r = \left(\frac{r^2 \sin^2 \theta \cos^2 \phi}{r^2 \sin^2 \theta} - r^2 \sin^2 \theta \sin^2 \phi \right) \sin \theta \cos \phi$$

$$+ (\cos \phi \sin \phi + r^2 \sin^2 \theta \sin \phi \cos \phi) \sin \theta \sin \phi + \cos \theta$$

$$= \sin \theta \cos \phi + \cos \theta,$$

$$T_\theta = (\cos^2 \phi - r^2 \sin^2 \theta \sin^2 \phi) \cos \theta \sin \phi$$

$$+ (\cos \phi \sin \phi + r^2 \sin^2 \theta \sin \phi \cos \phi) \cos \theta \sin \phi - \sin \theta$$

$$= \cos \theta \cos \phi - \sin \theta$$

$$T_\phi = -(\cos^2 \phi - r^2 \sin^2 \theta \sin^2 \phi) \sin \phi$$

$$+ (\cos \phi \sin \phi + r^2 \sin^2 \theta \sin \phi \cos \phi) \cos \phi$$

$$= -\cos^2 \phi \sin \phi + r^2 \sin^2 \theta \sin^3 \phi + \sin \phi \cos \phi$$

$$+ r^2 \sin^2 \theta \sin \phi \cos^2 \phi = r^2 \sin^2 \theta \sin \phi$$

$$\vec{T}(r, \theta, \phi) = (\sin \theta \cos \phi + \cos \theta) \vec{a}_r$$

$$+ (\cos \theta \cos \phi - \sin \theta) \vec{a}_\theta$$

$$+ r^2 \sin^2 \theta \sin \phi \vec{a}_\phi.$$

$$(d) \begin{bmatrix} s_p \\ s_\phi \\ s_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{y}{x^2 + y^2} \\ -\frac{x}{x^2 + y^2} \\ 1 \end{bmatrix}$$

$$= \left(\frac{p \sin \phi}{p^2} \cos \phi - \frac{p \cos \phi}{p^2} \sin \phi \right) \vec{a}_p$$

$$+ \left(-\frac{\rho \sin \phi}{\rho^2} \sin \phi - \frac{\rho \cos \phi}{\rho^2} \cos \phi \right) \bar{a}_\phi + 10 \bar{a}_z$$

$$\underline{\underline{\vec{S}(\rho, \phi, z) = -\frac{1}{\rho} \bar{a}_\phi + 10 \bar{a}_z.}}$$

$$\begin{bmatrix} S_r \\ S_\theta \\ S_\phi \end{bmatrix} = \begin{bmatrix} \sin^2 \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \frac{y}{x^2 + y^2} \\ \frac{-x}{x^2 + y^2} \\ 10 \end{bmatrix}$$

$$= \left(\frac{r \sin \theta \sin \phi}{r^2 \sin^2 \theta} \sin \theta \cos \phi - \frac{r \sin \theta \cos \phi}{r^2 \sin^2 \theta} \sin \theta \cos \phi \right.$$

$$\left. + 10 \cos \theta \right) \bar{a}_r + \left(\frac{r \sin \theta \sin \phi}{r^2 \sin^2 \theta} \cos \theta \cos \phi \right.$$

$$\left. - \frac{r \sin \theta \cos \phi}{r^2 \sin^2 \theta} \cos \theta \sin \phi - 10 \sin \theta \right) \bar{a}_\theta$$

$$+ \left(-\frac{r \sin \theta \sin \phi}{r^2 \sin^2 \theta} \sin \phi - \frac{r \sin \theta \cos \phi}{r^2 \sin^2 \theta} \cos \phi \right) \bar{a}_\phi$$

$$\underline{\underline{\vec{S}(r, \theta, \phi) = 10 \cos \theta \bar{a}_r - 10 \sin \theta \bar{a}_\theta - \frac{1}{r \sin \theta} \bar{a}_\phi.}}$$

Prob. 2.4

$$\begin{aligned} \text{(a) } \bar{A} &= (x^2 z^2 \sin \phi + yz \sin \phi) \bar{a}_\rho + (yz \cos \phi - y^2 z^2 \cos \phi) \bar{a}_\phi \\ &\quad + yx \bar{a}_z \\ &= (x^2 z^2 + yz) \sin \phi \bar{a}_\rho + yz(1 - yz) \cos \phi \bar{a}_\phi \\ &\quad + xy \bar{a}_z \end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{aligned} A_x &= A_p \cos \phi - A_\phi \sin \phi = x^2 z^2 \cos \phi \sin \phi + y z \cancel{\sin \phi \cos \phi} \\ &\quad - y z \cancel{\sin \phi \cos \phi} + y^2 z^2 \sin \phi \cos \phi \\ &= (x^2 + y^2) z^2 \cos \phi \sin \phi = (x^2 + y^2) \cdot z^2 \cdot \frac{xy}{x^2 + y^2} = xy z^2 \end{aligned}$$

$$\begin{aligned} A_y &= A_p \sin \phi + A_\phi \cos \phi \\ &= x^2 z^2 \sin^2 \phi + y z \sin^2 \phi + y z \cos^2 \phi - y^2 z^2 \cos^2 \phi \\ &= y z + z^2 \left(\frac{x^2 \cdot y^2}{x^2 + y^2} - y^2 \cdot \frac{x^2}{x^2 + y^2} \right) = y z \end{aligned}$$

$$\vec{A}(x, y, z) = \underline{\underline{xy z^2 \vec{a}_x + y z \vec{a}_y + xy \vec{a}_z}}$$

$$\begin{aligned} (b) \vec{B} &= r^3 (-\sin \phi \vec{a}_x + \cos \phi \vec{a}_y) \\ &\quad + 4r [-\sin \theta \vec{a}_y + \cos \theta (\cos \phi \vec{a}_x + \sin \phi \vec{a}_y)] \cos \theta \\ &\quad + 6r^2 \sin \theta \cos \phi [\cos \theta \vec{a}_z + \sin \theta (\cos \phi \vec{a}_x + \sin \phi \vec{a}_y)] \\ &= \vec{a}_x (-r^3 \sin \phi + 4r \cos^2 \theta \sin \phi \cos \phi + 6r^2 \sin^2 \theta \cos \phi) \\ &\quad + \vec{a}_y (r^3 \cos \phi + 4r \cos^2 \theta \sin^2 \phi + 6r^2 \sin^2 \theta \sin \phi \cos \phi) \\ &\quad + \vec{a}_z (-4r \sin \theta + 6r^2 \sin \theta \cos \theta \cos \phi) \end{aligned}$$

$$-32 =$$

$$\text{But } r = \sqrt{x^2 + y^2 + z^2}, \tan \theta = \frac{\sqrt{x^2 + y^2}}{z}, \tan \phi = \frac{y}{x}$$

$$\sin \theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\vec{B}(x, y, z) = \left[6x^2 + \frac{4xyz^2}{(x^2 + y^2)(x^2 + y^2 + z^2)^{3/2}} - \frac{y(x^2 + y^2 + z^2)^{3/2}}{(x^2 + y^2)^{3/2}} \right] \vec{a}_x$$

$$+ \left[6xy + \frac{4y^2z^2}{(x^2 + y^2)(x^2 + y^2 + z^2)^{3/2}} + \frac{x(x^2 + y^2 + z^2)^{3/2}}{(x^2 + y^2)^{3/2}} \right] \vec{a}_y$$

$$+ \left[6xz - 4(x^2 + y^2)^{3/2} \right] \vec{a}_z.$$

Prob. 2.5

$$(a) \vec{a}_x \cdot \vec{a}_\rho = (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi) \cdot \vec{a}_\rho = \cos \phi$$

$$\vec{a}_x \cdot \vec{a}_\phi = (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi) \cdot \vec{a}_\phi = -\sin \phi$$

$$\vec{a}_y \cdot \vec{a}_\rho = (\sin \phi \vec{a}_\rho + \cos \phi \vec{a}_\phi) \cdot \vec{a}_\rho = \sin \phi$$

$$\vec{a}_y \cdot \vec{a}_\phi = (\sin \phi \vec{a}_\rho + \cos \phi \vec{a}_\phi) \cdot \vec{a}_\phi = \cos \phi$$

(b) Since $\vec{a}_\rho$, $\vec{a}_\phi$, and $\vec{a}_z$ are mutually orthogonal,

$$\vec{a}_z \cdot \vec{a}_z = 1, \quad \vec{a}_z \cdot \vec{a}_\rho = 0 = \vec{a}_z \cdot \vec{a}_\phi.$$

Also, $\vec{a}_x \cdot \vec{a}_z = 0 = \vec{a}_y \cdot \vec{a}_z$. Hence

$$\begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{a}_x \cdot \vec{a}_\rho & \vec{a}_x \cdot \vec{a}_\phi & \vec{a}_x \cdot \vec{a}_z \\ \vec{a}_y \cdot \vec{a}_\rho & \vec{a}_y \cdot \vec{a}_\phi & \vec{a}_y \cdot \vec{a}_z \\ \vec{a}_z \cdot \vec{a}_\rho & \vec{a}_z \cdot \vec{a}_\phi & \vec{a}_z \cdot \vec{a}_z \end{bmatrix}$$

(c) In spherical system,

$$\vec{a}_x = \sin \theta \cos \phi \vec{a}_r + \cos \theta \cos \phi \vec{a}_\theta - \sin \phi \vec{a}_\phi$$

$$\vec{a}_y = \sin \theta \sin \phi \vec{a}_r + \cos \theta \sin \phi \vec{a}_\theta + \cos \phi \vec{a}_\phi$$

$$\vec{a}_z = \cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta.$$

Hence $\vec{a}_x \cdot \vec{a}_r = \sin \theta \cos \phi,$

$$\vec{a}_x \cdot \vec{a}_\theta = \cos \theta \cos \phi,$$

$$\vec{a}_y \cdot \vec{a}_r = \sin \theta \sin \phi,$$

$$\vec{a}_y \cdot \vec{a}_\theta = \cos \theta \sin \phi,$$

$$\vec{a}_z \cdot \vec{a}_r = \cos \theta,$$

$$\vec{a}_z \cdot \vec{a}_\theta = -\sin \theta.$$

Prob. 2.6

$$(a) \quad r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$$

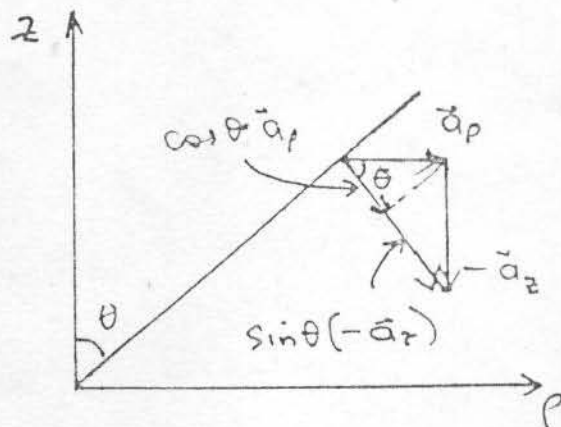
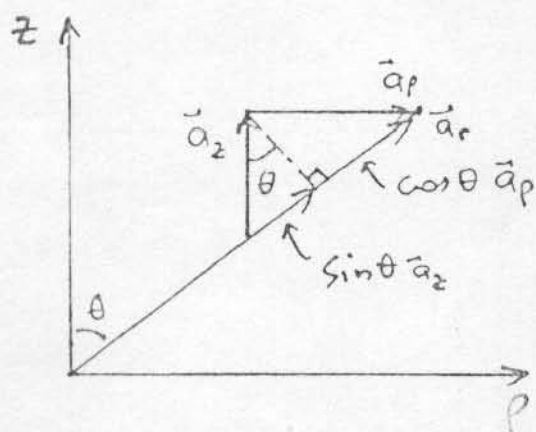
$$\theta = \tan^{-1} \frac{\rho}{z}, \quad \phi = \phi$$

$$\text{or} \quad \rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi}$$

$$= r \sin \theta,$$

$$z = r \cos \theta, \quad \phi = \phi.$$

(b) from the figures below,



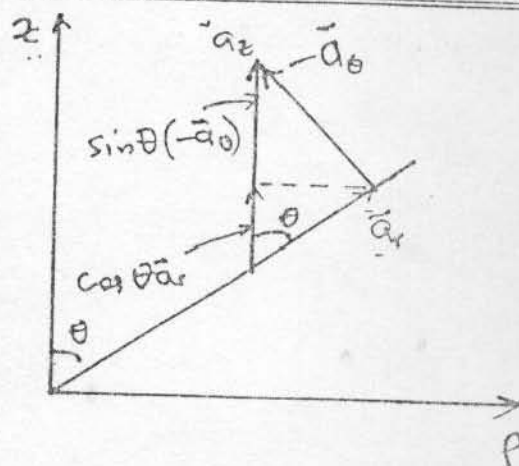
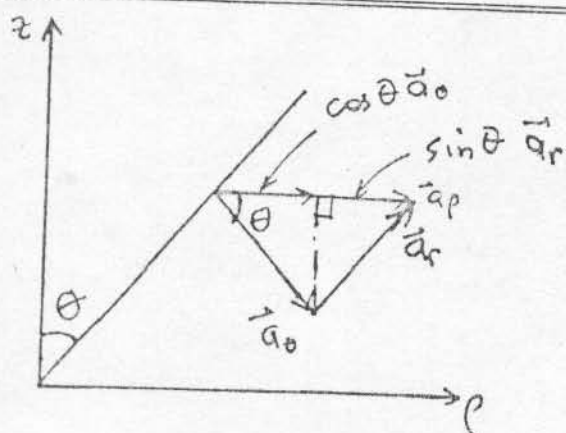
$$\vec{a}_r = \sin \theta \vec{a}_z + \cos \theta \vec{a}_\rho, \quad \vec{a}_\theta = \cos \theta \vec{a}_\rho - \sin \theta \vec{a}_z, \quad \vec{a}_\phi = \vec{a}_\phi$$

Hence

$$\begin{bmatrix} \vec{a}_r \\ \vec{a}_\theta \\ \vec{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{a}_\rho \\ \vec{a}_\phi \\ \vec{a}_z \end{bmatrix}$$

or from the figures on the next page,

$$\vec{a}_\rho = \cos \theta \vec{a}_\theta + \sin \theta \vec{a}_r, \quad \vec{a}_z = \cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta, \quad \vec{a}_\phi = \vec{a}_\phi$$



Hence

$$\begin{bmatrix} \vec{a}_\rho \\ \vec{a}_\phi \\ \vec{a}_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} \vec{a}_r \\ \vec{a}_\theta \\ \vec{a}_\phi \end{bmatrix}$$

Prob. 2.7

(a) Using the results in Prob. 2.6,

$$A_\rho = \rho z \sin \phi = r^2 \sin \theta \cos \theta \sin \phi,$$

$$A_\phi = 3\rho \cos \phi = 3r \sin \theta \cos \phi,$$

$$A_z = \rho \cos \phi \sin \phi = r \sin \theta \cos \phi \sin \phi.$$

Hence

$$\begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2 \sin \theta \cos \theta \sin \phi \\ 3r \sin \theta \cos \phi \\ r \sin \theta \cos \phi \sin \phi \end{bmatrix}$$

$$\vec{A}(r, \theta, \phi) = r \sin \theta \left[\sin \phi \cos \theta (r \sin \theta + \cos \phi) \vec{a}_r + \sin \phi (r \cos^2 \theta - \sin \theta \cos \phi) \vec{a}_\theta + 3 \cos \phi \vec{a}_\phi \right].$$

At $(10, \pi/2, 3\pi/4)$, $r=10$, $\theta = \pi/2$, $\phi = 3\pi/4$,

$$\vec{A} = 10 \left(0\vec{a}_r + \frac{1}{2}\vec{a}_\theta - \frac{3}{\sqrt{2}}\vec{a}_\phi \right) = \underline{\underline{5\vec{a}_\theta - 21.21\vec{a}_\phi}}$$

$$(b) \vec{B}_r = r^2 = (\rho^2 + z^2), \quad B_\theta = 0, \quad B_\phi = \sin\theta = \frac{\rho}{\sqrt{\rho^2 + z^2}}$$

$$\begin{bmatrix} B_r \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\underline{\underline{\vec{B}(\rho, \phi, z) = \sqrt{\rho^2 + z^2} \left(\rho \vec{a}_\rho + \frac{\rho}{\rho^2 + z^2} \vec{a}_\theta + z \vec{a}_z \right)}}$$

At $(2, \pi/6, 1)$, $\rho=2$, $\phi = \pi/6$, $z=1$,

$$\vec{B} = \sqrt{5} \left(2\vec{a}_\rho + \frac{2}{5}\vec{a}_\theta + \vec{a}_z \right)$$

$$= \underline{\underline{4.472\vec{a}_\rho + 0.8944\vec{a}_\theta + 2.236\vec{a}_z}}$$

Prob. 2.8

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Prob. 2.9

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b) $d^2 = 3^2 + 5^2 - 2(3)(5) \cos \pi + (-1-5)^2 = 100$
 $d = \sqrt{100} = \underline{\underline{10}}$

(c) $d^2 = 10^2 + 5^2 - 2(10)(5) \cos \frac{\pi}{4} \cos \frac{\pi}{6}$
 $- 2(10)(5) \sin \frac{\pi}{4} \sin \frac{\pi}{6} \cos \left(\frac{7\pi}{4} - \frac{3\pi}{4} \right)$
 $d = \sqrt{99.12} = \underline{\underline{9.956}}$

Prob. 2.10

(a) $r = \sqrt{8^2 + (-15)^2 + 12^2} = 20.81$
 $\theta = \tan^{-1} \sqrt{x^2+y^2}/z = \tan^{-1} \frac{17}{12} = 54.78^\circ$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-15}{8} = 298^\circ$$

$$P(r, \theta, \phi) = \underline{P(20.81, 54.78^\circ, 298^\circ)}.$$

$$(b) \begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xy \\ -x^2 \\ 0 \end{bmatrix}$$

$$F_r = 2xy \cos \phi - x^2 \sin \phi, \quad F_\phi = -2xy \sin \phi - x^2 \cos \phi,$$

$$F_z = 0.$$

$$\text{But } x = r \cos \phi, \quad y = r \sin \phi,$$

$$F_r = r^2 (2 \cos \phi \sin \phi - \cos^2 \phi \sin \phi) = r^2 \cos^2 \phi \sin \phi$$

$$F_\phi = -r^2 (2 \cos \phi \sin^2 \phi + \cos^3 \phi) = -r^2 \cos \phi (1 + \sin^2 \phi)$$

$$\underline{\underline{\vec{F} = r^2 \cos \phi [\cos \phi \sin \phi \hat{a}_r - (1 + \sin^2 \phi) \hat{a}_\phi]}}$$

Prob. 2.11

(a) An infinite line parallel to the z -axis.

(b) Point $(2, -1, 10)$.

(c) A circle of radius $r \sin \theta = 5$, i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z -axis.

(e) A semi-infinite line parallel to the x - y plane.

(f) A semi-circle of radius 5 in the x-y plane.

Prob. 2.12

At $T(2, 3, -4)$,

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\sqrt{13}}{-4} = 137.97^\circ$$

$$\cos \theta = -\frac{4}{\sqrt{29}} = -0.7428, \quad \sin \theta = \frac{\sqrt{13}}{\sqrt{29}} = 0.6695$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{3}{2} = 56.31^\circ$$

$$\cos \phi = \frac{2}{\sqrt{13}}, \quad \sin \phi = \frac{3}{\sqrt{13}}$$

$$\bar{a}_z = \cos \theta \bar{a}_r - \sin \theta \bar{a}_\theta = \underline{\underline{-0.7428 \bar{a}_r - 0.6695 \bar{a}_\theta}}$$

$$\begin{aligned} \bar{a}_r &= \sin \theta \cos \phi \bar{a}_x + \sin \theta \sin \phi \bar{a}_y + \cos \theta \bar{a}_z \\ &= \underline{\underline{0.3714 \bar{a}_x + 0.5571 \bar{a}_y - 0.7428 \bar{a}_z}} \end{aligned}$$

Prob. 2.13

At $P(0, 2, -5)$, $\phi = 90^\circ$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_p \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\vec{B} = -\bar{a}_x - 5\bar{a}_y - 3\bar{a}_z$$

$$(c) \vec{A} + \vec{B} = (2, 4, 10) + (-1, -5, -3) = \underline{\underline{\bar{a}_x - \bar{a}_y + 7\bar{a}_z}}$$

$$(b) \cos \theta_{AB} = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{-52}{\sqrt{4200}} \rightarrow \underline{\underline{\theta_{AB} = 143.26^\circ}}$$

$$(c) A_B = \vec{A} \cdot \vec{a}_B = \frac{\vec{A} \cdot \vec{B}}{B} = \frac{-52}{\sqrt{35}} = \underline{\underline{-8.789}}$$

Prob. 2.14

$$\text{At } P(8, 30^\circ, 60^\circ) = P(r, \theta, \phi),$$

$$x = r \sin \theta \cos \phi = 8 \sin 30^\circ \cos 60^\circ = 2$$

$$y = r \sin \theta \sin \phi = 8 \sin 30^\circ \sin 60^\circ = 2\sqrt{3}$$

$$z = r \cos \theta = 4\sqrt{3}$$

$$\vec{G} = 14 \vec{a}_x + 8\sqrt{3} \vec{a}_y + (48 + 24) \vec{a}_z = (14, 13.86, 72)$$

$$\vec{a}_\phi = -\sin \phi \vec{a}_x + \cos \phi \vec{a}_y = -\frac{\sqrt{3}}{2} \vec{a}_x + \frac{1}{2} \vec{a}_y$$

$$\begin{aligned} \vec{G}_\phi &= (\vec{G} \cdot \vec{a}_\phi) \vec{a}_\phi = (-7\sqrt{3} + 4\sqrt{3}) \frac{1}{2} (-\sqrt{3} \vec{a}_x + \vec{a}_y) \\ &= \underline{\underline{4.5 \vec{a}_x - 2.598 \vec{a}_y}} \end{aligned}$$

Prob. 2.15

$$(a) \vec{T}_z = (\vec{T} \cdot \vec{a}_z) \vec{a}_z.$$

$$\text{At } (2, \pi/2, 3\pi/2), \vec{a}_z = \cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta = -\vec{a}_\theta.$$

$$\vec{T}_z = -\cos 2\theta \sin \phi \vec{a}_\theta = -\cos \pi \sin \frac{3\pi}{2} \vec{a}_\theta = \underline{\underline{-\vec{a}_\theta}}$$

$$(b) \vec{T}_\phi = \tan \frac{\theta}{2} \ln r \vec{a}_\phi = \tan \frac{\pi}{4} \ln 2 \vec{a}_\phi = \ln 2 \vec{a}_\phi = \underline{\underline{0.6931 \vec{a}_\phi}}$$

$$\begin{aligned} (c) \vec{T}_t &= \vec{T} - \vec{T}_n = \vec{T} - \vec{T}_r = -\vec{a}_\theta + \ln 2 \vec{a}_\phi \\ &= \underline{\underline{-\vec{a}_\theta + 0.6931 \vec{a}_\phi}} \end{aligned}$$

$$(d) \vec{T}_\rho = (\vec{T} \cdot \vec{a}_\rho) \vec{a}_\rho,$$

$$\vec{a}_\rho = \sin \theta \cos \phi \vec{a}_r + \cos \theta \cos \phi \vec{a}_\theta - \sin \phi \vec{a}_\phi = \vec{a}_\phi$$

at $(2, \frac{1}{2}, 3\frac{1}{2})$,

$$\vec{J}_p = \ln 2 \vec{a}_\phi = \underline{\underline{0.6931 \vec{a}_\phi}}$$

Prob. 2.16

At $P(3, -2, 6)$, $x=3$, $y=-2$, $z=6$,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{13}, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-2}{3}$$

$$r = \sqrt{x^2 + y^2 + z^2} = 7, \quad \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{\sqrt{13}}{6}$$

$$\begin{aligned} \text{(a)} \quad \vec{A} &= \sqrt{13} \frac{3}{\sqrt{13}} \vec{a}_\rho + 6 \left(\frac{-2}{\sqrt{13}} \right) \vec{a}_\phi - \sqrt{13} (36) \vec{a}_z \\ &= \underline{\underline{3 \vec{a}_\rho - 3.328 \vec{a}_\phi - 129.8 \vec{a}_z}} \end{aligned}$$

$$\vec{B} = 7 \left(\frac{\sqrt{13}}{7} \right) \vec{a}_r + 49 \left(\frac{6}{7} \right) \left(\frac{-2}{\sqrt{13}} \right) \vec{a}_\phi = \underline{\underline{3.606 \vec{a}_r - 23.3 \vec{a}_\phi}}$$

$$\text{(b)} \quad \vec{A}_B = (\vec{A} \cdot \vec{a}_B) \vec{a}_B = \frac{(\vec{A} \cdot \vec{B}) \vec{B}}{B^2}$$

In spherical system,

$$\begin{aligned} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} &= \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \\ \frac{6}{7} & 0 & -\frac{\sqrt{13}}{7} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -12 \\ \frac{\sqrt{13}}{7} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{13}}{7} - \frac{216\sqrt{13}}{7} \\ \frac{18}{7} + \frac{36 \times 13}{7} \\ -\frac{12}{\sqrt{13}} \end{bmatrix} \end{aligned}$$

$$\vec{A} = -109.71 \vec{a}_r + 69.43 \vec{a}_\theta - 3.328 \vec{a}_\phi$$

Hence $\vec{A}_B = \frac{-2013.33}{555.9} (3.606 \vec{a}_r - 23.3 \vec{a}_\theta)$
 $= \underline{-13.06 \vec{a}_r + 84.39 \vec{a}_\theta}$

(c) In cylindrical system,

$$\begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{13}}{7} & \frac{6}{7} & 0 \\ 0 & 0 & 1 \\ \frac{6}{7} & -\frac{\sqrt{13}}{7} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{13} \\ -\frac{12 \times 49}{7\sqrt{13}} \\ 0 \end{bmatrix}$$

$$\vec{B} = \left(\frac{13}{7} - \frac{72}{\sqrt{13}} \right) \vec{a}_\theta + \left(\frac{6\sqrt{13}}{7} + 12 \right) \vec{a}_z = -18.11 \vec{a}_\theta + 15.09 \vec{a}_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 3 & -3.328 & -129.8 \\ -18.11 & 0 & 15.09 \end{vmatrix}$$

$$= -52.92 \vec{a}_\rho + 2396 \vec{a}_\theta - 60.27 \vec{a}_z$$

$$\hat{a}_{A \times B} = \underline{\pm (0.0221 \vec{a}_\rho - 0.9994 \vec{a}_\theta + 0.025 \vec{a}_z)}$$

Prob. 2.17

(a) $F_z = F \cos \gamma = F \cos \theta \rightarrow \gamma = \theta,$

$F_x = F \sin \theta \cos \phi = F \cos \alpha \rightarrow \cos \alpha = \sin \theta \cos \phi,$

$F_y = F \sin \theta \sin \phi = F \sin \beta \rightarrow \cos \beta = \sin \theta \sin \phi,$

Hence,

$$\alpha = \cos^{-1}(\sin \theta \cos \phi), \beta = \cos^{-1}(\sin \theta \sin \phi), \gamma = \theta$$

$$(b) \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta = \sin^2 \theta + \cos^2 \theta = 1$$

Alternatively,

$$\frac{\vec{F} \cdot \vec{F}}{|\vec{F}|^2} = 1 \rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Prob. 2.18

$$(a) \text{ for } (x, y, z) = (-2, 3, 6),$$

$$r = \sqrt{x^2 + y^2 + z^2} = 7$$

$$x = r \cos \alpha \rightarrow \cos \alpha = \frac{x}{r} = -\frac{2}{7} \rightarrow \alpha = 106.6^\circ$$

$$y = r \cos \beta \rightarrow \cos \beta = \frac{y}{r} = \frac{3}{7} \rightarrow \beta = 64.6^\circ$$

$$z = r \cos \gamma \rightarrow \cos \gamma = \frac{z}{r} = \frac{6}{7} \rightarrow \gamma = 31^\circ$$

Hence,

$$(r, \alpha, \beta, \gamma) = \underline{(7, 106.6^\circ, 64.6^\circ, 31^\circ)}$$

$$(b) \text{ for } (p, \phi, z) = (4, 30^\circ, -3),$$

$$r = \sqrt{p^2 + z^2} = 5,$$

$$\cos \gamma = \frac{z}{r} = -\frac{3}{5} \rightarrow \gamma = 126.9^\circ$$

$$\cos \alpha = \frac{x}{r} = \frac{p \cos \phi}{r} = \frac{4 \cos 30^\circ}{5} \rightarrow \alpha = 46.15^\circ$$

$$\cos \beta = \frac{y}{r} = \frac{\rho \sin \phi}{r} = \frac{4}{5} \sin 30^\circ \rightarrow \beta = 66.42^\circ$$

$$(r, \alpha, \beta, \gamma) = \underline{(5, 46.15^\circ, 66.42^\circ, 126.9^\circ)}.$$

e) for $(r, \theta, \phi) = (3, 30^\circ, 60^\circ),$

$$r=3, \quad \gamma = \theta = 30^\circ,$$

$$\cos \alpha = \frac{x}{r} = \frac{r \sin \theta \cos \phi}{r} = \frac{1}{4} \rightarrow \alpha = 75.52^\circ;$$

$$\cos \beta = \frac{y}{r} = \sin \theta \sin \phi = 0.433 \rightarrow \beta = 64.34^\circ,$$

$$(r, \alpha, \beta, \gamma) = \underline{(3, 75.52^\circ, 64.34^\circ, 30^\circ)}.$$

Prob. 2.19

$$J_x = 3 \cos \theta \sin \theta \cos \phi, \quad J_y = 3 \cos \theta \sin \theta \sin \phi,$$

$$J_z = 2 - 3 \sin^2 \theta \cos^2 \phi - 3 \sin^2 \theta \sin^2 \phi = 2 - 3 \sin^2 \theta,$$

$$\begin{bmatrix} J_r \\ J_\theta \\ J_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}$$

$$J_r = 3 \cos \theta \sin^2 \theta \cos^2 \phi + 3 \cos \theta \sin^2 \theta \sin^2 \phi + 2 \cos \theta$$

$$- 3 \cos \theta \sin^2 \theta = 2 \cos \theta$$

$$J_\theta = 3 \cos^2 \theta \sin \theta \cos^2 \phi + 3 \cos^2 \theta \sin \theta \sin^2 \phi$$

$$- 2 \sin \theta + 3 \sin^3 \theta = \sin \theta$$

$$J_\phi = -3 \cos \theta \sin \theta \cos \phi \sin \phi + 3 \cos \theta \sin \theta \sin \phi \cos \phi = 0$$

Hence $\underline{\underline{\vec{J} = 2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta}}$

CHAPTER 3

P.E. 3.1

$$(a) \quad DH = \int_{\phi=45^{\circ}}^{60^{\circ}} r \sin \phi \, d\phi \Big|_{r=3, \theta=90^{\circ}} = 3(1) \left(\frac{\pi}{4} - \frac{\pi}{3} \right) = -\frac{\pi}{4}$$

$$= \underline{\underline{0.7854}}$$

$$(b) \quad FG = \int_{\theta=60^{\circ}}^{90^{\circ}} r \, d\theta \Big|_{r=5} = 5 \left(\frac{\pi}{3} - \frac{\pi}{2} \right) = \frac{5\pi}{6} = \underline{\underline{2.618}}$$

$$(c) \quad AEHD = \int_{\theta=60^{\circ}}^{90^{\circ}} \int_{\phi=45^{\circ}}^{60^{\circ}} r^2 \sin \theta \, d\theta \, d\phi \Big|_{r=3} = 9(-\cos \theta) \Big|_{60^{\circ}}^{90^{\circ}} \phi \Big|_{45^{\circ}}^{60^{\circ}}$$

$$= 9 \left(\frac{1}{2} \right) \left(\frac{\pi}{12} \right) = \frac{3\pi}{8} = \underline{\underline{1.178}}$$

$$(d) \quad ABCD = \int_{r=3}^5 \int_{\theta=60^{\circ}}^{90^{\circ}} r \, d\theta \, dr = \frac{r^2}{2} \Big|_3^5 \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{4\pi}{3}$$

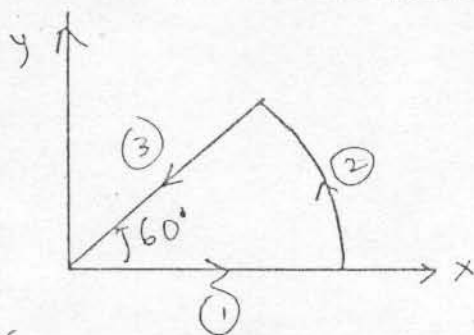
$$= \underline{\underline{4.189}}$$

$$(e) \quad \text{volume} = \int_{r=3}^5 \int_{\phi=45^{\circ}}^{60^{\circ}} \int_{\theta=60^{\circ}}^{90^{\circ}} r^2 \sin \theta \, d\theta \, d\phi \, dr$$

$$= \frac{r^3}{3} \Big|_3^5 (-\cos \theta) \Big|_{60^{\circ}}^{90^{\circ}} \phi \Big|_{45^{\circ}}^{60^{\circ}} = \frac{1}{3} (98) \left(\frac{1}{2} \right) \frac{\pi}{12}$$

$$= \frac{49\pi}{36} = \underline{\underline{4.276}}$$

P.E. 3.2



$$\text{Let } \oint_L \vec{A} \cdot d\vec{l} = \left(\int_{(1)} + \int_{(2)} + \int_{(3)} \right) \vec{A} \cdot d\vec{l} = C_1 + C_2 + C_3$$

$$\text{Along (1), } C_1 = \int \vec{A} \cdot d\vec{l} = \int_0^2 \rho \cos \phi d\rho \Big|_{\phi=0} = \frac{\rho^2}{2} \Big|_0^2 = 2$$

$$\text{Along (2), } d\vec{l} = \rho d\phi \hat{a}_\phi, \quad \vec{A} \cdot d\vec{l} = 0 \rightarrow C_2 = 0$$

$$\text{Along (3), } C_3 = \int_2^0 \rho \cos \phi d\rho \Big|_{\phi=60^\circ} = \frac{\rho^2}{2} \Big|_0^2 \left(\frac{1}{2} \right) = -1$$

$$\oint_L \vec{A} \cdot d\vec{l} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

P.E. 3.3

$$\begin{aligned} \text{(a) } \nabla U &= \frac{\partial U}{\partial x} \vec{a}_x + \frac{\partial U}{\partial y} \vec{a}_y + \frac{\partial U}{\partial z} \vec{a}_z \\ &= \underline{\underline{y(2x+z) \vec{a}_x + x(x+z) \vec{a}_y + xy \vec{a}_z}} \end{aligned}$$

$$\begin{aligned} \text{(b) } \nabla V &= \frac{\partial V}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \vec{a}_\phi + \frac{\partial V}{\partial z} \vec{a}_z \\ &= (z \sin \phi + 2\rho) \vec{a}_\rho + \left(z \cos \phi - \frac{z^2}{\rho} \sin 2\phi \right) \vec{a}_\phi \end{aligned}$$

$$\underline{\underline{+ (p \cos \phi + 2z \cos^2 \phi) \vec{a}_z.}}$$

$$\begin{aligned} \text{(c) } \nabla f &= \frac{\partial f}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \vec{a}_\phi \\ &= \left(\frac{\cos \theta \sin \phi}{r} + 2r \phi \right) \vec{a}_r - \frac{\sin \theta \sin \phi}{r} \ln r \vec{a}_\theta \\ &\quad + \left(\frac{\cot \theta \cos \phi}{r} \ln r + r \operatorname{cosec} \theta \right) \vec{a}_\phi. \end{aligned}$$

P.E. 3.4

$$\nabla \Phi = (y+z) \vec{a}_x + (x+z) \vec{a}_y + (y+x) \vec{a}_z$$

$$\text{At } (1, 2, 3), \nabla \Phi = \underline{\underline{(5, 4, 3)}}.$$

$$\begin{aligned} \nabla \Phi \cdot \vec{a}_1 &= (5, 4, 3) \cdot \frac{(2, 2, 1)}{3}, \text{ where } (2, 2, 1) \\ &= (3, 4, 4) - (1, 2, 3) \\ &= \frac{21}{3} = \underline{\underline{7}}. \end{aligned}$$

P.E. 3.5

$$\text{Let } f = x^2 y + z - 3, g = x \log z - y^2 + 4,$$

$$\nabla f = 2xy \vec{a}_x + x^2 \vec{a}_y + \vec{a}_z,$$

$$\nabla g = \log z \vec{a}_x - 2y \vec{a}_y + \frac{x}{z} \vec{a}_z.$$

$$\text{At } P(-1, 2, 1),$$

$$\vec{n}_f = \pm \frac{\nabla f}{|\nabla f|} = - \frac{(-4 \vec{a}_x + \vec{a}_y + \vec{a}_z)}{\sqrt{18}}.$$

$$\cos \theta = \vec{n}_f \cdot \vec{n}_g = \pm \frac{(-5)}{\sqrt{18 \times 17}} \rightarrow \theta = \cos^{-1} \frac{5}{17.493}$$

$$\theta = \underline{\underline{73.39^\circ}}$$

P.E. 3.6

$$(a) \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}$$

$$\text{At } (1, -2, 3), \nabla \cdot \vec{A} = \underline{\underline{4}}$$

$$\begin{aligned} (b) \nabla \cdot \vec{B} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \\ &= \frac{1}{\rho} 2\rho z \sin \phi - \frac{1}{\rho} 3\rho z^2 \sin \phi \\ &\quad + 2z \sin \phi - 3z^2 \sin \phi \\ &= \underline{\underline{(2-3z)z \sin \phi}} \end{aligned}$$

$$\text{At } (5, \pi/2, 1), \nabla \cdot \vec{B} = (2-3)(1) = \underline{\underline{-1}}$$

$$\begin{aligned} (c) \nabla \cdot \vec{C} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} = \frac{1}{r^2} 6r^2 \cos \theta \cos \phi \\ &= \underline{\underline{6 \cos \theta \cos \phi}} \end{aligned}$$

$$\text{At } (1, \pi/6, \pi/3), \nabla \cdot \vec{C} = 6 \cos \frac{\pi}{6} \cos \frac{\pi}{3} = \underline{\underline{2.598}}$$

P.E. 3.7

This is similar to Example 3.7.

$$\Psi = \oint_S \vec{A} \cdot d\vec{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\vec{A}$ has no z -component.

$$\Psi_c = \iiint \vec{r} \cos^2 \phi \, \rho \, d\phi \, dz = \rho^3 \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi \int_{z=0}^1 dz \Big|_{\rho=4}$$

$$= (4)^3 \pi (1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{64\pi}$$

By the divergence theorem,

$$\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot \vec{A} \, dv$$

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^3 \cos^2 \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} z \sin \phi + \frac{\partial A_z}{\partial z}$$

$$= 3\rho \cos^2 \phi + \frac{1}{\rho} \cos \phi$$

$$\Psi = \int_V \nabla \cdot \vec{A} \, dv = \int_V (3\rho \cos^2 \phi + \frac{1}{\rho} \cos \phi) \rho \, d\phi \, dz \, d\rho$$

$$= 3 \int_0^4 \rho^2 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cancel{\cos \phi \, d\phi} \int_0^1 dz$$

$$= 3 \left(\frac{4^3}{3} \right) \pi (1) = \underline{64\pi}$$

P.E. 3.8

$$(a) \nabla \times \vec{A} = \vec{a}_x (1-0) + \vec{a}_y (y-0) + \vec{a}_z (4y-z) \\ = \underline{\underline{\vec{a}_x + y\vec{a}_y + (4y-z)\vec{a}_z}}$$

$$\text{At } (1, -2, 3), \nabla \times \vec{A} = \underline{\underline{\vec{a}_x - 2\vec{a}_y - 11\vec{a}_z}}$$

$$(b) \nabla \times \vec{B} = \vec{a}_\rho (0 - 6\rho z \cos \phi) + \vec{a}_\phi (\rho \sin \phi - 0) \\ + \vec{a}_z \frac{1}{\rho} (6\rho z \cos \phi - \rho z \cos \phi) \\ = \underline{\underline{-6\rho z \cos \phi \vec{a}_\rho + \rho \sin \phi \vec{a}_\phi + (6z-1)z \cos \phi \vec{a}_z}}$$

$$\text{At } (5, \pi/2, -1), \nabla \times \vec{B} = \underline{\underline{5\vec{a}_\phi}}$$

$$(c) \nabla \times \vec{C} = \vec{a}_r \frac{1}{r \sin \theta} (r^{3/2} \cos \theta - 0) \\ + \frac{\vec{a}_\theta}{r} \left(-\frac{2r \cos \theta \sin \phi}{\sin \theta} - \frac{3}{2} r^{1/2} \right) \\ + \frac{\vec{a}_\phi}{r} (0 - 2r \sin \theta \cos \phi) \\ = r^{1/2} \cos \theta \vec{a}_r - \left(2 \cot \theta \sin \phi + \frac{3}{2} r^{1/2} \right) \vec{a}_\theta \\ - 2 \sin \theta \cos \phi \vec{a}_\phi$$

$$\text{At } (1, \pi/6, \pi/3),$$

$$\nabla \times \vec{C} = \underline{\underline{1.732\vec{a}_r - 4.5\vec{a}_\theta - 0.5\vec{a}_\phi}}$$

P.E. 3.9

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S}$$

$$\nabla \times \vec{A} = \sin \phi \vec{a}_z + \frac{z \cos \phi}{\rho} \vec{a}_\rho, \quad d\vec{S} = \rho d\phi d\rho \vec{a}_z.$$

$$\begin{aligned} \int_S (\nabla \times \vec{A}) \cdot d\vec{S} &= \iint \rho \sin \phi d\phi d\rho = \frac{\rho^2}{2} \Big|_0^2 (-\cos \phi) \Big|_0^{60^\circ} \\ &= 2 \left(-\frac{1}{2} + 1 \right) = \underline{\underline{1}}. \end{aligned}$$

P.E. 3.10

$$\begin{aligned} \nabla \times \nabla V &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \right) \vec{a}_x \\ &+ \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x} \right) \vec{a}_y + \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right) \vec{a}_z = 0. \end{aligned}$$

P.E. 3.11

$$\begin{aligned} (a) \nabla^2 U &= \frac{\partial}{\partial x} (2xy + yz) + \frac{\partial}{\partial y} (x^2 + xz) + \frac{\partial}{\partial z} (xy) \\ &= \underline{\underline{2y}}. \end{aligned}$$

$$\begin{aligned} (b) \nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho (z \sin \phi + 2\rho) + \frac{1}{\rho} (-\rho z \sin \phi \\ &\quad - 2z^2 \frac{\partial}{\partial \phi} \sin \phi \cos \phi) + \frac{\partial}{\partial z} (\rho \sin \phi + 2z \cos \phi) \\ &= \frac{1}{\rho} (z \sin \phi + 4\rho) - \frac{1}{\rho} (2\rho \sin \phi + 2z^2 \cos 2\phi) \end{aligned}$$

$$+ 2 \cos^2 \phi$$

$$= 4 + 2 \cos^2 \phi - \frac{2z^2}{r^2} \cos 2\phi.$$

$$(c) \nabla \cdot \vec{f} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{1}{r^2} \cdot \frac{1}{r} \cos \theta \sin \phi + 2r^2 \phi \right]$$

$$+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (-\sin^2 \theta \sin \phi \ln r)$$

$$+ \frac{1}{r^2 \sin^2 \theta} (-\cos \theta \sin \theta \ln r)$$

$$= \frac{1}{r^2} \cos \theta \sin \phi (1 - 2 \ln r - \cos \theta \sec^2 \theta \ln r) + 6\phi.$$

P.E. 3.12

If $\vec{B}$ is conservative, $\nabla \times \vec{B} = 0$ must be satisfied.

$$\nabla \times \vec{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z \cos xz & x & x \cos xz \end{vmatrix}$$

$$= 0 \vec{a}_x + (\cos xz - xz \sin xz - \cos xz + xz \sin xz) \vec{a}_y + (1-1) \vec{a}_z$$

$$= 0$$

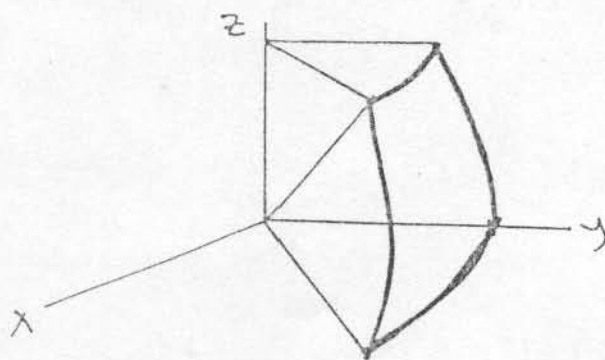
Hence $\vec{B}$ is a conservative field.

Prob. 3.1

$$(a) L = \int dl = \int_{30^\circ}^{90^\circ} p d\phi \Big|_{p=10} = 10 \left(\frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{10\pi}{3} = \underline{\underline{10.47}}$$

$$(b) L = \int r d\theta = 5 \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) = \frac{5\pi}{2} = \underline{\underline{7.854}}$$

(c)



$$L = 2 \int_{\pi/6}^{\pi/2} r d\theta \Big|_{r=10} + r \left(\sin \frac{\pi}{2} + \sin \frac{\pi}{6} \right) \int_{\pi/4}^{\pi/2} d\phi \Big|_{r=10}$$

$$= \frac{20\pi}{3} + \frac{15\pi}{4} = \frac{125\pi}{12} = \underline{\underline{32.72}}$$

Prob. 3.2

$$(a) S = \int_{p=0}^{10} \int_{\phi=0}^{\pi} p d\phi dp = \pi \frac{p^2}{2} \Big|_0^{10} = 50\pi = \underline{\underline{157.1}}$$

$$(b) ds = p d\phi dz, S = \int_{z=0}^4 \int_{\phi=\pi/4}^{\pi/2} p d\phi dz \Big|_{p=50}$$

$$= 50(4) \left(\frac{\pi}{4} \right) = 50\pi = \underline{\underline{157.1}}$$

$$(c) ds = r^2 \sin\theta d\theta d\phi,$$

$$S = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^2 \sin \theta d\theta d\phi \Big|_{r=5}$$

$$= 25(2\pi)(-\cos \theta) \Big|_0^{\pi/2} = 50\pi = \underline{\underline{157.1}}$$

(d) $ds = r \sin \theta dr d\phi$,

$$S = \int_{r=0}^{10} \int_{\phi=0}^{2\pi} r \sin \theta dr d\phi \Big|_{\theta=30^\circ}$$

$$= \sin 30^\circ (2\pi) \left(\frac{100}{2} \right) = 50\pi = \underline{\underline{157.1}}$$

(e) $ds = r^2 \sin \theta d\theta d\phi$,

$$S = (8.26)^2 \int_0^{2\pi} d\phi \int_{30^\circ}^{60^\circ} \sin \theta d\theta = (2\pi)^2 (2\pi) \left(-0.5 + 0.866 \right)$$

$$= 50\pi = \underline{\underline{157.1}}$$

10b.3.3

(a) $dv = \rho d\rho d\phi dz$,

$$V = \int_1^2 \rho d\rho \int_{-1}^1 dz \int_{\pi/6}^{3\pi/4} d\phi = \left(\frac{\rho^2}{2} \right) \Big|_1^2 (2) \left(\frac{3\pi}{4} - \frac{\pi}{6} \right)$$

$$= \frac{7\pi}{4} = \underline{\underline{5.498}}$$

(b) $dv = r^2 \sin \theta dr d\theta d\phi$,

$$V = \int_3^5 r^2 dr \int_{\pi/3}^{\pi/2} \sin \theta d\theta \int_{5\pi/6}^{\pi} d\phi$$

$$= \frac{r^3}{3} \Big|_3^5 \left(-\cos \theta \Big|_{\pi/3}^{\pi/2} \right) \left(\pi - \frac{5\pi}{6} \right) = \frac{49\pi}{18} = \underline{\underline{-8.552}}$$

(c) $dV = dx dy dz$,

$$V = \int_{-4}^4 dz \int_{x=0}^2 \int_{y=0}^x dy dx = 8 \int_{x=0}^2 y \Big|_0^x dx$$

$$= 8 \int_0^2 x dx = 8 \frac{x^2}{2} \Big|_0^2 = \underline{\underline{16}}$$

Prob. 3.4

$\oint_L \vec{F} \cdot d\vec{l} = \int (\nabla \times \vec{F}) \cdot d\vec{s}$ - Stokes's theorem

$\nabla \times \vec{F} = -7\vec{a}_z$, $d\vec{s} = \pm dx dy \vec{a}_z$

(a) $\oint \vec{F} \cdot d\vec{l} = \pm \iint (-7) dx dy = \pm 7\pi^2 = \pm \underline{\underline{28\pi}}$

(b) $\oint \vec{F} \cdot d\vec{l} = \pm 7 \int_{-1}^1 \int_{-1}^1 dx dy = \pm 7(2)(2) = \pm \underline{\underline{28}}$

Prob. 3.5

(a) $\int \vec{F} \cdot d\vec{l} = \int_{y=0}^1 (x^2 - z^2) dy \Big|_{\substack{x=0 \\ z=0}} + \int_{x=0}^2 2xy dx \Big|_{\substack{y=1 \\ z=0}} \\ + \int_{z=0}^3 (-3xz^2) dz \Big|_{x=0, y=1}$

$$= 0 + 2(1) \frac{x^2}{2} \Big|_0^2 - 3(2) \frac{z^3}{3} \Big|_0^3$$

$$= 0 + 4 - 54 = \underline{\underline{-50}}$$

(b) Let $x=2t$, $y=t$, $z=3t$,
 $dx=2dt$, $dy=dt$, $dz=3dt$.

$$\int \vec{F} \cdot d\vec{l} = \int_0^1 (8t^2 - 5t^2 - 162t^3) dt = -\frac{79}{2}$$

$$= \underline{\underline{-39.5}}$$

Prob. 3.6

$$\int \vec{H} \cdot d\vec{l} = \int_{x=0}^1 (x-y) dx \Big|_{y=0} + \int_{z=0}^1 (x^2 + zy) dz \Big|_{y=0, z=0}$$

$$+ \int (x^2 + zy) dy + 5yz dz \Big|_{x=0, z=1-\frac{y}{2}}$$

$$= \frac{x^2}{2} \Big|_0^1 + 0 + \int_{y=0}^2 y(1-\frac{y}{2}) dy + 5y(1-\frac{y}{2})(-\frac{dy}{2})$$

$$= -\frac{1}{2} + \int_0^2 \left(-\frac{3}{2}y + \frac{3y^2}{4}\right) dy$$

$$= -\frac{1}{2} + \left(-\frac{3}{4}y^2 + \frac{y^3}{4}\right) \Big|_0^2 = -\frac{1}{2} - 3 + 2$$

$$= \underline{\underline{-1.5}}$$

Prob. 3.7

Let the integrand be $\vec{F}$

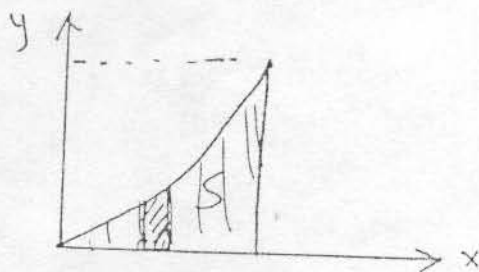
$$\vec{F}(x, y, z) \rightarrow \vec{F}(\rho, \phi, z) = -\frac{\rho \sin \phi}{\rho} \vec{a}_x + \frac{\rho \cos \phi}{\rho} \vec{a}_y$$

$$\begin{bmatrix} F_\rho \\ F_\phi \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} -\sin \phi \\ \cos \phi \end{bmatrix}$$

$$\vec{F} = \vec{a}_\phi, \quad \vec{F} \cdot d\vec{l} = \rho d\phi$$

$$\oint \vec{F} \cdot d\vec{l} = \int_0^{2\pi} \rho d\phi = \underline{\underline{2\pi a}}$$

Prob. 3.8



$$\int_R f ds = \int_{x=0}^1 \int_{y=0}^{x^2} (x^2 + xy) dy dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{2} \right) \Big|_0^{x^2} dx = \int_0^1 \left(x^4 + \frac{x^5}{2} \right) dx$$

$$= \frac{1}{5} + \frac{1}{12} = \frac{17}{60} = \underline{\underline{0.2833}}$$

Prob. 3.9

$$\Psi = \oint \vec{A} \cdot d\vec{s} = \int_V \nabla \cdot \vec{A} dv = \iiint (4z - y) dx dy dz$$

$$= 4 \int_0^1 \int_0^1 \int_0^1 z \, dx \, dy \, dz - \int_0^1 \int_0^1 \int_0^1 y \, dx \, dy \, dz$$

$$= 3(1)(1)\left(\frac{1}{2}\right) = \underline{\underline{1.5}}$$

Prob. 3.10

$$\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot \vec{A} \, dv$$

$$= \int_V \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + 0 + 0 \right) dv$$

$$= \iiint \left(\frac{1}{r^2} \cdot 4r^3 \right) r^2 \sin \theta \, d\theta \, d\phi \, dr$$

$$= 4 \int_0^5 r^3 \, dr \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta \, d\theta$$

$$= (5)^4 (2\pi) (1) = \underline{\underline{1250\pi}}$$

Prob. 3.11

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} \int_{r=0}^a r^2 \sin \theta \, d\theta \, dr \, d\phi$$

$$= \frac{2\pi a^3}{3} (1 - \cos \alpha)$$

$$V(\alpha = \pi/3) = \frac{2\pi a^3}{3} \left(1 - \frac{1}{2}\right) = \underline{\underline{\pi a^3/3}}$$

$$V(\alpha = \pi/2) = \frac{2\pi a^3}{3} (1 - 0) = \underline{\underline{2\pi a^3/3}}$$

Prob. 3.12

$$\begin{aligned} \text{volume} &= \int dv = \int_{\phi=0}^{2\pi} \int_{\rho=0}^1 \int_{z=0}^{-e^{-\rho^2}} \rho d\rho d\phi dz \\ &= 2\pi \int_0^1 \rho e^{-\rho^2} d\rho = 2\pi \left(-\frac{1}{2} e^{-\rho^2} \right) \Big|_0^1 \\ &= \pi(1 - e^{-1}) = \underline{\underline{1.986}} \end{aligned}$$

Prob. 3.13

$$\begin{aligned} \int g dv &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \int_{r=0}^1 r^4 \sin \theta dr d\phi d\theta \\ &= 2\pi \frac{r^5}{5} \Big|_0^1 (-\cos \theta) \Big|_0^{\pi/2} = \frac{2\pi}{5} = \underline{\underline{1.257}} \end{aligned}$$

Prob. 3.14

$$\begin{aligned} (a) \nabla &= \frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z \\ &= \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} \right) (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi) \\ &\quad + \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi} \right) (\sin \phi \vec{a}_\rho + \cos \phi \vec{a}_\phi) \\ &\quad + \frac{\partial}{\partial z} \vec{a}_z \\ &= \frac{\partial}{\partial \rho} (\cos^2 \phi + \sin^2 \phi) \vec{a}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\sin^2 \phi + \cos^2 \phi) \vec{a}_\phi \\ &\quad + \frac{\partial}{\partial z} \vec{a}_z = \frac{\partial}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} \vec{a}_\phi + \frac{\partial}{\partial z} \vec{a}_z \end{aligned}$$

$$\begin{aligned}
 (b) \nabla &= \frac{\partial}{\partial x} \tilde{a}_x + \frac{\partial}{\partial y} \tilde{a}_y + \frac{\partial}{\partial z} \tilde{a}_z \\
 &= \left(\sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi} \right) \times \\
 &\quad \left(\sin\theta \cos\phi \tilde{a}_r + \cos\theta \cos\phi \tilde{a}_\theta - \sin\phi \tilde{a}_\phi \right) \\
 &+ \left(\sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r} \frac{\partial}{\partial \phi} \right) \times \\
 &\quad \left(\sin\theta \sin\phi \tilde{a}_r + \cos\theta \sin\phi \tilde{a}_\theta + \cos\phi \tilde{a}_\phi \right) \\
 &+ \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos\theta \tilde{a}_r - \sin\theta \tilde{a}_\theta \right) \\
 &= \tilde{a}_r \left(\sin^2\theta \cos^2\phi \frac{\partial}{\partial r} + \frac{\sin\theta \cos\theta \cos^2\phi}{r} \frac{\partial}{\partial \theta} \right. \\
 &\quad \left. - \frac{\sin\theta \sin\phi \cos\phi}{r} \frac{\partial}{\partial \phi} + \frac{\sin^2\theta \sin^2\phi}{r} \frac{\partial}{\partial r} \right. \\
 &\quad \left. + \frac{\sin\theta \cos\theta \sin^2\phi}{r} \frac{\partial}{\partial \theta} + \frac{\sin\theta \cos\phi \sin\phi}{r} \frac{\partial}{\partial \phi} \right. \\
 &\quad \left. + \cos^2\theta \frac{\partial}{\partial r} - \frac{\sin\theta \cos\theta}{r} \frac{\partial}{\partial \theta} \right) \\
 &+ \tilde{a}_\theta \left(\sin\theta \cos\theta \cos^2\phi \frac{\partial}{\partial r} + \cos^2\theta \cos^2\phi \frac{\partial}{\partial \theta} \right. \\
 &\quad \left. - \frac{\cos\theta \sin\phi \cos\phi}{r} \frac{\partial}{\partial \phi} + \sin\theta \cos\theta \sin^2\phi \frac{\partial}{\partial r} \right. \\
 &\quad \left. + \cos^2\theta \sin^2\phi \frac{\partial}{\partial \theta} + \frac{\cos\theta \sin\phi \cos\phi}{r} \frac{\partial}{\partial \phi} \right)
 \end{aligned}$$

$$\begin{aligned}
 & -\sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\sin^2\theta}{r} \frac{\partial}{\partial\theta} \Big) \\
 & + \tilde{a}_\phi \left(-\sin\theta \cos\phi \sin\phi \frac{\partial}{\partial r} - \frac{\cos\theta \cos\phi \sin\phi}{r} \frac{\partial}{\partial\theta} \right. \\
 & + \frac{\sin^2\phi}{r} \frac{\partial}{\partial\phi} + \sin\theta \sin\phi \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi \cos\phi}{r} \frac{\partial}{\partial\theta} \\
 & \left. + \frac{\cos^2\phi}{r} \frac{\partial}{\partial\phi} \right) \\
 & = \frac{\partial}{\partial r} \tilde{a}_r + \frac{1}{r} \frac{\partial}{\partial\theta} \tilde{a}_\theta + \frac{1}{r \sin\theta} \frac{\partial}{\partial\phi} \tilde{a}_\phi.
 \end{aligned}$$

Prob. 3.15

$$\begin{aligned}
 (a) \nabla U &= \frac{\partial U}{\partial x} \tilde{a}_x + \frac{\partial U}{\partial y} \tilde{a}_y + \frac{\partial U}{\partial z} \tilde{a}_z \\
 &= \underline{4z^2 \tilde{a}_x + 3z \tilde{a}_y + (8xz + 3y) \tilde{a}_z}.
 \end{aligned}$$

$$\begin{aligned}
 (b) \nabla V &= 2e^{(2x+3y)} \cos 5z \tilde{a}_x + 3e^{(2x+3y)} \cos 5z \tilde{a}_y \\
 &\quad - 5e^{(2x+3y)} \sin 5z \tilde{a}_z.
 \end{aligned}$$

$$\begin{aligned}
 (c) \nabla T &= \frac{\partial T}{\partial \rho} \tilde{a}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \tilde{a}_\phi + \frac{\partial T}{\partial z} \tilde{a}_z \\
 &= \underline{5e^{-2z} \sin\phi \tilde{a}_\rho + 5e^{-2z} \cos\phi \tilde{a}_\phi - 10\rho e^{-2z} \sin\phi \tilde{a}_z}.
 \end{aligned}$$

$$\begin{aligned}
 (d) \nabla W &= 2(z^2+1) \cos\phi \tilde{a}_\rho - 2(z^2+1) \sin\phi \tilde{a}_\phi \\
 &\quad + 4\rho z \cos\phi \tilde{a}_z.
 \end{aligned}$$

$$(e) \nabla H = \frac{\partial H}{\partial r} \bar{a}_r + \frac{1}{r} \frac{\partial H}{\partial \theta} \bar{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial H}{\partial \phi} \bar{a}_\phi$$

$$= \underline{2r \cos \theta \cos \phi \bar{a}_r - r \sin \theta \cos \phi \bar{a}_\theta - r \cot \theta \sin \phi \bar{a}_\phi}$$

$$(f) \nabla Q = \underline{\underline{\frac{-3}{r^4} \sin \theta \sin \phi \bar{a}_r + \frac{\cos \theta \sin \phi}{r^4} \bar{a}_\theta + \frac{\cos \phi}{r^4} \bar{a}_\phi}}$$

Prob. 3.16

Let $f = xy + yz + zx - 5 = 0$

$$\nabla f = \frac{\partial f}{\partial x} \bar{a}_x + \frac{\partial f}{\partial y} \bar{a}_y + \frac{\partial f}{\partial z} \bar{a}_z$$

$$= (y+z) \bar{a}_x + (x+z) \bar{a}_y + (x+y) \bar{a}_z$$

At $(-1, 3, 4)$, $\nabla f = 7 \bar{a}_x + 3 \bar{a}_y + 2 \bar{a}_z$

$$\bar{a}_n = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{(7, 3, 2)}{\sqrt{62}}$$

$$= \underline{\underline{\pm (0.889 \bar{a}_x + 0.381 \bar{a}_y + 0.254 \bar{a}_z)}}$$

Prob. 3.17

$$\nabla T = 2x \bar{a}_x + 2y \bar{a}_y - \bar{a}_z$$

At $(1, 1, 2)$, $\nabla T = (2, 2, -1)$. The mosquito should move in the direction of

$$\underline{\underline{2 \bar{a}_x + 2 \bar{a}_y - \bar{a}_z}}$$

Prob. 3.18

$$(a) \nabla \cdot \vec{A} = \underline{y e^{xy} + x \cos xy - 2x \cos z x \sin z x}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2 xz \end{vmatrix}$$

$$= (0-0)\vec{a}_x + (0+2z \cos xz \sin xz)\vec{a}_y + (y \cos xy - x e^{xy})\vec{a}_z$$

$$= \underline{z \sin 2xz \vec{a}_y + (y \cos xy - x e^{xy})\vec{a}_z}$$

$$(b) \nabla \cdot \vec{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 z \cos \phi) + 0 + \sin^2 \phi$$

$$= \underline{2z \cos \phi + \sin^2 \phi}$$

$$\nabla \times \vec{B} = \left(\frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - 0 \right) \vec{a}_\rho + \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right) \vec{a}_\phi$$

$$+ \frac{1}{\rho} \left(0 - \frac{\partial B_\rho}{\partial \phi} \right) \vec{a}_z$$

$$= \underline{\frac{z \sin 2\phi}{\rho} \vec{a}_\rho + 2\rho z \cos \phi \vec{a}_\phi + z \sin \phi \vec{a}_z}$$

$$(c) \nabla \cdot \vec{C} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(-\frac{1}{r} \sin^2 \theta \right) + 0$$

$$= \underline{3 \cos \theta - \frac{2 \cos \theta}{r^2}}$$

$$\begin{aligned}\nabla \times \vec{C} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (2r^2 \sin^2 \theta) - 0 \right] \vec{a}_r \\ &+ \frac{1}{r} \left[0 - \frac{\partial}{\partial r} (2r^3 \sin \theta) \right] \vec{a}_\theta \\ &+ \frac{1}{r} \left[\frac{\partial}{\partial r} (-\sin \theta) + r \sin \theta \right] \vec{a}_\phi \\ &= \underline{\underline{4r \cos \theta \vec{a}_r - 6r \sin \theta \vec{a}_\theta + \sin \theta \vec{a}_\phi}}\end{aligned}$$

Prob. 3.19

$$\begin{aligned}\text{(a)} \quad \nabla \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1 + yz + 2y \\ &= \underline{\underline{1 + 2y + yz}}\end{aligned}$$

$$\nabla \times \vec{A} = \underline{\underline{(z-x)\vec{a}_x + (xy-2x)\vec{a}_y + (2x-z-x^2)\vec{a}_z}}$$

$$\begin{aligned}\text{(b)} \quad \nabla \cdot \vec{B} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0 + \frac{1}{\rho} 2\rho \cos \phi \\ &= \underline{\underline{2 \cos \phi}}\end{aligned}$$

$$\begin{aligned}\nabla \times \vec{B} &= \vec{a}_\rho \left(-\frac{1}{\rho} 2 \sin \phi \cos \phi - \rho \sin \phi \right) + \vec{a}_\phi \left(\frac{1}{\rho} - 0 \right) \\ &+ \frac{\vec{a}_z}{\rho} (\cos^2 \phi - 0) \\ &= \underline{\underline{-\vec{a}_\rho \left(\frac{\sin 2\phi}{\rho} - \rho \sin \phi \right) + \frac{\vec{a}_\phi}{\rho} + \vec{a}_z \frac{\cos^2 \phi}{\rho}}}\end{aligned}$$

$$\begin{aligned} (c) \quad \nabla \cdot \vec{C} &= \frac{1}{r^2} (4r^3 + 2r \sin \theta \cos \phi) \\ &\quad + \frac{1}{r \sin \theta} (2r \sin \theta \cos \theta \cos \phi) + \frac{1}{r \sin \theta} \cos \phi \\ &= 4r + \frac{2 \sin \theta \cos \phi}{r} + 2 \cos \theta \cos \phi + \frac{\cos \phi}{r \sin \theta} \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{C} &= \vec{a}_r \frac{1}{r \sin \theta} \left[\frac{1}{r} \sec^2 \theta \sin \theta + \frac{1}{r} \tan \theta \cos \theta \right. \\ &\quad \left. + \sin \phi \cos \theta - r \sin \theta \sin \phi \right] \\ &\quad + \frac{\vec{a}_\theta}{r} \left[-\frac{1}{\sin \theta} \sin \theta \sin \phi - \sin \phi \right] \\ &\quad + \frac{\vec{a}_\phi}{r} \left[2r \sin \theta \cos \phi - \cos \theta \cos \phi \right] \\ &= \vec{a}_r \left[\frac{\sec^2 \theta}{r^2} + \frac{1}{r^2} + \sin \phi \cot \theta - \sin \phi \right] \\ &\quad - \frac{\vec{a}_\theta}{r} 2 \sin \phi + \vec{a}_\phi \left[2 \sin \theta \cos \phi - \frac{\cos \theta \cos \phi}{r} \right] \end{aligned}$$

Prob. 3.20

$$\begin{aligned} (a) \quad \text{grad}(r) &= \text{grad} (x^2 + y^2 + z^2)^{1/2} \\ &= (x^2 + y^2 + z^2)^{-1/2} \left(\frac{1}{2} 2x \vec{a}_x + \frac{1}{2} 2y \vec{a}_y + \frac{1}{2} 2z \vec{a}_z \right) \\ &= \frac{\vec{r}}{r} \end{aligned}$$

$$\begin{aligned} (b) \quad \text{grad} \left(\frac{1}{r} \right) &= \text{grad} (x^2 + y^2 + z^2)^{-1/2} \\ &= (x^2 + y^2 + z^2)^{-3/2} \left(-\frac{1}{2} 2x \vec{a}_x - \frac{1}{2} 2y \vec{a}_y - \frac{1}{2} 2z \vec{a}_z \right) \end{aligned}$$

$$= \frac{-\vec{r}}{r^3}$$

$$(c) \text{grad} (\vec{A} \cdot \vec{r}) = \text{grad} (A_x x + A_y y + A_z z) \\ = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z = \vec{A}$$

since $\vec{A}$ is a uniform vector meaning that A_x , A_y , and A_z are independent of x , y , and z .

$$(d) (\vec{U} \cdot \text{grad}) \vec{r} = \left(U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z} \right) \vec{r} \\ = U_x \vec{a}_x + U_y \vec{a}_y + U_z \vec{a}_z = \vec{U}$$

$$(e) \text{div} \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$(f) \text{div} (r^n \vec{r}) = \text{div} \left[(x^2 + y^2 + z^2)^{n/2} (x \vec{a}_x + y \vec{a}_y + z \vec{a}_z) \right] \\ = (x^2 + y^2 + z^2)^{n/2} + 2x^2 \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \\ + (x^2 + y^2 + z^2)^{n/2} + 2y^2 \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \\ + (x^2 + y^2 + z^2)^{n/2} + 2z^2 \cdot \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \\ = 3r^n + n(x^2 + y^2 + z^2) (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \\ = 3r^n + nr^n = (n+3)r^n$$

$$(g) \text{curl} \vec{r} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

$$(h) \text{Laplacian } \frac{1}{r} = \nabla \cdot \nabla \left(\frac{1}{r} \right)$$

from (b), $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$, hence

$$\nabla^2\left(\frac{1}{r}\right) = \operatorname{div}(-r^{-3}\vec{r})$$

which is (A) if $n=3$. Hence

$$\nabla^2\left(\frac{1}{r}\right) = -(-3+3)r^{-3} = 0.$$

Prob. 3.21

(a) Let $\vec{F} = \vec{r} \Phi(r) = r \Phi \vec{a}_r$

since $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z = r\vec{a}_r$.

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \Phi) \\ &= \frac{3r^2}{r^2} \Phi + \frac{r^3}{r^2} \Phi' = 3\Phi(r) + r\Phi' .\end{aligned}$$

(b) Let $\vec{F} = \frac{\vec{r}}{r^3} = \frac{\vec{a}_r}{r^2}$

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0 .$$

(c) Let $V = r^n$

$$\begin{aligned}\nabla^2 V &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n r^{n-1}) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (n r^{n+1}) = n(n+1) r^{n-2}\end{aligned}$$

(d) Let $\vec{A} = r^n \vec{r} = x r^n \vec{a}_x + y r^n \vec{a}_y + z r^n \vec{a}_z$

$$\nabla^2 \vec{A} = \nabla^2 (x r^n) \vec{a}_x + \nabla^2 (y r^n) \vec{a}_y + \nabla^2 (z r^n) \vec{a}_z$$

$$\begin{aligned}
 \text{But } \nabla^2(xr^n) &= \nabla^2 \left[x(x^2+y^2+z^2)^{\frac{n}{2}} \right] \\
 &= \frac{\partial^2}{\partial x^2} \left[x(x^2+y^2+z^2)^{\frac{n}{2}} \right] \\
 &\quad + \frac{\partial^2}{\partial y^2} \left[x(x^2+y^2+z^2)^{\frac{n}{2}} \right] \\
 &\quad + \frac{\partial^2}{\partial z^2} \left[x(x^2+y^2+z^2)^{\frac{n}{2}} \right] \\
 &= \frac{\partial}{\partial x} \left[(x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2} (2x) x (x^2+y^2+z^2)^{\frac{n}{2}-1} \right] \\
 &\quad + \frac{\partial}{\partial y} \left[\frac{n}{2} (2y) x (x^2+y^2+z^2)^{\frac{n}{2}-1} \right] \\
 &\quad + \frac{\partial}{\partial z} \left[\frac{n}{2} (2z) x (x^2+y^2+z^2)^{\frac{n}{2}-1} \right] \\
 &= \frac{n}{2} (2x) (x^2+y^2+z^2)^{\frac{n}{2}-1} + 2nx (x^2+y^2+z^2)^{\frac{n}{2}-1} \\
 &\quad + nx^2 (2x) \left(\frac{n}{2}-1\right) (x^2+y^2+z^2)^{\frac{n}{2}-2} \\
 &\quad + nx (x^2+y^2+z^2)^{\frac{n}{2}-1} + nxy (2y) \left(\frac{n}{2}-1\right) (x^2+y^2+z^2)^{\frac{n}{2}-2} \\
 &\quad + nx (x^2+y^2+z^2)^{\frac{n}{2}-1} + nxz (2z) \left(\frac{n}{2}-1\right) (x^2+y^2+z^2)^{\frac{n}{2}-2} \\
 &= 5nx (x^2+y^2+z^2)^{\frac{n}{2}-1} \\
 &\quad + nx (x^2+y^2+z^2) (n-2) (x^2+y^2+z^2)^{\frac{n}{2}-2}
 \end{aligned}$$

$$= 5nx(x^2+y^2+z^2)^{\frac{n}{2}-1} + n(n-2)x(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$= (5nx - n^2x - 2nx)(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\nabla^2(xr^n) = n(n+3)xr^{n-2}$$

Similarly,

$$\nabla^2(yr^n) = n(n+3)yr^{n-2}$$

$$\nabla^2(zr^n) = n(n+3)zr^{n-2}$$

Hence,

$$\nabla^2 \vec{A} = n(n+3)r^{n-2}(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z)$$

$$= n(n+3)r^{n-2}\vec{r}$$

Prob. 3.22

$$\nabla \cdot \vec{H} = k \nabla \cdot \nabla T = k \nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50 \sin \frac{\pi x}{2} \cosh \frac{\pi y}{2} \left(-\frac{\pi^2}{4} + \frac{\pi^2}{4} \right)$$

$$= 0$$

Hence

$$\nabla \cdot \vec{H} = 0$$

Prob. 3.23

$$(a) (\nabla \cdot \vec{r}) \vec{T} = 3\vec{T} = \underline{6yz \vec{a}_x + 3xy^2 \vec{a}_y + 3x^2yz \vec{a}_z}$$

$$(b) (\vec{r} \cdot \nabla) \vec{T} = x \frac{\partial \vec{T}}{\partial x} + y \frac{\partial \vec{T}}{\partial y} + z \frac{\partial \vec{T}}{\partial z}$$

$$= x(y^2 \vec{a}_y + 2xyz \vec{a}_z)$$

$$+ y(2z \vec{a}_x + 2xy \vec{a}_y + x^2z \vec{a}_z)$$

$$+ z(2y \vec{a}_x + x^2y \vec{a}_z)$$

$$= \underline{4yz \vec{a}_x + 3xy^2 \vec{a}_y + 4x^2yz \vec{a}_z}$$

$$(c) \nabla \cdot \vec{r} (\vec{r} \cdot \vec{T}) = 3(2xyz + xy^3 + x^2yz^2)$$

$$= \underline{6xyz + 3xy^3 + 3x^2yz^2}$$

$$(d) (\vec{r} \cdot \nabla) \vec{r} = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= x \cdot 2x + y \cdot 2y + z \cdot 2z$$

$$= \underline{2(x^2 + y^2 + z^2) = 2r^2}$$

Prob. 3.24

$$(a) \frac{\partial V_1}{\partial x} = -3x(x^2 + y^2 + z^2)^{-5/2}$$

$$\frac{\partial^2 V_1}{\partial x^2} = 15x^2(x^2 + y^2 + z^2)^{-7/2} - 3(x^2 + y^2 + z^2)^{-5/2}$$

Hence,

$$\begin{aligned}\nabla^2 V_1 &= 15(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-7/2} \\ &\quad - 9(x^2 + y^2 + z^2)^{-5/2} \\ &= \underline{\underline{6(x^2 + y^2 + z^2)^{-5/2}}}\end{aligned}$$

$$(b) \frac{\partial V_2}{\partial x} = ze^{-y} + 2x \ln y, \quad \frac{\partial^2 V_2}{\partial x^2} = 2 \ln y$$

$$\frac{\partial V_2}{\partial y} = -xze^{-y} + \frac{2x}{y}, \quad \frac{\partial^2 V_2}{\partial y^2} = xze^{-y} - \frac{2x}{y^2}.$$

$$\nabla^2 V_2 = \underline{\underline{2 \ln y + xze^{-y} - \frac{2x}{y^2}}}.$$

$$\begin{aligned}(c) \nabla^2 V_3 &= \frac{1}{\rho} \frac{\partial}{\partial \phi} (20\rho^2 \sin 2\phi) - \frac{40\rho^2}{\rho^2} \sin 2\phi \\ &= 40 \sin 2\phi - 40 \sin 2\phi = \underline{\underline{0}}.\end{aligned}$$

$$\begin{aligned}(d) \nabla^2 V_4 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho z) (\cos \phi + \sin \phi) - \frac{\rho z}{\rho^2} (\cos \phi + \sin \phi) \\ &= \frac{z}{\rho} (\cos \phi + \sin \phi) - \frac{z}{\rho} (\cos \phi + \sin \phi) \\ &= \underline{\underline{0}}.\end{aligned}$$

$$\begin{aligned}(e) \nabla^2 V_5 &= \frac{1}{r^2} \frac{\partial}{\partial r} (10r^3) \sin \theta \cos \phi + \frac{5r^2 \cos \phi}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cos \phi) \\ &\quad - \frac{5r^2 \sin \theta \cos \phi}{r^2 \sin^2 \theta}\end{aligned}$$

$$= 30 \sin \theta \cos \phi - 10 \sin \theta \cos \phi$$

$$= \underline{\underline{20 \sin \theta \cos \phi}}$$

$$\begin{aligned} (f) \quad \nabla^2 V_6 &= \frac{\sin \theta}{r^2} \frac{\partial}{\partial r} \left(-\frac{2}{r} \right) + \frac{1}{r^4 \sin \theta} (1 - 2 \sin^2 \theta) \\ &= \underline{\underline{\frac{1}{r^4 \sin \theta}}} \end{aligned}$$

Prob. 3.25

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$$

$$(a) \quad \nabla \cdot \vec{A} = 4x$$

$$\nabla \times \vec{A} = \vec{a}_x + y \vec{a}_y + (4y - z) \vec{a}_z$$

$$\nabla^2 \vec{A} = 4 \vec{a}_x - \vec{a}_x (4 - 0) = \underline{\underline{0}}$$

$$(b) \quad \nabla \cdot \vec{B} = (2 - 3z) \sin \phi,$$

$$\nabla \times \vec{B} = -6\rho z \cos \phi \vec{a}_\rho + \rho \sin \phi \vec{a}_\phi + (6z - 1)z \cos \phi \vec{a}_z$$

$$\nabla^2 \vec{B} = 0 \vec{a}_\rho + \frac{1}{\rho} (2 - 3z) z \cos \phi \vec{a}_\rho + (2 - 6z) \sin \phi \vec{a}_z$$

$$- \vec{a}_\phi \left[-\frac{1}{\rho} (6z - 1) z \sin \phi - 0 \right] - \vec{a}_\phi (-6\rho \cos \phi)$$

$$- \frac{\vec{a}_z}{\rho} (2\rho \sin \phi - 6\rho z \sin \phi)$$

$$= \frac{1}{\rho} (6z - 1) z \sin \phi \vec{a}_\rho + \left[(2 - 3z) \frac{z}{\rho} + 6\rho \right] \cos \phi \vec{a}_\phi$$

At $(5, \pi/4, 1)$, $\nabla^2 \vec{B} = \underline{\vec{a}_\phi}$

(c) $\nabla \cdot \vec{C} = 6 \cos \theta \cos \phi$,

$$\nabla(\nabla \cdot \vec{C}) = \frac{-6 \sin \theta \cos \phi}{r} \vec{a}_\theta - \frac{6 \cot \theta \sin \phi}{r} \vec{a}_\phi$$

$$\nabla \times \vec{C} = r^{-1/2} \cot \theta \vec{a}_r - \left(2 \cot \theta \sin \phi + \frac{3}{2} r^{-1/2} \right) \vec{a}_\theta + 2 \sin \theta \cos \phi \vec{a}_\phi$$

$$\begin{aligned} \nabla \times \nabla \times \vec{C} = & \left[\frac{4 \cos \theta \cos \phi}{r} + \frac{2 \cos \theta \cos \phi}{r \sin^2 \theta} \right] \vec{a}_r \\ & - \frac{2 \sin \theta \cos \phi}{r} \vec{a}_\theta \\ & + \left(r^{-3/2} \csc^2 \theta - \frac{2 \cos \theta \sin \phi}{r \sin \theta} - \frac{3}{4} r^{-3/2} \right) \vec{a}_\phi \end{aligned}$$

$$\nabla^2 \vec{C} = \nabla(\nabla \cdot \vec{C}) - \nabla \times \nabla \times \vec{C}$$

$$\begin{aligned} = & - \left[\frac{4 \cos \theta \cos \phi}{r} + \frac{2 \cos \theta \cos \phi}{r \sin^2 \theta} \right] \vec{a}_r \\ & - \frac{4 \sin \theta \cos \phi}{r} \vec{a}_\theta \end{aligned}$$

$$- \left[\frac{1}{r^{3/2} \sin^2 \theta} + \frac{4 \cos \theta \sin \phi}{r \sin \theta} - \frac{3}{4 r^{3/2}} \right] \vec{a}_\phi$$

At $(1, \pi/6, \pi/3)$,

$$\nabla^2 \vec{C} = -(\sqrt{3} + 2\sqrt{3}) \vec{a}_r + (0.5 - 1.5) \vec{a}_\theta + (4 + 6 - 0.75) \vec{a}_\phi$$

$$\nabla^2 \vec{C} = \underline{5.196 \vec{a}_r - \vec{a}_\theta - 9.25 \vec{a}_\phi.}$$

Prob. 3.26

$$(a) \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \underline{2(y^2 z^2 + x^2 z^2 + x^2 y^2)}$$

$$(b) \nabla^2 \vec{A} = \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$$

$$= (2y + 0 + 0) \vec{a}_x + (0 + 0 + 6xz) \vec{a}_y + (0 - 2z^2 - 2y^2) \vec{a}_z$$

$$= \underline{2y \vec{a}_x + 6xz \vec{a}_y - 2(y^2 + z^2) \vec{a}_z.}$$

$$(c) \text{grad div } \vec{A} = \nabla (\nabla \cdot \vec{A})$$

$$= \nabla (2xy + 0 - 2y^2 z)$$

$$= \underline{2y \vec{a}_x + 2(x - 2yz) \vec{a}_y - 2y^2 \vec{a}_z.}$$

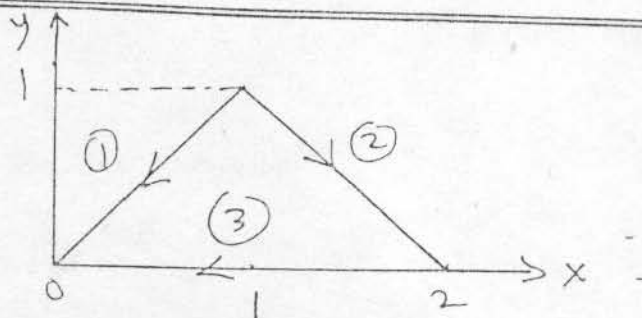
$$(d) \text{curl curl } \vec{A} = \nabla \times \nabla \times \vec{A}$$

$$= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

from parts (b) and (c),

$$\nabla \times \nabla \times \vec{A} = \underline{2(x - 2yz - 3xz) \vec{a}_y + 2z^2 \vec{a}_z.}$$

Prob. 3.27



$$(a) \oint_L \vec{F} \cdot d\vec{l} = \left(\int_{(1)} + \int_{(2)} + \int_{(3)} \right) \vec{F} \cdot d\vec{l}$$

for (1), $y = x \rightarrow dy = dx$, $d\vec{l} = dx\vec{a}_x + dy\vec{a}_y$,

$$\int_{(1)} \vec{F} \cdot d\vec{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}.$$

for (2), $y = -x + 2$, $dy = -dx$, $d\vec{l} = dx\vec{a}_x + dy\vec{a}_y$,

$$\int_{(2)} \vec{F} \cdot d\vec{l} = \int_1^2 (-x^3 + 2x^2 - x + 2) dx = \frac{17}{12}.$$

$$\int_{(3)} \vec{F} \cdot d\vec{l} = \int_2^0 x^2 y dx \Big|_{y=0} = 0.$$

$$\oint_L \vec{F} \cdot d\vec{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

$$(b) \nabla \times \vec{F} = -x^2 \vec{a}_z, \quad d\vec{S} = dx dy (-\vec{a}_z)$$

$$\int (\nabla \times \vec{F}) \cdot d\vec{S} = - \iint (-x^2) dx dy$$

$$= \int_0^1 \int_0^x x^2 dy dx + \int_1^2 \int_0^{-x+2} x^2 dy dx.$$

$$= \int_0^1 x^2 y \Big|_0^x dx + \int_1^2 x^2 y \Big|_0^{-x+2} dx$$

$$= \frac{x^4}{4} \Big|_0^1 + \int_1^2 x^2 (-x+2) dx = \underline{\underline{\frac{7}{6}}}$$

(c) Yes.

Prob. 3.28

(a) $\nabla \times \vec{A} = 2 \vec{a}_z$

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \int 2 dS = 2 \int dS$$

$$= 2 \times \text{Area of } S.$$

(b) $\psi = \int \vec{r} \cdot d\vec{S} = \int \nabla \cdot \vec{r} dv = \int 3 dv = 3v.$

Prob. 3.29

(a) for $x=0$, $\int \vec{F} \cdot d\vec{S} = - \int x^2 dz dy \Big|_{x=0} = 0$

for $x=1$, $\int \vec{F} \cdot d\vec{S} = \int x^2 dx dy \Big|_{x=1} = (1)(1) = 1.$

for $y=0$, $\int \vec{F} \cdot d\vec{S} = - \int y dx dz \Big|_{y=0} = 0.$

for $y=1$, $\int \vec{F} \cdot d\vec{S} = \int y dx dz \Big|_{y=1} = (1)(1) = 1.$

For $z=0$, $\int \vec{F} \cdot d\vec{S} = - \int z \, dx \, dy \Big|_{z=0} = 0$

for $z=1$, $\int \vec{F} \cdot d\vec{S} = \int z \, dx \, dy \Big|_{z=1} = (1)(1) = 1$

Hence,

$$\oint \vec{F} \cdot d\vec{S} = 1 + 0 + 1 + 0 + 1 + 0 = \underline{\underline{3}}.$$

(b) $\nabla \cdot \vec{F} = 2x + 1 + 1 = 2(x+1),$

$$\begin{aligned} \int_V \nabla \cdot \vec{F} \, dV &= 2 \int_0^1 (x+1) \, dx \int_0^1 dy \int_0^1 dz \\ &= 2 \left(\frac{x^2}{2} + 1 \right) \Big|_0^1 (1)(1) = \underline{\underline{3}}. \end{aligned}$$

Prob. 3.30

$$\begin{aligned} (a) \oint_S \vec{E} \cdot d\vec{S} &= \int \frac{1}{r^4} \sin^2 \phi \, r^2 \sin \theta \, d\theta \, d\phi \Big|_{r=4} \\ &\quad - \int \frac{1}{r^4} \sin^2 \phi \, r^2 \sin \theta \, d\theta \, d\phi \Big|_{r=2} \\ &= \left(\frac{1}{4^2} - \frac{1}{2^2} \right) \int_{\phi=0}^{2\pi} \sin^2 \phi \, d\phi \int_0^\pi \sin \theta \, d\theta \\ &= \underline{\underline{-\frac{3\pi}{8}}}. \end{aligned}$$

(b) $\nabla \cdot \vec{E} = -\frac{2}{r^5} \sin \phi$

$$\begin{aligned} \int_V \nabla \cdot \vec{E} \, dV &= -2 \int \frac{1}{r^3} \sin^2 \phi \sin \theta \, d\theta \, dr \, d\phi \\ &= -2 \int_0^{2\pi} \sin^2 \phi \, d\phi \int_0^\pi \sin \theta \, d\theta \int_2^4 \frac{dr}{r^3} \end{aligned}$$

$$= -2(\pi)(2) \left(\frac{1}{4} - \frac{1}{2} \right) = \underline{\underline{-\frac{3\pi}{8}}}$$

Prob. 3.31

$$\int_V \rho_v dv = \oint_S \vec{A} \cdot d\vec{s} \text{ — divergence theorem}$$

where $\rho_v = \nabla \cdot \vec{A} = x^2 + y^2$

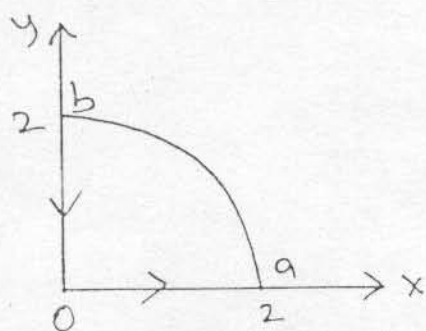
$$\frac{\partial A_x}{\partial x} = x^2 \longrightarrow A_x = \frac{x^3}{3} + c_1$$

$$\frac{\partial A_y}{\partial y} = y^2 \longrightarrow A_y = \frac{y^3}{3} + c_2$$

Hence,
$$\underline{\underline{\vec{A} = \left(\frac{x^3}{3} + c_1 \right) \vec{a}_x + \left(\frac{y^3}{3} + c_2 \right) \vec{a}_y}}$$

Prob. 3.32

(a)



$$d\vec{l} = dr \vec{a}_r + r d\phi \vec{a}_\phi, \quad \vec{A} \cdot d\vec{l} = r \sin \phi dr + r^3 d\phi$$

Along oa, $d\phi = 0$, $\phi = 0$, $\vec{A} \cdot d\vec{l} = 0$, $\int_0^a \vec{A} \cdot d\vec{l} = 0$

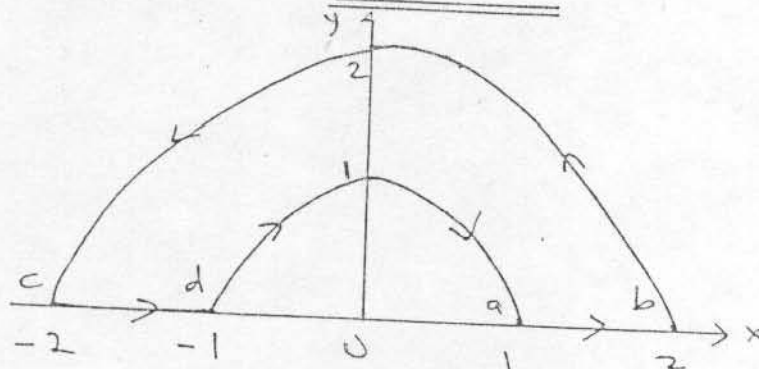
Along ab, $dr = 0$, $r = 2$, $\int_a^b \vec{A} \cdot d\vec{l} = \int_0^{\pi/2} 8 d\phi = 4\pi$

Along bo, $d\phi = 0$, $\phi = \pi/2$, $\int_b^0 \vec{A} \cdot d\vec{l} = \int_2^0 r dr = -2$

Hence,

$$\oint \vec{A} \cdot d\vec{l} = \underline{\underline{4\pi - 2}}$$

(b)



Along ab , $d\phi = 0$, $\phi = 0$, $\vec{A} \cdot d\vec{l} = 0$, $\int_a^b \vec{A} \cdot d\vec{l} = 0$.

Along bc , $d\rho = 0$, $\vec{A} \cdot d\vec{l} = \rho^3 d\phi$,

$$\int_b^c \vec{A} \cdot d\vec{l} = \int_0^\pi \rho^3 d\phi = (2)^3 (\pi - 0) = 8\pi$$

Along cd , $d\phi = 0$, $\phi = \pi$, $\vec{A} \cdot d\vec{l} = 0$, $\int_c^d \vec{A} \cdot d\vec{l} = 0$

Along da , $d\rho = 0$, $\vec{A} \cdot d\vec{l} = \rho^3 d\phi$,

$$\int_d^a \vec{A} \cdot d\vec{l} = \rho^3 \int_\pi^0 d\phi = (1)^3 (0 - \pi) = -\pi$$

Hence,

$$\oint \vec{A} \cdot d\vec{l} = 0 + 8\pi + 0 - \pi = \underline{\underline{7\pi}}$$

The result may be checked using Stokes's theorem.

Prob. 3.33

Let $\Psi = \oint \vec{F} \cdot d\vec{S} = \Psi_t + \Psi_b + \Psi_o + \Psi_i$

where Ψ_t , Ψ_b , Ψ_o , and Ψ_i are the fluxes through the top surface, bottom surface, outer surface ($\rho=3$), and inner surface ($\rho=2$) respectively (see Fig. 3.17).

for the top surface, $d\vec{S} = \rho d\phi d\rho \vec{a}_z$, $z=5$,

$\vec{F} \cdot d\vec{S} = \rho^2 z d\phi dz$. Hence

$\Psi_t = \int_{\rho=2}^3 \int_{\phi=0}^{2\pi} \rho^2 z d\phi d\rho \Big|_{z=5} = \frac{190\pi}{3}$

for the bottom surface, $z=0$, $d\vec{S} = \rho d\phi d\rho (-\vec{a}_z)$

$\vec{F} \cdot d\vec{S} = \rho^2 z d\phi d\rho = 0$. Hence

$\Psi_b = 0$

for the outer curved surface, $\rho=3$, $d\vec{S} = \rho d\phi dz \vec{a}_\rho$

$\vec{F} \cdot d\vec{S} = \rho^3 \sin\phi d\phi dz$. Hence

$\Psi_o = \int_{z=0}^5 dz \rho^3 \int_{\phi=0}^{2\pi} \sin\phi d\phi \Big|_{\rho=3} = 0$

for the inner curved surface, $\rho=2$, $d\vec{S} = \rho d\phi dz (-\vec{a}_\rho)$

$\Psi_i = - \int_{z=0}^5 dz \rho^3 \int_{\phi=0}^{2\pi} \sin\phi d\phi \Big|_{\rho=2} = 0$

$$\Psi = \frac{190\pi}{3} + 0 + 0 + 0 = \underline{\underline{\frac{190\pi}{3}}}$$

Using divergence theorem,

$$\Psi = \oint \vec{F} \cdot d\vec{s} = \int \nabla \cdot \vec{F} \, dv$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^3 \sin \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (z \cos \phi) + \rho \\ &= 3\rho \sin \phi - \frac{z}{\rho} \sin \phi + \rho \end{aligned}$$

$$\begin{aligned} \int_V \nabla \cdot \vec{F} \, dv &= \iiint (3\rho \sin \phi - \frac{z}{\rho} \sin \phi + \rho) \rho \, d\phi \, d\rho \, dz \\ &= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 \, d\rho \\ &= \underline{\underline{\frac{190\pi}{3}}} \end{aligned}$$

Prob. 3.34

$$\text{Let } \vec{B} = \nabla \times \vec{T}$$

$$\Psi = \oint_S \vec{B} \cdot d\vec{s} = \int_V \nabla \cdot \vec{B} \, dv = \int_V \nabla \cdot \nabla \times \vec{T} \, dv = 0$$

Prob. 3.35

$$\begin{aligned} \vec{Q} &= \frac{r}{r \sin \theta} \left[(\cos \phi - \sin \phi) \vec{a}_x + (\cos \phi + \sin \phi) \vec{a}_y \right] \\ &= r (\cos \phi - \sin \phi) \vec{a}_x + r (\cos \phi + \sin \phi) \vec{a}_y \end{aligned}$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\vec{Q} = r \sin\theta \vec{a}_r + r \cos\theta \vec{a}_\theta + r \vec{a}_\phi$$

(a) $d\vec{l} = \rho d\phi \vec{a}_\phi$, $\rho = r \sin 30^\circ = 2(\frac{1}{2}) = 1$

$$z = r \cos 30^\circ = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \vec{Q} \cdot d\vec{l} = \int_0^{2\pi} \sqrt{\rho^2 + z^2} \rho d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b) $\nabla \times \vec{Q} = \cot\theta \vec{a}_r - 2\vec{a}_\theta + \cos\theta \vec{a}_\phi$

for S_1 , $d\vec{s} = r^2 \sin\theta d\theta d\phi \vec{a}_r$,

$$\begin{aligned} \int_{S_1} (\nabla \times \vec{Q}) \cdot d\vec{s} &= \int r^2 \sin\theta \cot\theta d\theta d\phi \Big|_{r=2} \\ &= 4 \int_0^{2\pi} d\phi \int_0^{30^\circ} \cos\theta d\theta = \underline{\underline{4\pi}} \end{aligned}$$

(c) for S_2 , $d\vec{s} = r \sin\theta d\theta dr \vec{a}_\theta$,

$$\begin{aligned} \int_{S_2} (\nabla \times \vec{Q}) \cdot d\vec{s} &= -2 \int r \sin\theta d\phi dr \Big|_{\theta=30^\circ} \\ &= -2 \sin 30^\circ \int_0^2 r dr \int_0^{2\pi} d\phi \\ &= \underline{\underline{-4\pi}} \end{aligned}$$

(d) for S_1 , $d\vec{s} = r^2 \sin \theta d\phi d\theta \vec{a}_r$,

$$\begin{aligned} \int_{S_1} \vec{Q} \cdot d\vec{s} &= r^3 \int \sin^2 \theta d\theta d\phi \Big|_{r=2} \\ &= 8 \int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2 \theta d\theta \\ &= 4\pi \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right]. \end{aligned}$$

(e) for S_2 , $d\vec{s} = r \sin \theta d\phi dr \vec{a}_\theta$

$$\begin{aligned} \int_{S_2} \vec{Q} \cdot d\vec{s} &= \int r^2 \sin \theta \cot \theta d\phi dr \Big|_{\theta=30^\circ} \\ &= \frac{4\pi\sqrt{3}}{3}. \end{aligned}$$

(f) $\nabla \cdot \vec{Q} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \sin \theta) + \frac{r}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cot \theta)$

$+ 0$

$= 2 \sin \theta + \cot \theta \cot \theta$

$$\int \nabla \cdot \vec{Q} dv = \int (2 \sin \theta + \cot \theta \cot \theta) r^2 \sin \theta d\theta d\phi dr$$

$$= \frac{r^3}{3} \Big|_0^2 (2\pi) \int_0^{30^\circ} (1 + \sin^2 \theta) d\theta$$

$$= \frac{4\pi}{3} \left(\pi - \frac{\sqrt{3}}{2} \right).$$

Check: $\int \nabla \cdot \vec{Q} dv = \left(\int_{S_1} + \int_{S_2} \right) (\nabla \times \vec{Q}) \cdot d\vec{s}$

$$= 4\pi \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} \right]$$

$$= \frac{4\pi}{3} \left[\pi - \frac{\sqrt{3}}{2} \right]$$

It checks.

Prob. 3.36

Since $\vec{u} = \vec{w} \times \vec{r}$, $\nabla \times \vec{u} = \nabla \times (\vec{w} \times \vec{r})$

from Appendix A.10,

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B}$$

$$\nabla \times \vec{u} = \vec{w} \nabla \cdot \vec{r} - \vec{r} \nabla \cdot \vec{w} + (\vec{r} \cdot \nabla) \vec{w} - (\vec{w} \cdot \nabla) \vec{r}$$

$$= \vec{w} (3) - \vec{w} = 2\vec{w}$$

$$\therefore \vec{w} = \frac{1}{2} \nabla \times \vec{u}$$

Alternatively, let $x = r \cos \omega t$, $y = r \sin \omega t$,

$$\vec{u} = \frac{\partial x}{\partial t} \vec{a}_x + \frac{\partial y}{\partial t} \vec{a}_y$$

$$= -\omega r \sin \omega t \vec{a}_x + \omega r \cos \omega t \vec{a}_y$$

$$= -\omega y \vec{a}_x + \omega x \vec{a}_y$$

$$\nabla \times \vec{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial t} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \vec{a}_z = 2\vec{w}$$

i.e. $\vec{w} = \frac{1}{2} \nabla \times \vec{u}$

Prob. 3.37

$$\begin{aligned}
 (a) \quad \nabla (V \vec{A}) &= \frac{\partial}{\partial x} V A_x + \frac{\partial}{\partial y} V A_y + \frac{\partial}{\partial z} V A_z \\
 &= A_x \frac{\partial V}{\partial x} + V \frac{\partial A_x}{\partial x} + A_y \frac{\partial V}{\partial y} + V \frac{\partial A_y}{\partial y} \\
 &\quad + A_z \frac{\partial V}{\partial z} + V \frac{\partial A_z}{\partial z} \\
 &= \vec{A} \cdot \nabla V + V \nabla \cdot \vec{A}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \text{curl curl } \vec{A} &= \nabla \times (\nabla \times \vec{A}) \\
 &= \nabla \times \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
 &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{vmatrix} \\
 &= \frac{\partial}{\partial x} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \vec{a}_x + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \vec{a}_y \\
 &\quad + \frac{\partial}{\partial z} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \vec{a}_z \\
 &\quad - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z) \\
 &= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}.
 \end{aligned}$$

Prob. 3.38.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^2y & 3xy^2 & 0 \end{vmatrix} = (3y^2 - 5x^2) \vec{a}_z$$

$$\begin{aligned} \int \vec{\nabla} \times \vec{A} \, dv &= \iiint (3y^2 - 5x^2) \, dx \, dy \, dz \, \vec{a}_z \\ &= \vec{a}_z \left[3 \int_0^2 dx \int_{-1}^1 y^2 dy \int_{-5}^5 dz - 5 \int_0^2 x^2 dx \int_{-1}^1 dy \int_{-5}^5 dz \right] \\ &= \vec{a}_z \left[40 - \frac{800}{3} \right] = -226.67 \vec{a}_z \end{aligned}$$

$$\begin{aligned} \oint \vec{A} \times d\vec{s} &= \int \vec{A} \times dy \, dz \, \vec{a}_x \Big|_{x=2} - \int \vec{A} \times dy \, dz \, \vec{a}_x \Big|_{x=0} \\ &\quad + \int \vec{A} \times dx \, dz \, \vec{a}_y \Big|_{y=1} - \int \vec{A} \times dx \, dz \, \vec{a}_y \Big|_{y=-1} \\ &\quad + \int \vec{A} \times dx \, dy \, \vec{a}_z \Big|_{z=5} - \int \vec{A} \times dx \, dy \, \vec{a}_z \Big|_{z=-5} \\ &= - \iint 3xy^2 \, dy \, dz \Big|_{x=2} \vec{a}_z + \iint 3xy^2 \, dy \, dz \Big|_{x=0} \vec{a}_z \\ &\quad + \iint 5x^2y \, dx \, dz \Big|_{y=1} \vec{a}_z - \iint 5x^2y \, dx \, dz \Big|_{y=-1} \vec{a}_z \\ &\quad + \iint (3xy^2 \vec{a}_x - 5x^2y \vec{a}_y) \, dx \, dy \Big|_{z=5} \end{aligned}$$

$$\begin{aligned}
 & - \iint (3xy^2 \vec{a}_x - 5x^2y \vec{a}_y) dx dy \Big|_{z=5} \\
 & = -6 \int_{-1}^1 y^2 dy \int_{-5}^5 dz \vec{a}_z + 2(5) \int_0^2 x^2 dx \int_{-5}^5 dz \vec{a}_z \\
 & = \left(-40 + \frac{800}{3} \right) \vec{a}_z = 226.67 \vec{a}_z.
 \end{aligned}$$

Thus $\int \nabla \times \vec{A} dv = - \oint \vec{A} \times d\vec{S}.$

Prob. 3.39

$$\nabla \cdot \vec{R} = \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z} = 4xy + 2xy + cxy + 0$$

$$0 = xy(6+c) \rightarrow \underline{\underline{c = -6.}}$$

Prob. 3.40

$$\nabla \times \vec{T} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x + \beta z^2 & 3x^2 - \gamma z & 3xz^2 - y \end{vmatrix}$$

$$= (-1 + \gamma) \vec{a}_x + (3\beta z^2 - 3z^2) \vec{a}_y + (6x - \alpha x) \vec{a}_z$$

If $\vec{T}$ is irrotational, $\nabla \times \vec{T} = 0$, i.e.

$$\underline{\underline{\alpha = 1 = \beta = \gamma.}}$$

$$\nabla \cdot \vec{T} = \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z} = xy + 0 + 6xz$$

At $(2, -1, 0)$, $\nabla \cdot \vec{T} = -1 + 0 = \underline{\underline{-1}}$.

Prob. 3.41

(a) Let $\vec{A} = (x-yz)\vec{a}_x + (y^2-xz)\vec{a}_y + (z^2-xy)\vec{a}_z$

$$\nabla \times \vec{A} = 0$$

Hence $\int \vec{A} \cdot d\vec{l}$ is independent of path taken.

Similarly, if $\vec{B} = z^2\vec{a}_x + 2y\vec{a}_y + 2xz\vec{a}_z$,

$\nabla \times \vec{B} = 0$. Hence $\int \vec{B} \cdot d\vec{l}$ is independent of path.

(b) Let $I_1 = \int \vec{A} \cdot d\vec{l}$, $I_2 = \int \vec{B} \cdot d\vec{l}$

(i) $x=y=z \rightarrow dx=dy=dz$

$$I_1 = \int_0^1 (x - x^2) dx = \frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}}$$

$$I_2 = \int_0^1 (2x + 3x^2) dx = 1 + 1 = \underline{\underline{2}}$$

(ii) $I_1 = \int_0^1 x dx + \int_0^1 y^2 dy + \int_0^1 (z^2 - 1) dz$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - 1 = \underline{\underline{\frac{1}{6}}}$$

$$I_2 = \int_0^1 0 dx + \int_0^1 2y dy + \int_0^1 2z dz = 1 + 1 = \underline{\underline{2}}$$

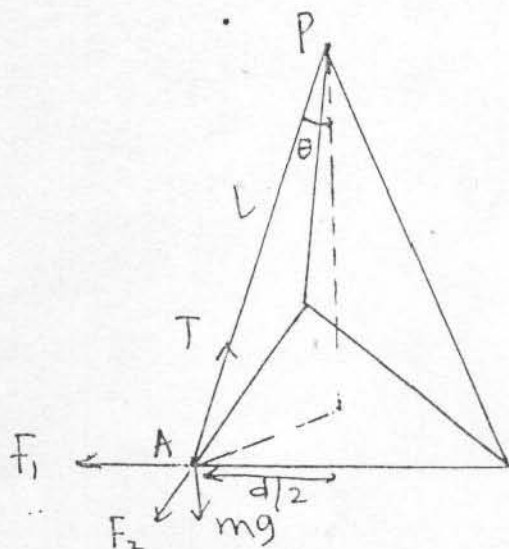
CHAPTER 4

P.E. 4.1

$$\begin{aligned}
 (a) \quad \vec{F} &= \frac{1 \times 10^{-9}}{4\pi \times 10^{-9} / 36\pi} \left[\frac{5 \times 10^{-9} \{ (1, -3, 7) - (2, 0, 4) \}}{|(1, -3, 7) - (2, 0, 4)|^3} \right. \\
 &\quad \left. - \frac{2 \times 10^{-9} \{ (1, -3, 7) - (-3, 0, 5) \}}{|(1, -3, 7) - (-3, 0, 5)|^3} \right] \text{ nN} \\
 &= \left[\frac{45(-1, -3, 3)}{19^{3/2}} - \frac{18(4, -3, 2)}{29^{3/2}} \right] \text{ nN} \\
 &= \underline{\underline{-1.004 \vec{a}_x - 1.284 \vec{a}_y + 1.4 \vec{a}_z \text{ nN}}}
 \end{aligned}$$

$$(b) \quad \vec{E} = \frac{\vec{F}}{Q} = \underline{\underline{-1.004 \vec{a}_x - 1.284 \vec{a}_y + 1.4 \vec{a}_z \text{ V/m}}}$$

P.E. 4.2



At point A,

$$\begin{aligned} T \sin \theta \cos 30^\circ &= F_1 + F_2 \cos 60^\circ \\ &= \frac{q^2}{4\pi\epsilon_0 d^2} + \frac{q^2}{4\pi\epsilon_0 d^2} \cdot \frac{1}{2} \\ &= \frac{3q^2}{8\pi\epsilon_0 d^2} \end{aligned}$$

$$T \cos \theta = mg$$

$$\text{Hence } \tan \theta \cos 30^\circ = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

$$\text{But } \sin \theta = \frac{h}{l} = \frac{d}{\sqrt{3}l} \rightarrow \tan \theta = \frac{d/\sqrt{3}}{\sqrt{l^2 - d^2/3}}$$

$$\text{Thus } \frac{\frac{d}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}}{\sqrt{l^2 - d^2/3}} = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

$$\text{or } q^2 = \frac{4\pi\epsilon_0 d^3 mg}{3\sqrt{l^2 - d^2/3}}$$

$$\text{But } q = Q/3 \rightarrow q^2 = Q^2/9. \text{ Hence}$$

$$Q^2 = \frac{12\pi\epsilon_0 d^3 mg}{\sqrt{l^2 - d^2/3}}$$

P.E. 4.3

$$e\vec{E} = m \frac{d^2 \vec{l}}{dt^2}$$

$$eE_0(-2\ddot{x} + \ddot{y}) = m \left(\frac{d^2x}{dt^2} \ddot{x} + \frac{d^2y}{dt^2} \ddot{y} + \frac{d^2z}{dt^2} \ddot{z} \right)$$

where $E_0 = 200 \text{ kV/m}$.

$$\frac{d^2z}{dt^2} = 0 \longrightarrow z = c_1 t + c_2$$

$$m \frac{d^2x}{dt^2} = -2eE_0 \longrightarrow x = -\frac{2eE_0 t^2}{2m} + c_3 t + c_4$$

$$m \frac{d^2y}{dt^2} = eE_0 \longrightarrow y = \frac{eE_0 t^2}{2m} + c_5 t + c_6$$

At $t=0$, $(x, y, z) = (0, 0, 0) \longrightarrow c_1 = 0 = c_4 = c_6$

Also, $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = (0, 0, 0)$ at $t=0 \Rightarrow c_1 = 0 = c_3 = c_5$.

Hence

$$(x, y) = \frac{eE_0 t^2}{2m} (-2, 1)$$

i.e. $2|y| = |x|$. Thus the largest horizontal distance is $80 \text{ cm} = \underline{\underline{0.8 \text{ m}}}$.

P.E. 4.4

(a) Consider an element of area ds of the disk. The contribution due to $ds = \rho d\phi dp$ is

$$dE = \frac{\rho_s ds}{4\pi\epsilon_0 r^2} = \frac{\rho_s ds}{4\pi\epsilon_0 (p^2 + h^2)}$$

The sum of the contributions along ρ gives zero.

$$dE_z = dE \cos \alpha = \frac{\rho_s ds}{4\pi\epsilon_0 (\rho^2 + h^2)} \cdot \frac{h}{(\rho^2 + h^2)^{3/2}}$$

$$E_z = \frac{\rho_s}{4\pi\epsilon_0} \int_0^a \int_0^{2\pi} \frac{z \rho d\rho d\phi}{(\rho^2 + h^2)^{3/2}} = \frac{h\rho_s}{2\epsilon_0} \int_0^a \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{h\rho_s}{4\epsilon_0} \int_0^a (\rho^2 + h^2)^{-3/2} d(\rho^2) = \frac{h\rho_s}{2\epsilon_0} \left(-2(\rho^2 + h^2)^{-1/2} \right) \Big|_0^a$$

$$= \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{h}{(h^2 + a^2)^{1/2}} \right]$$

(b) As $a \rightarrow \infty$, $\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a}_z$

P.E. 4.5

$$Q = \int \rho_s ds = \int_{-2}^2 \int_{-2}^2 12|y| dx dy$$

$$= 12(4) \int_0^3 2y dy = \underline{\underline{192 \text{ mC}}}$$

$$\vec{E} = \int \frac{\rho_s ds}{4\pi\epsilon_0 r^2} \vec{a}_r = \int \frac{\rho_s ds}{4\pi\epsilon_0} \frac{|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|^3}$$

where $\vec{r} - \vec{r}' = (0, 0, 10) - (x, y, z) = (-x, -y, 10)$.

$$\vec{E} = \int_{x=-2}^2 \int_{y=-2}^2 \frac{|2y| \cdot 10^{-3} (-x, -y, 10)}{4\pi \cdot \frac{10^{-9}}{36\pi} (x^2 + y^2 + 100)^{3/2}}$$

$$= 108 \times 10^6 \left[\int_{-2}^2 |y| \int_{-2}^2 \frac{-x dx \vec{a}_x dy}{(x^2 + y^2 + 100)^{3/2}} \right.$$

$$+ \int_{-2}^2 x \int_{-2}^2 \frac{-y |y| dy \vec{a}_y dx}{(x^2 + y^2 + 100)^{3/2}} \left.$$

$$+ 10 \vec{a}_z \int_{-2}^2 \int_{-2}^2 \frac{|y| dx dy}{(x^2 + y^2 + 100)^{3/2}} \right]$$

$$= 108 \vec{a}_z \cdot 10^7 \int_{-2}^2 \left[2 \int_0^2 \frac{\frac{1}{2} d(y^2)}{(x^2 + y^2 + 100)^{3/2}} \right] dx$$

$$= -216 \vec{a}_z \cdot 10^7 \int_{-2}^2 \left[\frac{1}{(x^2 + 104)^{1/2}} - \frac{1}{(x^2 + 100)^{1/2}} \right] dx$$

$$= -216 \vec{a}_z \cdot 10^7 \left| \ln \left| \frac{x + \sqrt{x^2 + 100}}{x + \sqrt{x^2 + 104}} \right| \right|_{-2}^2$$

$$= -216 \vec{a}_z \cdot 10^7 \left(\ln \left(\frac{2 + \sqrt{108}}{2 + \sqrt{104}} \right) - \ln \left(\frac{-2 + \sqrt{108}}{-2 + \sqrt{104}} \right) \right)$$

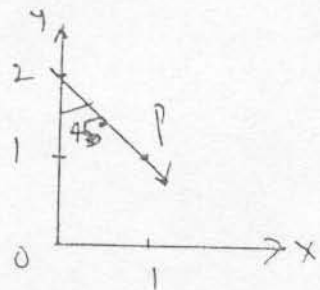
$$= -216 \vec{a}_z \cdot 10^7 (-7.6202 \times 10^{-3})$$

$$\vec{E} = 16.46 \vec{a}_z \text{ MV/m.}$$

P.E. 4.6

$$\vec{E}_3 = \frac{\rho_L}{2\pi\epsilon_0 r} \vec{a}_\rho$$

To get $\vec{a}_\rho$, consider the $z = -1$ plane



$$\rho = \sqrt{2}$$

$$\begin{aligned} \vec{a}_\rho &= \vec{a}_x \cos 45^\circ - \vec{a}_y \sin 45^\circ \\ &= \frac{1}{\sqrt{2}} (\vec{a}_x - \vec{a}_y) \end{aligned}$$

$$\begin{aligned} \vec{E}_3 &= \frac{10 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \cdot \frac{1}{2} (\vec{a}_x - \vec{a}_y) \\ &= 90\pi (\vec{a}_x - \vec{a}_y). \text{ Hence,} \end{aligned}$$

$$\begin{aligned} \vec{E} &= \vec{E}_1 + \vec{E}_2 + \vec{E}_3 \\ &= -180\pi \vec{a}_x + 270\pi \vec{a}_y + 90\pi \vec{a}_x - 90\pi \vec{a}_y \\ &= \underline{\underline{-282.7 \vec{a}_x + 565.5 \vec{a}_y \text{ V/m.}}} \end{aligned}$$

P.E. 4.7

from Example 4.7, $E = \frac{2\rho_L l}{2\pi\epsilon_0 r \sqrt{l^2 + 4r^2}}$

But $r = l/2$, $\rho_L = Q/l$. Hence

$$E = \frac{Q}{\pi\epsilon_0 \frac{l}{2} \sqrt{l^2 + l^2}} = \frac{\sqrt{2} Q}{\pi\epsilon_0 l^2}$$

P.E. 4.8

$$\begin{aligned}
 \vec{D} &= \vec{D}_q + \vec{D}_p = \frac{Q}{4\pi r^2} \vec{a}_r + \frac{\beta}{2} \vec{a}_n \\
 &= \frac{30 \times 10^{-9}}{4\pi (5)^2} \left[\frac{(0, 4, 3) - (0, 0, 0)}{5} \right] + \frac{10 \times 10^{-9}}{2} \vec{a}_y \\
 &= \frac{30}{500\pi} (0, 4, 3) + 5 \vec{a}_y \text{ nC/m}^2 \\
 &= \underline{\underline{5.076 \vec{a}_y + 0.0573 \vec{a}_z \text{ nC/m}^2}}
 \end{aligned}$$

P.E. 4.9

$$\begin{aligned}
 (a) \quad \rho_v &= \nabla \cdot \vec{D} = 4x \\
 \rho_v(-1, 0, 3) &= \underline{\underline{-4 \text{ C/m}^3}}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \psi = Q &= \int \rho_v dv = \int_0^1 \int_0^1 \int_0^1 4x dx dy dz \\
 &= 4(1)(1)\left(\frac{1}{2}\right) = \underline{\underline{2 \text{ C}}}
 \end{aligned}$$

$$(c) \quad Q = \psi = \underline{\underline{2 \text{ C}}}$$

P.E. 4.10

$$Q = \int \rho_v dv = \psi = \oint \vec{D} \cdot d\vec{s}$$

$$\text{for } 0 \leq r \leq 10,$$

$$D_r \cdot 4\pi r^2 = \iiint 2r \cdot r^2 \sin\theta d\theta dr d\phi$$

$$\text{Dr. } 4\pi r^2 = 4\pi \cdot \frac{2r^4}{4} \Big|_0^r = 2\pi r^4$$

$$D = \frac{r^2}{2} \rightarrow \vec{E} = \frac{r^2}{2\epsilon_0} \vec{a}_r \text{ nV/m}$$

$$\vec{E}(r=2) = \frac{4 \times 10^{-9}}{2 \times 10^{-9} \cdot 36\pi} \vec{a}_r = 72\pi \vec{a}_r = \underline{\underline{226 \vec{a}_r \text{ V/m.}}}$$

$$\text{for } r \geq 10, \text{ Dr. } 4\pi r^2 = 2\pi r_0^4, r_0 = 10 \text{ m.}$$

$$D = \frac{r_0^4}{2r^2} \rightarrow \vec{E} = \frac{r_0^4}{2\epsilon_0 r^2} \vec{a}_r \text{ nV/m}$$

$$\vec{E}(r=12) = \frac{10^4 \times 10^{-9}}{2 \times 10^{-9} \cdot 36\pi \cdot 144} \vec{a}_r = 1250\pi \vec{a}_r = \underline{\underline{3.927 \vec{a}_r \text{ kV/m.}}}$$

P.E. 4.11

$$V(\vec{r}) = \sum_{k=1}^3 \frac{Q_k}{4\pi\epsilon_0 |\vec{r} - \vec{r}_k|} + C$$

$$\text{If } V(\infty) = 0, C = 0$$

$$|\vec{r} - \vec{r}_1| = |(-1, 5, 2) - (2, -1, 3)| = \sqrt{46}$$

$$|\vec{r} - \vec{r}_2| = |(-1, 5, 2) - (0, 4, -2)| = \sqrt{18}$$

$$|\vec{r} - \vec{r}_3| = |(-1, 5, 2) - (0, 0, 0)| = \sqrt{30}$$

$$V(-1, 5, 2) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}} \right]$$

$$= \underline{\underline{10.3 \text{ kV.}}}$$

P.E. 4.12

$$V = \frac{Q}{4\pi\epsilon_0 r} + C$$

If $V(0, 6, -8) = V(r=10) = 2,$

$$2 = \frac{5 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} + C \rightarrow C = -2.5$$

$$(a) V_A = \frac{5 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} |(-3, 2, 6) - (0, 0, 0)|} - 2.5$$

$$= \underline{\underline{3.929 \text{ V}}}$$

$$(b) V_B = \frac{45}{\sqrt{7^2 + 1^2 + 5^2}} - 2.5 = \underline{\underline{2.696 \text{ V.}}}$$

$$(c) V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-1.233 \text{ V.}}}$$

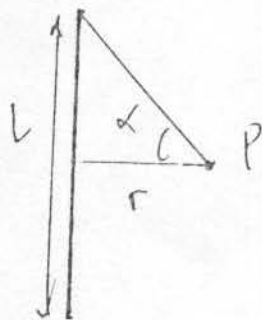
P.E. 4.13

$$V = 4V_1$$

where V_1 is the potential due to one side.

from Example 4.13,

$$V_1 = V_p = \frac{\rho_L}{4\pi\epsilon_0} \ln \frac{1 + \sin \alpha}{1 - \sin \alpha}$$



where $\tan \alpha = \frac{L/2}{r} \rightarrow \sin \alpha = \frac{L/2}{\sqrt{r^2 + L^2/4}}$

$$\begin{aligned} V = 4V_1 &= \frac{\rho_L}{\pi\epsilon_0} \ln \frac{1 + \sin 45^\circ}{1 - \sin 45^\circ} = \frac{\rho_L}{\pi\epsilon_0} \ln \left(\frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} \right) \\ &= \frac{\rho_L}{\pi\epsilon_0} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \left(\frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right) = \frac{2\rho_L}{\pi\epsilon_0} \ln (\sqrt{2} + 1) \end{aligned}$$

P.E. 4.14

$$\begin{aligned} (a) \quad -\frac{W}{Q} &= \int \vec{E} \cdot d\vec{l} = \int (3x^2 + y) dx + x dy \\ &= \int_0^2 (3x^2 + y) dx \Big|_{y=5} + \int_5^{-1} x dy \Big|_{x=2} \\ &= 18 - 12 = 6 \end{aligned}$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}$$

$$\begin{aligned} (b) \quad dy &= -3dx, \\ -\frac{W}{Q} &= \int \vec{E} \cdot d\vec{l} = \int_0^2 (3x^2 + 5 - 3x) dx + x(-3) dx \\ &= \int_0^2 (3x^2 - 6x + 5) dx = 8 - 12 + 10 = 6 \end{aligned}$$

$$W = \underline{\underline{12 \text{ m J.}}}$$

P.E. 4.15

$$(a) (0, 0, 10) \rightarrow (r=10, \theta=0, \phi=0)$$

$$V = \frac{100 \cos 0}{4\pi\epsilon (10)} \cdot 10^{-12} = \frac{10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi}} = \underline{\underline{9 \text{ mV.}}}$$

$$\begin{aligned} \vec{E} &= \frac{100 \times 10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi} (10)^3} [2 \cos 0 \vec{a}_r + \sin 0 \vec{a}_\theta] \\ &= \underline{\underline{1.8 \vec{a}_r \text{ mV/m.}}} \end{aligned}$$

$$(b) \text{ At } (1, \pi/3, \pi/2),$$

$$V = \frac{100 \cos \pi/3 \cdot 10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi} (1)^2} = \underline{\underline{0.45 \text{ V.}}}$$

$$\begin{aligned} \vec{E} &= \frac{100 \times 10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi} (1)^3} (2 \cos \frac{\pi}{3} \vec{a}_r + \sin \frac{\pi}{3} \vec{a}_\theta) \\ &= \underline{\underline{0.9 \vec{a}_r + 0.7794 \vec{a}_\theta \text{ V/m.}}} \end{aligned}$$

P.E. 4.16

$$(a) \frac{dy}{dx} = \frac{E_y}{E_x} = \frac{2xy}{y^2} = \frac{2x}{y}$$

$$\int y dy = \int 2x dx \rightarrow \frac{y^2}{2} = x^2 + c$$

$$\underline{y^2 = 2x^2 + c.}$$

which are hyperbolas. At $(2, -1, 3)$,

$$x=2, y=-1 \rightarrow 1 = 8 + c. \text{ or } c = -7,$$

$$\underline{y^2 = 2x^2 - 7.}$$

(b) $V = x^2 + y^2 = \text{constant}$

$$\text{i.e. } x^2 + y^2 = \rho^2 = c,$$

which are cylinders of radius ρ .

$$\frac{dy}{dx} = -\frac{1}{E_y/E_x} = -\frac{y}{2x} \rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\frac{1}{2} \ln x + \ln c \rightarrow c = y\sqrt{x}$$

$$\text{or } \underline{c_2 = xy^2.}$$

P. E. 4.17

After Q_1 , $W_1 = 0$

$$\begin{aligned} \text{After } Q_2, W_2 &= Q_2 V_{21} = \frac{Q_2 Q_1}{4\pi\epsilon_0 |(1,0,0) - (0,0,0)|} \\ &= \frac{(1)(-2) \times 10^{-18}}{4\pi \times 10^{-9} / 36\pi} = \underline{\underline{-18 \text{ nJ}.}} \end{aligned}$$

-10-

After Q_3 , $W_3 = Q_3 (V_{31} + V_{32}) + Q_2 V_{21}$

$$= 3 \times 9 \times 10^{-9} \left(\frac{1}{|(0,0,-1)-(0,0,0)|} + \frac{-2}{|(0,0,-1)-(1,0,0)|} \right) - 18 \text{ nJ}$$
$$= 27 \left(1 - \frac{2}{\sqrt{2}} \right) - 18 = \underline{\underline{-29.18 \text{ nJ}}}$$

After Q_4 ,

$$W_4 = Q_4 (V_{41} + V_{42} + V_{43}) + Q_3 (V_{31} + V_{32}) + Q_2 V_{21}$$
$$= -4 \times 9 \times 10^{-9} \left[\frac{1}{|(0,0,1)-(0,0,0)|} + \frac{-2}{|(0,0,1)-(1,0,0)|} + \frac{3}{|(0,0,1)-(0,0,-1)|} \right] + W_3$$
$$= -36 \left(1 - \frac{2}{\sqrt{2}} + \frac{3}{2} \right) + W_3$$
$$= -39.09 - 29.18 \text{ nJ} = \underline{\underline{-68.27 \text{ nJ}}}$$

P.E 4.18

$$\vec{E} = -\nabla V = -(y+1)\vec{a}_x + (1-x)\vec{a}_y - 2\vec{a}_z \text{ V/m}$$

At (1,2,3), $\vec{E} = \underline{-3\vec{a}_x - 2\vec{a}_z} \text{ V/m.}$

$$W = \frac{1}{2} \epsilon_0 \int \vec{E} \cdot \vec{E} dv$$

$$= \frac{1}{2} \epsilon_0 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x^2 + y^2 - 2x + 2y + 6) dx dy dz$$

$$= \frac{1}{2} \epsilon_0 \left[\int_{-1}^1 x^2 dx \int \int dy dz + \int_{-1}^1 y^2 dy \int \int dx dz \right.$$

$$\left. - 2 \int_{-1}^1 x dx \int \int dy dz + 2 \int_{-1}^1 y dy \int \int dx dz + 6(2)(2)(2) \right]$$

$$= \frac{1}{2} \epsilon_0 \left[2 \frac{x^3}{3} \Big|_{-1}^1 (2)(2) + 6(8) \right] = \frac{80\epsilon_0}{3}$$

$W = \underline{0.2358 \text{ nJ.}}$

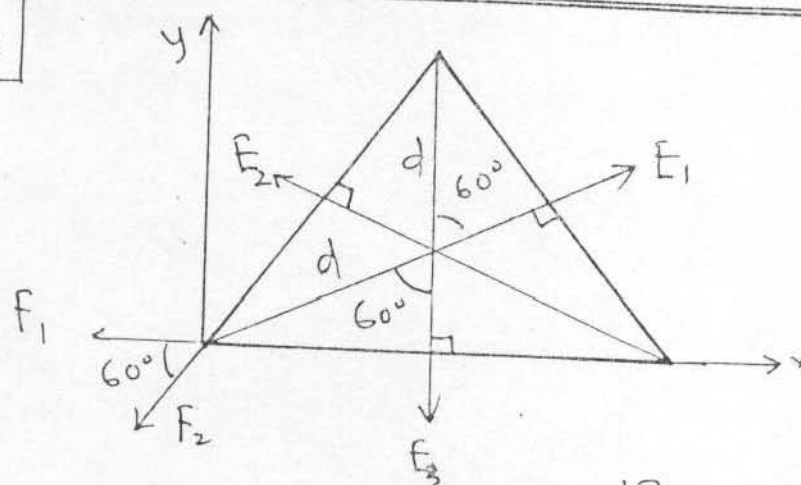
Prob. 4.1

(a) See Text.

$$\begin{aligned} (b) \vec{F}_1 &= \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{[(0,0,4) - (-2,6,1)]}{|(0,0,4) - (-2,6,1)|^3} \\ &\quad + \frac{Q_1 Q_3}{4\pi\epsilon_0} \frac{[(0,0,4) - (3,-4,-8)]}{|(0,0,4) - (3,-4,-8)|^3} \\ &= 9 \left[\frac{(4, -12, 6)}{7^3} + \frac{(9, -12, -36)}{13^3} \right] \text{ kN} \end{aligned}$$

$\vec{F}_1 = \underline{142 \vec{a}_x - 364 \vec{a}_y + 10 \vec{a}_z} \text{ N.}$

Prob. 4.2



$$a) |\vec{F}| = |\vec{F}_2| = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} = \frac{100 \times 10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi} \cdot 100 \times 10^{-4}} = 9 \times 10^{-5} \text{ N}$$

$$F_x = F_1 + F_2 \cos 60^\circ = \frac{3}{2} F_1$$

$$F_y = F_2 \sin 60^\circ = \frac{\sqrt{3}}{2} F_1$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{3} F_1 = 9\sqrt{3} \cdot 10^{-5}$$

$$F = \underline{\underline{156 \mu\text{N}}}$$

$$b) E_1 = E_2 = E_3 = \frac{Q}{4\pi\epsilon_0 d^2}$$

$$E_x = E_1 \sin 60^\circ - E_2 \sin 60^\circ = 0$$

$$E_y = E_1 \cos 60^\circ + E_2 \cos 60^\circ - E_3 = 0$$

Hence $\underline{\underline{\vec{E} = 0}}$ at the center.

Prob. 4.3

$$\vec{F} = \frac{Q}{4\pi\epsilon_0} \left[\frac{Q_1 \{ (0, -3, 4) - (0, -4, 3) \}}{|(0, -3, 4) - (0, -4, 3)|^3} + \frac{Q_2 \{ (0, -3, 4) - (0, 1, 1) \}}{|(0, -3, 4) - (0, 1, 1)|^3} \right]$$

$$= \frac{Q}{4\pi\epsilon_0} \left[Q_1 \frac{(0, 1, 1)}{2\sqrt{2}} + Q_2 \frac{(0, -4, 3)}{125} \right]$$

(a) If $F_z = 0$, then

$$\frac{Q_1}{2\sqrt{2}} + \frac{3Q_2}{125} = 0 \rightarrow Q_2 = -\frac{125}{3} \frac{Q_1}{2\sqrt{2}} = -\frac{125}{3\sqrt{2}}$$

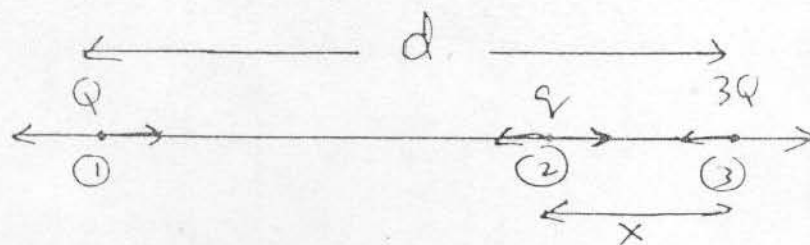
$$Q_2 = \underline{\underline{-29.46 \text{ nC}}}$$

(b) If $E_y = 0$, then

$$\frac{Q_1}{2\sqrt{2}} - \frac{4Q_2}{125} = 0 \rightarrow Q_2 = \frac{125}{4} \frac{Q_1}{2\sqrt{2}} = \frac{125}{4\sqrt{2}} \text{ nC}$$

$$Q_2 = \underline{\underline{22.1 \text{ nC}}}$$

Prob. 4.4



for the system to be in equilibrium, q must be negative and

$$\vec{F}_{12} = \vec{F}_{23} = \vec{F}_{13}$$

$$\therefore -\frac{Qq}{4\pi(d-x)^2} = -\frac{3Qq}{4\pi x^2} = \frac{4Q^2}{4\pi d^2}$$

$$\therefore 3(d-x)^2 = x^2 \rightarrow 3d^2 - 6dx + 3x^2 = x^2$$

$$2x^2 - 6dx + 3d^2 = 0$$

$$x = \frac{6d \pm \sqrt{36d^2 - 24d^2}}{4} = \frac{6d \pm d\sqrt{2}}{4}$$

$$x = 3 \pm \sqrt{3} = \underline{\underline{4.732\text{m}, 1.268\text{m}}}$$

Also, $3q d^2 = -4Q x^2$

$$q = -\frac{4Q x^2}{d^2} = -Q(1.268)^2 \text{ or } -Q(4.732)^2$$

$$= \underline{\underline{-1.6078Q, -22.392Q}}$$

Prob. 4.5

$$(a) Q = \int \rho_L dl = \int_0^5 12x^2 dx = 4x^3 \Big|_0^5 \text{ mC} = \underline{\underline{0.5\text{C}}}$$

$$(b) Q = \int \rho_s ds = \int_{z=0}^4 \int_{\phi=0}^{2\pi} \rho r^2 r d\phi dz \Big|_{\rho=3} = 9(2\pi) \frac{r^3}{3} \Big|_0^4 \text{ nC}$$

$$= \underline{\underline{1.206 \mu\text{C}}}$$

$$\begin{aligned}
 (c) \quad Q &= \int \rho \, dv = \iiint \frac{10}{r \sin \theta} r^2 \sin \theta \, d\theta \, d\phi \, dr \\
 &= 10 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^4 r \, dr = 10(2\pi)(\pi) \frac{4^2}{2} \\
 Q &= \underline{\underline{157.91 \, C.}}
 \end{aligned}$$

Prob. 4.6

$$\begin{aligned}
 Q_A &= \int_S \rho \, ds = \int_{x=0}^2 \int_{y=0}^{x/2} 2 \times 10^{-3} \, dx \, dy \\
 &= 2 \times 10^{-3} \left. \frac{x^2}{4} \right|_0^2 = \underline{\underline{2 \, mC.}}
 \end{aligned}$$

$$\begin{aligned}
 Q_B &= \int \rho \, ds = \int_{\phi=0}^{\pi} \int_{\phi=\frac{5\pi}{6}}^{\pi} 10^{-3} \rho \, d\rho \, d\phi \\
 &= \frac{\pi}{6} \cdot 10^{-3} \left. \frac{\rho^2}{2} \right|_0^2 = \underline{\underline{\frac{\pi}{3} \, mC.}}
 \end{aligned}$$

$$Q_C = \int \rho \, dl = \int_{\phi=\pi/6}^{3\pi/2} 10^{-6} \rho \, d\phi = \underline{\underline{\frac{2\pi}{3} \, \mu C.}}$$

Prob. 4.7

(a) from Example 4.4,

$$\begin{aligned}
 \vec{E} &= \frac{\rho_L a h \vec{a}_x}{2\epsilon_0 (h^2 + a^2)^{3/2}} = \frac{5 \times 10^{-9} \times 4 \times 3 \vec{a}_x}{2 \times 10^{-9} / 36\pi (5)^{3/2}} \\
 &= 18\pi (12) / 25 \vec{a}_x = \underline{\underline{27.14 \vec{a}_x \, V/m.}}
 \end{aligned}$$

(b) $\vec{E}$ due to the charges is

$$\vec{E}_Q = \frac{Q}{4\pi\epsilon_0} \left[\frac{(4, 0, 0) - (0, 4, 0)}{|(4, 0, 0) - (0, 4, 0)|^3} + \frac{(4, 0, 0) - (0, -4, 0)}{|(4, 0, 0) - (0, -4, 0)|^3} \right] = \frac{2Q}{\pi\epsilon_0 32^{3/2}} \vec{a}_x$$

$$\text{If } \vec{E}_Q + \vec{E}_L = 0 \rightarrow Q = -\frac{32^{3/2} \pi\epsilon_0 (27.14)}{2}$$

$$Q = \underline{\underline{-68.23 \text{ nC.}}}$$

Prob. 4.8

Let $\rho_s = k\rho$

$$Q = \int k\rho dS = \int_{\rho=0}^a \int_{\phi=0}^{2\pi} k\rho \cdot \rho d\phi d\rho = 2\pi k \frac{a^3}{3}$$

Hence $k = \frac{3Q}{2\pi a^3}$ and $\rho_s = \frac{3Q\rho}{2\pi a^3}$

Due to symmetry, $\vec{E}$ has only z -component.

$$dE_z = \frac{\rho_s dS \cos\alpha}{4\pi\epsilon_0 r} = \frac{\rho_s dS}{4\pi\epsilon_0 (\rho^2 + h^2)} \cdot \frac{h}{(\rho^2 + h^2)^{1/2}}$$

$$E_z = \frac{3Q}{2\pi a^3} \cdot \frac{1}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^a \frac{\rho h \cdot \rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{3Qh}{4\pi\epsilon_0 a^3} \int_0^a \frac{\rho^2 d\rho}{(\rho^2 + h^2)^{3/2}}$$

Let $p = h \tan \theta$, $dp = h \sec^2 \theta d\theta$.

$$\int_0^a \frac{p^2 dp}{(p^2 + h^2)^{3/2}} = \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int (\sec \theta - \cos \theta) d\theta$$

$$= \ln (\sec \theta + \tan \theta) - \sin \theta$$

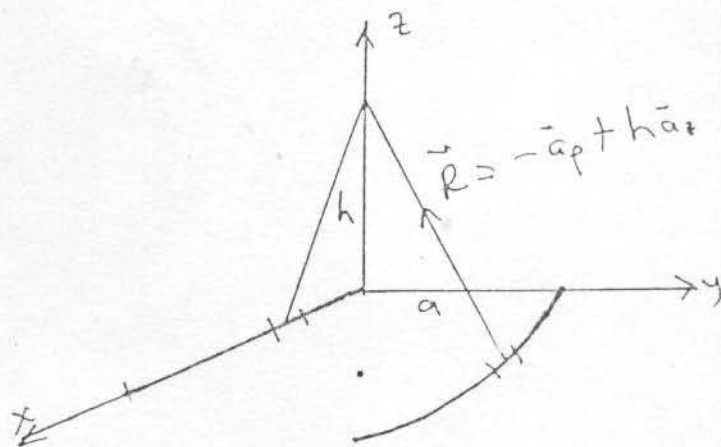
Hence,

$$E = \frac{3Qh}{4\pi\epsilon_0 a^3} \left[\ln \left(\frac{\sqrt{p^2 + h^2} + p}{h} \right) - \frac{p}{\sqrt{p^2 + h^2}} \right]_0^a$$

$$\vec{E} = \frac{3Qh}{4\pi\epsilon_0 a^3} \left[\ln \frac{a + \sqrt{a^2 + h^2}}{h} - \frac{a}{\sqrt{a^2 + h^2}} \right] \vec{a}_z$$

as required.

Prob. 4.9



(a) $\rho_L = \frac{Q}{L} = \frac{2}{3} nC/m$.

$$\vec{E} = \int \frac{\rho_L dl}{4\pi\epsilon_0 R^2} \vec{a}_R = \frac{\frac{2}{3} \times 10^{-9}}{4\pi \times 10^{-9} \times 36\pi} \int_0^3 \frac{(4\vec{a}_z + x\vec{a}_x) dx}{(x^2 + 16)^{3/2}}$$

$$= \frac{-6}{\sqrt{x^2 + 16}} \Big|_0^3 \vec{a}_x + \frac{24x}{\sqrt{x^2 + 16}} \Big|_0^3 \vec{a}_z$$

$$\vec{E} = \underline{\underline{-0.3\vec{a}_x + 0.9\vec{a}_z \text{ V/m.}}}$$

$$(1) \vec{E} = -E_\rho \vec{a}_\rho + E_z \vec{a}_z, \quad \rho_L = \frac{Q}{3\pi/4} = \frac{4Q}{3\pi}$$

$$E_z = \frac{\rho_L a h}{4\pi\epsilon_0 (h^2 + a^2)^{3/2}} \int_{\phi=\pi/4}^{\pi/2} d\phi = \frac{\rho_L a h}{16\epsilon_0 (a^2 + h^2)^{3/2}}$$

$$= \frac{8}{3\pi} \cdot \frac{10^{-9}(3)(4)}{16 \times 10^{-9} / 36\pi (125)} = \frac{72}{125}$$

$$\text{Since } \vec{a}_\rho = \cos\phi \vec{a}_x + \sin\phi \vec{a}_y,$$

$$E_\rho = \frac{-\rho_L a^2}{4\pi\epsilon_0 (h^2 + a^2)^{3/2}} \int_{\pi/4}^{\pi/2} (\cos\phi \vec{a}_x + \sin\phi \vec{a}_y) d\phi$$

$$= \frac{-\rho_L a^2}{16\epsilon_0 (h^2 + a^2)^{3/2}} \left[\left(1 - \frac{1}{\sqrt{2}}\right) \vec{a}_x + \frac{1}{\sqrt{2}} \vec{a}_y \right]$$

$$= -0.1611 \vec{a}_x - 0.3889 \vec{a}_y$$

$$\therefore \vec{E} = \underline{\underline{-0.1611 \vec{a}_x - 0.3889 \vec{a}_y + 0.576 \vec{a}_z \text{ V/m.}}}$$

Prob 4.10

Due to symmetry, $\vec{E}$ has only z-component
given by

$$dE_z = dE \cos \alpha$$

$$dE_z = \frac{\rho_s dx dy}{4\pi\epsilon_0 (x^2 + y^2 + h^2)} \cdot \frac{h}{(x^2 + y^2 + h^2)^{3/2}}$$

$$E_z = \frac{\rho_s h}{4\pi\epsilon_0} \int_{-a}^a \int_{-b}^b \frac{dx dy}{(x^2 + y^2 + h^2)^{3/2}}$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a \int_0^b \frac{dx dy}{(x^2 + y^2 + h^2)^{3/2}}$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a \frac{y dx}{(x^2 + h^2) (x^2 + y^2 + h^2)^{1/2}} \Big|_0^b$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a \frac{b dx}{(x^2 + h^2) (x^2 + b^2 + h^2)^{1/2}}$$

By changing variables, we finally obtain

$$E_z = \frac{\rho_s}{\pi\epsilon_0} \tan^{-1} \left[\frac{ab}{h(a^2 + b^2 + h^2)^{1/2}} \right]$$

$$(b) Q = \int \rho_s ds = \rho_s (2a)(2b) = 10^{-5} (4)(10) = \underline{\underline{0.4 \text{ mC}}}$$

$$\vec{E} = \frac{10^{-5}}{15 \times 10^{-9} / 36\pi} \tan^{-1} \left[\frac{10}{10(4 + 25 + 100)^{1/2}} \right] \vec{a}_z$$

$$= 36 \times 10^{-3} (0.0878 \text{ radian}) \vec{a}_z = \underline{\underline{31.61 \vec{a}_z \mu\text{V/m}}}$$

Prob. 4.11

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r + \frac{\rho_L}{2\pi\epsilon_0 \rho} \vec{a}_\rho + \frac{\rho_S}{2\epsilon_0} \vec{a}_n$$

$$= \frac{100 \times 10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi}} \frac{[(1,1,1) - (4,1,-3)]}{|(1,1,1) - (4,1,-3)|^3}$$

$$+ \frac{2 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \frac{[(1,1,1) - (1,0,0)]}{|(1,1,1) - (1,0,0)|^2} + \frac{5 \times 10^{-9} (-\vec{a}_z)}{2 \times \frac{10^{-9}}{36\pi}}$$

$$= (-0.0216, 0, 0.0288) + (0, 18, 18) - 90\pi (0, 0, 1)$$

$$= \underline{\underline{-0.0216 \vec{a}_x + 18 \vec{a}_y - 264.7 \vec{a}_z \text{ V/m.}}}$$

Prob. 4.12

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_L}{2\pi\epsilon_0 \rho} \vec{a}_\rho + \frac{\rho_S}{2\epsilon_0} \vec{a}_n$$

$$= \frac{30 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \frac{[(0,0,0) - (0,1,-3)]}{|(0,0,0) - (0,1,-3)|^2} + \frac{20 \times 10^{-9} (-\vec{a}_x)}{2 \times \frac{10^{-9}}{36\pi}}$$

$$= \underline{\underline{-1131 \vec{a}_x - 54 \vec{a}_y + 162 \vec{a}_z \text{ V/m.}}}$$

Prob. 4.13

$$\vec{D} = \sum_{k=1}^4 \frac{Q_k (\vec{r} - \vec{r}_k)}{4\pi |\vec{r} - \vec{r}_k|^3}$$

$$\begin{aligned}
 \vec{D} &= \frac{Q}{4\pi} \left[\frac{2\{(0,0,6) - (2,2,0)\}}{|(0,0,6) - (2,2,0)|^3} - \frac{2\{(0,0,6) - (-2,-2,0)\}}{|(0,0,6) - (-2,-2,0)|^3} \right. \\
 &\quad \left. + \frac{\{(0,0,6) - (-2,2,0)\}}{|(0,0,6) - (-2,2,0)|^3} - \frac{\{(0,0,6) - (2,-2,0)\}}{|(0,0,6) - (2,-2,0)|^3} \right] \\
 &= \frac{15}{4\pi} \left[\frac{2(-2,-2,6)}{44^{3/2}} - \frac{2(2,2,6)}{44^{3/2}} + \frac{(2,-2,6)}{44^{3/2}} - \frac{(-2,2,6)}{44^{3/2}} \right] \\
 &= \frac{15}{4\pi 44^{3/2}} (-4, -12, 0) \text{ nC/m}^2 \\
 &= \underline{\underline{-16.36 \vec{a}_x - 49.08 \vec{a}_y \text{ nC/m}^2}}
 \end{aligned}$$

Prob. 4.14

$$\begin{aligned}
 \vec{D} &= \sum_{k=1}^3 \frac{Q_k (\vec{r} - \vec{r}_k)}{4\pi |\vec{r} - \vec{r}_k|^3} \\
 &= \frac{1}{4\pi} \left[\frac{Q_1(0,0,-1)}{|(0,0,-1)|^3} + \frac{Q_2(6,-8,0)}{|(6,-8,0)|^3} + \frac{Q_3(0,-4,3)}{|(0,-4,3)|^3} \right] \\
 \vec{D} &= \underline{\underline{-2.39 \vec{a}_x - 8.276 \vec{a}_y - 314.5 \vec{a}_z \text{ nC/m}^2}}
 \end{aligned}$$

Prob. 4.15

(a) $D = \frac{p_L}{2\pi r}$, $p = \sqrt{6^2 + 8^2} = 10$

$$D = \frac{p_L}{2\pi(10)} \rightarrow p_L = 20\pi D = 60\pi \text{ nC/m} \\
 = \underline{\underline{188.5 \text{ nC/m}}}$$

$$(b) \vec{D}(0,0,4) = \frac{\rho_L}{2\pi\ell} \vec{a}_z = \frac{60\pi}{2\pi(4)} \vec{a}_z = \underline{\underline{7.5 \vec{a}_z \text{ nC/m}^2}}$$

Prob. 4.16

Let Q_1 be located at the origin. At the spherical surface of radius r ,

$$Q_1 = \oint \vec{D} \cdot d\vec{s} = \epsilon E_r \cdot 4\pi r^2$$

$$\therefore \vec{E} = \frac{Q_1}{4\pi\epsilon r^2} \vec{a}_r$$

by Gauss's law. If a second charge Q_2 is placed on the spherical surface, Q_2 experiences a force

$$\vec{F} = Q_2 \vec{E} = \frac{Q_1 Q_2}{4\pi\epsilon r^2} \vec{a}_r$$

which is Coulomb's law.

Prob. 4.17

$$(a) \rho_v = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 8y + 0 = \underline{\underline{8y \text{ C/m}^3}}$$

$$(b) \rho_v = \nabla \cdot \vec{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 \sin \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (2\rho \cos \phi) + \frac{\partial}{\partial z} (2z^2) \\ = 2 \sin \phi - 2 \sin \phi + 4z = \underline{\underline{4z \text{ C/m}^2}}$$

$$(c) \rho_v = \nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{2}{r} \cos \theta \right) + \frac{1}{r^4 \sin \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta)$$

$$\rho_v = \frac{-2}{r^3} \cos \theta + \frac{1}{r^4 \sin \theta} 2 \sin \theta \cos \theta = \underline{\underline{0}}$$

Prob. 4.18

for $r > a = 10 \text{ cm}$,

$$Q_T = \oint \vec{D} \cdot d\vec{S} = Q + \int \rho_v dv$$

If $\vec{D} = 0$ for $r > a$, then

$$\begin{aligned} Q &= - \int \rho_v dv = - \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{0.1} 10^{-2} r^3 r^2 \sin \theta d\theta d\phi dr \\ &= -4\pi 10^{-2} \frac{r^6}{6} \Big|_0^{0.1} = \underline{\underline{-21 \text{ nC}}} \end{aligned}$$

Prob. 4.19

$$(a) \oint \vec{D} \cdot d\vec{S} = Q_T \rightarrow \vec{E} = \frac{Q_T}{4\pi\epsilon_0 r^2} \vec{a}_r$$

for $r = 4 \text{ cm}$,

$$Q_T = \int \rho_v dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\int_{r=0}^3 \rho_{v1} + \int_{r=3}^4 \rho_{v2} \right] r^2 \sin \theta d\theta d\phi dr$$

where $\rho_{v1} = 10$, $\rho_{v2} = 20$, $\rho_{v3} = 0$.

$$\begin{aligned} Q_T &= \frac{4\pi}{3} [27\rho_{v1} + (64-27)\rho_{v2}] \cdot 10^{-6} \\ \vec{E}(r=4 \text{ cm}) &= \frac{4\pi}{3} [270 + 37 \times 20] \times 10^{-6} \times 10^{-9} \vec{a}_r \\ &= \underline{\underline{23.8 \vec{a}_r \text{ V/m}}} \end{aligned}$$

(b) for $r = 6\text{cm}$,

$$Q_T = \frac{4\pi}{3} \left[\rho_{v1} r^3 \Big|_0^3 + \rho_{v2} r^3 \Big|_3^5 + \underbrace{\rho_{v3} r^3 \Big|_5^6}_{\times 20} \right]$$

$$= \frac{4\pi}{3} [27\rho_{v1} + 98\rho_{v2}]$$

$$\vec{E}(r=6\text{cm}) = \frac{4\pi}{3} \frac{(270 + 98 \times 20) \times 10^{-6} \times 10^{-9} \vec{a}_r}{4\pi \times \frac{10^{-9}}{36\pi} \times 36 \times 10^{-4}}$$

$$= \underline{\underline{23.35 \vec{a}_r \text{ V/m}}}$$

Prob. 4.20

from Gauss's law, $\oint \vec{D} \cdot d\vec{S} = Q = \Psi$. Since the faces are equidistant from the center, they must have the flux Ψ_i thru them. Hence

$$6\Psi_i = Q \rightarrow \Psi_i = \frac{Q}{6} = \underline{\underline{10 \text{ mC}}}$$

Prob. 4.21

Let $Q_1 = 5\mu\text{C}$, $Q_2 = -3\mu\text{C}$, $Q_3 = 2\mu\text{C}$, and $Q_4 = 10\mu\text{C}$. The corresponding distances of the charge from the origin are:

$$r_1 = \sqrt{12^2 + 5^2} = 13\text{m},$$

$$r_2 = \sqrt{3^2 + 4^2} = 5\text{m},$$

$$r_3 = \sqrt{2^2 + 6^2 + 3^2} = 7 \text{ m},$$

$$r_4 = \sqrt{3^2} = 3 \text{ m}.$$

According to Gauss's Law, $\psi = Q_{\text{enc}}$.

(a) For $r=1 < r_1, r_2, r_3, r_4$, $Q_{\text{enc}} = 0$

$$\psi = \underline{\underline{0}}.$$

(b) For $r=10 > r_2, r_3, r_4$,

$$Q_{\text{enc}} = Q_2 + Q_3 + Q_4 = -3 + 2 + 10 = 9 \mu\text{C}$$

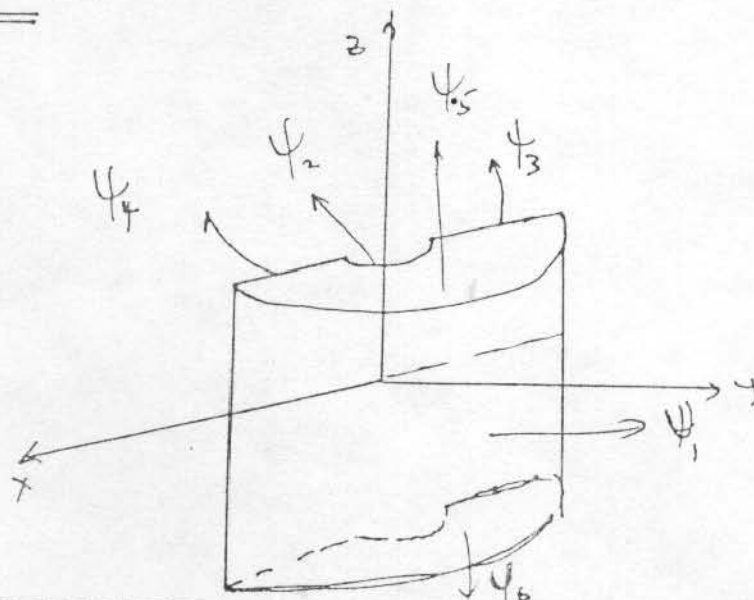
$$\psi = \underline{\underline{9 \mu\text{C}}}.$$

(c) For $r=15 > r_1, r_2, r_3, r_4$

$$Q_{\text{enc}} = Q_1 + Q_2 + Q_3 + Q_4 = 14 \mu\text{C}$$

$$\psi = \underline{\underline{14 \mu\text{C}}}.$$

Prob. 4.22



Method 1

Let $\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6$

where ψ_n 's are shown on the previous page.

$$\psi = \left. \iint 2z^2 \sin \frac{\phi}{2} \rho d\phi dz \right|_{\rho=4} - \left. \iint 2z^2 \sin \frac{\phi}{2} \rho d\phi dz \right|_{\rho=1}$$

$$+ \left. \iint z^2 \cos \frac{\phi}{2} d\rho dz \right|_{\phi=\pi} - \left. \iint z^2 \cos \frac{\phi}{2} d\rho dz \right|_{\phi=0}$$

$$+ \left. \iint 4z\rho \sin \frac{\phi}{2} \rho d\phi d\rho \right|_{z=1} - \left. \iint 4z\rho \sin \frac{\phi}{2} \rho d\phi d\rho \right|_{z=-2}$$

$$= 6 \int_0^\pi \sin \frac{\phi}{2} \int_{-2}^1 z^2 dz - \int_1^4 d\rho \int_{-2}^1 z^2 dz + 12 \int_1^4 \rho^2 d\rho \int_0^\pi \sin \frac{\phi}{2} d\phi$$

$$= 4(9) - 9 + 8(63) = 531 \text{ mC}$$

$$\psi = \underline{\underline{0.531 \text{ C}}}$$

Method 2

$$\begin{aligned} \nabla \cdot \vec{D} &= \frac{2z^2 \sin \frac{\phi}{2}}{\rho} - \frac{z^2}{2\rho} \sin \frac{\phi}{2} + 4\rho \sin \frac{\phi}{2} \\ &= \sin \frac{\phi}{2} \left(4\rho + \frac{3}{2} z^2 \right) \end{aligned}$$

$$\psi = \int \nabla \cdot \vec{D} dv = \int_0^\pi \sin \frac{\phi}{2} d\phi \iint \left(\frac{3z^2}{2\rho} + 4\rho \right) \rho d\rho dz$$

$$= 2 \left[4 \int_{-2}^1 dz \int_1^4 \rho^2 d\rho + \frac{3}{2} \int_1^4 d\rho \int_{-2}^1 z^2 dz \right]$$

$$\psi = 2 \left[4(63) + \frac{3}{2}(a) \right] = 531 \text{ mC}$$

$$= \underline{\underline{0.531 \text{ C}}}$$

Prob. 4.23

$$(a) \psi = \int \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{z=0}^5 2\rho z \cos^2 \phi \rho d\phi dz \Big|_{\rho=3}$$

$$= 2(3)^2 \int_0^{2\pi} \cos^2 \phi d\phi \int_0^5 z dz = 225\pi$$

$$\psi = \underline{\underline{706.9 \text{ C}}}$$

(b) Since $d\vec{s}$ is normal to $-\vec{a}_z$ due to the fact that the surface is at the bottom of the cylinder,

$$\psi = - \int_{\rho=0}^3 \int_{\phi=0}^{2\pi} \rho^2 \cos^2 \phi \rho d\phi d\rho$$

$$= - \int_0^{2\pi} \cos^2 \phi d\phi \int_0^3 \rho^3 d\rho = -\frac{81\pi}{4}$$

$$\psi = \underline{\underline{-63.6 \text{ C}}}$$

(c) Same as in (b) except that $d\vec{s}$ is along $+\vec{a}_z$ (top of the cylinder). Hence

$$\psi = \underline{\underline{63.6 \text{ C}}}$$

$$(d) \psi = \oint \vec{D} \cdot d\vec{s} = \psi = 706.9 - 63.6 + 63.6$$

$$= 706.9 \text{ C}$$

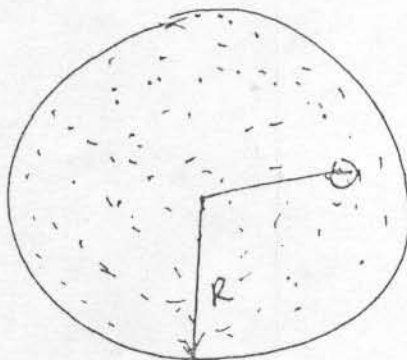
$$\alpha \quad Q = \int \rho_r dv = \int \nabla \cdot \vec{D} dv$$

$$\nabla \cdot \vec{D} = 4z \cos^2 \phi - z \cos 2\phi, \quad dv = \rho d\phi d\rho dz$$

$$Q = 4 \int_0^5 z dz \int_0^{2\pi} \cos^2 \phi d\phi \int_0^3 \rho d\rho - 0 = 225\pi$$

$$Q = \underline{\underline{706.9 \text{ C}}}$$

Prob. 4.24



$$F = eE, \quad \rho_0 = \frac{e}{4\pi R^3/3} = \frac{3e}{4\pi R^3}$$

$$\rho_r = \begin{cases} \rho_0, & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$$

$$\oint \vec{D} \cdot d\vec{S} = Q_{enc} = \int \rho_r dv = \frac{3e}{4\pi R^3} \cdot \frac{4\pi R^3}{3} = 4\pi r^2$$

$$E_r = \frac{3er}{12\pi\epsilon_0 R^3}$$

$$f = eE = \frac{e^2 r}{4\pi\epsilon_0 R^3}$$

Prob. 4.25

$$\rho_r = \nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (10r) = \frac{10}{r^2} \text{ nc/m}^3$$

$$Q = \int \rho_v dv = \iiint \frac{10}{r^2} \cdot r^2 \sin \theta d\theta dr d\phi$$

$$= 10 \int_0^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = 10(2)(2\pi)(2)$$

$$= 80\pi = \underline{\underline{251.3 \text{ nC}}}$$

$$^{\circ} Q = \oint \vec{D} \cdot d\vec{S} = \iint \frac{10}{r} \cdot r^2 \sin \theta d\theta d\phi \Big|_{r=2}$$

$$= 10(2) \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = 20(2)(2\pi) = 80\pi$$

$$= \underline{\underline{251.3 \text{ nC}}}$$

Prob. 4.26

Break up the path $P(1, 2, -4) \rightarrow R(3, -5, 6)$ as:

$$P(1, 2, -4) \quad R(3, -5, 6)$$

$$\downarrow \quad \uparrow$$

$$P'(3, 2, -4) \longrightarrow R'(3, -5, -4)$$

$$-\frac{W}{Q} = \int \vec{E} \cdot d\vec{l} = \left(\int_P^{P'} + \int_{P'}^{R'} + \int_{R'}^R \right) \vec{E} \cdot d\vec{l}$$

$$= \int_{x=1}^3 dx + \int_{y=2}^{-5} z^2 dy \Big|_{z=-4} + \int_{z=-4}^6 2yz dz \Big|_{y=-5}$$

$$= 2 + 16(-7) + 2(-5) \frac{z^2}{2} \Big|_{-4}^6 = 2 - 112 - 100 = -210$$

$$W = 210Q = 210(5) = \underline{\underline{1050 \text{ J}}}$$

Prob. 4.27

$$W = -Q \int \vec{E} \cdot d\vec{l}$$

(a) from A to B, $d\vec{l} = \rho d\phi \vec{a}_\phi$, $z=1$, $\phi=0$.

$$W = -Q \int_0^2 20\rho z \cos\phi d\phi \Big|_{z=1, \phi=0} = \underline{\underline{-40 \mu J.}}$$

(b) from B to C, $d\vec{l} = \rho d\phi \vec{a}_\phi$, $\rho=2$, $z=1$

$$W = -Q \int_{\phi=0}^{\pi/2} (-10\rho z \sin\phi) \rho d\phi = \underline{\underline{20 \mu J.}}$$

(c) from C to D, $d\vec{l} = dz \vec{a}_z$, $\rho=2$, $\phi = \pi/3$,

$$W = -Q \int_1^3 10\rho^2 \cos\phi dz = \underline{\underline{-40 \mu J.}}$$

(d) From A to D,

$$W = -40 + 20 - 40 = \underline{\underline{-60 \mu J.}}$$

Prob. 4.28

(a) from A to B, $d\vec{l} = r d\theta \vec{a}_\theta$,

$$W_{AB} = -Q \int_{\theta=30^\circ}^{90^\circ} 10r \cos\theta r d\theta \Big|_{r=5} = \underline{\underline{-1250 nJ.}}$$

(b) from A to C, $d\vec{l} = dr \vec{a}_r$,

$$W_{AC} = -Q \int_{r=5}^{10} 20r \sin\theta dr \Big|_{\theta=30^\circ} = \underline{\underline{-3750 nJ.}}$$

(c) from A to D, $d\vec{l} = r \sin\theta d\phi \vec{a}_\phi$,

$$W_{AD} = -Q \int 0 \cdot r \sin \theta d\theta = \underline{\underline{0 J.}}$$

$$(d). W_{AE} = \cancel{W_{AD}} + W_{DF} + W_{FE}$$

where f is $(10, 30^\circ, 60^\circ)$. Hence

$$W_{AE} = -Q \left[\int_{r=5}^{10} 20r \sin \theta dr \Big|_{\theta=30^\circ} + 10 \int_{\theta=30^\circ}^{90^\circ} 10r \cos \theta d\theta \Big|_{r=10} \right]$$

$$= -100 \left[\frac{75}{2} + \frac{100}{2} \right] nJ = \underline{\underline{-8750 nJ}}$$

Prob. 4.29

$$V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r} - \vec{r}_2|}$$

$$(a) |\vec{r} - \vec{r}_1| = |(0, 1, 0) - (0, 0, 0)| = 1,$$

$$|\vec{r} - \vec{r}_2| = |(0, 1, 0) - (0, 0, 1)| = \sqrt{2}$$

$$V = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{3}{1} - \frac{2}{\sqrt{2}} \right] = \underline{\underline{14.27 V.}}$$

$$(b) |\vec{r} - \vec{r}_1| = |(1, 1, 1) - (0, 0, 0)| = \sqrt{3}$$

$$|\vec{r} - \vec{r}_2| = |(1, 1, 1) - (0, 0, 1)| = \sqrt{2}$$

$$V = \frac{10^{-9}}{4\pi \times 10^{-9} / 36\pi} \left[\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{2}} \right] = \underline{\underline{2.86 \text{ V}}}$$

Prob. 4.30

$$V = - \int \vec{E} \cdot d\vec{l} = - \int \frac{\rho_L}{2\pi\epsilon_0 r} dr = - \frac{\rho_L}{2\pi\epsilon_0} \ln r + C$$

$$V_A - V_P = - \frac{\rho_L}{2\pi\epsilon_0} \ln \frac{\rho_A}{\rho_P}$$

where $\rho_A = \sqrt{x_A^2 + z_A^2} = 5;$

$\rho_P = \sqrt{x_P^2 + z_P^2} = 6.$

$$(a) \quad V_A - 5 = - \frac{10 \times 10^{-9}}{2\pi \times 10^{-9} / 36\pi} \ln \frac{5}{6}$$

$$V_A = 32.82 + 5 = \underline{\underline{37.82 \text{ V}}}$$

$$(b) \quad 5 - V_P = -180 \ln 5/6$$

$$V_P = 5 - 32.82 = \underline{\underline{-27.82 \text{ V}}}$$

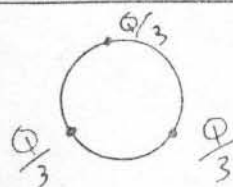
Prob. 4.31

$$(a) \quad V = \frac{2(Q/2)}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r}$$

$$= \frac{60 \times 10^{-6}}{4\pi \times 10^{-9} / 36\pi} \cdot 4 = \underline{\underline{15 \text{ kV}}}$$



$$(b) V = \frac{3 \left(\frac{Q}{3} \right)}{4\pi\epsilon_0 r} = \underline{\underline{15 \text{ kV.}}}$$



$$(c) V = \int \frac{\rho_L dl}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} = \underline{\underline{15 \text{ kV.}}}$$

Prob. 4.32

$$V = \int \frac{\rho_s ds}{4\pi\epsilon_0 R} = \frac{\rho_s}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^a \frac{\rho d\phi d\rho}{(h^2 + \rho^2)^{3/2}}$$

$$= \frac{\rho_s}{4\pi\epsilon_0} (2\pi) \int_0^a (h^2 + \rho^2)^{-3/2} \frac{1}{2} d(\rho^2)$$

$$= \frac{\rho_s}{4\epsilon_0} 2 (h^2 + \rho^2)^{1/2} \Big|_0^a$$

$$V = \frac{\rho_s}{2\epsilon_0} \left[\sqrt{h^2 + a^2} - h \right].$$

Prob. 4.33

$$(a) \nabla \times \vec{E} = (-5\rho \sin\phi + 5\rho \sin\phi) \vec{a}_\rho \\ + (10\rho \cos\phi - 10\rho \cos\phi) \vec{a}_\phi \\ + \frac{1}{\rho} (-10\rho z \sin\phi + 10\rho z \sin\phi) \vec{a}_z = 0.$$

$$(b) d\vec{l} = \rho d\phi \vec{a}_\phi, \rho=1, z=1 \\ \oint \vec{E} \cdot d\vec{l} = \int_0^{2\pi} (-5 \rho z \sin\phi \rho d\phi) = -5 \int_0^{2\pi} \sin\phi d\phi = 0.$$

Prob. 4.34.

from Eq. (4.12),

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum \frac{Q_k (\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3}.$$

Using $\nabla \times V\vec{A} = V \nabla \times \vec{A} + \nabla V \times \vec{A}$ where

$$V = \frac{1}{|\vec{r} - \vec{r}_k|^3}, \quad \vec{A} = \vec{r} - \vec{r}_k,$$

$$\nabla \times \frac{\vec{r} - \vec{r}_k}{|\vec{r} - \vec{r}_k|^3} = \frac{1}{|\vec{r} - \vec{r}_k|^3} \nabla \times (\vec{r} - \vec{r}_k) + \nabla \frac{1}{|\vec{r} - \vec{r}_k|^3} \times (\vec{r} - \vec{r}_k)$$

from Prob. 3.20(g), $\nabla \times (\vec{r} - \vec{r}_k) = 0$. It can also be shown that $\text{grad } r^n = n r^{n-2} \vec{r}$ so that

$$\nabla \frac{1}{|\vec{r} - \vec{r}_k|^3} = \frac{-3(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^5}.$$

Hence

$$\nabla \times \frac{\vec{r} - \vec{r}_k}{|\vec{r} - \vec{r}_k|^3} = 0 - \frac{3(\vec{r} - \vec{r}_k) \times (\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3} = 0$$

and thus $\nabla \times \vec{E} = 0$ as required.

Prob. 4.35

$$(a) \vec{E} = -\nabla V = e^{-x} (-\vec{a}_x \sinh y \sin z + \vec{a}_y \cosh y \sin z + \vec{a}_z \sinh y \cos z) \text{ V/m.}$$

$$\begin{aligned}
 (b) \quad \vec{E} &= -\nabla V = -\frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-3/2} \vec{a}_x + \dots \\
 &= \frac{3}{2} (x^2 + y^2 + z^2)^{-5/2} (2x) \vec{a}_x + \dots \\
 &= \frac{3(x \vec{a}_x + y \vec{a}_y + z \vec{a}_z)}{(x^2 + y^2 + z^2)^{5/2}} \quad V/m.
 \end{aligned}$$

$$(c) \quad \vec{E} = -\nabla V = \underline{\underline{e^{-z} (-\cos \phi \vec{a}_\rho + \sin \phi \vec{a}_\phi + \rho \cos \phi \vec{a}_z) V/m}}}$$

$$(d) \quad \vec{E} = -\nabla V = \underline{\underline{\frac{1}{r^3} (2 \cos \phi \sin \theta \vec{a}_r - \cos \theta \cos \phi \vec{a}_\theta + \sin \phi \vec{a}_\phi) \quad V/m.}}}$$

Prob. 4.36

$$(a) \quad V = -\int \vec{E} \cdot d\vec{l} = -\int \vec{E} \cdot dx \vec{a}_x = -\int 2xy dx$$

$$V = -x^2 y + V_1(y, z) \quad (1)$$

$$\text{or } V = -\int \vec{E} \cdot dy \vec{a}_y = -\int x^2 dy = -x^2 y + V_2(x, z) \quad (2)$$

$$\text{or } V = -\int \vec{E} \cdot dz \vec{a}_z = \int dz = z + V_3(x, y) \quad (3)$$

from (1) to (3),

$$V = \underline{\underline{z - x^2 y + C_1}}$$

$$(b) \quad V = -\int \vec{E} \cdot d\rho \vec{a}_\rho = -\rho e^{-z} \sin \phi + V_1(z, \phi) \quad (1)$$

$$\text{or } V = -\int \vec{E} \cdot \rho d\phi \vec{a}_\phi = -\rho e^{-z} \sin \phi + V_2(\rho, z) \quad (2)$$

$$\text{or } V = - \int \vec{E} \cdot d\vec{z} \vec{a}_z = - \rho e^{-z} \sin \phi + V_3(\rho, \phi) \quad (3)$$

From (1) to (3),

$$V = \underline{\underline{C_2 - \rho e^{-z} \sin \phi}}$$

$$(c) \quad V = - \int \vec{E} \cdot d\vec{r} \vec{a}_r = - \ln r \sin \theta + V_1(\phi, \theta) \quad (1)$$

$$\text{or } V = - \int \vec{E} \cdot r d\theta \vec{a}_\theta = - \ln r \sin \theta + V_2(r; \phi) \quad (2)$$

from (1) and (2),

$$V = \underline{\underline{C_3 - \ln r \sin \theta}}$$

Prob. 4.37

$$(a) \quad m \frac{dy}{dt^2} = eE \rightarrow u \frac{dy}{dt} = \frac{eEt}{m} + C_0$$

$$y = \frac{eEt^2}{2m} + C_0 t + C_1$$

"from rest" implies $C_1 = 0 = C_0$.

$$\text{At } t = t_0, \quad y = d, \quad E = \frac{V}{d} \quad \text{or} \quad V = Ed.$$

$$t^2 = \frac{2md}{eE}$$

Hence

$$u = \frac{eE}{m} \sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eV}{m}}$$

$$\text{i.e.} \quad u \propto \sqrt{V}$$

$$\sim \quad u = k\sqrt{V}$$

$$(b) \quad k = \sqrt{\frac{2e}{m}} = \sqrt{\frac{2 \times 1.603 \times 10^{-19}}{9.1066 \times 10^{-31}}} \\ = \underline{\underline{5.933 \times 10^5}}$$

$$(c) \quad V = \frac{U_m^2}{2e} = \frac{9 \times 10^{16} \times \frac{1}{100}}{2 \times 1.76 \times 10^{11}} = 2557$$

$$V = \underline{\underline{2.557 \text{ kV}}}$$

Prob. 4.38

(a) This is similar to Example 4.3.

$$U_y = \frac{eEt}{m}, \quad U_x = U_0$$

$$y = \frac{eEt^2}{2m}, \quad x = U_0 t$$

$$t = \frac{x}{U_0} = \frac{10 \times 10^{-2}}{10^7} = 10 \text{ ns.}$$

Since $x = 10 \text{ cm}$ when $y = 1 \text{ cm}$,

$$E = \frac{2my}{et^2} = \frac{2 \times 10^{-2}}{1.76 \times 10^{11} \times 10^{-16}} = 1.136 \text{ kV/m}$$

$$\vec{E} = \underline{\underline{-1.136 \hat{a}_y \text{ kV/m}}}$$

$$(b) \quad U_x = U_0 = 10^7, \quad U_y = \frac{2000}{1.76} \times 1.76 \times 10^{11} \times 10^{-8} = 2 \times 10^6$$

$$\vec{U} = \underline{\underline{(\hat{a}_x + 0.2 \hat{a}_y) \times 10^7 \text{ m/s}}}$$

Prob. 4.39

(a) Applying Gauss's law,

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

$$E_r \cdot 4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$$

$$\begin{aligned} Q_{enc} &= \int_0^a 4\pi r^2 \rho_r dv = 4\pi \rho_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r^2 dr \\ &= \frac{8\pi a^3 \rho_0}{15} = E_r \cdot 4\pi r^2 \epsilon_0 \end{aligned}$$

$$\vec{E} = \frac{2 a^3 \rho_0}{15 \epsilon_0 r^2} \vec{a}_r, \quad r \geq a.$$

$$V = - \int E_r dr = \frac{2 a^3 \rho_0}{15 \epsilon_0 r} + c_1$$

But $V(r \rightarrow \infty) = 0$, hence $c_1 = 0$ and

$$V = \frac{2 a^3 \rho_0}{15 \epsilon_0 r}, \quad r \geq a.$$

(b) When $r < a$, i.e. inside the sphere,

$$\begin{aligned} Q_{enc} &= \int_0^r \rho_0 \left(1 - \frac{r^2}{a^2}\right) 4\pi r^2 dr \\ &= 4\pi \rho_0 \left(\frac{r^3}{3} - \frac{r^5}{15 a^2}\right) = E_r \cdot 4\pi r^2 \epsilon_0 \end{aligned}$$

$$\vec{E} = \frac{\rho_0}{\epsilon_0} \left[\frac{r}{3} - \frac{r^3}{5a^2} \right] \vec{a}_r \quad \forall m, r \leq a.$$

$$V = - \int E_r dr = - \frac{\rho_0}{\epsilon_0} \left(\frac{r^2}{6} - \frac{r^4}{20a^2} \right) + C_2$$

from (a), $V(r=a) = \frac{2a^2\rho_0}{15\epsilon_0}$, hence

$$\frac{2a^2\rho_0}{15\epsilon_0} = \frac{\rho_0}{\epsilon_0} \left(\frac{a^2}{20} - \frac{a^2}{6} \right) + C_2$$

i.e. $C_2 = \frac{a^2\rho_0}{4\epsilon_0}$

Thus

$$V = \frac{\rho_0}{\epsilon_0} \left[\frac{a^2}{4} + \frac{r^4}{20a^2} - \frac{r^2}{6} \right], r \leq a$$

(c) $\vec{E}$ is maximum for $r \leq a$.

$$\frac{dE_r}{dr} = \frac{\rho_0}{\epsilon_0} \left(\frac{1}{3} - \frac{3r^2}{5a^2} \right) = 0 \rightarrow r^2 = \frac{5a^2}{9}$$

$$r = a \sqrt{\frac{5}{9}} = \underline{\underline{0.745a}}$$

(d) V is maximum for $r \leq a$ where

$$\frac{dV}{dr} = 0 \rightarrow \frac{4r^3}{20a^2} - \frac{2r}{6} = 0$$

i.e. $r = 0, a\sqrt{\frac{5}{3}}$

$$V(r=0) = \frac{\rho_0 a^2}{4\epsilon_0}, \quad V(r=a\sqrt{\frac{5}{3}}) = \frac{a^2\rho_0}{9\epsilon_0}$$

Hence

$$V_{\max} = \frac{\rho_0 a^2}{4\epsilon_0} \text{ at } r=0.$$

Prob. 4.40

$$V = \frac{p \cos \theta}{4\pi\epsilon_0 r} = \frac{k \cos \theta}{r}$$

At $(0, 1\text{nm})$, $\theta=0$, $r=1\text{nm}$, $V=9$, i.e.

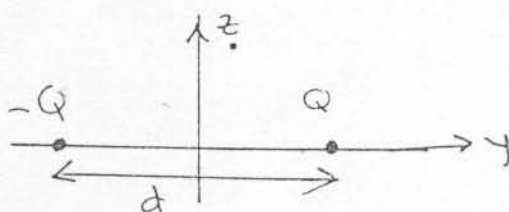
$$9 = \frac{k \cdot 1}{1 \times 10^{-18}} \rightarrow k = 9 \times 10^{-18}$$

$$V = 9 \times 10^{-18} \frac{\cos \theta}{r}$$

At $(1, 1)\text{nm}$, $r = \sqrt{2}\text{nm}$, $\theta = 45^\circ$,

$$V = \frac{9 \times 10^{-18} \cos 45^\circ}{10^{-18} \sqrt{2}} = \frac{9}{2\sqrt{2}} = \underline{\underline{3.182\text{ V}}}$$

Prob. 4.41



The dipole is oriented along y-axis.

$$V = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}, \quad \vec{p} \cdot \vec{r} = Qd \vec{a}_y \cdot \vec{a}_r = Qd \sin \theta \sin \phi$$

$$V = \frac{Qd \sin \theta \sin \phi}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} \vec{a}_r - \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{a}_\theta - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \vec{a}_\phi$$

$$= \frac{Qd}{4\pi\epsilon_0} \left[\frac{2 \sin \theta \sin \phi}{r^3} \vec{a}_r - \frac{\cos \theta \sin \phi}{r^3} \vec{a}_\theta - \frac{\cos \phi}{r^3} \vec{a}_\phi \right]$$

$$\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3} (2 \sin \theta \sin \phi \vec{a}_r - \cos \theta \sin \phi \vec{a}_\theta - \cos \phi \vec{a}_\phi)$$

Prob. 4.42

(a) $\frac{dr}{E_r} = \frac{r d\theta}{E_\theta} = \frac{r \sin \theta d\phi}{E_\phi}$. Hence

$$\frac{dr}{2 \cos \theta} = \frac{r d\theta}{\sin \theta} \rightarrow \frac{dr}{r} = \frac{2 \cos \theta}{\sin \theta} d\theta$$

Integrating both sides,

$$\ln r = 2 \int \frac{d(\sin \theta)}{\sin \theta} = 2 \ln \sin \theta + \ln c$$

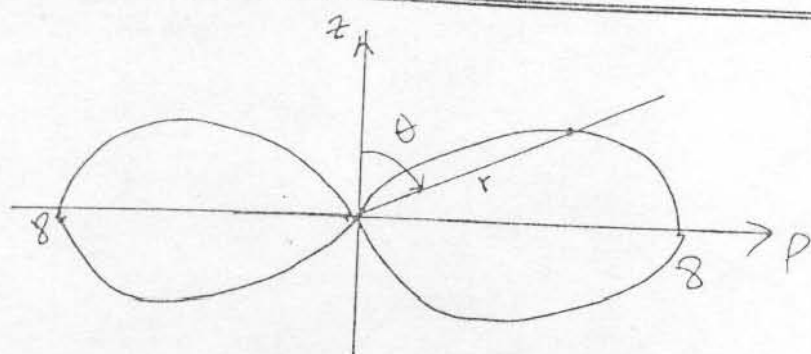
$$\text{or } r = c \sin^2 \theta$$

At $(4, \pi/4, \pi/2)$, $r=1$, $\theta = \pi/4$,

$$4 = c \cdot \frac{1}{2} \rightarrow c = 8$$

$$r = 8 \sin^2 \theta.$$

The flux line is as sketched on the next page.



Since $\vec{E}$ is tangential to the line,

$$\vec{E}(1, \frac{1}{4}, \frac{1}{2}) = \frac{(2\vec{a}_r + \vec{a}_z)}{4^3 \sqrt{2}}$$

A unit vector along $\vec{E}$ is

$$\vec{a} = \frac{1}{\sqrt{5}}(2\vec{a}_r + \vec{a}_z) = \underline{\underline{0.8944\vec{a}_r + 0.4472\vec{a}_z}}$$

(b) The equipotential line is $V=0$, i.e.

$$0 = V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2}$$

At an arbitrary point $(x, y, 0)$,

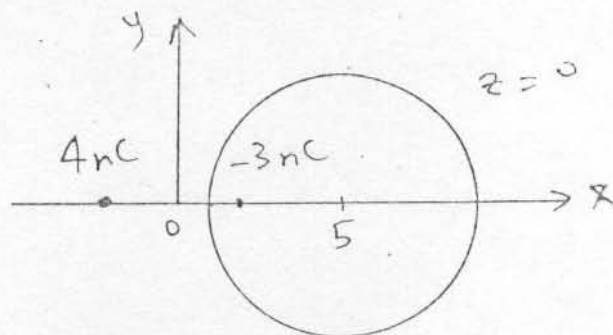
$$0 = \frac{10^{-9}}{4\pi\epsilon_0} \left[\frac{-3}{|(x, y) - (1, 0)|} + \frac{4}{|(x, y) - (-1, 0)|} \right]$$

$$\frac{3}{\sqrt{(x-1)^2 + y^2}} = \frac{4}{\sqrt{(x+1)^2 + y^2}}$$

$$16(x-1)^2 + 16y^2 = 9(x+1)^2 + 9y^2$$

$$\therefore (x-5)^2 + y^2 = (4.86)^2$$

i.e. a circle of radius 4.86 centered at (5,0). Thus the equipotential line is sketched below



Prob. 4.43

$$(a) \quad W = Q_3 (V_{31} + V_{32}) = Q_3 \left(\frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} \right)$$

$$= \frac{2 \times 10^{-18}}{4\pi \times 10^{-9} \times 36\pi} \left[\frac{-3}{|(0,0,0) - (1,0,0)|} + \frac{4}{|(0,0,0) - (-1,0,0)|} \right]$$

$$W = \underline{\underline{18 \text{ nJ}}}$$

$$(b) \quad W = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{Q_2 Q_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{Q_3 Q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{Q_3 Q_2}{|\vec{r}_3 - \vec{r}_2|} \right]$$

$$= 9 \times 10^{-9} \times 10^{-18} \left[\frac{-12}{2} - \frac{6}{1} + \frac{8}{1} \right]$$

$$= \underline{\underline{-36 \text{ nJ}}}$$

Prob. 4.44

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, dv = \frac{\epsilon_0}{2} \int |\vec{E}|^2 \, dv, \quad \vec{E} = \frac{Q \vec{a}_r}{4\pi\epsilon_0 r^2}$$

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \iiint \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} \cdot r^4 \sin\theta \, dr \, d\theta \, d\phi \\ &= \frac{Q^2}{32\pi^2\epsilon_0} \cdot 4\pi \int_a^\infty \frac{1}{r^2} \, dr = \underline{\underline{\frac{Q^2}{8\pi\epsilon_0 a}}} \end{aligned}$$

Prob. 4.45

$$\begin{aligned} W &= \frac{1}{2} \int \vec{D} \cdot \vec{E} \, dv = \frac{1}{2\epsilon_0} \int \vec{D} \cdot \vec{D} \, dv \\ &= \frac{10^{-18}}{2\epsilon_0} \iiint (y^2 z^2 + x^2 z^2 + x^2 y^2) \, dx \, dy \, dz \\ &= \frac{3 \times 10^{-18}}{2 \times 10^{-9} \cdot 36\pi} \left[\int_0^1 dx \int_0^1 y^2 dy \int_0^1 z^2 dz \right] \\ &= 6\pi \, \text{nJ} = \underline{\underline{18.85 \, \text{nJ}}} \end{aligned}$$

Prob. 4.46

$$\begin{aligned} W &= \frac{1}{2}\epsilon_0 \int |\vec{E}|^2 \, dv = \frac{1}{2}\epsilon_0 \iiint (4r^2 \sin^2\theta \cos^2\phi \\ &\quad + r^2 \cos^2\theta \cos^2\phi + r^2 \sin^2\phi) r^2 \sin\theta \, d\theta \, d\phi \, dr \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \epsilon_0 \int_0^2 r^4 dr \left[4 \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \sin^3 \theta d\theta \right. \\
 &\quad \left. + \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \cos^2 \theta \sin \theta d\theta + \int_0^{2\pi} \sin^2 \phi d\phi \int_0^\pi \sin \theta d\theta \right] \\
 &= \frac{1}{2} \epsilon_0 \frac{r^5}{5} \Big|_0^2 \left[4 \cdot \frac{1}{2} (2\pi) \int_0^\pi (1 - \cos^2 \theta) d(-\cos \theta) \right. \\
 &\quad \left. + \frac{1}{2} (2\pi) \int_0^\pi \cos^2 \theta d(-\cos \theta) + \frac{1}{2} (2\pi) (-\cos \theta) \Big|_0^\pi \right] \\
 &= 3.2 \epsilon_0 \left[4\pi \left(\frac{\cos^3 \theta}{3} - \cos \theta \right) \Big|_0^\pi + \pi \left(-\frac{\cos^3 \theta}{3} \right) \Big|_0^\pi + 2\pi \right] \\
 &= 3.2 \epsilon_0 (8\pi) = 25.6 \pi \times 10^{-9} = \underline{\underline{0.7111 \text{ nJ}}}.
 \end{aligned}$$

Prob. 4.47 $\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \vec{a}_\phi + \frac{\partial V}{\partial z} \vec{a}_z \right)$

$$= -(2\rho z \sin \phi \vec{a}_\rho + \rho z \cos \phi \vec{a}_\phi + \rho^2 \sin \phi \vec{a}_z)$$

$$W = \frac{1}{2} \epsilon_0 \int |\vec{E}|^2 dv = \iiint (4\rho^2 z^2 \sin^2 \phi + \rho^2 z^2 \cos^2 \phi + \rho^4 \sin^2 \phi) \rho d\phi dz d\rho$$

$$\begin{aligned}
 \frac{2W}{\epsilon_0} &= 4 \int_1^4 \rho^3 d\rho \int_{-2}^2 z^2 dz \int_0^{\pi/3} \sin^2 \phi d\phi + \int_1^4 \rho^3 d\rho \int_{-2}^2 z^2 dz \int_0^{\pi/3} \cos^2 \phi d\phi \\
 &\quad + \int_1^4 \rho^5 d\rho \int_{-2}^2 dz \int_0^{\pi/3} \sin^2 \phi d\phi
 \end{aligned}$$

But $\int_0^{\pi/3} \cos^2 \phi d\phi = \frac{1}{2} \int_0^{\pi/3} [1 + \cos 2\phi] d\phi = \frac{\pi}{6} + \frac{1}{4} \sin \frac{2\pi}{3}$

$= 0.7401$

$$\int_0^{\pi/2} \sin^2 \phi d\phi = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2\phi) d\phi = \frac{\pi}{6} - \frac{1}{4} \sin^2 \frac{\pi}{3}$$

$$\begin{aligned} \frac{2W}{\epsilon} &= \frac{4}{4} \rho^4 \left|_0^1 \frac{2z^3}{3} \right|_0^2 (0.3071) + \frac{\rho^4}{4} \left|_0^1 \frac{2z^3}{3} \right|_0^2 (0.7401) \\ &\quad + \frac{\rho^6}{6} \left|_1^4 (4)(0.3071) \right. \end{aligned}$$

$$= 255 \times \frac{16}{3} \times 0.3071 + \frac{225}{4} \times \frac{16}{3} \times 0.7401 + \frac{4096}{6} (0.3071)$$

$$= 417.67 + 239.394 + 838.59 = 1495.6$$

$$W = \frac{1495.6 \times 10^9}{2 \times 36\pi} = \underline{\underline{6.612 \text{ nJ}}}$$

CHAPTER 5

P.E. 5.1

$$d\vec{S} = \rho d\phi dz \vec{a}_\rho$$

$$I = \int \vec{J} \cdot d\vec{S} = \int_{\phi=0}^{2\pi} \int_{z=1}^5 10z \sin^2 \phi \rho dz d\phi \Big|_{\rho=2}$$

$$= 10(2) \frac{z^2}{2} \Big|_1^5 \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\phi) d\phi = 240\pi$$

$$I = \underline{\underline{754 \text{ A}}}$$

P.E. 5.2

$$I = p_s w u = 0.5 \times 10^{-6} \times 0.1 \times 10 = 0.5 \mu\text{A}$$

$$V = IR = 10^{14} \times 0.5 \times 10^{-6} = \underline{\underline{50 \text{ mV}}}$$

P.E. 5.3

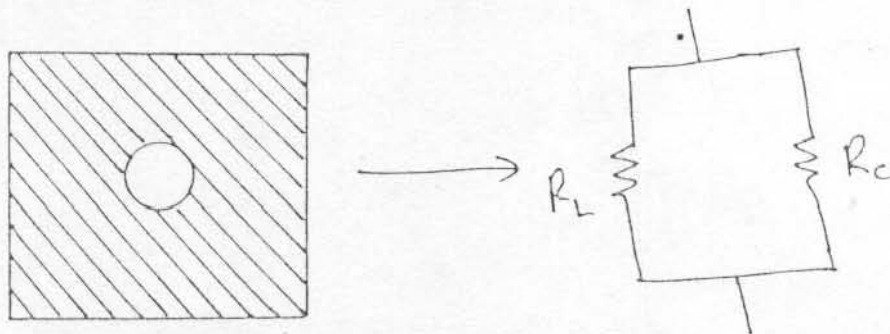
$$\sigma = 5.8 \times 10^7 \text{ V/m}$$

$$J = \sigma E \rightarrow E = \frac{J}{\sigma} = \frac{8 \times 10^6}{5.8 \times 10^7} = \underline{\underline{0.138 \text{ V/m}}}$$

$$J = \rho_v u \rightarrow u = \frac{J}{\rho_v} = \frac{8 \times 10^6}{1.81 \times 10^{10}} = \underline{\underline{4.42 \times 10^{-4} \text{ m/s}}}$$

P.E. 5.4

The composite bar can be modeled by a parallel combination of resistors as shown below.



for the lead, $R_L = \frac{L}{\sigma_L S_L}$, $S_L = d^2 - \pi r^2 = 9 - \frac{\pi}{4} \text{ cm}^2$

$$R_L = 974 \mu\Omega = 0.974 \text{ m}\Omega$$

for copper, $R_c = \frac{L}{\sigma_c S_c}$, $S_c = \pi r^2 = \frac{\pi}{4} \text{ cm}^2$.

$$R_c = \frac{4}{5.8 \times 10^6 \times \frac{\pi}{4} \times 10^{-4}} = 8.781 \text{ m}\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{0.974 \times 8.781}{0.974 + 8.781}$$

$$= 0.8767 \text{ m}\Omega = \underline{\underline{876.7 \mu\Omega}}$$

P.E.5.5

$$P_{ps} = P_x = \vec{P} \cdot \vec{a}_x = ax^2 + b$$

$$P_{ps}|_{x=0} = \vec{P} \cdot (-\vec{a}_x)|_{x=0} = \underline{\underline{-b}}$$

$$P_{ps}|_{x=L} = \vec{P} \cdot \vec{a}_x|_{x=L} = \underline{\underline{aL^2 + b}}$$

$$Q_s = \int P_{ps} dS = -b \cdot A + (aL^2 + b) \cdot A = AaL^2$$

$$P_{pv} = -\nabla \cdot \vec{P} = -\frac{d}{dx} (ax^2 + b) = -2ax$$

$$P_{pv}|_{x=0} = \underline{\underline{0}}, \quad P_{pv}|_{x=L} = \underline{\underline{-2aL}}$$

$$Q_v = \int P_{pv} dV = \int_0^L (-2ax) A dx = -AaL^2$$

Hence $Q_T = Q_v + Q_s = -AaL^2 + AaL^2 = 0$

P.E. 5.6

$$\vec{E} = \frac{V}{d} \vec{a}_x = \frac{1 \times 10^3}{2 \times 10^{-3}} \vec{a}_x = \underline{\underline{500 \vec{a}_x \text{ kV/m.}}}$$

$$\vec{P} = \chi_e \epsilon_0 \vec{E} = (2.25 - 1) \times \frac{10^{-9}}{36\pi} \times 0.5 \times 10^6$$

$$= \underline{\underline{6.853 \vec{a}_x \mu\text{C/m}^2.}}$$

$$P_{ps} = \vec{P} \cdot \vec{a}_x = \underline{\underline{6.853 \mu\text{C/m}^2.}}$$

P.E. 5.7

(a) Since $\vec{P} = \epsilon_0 \chi_e \vec{E}$, $P_x = \chi_e \epsilon_0 E_x$

$$\chi_e = \frac{P_x}{\epsilon_0 E_x} = \frac{3 \times 10^{-9}}{10\pi} \cdot \frac{1}{5} \times 36\pi \times 10^9 = \underline{\underline{2.16}}$$

$$(b) \vec{E} = \frac{\vec{P}}{\chi_e \epsilon_0} = \frac{36\pi \times 10^9}{2.16} \cdot \frac{1}{10\pi} (3, -1, 4) \cdot 10^{-9}$$

$$= \underline{\underline{5 \vec{a}_x - 1.67 \vec{a}_y + 6.67 \vec{a}_z \text{ V/m.}}}$$

$$(c) \vec{D} = \epsilon_0 \epsilon_r \vec{E} = \frac{\epsilon_r \vec{P}}{\chi_e} = \frac{3.16}{2.16} \left(\frac{1}{10\pi} \right) (3, -1, 4) \text{ nC/m}^2$$

$$= \underline{\underline{139.7 \vec{a}_x - 46.6 \vec{a}_y + 186.3 \vec{a}_z \text{ pC/m}^2.}}$$

P.E. 5.8

from Example 5.8,

$$F = \frac{P_s^2 S}{2 \epsilon_0}$$

i.e. $\rho_s^2 = \frac{2\epsilon_0 F}{S}$

But $\rho_s = \epsilon_0 E = \epsilon_0 \frac{V_d}{d}$. Hence

$$\rho_s^2 = \frac{\epsilon_0^2 V_d^2}{d^2} = \frac{2\epsilon_0 F}{S} \rightarrow V_d = \frac{2Fd}{\epsilon_0 S}$$

i.e. $V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\epsilon_0 S}}$ as required.

P. E. 5.9

(a) Since $\vec{a}_n = \vec{a}_x$,

$$\vec{D}_{1n} = 12\vec{a}_x, \quad \vec{D}_{1t} = -10\vec{a}_y + 4\vec{a}_z$$

$$\vec{D}_{2n} = \vec{D}_{1n} = 12\vec{a}_x$$

$$\vec{E}_{2t} = \vec{E}_{1t} \rightarrow \vec{D}_{2t} = \frac{\epsilon_2 \vec{D}_{1t}}{\epsilon_1} = \frac{1}{2.5} (-10\vec{a}_y + 4\vec{a}_z)$$

$$= -4\vec{a}_y + 1.6\vec{a}_z$$

$$\vec{D}_2 = \vec{D}_{2n} + \vec{D}_{2t} = \underline{-12\vec{a}_x - 4\vec{a}_y + 1.6\vec{a}_z \text{ nC/m}^2}$$

(b) $\tan \theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359$

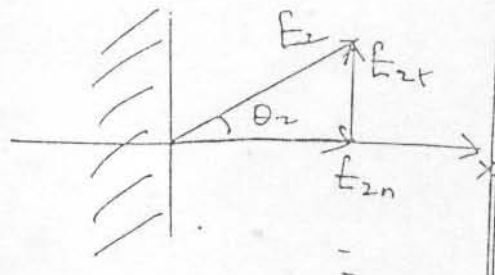
$$\rightarrow \underline{\theta_2 = 19.75^\circ}$$

(b)

$$E_{1t} = E_{2t}$$

$$= E_2 \sin \theta_2 = 12 \sin 60^\circ$$

$$= 10.392$$



$$E_{1n} = \frac{E_{r2}}{E_{r1}} E_{2n} = \frac{1}{2.5} 12 \cos 60^\circ = 2.4 \cdot 157$$

$$E_1 = \sqrt{E_{1t}^2 + E_{1n}^2} = \underline{\underline{10.67}}$$

$$\tan \theta_1 = \frac{E_{r1}}{E_{r2}} \tan \theta_2 = \frac{2.5}{1} \tan 60^\circ = 4.33$$

$$\rightarrow \theta_1 = \underline{\underline{77^\circ}}$$

Note that $\theta_1 > \theta_2$.

P.E. 5.10

$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} = \frac{10^{-9}}{36\pi} (60, 20, -30) \cdot 10^{-3} \\ &= \underline{\underline{0.531 \bar{a}_x + 0.177 \bar{a}_y - 0.265 \bar{a}_z \text{ pC/m}^2}} \end{aligned}$$

$$\begin{aligned} \rho = D_n = |\vec{D}| &= \frac{10^{-9}}{36\pi} (10) \sqrt{36+4+9} \cdot 10^{-3} \\ &= \underline{\underline{0.619 \text{ pC/m}^2}} \end{aligned}$$

Prob. 5.1

$$\begin{aligned} I &= \int \vec{J} \cdot d\vec{s} = \int_{\rho=0}^a \int_{\phi=0}^{2\pi} \frac{500}{\rho} \rho d\phi d\rho = 500 (2\pi a) \\ &= 1000\pi \times 1.6 \times 10^{-3} = 1.6\pi \\ &= \underline{\underline{5.026 \text{ A}}} \end{aligned}$$

Prob. 5.2

$$I = \int \vec{J} \cdot d\vec{s} = 10 \int_{\rho=0}^a \int_{\phi=0}^{2\pi} e^{-(1-\rho/a)} \rho d\phi d\rho$$

$$= 20\pi \int_0^a \rho e^{-(1-\rho/a)} d\rho$$

But $\int x e^{ax} = \frac{e^{ax}}{a^2} (ax - 1)$,

$$I = 20\pi e^{-1} a^2 \left(\frac{\rho}{a} - 1 \right) e^{\rho/a} \Big|_0^a = \frac{20\pi a^2}{e} (1 + 0)$$

$$= \underline{\underline{23.11 a^2 \text{ A}}}$$

Prob. 5.3

$$I = \frac{dQ}{dt} = -3 \times 10^{-4} e^{-3t}$$

$$I(t=0) = \underline{\underline{-0.3 \text{ mA}}}$$

$$I(t=2.5) = -0.3 e^{-7.5} = \underline{\underline{-166 \text{ nA}}}$$

Prob. 5.4

$$Q = It = \underline{\underline{12 \text{ C}}}$$

$$n = \frac{Q}{e} = \frac{12}{1.6 \times 10^{-19}} = \underline{\underline{7.5 \times 10^{19} \text{ electrons}}}$$

Prob. 5.5

$$(a) \quad \frac{Q}{t} = I = \underline{\underline{10 \mu\text{C/s}}}$$

$$(b) \quad \rho_s = \frac{I}{WU} = \frac{10 \times 10^{-6}}{25 \times 0.5} = \underline{\underline{0.8 \mu C/m^2}}$$

Prob. 5.6

$$(a) \quad R = \frac{\rho L}{S} \rightarrow \rho = \frac{RS}{L} = \frac{4.04}{10^3} \cdot \frac{\pi d^2}{4} = 2.855 \times 10^{-8}$$

$$\sigma = \frac{1}{\rho} = \underline{\underline{3.5 \times 10^7 \text{ mhos/m}}} \quad (\text{Aluminium}).$$

$$(b) \quad J = \frac{I}{S} = \frac{40}{\frac{\pi}{4} \times 90 \times 10^{-6}} = 5.66 \times 10^6 \text{ A/m}^2$$

$$\text{or } J = \sigma E = 3.5 \times 10^7 \times 1.616 \times 10^7 \\ = \underline{\underline{5.66 \times 10^6 \text{ A/m}^2}}$$

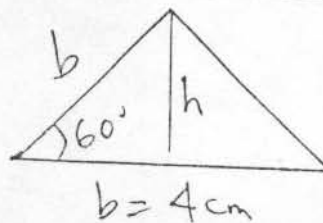
Prob. 5.7

$$P = I^2 R = I^2 \frac{\rho L}{S} = \left(\frac{I}{S}\right)^2 \rho L S = J^2 \rho L S$$

But $\rho_m = \frac{M}{V} = \frac{M}{LS}$, hence

$$P = J^2 \rho \frac{M}{\rho_m} = \frac{(0.8 \times 10^6)^2 \times 0.176 \times 10^{-6} \times 1}{8.9 \times 10^{-3} \times 10^6} \\ = \underline{\underline{1.26 \text{ W}}}$$

Prob. 5.8



$$h = b \sin 60^\circ, \quad S = \frac{1}{2} h b = \frac{1}{2} b^2 \sin 60^\circ$$

$$R = \frac{L}{\sigma S} = \frac{2L}{\sigma b^2 \sin 60^\circ} = \frac{2 \times 3}{10^{-15} \times 16 \times 10^{-4} \sin 60^\circ}$$

$$= \underline{\underline{4.33 \times 10^{18} \Omega}}$$

Prob. 5.9

$$(a) \quad R = \frac{R_i L}{S}, \quad S_i = \pi r_i^2 = \pi (1.5)^2 \times 10^{-4}$$

$$= 2.25 \pi \times 10^{-4}$$

$$S_o = \pi (r_o^2 - r_i^2) = \pi (4 - 2.25) \times 10^{-4}$$

$$= 1.75 \pi \times 10^{-4}$$

$$R = R_i \parallel R_o = \frac{R_i R_o}{R_i + R_o} = \left(\frac{\frac{\rho_i}{S_i} \cdot \frac{\rho_o}{S_o}}{\frac{\rho_i}{S_i} + \frac{\rho_o}{S_o}} \right) L$$

$$= 10 \left(\frac{1.77 \times 11.8 \times 10^{-16}}{2.25 \pi \times 1.75 \pi \times 10^{-8}} \right) = 27 \times 10^{-4}$$

$$\frac{1.77 \times 10^{-8}}{1.75 \pi \times 10^{-4}} + \frac{11.8 \times 10^{-8}}{2.25 \pi \times 10^{-4}}$$

$$= \underline{\underline{0.27 \text{ m}\Omega}}$$

$$(b) \quad V = I_i R_i = I_o R_o \rightarrow \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$$

$$I_i + I_o = 1.1929 I_o = 60 \text{ A}$$

$$I_o = \underline{50.3 \text{ A}} \quad (\text{copper})$$

$$I_i = \underline{9.7 \text{ A}} \quad (\text{steel})$$

$$(c) \quad R = \frac{10 \times 1.77 \times 10^{-8}}{1.75 \pi \times 10^{-4}} = \underline{0.322 \text{ m}\Omega}$$

Prob. 5.10

$$R = \frac{L}{GS} = \frac{h}{\sigma \pi (b^2 - a^2)} = \frac{2}{10^5 \pi (25 - 9) \times 10^{-4}} \\ = \underline{4 \text{ m}\Omega}$$

Prob. 5.11

$$|\vec{P}| = n |\vec{p}| = n Q d = 2 n e d \quad (\phi = 2e)$$

$$= \chi_e \epsilon_0 E$$

$$\chi_e = \frac{2 n e d}{\epsilon_0 E} = \frac{2 \times 5 \times 10^{25} \times 1.602 \times 10^{-19} \times 10^{-18}}{10^{-9} \times 10^4 \times 36 \pi}$$

$$= 0.000182$$

$$\epsilon_r = 1 + \chi_e = \underline{1.000182}$$

Prob. 5.12

$$\vec{P} = \frac{\sum_{i=1}^N q_i \vec{d}_i}{V} = \frac{\sum_{i=1}^N \vec{p}_i}{V}$$

$$|\vec{P}| = \frac{N}{V} |\vec{p}| = 2 \times 10^{19} \times 1.8 \times 10^{-27} = 3.6 \times 10^{-8}$$

$$\vec{P} = |\vec{P}| \vec{a}_x = \underline{\underline{3.6 \times 10^{-18} \vec{a}_x \text{ C/m}^2}}$$

But $\vec{P} = \chi_e \epsilon_0 \vec{E}$

$$\text{or } \chi_e = \frac{P}{\epsilon_0 E} = \frac{3.6 \times 36\pi \times 10^9 \times 10^{-18}}{10^5}$$

$$= 0.0407$$

$$\epsilon_r = 1 + \chi_e = \underline{\underline{1.0407}}$$

Prob. 5.13

(a) $\vec{E} = -\nabla V = -\frac{dV}{dz} \vec{a}_z = 600z \vec{a}_z$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \frac{10^{-9}}{36\pi} (2.4) 600z \vec{a}_z$$

$$= \underline{\underline{12.73z \vec{a}_z \text{ nC/m}^2}}$$

$$\rho_v = \nabla \cdot \vec{D} = \frac{\partial D_z}{\partial z} = \underline{\underline{12.73 \text{ nC/m}^3}}$$

(b) $\chi_e = \epsilon_r - 1 = 1.4$

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{\chi_e D}{\epsilon_r}$$

$$\vec{P} = \frac{1.4}{2.4} (12.732) \vec{a}_z = \underline{\underline{7.4272 \vec{a}_z \text{ nC/m}^2}}$$

$$P_{PV} = -\vec{\nabla} \cdot \vec{P} = \underline{\underline{-7.427 \text{ nC/m}^3}}$$

Prob. 5.14

$$(a) \vec{P} = \chi_e \epsilon_0 \vec{E} = (\epsilon_r - 1) \epsilon_0 \vec{E} = (\epsilon_r - \epsilon_0) \vec{E}$$

$$\vec{D} = \epsilon \vec{E} = \frac{\epsilon}{\epsilon - \epsilon_0} \vec{P} = \frac{\epsilon_r}{\epsilon_r - 1} \vec{P}$$

$$(b) \epsilon_r = 1 + \chi_e = \underline{\underline{3.4}}$$

$$E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{300 \times 10^{-6}}{3.4} \cdot 36\pi \times 10^9$$

$$= \underline{\underline{9.979 \text{ MV/m}}}$$

$$P = \epsilon_0 \chi_e E = \epsilon_0 \chi_e \frac{D}{\epsilon_r \epsilon_0} = \frac{\chi_e}{\epsilon_r} D$$

$$= \frac{2.4}{3.4} (300) = \underline{\underline{211.8 \mu\text{C/m}^2}}$$

Prob. 5.15

(a) Applying Coulomb's law,

$$E_r = \begin{cases} \frac{D_r}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}, & b < r < a \\ \frac{D_r}{\epsilon} = \frac{Q}{4\pi\epsilon r^2}, & a < r < b \end{cases}$$

From the previous problem,

$$\vec{P} = \frac{\epsilon_r - 1}{\epsilon_r} \vec{D} \quad (= \vec{D} - \epsilon \vec{E}) \quad \text{Hence}$$

$$P_r = \frac{\epsilon_r - 1}{\epsilon_r} \cdot \frac{Q}{4\pi r^2}, \quad a < r < b.$$

$$(b) \quad P_{pr} = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{d}{dr} (r^2 P_r) = \underline{\underline{0}}.$$

$$(c) \quad P_{ps} = \vec{P} \cdot (-\vec{a}_r) = -\frac{Q}{4\pi a^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right), \quad r = a$$

$$P_{ps} = \vec{P} \cdot \vec{a}_r = \frac{Q}{4\pi b^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right), \quad r = b.$$

Prob. 5.16

$$(a) \quad \vec{J} = 10^4 (9 + 16) \vec{a}_z = \underline{\underline{250 \vec{a}_z \text{ kA/m}^2}}.$$

$$(b) \quad \frac{\partial P_r}{\partial r} = -\nabla \cdot \vec{J} = \underline{\underline{0}}.$$

$$\begin{aligned} (c) \quad I &= \int \vec{J} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{0.005} 10^4 \rho^2 \rho d\phi d\rho \\ &= 10^4 (2\pi) \frac{\rho^4}{4} \Big|_0^{0.005} \\ &= \underline{\underline{9.817 \mu A}}. \end{aligned}$$

Prob. 5.17

$$(a) I = \int \vec{J} \cdot d\vec{S} = \int_{\phi=0}^{2\pi} \int_{z=0}^3 \frac{e^{-10^3 t}}{\rho^2} \rho d\phi dz$$

$$= (2\pi)(3) \frac{e^{-10}}{2} = \underline{\underline{428 \mu A.}}$$

$$(b) \frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{J} = -\frac{e^{-10^3 t}}{\rho} \frac{d}{d\rho} \left(\frac{1}{\rho} \right) = \frac{e^{-10^3 t}}{\rho^3}$$

$$\rho_v = \frac{-10^{-3} e^{-10^3 t}}{\rho^3} + c(t)$$

It we let $\rho_v \rightarrow 0$ as $t \rightarrow \infty$, $c(t) = 0$.

$$\rho_v(\rho=2, t=10 \text{ ms}) = \frac{-10^{-3} e^{-10}}{8} = \underline{\underline{-5.67 \text{ nC/m}^3.}}$$

Prob. 5.18

$$(a) T = \frac{\epsilon_0 E_r}{\sigma} = \frac{2.55 \times 10^{-9}}{10^{-16} \times 36\pi} = 2.255 \times 10^5 \text{ s}$$

$$= 62.64 \text{ hours} = \underline{\underline{2.61 \text{ days.}}}$$

$$(b) T = \frac{\epsilon_0 E_r}{\sigma} = \frac{20 \times 10^{-9}}{10^{-4} \times 36\pi} = \underline{\underline{1.76 \mu s}}$$

$$(c) T = \frac{10^{-9}}{1.6 \times 10^7 \times 36\pi} = \underline{\underline{5.53 \times 10^{-19} \text{ s.}}}$$

Prob. 5.19

$$T = \frac{\epsilon}{\sigma} \rightarrow \sigma = \frac{\epsilon}{T} = \frac{6 \times 10^{-9}}{53 \times 36\pi} = \underline{\underline{10^{-12} \text{ C/m}}}$$

Glass or Porcelain.

Prob. 5.20

$$T_r = \frac{\epsilon}{\sigma} = \frac{2.5 \times 10^{-9}}{5 \times 10^{-6} \times 36\pi} = 4.42 \mu\text{s}$$

$$\rho_{v0} = \frac{Q}{V} = \frac{10 \times 10^{-6}}{\frac{4\pi}{3} \times 10^{-6} \times 8} = \underline{\underline{0.2984 \text{ C/m}^3}}$$

$$\rho_v = \rho_{v0} e^{-t/T_r} = 0.2984 e^{-2/4.42} = \underline{\underline{0.1898 \text{ C/m}^3}}$$

Prob. 5.21

$$(a) \vec{E}_2 = \vec{E}_1 = -300\vec{a}_x + 50\vec{a}_y, \vec{E}_{1n} = 70\vec{a}_z$$

$$\vec{D}_2 = \vec{D}_1 \rightarrow \epsilon_2 \vec{E}_{2n} = \epsilon_1 \vec{E}_{1n}$$

$$\vec{E}_{2n} = \frac{\epsilon_1}{\epsilon_2} \vec{E}_{1n} = \frac{2.5}{4} (70\vec{a}_z) = 43.75\vec{a}_z$$

$$\vec{E}_2 = -30\vec{a}_x + 50\vec{a}_y + 43.75\vec{a}_z$$

$$\vec{D}_2 = \epsilon_0 \epsilon_r \vec{E}_2 = \frac{4 \times 10^{-9}}{36\pi} (-30, 50, 43.75)$$

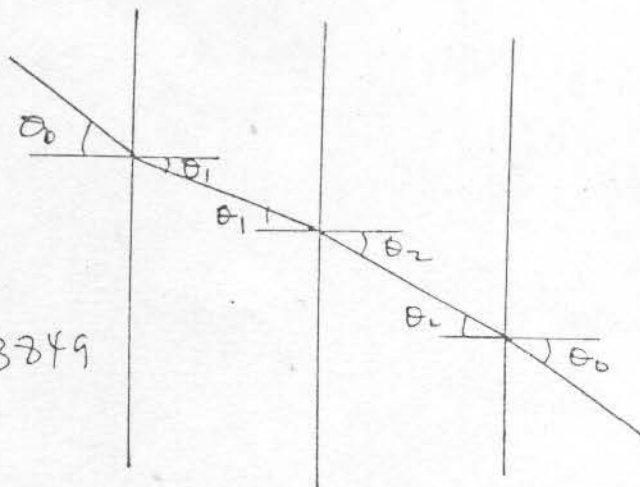
$$= \underline{\underline{-1.061\vec{a}_x + 1.768\vec{a}_y + 1.547\vec{a}_z \text{ nC/m}^2}}$$

$$(b) \vec{P}_2 = \epsilon_0 \epsilon_2 \vec{E}_2 = \frac{3 \times 10^{-9}}{36\pi} (-30, 50, 43.75) \\ = \underline{0.7958 \vec{a}_x + 1.326 \vec{a}_y + 1.161 \vec{a}_z \text{ nC/m}^2}$$

$$(c) \vec{E}_1 \cdot \vec{a}_2 = E_1 \cos \theta_n \\ \cos \theta_n = \frac{70}{\sqrt{30^2 + 50^2 + 70^2}} = 0.7683 \\ \theta_n = \underline{\underline{39.79^\circ}}$$

Prob. 5.22

$$(b) \frac{\tan \theta_0}{\tan \theta_1} = \frac{\epsilon_0}{\epsilon_1} = \frac{1}{1.5} \\ \tan \theta_0 = \frac{\tan 30^\circ}{1.5} = 0.3849 \\ \theta_0 = \underline{\underline{21.1^\circ}}$$



$$(a) \frac{\tan \theta_2}{\tan \theta_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}} = \frac{3}{1.5} = 2, \\ \tan \theta_2 = 2 \tan 30^\circ = 1.155 \\ \theta_2 = \underline{\underline{49.1^\circ}}$$

Prob. 5.23

$$(a) \vec{P}_1 = \epsilon_0 \epsilon_1 \vec{E}_1 = \frac{1.5 \times 10^{-9}}{36\pi} (2, 5, -4) \cdot 10^3 \\ = \underline{\underline{26.53 \vec{a}_x + 66.31 \vec{a}_y - 53.05 \vec{a}_z \text{ nC/m}^2}}$$

$$\begin{aligned} \rho_{pv1} &= -\nabla \cdot \vec{P}_1 = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (26.53 \rho) \\ &= -\frac{26.53}{\rho} \text{ nC/m}^3. \end{aligned}$$

$$(b) \vec{E}_{2t} = \vec{E}_{1t} = 5\vec{a}_\phi - 4\vec{a}_z$$

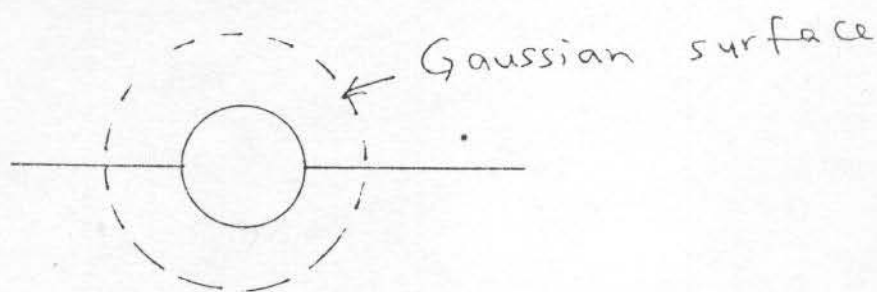
$$D_{2n} = D_{1n} \rightarrow E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n}$$

$$E_{2n} = \frac{2.5}{1.0} (2) = 5$$

$$\vec{E}_2 = \underline{5\vec{a}_\phi + 5\vec{a}_\rho - 4\vec{a}_z \text{ kV/m.}}$$

$$\begin{aligned} \vec{D}_2 &= \epsilon_2 \vec{E}_2 = \frac{2.5 \times 10^{-9}}{36\pi} (5, 5, -4) \cdot 10^3 \\ &= \underline{110.52 \vec{a}_\phi + 110.52 \vec{a}_\rho - 88.42 \vec{a}_z \text{ nC/m}^2.} \end{aligned}$$

Prob. 5.24

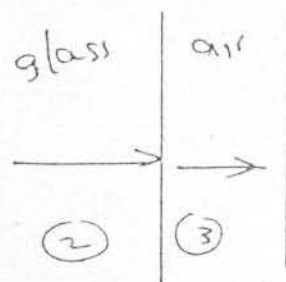
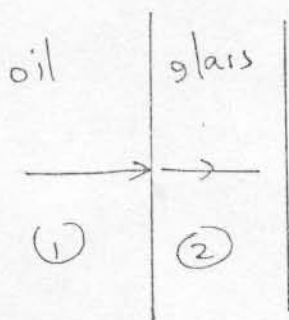


$$\begin{aligned} Q &= \int \vec{D} \cdot d\vec{s} = \epsilon_1 E_r \cdot \frac{4\pi r^2}{2} + \epsilon_2 E_r \cdot \frac{4\pi r^2}{2} \\ &= 2\pi r^2 (\epsilon_1 + \epsilon_2) E_r. \end{aligned}$$

$$E_r = \begin{cases} \frac{Q}{2\pi(\epsilon_1 + \epsilon_2)} & , r > a \\ 0 & , r < a. \end{cases}$$

Prob. 5.25

(a).



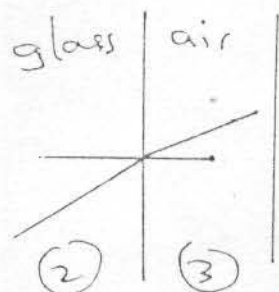
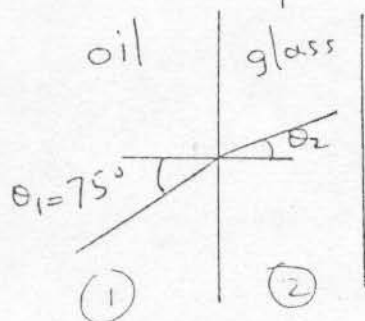
$$E_{1n} = 2000, E_{1t} = 0 = E_{2t} = E_{3t}$$

$$D_{1n} = D_{2n} = D_{3n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} = \epsilon_3 E_{3n}$$

$$\rightarrow E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3.0}{8.5} (2000) = \underline{\underline{705.9 \text{ V/m}, \theta_2 = 0^\circ}}$$

$$E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3.0}{1.0} (2000) = \underline{\underline{6000 \text{ V/m}, \theta_3 = 0^\circ}}$$

(b)



$$E_{1n} = 2000 \cos 75^\circ = 517.63$$

$$E_{1t} = 2000 \sin 75^\circ = E_{2t} = E_{3t} = 1931.85$$

$$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3}{8.5} (517.63) = 182.7$$

$$E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3}{1} (517.63) = 1552.89$$

$$E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} = 1940.5, \theta_2 = \tan^{-1} \frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^\circ}}$$

$$E_3 = \sqrt{E_{3n}^2 + E_{3t}^2} = 2478.6, \theta_3 = \tan^{-1} \frac{E_{3t}}{E_{3n}} = \underline{\underline{51.2^\circ}}$$

Prob. 5.26

$$(a) \rho_s = D_n = \epsilon_0 E_n = \frac{10^{-9}}{36\pi} \sqrt{15^2 + 8^2} = \frac{17}{36\pi} \text{ nC/m}^2$$

$$= \underline{\underline{0.1503 \text{ nC/m}^2}}$$

$$(b) D_n = \rho_n = -20 \text{ nC}$$

$$\vec{D} = D_n \vec{a}_n = (-20 \text{ nC}) (-\vec{a}_y) = \underline{\underline{20 \vec{a}_y \text{ nC/m}^2}}$$

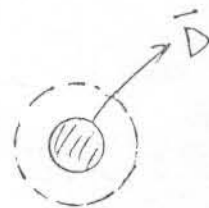
Prob. 5.27

$$(a) D_n = \rho_s = \frac{Q}{4\pi a^2} = \frac{12 \times 10^{-9}}{4\pi \times 25 \times 10^{-4}} = \frac{1200}{\pi} \text{ nC/m}^2$$

$$|\vec{D}| = \underline{\underline{381.97 \text{ nC/m}^2}}$$

(b) Using Gauss's law,

$$D_r \cdot 4\pi r^2 = Q \rightarrow D_r = \frac{Q}{4\pi r^2}$$



$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{12}{4\pi r^2} \vec{a}_r \text{ nC/m}^2 = \frac{0.955}{r^2} \vec{a}_r \text{ nC/m}^2$$

$$(c) W = \frac{1}{2\epsilon_0} \int |\vec{D}| dV = \frac{Q^2}{2\epsilon_0 \cdot 16\pi^2} \iiint \frac{1}{r^4} \cdot r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2 \epsilon_0} \cdot 4\pi \int_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi \epsilon_0 a}$$

$$= \frac{144 \times 10^{-18}}{8\pi \times \frac{10^{-9}}{36\pi} \times 5 \times 10^{-2}} = \underline{\underline{12.96 \mu\text{J}}}$$

Prob. 5.28

(a) $\vec{J} = \sigma \vec{E} = \underline{\underline{0}}$ since $\vec{E} = 0$.

(b) $\rho_s = D_n = \frac{Q}{S} = \frac{Q}{4\pi r^2} = \frac{50 \times 10^{-9}}{4\pi \times 1600 \times 10^{-4}}$
 $= \underline{\underline{24.37 \text{ nC/m}^2}}$

Prob. 5.29

for steady state current,

$\nabla \cdot \vec{J} = 0 \rightarrow \oint \vec{J} \cdot d\vec{S} = 0$.

Applying the integral form to a thin disc at the interface,

$J_{1n} = J_{2n}$.

Also, at the interface,

$E_{1t} = E_{2t} \rightarrow \sigma_1 \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$

CHAPTER 6

P.E. 6.1

$\nabla^2 V = -\frac{\rho}{\epsilon} \rightarrow \frac{d^2 V}{dx^2} = -\frac{\rho_0 x}{\epsilon a}$

$V = -\frac{\rho_0 x^3}{6\epsilon a} + Ax + B$

$\vec{E} = -\frac{dV}{dx} \vec{a}_x = \left(\frac{\rho_0 x^2}{2\epsilon a} - A \right) \vec{a}_x$

If $\vec{E} = 0$ at $x=0$, then

$$0 = 0 - A \rightarrow A = 0$$

If $V = 0$ at $x=a$, then

$$0 = -\frac{\rho_0 a^3}{6\epsilon a} + B \rightarrow B = \frac{\rho_0 a^2}{6\epsilon}$$

Thus

$$V = \frac{\rho_0}{6\epsilon a} (a^3 - x^3), \quad \vec{E} = \frac{\rho x^2}{2\epsilon a}$$

P.E. 6.2

$$V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$$

$$V_1(x=d) = V_0 = A_1 d + B_1 \rightarrow B_1 = V_0 - A_1 d$$

$$V_2(x=0) = 0 = 0 + B_2 \rightarrow B_2 = 0$$

$$V_1(x=a) = V_2(x=a) \rightarrow A_1 a + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \rightarrow \epsilon_1 A_1 = \epsilon_2 A_2 \rightarrow A_2 = \frac{\epsilon_1}{\epsilon_2} A_1$$

$$A_1 a + V_0 - A_1 d = \frac{\epsilon_1}{\epsilon_2} a A_1$$

$$V_0 = A_1 \left(-a + d + \frac{\epsilon_1}{\epsilon_2} a \right)$$

$$\text{or } A_1 = \frac{V_0}{d - a + \frac{\epsilon_1}{\epsilon_2} a}, \quad A_2 = \frac{\epsilon_1}{\epsilon_2} A_1 = \frac{\epsilon_1 V_0}{\epsilon_2 d - \epsilon_2 a + \epsilon_1 a}$$

Hence

$$\vec{E}_1 = -A_1 \vec{a}_x = \frac{-V_0 \vec{a}_x}{d - a + \frac{\epsilon_1}{\epsilon_2} a}, \quad \vec{E}_2 = -A_2 \vec{a}_x = \frac{-V_0 \vec{a}_x}{a + \frac{\epsilon_2}{\epsilon_1} d - \frac{\epsilon_2}{\epsilon_1} a}$$

P.E. 6.3

from Example 6.3,

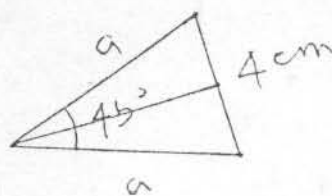
$$\vec{E} = -\frac{V_0}{\rho\phi_0} \vec{a}_\rho, \quad \vec{D} = \epsilon_0 \vec{E}$$

$$\rho_s = D_n(\phi=0) = -\frac{V_0 \epsilon}{\rho\phi_0}$$

The charge on the plate $\phi=0$ is

$$Q = \int \rho_s dS = -\frac{V_0 \epsilon}{\phi_0} \int_{z=0}^L \int_{\rho=a}^b \frac{1}{\rho} d\rho dz = -\frac{V_0 \epsilon L}{\phi_0} \ln \frac{b}{a}$$

$$C = \frac{|Q|}{V_0} = \frac{\epsilon L}{\phi_0} \ln \frac{b}{a}$$



$$a \sin \frac{45^\circ}{2} = 2 \rightarrow a = \frac{2}{\sin 22.5^\circ} = 5.226 \text{ mm}$$

$$C = \frac{1.5 \times 10^{-9}}{36\pi \times \frac{1}{4}} 5 \ln \frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_0 = 444 \times 10^{-12} \times 50 = \underline{\underline{22.2 \text{ nC}}}$$

P.E. 6.4

from Example 6.4, $V_0 = 50$, $\theta_1 = \frac{\pi}{2}$

$$\theta_2 = 45^\circ, r = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}, \quad \phi = \tan^{-1} \frac{\rho}{z} = \frac{5}{2} \rightarrow \theta = 68.2^\circ$$

$$V = \frac{50 \ln(\tan 34.1^\circ)}{\tan(22.5^\circ)} = \underline{\underline{22.13 \text{ V}}}$$

$$\vec{E} = \frac{-50 \vec{a}_0}{\sqrt{29} \sin 68.2^\circ \ln(\tan 22.5^\circ)} = \underline{\underline{11.364 \vec{a}_0 \text{ V/m}}}$$

P.E-6.5

$$\begin{aligned}\vec{E} &= -\nabla V = -\frac{\partial V}{\partial x} \vec{a}_x - \frac{\partial V}{\partial y} \vec{a}_y \\ &= -\frac{4V_0}{b} \sum_{n=\text{odd}}^{\infty} \frac{1}{\sinh \frac{n\pi a}{b}} \left[\cos \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \vec{a}_x \right. \\ &\quad \left. + \sin \frac{n\pi x}{b} \cosh \frac{n\pi y}{b} \vec{a}_y \right]\end{aligned}$$

(a) At $(x, y) = (a, a/2)$,

$$\begin{aligned}V &= \frac{400}{\pi} (0.3775 - 0.0313 + 0.00394 - 0.000585 + \dots) \\ &= \underline{\underline{44.51 \text{ V}}}\end{aligned}$$

$$\begin{aligned}\vec{E} &= 0 \vec{a}_x + (-115.118 + 19.127 - 3.9431 + 0.8192 \\ &\quad - 0.1703 + 0.035 - 0.0094 + \dots) \vec{a}_y \\ &= \underline{\underline{-99.25 \vec{a}_y \text{ V/m}}}\end{aligned}$$

(b) At $(x, y) = (3a/2, a/4)$,

$$\begin{aligned}V &= \frac{400}{\pi} (0.1238 + 0.00626 - 0.00383 - 0.000264 + \dots) \\ &= \underline{\underline{16.50 \text{ V}}}\end{aligned}$$

$$\begin{aligned}\vec{E} &= (24.757 - 3.7358 - 0.3834 + 0.0369 \\ &\quad + 0.00351 - 0.00033 + \dots) \vec{a}_x \\ &\quad + (-66.25 - 4.518 + 0.3988 + 0.03722 \\ &\quad - 0.00352 - 0.000333 + \dots) \vec{a}_y \\ &= \underline{\underline{20.68 \vec{a}_x - 70.34 \vec{a}_y \text{ V/m}}}\end{aligned}$$

P.E. 6.6

$$V(y=a) = V_0 \sin \frac{7\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

By equating coefficients, we notice that

$$c_n = 0 \text{ for } n \neq 7.$$

for $n=7$,

$$V_0 \sin \frac{7\pi x}{b} = c_7 \sin \frac{7\pi x}{b} \sinh \frac{7\pi a}{b}$$

$$\text{or } c_7 = V_0 \left| \sinh \frac{7\pi a}{b} \right|$$

Hence

$$V(x,y) = \frac{V_0 \sin \frac{7\pi x}{b} \sinh \frac{7\pi y}{b}}{\sinh \frac{7\pi a}{b}}$$

P.E. 6.7

Let $V(r, \theta, \phi) = R(r) F(\theta) \Phi(\phi)$,

substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dF}{d\theta} \right) + \frac{R F}{r^2 \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0$$

Dividing thru by $R F \Phi / r^2 \sin^2 \theta$ gives

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} (r^2 R') + \frac{\sin \theta}{F} \frac{d}{d\theta} (\sin \theta F') = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \lambda^2$$

$$\underline{\Phi'' + \lambda^2 \Phi = 0.}$$

$$\frac{1}{R} \frac{d}{dr} (r^2 R') + \frac{1}{f \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \frac{\lambda^2}{\sin^2 \theta}$$

$$\frac{1}{R} \frac{d}{dr} (r^2 R') = \frac{\lambda^2}{\sin^2 \theta} - \frac{1}{F \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \mu^2$$

$$2R' + r^2 R'' = \mu^2 R$$

$$\therefore \underline{R'' + \frac{2}{r} R' - \frac{\mu^2}{r^2} R = 0.}$$

$$\frac{\sin \theta}{F} \frac{d}{d\theta} (\sin \theta F') - \lambda^2 + \mu^2 \sin^2 \theta = 0$$

$$\therefore \underline{F'' + \cot \theta F' + (\mu^2 - \lambda^2 \csc^2 \theta) F = 0.}$$

P.E. 6.8

(a) This is similar to Example 6.8 (a) except that here, $0 < \phi < 2\pi$ instead of $0 < \phi < \pi/2$. Hence

$$I = \frac{2\pi t V_0 \delta}{\ln \frac{b}{a}}$$

$$\text{and } R = \frac{V_0}{I} = \frac{\ln \frac{b}{a}}{2\pi t \delta}$$

(b) This is similar to Example 6.8 (b) except that here $0 < \phi < 2\pi$. Hence

$$I = \frac{V_0 \delta}{t} \int_a^b \int_0^{2\pi} \rho d\rho d\phi = \frac{V_0 \delta \pi (b^2 - a^2)}{t}$$

$$\text{and } R = \frac{V_0}{I} = \frac{t}{\sigma \pi (b^2 - a^2)}$$

P.E. 6.9

from Example 6.9,

$$J_1 = \frac{\sigma_1 V_0}{\rho \ln \frac{b}{a}}, \quad J_2 = \frac{\sigma_2 V_0}{\rho \ln \frac{b}{a}}$$

$$I = \int \vec{J} \cdot d\vec{s} = \int_{z=0}^L \left[\int_{\phi=0}^{\pi} J_1 \rho d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho d\phi \right] dz$$

$$= \frac{V_0 L}{\ln \frac{b}{a}} [\pi \sigma_1 + \pi \sigma_2]$$

$$R = \frac{V_0}{I} = \frac{\ln \frac{b}{a}}{\pi L (\sigma_1 + \sigma_2)}$$

11.6.10

(a) $C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in series.

$$C_1 = 4\pi \times 10^{-9} / (36\pi) \left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}} \right) = \frac{5}{3} \text{ pF}$$

$$C_2 = 4\pi \times 10^{-9} / (36\pi) \left(\frac{3.5}{\frac{10^3}{2} - \frac{10^3}{3}} \right) = \frac{7}{9} \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\frac{5}{3} \left(\frac{7}{9} \right)}{\frac{5}{3} + \frac{7}{9}} = \frac{35}{36} \text{ pF} = \underline{\underline{0.53 \text{ pF}}}$$

(b) $C = \frac{2\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in parallel.

$$C_1 = \frac{2\pi \times 10^{-9}}{36\pi} \left(\frac{2.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = \frac{5}{24} \text{ pF},$$

$$C_2 = \frac{2\pi \times 10^{-9}}{36\pi} \left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = \frac{7}{24} \text{ pF},$$

$$C = C_1 + C_2 = \frac{12}{24} \text{ pF} = \underline{\underline{0.5 \text{ pF}}}$$

P.E. 6.11

As in Example 6.8, assuming $V(p=0)=0$, $V(p=b)=V_0$,

$$V = V_0 \frac{\ln p/a}{\ln b/a}, \quad \vec{E} = -\nabla V = \frac{-V_0 \vec{a}_p}{p \ln b/a}$$

$$Q = \oint \vec{E} \cdot d\vec{s} = \frac{V_0 \epsilon}{\ln \frac{b}{a}} \int_{z=0}^L \int_{\phi=0}^{2\pi} \frac{1}{p} dz p d\phi = V_0 \cdot \frac{2\pi \epsilon L}{\ln \frac{b}{a}}$$

$$C = \frac{Q}{V_0} = \underline{\underline{\frac{2\pi \epsilon L}{\ln \frac{b}{a}}}}$$

P.E. 6.12

(a) C_1 and C_2 are in series

$$C_1 = \frac{2\pi \epsilon_r \epsilon_0}{\ln \frac{c}{a}} = \frac{2\pi \times 2.5 \times 10^{-9}}{\ln \frac{7}{1} \times 36\pi} = 200 \text{ pF/m}$$

$$C_2 = \frac{2\pi \epsilon_r \epsilon_0}{\ln \frac{b}{c}} = \frac{2\pi \times 3.5 \times 10^{-9}}{\ln \frac{3}{1/2}} = 480 \text{ pF/m}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{200 \times 480}{680} = 141.12 \text{ pF/m.}$$

$$C_T = Cl = \underline{\underline{1.41 \text{ nF.}}}$$

(b) C_1 and C_2 are in parallel.

$$C = C_1 + C_2 = \frac{\pi \epsilon_0 \epsilon_r r_1}{\ln \frac{b}{a}} + \frac{\pi \epsilon_0 \epsilon_r r_2}{\ln \frac{b}{a}} = \frac{\pi \epsilon_0 (\epsilon_r + \epsilon_r)}{\ln \frac{b}{a}}$$

$$= \frac{6\pi \times \frac{10^{-9}}{36\pi}}{\ln \frac{3}{1}} = 151.7 \text{ pF/m}$$

$$C = Cl = \underline{\underline{1.52 \text{ nF.}}}$$

P.E. 6.13

Instead of eq. (6.31), we now have

$$V = - \int_b^a \frac{Q dr}{4\pi \epsilon r^2} = - \int_b^a \frac{Q}{4\pi \frac{10\epsilon}{r} r^2} dr$$

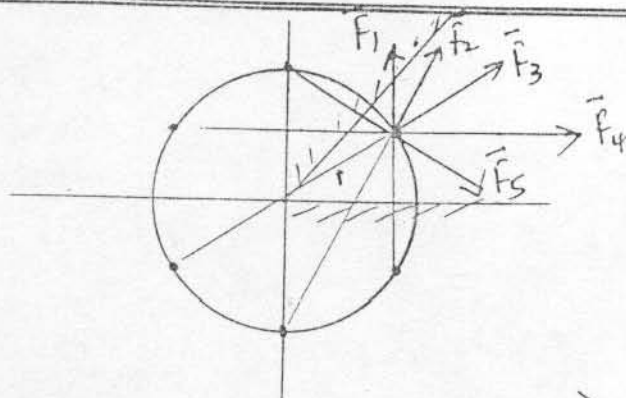
$$= \frac{-Q}{40\pi \epsilon} \ln \frac{b}{a}$$

$$C = \frac{Q}{|V|} = \frac{40\pi \cdot \frac{10^{-9}}{36\pi}}{\ln \frac{4}{1.5}} = \underline{\underline{1.13 \text{ nF.}}}$$

P.E. 6.14

$$\text{Let } \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5$$

where $\vec{F}_i, i=1, \dots, 5$ are as shown on the next page.



$$\vec{F} = \frac{-Q^2 \vec{g}_y}{4\pi\epsilon_0 r^2} + \frac{Q^2 (\vec{a}_x \sin 30^\circ + \vec{a}_y \cos 30^\circ)}{4\pi\epsilon_0 (r \cos 30^\circ)^2}$$

$$- \frac{Q^2}{4\pi\epsilon_0 (2r)^2} (\vec{a}_x \cos 30^\circ + \vec{a}_y \sin 30^\circ) + \frac{Q^2 \vec{a}_x}{4\pi\epsilon_0 (r \cos 30^\circ)^2}$$

$$- \frac{Q^2}{4\pi\epsilon_0 r^2} (\vec{a}_x \cos 30^\circ - \vec{a}_y \sin 30^\circ)$$

$$= \frac{Q^2}{4\pi\epsilon_0 r^2} \left[-\vec{a}_y + \frac{4}{3} \left(\frac{\vec{a}_x}{2} + \vec{a}_y \frac{\sqrt{3}}{2} \right) - \frac{1}{4} \left(\vec{a}_x \frac{\sqrt{3}}{2} + \frac{\vec{a}_y}{2} \right) + \frac{4}{3} \vec{a}_x - \vec{a}_x \frac{\sqrt{3}}{2} + \frac{\vec{a}_y}{2} \right]$$

$$= 9 \times 10^{-5} \left[\vec{a}_x \left(2 - \frac{5\sqrt{3}}{8} \right) + \vec{a}_y \left(\frac{4\sqrt{3}-5}{8} \right) \right]$$

$$= 82.57 \vec{a}_x + 21.69 \vec{a}_y \mu N$$

$$|\vec{F}| = \underline{\underline{85.37 \mu N}}$$

Prob. 6.1

If

$$V'' = f$$

$$V' = \int_0^x f(x) dx + c_1$$

$$V = \int_0^x \int_0^\lambda f(\mu) d\mu d\lambda + c_1 x + c_2$$

$$V(x=0) = V_1 = c_2 \longrightarrow c_2 = V_1$$

$$V(x=L) = V_2 = \int_0^L \int_0^\lambda f(\mu) d\mu d\lambda + c_1 L + c_2$$

$$c_1 = \frac{1}{L} \left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu) d\mu d\lambda \right]$$

Thus,

$$V = \frac{x}{L} \left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu) d\mu d\lambda \right] + V_1 + \int_0^x \int_0^\lambda f(\mu) d\mu d\lambda$$

Prob. 6.2

$$\nabla^2 V = \frac{d^2 V}{dy^2} = -\frac{\rho}{\epsilon} = -\frac{50(1-y^2)}{\epsilon} \cdot 10^{-6}$$

$$= -k(1-y^2)$$

$$\text{where } k = \frac{50 \times 10^{-6}}{\frac{3 \times 10^{-9}}{36\pi}} = 600\pi \cdot 10^3$$

$$\frac{dV}{dy} = -k \left(y - \frac{y^3}{3} \right) + A$$

$$V = -k \left(\frac{y^2}{2} - \frac{y^4}{12} \right) + Ay + B$$

$$= \frac{ky^4}{12} - \frac{ky^2}{2} + Ay + B$$

$$V = 50\pi \cdot 10^3 y^4 - 300\pi \cdot 10^3 y^2 + Ay + B$$

When $y = 2 \text{ cm}$, $V = 30 \times 10^3$,

$$30 \times 10^3 = 50\pi \times 10^3 \times 16 \times 10^{-6} - 30\pi \cdot 10^3 \times 4 \times 10^{-4} + Ay + B$$

or $30,374.5 = 0.02A + B \quad \text{---(1)}$

When $y = -2 \text{ cm}$, $V = 30 \times 10^3$,

$$30,374.5 = -0.02A + B \quad \text{---(2)}$$

from (1) and (2), $A = 0$, $B = 30,374.5$. Then

$$V = \underline{\underline{157.08 y^4 - 942.5 y^2 + 30.374 \text{ kV}}}$$

Prob. 6.3

$$(a) \nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) = -\frac{\rho_v}{\epsilon} = \frac{-10 \times 10^{-12}}{\rho \times \frac{10^{-9}}{36\pi}}$$

$$\frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right) = -1.131$$

$$\rho \frac{dV}{d\rho} = -1.131\rho + A \rightarrow \frac{dV}{d\rho} = -1.131 + \frac{A}{\rho}$$

$$V = -1.131\rho + A \ln \rho + B$$

If $V = 0$ at $\rho = 1$, $0 = -1.131 + B \rightarrow B = 1.131$

If $V = 100$ at $\rho = 4$, $100 = -1.131 \times 4 + A \ln 4 + 1.131$
 $\rightarrow A = 74.58$

Hence,

$$V = -1.131\rho + 74.58 \ln \rho + 1.131$$

$$V(\rho=3) = \underline{\underline{79.67 \text{ V}}}$$

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$$(b) \vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \vec{a}_\rho = (1.131 - \frac{A}{\rho}) \vec{a}_\rho$$

$$\vec{E}(\rho=2) = \underline{\underline{-36.14 \vec{a}_\rho \text{ V/m.}}}$$

Prob. 6.4

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = -\frac{\rho_v}{\epsilon} = -\frac{\rho_0 a^4}{\epsilon_0 r^4}$$

$$r^2 \frac{dV}{dr} = \frac{\rho_0 a^4}{\epsilon_0 r} + A$$

$$V = -\frac{\rho_0 a^4}{2\epsilon_0 r^2} - \frac{A}{r} + B$$

$$V(r \rightarrow \infty) = 0 \rightarrow B = 0$$

$$V(r=a) = 1.5 \times 10^6 = V_0$$

$$V_0 = -\frac{\rho_0 a^2}{2\epsilon_0} - \frac{A}{a} \Rightarrow A = -a \left(V_0 + \frac{\rho_0 a^2}{2\epsilon_0} \right)$$

$$\vec{E} = -\nabla V = -\left(\frac{\rho_0 a^4}{\epsilon_0 r^3} + \frac{A}{r^2} \right) \vec{a}_r$$

$$\vec{E}(r=a) = \left(\frac{V_0}{a} - \frac{\rho_0 a}{2\epsilon_0} \right) \vec{a}_r$$

$$= \left[\frac{1.5 \times 10^6}{0.1} - \frac{1.2 \times 10^{-3} \times 0.1}{2 \times 10^{-9} / 36\pi} \right] \vec{a}_r$$

$$= 8.214 \vec{a}_r \text{ MV/m.}$$

With $\rho_0 = a$, $\vec{E} = 15 \text{ MV/m}$ which is greater. No!

Prob. 6.5

from eq. (6.3), $\nabla \cdot (\epsilon \nabla V) = -\rho_v$.

Let $f = \epsilon$ and $\vec{A} = \nabla V$. Using the vector identity

$$\nabla \cdot (f \vec{A}) = f \nabla \cdot \vec{A} + (\vec{A} \cdot \nabla) f$$

gives

$$\begin{aligned} \nabla \cdot (\epsilon \nabla V) &= \epsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \epsilon \\ &= \epsilon \nabla^2 V + \nabla V \cdot \nabla \epsilon \end{aligned}$$

Hence

$$\epsilon \nabla^2 V + \nabla V \cdot \nabla \epsilon = -\rho_v.$$

Prob. 6.6

$$\begin{aligned} (a) \quad \nabla^2 V &= V_0 \left(-\frac{n^2 \pi^2}{a^2} - \frac{n^2 \pi^2}{a^2} \right) \sin \left(\frac{n\pi x}{a} \right) \cos \left(\frac{n\pi y}{a} \right) \\ &= -\frac{2\pi^2 n^2}{a^2} V \neq 0 \end{aligned}$$

$$(b) \quad \nabla^2 V = V_0 \left(-\frac{n^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{a^2} \right) \sin \left(\frac{n\pi x}{a} \right) \cosh \left(\frac{n\pi y}{a} \right) = 0$$

$$(c) \quad \nabla^2 V = -\frac{2\pi^2 n^2}{a^2} V \neq 0$$

$$(d) \quad V = (x+y) r^{-3/2}$$

$$\frac{\partial V}{\partial x} = r^{-3/2} - \frac{3}{2} (x+y) r^{-5/2} (2x) = r^{-3/2} - 3x(x+y) r^{-5/2}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{3}{2} r^{-5/2} (2x) - 3(2x+y) r^{-5/2} + \frac{15}{2} (x^2+xy) r^{-7/2} (2x)$$

$$\frac{\partial^2 V}{\partial x^2} = -3(3x+y)r^{-5/2} + 15(x^3+x^2y)r^{-7/2}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = -3(x+3y)r^{-5/2} + 15(y^3+y^2x)r^{-7/2}$$

$$\frac{\partial V}{\partial z} = -\frac{3}{2}(x+y)r^{-5/2}(2z) = -3z(x+y)r^{-5/2}$$

$$\begin{aligned}\frac{\partial^2 V}{\partial z^2} &= -3(x+y)r^{-5/2} - 3z(x+y)\left(-\frac{5}{2}\right)r^{-7/2}(2z) \\ &= -3(x+y)r^{-5/2} + 15z^2(x+y)r^{-7/2}\end{aligned}$$

$$\begin{aligned}\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ &= -3(5x+5y)r^{-5/2} + 15(x+y)(x^2+y^2+z^2)r^{-7/2} \\ &= -15(x+y)r^{-5/2} + 15(x+y)r^{-5/2} = 0\end{aligned}$$

Or expressing V in spherical system,

$$V = \frac{\sin \theta}{r^2} (\cos \phi + \sin \phi)$$

$$\begin{aligned}\nabla^2 V &= \frac{\sin \theta}{r^2} \frac{\partial}{\partial r} \left[r^2 \cdot \left(-\frac{2}{r^3}\right) \right] (\cos \phi + \sin \phi) \\ &\quad + \frac{(\cos \phi + \sin \phi)}{r^4 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cos \theta) \\ &\quad - \frac{1}{r^4 \sin \theta} (\cos \phi + \sin \phi)\end{aligned}$$

$$\begin{aligned}\nabla^2 V &= \frac{(\cos \phi + \sin \phi)}{r^4} \left[2 \sin \theta + \frac{1}{\sin \theta} (-\sin^2 \theta + \cos^2 \theta) - \frac{1}{\sin \theta} \right] \\ &= \frac{(\cos \phi + \sin \phi)}{r^4} \left(2 \cancel{\sin \theta} + \frac{1}{\cancel{\sin \theta}} - 2 \cancel{\sin \theta} - \frac{1}{\cancel{\sin \theta}} \right) = 0\end{aligned}$$

Thus,
(a) and (c) do not satisfy Laplace's equation
while (b) and (d) satisfy it.

Prob. 6.7

$$(a) \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 30yz - 30yz + 0 = 0.$$

$$\begin{aligned}(b) \nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(-\frac{\cos \phi}{\rho} \right) + \frac{1}{\rho^2} \left(-\frac{\cos \phi}{\rho} \right) \\ &= \frac{\cos \phi}{\rho^3} - \frac{\cos \phi}{\rho^3} = 0.\end{aligned}$$

$$\begin{aligned}(c) \nabla^2 V &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \\ &= \frac{10 \sin \theta \sin \phi}{r^2} \frac{\partial}{\partial r} \left(-\frac{2}{r} \right) + \frac{10 \sin \phi}{r^4 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cos \theta) \\ &\quad - \frac{10}{r^4 \sin^3 \theta} \sin \theta \sin \phi \\ &= \frac{20 \sin \theta \sin \phi}{r^4} + \frac{10 \sin \phi (1 - 2 \sin^2 \theta)}{r^4 \sin \theta} - \frac{10 \sin \phi}{r^4 \sin \theta} = 0\end{aligned}$$

Prob. 6.8

If $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, then

$$\begin{aligned} 0 &= -\frac{\partial}{\partial x} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) \\ &= \frac{\partial^2}{\partial x^2} \left(-\frac{\partial V}{\partial x} \right) + \frac{\partial^2}{\partial y^2} \left(-\frac{\partial V}{\partial x} \right) + \frac{\partial^2}{\partial z^2} \left(-\frac{\partial V}{\partial x} \right) \\ &= \frac{\partial^3}{\partial x^3} E_x + \frac{\partial^3 E_x}{\partial y^2} + \frac{\partial^3 E_x}{\partial y} = \nabla^2 E_x \end{aligned}$$

i.e. $\nabla^2 E_x = 0$.

The same holds for E_y and E_z .

Prob. 6.9

$$\nabla^2 V = \frac{d^2 V}{dz^2} = 0 \rightarrow V = Az + B$$

when $z=0$, $V=0 \rightarrow B=0$

when $z=d$, $V=V_0 \rightarrow V_0 = Ad$ or $A = \frac{V_0}{d}$

Hence $V = \frac{V_0}{d} z$

$$\vec{E} = -\nabla V = -\frac{dV}{dz} \hat{a}_z = -\frac{V_0}{d} \hat{a}_z$$

$$\vec{D} = \epsilon \vec{E} = -\epsilon \epsilon_r \frac{V_0}{d} \hat{a}_z$$

Since $V_0 = 50V$ and $d = 2mm$,

$$V = \underline{252 \text{ kV}}, \quad \vec{E} = \underline{-25 \hat{a}_z \text{ kV/m}},$$

$$\vec{D} = \frac{-10^{-9}}{36\pi} (1.5) 25 \times 10^3 \vec{a}_z = \underline{\underline{-332 \vec{a}_z \text{ nC/m}^2}}$$

$$\rho_s = D_n = \underline{\underline{\pm 332 \text{ nC/m}^2}}$$

The surface charge density is positive on the plate at $z=d$ and negative on the plate at $z=0$.

Prob 6.10

From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_0 \ln \rho/a}{\ln b/a}$$

$$\vec{E} = -\nabla V = -\frac{V_0}{\rho \ln b/a} \vec{a}_\rho$$

$$\vec{D} = \epsilon \vec{E} = -\frac{V_0 \epsilon_0 \epsilon_r}{\rho \ln b/a} \vec{a}_\rho$$

$$\rho_s = D_n = \pm \left. \frac{V_0 \epsilon_0 \epsilon_r}{\rho \ln b/a} \right|_{\rho=a,b}$$

In this case, $V_0 = 100\text{V}$, $b = 5\text{mm}$, $a = 15\text{mm}$, $\epsilon_r = 2.0$.
Hence at $\rho = 10\text{mm}$,

$$V = \frac{100 \ln \frac{10}{5}}{\ln \frac{15}{5}} = \underline{\underline{36.91 \text{ V}}}$$

$$\vec{E} = \frac{100}{10 \times 10^3 \ln 3} \vec{a}_\rho = \underline{\underline{9.102 \vec{a}_\rho \text{ kV/m}}}$$

$$\vec{D} = 9.102 \times 10^3 \times \frac{10^{-9}}{36\pi} \times 2 \vec{a}_\rho = \underline{\underline{161 \vec{a}_\rho \text{ nC/m}^2}}$$

$$\rho_s (\rho = 5 \text{ mm}) = \frac{10^{-9}}{36\pi} (2) \frac{10^5}{5 \ln 3} = \underline{\underline{322 \text{ nC/m}^2}}$$

$$\rho_s (\rho = 15 \text{ mm}) = -\frac{10^{-9}}{36\pi} (2) \frac{10^5}{15 \ln 3} = \underline{\underline{-107.33 \text{ nC/m}^2}}$$

Prob. 6.11

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0 \rightarrow V = -\frac{A}{r} + B$$

$$V(r=0.1) = 0 \rightarrow 0 = -\frac{A}{0.1} + B \rightarrow B = 10A$$

$$V(r=2) = 100 \rightarrow 100 = -\frac{A}{2} + B = \frac{19}{2} A$$

$$\therefore A = \frac{200}{19} \quad B = \frac{2000}{19}$$

$$V = \underline{\underline{-\frac{10.53}{r} + 105.3 \text{ V}}}$$

$$\vec{E} = -\nabla V = -\frac{dV}{dr} \vec{a}_r = \underline{\underline{-\frac{10.53}{r^2} \vec{a}_r \text{ V/m}}}$$

$$\vec{D} = \epsilon_0 \vec{E} = \underline{\underline{-\frac{9.32 \times 10^{-11}}{r^2} \vec{a}_r \text{ C/m}^2}}$$

Prob. 6.12

$$\nabla^2 V = 0 \rightarrow V = -\frac{A}{r} + B$$

$$\text{At } r=0.5, V=-50 \rightarrow -50 = -\frac{A}{0.5} + B$$

$$\text{or } -50 = -2A + B \quad \text{---(1)}$$

$$\text{At } r=1, V=50 \rightarrow 50 = -A + B \quad \text{---(2)}$$

From (1) and (2), $A=100$, $B=150$. Hence

$$V = \underline{\underline{-\frac{100}{r} + 150}}$$

$$\vec{E} = -\nabla V = -\frac{A}{r^2} \vec{a}_r = \underline{\underline{-\frac{100}{r^2} \vec{a}_r \text{ V/m.}}}$$

Prob. 6.13

From Example 6.4,

$$V = \frac{V_0 \ln \left(\frac{\tan \theta / 2}{\tan \theta_1 / 2} \right)}{\ln \left(\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right)}$$

$$V_0 = 100, \theta_1 = 30^\circ, \theta_2 = 120^\circ, r = \sqrt{3^2 + 0^2 + 4^2} = 5.$$

$$\theta = \tan^{-1} \frac{p}{z} = \tan^{-1} \frac{3}{4} = 36.87^\circ$$

$$V = 100 \ln \left(\frac{\tan 18.435^\circ}{\tan 15^\circ} \right) / \ln \left(\frac{\tan 60^\circ}{\tan 15^\circ} \right) = \underline{\underline{11.7 \text{ V}}}$$

$$\vec{E} = \frac{-V_0 \vec{a}_\theta}{r \sin \theta \ln \left(\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right)} = \frac{-100 \vec{a}_\theta}{5 \sin 36.87^\circ \ln 6.464}$$

$$= \underline{\underline{-17.86 \vec{a}_\theta \text{ V/m.}}}$$

Prob. 6.14

$$(a) \nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) = 0 \rightarrow V = A \ln \rho + B$$

$$V(\rho=b)=0 \rightarrow 0 = A \ln b + B \rightarrow B = -A \ln b$$

$$V(\rho=a)=V_0 \rightarrow V_0 = A \ln \frac{a}{b} \rightarrow A = \frac{V_0}{\ln \frac{b}{a}}$$

$$V = -\frac{V_0}{\ln \frac{b}{a}} \ln \frac{\rho}{b} = \frac{V_0 \ln \frac{b}{\rho}}{\ln \frac{b}{a}}$$

$$V(\rho=15 \text{ mm}) = \frac{70 \ln 2}{\ln 50} = \underline{\underline{12.4 \text{ V.}}}$$

(b) As the electron decelerates, potential energy gained = $k \cdot E \cdot \text{loss}$

$$e[70 - 12.4] = \frac{1}{2} m [(10^7)^2 - u^2]$$

$$10^{14} - u^2 = \frac{2e}{m} \times 57.6$$

$$u^2 = 10^{14} - \frac{2 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}} \times 57.6 = 10^{12} (100 - 20.25)$$

$$u = \underline{\underline{8.93 \times 10^6 \text{ m/s.}}}$$

Prob. 6.15

(a) for the parallel-plate capacitor,

$$\vec{E} = -\frac{V_0}{d} \vec{a}_x$$

from example 6.11.

$$C = \frac{1}{V_0^2} \int \epsilon |\vec{E}|^2 dV = \frac{1}{V_0^2} \int \epsilon \frac{V_0^2}{d^2} dV$$

$$= \frac{\epsilon}{d^2} \cdot Sd = \frac{\epsilon S}{d}$$

(b) for the cylindrical capacitor,

$$\vec{E} = -\frac{V_0}{\rho \ln \frac{b}{a}} \vec{a}_\rho$$

from example 6.8.

$$C = \frac{1}{V_0^2} \iiint \frac{\epsilon V_0^2}{\rho^2 (\ln \frac{b}{a})^2} \rho d\rho d\phi dz$$

$$= \frac{2\pi L \epsilon}{(\ln \frac{b}{a})^2} \int_a^b \frac{d\rho}{\rho} = \frac{2\pi \epsilon L}{\ln \frac{b}{a}}$$

(c) for the spherical capacitor,

$$\vec{E} = \frac{V_0}{r^2 (\frac{1}{a} - \frac{1}{b})} \vec{a}_r$$

from example 6.10.

$$C = \frac{1}{V_0^2} \iiint \frac{\epsilon V_0^2}{r^2 (\frac{1}{a} - \frac{1}{b})^2} r^2 \sin \theta d\theta d\phi dr$$

$$C = \frac{\epsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)^2} \cdot 4\pi \int_a^b \frac{dr}{r^2} = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Prob. 6.16

This is similar to case 1 in example 6.5.

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

But $X(0) = 0 \rightarrow 0 = c_2$
 $Y(0) = 0 \rightarrow 0 = c_4$

Hence, $V(x, y) = XY = a_0 xy$, $a_0 = c_1 c_3$

Also, $V(xy=4) = 20 \rightarrow 20 = a_0 \cdot 4 \rightarrow a_0 = 5$.

Thus $V(x, y) = 5xy$ and $\vec{E} = -\nabla V = -5y \vec{a}_x - 5x \vec{a}_y$

At $(x, y) = (1, 2)$,

$V = 10V$, $\vec{E} = \underline{\underline{-10\vec{a}_x - 5\vec{a}_y \text{ V/m.}}}$

Prob. 6.17

(a) As in example 6.5, $X(x) = A \sin \frac{n\pi x}{b}$

for Y ,

$$Y(y) = c_1 \cosh \frac{n\pi y}{b} + c_2 \sinh \frac{n\pi y}{b}$$

$$Y(a) = 0 \rightarrow 0 = c_1 \cosh \frac{n\pi a}{b} + c_2 \sinh \frac{n\pi a}{b}$$

$$\text{or } c_1 = -c_2 \tanh \frac{n\pi a}{b}$$

$$V = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \left(\sinh \frac{n\pi y}{b} - \tanh \frac{n\pi a}{b} \cosh \frac{n\pi y}{b} \right)$$

$$V(x, y=0) = V_0 = - \sum_{n=1}^{\infty} a_n \tanh \frac{n\pi a}{b} \sin \frac{n\pi x}{b}$$

$$-a_n \tanh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b V_0 \sin \frac{n\pi y}{b} dy = \begin{cases} \frac{4V_0}{n\pi}, & n=\text{odd} \\ 0, & n=\text{even} \end{cases}$$

Hence,

$$V = - \frac{4V_0}{\pi} \sum_{n=\text{odd}} \sin \frac{n\pi x}{b} \left[\frac{\sin \frac{n\pi y}{b}}{n \tanh \frac{n\pi a}{b}} - \frac{\cosh \frac{n\pi y}{b}}{n} \right]$$

$$= - \frac{4V_0}{\pi} \sum \frac{\sin \frac{n\pi x}{b}}{n \sinh \frac{n\pi a}{b}} \left[\sin \frac{n\pi y}{b} \cosh \frac{n\pi a}{b} - \cosh \frac{n\pi y}{b} \sinh \frac{n\pi a}{b} \right]$$

$$= \frac{4V_0}{\pi} \sum_{n=\text{odd}} \frac{\sin \frac{n\pi x}{a} \sinh \frac{n\pi}{b} (a-y)}{n \sinh \frac{n\pi a}{b}}$$

Alternatively, for ψ ,

$$\psi(y) = C_1 \sinh \frac{n\pi}{b} (y - c_2)$$

$$\psi(a) = 0 \rightarrow 0 = C_1 \sinh \frac{n\pi}{b} (a - c_2) \rightarrow c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi}{b} (y-a)$$

where

$$b_n = \begin{cases} -\frac{4V_0}{n\pi \sinh \frac{n\pi a}{b}}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as example 6.5 except that we exchange y and x . Hence,

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi y}{a} \sinh \frac{n\pi x}{a}}{n \sinh \frac{n\pi b}{a}}$$

(c) This is the same as part (a) except that we must exchange x and y . Hence,

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi y}{b} \sinh \frac{n\pi}{b} (a-x)}{n \sinh \frac{n\pi a}{b}}$$

Prob. 6.18

(a) $X(x)$ is the same as in example 6.5. Hence

$$V(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{b} \left(a_n \sinh \frac{n\pi y}{b} + b_n \cosh \frac{n\pi y}{b} \right)$$

At $y=0$, $V=V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{b} \rightarrow b_n = \begin{cases} \frac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

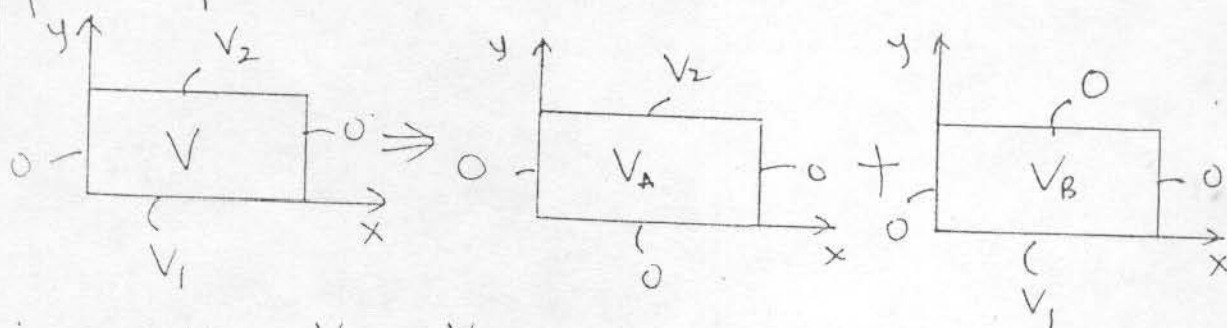
At $y=b$, $V=V_2$

$$V_2 = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left(a_n \sinh \frac{n\pi a}{b} + b_n \cosh \frac{n\pi a}{b} \right)$$

$$\Rightarrow a_n \sinh \frac{n\pi a}{b} + b_n \cosh \frac{n\pi a}{b} = \begin{cases} \frac{4V_2}{n\pi} & , n = \text{odd} \\ 0 & , n = \text{even} \end{cases}$$

$$\text{or } a_n = \begin{cases} \frac{4}{n\pi \sinh \frac{n\pi a}{b}} (V_2 - V_1 \cosh \frac{n\pi a}{b}) & , n = \text{odd} \\ 0 & , n = \text{even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$

V_A is exactly the same as example 6.5 with $V_0 = V_2$ while V_B is exactly the same as Prob. 6.17(a). Hence,

$$V = \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{b}}{n \sinh \frac{n\pi a}{b}} \left[V_1 \sinh \frac{n\pi}{b}(a-y) + V_2 \sinh \frac{n\pi y}{b} \right]$$

(b) $V(x,y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x}) (a_3 \sin \alpha y + a_4 \cos \alpha y)$
 $\lim_{x \rightarrow \infty} V(x,y) = 0 \rightarrow a_2 = 0$

$$V(x, y=0) = 0 \rightarrow a_4 = 0$$

$$V(x, y=a) = 0 \rightarrow \alpha = \frac{n\pi}{a}, \quad n=1, 2, 3.$$

Hence,

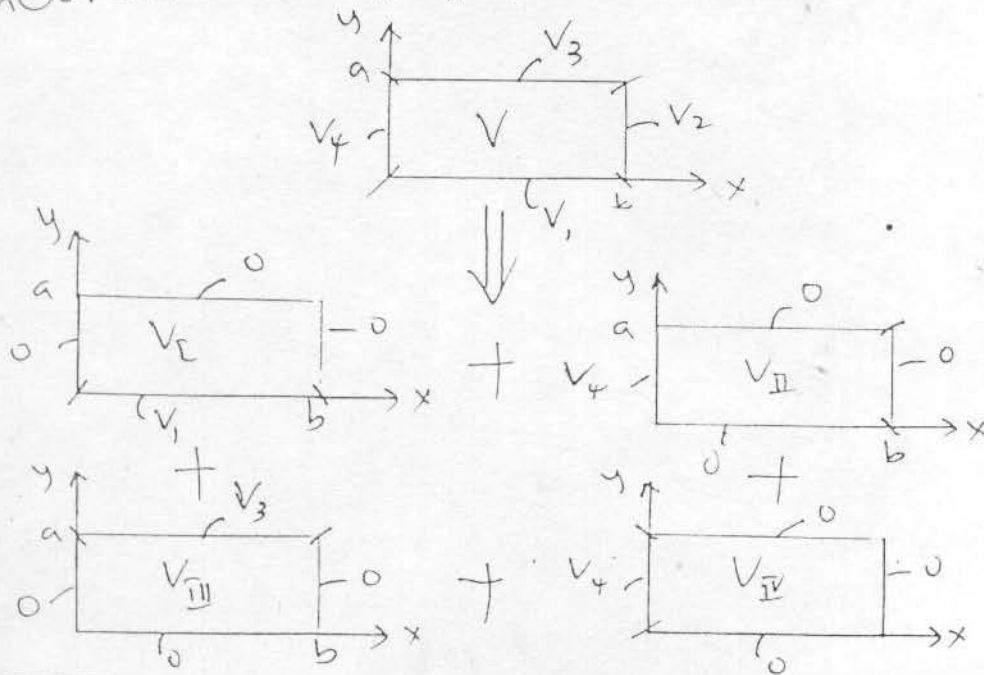
$$V(x, y) = \sum_{n=1}^{\infty} a_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a}.$$

$$V(x=0, y) = V_0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{a}$$

$$\Rightarrow a_n = \begin{cases} \frac{4V_0}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even}. \end{cases}$$

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{a} \exp\left(-\frac{n\pi x}{a}\right).$$

(c) The problem is easily solved using superposition theorem as illustrated below.



Therefore,

$$V = V_I + V_{II} + V_{III} + V_{IV}$$

where

$$V_I = \frac{4V_1}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi}{b} (a-y)}{n \sinh \frac{n\pi a}{b}},$$

$$V_{II} = \frac{4V_2}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}}{n \sinh \frac{n\pi b}{a}},$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}}{n \sinh \frac{n\pi a}{b}},$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi y}{a} \sinh \frac{n\pi}{a} (b-x)}{n \sinh \frac{n\pi b}{a}}$$

Prob. 6.19

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

If we let $V(\rho, \phi) = R(\rho) \Phi(\phi)$

$$\frac{\phi}{\rho} \frac{\partial}{\partial \rho} (\rho R') + \frac{1}{\rho^2} R \Phi'' = 0$$

$$\text{or } \frac{\rho}{R} \frac{d}{d\rho} (\rho R') = -\frac{\Phi''}{\Phi} = \lambda$$

Hence

$$\underline{\underline{\Phi'' + \lambda \Phi = 0}}$$

and $\frac{d}{dp} (pR') - \frac{\lambda R}{p} = 0$

$\therefore \underline{R'' + \frac{R'}{p} - \frac{\lambda R}{p^2} = 0}$

Prob. 6.20

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

If $V(r, \theta) = R(r) F(\theta)$, $r \neq 0$,

$$F \frac{d}{dr} (r^2 R') + \frac{R}{\sin \theta} \frac{d}{d\theta} (\sin \theta F') = 0$$

Dividing thru by RF gives

$$\frac{1}{R} \frac{d}{dr} (r^2 R') = - \frac{1}{F \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \lambda$$

Hence,

$$\sin \theta F' + \cos \theta F' + \lambda F \sin \theta = 0$$

$\therefore \underline{F'' + \cot \theta F' + \lambda F = 0}$

Also, $\frac{d}{dr} (r^2 R') - \lambda R = 0$

$\therefore \underline{R'' + \frac{2R'}{r} - \frac{\lambda}{r^2} R = 0}$

Prob. 6.21

If the ends at $\phi = 0$ and $\phi = \pi/2$ are maintained at a potential difference of V_0 ,

from example 6.3,

$$E_p = \frac{2V_0}{\pi p}, \quad J = \sigma E.$$

Hence,

$$I = \int \vec{J} \cdot d\vec{s} = \frac{2V_0\sigma}{\pi} \int_{p=a}^b \int_{z=0}^t \frac{1}{p} dp dz = \frac{2V_0\sigma t}{\pi} \ln \frac{b}{a}$$

and

$$R = \frac{V_0}{I} = \frac{\pi}{2\sigma t \ln \frac{b}{a}}.$$

Prob. 6.22

If $V(r=a)=0$, $V(r=b)=V_0$, from example 6.9,

$$E = \frac{V_0}{r^2 \left(\frac{1}{a} - \frac{1}{b} \right)}, \quad J = \sigma E.$$

Hence,

$$I = \int \vec{J} \cdot d\vec{s} = \frac{\sigma V_0}{\frac{1}{a} - \frac{1}{b}} \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \frac{1}{r^2} r^2 \sin \theta d\theta d\phi$$

$$= \frac{\sigma V_0 2\pi}{\frac{1}{a} - \frac{1}{b}} (-\cos \theta) \Big|_0^{\alpha}$$

$$R = \frac{V_0}{I} = \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma(1 - \cos \alpha)}$$

Prob. 6.23

for a spherical capacitor, from eq. (6.38),

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\epsilon_0}$$

for the hemisphere, $R' = 2R$. Since the sphere consists of two hemispheres in parallel. As $b \rightarrow \infty$,

$$R' = \lim_{b \rightarrow \infty} \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\epsilon_0} = \frac{1}{2\pi a\epsilon_0}$$

$$C = \frac{1}{R'} = 2\pi a\epsilon_0$$

Alternatively, for an isolated sphere,

$$C = 4\pi\epsilon_0 a$$

But $RC = \frac{\epsilon}{\epsilon_0} \rightarrow R = \frac{1}{4\pi a\epsilon_0}$

$$R' = 2R = \frac{1}{2\pi a\epsilon_0} \quad \text{or} \quad C = 2\pi a\epsilon_0$$

Prob. 6.24

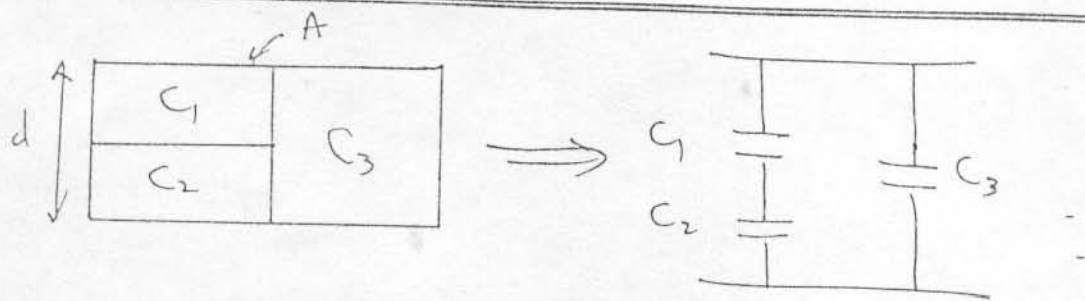
$$C = \frac{\epsilon S}{d} \rightarrow S = \frac{Cd}{\epsilon_0 \epsilon_r} = \frac{2 \times 10^{-9} \times 10^{-6}}{4 \times \frac{10^{-9}}{36\pi}} \text{ m}^2$$

$$S = 0.18\pi \text{ cm}^2 = \underline{\underline{0.5655 \text{ cm}^2}}$$

Prob. 6.25

from the figure on the next page,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3, \quad \text{where}$$



$$C_1 = \frac{\epsilon_0 A/2}{d/2} = \frac{\epsilon_0 A}{d}, \quad C_2 = \frac{\epsilon_0 \epsilon_r A}{d}, \quad C_3 = \frac{\epsilon_0 A}{2d}$$

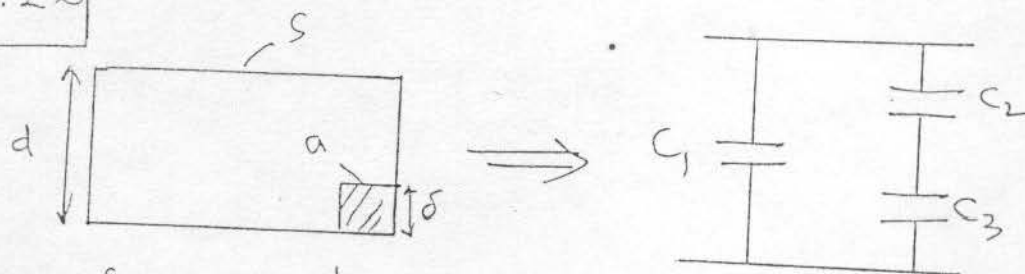
Hence,
$$C = \frac{\epsilon_0 \epsilon_r A^2/d^2}{\epsilon_r (\epsilon_r + 1) A/d} + \frac{\epsilon_0 A}{2d}$$

$$= \frac{\epsilon_0 A}{d} \left(\frac{1}{2} + \frac{\epsilon_r}{\epsilon_r + 1} \right)$$

$$= \frac{10^{-9}}{36\pi} \cdot \frac{10 \times 10^{-4}}{2 \times 10^{-3}} \left(\frac{1}{2} + \frac{6}{7} \right) = 5.99 \times 10^{-12}$$

$$C \approx \underline{\underline{6 \text{ pF}}}$$

Prob. 6.26



from the figure above,

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3}$$

where $C_1 = \frac{\epsilon_0 (s-a)}{d}$, $C_2 = \frac{\epsilon_0 \epsilon_r a}{d}$, $C_3 = \frac{\epsilon_0 a}{d}$

$$C_1 = \frac{10^{-9}}{36\pi} \frac{(2000-9) \times 10^{-6}}{5 \times 10^{-3}} = 3.52 \text{ pF.}$$

$$C_2 = \frac{10^{-9}}{36\pi} \frac{9 \times 10^{-6}}{2 \times 10^{-3}} = 0.04 \text{ pF.}$$

$$C_3 = \frac{10^{-9}}{36\pi} \frac{4.6 \times 9 \times 10^{-6}}{3 \times 10^{-3}} = 0.122 \text{ pF.}$$

Hence,

$$C = 3.52 + \frac{0.04 \times 0.122}{0.162} = \underline{\underline{3.55 \text{ pF}}}$$

Prob. 6.27

$$f dx = dW_E \rightarrow f = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2} \epsilon |\vec{E}|^2 dv = \frac{1}{2} \epsilon_0 \epsilon_r E^2 x ad + \frac{1}{2} \epsilon_0 E^2 da (1-x)$$

where $E = V_0/d$

$$\frac{dW_E}{dx} = \frac{1}{2} \epsilon_0 \frac{V_0^2}{d^2} (\epsilon_r - 1) da \rightarrow f = \frac{\epsilon_0 (\epsilon_r - 1) V_0^2 a}{2d}$$

Alternatively, $W_E = \frac{1}{2} C V_0^2$

where $C = C_1 + C_2 = \epsilon_0 \epsilon_r \frac{ax}{d} + \epsilon_0 \epsilon_r \frac{(1-x)}{d}$

$$\frac{dW_E}{dx} = \frac{1}{2} V_0^2 \frac{\epsilon_0 a}{d} (\epsilon_r - 1)$$

$$F = \underline{\underline{\frac{\epsilon_0 (\epsilon_r - 1) V_0^2 a}{d}}}$$

Prob. 6.28

$$(a) \quad C = \frac{\epsilon_0 S}{d} = \frac{10^{-9}}{36\pi} \cdot \frac{200 \times 10^{-4}}{3 \times 10^{-3}} = \underline{\underline{59 \text{ pF}}}$$

$$(b) \quad p_s = D_n = 10^{-6} \text{ nC/m}^2$$

$$\text{But } D_n = \epsilon \bar{E}_n = \epsilon_0 \frac{V_0}{d} = p_s$$

$$\text{or } V_0 = \frac{p_s d}{\epsilon_0} = 10^{-6} \times 3 \times 10^{-3} \times 36\pi \times 10^9$$

$$= \underline{\underline{339.3 \text{ V}}}$$

$$(c) \quad F = \frac{Q^2}{2S\epsilon_0} = \frac{p_s^2 S}{2\epsilon_0} = \frac{10^{-12} \times 200 \times 10^{-4} \times 36\pi \times 10^9}{2}$$

$$F = 36\pi \times 10^{-5} = \underline{\underline{1.131 \text{ mN}}}$$

Prob. 6.29

(a) $\vec{E}$ can be found in 2 ways,

method 1: $\vec{E} = \frac{p_s}{\epsilon} (-\hat{a}_x)$, where p_s is to be determined.

$$V_0 = -\int \vec{E} \cdot d\vec{l} = -\int_0^d -\frac{p_s}{\epsilon} dx$$

$$= p_s \int_0^d \left(\frac{x+d}{2d\epsilon_0} \right) dx = \frac{3p_s d}{4\epsilon_0} \rightarrow p_s = \frac{4\epsilon_0 V_0}{3d}$$

$$\vec{E} = -\frac{4V_0 \epsilon_0}{3d} \frac{x+d}{2d\epsilon_0} \hat{a}_x = \underline{\underline{-\frac{2V_0 (x+d)}{3d^2} \hat{a}_x}}$$

Method 2: we solve Laplace's equation for inhomogeneous media.

$$\nabla \cdot (\epsilon \nabla V) = \frac{d}{dx} \left(\epsilon \frac{dV}{dx} \right) = 0$$

$$\epsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\epsilon} = \frac{A}{2d\epsilon_0} (x+d) = c_1 (x+d)$$

$$V = c_1 \left(\frac{x^2}{2} + dx \right) + c_2$$

$$V(x=0) = 0 \rightarrow c_2 = 0$$

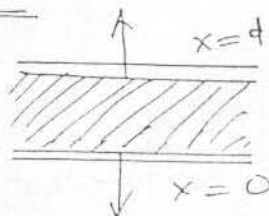
$$V(x=d) = V_0 \rightarrow c_1 = \frac{V_0}{d^2/2 + d^2} = \frac{2V_0}{3d^2}$$

$$V = \frac{2V_0}{3d^2} \left(\frac{x^2}{2} + dx \right)$$

$$\vec{E} = -\nabla V = -\frac{2V_0}{3d^2} (x+d) \vec{a}_x$$

$$\begin{aligned} (b) \quad \vec{P} &= (\epsilon_r - 1) \epsilon_0 \vec{E} = -\frac{2V_0}{3d^2} (x+d) \left(\frac{2d}{x+d} - 1 \right) \epsilon_0 \vec{a}_x \\ &= \frac{-2V_0 \epsilon_0 (d-x)}{3d^2} \vec{a}_x \end{aligned}$$

$$\begin{aligned} (c) \quad \rho_{ps} \Big|_{x=0} &= \vec{P} \cdot (-\vec{a}_x) \Big|_{x=0} \\ &= \frac{2V_0 \epsilon_0}{3d} \end{aligned}$$



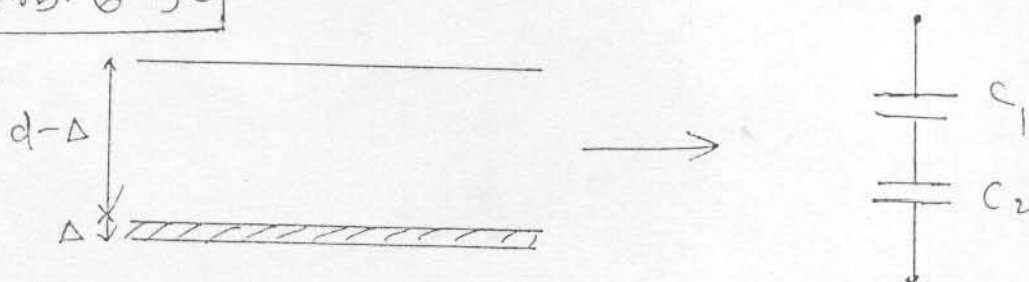
$$P_s|_{x=d} = \vec{P} \cdot \vec{a}_x|_{x=d} = \underline{\underline{0}}$$

$$(d) \quad Q = \int P_s dS = P_s S = \frac{4V_0 \epsilon_0 S}{3d}$$

$$C = \frac{Q}{V_0} = \frac{4\epsilon_0 S}{3d} = \frac{4}{3} \frac{10^{-9}}{36\pi} \cdot \frac{200 \times 10^{-4}}{2.5 \times 10^{-3}}$$

$$= \frac{8}{27\pi} \text{ nF} = \underline{\underline{94.31 \text{ pF}}}$$

Prob. 6.30



From the figure shown above,

$$C = \frac{C_1 C_2}{C_1 + C_2}, \text{ where } C_1 = \frac{\epsilon_0 S}{d - \Delta}, \quad C_2 = \frac{\epsilon_0 \epsilon_r S}{\Delta}$$

$$C = \frac{\epsilon_0 S \left(\frac{\epsilon_r}{\Delta} \cdot \frac{1}{d - \Delta} \right)}{\frac{\epsilon_r}{\Delta} + \frac{1}{d - \Delta}} = \frac{\epsilon_0 \epsilon_r S}{\epsilon_r (d - \Delta) + \Delta}$$

$$= \frac{\epsilon_r d C_0}{\epsilon_r d + (1 - \epsilon_r) \Delta} = \frac{\epsilon_r d C}{\epsilon_r d - \chi_e \Delta}$$

Prob. 6.31

$$C = \frac{2\pi\epsilon_0\epsilon_r L}{\ln \frac{b}{a}} = \frac{2\pi \times 3.5 \times 10^{-9} \times 10^3}{36\pi \ln \frac{2}{1}} = \frac{3.5 \times 10^{-6}}{1.8 \ln 2}$$

$$= \underline{\underline{280.5 \text{ nF/km.}}}$$

Prob. 6.32

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Since $b \rightarrow \infty$, $C = 4\pi a \epsilon_0 \epsilon_r$

$$= 4\pi \times 5 \times 10^{-2} \times 80 \times \frac{10^{-9}}{36\pi}$$

$$= \underline{\underline{444 \text{ pF.}}}$$

Prob. 6.33

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times \frac{10^{-9}}{36\pi} \times 5.9}{\left(\frac{1}{2} - \frac{1}{5}\right) \cdot 10^2} = \frac{5.9 \times 10^{-11}}{2.7}$$

$$= \underline{\underline{21.85 \text{ pF}}}$$

Prob. 6.34

$$C = \frac{2\pi\epsilon_0 L}{\ln \frac{b}{a}} = \frac{2\pi \times 10^{-9}}{36\pi} \times \frac{100 \times 10^{-6}}{\ln \frac{600}{20}} = 1.633 \times 10^{-15}$$

$$V = \frac{Q}{C} = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.35

$$V = V_0 e^{-t/T_r}$$

where $T_r = RC = 10 \times 10^{-6} \times 100 \times 10^6 = 1000$

$$50 = 100 e^{-t/T_r} \rightarrow 2 = e^{t/T_r}$$

$$t = 1000 \ln 2 = \underline{\underline{693.1 \text{ s}}}$$

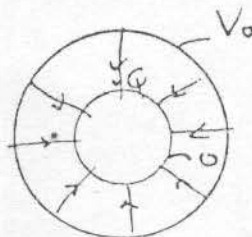
Prob. 6.36

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma} \rightarrow G = \frac{C \sigma}{\epsilon}$$

$$G = \frac{\pi \sigma}{\cosh^{-1}(d/2a)}$$

Prob. 6.37

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2} \vec{a}_r$$

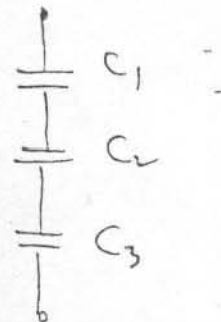
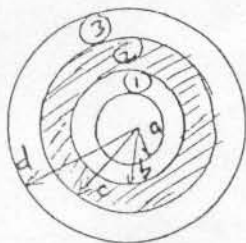


$$W = \frac{1}{2} \int \epsilon |\vec{E}|^2 dv = \iiint \frac{Q^2}{32\pi^2 \epsilon^2 r^2} \epsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2 \epsilon} (2\pi)(2) \int_b^c \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon} \left(\frac{1}{c} - \frac{1}{b} \right)$$

$$W = \frac{Q^2(b-c)}{8\pi\epsilon bc}$$

Prob. 6.38



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_1 = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}, \quad C_2 = \frac{4\pi\epsilon_0\epsilon_r}{\frac{1}{b} - \frac{1}{c}}, \quad C_3 = \frac{4\pi\epsilon_0}{\frac{1}{c} - \frac{1}{d}}$$

$$\frac{4\pi\epsilon_0}{C} = \frac{1}{a} - \frac{1}{b} + \frac{1}{\epsilon_r} \left(\frac{1}{b} - \frac{1}{c} \right) + \frac{1}{c} - \frac{1}{d}$$

$$\frac{4\pi\epsilon_0}{C} = \frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r} \left(\frac{1}{c} - \frac{1}{b} \right)$$

$$C = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r} \left(\frac{1}{c} - \frac{1}{b} \right)}$$

Prob. 6.39

$$C = 4\pi\epsilon_0 a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^6 = \frac{6.37}{9} \text{ mF}$$

$$= \underline{\underline{0.7078 \text{ mF}}}$$

Prob. 6.40

(a)

$$V = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{|(6,3,2)|} - \frac{1}{|(6,3,8)|} \right]$$

$$= \frac{10 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{1}{7} - \frac{1}{\sqrt{109}} \right] = \underline{4.237 \text{ V}}$$

$$\vec{E} = \frac{10 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{(6,3,2)}{7^3} - \frac{(6,3,8)}{109^{3/2}} \right]$$

$$= \underline{1.1\vec{a}_x + 0.55\vec{a}_y - 0.108\vec{a}_z \text{ V/m}}$$

$$(b) \vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2} \vec{a}_r = \frac{-10 \times 10 \times 10^{-18} [(0,0,-3) - (0,0,3)]}{4\pi \times \frac{10^{-9}}{36\pi} |(0,0,-3) - (0,0,3)|^3}$$

$$= -900 \times 10^{-9} \frac{(0,0,-6)}{6^3} = \underline{-25\vec{a}_z \text{ N}}$$

Prob. 6.41



(a) $Q_i = -(3nC - 4nC) = \underline{1nC}$

(b) The force of attraction between the

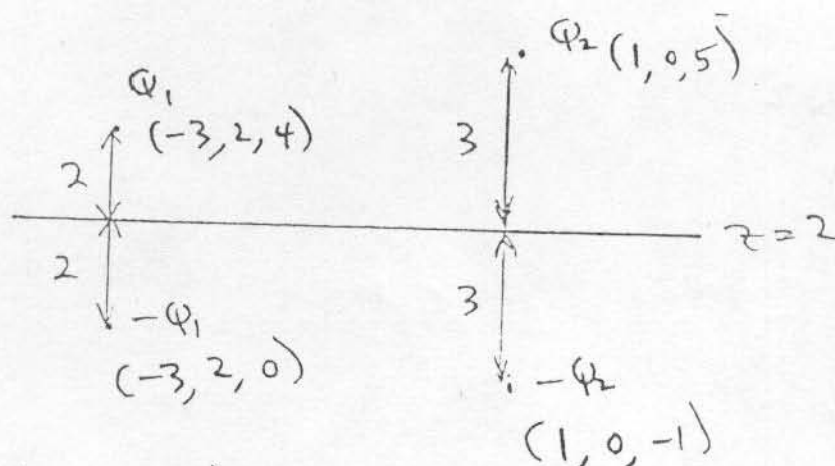
charges and the plates is

$$\vec{F} = \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{23} + \vec{F}_{24}$$

$$|\vec{F}| = \frac{10^{-18}}{4\pi \times 10^{-9} / 36\pi} \left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2} \right]$$

$$= \frac{81}{4} - 24 + 9 \text{ nN} = \underline{\underline{5.25 \text{ nN}}}$$

Prob. 6.42



$$\begin{aligned} \vec{D}(x, y, z) &= \frac{Q_1}{4\pi} \left[\frac{(x, y, z) - (-3, 2, 4)}{|(x+3)^2 + (y-2)^2 + (z-4)^2|^{3/2}} - \frac{(x, y, z) - (-3, 2, 0)}{|(x+3)^2 + (y-2)^2 + z^2|^{3/2}} \right] \\ &+ \frac{Q_2}{4\pi} \left[\frac{(x, y, z) - (1, 0, 5)}{|(x-1)^2 + y^2 + (z-5)^2|^{3/2}} - \frac{(x, y, z) - (1, 0, -1)}{|(x-1)^2 + y^2 + (z+1)^2|^{3/2}} \right] \\ &= \frac{50}{4\pi} \left[\frac{(x+3, y-2, z-4)}{|(x+3)^2 + (y-2)^2 + (z-4)^2|^{3/2}} - \frac{(x+3, y-2, z)}{|(x+3)^2 + (y-2)^2 + z^2|^{3/2}} \right] \\ &- \frac{20}{4\pi} \left[\frac{(x-1, y, z-5)}{|(x-1)^2 + y^2 + (z-5)^2|^{3/2}} - \frac{(x-1, y, z+1)}{|(x-1)^2 + y^2 + (z+1)^2|^{3/2}} \right] \end{aligned}$$

(a) At $(x, y, z) = (7, -2, 2)$,

$$\rho_s = D_z|_{z=2} = \frac{50}{4\pi} \left[\frac{2-4}{(10^2 + 4^2 + 2^2)^{3/2}} - \frac{2}{(10^2 + 4^2 + 2^2)^{3/2}} \right]$$

$$- \frac{20}{4\pi} \left[\frac{-3}{(6^2 + 4^2 + 3^2)^{3/2}} - \frac{3}{(6^2 + 4^2 + 3^2)^{3/2}} \right]$$

$$= \frac{-200}{4\pi (120)^{3/2}} + \frac{120}{4\pi (61)^{3/2}}$$

$$= -0.01211 + 0.02004 \text{ nC/m}^2$$

$$\rho_s = \underline{\underline{7.934 \text{ pC/m}^2}}.$$

(b) At $(3, 4, 8)$,

$$\vec{D} = \frac{50}{4\pi} \left[\frac{(6, 2, 4)}{(36 + 4 + 16)^{3/2}} - \frac{(6, 2, 8)}{(36 + 4 + 64)^{3/2}} \right]$$

$$- \frac{20}{4\pi} \left[\frac{(2, 4, 3)}{(4 + 16 + 9)^{3/2}} - \frac{(2, 4, 9)}{(4 + 16 + 81)^{3/2}} \right] \text{ nC/m}^2$$

$$= \frac{25}{\pi} \left[\frac{(3, 1, 2)}{56^{3/2}} - \frac{(3, 1, 4)}{104^{3/2}} \right] - \frac{5}{\pi} \left[\frac{(2, 4, 3)}{29^{3/2}} - \frac{(2, 4, 9)}{101^{3/2}} \right]$$

$$= (34.46, 18.24, 7.974) - (17.25, 34.53, 16.46) \text{ pC/m}^2$$

$$= \underline{\underline{17.21 \vec{a}_x - 16.29 \vec{a}_y - 8.486 \vec{a}_z \text{ pC/m}^2}}.$$

(c) Since $(1, 1, 1)$ is below the ground plane,

$$\vec{D} = \underline{\underline{0}}$$

Prob. 6.43

We have 7 images as follows: $-Q$ at $(-1, 1, 1)$, $-Q$ at $(1, -1, 1)$, $-Q$ at $(1, 1, -1)$, $-Q$ at $(-1, -1, -1)$, Q at $(1, -1, -1)$, Q at $(-1, -1, 1)$, and Q at $(-1, 1, -1)$.

$$\begin{aligned}\vec{F} &= \frac{Q}{4\pi\epsilon_0} \left[\frac{-2\vec{a}_x}{2^3} - \frac{2\vec{a}_y}{2^3} - \frac{2\vec{a}_z}{2^3} - \frac{(2\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z)}{12^{3/2}} \right. \\ &\quad \left. + \frac{(2\vec{a}_y + 2\vec{a}_z)}{8^{3/2}} + \frac{(2\vec{a}_x + 2\vec{a}_z)}{8^{3/2}} + \frac{(2\vec{a}_x + 2\vec{a}_y)}{8^{3/2}} \right] \\ &= 0.9(\vec{a}_x + \vec{a}_y + \vec{a}_z) \left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}} \right) \\ &= \underline{\underline{-0.1891(\vec{a}_x + \vec{a}_y + \vec{a}_z) \text{ N.}}}\end{aligned}$$

Prob. 6.44

$$\begin{aligned}\text{(a)} \quad \vec{E} &= \vec{E}_+ + \vec{E}_- = \frac{\rho_L}{2\pi\epsilon_0} \left(\frac{\vec{a}_{p_1}}{r_1} - \frac{\vec{a}_{p_2}}{r_2} \right) \\ &= \frac{20 \times 10^{-9}}{2\pi \times 10^{-9}} \left[\frac{(1, 2, 3) - (1, 1, 5)}{1^2 + 1^2} - \frac{(1, 2, 3) - (1, 1, -3)}{1^2 + 1^2} \right] \\ &= 360 \left[\frac{(0, 1, -2)}{5} - \frac{(0, 1, 8)}{65} \right] = \underline{\underline{66.46\vec{a}_y - 188.3\vec{a}_z \text{ V/m.}}}\end{aligned}$$

(b) Since $\vec{E}$ does not exist below plane $z=0$,
 $\underline{\underline{\vec{E} = 0.}}$

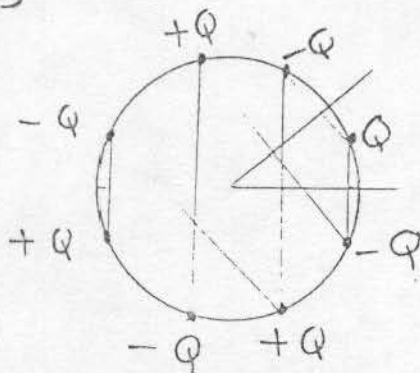
Prob. 6.45

$$\vec{E} = \sum \frac{\rho_s}{2\epsilon_0} \vec{a}_n = \frac{10^{-9}}{2 \times 10^{-9} \frac{36\pi}{36\pi}} [-5\vec{a}_x + 10\vec{a}_x - 10\vec{a}_x - 5\vec{a}_x]$$

$$= -180\pi \vec{a}_x = \underline{\underline{-565.5 \vec{a}_x \text{ V/m}}}$$

Prob. 6.46

$$N = \left(\frac{360^\circ}{45^\circ} - 1 \right) = 8 - 1 = \underline{\underline{7}}$$



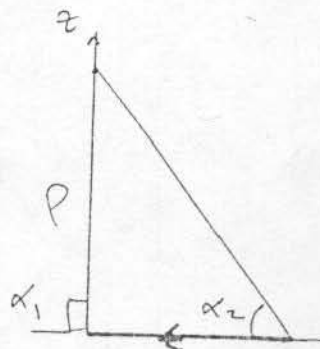
CHAPTER 7

P.E. 7.1

$$\rho = 5, \cos \alpha_1 = 0, \cos \alpha_2 = \sqrt{\frac{2}{27}}$$

$$\begin{aligned} \vec{a}_\phi &= \vec{a}_1 \times \vec{a}_\rho \\ &= \left(\frac{-\vec{a}_x - \vec{a}_y}{\sqrt{2}} \right) \times \vec{a}_z = \frac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}} \end{aligned}$$

$$\vec{H}_3 = \frac{10}{4\pi(5)} \left(\sqrt{\frac{2}{27}} - 0 \right) \left(\frac{-\vec{a}_x + \vec{a}_y}{2} \right) = \underline{\underline{-30.63\vec{a}_x + 30.63\vec{a}_y \text{ mA/m}}}$$



P.E. 7.2

$$(a) \vec{H} = \frac{2}{4\pi(2)} \left(1 + \frac{3}{\sqrt{13}}\right) \vec{a}_z = \underline{0.1458 \vec{a}_z \text{ A/m.}}$$

$$(b) \rho = \sqrt{3^2 + 4^2} = 5, \alpha_2 = 0, \cos \alpha_1 = -\frac{12}{13};$$

$$\vec{a}_\phi = -\vec{a}_y \times \left(\frac{3\vec{a}_x - 4\vec{a}_z}{5}\right) = \frac{4\vec{a}_x + 3\vec{a}_z}{5}$$

$$\vec{H} = \frac{2}{4\pi(5)} \left(1 + \frac{12}{13}\right) \left(\frac{4\vec{a}_x + 3\vec{a}_z}{5}\right) = \frac{1}{26\pi} (4\vec{a}_x + 3\vec{a}_z)$$

$$= \underline{48.97 \vec{a}_x + 36.73 \vec{a}_z \text{ mA/m.}}$$

P.E. 7.3

(a) from Example 7.3,

$$\vec{H} = \frac{I a^2}{2(a^2 + z^2)^{3/2}} \vec{a}_z$$

At $(0, 0, -1)$, $z = 2 \text{ cm}$,

$$\vec{H} = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 2^2)^{3/2} \times 10^{-6}} \vec{a}_z \cdot \text{A/m}$$

$$= \underline{400.2 \vec{a}_z \text{ mA/m.}}$$

(b) At $(0, 0, 10 \text{ cm})$, $z = 9 \text{ cm}$,

$$\vec{H} = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 9^2)^{3/2} \times 10^{-6}} \vec{a}_z \text{ A/m}$$

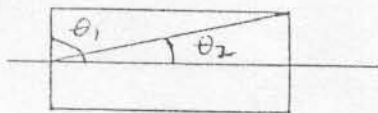
$$= \underline{57.3 \vec{a}_z \text{ mA/m.}}$$

P.E.7.4

$$\vec{H} = \frac{NI}{2L} (\cos \theta_2 - \cos \theta_1) \vec{a}_z = \frac{2 \times 10^3 \times 50 \times 10^{-3}}{2 \times 0.75} (\) \vec{a}_z$$

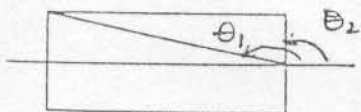
$$= \frac{100}{1.5} (\cos \theta_2 - \cos \theta_1) \vec{a}_z$$

(a) At $(0, 0, 0)$, $\theta_1 = 90^\circ$, $\cos \theta_2 = \frac{0.75}{\sqrt{0.75^2 + 0.05^2}} = 0.9978$



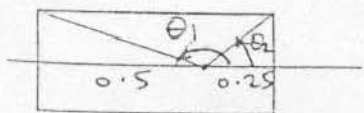
$$\vec{H} = \frac{100}{1.5} (0.9978 - 1) \vec{a}_z = \underline{\underline{66.52 \vec{a}_z \text{ A/m}}}$$

(b) At $(0, 0, 0.75)$, $\theta_2 = 90^\circ$, $\cos \theta_1 = -0.9978$



$$\vec{H} = \frac{100}{1.5} (0 + 0.9978) \vec{a}_z = \underline{\underline{66.52 \vec{a}_z \text{ A/m}}}$$

(c) At $(0, 0, 0.5)$, $\cos \theta_1 = \frac{-0.5}{\sqrt{0.5^2 + 0.05^2}} = -0.995$



$$\cos \theta_2 = \frac{0.25}{\sqrt{0.25^2 + 0.05^2}} = 0.9806$$

$$\vec{H} = \frac{100}{1.5} (0.9806 + 0.995) \vec{a}_z = \underline{\underline{131.7 \vec{a}_z \text{ A/m}}}$$

P.E. 7.5

$$\vec{H} = \frac{1}{2} \vec{F} \times \vec{a}_n$$

$$(a) \vec{H}(0,0,0) = \frac{1}{2} 50 \vec{a}_z \times (-\vec{a}_y) = \underline{\underline{25 \vec{a}_x \text{ mA/m}}}$$

$$(b) \vec{H}(1,5,-3) = \frac{1}{2} 50 \vec{a}_z \times \vec{a}_y = \underline{\underline{-25 \vec{a}_x \text{ mA/m}}}$$

P.E. 7.6

$$|\vec{H}| = \begin{cases} \frac{NI}{2\pi\rho}, & \rho - a < \rho < \rho + a = 9 < \rho < 11 \\ 0, & \text{otherwise} \end{cases}$$

$$(a) \text{ At } (3, -4, 0), \rho = \sqrt{3^2 + 4^2} = 5 \text{ cm} < 9 \text{ cm}$$

$$|\vec{H}| = \underline{\underline{0}}$$

$$(b) \text{ At } (6, 9, 0), \rho = \sqrt{6^2 + 9^2} = \sqrt{117} < 11$$

$$\vec{H} = \frac{10^3 \times 100 \times 10^{-3}}{2\pi \sqrt{117} \times 10^{-2}} = \underline{\underline{147.1 \text{ A/m}}}$$

P.E. 7.7

$$(a) \vec{B} = \nabla \times \vec{A} = (-4xz - 0)\vec{a}_x + (0 + 4yz)\vec{a}_y + (y^2 - x^2)\vec{a}_z$$

$$\vec{B}(-1, 2, 5) = \underline{\underline{20 \vec{a}_x + 40 \vec{a}_y + 3 \vec{a}_z \text{ Wb/m}^2}}$$

$$(b) \Psi = \int \vec{B} \cdot d\vec{S} = \int_{y=-1}^4 \int_{x=0}^1 (y^2 - x^2) dx dy = \int_{-1}^4 y^2 dy - 5 \int_0^1 x^2 dx$$

$$= \frac{1}{3} (64 + 1) - \frac{5}{3} = \underline{\underline{20 \text{ Wb}}}$$

Alternatively,

$$\begin{aligned}\Psi &= \oint \vec{A} \cdot d\vec{l} = \int_0^1 x^2(-1)dx + \int_{-1}^4 y^2(1)dy + \int_1^0 x^2(4)dx + 0 \\ &= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20 \text{ Wb}}}\end{aligned}$$

P.E. 7.8

$$\vec{H} = \int \frac{\vec{k} ds \times \vec{R}}{4\pi R^3},$$

$$ds = dx dy, \quad \vec{k} = k_y \vec{a}_y,$$

$$\vec{R} = (-x, -y, h),$$

$$\vec{k} \times \vec{R} = (h\vec{a}_x + x\vec{a}_z)k_y$$

$$\vec{H} = \int \frac{k_y (h\vec{a}_x + x\vec{a}_z) dx dy}{4\pi (x^2 + y^2 + h^2)^{3/2}}$$

$$= \frac{k_y h \vec{a}_x}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dy}{(x^2 + y^2 + h^2)^{3/2}} + \frac{k_y \vec{a}_z}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x dx dy}{(x^2 + y^2 + h^2)^{3/2}}$$

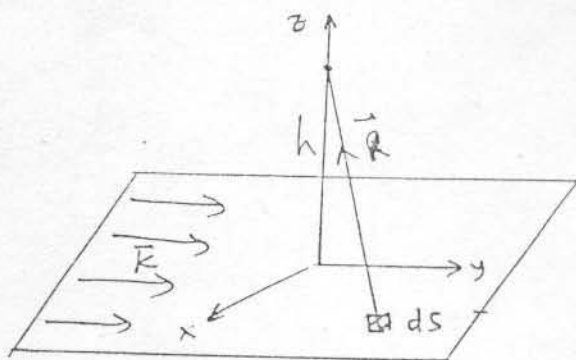
$$= \frac{k_y h \vec{a}_x}{4\pi} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{k_y h 2\pi \vec{a}_x}{4\pi} \int_0^{\infty} \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{k_y h}{2} \vec{a}_x \left(\frac{1}{(\rho^2 + h^2)^{1/2}} \right) \Big|_0^{\infty} = \frac{k_y}{2} \vec{a}_x$$

Similarly, for point $(0, 0, -h)$, $\vec{H} = -\frac{1}{2} k_y \vec{a}_x$

Hence,

$$\vec{H} = \begin{cases} \frac{1}{2} k_y \vec{a}_x, & z > 0 \\ -\frac{1}{2} k_y \vec{a}_x, & z < 0 \end{cases}$$



Prob. 7.1

Since the element is so small, from Biot-Savart law,

$$\vec{H} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3},$$

where $\vec{R} = (3, -6, 2) - (0, 0, 0) = (3, -6, 2),$

$R = |\vec{R}| = 7, \quad d\vec{l} = dy \vec{a}_y$

$$d\vec{l} \times \vec{R} = \begin{vmatrix} 0 & dy & 0 \\ 3 & -6 & 2 \end{vmatrix} = 2 dy \vec{a}_x - 3 dy \vec{a}_z$$

$$\vec{H} = \frac{16 \cdot 10^{-3} (2 \vec{a}_x - 3 \vec{a}_z) \cdot 2 \times 10^{-3}}{4\pi \times 7^3} \text{ A/m}$$

$$= \underline{14.85 \vec{a}_x - 22.27 \vec{a}_z \text{ nA/m}}$$

Prob. 7.2

(a) $\vec{H} = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1) \vec{a}_\phi = \frac{2}{4\pi(5)} \left(\frac{10}{5\sqrt{2}} - 0 \right) \vec{a}_y$

$$= \underline{28.47 \vec{a}_y \text{ mA/m}}$$

(b) $\vec{H} = \frac{2}{4\pi 5\sqrt{2}} \left(\frac{10}{5\sqrt{6}} - 0 \right) \vec{a}_\phi$, where $\vec{a}_\phi = \vec{a}_z \times \left(\frac{\vec{a}_x + \vec{a}_y}{\sqrt{2}} \right)$

$$= \frac{1}{5\pi\sqrt{2}} \left(\frac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}} \right) = \underline{-13 \vec{a}_x + 13 \vec{a}_y \text{ mA/m}}$$

(c) $\vec{H} = \frac{2}{4\pi(5)\sqrt{10}} \left(\frac{10}{5\sqrt{14}} - 0 \right) \vec{a}_\phi$, $\vec{a}_\phi = \vec{a}_z \times \left(\frac{\vec{a}_x + 3\vec{a}_y}{\sqrt{10}} \right)$

$$= \frac{1}{50\pi\sqrt{14}} (-3\vec{a}_x + \vec{a}_y) = \underline{-5.1 \vec{a}_x + 1.7 \vec{a}_y \text{ mA/m}}$$

(d) $\vec{H} = \underline{5.1\vec{a}_x + 1.7\vec{a}_y \text{ mA/m}^2}$.

Prob. 7.3

Let $\vec{H} = \vec{H}_z + \vec{H}_x$

for $\vec{H}_z = \frac{I}{2\pi\rho} \vec{a}_\phi$, $\rho = \sqrt{6^2 + 8^2} = 10$,

$\vec{a}_\phi = \vec{a}_z \times \left(\frac{6\vec{a}_y + 8\vec{a}_x}{10} \right)$

$H_z = \frac{20}{2\pi(100)} (-8\vec{a}_x + 6\vec{a}_y) = \frac{1}{\pi} (-0.8\vec{a}_x + 0.6\vec{a}_y)$

for $H_x = \frac{I}{2\pi\rho} \vec{a}_\phi$, $\rho = \sqrt{8^2 + 6^2} = 10$

$\vec{a}_\phi = \vec{a}_x \times \left(\frac{8\vec{a}_y - 6\vec{a}_z}{10} \right)$

$H_x = \frac{30}{2\pi(100)} (6\vec{a}_y + 8\vec{a}_z) = \frac{1}{\pi} (0.9\vec{a}_y + 1.2\vec{a}_z)$

$\vec{H} = \vec{H}_x + \vec{H}_z = \frac{1}{\pi} (-0.8\vec{a}_x + 1.5\vec{a}_y + 1.2\vec{a}_z)$

$= \underline{-0.255\vec{a}_x + 0.4775\vec{a}_y + 0.382\vec{a}_z \text{ A/m.}}$

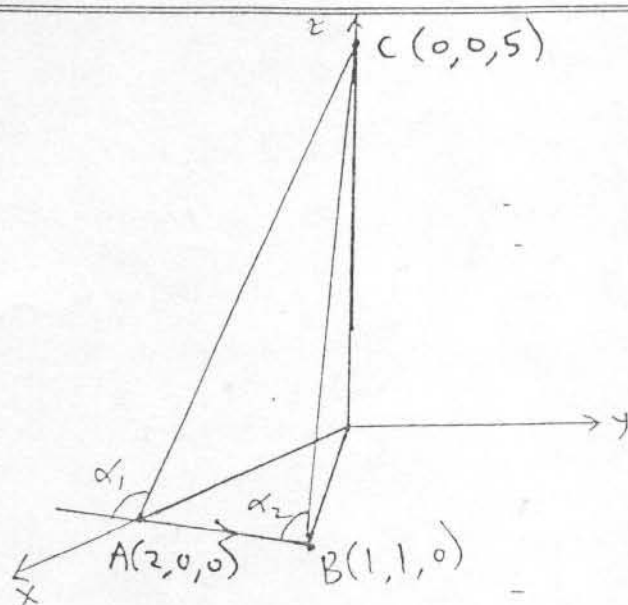
Prob. 7.4

(a) Consider the figure on the next page.

$\vec{AB} = (1, 1, 0) - (2, 0, 0) = (-1, 1, 0)$

$\vec{AC} = (0, 0, 5) - (2, 0, 0) = (-2, 0, 5)$

$\vec{AB} \cdot \vec{AC} = 2$, i.e. $\vec{AB}$ and $\vec{AC}$ are not



perpendicular.

$$\cos(180 - \alpha_1) = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{2}{\sqrt{2} \sqrt{29}} \rightarrow \cos \alpha_1 = -\sqrt{\frac{2}{29}}$$

$$\vec{BC} = (0, 0, 5) - (1, 1, 0) = (-1, -1, 5)$$

$$\vec{BA} = (1, -1, 0)$$

$$\cos \alpha_2 = \frac{\vec{BC} \cdot \vec{BA}}{|\vec{BC}| |\vec{BA}|} = \frac{-1 + 1}{|\vec{BC}| |\vec{BA}|} = 0$$

$$\text{i.e. } \vec{BC} = \vec{p} = (-1, -1, 5), \quad p = \sqrt{27}$$

$$\vec{a}_p = \vec{a}_1 \times \vec{a}_p = \frac{(-1, 1, 0)}{\sqrt{2}} \times \frac{(-1, -1, 5)}{\sqrt{27}} = \frac{(5, 5, 2)}{\sqrt{54}}$$

$$\vec{H}_2 = \frac{10}{4\pi\sqrt{27}} \left(0 + \sqrt{\frac{2}{29}}\right) \frac{(5, 5, 2)}{\sqrt{27}} = \frac{5}{2\pi\sqrt{29}} \cdot \frac{(5, 5, 2)}{27} \text{ A/m}$$

$$= 27.37 \vec{a}_x + 27.37 \vec{a}_y + 10.95 \vec{a}_z \text{ mA/m.}$$

$$(b) \vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 = (0, -59.1, 0) +$$

$$(27.37, 27.37, 10.95) + (-30.63, 30.63, 0) \\ = \underline{\underline{-3.26 \vec{a}_x - 1.1 \vec{a}_y + 10.95 \vec{a}_z \text{ mA/m}}}$$

Prob. 7.5

(a) Let $\vec{H} = \vec{H}_x + \vec{H}_y = 2\vec{H}_x$

$$\vec{H}_x = \frac{I}{4\pi r} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_\phi,$$

where $\vec{a}_\phi = -\vec{a}_x \times \vec{a}_y = -\vec{a}_z$, $\alpha_1 = 180^\circ$, $\alpha_2 = 45^\circ$.

$$\vec{H}_x = \frac{5}{4\pi(2)} (\cos 45^\circ - \cos 180^\circ) (-\vec{a}_z)$$

$$= \underline{\underline{-0.6792 \vec{a}_z \text{ A/m}}}$$

(b) $\vec{H} = \vec{H}_x + \vec{H}_y$

where $\vec{H}_x = \frac{5}{4\pi(2)} (1-0) \vec{a}_\phi$, $\vec{a}_\phi = -\vec{a}_x \times -\vec{a}_y = \vec{a}_z$

$$= 198.9 \vec{a}_z \text{ mA/m}.$$

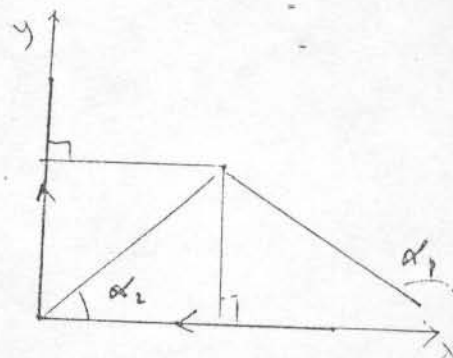
$\vec{H}_y = 0$ since $\alpha_1 = \alpha_2 = 0$.

$$\vec{H} = \underline{\underline{0.1989 \vec{a}_z \text{ A/m}}}$$

(c) $\vec{H} = \vec{H}_x + \vec{H}_y$

where $\vec{H}_x = \frac{5}{4\pi(2)} (1-0) (-\vec{a}_x \times \vec{a}_z) = 198.9 \vec{a}_y \text{ mA/m}$

$$\vec{H}_y = \frac{5}{4\pi(2)} (1-0) (\vec{a}_y \times \vec{a}_z) = 198.9 \vec{a}_x \text{ mA/m}$$



$$\vec{H} = \underline{0.1989 \vec{a}_x + 0.1989 \vec{a}_y \text{ A/m.}}$$

Prob. 7.6

for the side of the loop along y-axis,

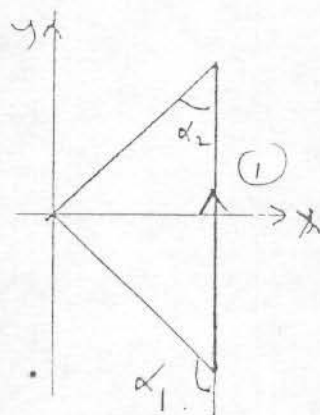
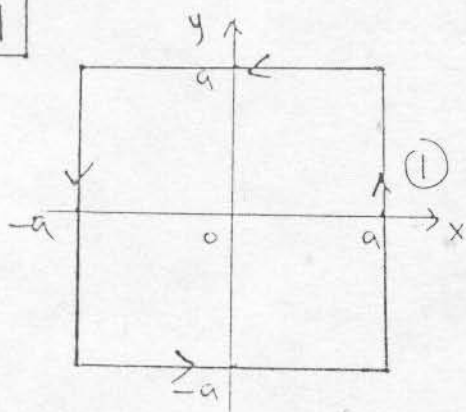
$$\vec{H}_1 = \frac{I}{4\pi p} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_q,$$

where $\vec{a}_q = -\vec{a}_x$, $p = 2 \tan 30^\circ = \frac{2}{\sqrt{3}}$, $\alpha_2 = 30^\circ$, $\alpha_1 = 150^\circ$

$$\vec{H}_1 = \frac{5}{4\pi} \frac{\sqrt{3}}{2} (\cos 30^\circ - \cos 150^\circ) (-\vec{a}_x) = -\frac{15}{8\pi} \vec{a}_x$$

$$\vec{H} = 3\vec{H}_1 = \underline{-1.79 \vec{a}_x \text{ A/m.}}$$

Prob. 7.7



$\vec{H} = 4\vec{H}_1$, where $\vec{H}_1$ is due to side 1.

$$\vec{H}_1 = \frac{I}{4\pi p} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_q,$$

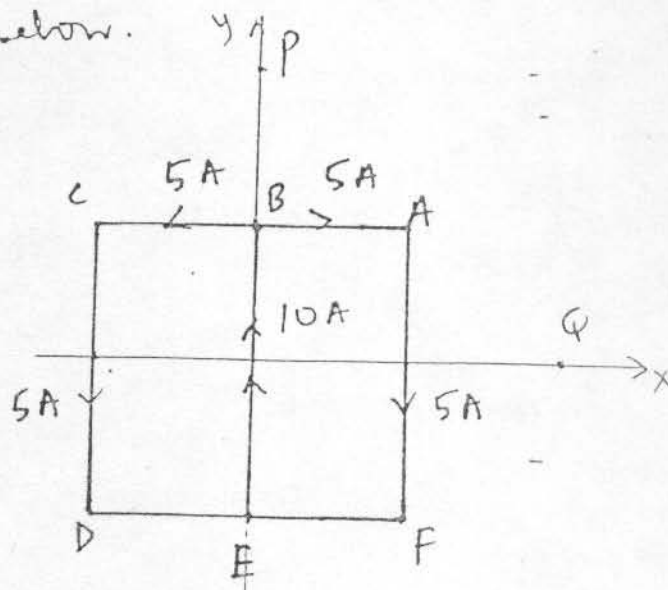
$p = a$, $\alpha_2 = 45^\circ$, $\alpha_1 = 135^\circ$, $\vec{a}_q = \vec{a}_y \times -\vec{a}_x = \vec{a}_z$

$$\vec{H}_1 = \frac{I}{4\pi p} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \vec{a}_z = \frac{2I}{4\pi a\sqrt{2}} \vec{a}_z$$

$$\vec{H} = 4\vec{H}_1 = \frac{\sqrt{2}I}{\pi a} \vec{a}_z.$$

Prob. 7.8

Consider the figure below.



(a) At P, due to symmetry, $\vec{B} = \underline{\underline{0}}$.

(b) At Q, the contribution by side AB cancels that due to side EF. Similarly, the contribution due to side BC cancels that due to side DE. Thus, at Q,

$$\vec{B} = \vec{B}_{AF} + \vec{B}_{BE} + \vec{B}_{CD}$$

we now apply

$$\vec{B} = \mu \vec{H} = \frac{\mu I}{4\pi r} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_\phi$$

$$\vec{B}_{AF} = \frac{5\mu_0}{4\pi(1)} \left(2 \frac{1}{\sqrt{2}} \right) (-\vec{a}_y \times \vec{a}_x) = \frac{10\mu_0}{4\pi\sqrt{2}} \vec{a}_z$$

$$\vec{B}_{BE} = \frac{10\mu_0}{4\pi(2)} \left(2 \frac{1}{\sqrt{2}} \right) (\vec{a}_y \times \vec{a}_z) = -\frac{10\mu_0}{4\pi\sqrt{2}} \vec{a}_x$$

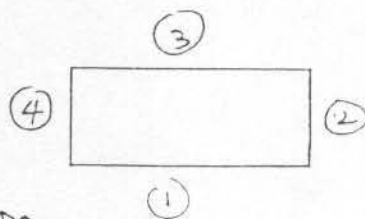
$$\vec{B}_{CD} = \frac{5 \mu_0}{4\pi(3)} \left(2 \frac{1}{\sqrt{10}} \right) (-\vec{a}_y \times \vec{a}_x) = \frac{10 \mu_0}{4\pi 3\sqrt{10}} \vec{a}_z$$

$$\begin{aligned} \vec{B} &= \frac{10 \mu_0}{4\pi} \vec{a}_z \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} + \frac{1}{3\sqrt{10}} \right) \\ &= \frac{10 \times 4\pi \times 10^{-7}}{4\pi} \vec{a}_z (0.707 - 0.4472 + 0.1054) \text{ Wb/m}^2 \\ &= \underline{\underline{0.3652 \vec{a}_z \text{ } \mu\text{Wb/m}^2}} \end{aligned}$$

Prob. 7.9

Let $\vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 + \vec{H}_4$

where $\vec{H}_n$ is the contribution by side n.



(a) $\vec{H} = 2\vec{H}_1 + \vec{H}_2 + \vec{H}_4$ since $\vec{H}_1 = \vec{H}_3$.

$$\vec{H}_1 = \frac{1}{4\pi a} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_z = \frac{10}{4\pi(2)} \left(\frac{6}{\sqrt{40}} + \frac{1}{\sqrt{2}} \right) \vec{a}_z$$

$$\vec{H}_2 = \frac{10}{4\pi(6)} \left(2 \times \frac{2}{\sqrt{40}} \right) \vec{a}_z, \quad \vec{H}_4 = \frac{10}{4\pi(2)} \left(2 \times \frac{1}{\sqrt{2}} \right) \vec{a}_z$$

$$\vec{H} = \left[\frac{5}{2\pi} \left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}} \right) + \frac{5}{6\pi\sqrt{10}} + \frac{5}{2\pi\sqrt{2}} \right] \vec{a}_z = \underline{\underline{1.964 \vec{a}_z \text{ A/m}}}$$

(b) At $(4, 2, 0)$, $\vec{H} = 2(\vec{H}_1 + \vec{H}_4)$

$$\vec{H}_1 = \frac{10}{4\pi(2)} \frac{3}{\sqrt{20}} \vec{a}_z, \quad \vec{H}_4 = \frac{10}{4\pi(4)} \frac{4}{\sqrt{20}} \vec{a}_z$$

$$\vec{H} = \frac{2\sqrt{5}}{\pi} \left(1 + \frac{1}{4}\right) \vec{a}_z = \underline{1.78 \vec{a}_z \text{ A/m.}}$$

(c) At $(4, 8, 0)$, $\vec{H} = \vec{H}_1 + 2\vec{H}_2 + \vec{H}_3$

$$\vec{H}_1 = \frac{10}{4\pi(8)} \left(2 - \frac{4}{4\sqrt{5}}\right) \vec{a}_z, \quad \vec{H}_2 = \frac{10}{4\pi(4)} \left(\frac{8}{4\sqrt{5}} - \frac{1}{\sqrt{2}}\right) \vec{a}_z$$

$$\vec{H}_3 = \frac{10}{4\pi(4)} \left(\frac{2}{\sqrt{2}}\right) (-\vec{a}_z)$$

$$\vec{H} = \frac{5}{8\pi} (\vec{a}_z) \left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} - \frac{4}{\sqrt{2}}\right) = \underline{-0.1178 \vec{a}_z \text{ A/m.}}$$

(d) At $(0, 0, 2)$,

$$\vec{H}_1 = \frac{10}{4\pi(2)} \left(\frac{8}{\sqrt{64}} - 0\right) (\vec{a}_x \times \vec{a}_z) = -\frac{10}{\pi\sqrt{68}} \vec{a}_y,$$

$$\vec{H}_2 = \frac{10}{4\pi\sqrt{64}} \left(\frac{4}{\sqrt{84}} - 0\right) \vec{a}_y \times \left(\frac{2\vec{a}_z - 8\vec{a}_x}{\sqrt{68}}\right) = \frac{5(\vec{a}_x + 4\vec{a}_y)}{17\pi\sqrt{84}}$$

$$\vec{H}_3 = \frac{10}{4\pi\sqrt{20}} \left(-\frac{8}{\sqrt{84}} - 0\right) \vec{a}_x \times \left(\frac{2\vec{a}_x - 4\vec{a}_y}{\sqrt{20}}\right) = \frac{\vec{a}_y + 2\vec{a}_z}{\pi\sqrt{21}}$$

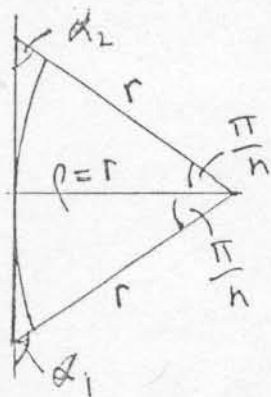
$$\vec{H}_4 = \frac{10}{4\pi(2)} \left(0 + \frac{4}{\sqrt{20}}\right) (-\vec{a}_y \times \vec{a}_z) = -\frac{5\vec{a}_x}{\pi\sqrt{20}}$$

$$\vec{H} = \left(\frac{5}{34\pi\sqrt{21}} - \frac{5}{\pi\sqrt{20}}\right) \vec{a}_x + \left(\frac{1}{\pi\sqrt{21}} - \frac{10}{\pi\sqrt{68}}\right) \vec{a}_y$$

$$+ \left(\frac{20}{34\pi\sqrt{21}} - \frac{2}{\pi\sqrt{21}}\right) \vec{a}_z$$

$$= \underline{-0.3457 \vec{a}_x - 0.3165 \vec{a}_y + 0.1798 \vec{a}_z \text{ A/m}}$$

Prob. 7.10



(a) Consider one side of the polygon as shown. The angle subtended by the side at the center of the circle is

$$\frac{360^\circ}{n} = \frac{2\pi}{n}.$$

The field due to this side is

$$H_1 = \frac{I}{4\pi p} (\cos \alpha_2 - \cos \alpha_1)$$

where $p = r$, $\cos \alpha_2 = \cos(90 - \frac{\pi}{n}) = \sin \frac{\pi}{n}$
 $\cos \alpha_1 = -\sin \frac{\pi}{n}$

$$H_1 = \frac{I}{4\pi r} 2 \sin \frac{\pi}{n}$$

$$H = n H_1 = \frac{n I}{2\pi r} \sin \frac{\pi}{n}$$

(b) for $n=3$, $H = \frac{3 I}{2\pi r} \sin \frac{\pi}{3}$

$r \cot 30^\circ = 2 \rightarrow r = \frac{2}{\sqrt{3}}$

$$H = \frac{3 \times 5}{2\pi \frac{2}{\sqrt{3}}} \cdot \frac{\sqrt{3}}{2} = \frac{48}{8\pi} = \underline{\underline{1.78 \text{ A/m}}}$$

for $n=4$, $H = \frac{4 I}{2\pi r} \sin \frac{\pi}{4} = \frac{4 \times 5}{2\pi (2)} \frac{1}{\sqrt{2}}$
 $= \underline{\underline{1.125 \text{ A/m}}}$

(i) As $n \rightarrow \infty$,

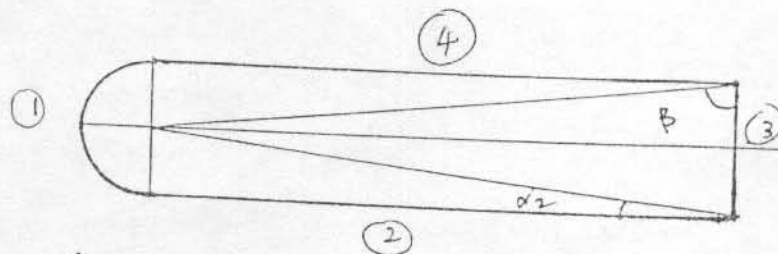
$$H = \lim_{n \rightarrow \infty} \frac{n I}{2\pi r} \sin \frac{\pi}{n} = \frac{n I}{2\pi r} \cdot \frac{\pi}{n} = \underline{\underline{\frac{I}{2r}}}$$

from Example 7-3, when $h = 0$,

$$H = \frac{I}{2r}$$

which agrees.

Prob. 7.11



$$\text{Let } \vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 + \vec{H}_4$$

$$\vec{H}_1 = \frac{I}{4a} \vec{a}_z = \frac{10}{4 \times 4 \times 10^{-2}} \vec{a}_z = 62.5 \vec{a}_z$$

$$\begin{aligned} \vec{H}_2 = \vec{H}_4 &= \frac{I}{4\pi \times 4 \times 10^{-2}} (\cos \alpha_2 - \cos 90^\circ) \vec{a}_z, \alpha_2 = \tan^{-1} \frac{4}{100} \\ &= 19.88^\circ = 2.29^\circ \end{aligned}$$

$$\vec{H}_3 = \frac{I}{4\pi(1)} 2 \cos \beta \vec{a}_z, \beta = \tan^{-1} \frac{100}{4} = 87.7^\circ$$

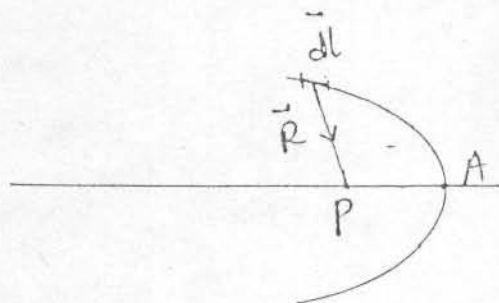
$$= \frac{10}{4\pi} 2 \cos 87.7^\circ \vec{a}_z = 0.06361 \vec{a}_z$$

$$\vec{H} = (62.5 + 2 \times 19.88 + 0.06361) \vec{a}_z$$

$$= \underline{\underline{102.32 \vec{a}_z \text{ A/m.}}}$$

Prob. 7.12

$$\vec{H} = \int \frac{I d\vec{l} \times \vec{R}}{4\pi R^3}$$



$$\left| \frac{\vec{R} \times d\vec{l}}{R^3} \right| = \frac{1}{r^2} r d\theta = \frac{d\theta}{r}$$

$$H = \frac{I}{4\pi} \int \frac{d\theta}{r}$$

If P is the focus of the parabola and the apex A is the point $r=d, \theta=0$, then $r(1+\cos\theta) = 2d$. Hence at P,

$$\begin{aligned} H &= \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{(1+\cos\theta)}{2d} d\theta = \frac{I}{8\pi d} (\theta - \sin\theta) \Big|_{-\pi}^{\pi} \\ &= \frac{I}{4d} = \frac{3}{0.1} = \underline{\underline{30 \text{ A/m}}} \end{aligned}$$

Prob. 7.13

from Example 7.3, $\vec{H}$ due to circular loop =

$$\vec{H} = \frac{I \rho^2}{2(\rho^2 + z^2)^{3/2}} \vec{a}_z$$

$$\begin{aligned} \text{(a) } \vec{H}(0,0,0) &= \frac{5 \times 2^2}{2(2^2 + 0^2)^{3/2}} \vec{a}_z + \frac{5 \times 2^2}{2(2^2 + 4^2)^{3/2}} \vec{a}_z \\ &= \underline{\underline{1.36 \vec{a}_z \text{ A/m}}} \end{aligned}$$

$$\begin{aligned} (b) \quad \vec{H}(0,0,2) &= 2 \frac{5 \times 2^2}{2(2^2 + 2^2)^{3/2}} \vec{a}_z \\ &= \underline{\underline{0.884 \vec{a}_z \text{ A/m}}} \end{aligned}$$

Prob. 7.14

$$\text{Let } \vec{H} = \vec{H}_L + \vec{H}_S$$

$$\text{where } \vec{H}_L = \frac{I}{2\pi\rho} \vec{a}_\phi, \quad I = 50\pi \text{ mA},$$

$$\rho = |(3,4,5) - (3,1,4)| = \sqrt{10}$$

$$\vec{a}_\phi = \vec{a}_1 \times \vec{a}_\rho = \vec{a}_x \times \left(\frac{3\vec{a}_y + \vec{a}_z}{\sqrt{10}} \right) = \frac{-\vec{a}_y + 3\vec{a}_z}{\sqrt{10}}$$

$$\vec{H}_L = \frac{50\pi}{2\pi \times 10} (-\vec{a}_y + 3\vec{a}_z) = -2.5\vec{a}_y + 7.5\vec{a}_z \text{ mA/m}$$

$$\vec{H}_S = \frac{1}{2} (20\vec{a}_x \times \vec{a}_z) = -10\vec{a}_y \text{ mA/m}$$

$$\text{Thus } \vec{H} = \vec{H}_L + \vec{H}_S = \underline{\underline{-12.5\vec{a}_y + 7.5\vec{a}_z \text{ mA/m}}}$$

Prob 7.15

(a) See text.

$$(b) \text{ for } 0 < \rho < a, \quad \oint \vec{H} \cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I \cdot \frac{\pi\rho^2}{\pi a^2}$$

$$H_\phi = \frac{I\rho}{2\pi a^2}$$

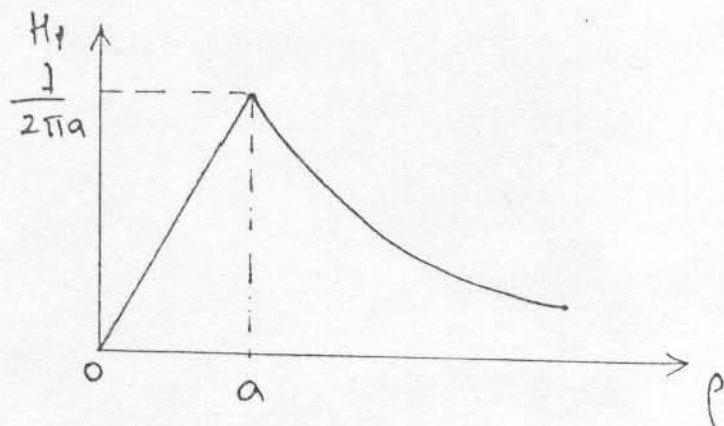
$$\text{for } \rho > a, \quad \oint \vec{H} \cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I$$

$$\text{or } H_\phi = \frac{I}{2\pi\rho}$$

Thus

$$\vec{H} = \begin{cases} \frac{J\rho}{2\pi a^2} \vec{a}_\phi, & 0 < \rho < a \\ \frac{J}{2\pi\rho} \vec{a}_\phi, & \rho > a \end{cases}$$

$|\vec{H}|$ is sketched below.



Prob. 7.16

Applying Ampere's law,

$$\oint \vec{H} \cdot d\vec{l} = I = \int \vec{J} \cdot d\vec{S}$$

for $\rho < a$, $H_\phi \cdot 2\pi\rho = J_0 \cdot \frac{2\pi\rho^3}{3} \rightarrow H_\phi = J_0 \frac{\rho^2}{3}$

for $\rho > a$, $H_\phi \cdot 2\pi\rho = J_0 \cdot \frac{2\pi a^3}{3} \rightarrow H_\phi = J_0 \frac{a^3}{3\rho}$

Thus,

$$H_\phi = \begin{cases} J_0 \frac{\rho^2}{3}, & \rho < a \\ \frac{J_0 a^3}{3\rho}, & \rho > a. \end{cases}$$

Prob. 7.17

(a) Applying Ampere's law,

$$H_\phi \cdot 2\pi r = I \cdot \frac{\pi r^2}{\pi a^2} \rightarrow H_\phi = \frac{I r}{2\pi a^2}$$

i.e. $\vec{H} = \frac{I r}{2\pi a^2} \vec{a}_\phi$

$$\begin{aligned} \vec{J} &= \nabla \times \vec{H} = -\frac{\partial H_\phi}{\partial z} \vec{a}_r + \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{a}_z \\ &= \frac{1}{r} \frac{I}{2\pi a^2} \cdot 2r \vec{a}_z = \frac{I}{\pi a^2} \vec{a}_z \end{aligned}$$

(b) from Prob. 7.15,

$$H_\phi = \begin{cases} \frac{I r}{2\pi a^2}, & r < a \\ \frac{I}{2\pi r}, & r > a \end{cases}$$

At (0, 1 cm, 0), $H_\phi = \frac{3 \times 1 \times 10^{-2}}{2\pi \times 4 \times 10^{-4}} = \frac{300}{8\pi}$

$\vec{H} = \underline{\underline{11.94 \vec{a}_\phi \text{ A/m}}}$

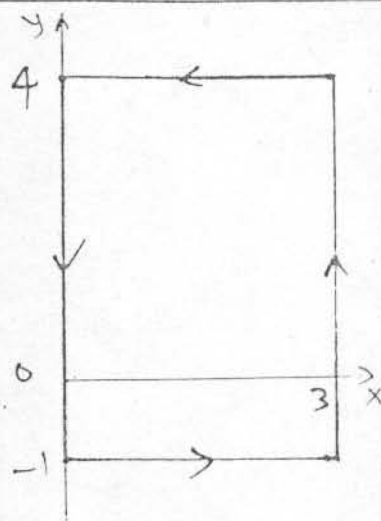
At (0, 4 cm, 0), $H_\phi = \frac{3}{2\pi \times 4 \times 10^{-2}} = \frac{300}{8\pi}$

$\vec{H} = \underline{\underline{11.94 \vec{a}_\phi \text{ A/m}}}$

Prob. 7.18

$$(a) \vec{J} = \nabla \times \vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$\vec{J} = \underline{\underline{-2\vec{a}_z \text{ A/m}^2}}$$



$$(b) \oint \vec{H} \cdot d\vec{l} = I_{enc}$$

$$I_{enc} = \int \vec{J} \cdot d\vec{S} = \int_{x=0}^3 \int_{y=-1}^4 (-2) dx dy = (-2)(3)(5) = -30 \text{ A}$$

$$\begin{aligned} \oint \vec{H} \cdot d\vec{l} &= \int_0^3 y dx \Big|_{y=-1} + \int_{y=-1}^4 (-x) dy \Big|_{x=3} + \int_3^0 y dx \Big|_{y=4} \\ &+ \int_4^{-1} (-x) dy \Big|_{x=0} = (-1)(3) + (-3)(5) + (4)(-3) \\ &= -30 \text{ A} \end{aligned}$$

$$\text{Thus } \oint \vec{H} \cdot d\vec{l} = I_{enc} = \underline{\underline{-30 \text{ A}}}$$

Prob. 7.19

$$(a) \vec{J} = \nabla \times \vec{H} = -2y(x+1)\vec{a}_x + (y^2 + z^2)\vec{a}_y$$

$$\vec{J}(1, 0, -3) = \underline{\underline{9\vec{a}_y \text{ A/m}^2}}$$

$$(b) I = \int \vec{J} \cdot d\vec{S} = \int_{x=0}^1 \int_{z=0}^1 (1+z^2) dx dz = (1) \left(z + \frac{z^3}{3} \right) \Big|_0^1 = \frac{4}{3}$$

$$I = \underline{\underline{1.333 \text{ A}}}$$

Prob. 7.20

$$(a) \vec{B} = \frac{\mu_0 I}{2\pi \rho} \vec{a}_\phi. \text{ At } (-3, 4, 5), \rho = 5$$

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 2}{2\pi(5)} \vec{a}_\phi = \underline{\underline{80 \vec{a}_\phi \text{ nWb/m}^2}}$$

$$\begin{aligned} (b) \Phi &= \int \vec{B} \cdot d\vec{S} = \frac{\mu_0 I}{2\pi} \iint \frac{d\rho dz}{\rho} \\ &= \frac{4\pi \times 10^{-7} \times 2}{2\pi} \ln \rho \Big|_2^6 z \Big|_0^4 = 16 \times 10^{-7} \ln 3 \\ &= \underline{\underline{1.756 \mu\text{Wb}}}. \end{aligned}$$

Prob. 7.21

$$\begin{aligned} \Psi &= \int \vec{B} \cdot d\vec{S} = \mu_0 \int_{\phi=0}^{\pi/2} \int_{z=0}^2 \frac{10^6}{\rho} \cos \phi \rho d\phi dz \\ &= 4\pi \times 10^{-7} \times 10^6 \int_0^2 dz \int_0^{\pi/2} \cos \phi d\phi = 0.4\pi(2)(1) \end{aligned}$$

$$\Psi = \underline{\underline{2.513 \text{ Wb}}}.$$

Prob. 7.22

$$\Psi = \int \vec{B} \cdot d\vec{S} = \mu_0 \int_{z=0}^{0.2} \int_{\phi=0}^{50^\circ} \frac{10^6}{\rho} \sin 2\phi \rho d\phi dz$$

$$\Psi = 4\pi \times 10^{-7} \times 10^6 (0.2) \left(-\frac{\cos 2\phi}{2} \right) \Big|_0^{50^\circ}$$

$$= 0.04\pi (1 - \cos 100^\circ) = \underline{\underline{0.1475 \text{ Wb.}}}$$

Prob 7.23

Let $\vec{H} = \vec{H}_1 + \vec{H}_2$

where $\vec{H}_1$ and $\vec{H}_2$ are due to the wires centered at $x=0$ and $x=10\text{ cm}$ respectively.

(a) for $\vec{H}_1$, $\rho = 50\text{ cm}$, $\vec{a}_\phi = \vec{a}_z \times \vec{a}_\rho = \vec{a}_z \times \vec{a}_x = \vec{a}_y$

$$\vec{H}_1 = \frac{5}{2\pi (5 \times 10^{-2})} \vec{a}_y = \frac{50}{\pi} \vec{a}_y$$

for $\vec{H}_2$, $\rho = 5\text{ cm}$, $\vec{a}_\phi = -\vec{a}_z \times -\vec{a}_x = \vec{a}_y$, $\vec{H}_2 = \vec{H}_1$

$$\vec{H} = 2\vec{H}_1 = \frac{100}{\pi} \vec{a}_y = \underline{\underline{31.43 \vec{a}_y \text{ A/m.}}}$$

(b) for $\vec{H}_1$, $\vec{a}_\phi = \vec{a}_z \times \left(\frac{2\vec{a}_x + \vec{a}_y}{\sqrt{5}} \right) = \frac{2\vec{a}_y - \vec{a}_x}{\sqrt{5}}$

$$\vec{H}_1 = \frac{5}{2\pi 5\sqrt{5} \times 10^{-2}} \left(\frac{-\vec{a}_x + 2\vec{a}_y}{\sqrt{5}} \right) = -3.183\vec{a}_x + 6.366\vec{a}_y$$

for $\vec{H}_2$, $\vec{a}_\phi = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$

$$\vec{H}_2 = \frac{5}{2\pi (5)} \vec{a}_x = 15.924\vec{a}_x$$

$$\vec{H} = \vec{H}_1 + \vec{H}_2 = \underline{\underline{12.79\vec{a}_x + 6.366\vec{a}_y \text{ A/m.}}}$$

Prob. 7.24

The stroke of lightning may be regarded as a semi-infinite filamentary current.

$$B = \frac{\mu_0 I}{4\pi r} = \frac{4\pi \times 10^{-7} \times 50 \times 10^3}{4\pi (100)}$$

$$= \underline{\underline{50 \mu Wb/m^2}}$$

Prob. 7.25

(a) $\vec{B} = \frac{\mu_0 N I}{2\pi r}, \rho_0 - a < r < \rho_0 + a.$

from the figure,

$$r = \rho_0 + r \cos \theta$$

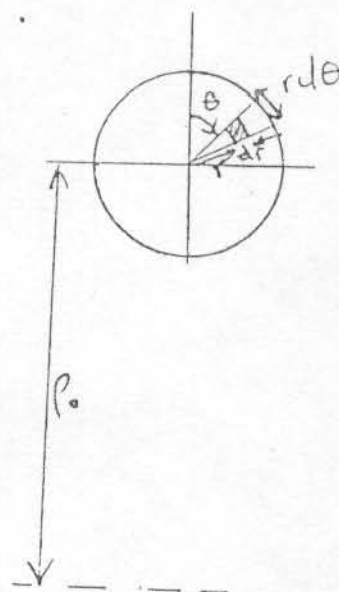
$$ds = r d\theta dr$$

$$\psi = \int \vec{B} \cdot d\vec{s} = \int_{r=0}^a \int_{\theta=0}^{2\pi} \frac{\mu_0 N I r d\theta dr}{2\pi (\rho_0 + r \cos \theta)}.$$

$$= \frac{\mu_0 N I}{2\pi} \int_0^a dr \int_0^{2\pi} \frac{r d\theta}{(\rho_0 + r \cos \theta)}$$

$$= \frac{\mu_0 N I}{2\pi} \int_0^a \frac{2\pi}{\sqrt{\rho_0^2 - r^2}} r dr = \mu_0 N I \left[-(\rho_0^2 - r^2)^{1/2} \right]_0^a$$

$$\psi = \mu_0 N I \left[\rho_0 - (\rho_0^2 - a^2)^{1/2} \right] \text{ as required.}$$



If $\rho_0 \gg a$, by binomial expansion

$$\rho_0 - (\rho_0^2 - a^2)^{1/2} = \rho_0 - \left(\rho_0 - \frac{1}{2} \frac{a^2}{\rho_0} + \dots \right) = \frac{a^2}{2\rho_0}$$

Hence

$$\psi = \frac{\mu_0 N I a^2}{2\rho_0}$$

$$\begin{aligned} (b) \psi &= \int \vec{B} \cdot d\vec{s} = \frac{\mu_0 N I}{2\pi} \int_{z=0}^5 \int_{\rho=\rho_0-\frac{a}{2}}^{\rho_0+\frac{a}{2}} \frac{d\rho dz}{\rho} \\ &= \frac{\mu_0 N I a}{2\pi} \ln \frac{\rho_0 + \frac{1}{2}a}{\rho_0 - \frac{1}{2}a} = \frac{\mu_0 N I a}{2\pi} \ln \frac{2\rho_0 + a}{2\rho_0 - a} \end{aligned}$$

Prob. 7.26

on the slant side of the ring, $z = \frac{h}{b} (\rho - a)$

$$\begin{aligned} \psi &= \int \vec{B} \cdot d\vec{s} = \int \frac{\mu_0 I}{2\pi \rho} d\rho dz \\ &= \frac{\mu_0 I}{2\pi} \int_{\rho=a}^{a+b} \int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz d\rho}{\rho} = \frac{\mu_0 I h}{2\pi b} \int_a^{a+b} \left(1 - \frac{a}{\rho}\right) d\rho \\ &= \frac{\mu_0 I h}{2\pi b} \left(b - a \ln \frac{a+b}{a}\right) \text{ as required.} \end{aligned}$$

If $a = 30\text{cm}$, $b = 10\text{cm}$, $h = 5\text{cm}$, $I = 10\text{A}$.

$$\begin{aligned} \psi &= \frac{4\pi \times 10^{-7} \times 10 \times 0.05}{2\pi \times 0.1} \left(0.1 - 0.3 \ln \frac{4}{3}\right) \\ &= 1.37 \times 10^{-8} \text{ Wb} \end{aligned}$$

Prob. 7.27

$$(a) \nabla \cdot \vec{A} = 0, \quad \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = 0$$

i.e. $\vec{A}$ is $\vec{E}$ -field in a charge-free region.

$$(b) \nabla \cdot \vec{B} = \frac{1}{(x^2 + y^2)^2} (-2xy + 2xy) = 0, \quad \nabla \times \vec{B} \neq 0$$

i.e. $\vec{B}$ is $\vec{H}$ -field.

$$(c) \nabla \cdot \vec{C} = e^{-y} (-\sin x + \sin x) = 0$$

$$\nabla \times \vec{C} = e^{-y} (-\cos x + \cos x) = 0$$

i.e. $\vec{C}$ is $\vec{E}$ -field in a charge-free region.

$$(d) \nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{D} = -10pe^{-2z} \neq 0$$

i.e. $\vec{D}$ is $\vec{H}$ -field.

$$(e) \nabla \cdot \vec{E} = \frac{4 \cos \theta}{r^2} \neq 0, \quad \nabla \times \vec{E} = \frac{2 \sin \theta}{r^2} \vec{a}_\phi \neq 0$$

i.e. $\vec{E}$ is neither electric nor magnetic.

Prob. 7.28

from eq. (7.44),
$$\vec{B} = \frac{\mu_0}{4\pi} \int_L \frac{I d\vec{l}' \times \vec{R}}{R^3}$$

Using this leads to eq. (7.47), namely

$$\vec{B} = -\frac{\mu_0}{4\pi} \int L d\vec{l}' \times \nabla\left(\frac{1}{r}\right)$$

$$\nabla \cdot \vec{B} = -\frac{\mu_0 I}{4\pi} \nabla \cdot \left[\int d\vec{l}' \times \nabla\left(\frac{1}{r}\right) \right]$$

Since $\int d\vec{l}'$ operates on (x', y', z') and ∇ operates on (x, y, z) , the two operators can be reversed in order, i.e.

$$\nabla \cdot \vec{B} = -\frac{\mu_0 I}{4\pi} \int \nabla \cdot \left[d\vec{l}' \times \nabla\left(\frac{1}{r}\right) \right]$$

$$\text{But } \nabla \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{C}).$$

$$\text{Taking } \vec{A} = d\vec{l}' \text{ and } \vec{C} = \nabla\left(\frac{1}{r}\right),$$

$$\nabla \cdot \vec{B} = -\frac{\mu_0 I}{4\pi} \int \left[\nabla\left(\frac{1}{r}\right) \cdot (\nabla \times d\vec{l}') - d\vec{l}' \cdot (\nabla \times \nabla\left(\frac{1}{r}\right)) \right]$$

$\nabla \times \nabla\left(\frac{1}{r}\right) = 0$

Since ∇ operates on (x, y, z) and $d\vec{l}'$ depends on (x', y', z') , $\nabla \times d\vec{l}' = 0$. Also, since $\nabla \times \nabla V = 0$, $\nabla \times \nabla\left(\frac{1}{r}\right) = 0$. Thus

$$\nabla \cdot \vec{B} = 0$$

Prob. 7.29

$$(a) \vec{B} = \nabla \times \vec{A} = - (e^{-z} \cos y + \cos x) \vec{a}_y + e^{-z} \sin y \vec{a}_z \text{ Wb/m}^2.$$

$$(b) \Psi = \int_0^\pi \int_0^\pi e^{-z} \sin y \, dx \, dy \Big|_{z=0} = (1)(\pi) (-\cos \pi + \cos 0) = 2\pi = \underline{6.283 \text{ Wb}}.$$

Prob. 7.30

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} \vec{a}_\rho - \frac{\partial A_z}{\partial \rho} \vec{a}_\phi$$

$$= \frac{15}{\rho} e^{-\rho} \cos \phi \vec{a}_\rho + 15 e^{-\rho} \sin \phi \vec{a}_\phi$$

$$\vec{B}(3, \frac{\pi}{4}, -10) = 5 e^{-3} \frac{1}{\sqrt{2}} \vec{a}_\rho + 15 e^{-3} \frac{1}{\sqrt{2}} \vec{a}_\phi$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{10^7}{4\pi} \frac{15}{\sqrt{2}} e^{-3} \left(\frac{1}{3} \vec{a}_\rho + \vec{a}_\phi \right)$$

$$\vec{H} = \underline{\underline{(14 \vec{a}_\rho + 42 \vec{a}_\phi) \cdot 10^4 \text{ A/m.}}}$$

$$\psi = \int \vec{B} \cdot d\vec{S} = \iint \frac{15}{\rho} e^{-\rho} \cos \phi \rho d\phi d\tau$$

$$= 15\tau \Big|_0^{10} (-\sin \phi) \Big|_0^{\pi/2} e^{-5} = -150 e^{-5}$$

$$\psi = \underline{\underline{-1.011 \text{ Wb.}}}$$

Prob. 7.31

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}}{R}$$

$\vec{R} = (x, y, z) - (0, 0, z')$, $R = \sqrt{x^2 + y^2 + (z - z')^2}$. At a distant point, $z \gg z'$ so that

$$R \approx \sqrt{x^2 + y^2 + z^2} = r$$

Hence,
$$\vec{A} = \frac{\mu_0 I}{4\pi r} \int_{-L}^L dz' \vec{a}_z = \frac{\mu_0 I L}{2\pi r} \vec{a}_z$$

$$\vec{A} = \frac{\mu_0 I L}{2\pi (x^2 + y^2 + z^2)^{3/2}} \vec{a}_z$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I L}{2\pi} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r} \end{vmatrix}$$

$$\frac{\partial}{\partial y} \left(\frac{1}{r} \right) = -\frac{y}{r^3}, \quad \frac{\partial}{\partial x} \left(\frac{1}{r} \right) = -\frac{x}{r^3}$$

$$\vec{B} = \frac{\mu_0 I L}{2\pi r^3} (-y \vec{a}_x + x \vec{a}_y) = \frac{\mu_0 I L (-y \vec{a}_x + x \vec{a}_y)}{2\pi (x^2 + y^2 + z^2)^{3/2}}$$

Prob. 7.32

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu_0} = \frac{I}{2\pi \rho} \vec{a}_\phi$$

$$\nabla \times \vec{A} = \frac{\mu_0 I}{2\pi \rho} \vec{a}_\phi = \left[\frac{\partial A_\phi}{\partial z} - \frac{\partial A_z}{\partial \phi} \right] \vec{a}_\phi$$

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi \rho} \rightarrow A_z = -\frac{\mu_0 I}{2\pi} \ln \rho$$

$$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln \rho \vec{a}_z$$

Prob. 7.33

$$\vec{A} = -\frac{\mu_0 \mu_0}{4\pi a^2} (x^2 + y^2) \vec{a}_z = -\frac{\mu_0 \mu_0 \rho^2}{4\pi a^2} \vec{a}_z$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 \mu_0 \rho}{2\pi a^2} \vec{a}_\phi = \mu_0 \vec{H}$$

$$\text{i.e. } \vec{H} = \frac{\mu_0 \rho}{2\pi a^2} \vec{a}_\phi = \frac{\mu_0 \sqrt{x^2 + y^2}}{2\pi a^2} \vec{a}_\phi$$

By Ampere's law, $\oint \vec{H} \cdot d\vec{l} = I_{enc}$

$$\mu_0 \cdot 2\pi \rho = I_0 \cdot \frac{\rho^2}{a^2}$$

$$\Rightarrow \vec{H} = \frac{I_0 \rho}{2\pi a^2} \vec{a}_\phi$$

Prob. 7.34

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \frac{\nabla \times \vec{A}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial \rho} \vec{a}_\phi = +\frac{20}{\rho^3} \vec{a}_\phi$$

$$\nabla \times \nabla \times \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) \vec{a}_z = -\frac{40}{\rho^4} \vec{a}_z$$

$$\vec{J} = -\frac{40}{\mu_0 \rho^4} \vec{a}_z \text{ A/m}^2$$

$$\propto \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\Rightarrow \vec{J} = -\frac{1}{\mu_0} \nabla^2 \vec{A} = -\frac{1}{\mu_0} \nabla^2 A_z \vec{a}_z$$

$$= -\frac{1}{\mu_0} \vec{a}_z \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} \right]$$

$$= \frac{1}{\mu_0} \vec{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(-\frac{20}{\rho^2} \right) = -\frac{40 \vec{a}_z}{\mu_0 \rho^4} \text{ A/m}^2$$

Prob. 7.35

From Fig. 7.5, $d\vec{A} = \frac{\mu_0 I d\vec{l}}{4\pi R}$

where $\vec{dl} = dz \vec{a}_z$, $R = \sqrt{\rho^2 + z^2}$.

$$d\vec{A} = \frac{\mu_0 I dz \vec{a}_z}{4\pi (\rho^2 + z^2)^{3/2}}$$

$$d\vec{B} = \nabla \times d\vec{A} = -\frac{\partial}{\partial \rho} dA_z \vec{a}_\rho = \frac{\mu_0 I dz}{4\pi} \frac{\rho}{(\rho^2 + z^2)^{3/2}} \vec{a}_\rho$$

$$\vec{B} = \frac{\mu_0 I \vec{a}_\rho}{4\pi} \int \frac{\rho dz}{(\rho^2 + z^2)^{3/2}}$$

Let $z = \rho \cot \alpha$, $dz = \rho \operatorname{cosec}^2 \alpha d\alpha$

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{I \vec{a}_\rho}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2 \alpha d\alpha}{\rho^3 \operatorname{cosec}^3 \alpha}$$

$$= \frac{I}{4\pi} (\cos \alpha_2 - \cos \alpha_1) \vec{a}_\rho.$$

Prob. 7.36

$$\vec{H} = -\nabla V_m \rightarrow V_m = -\int \vec{H} \cdot d\vec{l}$$

from Example 7.3, $\vec{H} = \frac{Ia^2}{2(z^2 + a^2)^{3/2}} \vec{a}_z$

$$V_m = -\frac{Ia^2}{2} \int (z^2 + a^2)^{-3/2} dz = \frac{-Iz}{2(z^2 + a^2)^{1/2}} + C$$

As $z \rightarrow \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + C \rightarrow C = \frac{I}{2}$$

Hence,

$$V_m = \frac{I}{2} \left[1 - \frac{z}{\sqrt{z^2 + a^2}} \right].$$

Prob. 7.37

for the outer conductor,

$$J_z = \frac{-I}{\pi(c^2 - b^2)} = \frac{-I}{\pi(16 - 9)a^2} = \frac{-I}{7\pi a^2}$$

Let $\vec{A} = A_z \vec{a}_z$. Using Poisson's equation,

$$\nabla^2 A_z = -\mu_0 J_z$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) = \frac{\mu_0 I}{7a^2 \pi}$$

$$\therefore \frac{d}{d\rho} \left(\rho \frac{dA_z}{d\rho} \right) = \frac{\mu_0 I \rho}{7\pi a^2}$$

Integrating once,

$$\rho \frac{dA_z}{d\rho} = \frac{\mu_0 I \rho^2}{14\pi a^2} + C_1$$

$$\text{or } \frac{dA_z}{d\rho} = \frac{\mu_0 I \rho}{14\pi a^2} + \frac{C_1}{\rho}$$

Integrating again,

$$A_z = \frac{\mu_0 I \rho^2}{28\pi a^2} + C_1 \ln \rho + C_2$$

But $A_z = 0$ when $\rho = 3a$.

$$0 = \frac{9}{28\pi} \mu_0 I + C_1 \ln 3a + C_2$$

$$C_2 = -C_1 \ln 3a - \frac{9}{28\pi} \mu_0 I$$

$$\text{i.e. } A_z = \frac{\mu_0 I}{28\pi} \left(\frac{\rho^2}{a^2} - 9 \right) + C_1 \ln \frac{\rho}{3a}$$

But $\nabla \times \vec{A} = \vec{B} = \mu_0 \vec{H}$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial \phi} \vec{a}_\phi = -\left(\frac{\mu_0 I \rho}{14\pi a^2} + \frac{C_1}{\rho}\right) \vec{a}_\phi$$

At $\rho = 3a$, $\int \vec{H} \cdot d\vec{l} = I \rightarrow 2\pi(3a) H_\phi = I$

or $H_\phi = \frac{I}{6\pi a}$

Thus $\nabla \times \vec{A}|_{\rho=3a} = \mu_0 \vec{H}(\rho=3a)$ implies that

$$-\left(\frac{3\mu_0 I}{14\pi a} + \frac{C_1}{3a}\right) = \frac{\mu_0 I}{6\pi a}$$

$$\text{or } C_1 = -\frac{1\mu_0 I}{2\pi} - \frac{9\mu_0 I}{14\pi} = -\frac{16\mu_0 I}{14\pi}$$

Thus,

$$A_z = \frac{\mu_0 I}{28\pi} \left(\frac{\rho^2}{a^2} - 9 \right) - \frac{8\mu_0 I}{7\pi} \ln \frac{\rho}{3a}$$

Prob. 7-38

for an infinite line current, $\vec{H} = \frac{I}{2\pi \rho} \vec{a}_\phi$

But $\vec{H} = -\nabla V_m$ ($\vec{J} = 0$)

$$\frac{I}{2\pi \rho} \vec{a}_\phi = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi} \vec{a}_\phi \rightarrow V_m = -\frac{I}{2\pi} \phi + C$$

(a) If $V_m = 0$ at $\phi = 0 \rightarrow C = 0$

$$V(6, \pi/4, 0) = -\frac{20}{2\pi} \cdot \frac{\pi}{4} = \underline{\underline{-2.5 \text{ A}}}$$

$$+ \frac{1}{\rho} \left(\frac{\partial^2 V}{\partial \rho \partial \phi} - \frac{\partial^2 V}{\partial \phi \partial \rho} \right) \bar{a}_z = 0$$

$$\begin{aligned} (b) \nabla \cdot (\nabla \times \vec{A}) &= \nabla \cdot \left[\left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \bar{a}_\rho \right. \\ &\quad \left. + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \bar{a}_\phi + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \bar{a}_z \right] \\ &= \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \rho \partial \phi} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_\phi}{\partial z} \right) + \frac{1}{\rho} \frac{\partial^2 A_\rho}{\partial \rho \partial z} - \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \phi \partial \rho} \\ &\quad + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) \right) - \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \\ &= -\frac{\partial^2 A_\phi}{\partial \rho \partial z} - \frac{1}{\rho} \frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial z \partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial z} = 0 \end{aligned}$$

Prob. 7.41

$$\begin{aligned} (a) \nabla \times \psi \vec{F} &= \left(\frac{\partial}{\partial y} (\psi F_z) - \frac{\partial}{\partial z} (\psi F_y) \right) \bar{a}_x \\ &\quad + \left(\frac{\partial}{\partial z} (\psi F_x) - \frac{\partial}{\partial x} (\psi F_z) \right) \bar{a}_y \\ &\quad + \left(\frac{\partial}{\partial x} (\psi F_y) - \frac{\partial}{\partial y} (\psi F_x) \right) \bar{a}_z \\ &= \psi \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \bar{a}_x + \psi \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \bar{a}_y \\ &\quad + \psi \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \bar{a}_z + \left(F_z \frac{\partial \psi}{\partial y} - F_y \frac{\partial \psi}{\partial z} \right) \bar{a}_x \\ &\quad + \left(F_x \frac{\partial \psi}{\partial z} - F_z \frac{\partial \psi}{\partial x} \right) \bar{a}_y \\ &\quad + \left(F_y \frac{\partial \psi}{\partial x} - F_x \frac{\partial \psi}{\partial y} \right) \bar{a}_z \end{aligned}$$

$$+ \frac{1}{\rho} \left(\frac{\partial^2 V}{\partial \rho \partial \phi} - \frac{\partial^2 V}{\partial \phi \partial \rho} \right) \bar{e}_z = 0$$

$$\begin{aligned} (b) \nabla \cdot (\nabla \times \vec{A}) &= \nabla \cdot \left[\left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \bar{e}_\rho \right. \\ &\quad \left. + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \bar{e}_\phi + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \bar{e}_z \right] \\ &= \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \rho \partial \phi} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_\phi}{\partial z} \right) + \frac{1}{\rho} \frac{\partial^2 A_\rho}{\partial \phi \partial z} - \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \phi \partial \rho} \\ &\quad + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) \right) - \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \\ &= -\frac{\partial^2 A_\phi}{\partial \rho \partial z} - \frac{1}{\rho} \frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial z \partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial z} = 0 \end{aligned}$$

Prob. 7.41

$$\begin{aligned} (a) \nabla \times \psi \vec{F} &= \left(\frac{\partial}{\partial y} (\psi F_z) - \frac{\partial}{\partial z} (\psi F_y) \right) \bar{e}_x \\ &\quad + \left(\frac{\partial}{\partial z} (\psi F_x) - \frac{\partial}{\partial x} (\psi F_z) \right) \bar{e}_y \\ &\quad + \left(\frac{\partial}{\partial x} (\psi F_y) - \frac{\partial}{\partial y} (\psi F_x) \right) \bar{e}_z \\ &= \psi \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \bar{e}_x + \psi \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \bar{e}_y \\ &\quad + \psi \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \bar{e}_z + \left(F_z \frac{\partial \psi}{\partial y} - F_y \frac{\partial \psi}{\partial z} \right) \bar{e}_x \\ &\quad + \left(F_x \frac{\partial \psi}{\partial z} - F_z \frac{\partial \psi}{\partial x} \right) \bar{e}_y + \left(F_y \frac{\partial \psi}{\partial x} - F_x \frac{\partial \psi}{\partial y} \right) \bar{e}_z \end{aligned}$$

$$+ \left(f_x \frac{\partial \psi}{\partial z} - f_z \frac{\partial \psi}{\partial x} \right) \bar{a}_y + \left(f_y \frac{\partial \psi}{\partial x} - f_x \frac{\partial \psi}{\partial y} \right) \bar{a}_z$$

$$= \nabla \times \vec{F} + \nabla \psi \times \vec{F}$$

$$(b) \nabla \times \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{vmatrix}$$

$$= \left(\frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial y^2} \right) \bar{a}_x$$

$$- \left(\frac{\partial^2 A_y}{\partial z^2} + \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_x}{\partial x \partial y} \right) \bar{a}_y$$

$$- \left(\frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_y}{\partial z \partial y} - \frac{\partial^2 A_z}{\partial x \partial z} \right) \bar{a}_z$$

$$\nabla(\nabla \cdot \vec{A}) = \left(\frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z \right) \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} \right) \bar{a}_x$$

$$+ \left(\frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z} \right) \bar{a}_y$$

$$+ \left(\frac{\partial^2 A_x}{\partial z \partial x} + \frac{\partial^2 A_y}{\partial z \partial y} + \frac{\partial^2 A_z}{\partial z^2} \right) \bar{a}_z$$

$$\nabla^2 \vec{A} = \nabla^2 A_x \bar{a}_x + \nabla^2 A_y \bar{a}_y + \nabla^2 A_z \bar{a}_z$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) \bar{a}_x +$$

$$\left(\frac{\partial^2 \tilde{A}_y}{\partial x^2} + \frac{\partial^2 \tilde{A}_y}{\partial y^2} + \frac{\partial^2 \tilde{A}_y}{\partial z^2} \right) \tilde{a}_y + \left(\frac{\partial^2 \tilde{A}_z}{\partial x^2} + \frac{\partial^2 \tilde{A}_z}{\partial y^2} + \frac{\partial^2 \tilde{A}_z}{\partial z^2} \right) \tilde{a}_z$$

$$\nabla(\nabla \cdot \tilde{\mathbf{A}}) - \nabla^2 \tilde{\mathbf{A}} = \left(\frac{\partial^2 \tilde{A}_y}{\partial x \partial y} + \frac{\partial^2 \tilde{A}_z}{\partial x \partial z} - \frac{\partial^2 \tilde{A}_x}{\partial y^2} - \frac{\partial^2 \tilde{A}_z}{\partial z^2} \right) \tilde{a}_x$$

$$+ \left(\frac{\partial^2 \tilde{A}_x}{\partial x \partial y} + \frac{\partial^2 \tilde{A}_z}{\partial z \partial y} - \frac{\partial^2 \tilde{A}_y}{\partial x^2} - \frac{\partial^2 \tilde{A}_y}{\partial z^2} \right) \tilde{a}_z$$

$$= \nabla \times \nabla \times \tilde{\mathbf{A}}.$$

Alternatively, we may use the bac-cab rule:

$$\tilde{\mathbf{A}} \times (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}}) = \tilde{\mathbf{B}} (\tilde{\mathbf{A}} \cdot \tilde{\mathbf{C}}) - \tilde{\mathbf{C}} (\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}})$$

and let $\tilde{\mathbf{A}} \rightarrow \nabla$, $\tilde{\mathbf{B}} \rightarrow \nabla$, $\tilde{\mathbf{C}} \rightarrow \tilde{\mathbf{A}}$, we obtain

$$\nabla \times \nabla \times \tilde{\mathbf{A}} = \nabla(\nabla \cdot \tilde{\mathbf{A}}) - \tilde{\mathbf{A}} (\nabla \cdot \nabla)$$

$$= \nabla(\nabla \cdot \tilde{\mathbf{A}}) - \nabla^2 \tilde{\mathbf{A}}.$$

Prob. 7.42

$$R = |\tilde{\mathbf{r}} - \tilde{\mathbf{r}}'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{1/2}$$

$$\nabla \frac{1}{R} = \left(\frac{\partial}{\partial x} \tilde{a}_x + \frac{\partial}{\partial y} \tilde{a}_y + \frac{\partial}{\partial z} \tilde{a}_z \right) \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-1/2}$$

$$= -\frac{1}{2} 2(x-x') \tilde{a}_x \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-3/2}$$

$$+ \dots + \dots$$

$$= - \left[(x-x') \tilde{a}_x + (y-y') \tilde{a}_y + (z-z') \tilde{a}_z \right] \frac{1}{R^3} = -\frac{\tilde{\mathbf{R}}}{R^3}.$$

$$\begin{aligned}\nabla' \frac{1}{R} &= \left(\frac{\partial}{\partial x'} \vec{a}_x + \frac{\partial}{\partial y'} \vec{a}_y + \frac{\partial}{\partial z'} \vec{a}_z \right) \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-1/2} \\ &= \left(-\frac{1}{2} \right) (-2) (x-x') \vec{a}_x \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-3/2} \\ &= \frac{\vec{R}}{R^3}.\end{aligned}$$

CHAPTER 8

P.E. 8.1

(a) $F = m \frac{d\vec{u}}{dt} = Q\vec{E} = \underline{\underline{6\vec{a}_z N}}$

(b) $\frac{d\vec{u}}{dt} = 6\vec{a}_z = \frac{d}{dt} (u_x, u_y, u_z) \Rightarrow$

$$\frac{du_x}{dt} = 0 \quad \rightarrow \quad u_x = A$$

$$\frac{du_y}{dt} = 0 \quad \rightarrow \quad u_y = B.$$

$$\frac{du_z}{dt} = 6 \quad \rightarrow \quad u_z = 6t + C$$

Since $\vec{u}(t=0) = 0$, $A = B = C = 0$

$$u_x = 0 = u_y, \quad u_z = 6t$$

$$u_x = \frac{dx}{dt} = 0 \quad \rightarrow \quad x = A_1$$

$$u_y = \frac{dy}{dt} = 0 \quad \rightarrow \quad y = B_1$$

$$u_z = \frac{dz}{dt} = 6t \rightarrow z = 3t^2 + c_1$$

At $t=0$, $(x, y, z) = (0, 0, 0) \rightarrow A_1 = 0 = B_1 = c_1$

Hence $(x, y, z) = (0, 0, 3t^2)$, $\vec{u} = 6t \vec{a}_z$ at any time. At $P(0, 0, 12)$, $z = 12 = 3t^2 \rightarrow t = 2s$

$t = 2s$

(c) $\vec{u} = 6t \vec{a}_z = \underline{12 \vec{a}_z \text{ m/s}}$.

$\vec{a} = \frac{d\vec{u}}{dt} = \underline{6 \vec{a}_z \text{ m/s}^2}$.

(d) $K \cdot E = \frac{1}{2} m |\vec{u}|^2 = \frac{1}{2} (1)(144) = \underline{72 \text{ J}}$.

P. E. 8.2

(a) $m \vec{a} = e \vec{u} \times \vec{B} = (e B_0 u_y, -e B_0 u_x, 0)$

$\frac{d^2 x}{dt^2} = \frac{e B_0}{m} \frac{dy}{dt} = \omega \frac{dy}{dt} \quad \text{--- (1)}$

$\frac{d^2 y}{dt^2} = -\frac{e B_0}{m} \frac{dx}{dt} = -\omega \frac{dx}{dt} \quad \text{--- (2)}$

$\frac{d^2 z}{dt^2} = 0 \rightarrow \frac{dz}{dt} = c_1 \quad \text{--- (3)}$

from (1) and (2),

$\frac{d^3 x}{dt^3} = \omega \frac{d^2 y}{dt^2} = -\omega^2 \frac{dx}{dt}$

$(D^3 + \omega^2 D)X = 0 \rightarrow DX = (0, \pm j\omega)X$

$$x = c_2 + c_3 \cos \omega t + c_4 \sin \omega t$$

$$\frac{dx}{dt} = -c_3 \omega \sin \omega t + c_4 \omega \cos \omega t$$

$$\frac{dy}{dt} = \frac{1}{\omega} \frac{d^2 x}{dt^2} = -c_3 \omega \cos \omega t - c_4 \omega \sin \omega t$$

At $t=0$, $\vec{u} = (\alpha, 0, \beta)$. Hence

$$c_1 = \beta, \quad c_3 = 0, \quad c_4 = \frac{\alpha}{\omega}$$

$$\underline{\underline{\frac{dx}{dt} = \alpha \cos \omega t, \quad \frac{dy}{dt} = -\alpha \sin \omega t, \quad \frac{dz}{dt} = \beta.}}$$

(b) Solving these yields

$$x = \frac{\alpha}{\omega} \sin \omega t, \quad y = \frac{\alpha}{\omega} \cos \omega t, \quad z = \beta t.$$

$$(c) \quad x^2 + y^2 = \frac{\alpha^2}{\omega^2}, \quad z = \beta t$$

showing that the particles move along a helix of radius α/ω placed along the z -axis.

P.E. 8.3

(a) From Example 8.3, $Q\omega B = QE$ regardless of the sign of the charge.

$$E = \omega B = 8 \times 10^6 \times 0.5 \times 10^{-3} = \underline{\underline{4 \text{ kV/m}}}.$$

(b) Yes, since $Q\omega B = QE$ holds for any Q and m .

P.E. 8.4

By Newton's 3rd law, $\vec{F}_{12} = -\vec{F}_{21}$, the force on the infinitely long wire is

$$\vec{F}_L = -\vec{F} = \frac{\mu_0 I_1 I_2 b}{2\pi} \left(\frac{1}{\rho_0} - \frac{1}{\rho_0 + a} \right) \vec{a}_\rho$$

$$= \frac{4\pi \times 10^{-7} \times 50 \times 3}{2\pi} \left(\frac{1}{2} - \frac{1}{3} \right) \vec{a}_\rho = \underline{\underline{5 \vec{a}_\rho \mu\text{N}}}$$

P.E. 8.5

$$\vec{m} = IS \vec{a}_n = 10 \times 10^{-4} \times 50 \frac{(2, 6, -3)}{7}$$

$$= 7.143 \times 10^{-3} (2, 6, -3)$$

$$= \underline{\underline{(-1.429 \vec{a}_x + 4.286 \vec{a}_y - 2.143 \vec{a}_z) \times 10^{-2} \text{ A}\cdot\text{m}^2}}$$

P.E. 8.6

$$(a) \vec{T} = \vec{m} \times \vec{B} = \frac{10 \times 10^{-4} \times 50}{7 \times 10} \begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= \underline{\underline{0.03 \vec{a}_x - 0.02 \vec{a}_y - 0.02 \vec{a}_z \text{ N}\cdot\text{m}}}$$

$$(b) |\vec{T}| = ISB \sin \theta \rightarrow |\vec{T}|_{\max} = ISB$$

$$|\vec{T}|_{\max} = \frac{50 \times 10^{-2}}{10} |6 \vec{a}_x + 4 \vec{a}_y + 5 \vec{a}_z| = 0.4387$$

$$\text{or } |\vec{T}|_{\max} = |\vec{m} \times \vec{B}| = |-0.3055 \vec{a}_x + 0.076 \vec{a}_y + 0.3055 \vec{a}_z|$$

$$= \underline{\underline{0.4387 \text{ N}\cdot\text{m}}}$$

P.E. 8.7

$$(a) \mu_r = \frac{\mu}{\mu_0} = 4.6, \quad \chi_m = \mu_r - 1 = \underline{\underline{3.6}}$$

$$(b) \vec{H} = \frac{\vec{B}}{\mu} = \frac{10 \times 10^{-3} e^{-y}}{4\pi \times 10^{-7} \times 4.6} \hat{a}_z \text{ A/m} = \underline{\underline{1730 e^{-y} \hat{a}_z \text{ A/m}}}$$

$$(c) \vec{M} = \chi_m \vec{H} = \underline{\underline{6228 e^{-y} \hat{a}_z \text{ A/m}}}$$

P.E. 8.8

$$\vec{a}_n = \frac{3\vec{a}_x + 4\vec{a}_y}{5}$$

$$\vec{B}_{1n} = (\vec{B}_1 \cdot \vec{a}_n) \vec{a}_n = \frac{(6+32)(6\vec{a}_x + 8\vec{a}_y)}{1000}$$

$$= 0.228\vec{a}_x + 0.304\vec{a}_y = \vec{B}_{2n}$$

$$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n} = -0.128\vec{a}_x + 0.096\vec{a}_y + 0.2\vec{a}_z$$

$$\vec{B}_{2t} = \frac{\mu_2}{\mu_1} \vec{B}_{1t} = 10\vec{B}_{1t} = -1.28\vec{a}_x + 0.96\vec{a}_y + 2\vec{a}_z \text{ Wb/m}^2$$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \underline{\underline{-1.052\vec{a}_x + 1.264\vec{a}_y + 2\vec{a}_z \text{ Wb/m}^2}}$$

P.E. 8.9

$$(a) \vec{B}_{1n} = \vec{B}_{2n} \rightarrow \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$$

$$\sim \mu_1 \vec{H}_1 \cdot \vec{a}_{n21} = \mu_2 \vec{H}_2 \cdot \vec{a}_{n21}$$

$$\mu_0 \frac{(60+2-36)}{7} = 2\mu_0 \frac{(6H_{2x} - 10-12)}{7}$$

$$35 = 6H_{2x}$$

$$\underline{H_{2x} = 5.833}$$

$$\begin{aligned} \text{(b)} \quad \vec{K} &= (\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \vec{a}_{n21} \times (\vec{H}_1 - \vec{H}_2) \\ &= \vec{a}_{n21} \times \left[(10, 1, 12) - \left(\frac{35}{6}, -5, 4 \right) \right] \\ &= \frac{1}{7} \begin{vmatrix} 6 & 2 & -3 \\ 25 & 6 & 8 \\ \frac{25}{6} & 6 & 8 \end{vmatrix} \end{aligned}$$

$$\underline{\vec{K} = 4.86 \vec{a}_x - 8.64 \vec{a}_y + 3.95 \vec{a}_z \text{ A/m.}}$$

(c) Since $\vec{B} = \mu \vec{H}$, $\vec{B}_1$ and $\vec{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos \theta_1 = \frac{\vec{H}_1 \cdot \vec{a}_{n21}}{|\vec{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\underline{\theta_1 = 76.27^\circ}$$

$$\cos \theta_2 = \frac{\vec{H}_2 \cdot \vec{a}_{n21}}{|\vec{H}_2|} = \frac{13}{7\sqrt{(5.833)^2 + 25 + 16}} = 0.2144$$

$$\underline{\theta_2 = 77.62^\circ}$$

P.E. 8.10

$$\begin{aligned} \text{(a)} \quad L' &= \mu_0 \mu_r n^2 S = 4\pi \times 10^{-7} \times 1000 \times 16 \times 10^6 \times 4 \times 10^{-4} \\ &= \underline{8.042 \text{ H/m}} \end{aligned}$$

$$\text{(b)} \quad W_m' = \frac{1}{2} L' I^2 = \frac{1}{2} (8.042) (6.5)^2 = \underline{1.605 \text{ J/m.}}$$

P.E. 8.11

from Example 8.11,

$$L_{in} = \frac{2l}{8\pi}$$

$$L_{ext} = \frac{2 W_m}{I^2} = \frac{1}{I^2} \iiint \frac{\mu I^2}{4\pi^2 \rho^2} \rho d\rho d\phi dz$$

$$= \frac{1}{4\pi^2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_0}{(1+\rho)\rho} d\rho$$

$$= \frac{2\mu_0}{4\pi^2} \cdot 2\pi l \int_a^b \left[\frac{1}{\rho} - \frac{1}{1+\rho} \right] d\rho$$

$$= \frac{\mu_0 l}{\pi} \left[\ln \frac{b}{a} - \ln \frac{1+b}{1+a} \right]$$

$$L = L_{in} + L_{ext} = \frac{\mu_0 l}{8\pi} + \frac{\mu_0 l}{\pi} \left[\ln \frac{b}{a} - \ln \frac{(1+b)}{(1+a)} \right]$$

P.E. 8.12

$$(a) L'_{in} = \frac{\mu_0}{8\pi} = \frac{4\pi \times 10^{-7}}{8\pi} = \underline{\underline{0.05 \mu H/m.}}$$

$$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{\underline{1.15 \mu H/m.}}$$

$$(b) L' = \frac{\mu_0}{2\pi} \left[\frac{1}{4} + \ln \frac{d-a}{a} \right]$$

$$\begin{aligned} \ln \frac{d-a}{a} &= \frac{2\pi l'}{\mu_0} - 0.25 = \frac{2\pi \times 1.2 \times 10^{-6}}{4\pi \times 10^{-7}} - 0.25 \\ &= 6 - 0.25 = 5.75 \end{aligned}$$

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$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d-a = 314.19 a = 314.19 \times \frac{2.588 \times 10^{-3}}{2} = 406.6 \text{ mm}$$

$$d = 407.9 \text{ mm} = \underline{\underline{40.79 \text{ cm.}}}$$

P.E. 8.13

This is similar to Example 8.13. In this case, however, $h=0$ so that

$$\bar{A}_1 = \frac{\mu_0 I_1 a^2 b}{4b^3} \bar{a}_1$$

$$\Phi_{12} = \frac{\mu_0 I_1 a^2}{4b^2} \cdot 2\pi b = \frac{\mu_0 \pi I_1 a^2}{2b}$$

$$M_{12} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 \pi a^2}{b} = \frac{4\pi \times 10^{-7} \times \pi \times 4}{2 \times 3}$$

$$= \underline{\underline{2.632 \mu\text{H.}}}$$

P.E. 8.14

$$L_{in} = \frac{\mu_0 l}{8\pi} = \frac{\mu_0 2\pi \rho_0}{8\pi} = \frac{4\pi \times 10^{-7} \times 10 \times 10^{-2}}{4}$$

$$= \underline{\underline{31.42 \text{ nH.}}}$$

P.E. 8.15

(a) from Example 7.6, $B_{ave} = \frac{\mu_0 N I}{L} = \frac{\mu_0 N I}{2\pi \rho_0}$

$$\Phi = B_{\text{ave}} \cdot S = \frac{\mu_0 N I}{2\pi r_0} \cdot \pi a^2$$

$$\begin{aligned} \therefore I &= \frac{2\pi r_0 \Phi}{\mu_0 a^2 N} = \frac{2 \times 10 \times 10^{-2} \times 0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-4} \times 10^3} \\ &= \underline{\underline{795.77 \text{ A}}} \end{aligned}$$

Alternatively, using circuit approach

$$R = \frac{L}{\mu S} = \frac{2\pi r_0}{\mu_0 S} = \frac{2\pi r_0}{\mu_0 \cdot \pi a^2}$$

$$\mathcal{F} = NI = \Phi R \rightarrow I = \frac{\Phi R}{N} = \frac{2\pi r_0 \Phi}{\mu_0 a^2 N}$$

as obtain before.

$$R = \frac{2\pi r_0}{\mu_0 a^2} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^{-4}} = 1.591 \times 10^9$$

$$\mathcal{F} = \Phi R = 0.5 \times 10^{-3} \times 1.591 \times 10^9 = 7.955 \times 10^5$$

$$I = \frac{\mathcal{F}}{N} = 795 \text{ A} \quad \text{as obtained before.}$$

(b) If $\mu = 500\mu_0$, $I = \frac{795.77}{500} = \underline{\underline{1.592 \text{ A}}}$.

P.E. 8.16

$$\begin{aligned} f &= \frac{B_a^2 S}{2\mu_0} = \frac{(1.5)^2 \times 10 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = \frac{22500}{8\pi} \\ &= \underline{\underline{895.25 \text{ N}}} \end{aligned}$$

Prob. 8.1

$$(a) \vec{F}_e = q\vec{E} = \underline{10\vec{a}_x - 30\vec{a}_y + 80\vec{a}_z \text{ N.}}$$

$$(b) \vec{F}_m = q\vec{u} \times \vec{B} = 10 \begin{vmatrix} 2 & 0 & -4 \\ 0.3 & 0.1 & 0 \end{vmatrix} = \underline{4\vec{a}_x - 12\vec{a}_y + 2\vec{a}_z \text{ N.}}$$

$$(c) \vec{F} = \vec{F}_e + \vec{F}_m = \underline{14\vec{a}_x - 42\vec{a}_y + 82\vec{a}_z \text{ N.}}$$

Prob. 8.2

from Lorentz force equation,

$$\vec{F} = q\vec{u} \times \vec{B}$$

$$dW = \vec{F} \cdot d\vec{l} = q \left(\frac{d\vec{E}}{dt} \times \vec{B} \right) \cdot d\vec{l} = 0$$

since $d\vec{l} \times \vec{B}$ is perpendicular to $d\vec{l}$. Thus a magnetic field does not contribute to the work done on a charged particle undergoing displacement $d\vec{l}$.

Prob. 8.3

$$(a) \vec{a} = \frac{q\vec{E}}{m} = \frac{3}{2} (12\vec{a}_x + 10\vec{a}_y) = \underline{18\vec{a}_x + 15\vec{a}_y \text{ m/s}^2}$$

$$(b) \vec{a} = \left(\frac{du_x}{dt}, \frac{du_y}{dt}, \frac{du_z}{dt} \right) = 18\vec{a}_x + 15\vec{a}_y$$

$$\frac{du_x}{dt} = 18 \rightarrow u_x = 18t + A$$

$$\frac{du_y}{dt} = 15 \rightarrow u_y = 15t + B$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C.$$

At $t=0$, $\vec{u} = (4, 0, 3)$, hence $A=4$, $B=0$, $C=3$

and $\vec{u} = (18t+4, 15t, 3)$

$$\vec{u}(t=1s) = \underline{22\vec{a}_x + 15\vec{a}_y + 3\vec{a}_z \text{ m/s.}}$$

$$(c) K.E. = \frac{1}{2} m |\vec{u}|^2 = \frac{1}{2} (2) (22^2 + 15^2 + 3^2) = \underline{718 J.}$$

$$(d) \frac{dx}{dt} = u_x = 18t + 4 \rightarrow x = 9t^2 + 4t + A_1$$

$$\frac{dy}{dt} = u_y = 15t \rightarrow y = 7.5t^2 + B_1$$

$$\frac{dz}{dt} = u_z = 3 \rightarrow z = 3t + C_1$$

At $t=0$, $(x, y, z) = (1, -2, 0)$. Hence

$$A_1 = 1, B_1 = -2, C_1 = 0.$$

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t)$$

$$\text{At } t=1, (x, y, z) = \underline{(14, 5.5, 3)}.$$

Prob. 8.4

$$(a) \vec{F} = m\vec{a} = Q(\vec{E} + \vec{u} \times \vec{B})$$

$$\begin{aligned} \frac{d}{dt}(u_x, u_y, u_z) &= 2 \left[-4\vec{a}_y + \begin{vmatrix} u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix} \right] \\ &= -8\vec{a}_y + 10u_z\vec{a}_y - 10u_y\vec{a}_z \end{aligned}$$

i.e.

$$\frac{du_x}{dt} = 0 \rightarrow u_x = A_1 \quad - (1)$$

$$\frac{du_y}{dt} = -8 + 10u_z \quad - (2)$$

$$\frac{du_z}{dt} = -10u_y \quad - (3)$$

$$\frac{d^2 u_y}{dt^2} = 0 + 10 \frac{du_z}{dt} = -100u_y$$

$$u_y + 100u_y = 0 \rightarrow u_y = B_1 \cos 10t + B_2 \sin 10t$$

from (2),
 $10u_z = 8 + \dot{u}_y = 8 - 10B_1 \sin 10t + 10B_2 \cos 10t$

$$u_z = 0.8 - B_1 \sin 10t + B_2 \cos 10t$$

At $t=0$, $\vec{u} = 0 \rightarrow A_1 = 0, B_1 = 0, B_2 = -0.8$

Hence,
 $\vec{u} = (0, -0.8 \sin 10t, 0.8 - 0.8 \cos 10t) \quad - (4)$

$$u_x = \frac{dx}{dt} = 0 \rightarrow x = c_1$$

$$u_y = \frac{dy}{dt} = -0.8 \sin 10t \rightarrow y = 0.08 \cos 10t + c_2$$

$$u_z = \frac{dz}{dt} = 0.8 - 0.8 \cos 10t \rightarrow z = 0.8t + c_3 - 0.08 \sin 10t$$

At $t=0$, $(x, y, z) = (2, 3, -4) \Rightarrow$

$$c_1 = 2, c_2 = 2.92, c_3 = -4.$$

Hence $(x, y, z) = (2, 2 + 0.08 \cos 10t, 0.8t - 0.08 \sin 10t - 4)$

At $t=1$,
 $(x, y, z) = \underline{(2, 1.933, -3.156)}$.

(b) from (4), at $t=1$, $\vec{u} = (0, 0.435, 1.471)$ m/s

$$K.E. = \frac{1}{2} m |\vec{u}|^2 = \frac{1}{2} (1) (0.435^2 + 1.471^2)$$

$$= \underline{1.177 \text{ J.}}$$

Prob. 8.5

$$m \vec{a} = q \vec{u} \times \vec{B}$$

$$10^{-3} \vec{a} = -2 \times 10^{-3} \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt} (u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $\frac{du_x}{dt} = 12u_z \quad \text{--- (1)}$

$\frac{du_y}{dt} = 0 \quad \longrightarrow \quad u_y = A_1 \quad \text{--- (2)}$

$\frac{du_z}{dt} = -12u_x \quad \text{--- (3)}$

from (1) and (2),

$$\ddot{u}_x = -12 \dot{u}_z = -144 u_x$$

or $\ddot{u}_x + 144 u_x = 0 \rightarrow u_x = C_1 \cos 12t + C_2 \sin 12t$

from (1), $u_z = -C_1 \sin 12t + C_2 \cos 12t$

At $t=0$, $u_x=2$, $u_y=0$, $u_z=0 \rightarrow A_1=0=C_2$,
 $C_1=5$

Hence $\vec{u} = (5 \cos 12t, 0, -5 \sin 12t)$
 $\vec{u}(t=10s) = (5 \cos 120, 0, -5 \sin 120)$
 $= \underline{4.071 \hat{a}_x - 2.903 \hat{a}_z} \text{ m/s.}$

$u_x = \frac{dx}{dt} = 5 \cos 12t \rightarrow x = \frac{5}{12} \sin 12t + B_1$

$u_y = \frac{dy}{dt} = 0 \rightarrow y = B_2$

$u_z = \frac{dz}{dt} = -5 \sin 12t \rightarrow z = \frac{5}{12} \cos 12t + B_3$

At $t=0$, $(x, y, z) = (0, 1, 2) \rightarrow B_1=0, B_2=1$,
 $B_3 = \frac{19}{12}$

$(x, y, z) = \left(\frac{5}{12} \sin 12t, 1, \frac{5}{12} \cos 12t + \frac{19}{12} \right) \text{---(4)}$

At $t=10s$, $(x, y, z) = \left(\frac{5}{12} \sin 120, 1, \frac{5}{12} \cos 120 + \frac{19}{12} \right)$
 $= \underline{(0.2419, 1, 1.923)}$

By eliminating t from (4),

$x^2 + \left(z - \frac{19}{12} \right)^2 = \left(\frac{5}{12} \right)^2, y=1$

which is a helix with axis on line
 $y=1, z = \frac{19}{12}$.

Prob. 8.6

(a) $m \vec{a} = e(\vec{v} \times \vec{B})$

$$\frac{m}{e} \frac{d}{dt} (u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_z \end{vmatrix}$$

$$= u_y B_0 \vec{a}_x - B_0 u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C = 0$$

$$\frac{du_x}{dt} = u_y \frac{B_0 e}{m} = u_y \omega, \text{ where } \omega = \frac{B_0 e}{m}$$

$$\frac{du_y}{dt} = -u_x \omega$$

Hence $\ddot{u}_x = \omega \dot{u}_y = -\omega^2 u_x$

or $\ddot{u}_x + \omega^2 u_x = 0 \rightarrow u_x = A \cos \omega t + B \sin \omega t$

$$u_y = \frac{\dot{u}_x}{\omega} = -A \sin \omega t + B \cos \omega t$$

At $t=0$, $u_x = u_0$, $u_y = 0 \rightarrow A = u_0$, $B = 0$.

Hence $u_x = u_0 \cos \omega t = \frac{dx}{dt} \rightarrow x = \frac{-u_0}{\omega} \sin \omega t + C_1$

$$u_y = -u_0 \sin \omega t = \frac{dy}{dt} \rightarrow y = \frac{-u_0}{\omega} \cos \omega t + C_2$$

At $t=0$, $x=0=y \rightarrow C_1 = 0$, $C_2 = \frac{u_0}{\omega}$. Hence

$$x = \frac{-u_0}{\omega} \sin \omega t, \quad y = \frac{u_0}{\omega} (1 - \cos \omega t)$$

$$\frac{u_0^2}{\omega^2} (\cos^2 \omega t + \sin^2 \omega t) = \left(\frac{u_0}{\omega}\right)^2 = x^2 + \left(y - \frac{u_0}{\omega}\right)^2$$

showing that the electron would move in a circle centered at $(0, \frac{u_0}{\omega})$. But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

$$(b) \quad d = \text{twice the radius of the semi-circle} \\ = \frac{2u_0}{\omega} = \frac{2u_0 m}{B_0 e}$$

Alternatively, due to the centrifugal force,

$$F = eu_0 B = \frac{mu_0^2}{r} \rightarrow r = \frac{mu_0}{B_0 e}$$

$$d = 2r = \frac{2mu_0}{B_0 e}$$

Prob. 8.7

(i) Let $\vec{F}_2$ be the force on 2 due to the field produced by 1,

$$d\vec{F}_2 = I_2 d\vec{l}_2 \times \vec{B}_1 = I_2 dz (-\vec{a}_z) \times \frac{\mu_0 I_1}{2ap} \vec{a}_\phi$$

The force per unit length on 2 is

$$\vec{f}_2 = \frac{d\vec{F}_2}{dz} = \frac{\mu_0 I_1 I_2}{2ap} \vec{a}_\phi \quad (\text{repulsive})$$

$$(b) f = \frac{4\pi \times 10^{-7} \times 5 \times 10}{2\pi \times 2} = 5 \mu\text{N/m}.$$

(i) $5 \mu\text{N/m}$, attractive, (ii) $5 \mu\text{N/m}$, repulsive.

Prob. 8.8

$$\vec{F} = I \vec{L} \times \vec{B} \rightarrow \vec{F} = \frac{\vec{F}}{L} = I_1 \vec{a}_1 \times \vec{B}_2 = \frac{\mu_0 I_1 I_2 \vec{a}_1 \times \vec{a}_2}{2\pi \rho}$$

$$(a) \vec{f}_{21} = \frac{\vec{a}_2 \times (-\vec{a}_x)}{2\pi} \frac{4\pi \times 10^{-7} \times (-2 \times 10^4)}{4}$$

$$= \underline{\underline{\vec{a}_y \text{ mN/m}}} \text{ (repulsive).}$$

$$(b) \vec{f}_{12} = -\vec{f}_{21} = \underline{\underline{-\vec{a}_y \text{ mN/m}}} \text{ (repulsive)}$$

$$(c) \vec{a}_1 \times \vec{a}_4 = \vec{a}_2 \times \left(-\frac{4}{5} \vec{a}_x + \frac{3}{5} \vec{a}_y\right) = -\frac{3}{5} \vec{a}_x - \frac{4}{5} \vec{a}_y, \rho=5$$

$$\vec{f}_{31} = \frac{4\pi \times 10^{-7}}{2\pi(5)} (-3 \times 10^4) \left(-\frac{3}{5} \vec{a}_x - \frac{4}{5} \vec{a}_y\right)$$

$$= \underline{\underline{0.72 \vec{a}_x + 0.96 \vec{a}_y \text{ mN/m}}} \text{ (repulsive).}$$

$$(d) \vec{f}_3 = \vec{f}_{31} + \vec{f}_{32}$$

$$\vec{f}_{32} = \frac{4\pi \times 10^{-7} \times 6 \times 10^4}{2\pi(3)} \vec{a}_2 \times \vec{a}_y = -4 \vec{a}_x \text{ mN/m} \text{ (attractive)}$$

$$\vec{f}_3 = \underline{\underline{-3.28 \vec{a}_x + 0.96 \vec{a}_y \text{ mN/m}}}$$

(attractive due to L_2 and repulsive due to L_1).

Prob. 8.9

$$W = - \int \vec{F} \cdot d\vec{l}, \quad \vec{F} = \int l d\vec{l} \times \vec{B} = 3(2\vec{a}_z) \times \cos \frac{\phi}{3} \vec{a}_\phi$$

$$= 6 \cos \frac{\phi}{3} \vec{a}_\phi \text{ mN}.$$

$$W = - \int_0^{2\pi} 6 \cos \frac{\phi}{3} \rho_0 d\phi = -6 \times 3 \sin \frac{\phi}{3} \Big|_0^{2\pi} \text{ mJ}$$

$$= -18 \sin \frac{2\pi}{3} = -\underline{\underline{15.59 \text{ mJ}}}.$$

Prob. 8.10

$$(a) \vec{F}_1 = \int_{\rho=2}^4 \frac{\mu_0 I_1 I_2}{2\pi \rho} d\rho \vec{a}_\rho \times \vec{a}_\phi = \frac{4\pi \times 10^{-7} (2)(5)}{2\pi} \ln \frac{4}{2} \vec{a}_z$$

$$= 2 \ln 2 \vec{a}_z \mu\text{N} = \underline{\underline{1.3863 \vec{a}_z \mu\text{N}}}.$$

$$(b) \vec{F}_2 = \int l_2 d\vec{l}_2 \times \vec{B}_1$$

$$= \frac{I_1 I_2 \mu_0}{2\pi} \int \frac{1}{\rho} [d\rho \vec{a}_\rho + dz \vec{a}_z] \times \vec{a}_\phi$$

$$= \frac{\mu_0 I_1 I_2}{2\pi} \int \frac{1}{\rho} [d\rho \vec{a}_z - dz \vec{a}_\rho]$$

But $\rho = z + 2, dz = d\rho.$

$$\vec{F}_2 = \frac{4\pi \times 10^{-7} (5)(2)}{2\pi} \int_{\rho=4}^2 \frac{1}{\rho} [d\rho \vec{a}_z - d\rho \vec{a}_\rho]$$

$$= 2 \ln \frac{2}{4} (\vec{a}_z - \vec{a}_\rho) \mu\text{N} = 1.386 \vec{a}_\rho - 1.386 \vec{a}_z \mu\text{N}$$

$$\vec{F}_3 = \frac{\mu_0 I_1 I_2}{2\pi} \int \frac{1}{\rho} [d\rho \vec{a}_z - dz \vec{a}_\rho]$$

But $z = -\rho + 6$, $dz = -d\rho$

$$\vec{F}_3 = \frac{4\pi \times 10^{-7} (5)(2)}{2\pi} \int_{\rho=6}^4 \frac{1}{\rho} [d\rho \vec{a}_z + d\rho \vec{a}_\rho]$$

$$= 2 \ln \frac{4}{6} (\vec{a}_\rho + \vec{a}_z) \mu N = -0.8109 \vec{a}_\rho - 0.8109 \vec{a}_z \mu N$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= 1.3863 \vec{a}_z + 1.386 \vec{a}_\rho - 1.386 \vec{a}_z - 0.8109 \vec{a}_\rho - 0.8109 \vec{a}_z$$

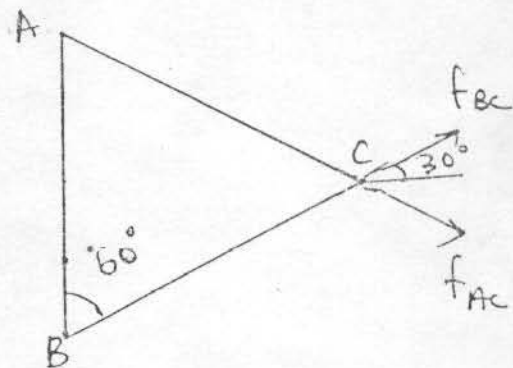
$$= \underline{\underline{0.5751 \vec{a}_\rho - 0.8109 \vec{a}_z \mu N}}$$

Prob. 8.11

from Prob. 8.7,

$$f = \frac{\mu_0 I_1 I_2}{2\pi \rho} \vec{a}_\rho \text{ N/m}$$

$$\vec{F} = \vec{F}_{AC} + \vec{F}_{BC}$$



$$f_{AC} = f_{BC} = \frac{4\pi \times 10^{-7} \times 75 \times 150}{2\pi \times 2} = 1.125 \times 10^{-3}$$

$$\vec{F} = 2 \times 1.125 \cos 30^\circ \vec{a}_x \text{ mN/m}$$

$$= \underline{\underline{1.949 \vec{a}_x \text{ mN/m}}}$$

Prob. 8.12

$$\vec{F} = \int L d\vec{l} \times \vec{B} = \int \vec{J} dv \times \vec{B}$$

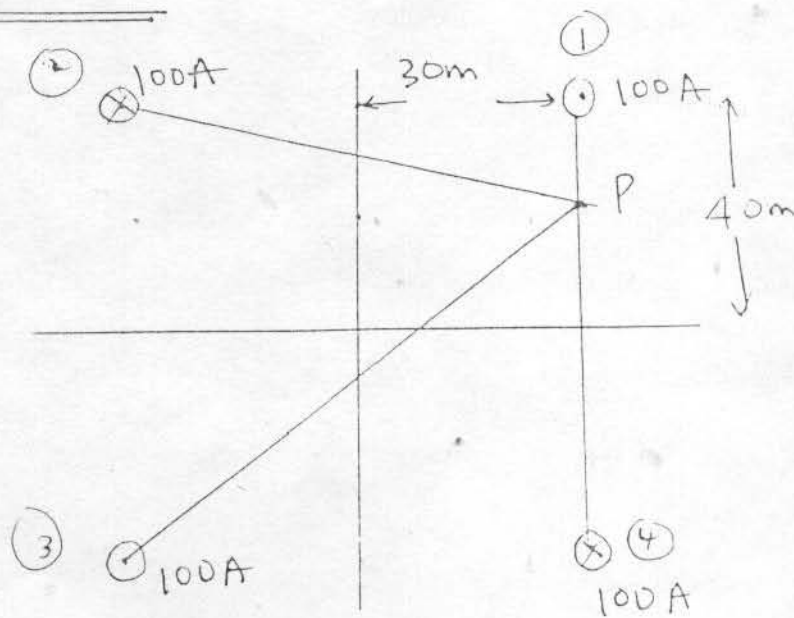
$$\vec{J} = \frac{I}{\pi(b^2 - a^2)} \hat{a}_z, \quad \vec{B} = B_0 \hat{a}_\phi$$

$$\vec{F} = \frac{I}{\pi(b^2 - a^2)} \int \hat{a}_z dv \times B_0 \hat{a}_\phi = \frac{IB_0 \hat{a}_\phi}{\pi(b^2 - a^2)} \int dV$$

$$= \frac{IB_0}{\pi(b^2 - a^2)} \pi(a^2 - b^2)L$$

$$\vec{F} = \frac{\vec{F}}{L} = \underline{IB_0 \hat{a}_\phi}$$

Prob. 8.13



$$\text{Let } \vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4$$

$$\text{where } \vec{B}_n = \frac{\mu_0 \mu_r I}{2\pi r} \hat{a}_\phi$$

for (1), $\vec{a}_q = \vec{a}_1 \times \vec{a}_p = \vec{a}_z \times (-\vec{a}_y) = \vec{a}_x$

$$\vec{B}_1 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 20 \times 10^{-3}} \vec{a}_x = 2\vec{a}_x$$

for (2), $\vec{p} = 6\vec{a}_x - 2\vec{a}_y$

$$\vec{a}_q = -\vec{a}_z \times \frac{(6\vec{a}_x - 2\vec{a}_y)}{\sqrt{40}} = \frac{(-2\vec{a}_x - 6\vec{a}_y)}{\sqrt{40}}$$

$$\begin{aligned} \vec{B}_2 &= \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 400 \times 10^{-3}} (-2\vec{a}_x - 6\vec{a}_y) \\ &= -0.2\vec{a}_x - 0.6\vec{a}_y \end{aligned}$$

for (3), $\vec{p} = 6\vec{a}_x + 6\vec{a}_y$

$$\vec{a}_q = \vec{a}_z \times \frac{(6\vec{a}_x + 6\vec{a}_y)}{\sqrt{72}} = \frac{-6\vec{a}_x + 6\vec{a}_y}{\sqrt{72}}$$

$$\begin{aligned} \vec{B}_3 &= \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 720 \times 10^{-3}} (-6\vec{a}_x + 6\vec{a}_y) \\ &= -0.3333\vec{a}_x + 0.3333\vec{a}_y \end{aligned}$$

for (4), $\vec{a}_q = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$

$$\vec{B}_4 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 60 \times 10^{-3}} \vec{a}_x = 0.6667\vec{a}_x$$

$$\vec{B} = \left(2 + \frac{2}{3} - \frac{1}{5} - \frac{1}{3}\right)\vec{a}_x + \left(-\frac{3}{5} + \frac{1}{3}\right)\vec{a}_y$$

$$= \underline{\underline{2.133\vec{a}_x - 0.2667\vec{a}_y \text{ Wb/m}^2}}$$

Prob. 8.14

$$\vec{m} = m \vec{a}_z = m (\cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta)$$

Hence

$$\frac{\vec{m}}{r^3} = \frac{m}{r^3} (\cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta)$$

$$\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} = \frac{3mr \cos \theta (r \vec{a}_r)}{r^5} = \frac{3m \cos \theta}{r^3} \vec{a}_r$$

Thus

$$\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} = \frac{2m \cos \theta \vec{a}_r + m \sin \theta \vec{a}_\theta}{r^3}$$

and

$$\vec{B} = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} \right] = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta)$$

Prob. 8.15

$$T = mB \sin \theta \rightarrow T_{\max} = mB$$

$$\begin{aligned} \text{or } B &= \frac{T_{\max}}{1.5 \text{ N}} = \frac{50 \times 10^{-6}}{10 \times 10^{-3} \times 800 \times 0.6 \times 10^{-4}} \\ &= \underline{\underline{104.2 \text{ mWb/m}^2}} \end{aligned}$$

Prob. 8.16

$$\begin{aligned} T = mB &= N I S B = 1000 \times 2 \times 10^{-3} \times 300 \times 10^{-6} \times 0.4 \\ &= \underline{\underline{240 \mu \text{N} \cdot \text{m}}} \end{aligned}$$

Prob. 8.17

$$(a) \vec{m} = IS \vec{a}_n = 1(\pi)(0.1)^2 \vec{a}_z = \underline{\underline{0.03142 \vec{a}_z \text{ A}\cdot\text{m}}}$$

$$(b) \vec{H} = \frac{m}{4\pi r^3} (2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta)$$

$$\text{where } r = \sqrt{4+4+4} = 2\sqrt{3}, \tan \theta = \frac{r_z}{r_\rho} = \frac{2\sqrt{2}}{2} = \sqrt{2},$$

$$\text{i.e. } \sin \theta = \sqrt{\frac{2}{3}}, \cos \theta = \frac{1}{\sqrt{3}}. \text{ Hence}$$

$$\begin{aligned} \vec{H} &= \frac{0.01\pi}{4\pi(24\sqrt{3})} \left(\frac{2}{\sqrt{3}} \vec{a}_r + \sqrt{\frac{2}{3}} \vec{a}_\theta \right) \\ &= \underline{\underline{69.44 \vec{a}_r + 49.1 \vec{a}_\theta \text{ } \mu\text{A/m.}}} \end{aligned}$$

$$(c) \vec{B} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta)$$

$$\text{where } r = \sqrt{36+64+100} = 10\sqrt{2}, \tan \theta = \frac{10}{10} = 1. \text{ Then}$$

$$\begin{aligned} \vec{B} &= \frac{4\pi \times 10^{-7} \times 0.01\pi}{4\pi(2000\sqrt{2})} \left(\frac{2}{\sqrt{2}} \vec{a}_r + \frac{1}{\sqrt{2}} \vec{a}_\theta \right) \\ &= \underline{\underline{1.571 \vec{a}_r + 0.7853 \vec{a}_\theta \text{ } \mu\text{Wb/m}^2.}} \end{aligned}$$

Prob. 8.18

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 10^{-2}}{288} (2 \vec{a}_r + \sqrt{2} \vec{a}_\theta)$$

$$\vec{m} = 4 \vec{a}_z = 4 (\cos \theta \vec{a}_r - \sin \theta \vec{a}_\theta) = 4 \left(\frac{\vec{a}_r}{\sqrt{3}} - \sqrt{\frac{2}{3}} \vec{a}_\theta \right)$$

$$\vec{T} = \vec{m} \times \vec{B} = \frac{4 \times 10^{-2} \times 4\pi \times 10^{-7}}{288} \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 2 & \sqrt{2} & 0 \end{vmatrix}$$

$$= -\sqrt{\frac{2}{3}} \cdot \frac{\pi}{8} \cdot 10^{-9} \hat{a}_\phi = \underline{\underline{-1.425 \times 10^{-10} \hat{a}_\phi \text{ N}\cdot\text{m}}}$$

Prob. 7.19

$$\vec{B} = \frac{k}{r^3} (2 \cos \theta \hat{a}_r + \sin \theta \hat{a}_\theta)$$

At $(10, 0, 0)$, $r=10$, $\theta = \pi/2$, $\hat{a}_r = \hat{a}_x$, $\hat{a}_\theta = -\hat{a}_z$

$$-0.5 \times 10^{-3} \hat{a}_z = \frac{k}{10^3} (0 - \hat{a}_z) \rightarrow k = 0.5$$

Thus $\vec{B} = \frac{0.5}{r^3} (2 \cos \theta \hat{a}_r + \sin \theta \hat{a}_\theta)$

(a) At $(0, 3, 0)$, $r=3$, $\theta = \pi/2$, $\hat{a}_r = \hat{a}_y$, $\hat{a}_\theta = -\hat{a}_z$

$$\vec{B} = \frac{0.5}{27} (0 - \hat{a}_z) = \underline{\underline{-18.52 \hat{a}_z \text{ mWb/m}^2}}$$

(b) At $(3, 4, 0)$, $r=5$, $\theta = \pi/2$, $\hat{a}_\theta = -\hat{a}_z$

$$\vec{B} = \frac{0.5}{125} (0 - \hat{a}_z) = \underline{\underline{-4 \hat{a}_z \text{ mWb/m}^2}}$$

(c) At $(1, -1, 1)$, $r = \sqrt{3}$, $\tan \theta = \frac{f_z}{-1} = \frac{\sqrt{2}}{-1}$, i.e.

$$\sin \theta = \frac{\sqrt{2}}{3}, \cos \theta = -\frac{1}{\sqrt{3}}$$

$$\vec{B} = \frac{0.5}{3\sqrt{3}} \left(-\frac{2}{\sqrt{3}} \hat{a}_r + \sqrt{\frac{2}{3}} \hat{a}_\theta \right) = \underline{\underline{-111 \hat{a}_r + 78.6 \hat{a}_\theta \text{ mWb/m}^2}}$$

Prob. 8.20

$$\vec{T}_2 = \vec{m}_2 \times \vec{B}_1$$

where $\vec{m}_2 = 3\vec{a}_y$, $m_1 = 5\vec{a}_x$. From Prob. 8.14,

$$\begin{aligned} \vec{B}_1 &= \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m}_1 \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}_1}{r^3} \right] \\ &= \frac{4\pi \times 10^{-7}}{4\pi} \left[\frac{3(20)(4, -3, 10)}{(5\sqrt{5})^3} - \frac{(5, 0, 0)}{(5\sqrt{5})^3} \right] \\ &= (-2.204, -1.03, 3.434) \cdot 10^{-10} \end{aligned}$$

Hence $\vec{T}_2 = 3\vec{a}_y \times \vec{B}_1$

$$= \underline{\underline{(10.302\vec{a}_x + 6.612\vec{a}_z) \cdot 10^{-10} \text{ N}\cdot\text{m}}}$$

Prob 8.21

(a) $\chi_m = \mu_r - 1 = \underline{\underline{3}}$.

(b) $\vec{H} = \frac{\vec{B}}{\mu} = \frac{(2, -5, 4) \times 10^{-3}}{4\pi \times 10^{-7} (4)}$

$$= \underline{\underline{398\vec{a}_x - 995\vec{a}_y + 796\vec{a}_z \text{ A/m}}}$$

(c) $\vec{M} = \chi_m \vec{H} = 3\vec{H} = \underline{\underline{1.19\vec{a}_x - 2.98\vec{a}_y + 2.39\vec{a}_z \text{ kA/m}}}$

(d) $w_m = \frac{1}{2} \vec{B} \cdot \vec{H} = \frac{|\vec{B}|^2}{2\mu}$

$$W_m = \frac{(4+25+16) \times 10^{-6}}{2 \times 4\pi \times 10^{-7} \times 4} = \underline{\underline{4.475 \text{ J/m}^3}}$$

Prob. 8.22

$$(a) \vec{M} = \chi_m \vec{H} = \underline{\underline{4z \hat{a}_y \text{ A/m}}}$$

$$(b) \vec{J} = \nabla \times \vec{M} = -\frac{\partial H_y}{\partial z} \hat{a}_x = \underline{\underline{-\hat{a}_x \text{ A/m}}}$$

$$(c) \vec{J}_b = \chi_m \vec{J} = \underline{\underline{-4 \hat{a}_x \text{ A/m}}}$$

Prob. 8.23

(a) for an infinitely long wire

$$H_\phi = \begin{cases} \frac{I\rho}{2\pi a^2}, & \rho < a = 2\text{m} \\ \frac{I}{2\pi\rho}, & \rho > a \end{cases}$$

$$\vec{m} = \chi_m \vec{H}, \vec{J}_b = \nabla \times \vec{m}, \vec{J}_f = \nabla \times \vec{H}, \vec{J} = \vec{J}_b + \vec{J}_f = 5 \nabla \times \vec{H}$$

$$\text{where } \nabla \times \vec{H} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \hat{a}_z$$

$$\text{At } \rho = 1\text{mm} < a, \nabla \times \vec{H} = \frac{1}{2\pi a^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2) \hat{a}_z = \frac{1}{\pi a^2} \hat{a}_z$$

$$\vec{J}_b = 4 \nabla \times \vec{H} = \frac{4 \times 4 \hat{a}_z}{\pi \times 4 \times 10^{-6}} = \underline{\underline{1.273 \hat{a}_z \text{ MA/m}^2}}$$

$$\vec{J} = 5 \nabla \times \vec{H} = \underline{\underline{1.592 \hat{a}_z \text{ MA/m}^2}}$$

$$\vec{M} = \frac{4 \times 1 \times 10^{-3}}{2\pi \times 4 \times 10^{-6}} \vec{a}_\phi = \underline{\underline{159.15 \vec{a}_\phi \text{ A/m}}}$$

At $\rho = 3 \text{ mm} > a$,

$$\vec{M} = 0 = \vec{J}_b = \vec{J}$$

$$\begin{aligned} (b) \vec{K}_b &= \vec{M} \times \vec{a}_n = 4 H_0 \vec{a}_\phi \times \vec{a}_\rho \Big|_{\rho=2 \text{ mm}} \\ &= \frac{4 \times 4 (-\vec{a}_z)}{2\pi \times 2 \times 10^{-3}} = \underline{\underline{-1.273 \vec{a}_z \text{ kA/m}^2}} \end{aligned}$$

Prob. 8.24

(a) from $H_{1t} - H_{2t} = k$ and $M = \chi_m H$, we obtain

$$\underline{\underline{\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = k}}$$

Also from $B_{1n} = B_{2n}$ and $B = \mu H = \frac{\mu}{\chi_m} M$, we get

$$\underline{\underline{\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}}}$$

(b) from $B_1 \cos \theta_1 = B_{1n} = B_{2n} = B_2 \cos \theta_2$ — (1)

and $\frac{B_1 \sin \theta_1}{\mu_1} = H_{2t} = k + H_{1t} = k + \frac{B_2 \sin \theta_2}{\mu_2}$ — (2),

dividing (2) by (1) gives

$$\frac{\tan \theta_1}{\mu_1} = \frac{k}{B_2 \cos \theta_2} + \frac{\tan \theta_2}{\mu_2} = \frac{\tan \theta_2}{\mu_2} \left(1 + \frac{k \mu_2}{B_2 \sin \theta_2} \right)$$

i.e.
$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2} \left(1 + \frac{\mu_2}{B_2 \sin \theta_2} \right)$$

Prob. 8.25

(a) $\vec{H}_{1t} = \vec{H}_{2t} = -6\vec{a}_y + 7\vec{a}_z$

$\vec{B}_{1n} = \vec{B}_{2n} \rightarrow \mu_0 \vec{H}_{1n} = 3\mu_0 \vec{H}_{2n} = 3\mu_0 (4\vec{a}_x)$

$\vec{H}_{1n} = 12\vec{a}_x$

Hence $\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t} = \underline{12\vec{a}_x - 6\vec{a}_y + 7\vec{a}_z \text{ kA/m}}$

(b) $\vec{B}_2 = \mu_2 \vec{H}_2 = 3(4\pi \times 10^{-7})(4, -6, 7) \cdot 10^3$
 $= \underline{15.1\vec{a}_x - 22.6\vec{a}_y + 26.4\vec{a}_z \text{ mWb/m}^2}$

(c) $\vec{H}_2 \cdot \vec{a}_x = H_2 \cos \theta_2 \rightarrow \cos \theta_2 = \frac{4}{\sqrt{4^2 + 6^2 + 7^2}}$
 $= 0.398$

$\therefore \theta_2 = \underline{66.55^\circ}$

$\vec{H}_1 \cdot \vec{a}_x = H_1 \cos \theta_1 \rightarrow \cos \theta_1 = \frac{12}{\sqrt{12^2 + 6^2 + 7^2}} = 0.793$

$\therefore \theta_1 = \underline{37.53^\circ}$

Prob. 8.26

(a) $\vec{B}_{1n} = \vec{B}_{2n} = 50\vec{a}_y$

$$\begin{aligned}\vec{H}_{1t} = \vec{H}_{2t} &\rightarrow \vec{B}_{2t} = \frac{\mu_2}{\mu_1} \vec{B}_{1t} = \frac{1}{2} (40\vec{a}_x - 30\vec{a}_z) \\ &= 20\vec{a}_x - 15\vec{a}_z \\ \vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} &= \underline{\underline{20\vec{a}_x + 50\vec{a}_y - 15\vec{a}_z \text{ mWb/m}^2}}\end{aligned}$$

$$\begin{aligned}(b) \frac{H_1}{H_2} &= \frac{B_1}{B_2} \cdot \frac{\mu_{r2}}{\mu_{r1}} = \frac{10\sqrt{4^2 + 5^2 + 3^2}}{10\sqrt{2^2 + 5^2 + 1.5^2}} \cdot \frac{1}{2} \\ &= \underline{\underline{0.6324}}\end{aligned}$$

$$(c) \frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2} = 2$$

$$\text{or } \tan \theta_1 = \frac{\sqrt{40^2 + 30^2}}{50} = 1 \rightarrow \theta_1 = 45^\circ$$

$$\tan \theta_2 = \frac{\sqrt{20^2 + 15^2}}{50} = 0.5 \rightarrow \theta_2 = 26.56^\circ$$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{1}{0.5} = \underline{\underline{2}}$$

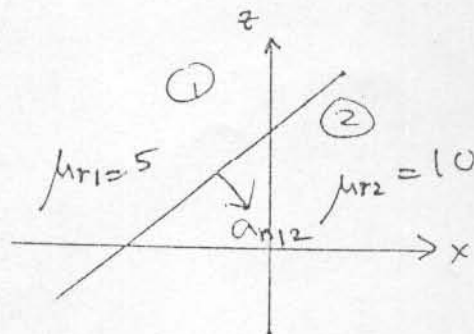
Prob. 8.27

Let $\vec{H}_2 = (H_x, H_y, H_z)$

$$(\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \vec{k}$$

where $f(x, z) = 5z - 4x = 0$ and

$$\vec{a}_{n12} = -\frac{\nabla f}{|\nabla f|} = \frac{4\vec{a}_x - 5\vec{a}_z}{\sqrt{41}}$$



$$(\vec{H}_1 - \vec{H}_2) \times \vec{a}_{112} = \frac{1}{\sqrt{41}} \begin{vmatrix} 25 - H_x & -30 - H_y & 45 - H_z \\ 4 & 0 & -5 \end{vmatrix}$$

$$= \frac{1}{\sqrt{41}} \begin{pmatrix} 150 + 5H_y, 180 - 4H_z + 125 - 5H_x, \\ 120 + 4H_y \end{pmatrix} = \vec{F} = 35\vec{a}_y.$$

Equating components,

$$\vec{a}_x: 150 + 5H_y = 0 \longrightarrow H_y = -30$$

$$\vec{a}_y: 305 - 4H_z - 5H_x = 35 \longrightarrow 4H_z + 5H_x = 270$$

$$\vec{a}_z: 120 + 4H_y = 0 \longrightarrow H_y = -30.$$

$$\text{Also, } \vec{B}_1 = \vec{B}_2 \longrightarrow \mu_1 \vec{H}_1 = \mu_2 \vec{H}_2$$

$$5\mu_0(25, -30, 45) \cdot \frac{(4, 0, 5)}{\sqrt{41}} = 10\mu_0(H_x, H_y, H_z) \cdot \frac{(4, 0, 5)}{\sqrt{41}}$$

$$100 - 225 = 8H_x - 10H_z$$

$$\text{or } 125 = 10H_z - 8H_x$$

$$= 10H_z - 8(54 - 0.8H_z)$$

$$\longrightarrow H_z = 33.96$$

$$\text{and } H_x = 54 - 0.8H_z = 26.83$$

Thus

$$\vec{H}_2 = \underline{\underline{26.83\vec{a}_x - 30\vec{a}_y + 33.96\vec{a}_z \text{ A/m.}}}$$

Prob. 8.28

$$\vec{H}_{1n} = -3\vec{a}_z, \quad \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$$

$$\vec{H}_{2t} = \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y,$$

$$\vec{H}_{2n} = \frac{\mu_1}{\mu_2} H_{1n} = \frac{1}{200} (-3\vec{a}_z) = -0.015\vec{a}_z$$

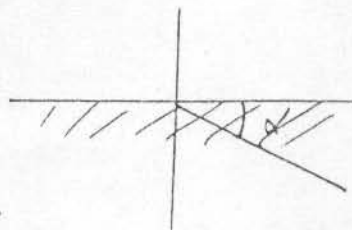
$$\vec{H}_2 = 10\vec{a}_x + 15\vec{a}_y - 0.015\vec{a}_z$$

$$\vec{B}_2 = \mu_2 \vec{H}_2 = 200 \times 4\pi \times 10^{-7} (10, 15, -0.015)$$

$$\vec{B}_2 = \underline{\underline{2.51\vec{a}_x + 3.77\vec{a}_y - 0.0037\vec{a}_z \text{ mWb/m}^2}}$$

$$\tan \alpha = \frac{B_{2n}}{B_{2t}}$$

$$\alpha = \tan^{-1} \frac{0.0037}{\sqrt{2.51^2 + 3.77^2}} = \underline{\underline{0.047^\circ}}$$



Prob. 8.29

$$(a) \vec{H} = \frac{1}{2} \vec{k} \times \vec{a}_n = \frac{1}{2} (30 - 40) \vec{a}_x \times (-\vec{a}_z) = \underline{\underline{-5\vec{a}_y \text{ A/m}}}$$

$$\vec{B} = \mu_0 \vec{H} = 4\pi \times 10^{-7} (-5\vec{a}_y) = \underline{\underline{-6.283\vec{a}_y \text{ mWb/m}^2}}$$

$$(b) \vec{H} = \frac{1}{2} (-30 - 40) \vec{a}_y = \underline{\underline{-35\vec{a}_y \text{ A/m}}}$$

$$\vec{B} = \mu_0 \mu_r \vec{H} = 4\pi \times 10^{-7} (-35\vec{a}_y) = \underline{\underline{-110\vec{a}_y \text{ mWb/m}^2}}$$

$$(c) \vec{H} = \frac{1}{2} (-30 + 40) \vec{a}_y = \underline{\underline{5 \vec{a}_y \text{ A/m}}}$$

$$\vec{B} = \mu_0 \vec{H} = \underline{\underline{6.283 \vec{a}_y \text{ mWb/m}^2}}$$

Prob. 8.30

$$(a) B = 70 + (210)^2 = 44.17 \text{ mWb/m}^2$$

$$\mu_r = \frac{B}{\mu_0 H} = \frac{44.17 \times 10^{-3}}{4\pi \times 10^{-7} \times 210} = \underline{\underline{167.4}}$$

$$(b) W_m = \int_0^{H_0} H dB = \int_0^{H_0} H \left(\frac{1}{3} + 2H \right) dH$$

$$= \frac{H_0^2}{6} + \frac{2}{3} H_0^3 = 7350 + 6174000$$

$$= \underline{\underline{6181.35 \text{ kJ/m}^3}}$$

Prob. 8.31

$$L' = \frac{L}{l} = \mu_0 n^2 S = \frac{\mu_0 N^2 S}{l^2} = \frac{\mu_0 N^2 \pi d^2}{4 l^2}$$

$$L = \frac{4\pi \times 10^{-7} \times 10^6 \pi \times 100 \times 10^{-4}}{4 (0.4)^2} = \underline{\underline{24.674 \text{ mH}}}$$

Prob. 8.32

$$(a) L = \frac{\lambda}{I} = \frac{N\psi}{I} = \frac{\mu_0 N^2 l a}{2\pi} \ln \left(\frac{2\rho_0 + a}{2\rho_1 - a} \right)$$

$$(b) L = \frac{N\psi}{I} = \mu_0 N^2 \left[\rho_0 - (\rho_0^2 - a^2)^{1/2} \right]$$

when $\rho_0 \gg a$, binomial series expansion gives

$$L = \frac{\mu_0 N^2 a^2}{2\rho_0}$$

or from Example 8.10,

$$L = L' l = \frac{\mu_0 N^2 L S}{l^2} = \frac{\mu_0 N^2 \pi a^2}{2\pi \rho_0} = \frac{\mu_0 N^2 a^2}{2\rho_0}$$

Prob. 8.33

$$(a) L' = \frac{L}{l} = \frac{\mu_0 \mu_r}{8\pi} = \frac{5 \times 4\pi \times 10^{-7}}{8\pi} = \underline{\underline{250 \text{ nH/m}}}$$

$$(b) W_H = \frac{W_H}{l} = \frac{1}{2} L' I^2 = \frac{1}{2} \times 250 \times 10^{-9} \times 16$$

$$= \underline{\underline{2 \mu J/m^3}}$$

Prob. 8.34

for $d \gg a$,

$$L' = \frac{L}{l} = \frac{\mu_0}{\pi} \ln \frac{d}{a} = \frac{4\pi \times 10^{-7}}{\pi} \ln \frac{d}{a} = 2.5 \times 10^{-6}$$

$$\text{or } \ln \frac{d}{a} = 6.25 \rightarrow \frac{d}{a} = e^{6.25} = 518.01$$

$$a = \frac{3}{518.01} = 5.78 \text{ mm}$$

$$D = 2a = \underline{\underline{11.58 \text{ mm}}}$$

Prob. 8.35

$$L = \frac{N^2 \mu S}{L} \rightarrow N^2 = \frac{L l}{\mu S} = \frac{L 2\pi \rho_0}{\mu_0 \mu r S}$$

$$= \frac{2.5 \times 2\pi \times 0.5}{4\pi \times 10^{-7} \times 200 \times 12 \times 10^{-4}}$$

$$= \frac{25}{96} \times 10^8$$

$$N = \underline{\underline{5103 \text{ turns}}}$$

Prob. 8.36

$$\psi_{12} = \int \vec{B}_1 \cdot d\vec{s} = \int_{\rho=\rho_0}^{\rho_0+a} \int_{z=0}^b \frac{\mu_0 I}{2\pi \rho} dz d\rho = \frac{\mu_0 I b}{2\pi} \ln \frac{a+\rho_0}{\rho_0}$$

for $N=1$,

$$M_{12} = \frac{N \psi_{12}}{l_1} = \frac{\mu_0 b}{2\pi} \ln \frac{a+\rho_0}{\rho_0}$$

$$= \frac{4\pi \times 10^{-7} (1) \ln 2}{2\pi} = \underline{\underline{0.1386 \mu H}}$$

Prob. 8.37

from Prob. 7.25,

$$\lambda = \psi N = \frac{\mu_0 N I}{2\pi} \int_0^{2\pi} \int_{\rho=0}^a \frac{r dr d\theta}{\rho_0 + r \cos \theta}$$

$$= \mu_0 N I \left[\rho_0 - (\rho_0^2 - a^2)^{\frac{1}{2}} \right]$$

$$M = \frac{\lambda}{I} = \underline{\underline{\mu_0 N \left[\rho_0 - (\rho_0^2 - a^2)^{1/2} \right]}}$$

Prob. 8.38

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_0 N_1 I_1}{L_1}$. The flux linking the second solenoid is

$$\Psi_2 = N_2 B_1 S_1 = \frac{\mu_0 N_1 I_1}{L_1} \pi r_1^2$$

$$M = \frac{\Psi_2}{I_1} = \underline{\underline{\frac{\mu_0 N_1 N_2}{L_1} \cdot \pi r_1^2}}$$

Prob. 8.39

(a) $\Psi = B \cdot S = \mu H S$

$$\rightarrow \mu = \frac{\Psi}{HS} = \frac{0.8 \times 10^{-3}}{700 \times 12 \times 10^{-4}} = \frac{8}{8400}$$

$$R = \frac{L}{\mu S} = \frac{0.6 \times 8400}{8 \times 12 \times 10^{-4}} = 525,000$$

$$\text{or } R = \frac{L}{S} \cdot \frac{HS}{\Psi} = \frac{LH}{\Psi} = \frac{0.6 \times 700}{0.8 \times 10^{-3}} \\ = \underline{\underline{5.25 \times 10^5 \text{ A}\cdot\text{t} / \text{Wb}}}$$

(b) $NI = \Psi R = HL = 700 \times 0.6 = 420 \text{ A}\cdot\text{t}$

$$\begin{aligned} \text{or } NI = \Psi R &= 0.8 \times 5.25 \times 10^5 \times 10^{-3} \\ &= \underline{\underline{420 \text{ A}\cdot\text{t}}} \end{aligned}$$

Prob. 8.40

$$F = NI = 400 \times 0.5 = 200 \text{ A}\cdot\text{t}.$$

$$I_a = \frac{R_a}{R_a + R_3 + R_1 \parallel R_2} = \frac{796 \times 10^3 \times 200}{(796 + 383) \times 10^3} = \underline{\underline{190.8 \text{ A}\cdot\text{t}}}$$

$$H_a = \frac{I_a}{l_a} = \frac{190.8}{1 \times 10^{-2}} = \underline{\underline{19,080 \text{ A/m}}}.$$

Prob. 8.41

$$\text{Total } F = NI = 2000 \times 10 = 20,000 \text{ A}\cdot\text{t}.$$

$$R_c = \frac{l_c}{\mu_r \mu_0 S} = \frac{(24 + 20 - 0.6) \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 2 \times 10^{-4}} = \underline{\underline{0.115 \times 10^7 \text{ A}\cdot\text{t/m}}}$$

$$R_a = \frac{l_a}{\mu_0 S} = \frac{0.6 \times 10^{-2}}{4\pi \times 10^{-7} (1) \times 2 \times 10^{-4}} = \underline{\underline{2.387 \times 10^7 \text{ A}\cdot\text{t/m}}}$$

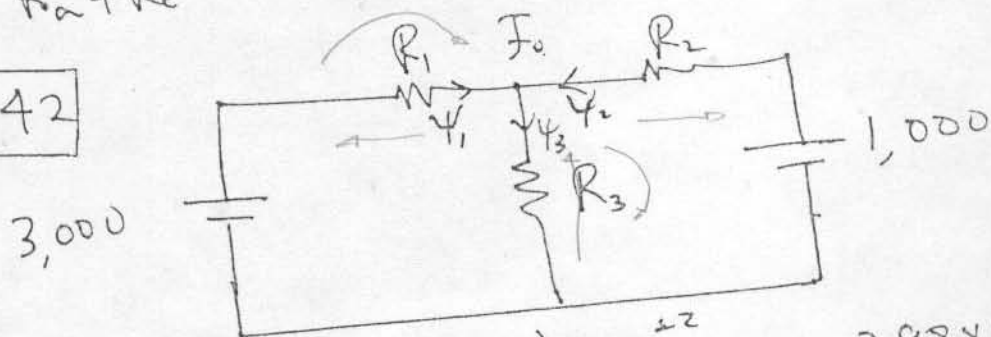
$$R = R_a + R_c = 2.502 \times 10^7 \text{ A}\cdot\text{t/m}$$

$$\begin{aligned} \Psi = \frac{F}{R} = \Psi_a = \Psi_c &= \frac{20,000}{2.502 \times 10^7} \\ &= \underline{\underline{8 \times 10^{-4} \text{ Wb/m}^2}}. \end{aligned}$$

$$I_a = \frac{R_c}{R_a + R_c} I = \frac{2.387 \times 20,000}{2.502} = \underline{\underline{19,081 \text{ A}\cdot\text{t}}}$$

$$I_c = \frac{R_a}{R_a + R_c} \cdot I = \frac{0.115}{2.502} \times 20,000 = \underline{\underline{919 \text{ A}\cdot\text{t}}}$$

Prob. 8.42



$$R_1 = R_2 = \frac{l}{\mu S} = \frac{(6+6+8) \times 10^{-2}}{4\pi \times 10^{-7} \times 1000 \times 4 \times 10^{-4}} = 3.98 \times 10^5$$

$$R_3 = \frac{8 \times 10^{-2}}{4\pi \times 10^{-7} \times 1000 \times 6 \times 10^{-4}} = 1.061 \times 10^5$$

$$V_1 + V_2 = V_3 \rightarrow \frac{I_0 - 3000}{R_1} + \frac{I_0 - 1000}{R_2} = \frac{0 - I_0}{R_3}$$

Since $R_1 = R_2$,

$$-I_0 = \frac{R_3}{R_1} (2I_0 - 4000) = \frac{1.061}{3.98} (2I_0 - 4000)$$

$$I_0 = 695.5 \text{ A}\cdot\text{t}$$

$$B_1 = \frac{V_1}{S_1} = \frac{I_0 - 3000}{R_1 S_1} = \frac{695.5 - 3000}{3.98 \times 10^5 \times 4 \times 10^{-4}} = -14.475$$

$$B_2 = \frac{\psi_2}{S_2} = \frac{I_0 - 1000}{R_2 S_2} = \frac{695.5 - 1000}{3.98 \times 10^5 \times 4 \times 10^{-4}} = -1.913$$

$$B_3 = \frac{\psi_3}{S_3} = \frac{0 - I_0}{R_3 S_3} = \frac{-695.5}{1.061 \times 10^5} = -10.925$$

Thus

$$B_1 = 14.475 \text{ Wb/m}^2, B_2 = 1.913 \text{ Wb/m}^2, B_3 = 10.925 \text{ Wb/m}^2.$$

Prob. 8.43

$$F = \frac{B^2 S}{2\mu_0} = \frac{\psi^2}{2\mu_0 S} = \frac{4 \times 10^{-6}}{2 \times 4\pi \times 10^{-7} \times 0.3 \times 10^{-4}} = 53.05 \text{ kN}$$

Prob. 8.44

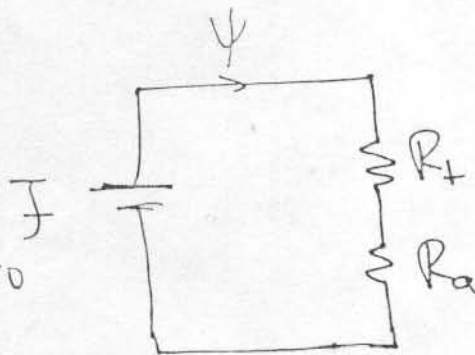
(a)

$$F = NI = 200 \times 10^{-3} \times 750 = 150 \text{ A-t}$$

$$R_a = \frac{l_a}{\mu_0 S} = \frac{10^{-3}}{25 \times 10^{-6} \mu_0} = 3.183 \times 10^7$$

$$R_t = \frac{l_t}{\mu_{\text{iron}} S} = \frac{2\pi \times 0.1}{\mu_0 \times 300 \times 25 \times 10^{-6}} = 20 \times 10^7$$

$$\psi = \frac{F}{R_a + R_t} = \frac{150}{10^7 (3.183 + 20)} = 6.47 \times 10^{-7}$$



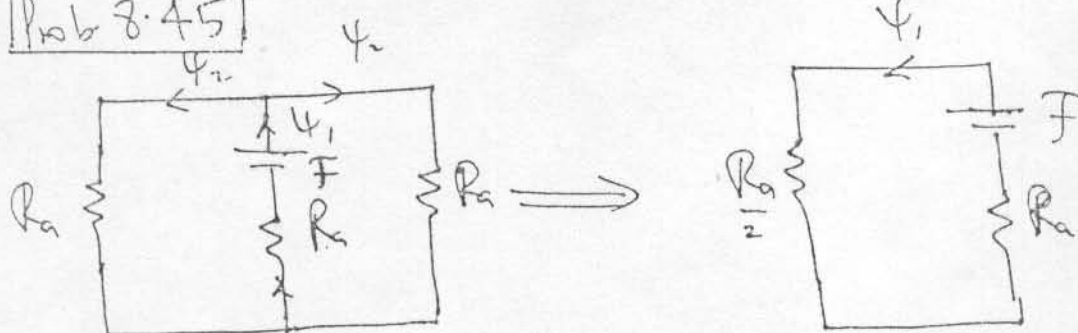
$$F = \frac{B^2 S}{2\mu_0} = \frac{\psi^2}{2\mu_0 S} = \frac{41.861 \times 10^{-14}}{2 \times 4\pi \times 10^{-7} \times 25 \times 10^{-6}} = \underline{\underline{6.66 \text{ mN}}}$$

(b) If $\mu_t \rightarrow \infty$, $R_t = 0$, $\psi = \frac{F}{R_a} = \frac{150}{3.183 \times 10^7}$

$$F_2 = L dl_1 \cdot B_1 = L dl_2 \frac{\psi_1}{S} = \frac{2 \times 10^{-3} \times 5 \times 10^{-3} \times 150}{3.183 \times 10^7 \times 25 \times 10^{-6}}$$

$$F_2 = \underline{\underline{1.885 \text{ nN}}}$$

Prob 8.45



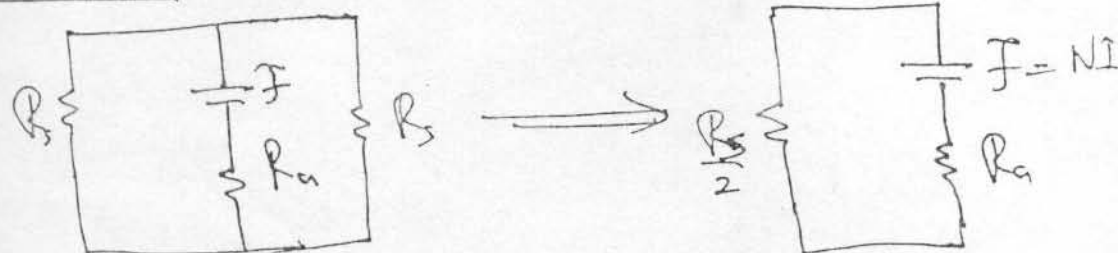
$$\psi_1 = 2\psi_2, \quad \psi_1 = \frac{F}{\frac{3}{2}R_a} = \frac{2F}{3R_a} \rightarrow \psi_2 = \frac{F}{3R_a}$$

$$F = 2 \left(\frac{\psi_2^2}{2\mu_0 S} \right) + \frac{\psi_1^2}{2\mu_0 S} = \frac{3\psi_1^2}{\mu_0 S} = \frac{F^2}{3R_a^2 \mu_0}$$

$$= \frac{\mu_0 S F^2}{3l_a^2} = \frac{4\pi \times 10^{-7} \times 210 \times 10^{-4} \times 9 \times 10^6}{3 \times 10^{-6}}$$

$$= 24\pi \times 10^3 = mg \rightarrow m = \frac{24\pi \times 10^3}{9.8} = \underline{\underline{7694 \text{ kg}}}$$

Prob. 8.46



Since $\mu \rightarrow \infty$ for the core, $R_c = 0$.

$$I = NI = \psi \left(R_a + \frac{R_s}{2} \right) = \frac{\psi (a/2 + x)}{\mu_0 S}$$

$$= \frac{\psi (2x + a)}{2\mu_0 S}$$

$$F = \frac{B^2 S}{2\mu_0} = \mu \frac{\psi^2}{2\mu_0 S} = \frac{1}{2\mu_0 S} \cdot \frac{N^2 I^2 4\mu_0 S^2}{(a + 2x)^2}$$

$$= \frac{2N^2 I^2 \mu_0 S}{(a + 2x)^2}$$

$\vec{F} = -F \hat{a}_x$ since the force is attractive, i.e.

$$\vec{F} = \frac{-2N^2 I^2 \mu_0 S \hat{a}_x}{(a + 2x)^2}$$

$$\frac{B^2 S}{2\mu_0} = \frac{\psi (a + 2x)}{2N\mu_0 S} = F$$

$$\frac{B^2 S^2}{\psi (a + 2x)}$$

CHAPTER 9

P.E. 9.1

$$(a) V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l} = uB\ell = 8(0.5)(0.1) = \underline{\underline{0.4 V}}$$

$$(b) \bar{I} = \frac{V_{emf}}{R} = \frac{0.4}{20} = \underline{\underline{20 mA}}$$

$$(c) \vec{F}_m = I\vec{\ell} \times \vec{B} = 0.2(0.1\vec{a}_y \times -0.5\vec{a}_z) = \underline{\underline{-\vec{a}_x mN}}$$

$$(d) P = f u = I^2 R = 8 mW$$

$$\cdot \quad \text{or } P = \frac{V_{emf}^2}{R} = \frac{(0.4)^2}{20} = \underline{\underline{8 mW}}$$

P.E. 9.2

$$(a) V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$\text{where } \vec{B} = B_0 \vec{a}_y = B_0(\sin \phi \vec{a}_\phi + \cos \phi \vec{a}_\theta), B_0 = 0.05$$

$$(\vec{u} \times \vec{B}) \cdot d\vec{l} = -\rho \omega B_0 \sin \phi dz = -0.2\pi \sin(\omega t + j/2) dz$$

$$V_{emf} = \int_0^{0.03} (\vec{u} \times \vec{B}) \cdot d\vec{l} = -6\pi \cos 100\pi t \text{ mV}$$

$$\text{At } t = 1 \text{ ms, } V_{emf} = -6\pi \cos 0.1\pi = \underline{\underline{-17.93 mV}}$$

$$i = \frac{V_{emf}}{R} = -60\pi \cos 100\pi t \text{ mA}$$

$$\text{At } t = 3 \text{ ms, } i = -60\pi \cos 0.3\pi = \underline{\underline{-0.1108 A}}$$

(b) Method 1:

$$\begin{aligned}\Psi &= \int \vec{B} \cdot d\vec{s} = \int B_0 t (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi) \cdot d\rho dz \vec{a}_\phi \\ &= - \int_0^{\rho_0} \int_0^{z_0} B_0 t \sin \phi d\rho dz = - B_0 \rho_0 z_0 t \sin \phi\end{aligned}$$

where $B_0 = 0.02$, $\rho_0 = 0.04$, $z_0 = 0.03$,
 $\phi = \omega t + \pi/2$.

$$\Psi = - B_0 \rho_0 z_0 t \cos \omega t$$

$$\begin{aligned}V_{emf} &= - \frac{\partial \Psi}{\partial t} = B_0 \rho_0 z_0 \cos \omega t - B_0 \rho_0 z_0 t \omega \sin \omega t \\ &= (0.02)(0.04)(0.03) [\cos \omega t - \omega t \sin \omega t] \\ &= 24 [\cos \omega t - \omega t \sin \omega t] \mu V\end{aligned}$$

Method 2:

$$V_{emf} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$\vec{B} = B_0 t \vec{a}_x = B_0 t (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi), \quad \phi = \omega t + \pi/2$$

$$\frac{\partial \vec{B}}{\partial t} = B_0 (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi)$$

Note that only explicit dependence of $\vec{B}$ on time is accounted for, i.e. we make $\phi = \text{constant}$ because it is transformer (stationary) emf. Thus,

$$V_{emf} = -B_0 \int_0^{\rho_0} \int_0^{z_0} (\cos \phi \vec{a}_\rho - \sin \phi \vec{a}_\phi) \cdot d\rho dz \vec{a}_\phi$$

$$+ \int_0^{z_0} -\rho_0 \omega B_0 t \cos \phi dz$$

$$= B_0 \rho_0 z_0 (\sin \phi + \omega t \cos \phi), \quad \phi = \omega t + \pi/2$$

$$= B_0 \rho_0 z_0 (\cos \omega t - \omega t \sin \omega t)$$

as obtained earlier.

At $t = 1 \text{ ms}$,

$$V_{emf} = 24 \left[\cos 18^\circ - 100\pi \times 10^{-3} \sin 18^\circ \right] \mu V$$

$$= \underline{\underline{20.5 \mu V.}}$$

At $t = 3 \text{ ms}$,

$$i = 240 \left[\cos 54^\circ - 0.3\pi \sin 54^\circ \right] \text{ mA}$$

$$= \underline{\underline{-41.92 \text{ mA.}}}$$

P.E. 9.3

$$V_1 = -N_1 \frac{d\psi}{dt}, \quad V_2 = -N_2 \frac{d\psi}{dt}$$

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \rightarrow V_2 = \frac{N_2}{N_1} V_1 = \frac{300 \times 120}{500} = \underline{\underline{72 \text{ V}}}$$

P.E. 9.4

$$(a) \vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \underline{\underline{-20\omega\epsilon_0 \sin(\omega t - 50x) \vec{a}_y \text{ A/m}^2}}$$

$$(b) \nabla \times \vec{H} = \vec{J}_d \rightarrow -\frac{\partial H_z}{\partial x} \vec{a}_y = -20 \omega \epsilon_0 \sin(\omega t - 50x) \vec{a}_y$$

$$\text{or } \vec{H} = \frac{20 \omega \epsilon_0}{50} \cos(\omega t - 50x) \vec{a}_z$$

$$= \underline{\underline{0.4 \omega \epsilon_0 \cos(\omega t - 50x) \vec{a}_z \text{ A/m.}}}$$

$$(c) \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x} \vec{a}_z = 0.4 \mu_0 \omega \epsilon_0 \sin(\omega t - 50x) \vec{a}_z$$

$$1000 = 0.4 \mu_0 \epsilon_0 \omega^2 = 0.4 \frac{u^2}{c^2}$$

$$\text{or } \underline{\underline{\omega = 1.5 \times 10^{10} \text{ rad/s.}}}$$

P.E. 9.5

$$(a) j^3 \left(\frac{1+j}{2-j} \right)^2 = -j \left[\frac{\sqrt{2} \angle 45^\circ}{\sqrt{5} \angle -26.56^\circ} \right]^2 = -j \left(\frac{2}{5} \angle 143.13^\circ \right)$$

$$= \underline{\underline{0.24 + j 0.32}}$$

$$(b) 6 \angle 30^\circ + j5 - 3 + e^{j45^\circ} = 5.196 + j3 + j5 - 3 + 0.7071(1+j)$$

$$= \underline{\underline{2.903 + j8.707}}$$

P.E. 9.6

$$\vec{P} = 2 \sin(10t + x - \pi/4) \vec{a}_y$$

$$= 2 \cos(10t + x - \pi/4 - \pi/2) \vec{a}_y, \omega = 10$$

$$= \text{Re} [2 e^{j(x - 3\pi/4)} \vec{a}_y e^{j\omega t}] = \text{Re} [\vec{P}_s e^{j\omega t}]$$

$$\text{ie. } \underline{\underline{\vec{P}_s = 2 e^{j(x - 3\pi/4)} \vec{a}_y}}$$

$$\begin{aligned}\tilde{Q} &= \text{Re} [\tilde{Q}_s e^{j\omega t}] = \text{Re} [e^{j(x+\omega t)} (\tilde{a}_x - \tilde{a}_z)] \sin \pi y \\ &= \underline{\underline{\sin \pi y \cos(\omega t + x) (\tilde{a}_x - \tilde{a}_z)}}.\end{aligned}$$

P.E. 9.7

$$\begin{aligned}-\int \frac{\partial \tilde{H}}{\partial t} = \nabla \times \tilde{E} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (E_\phi \sin \theta) \tilde{a}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \tilde{a}_\phi \\ &= \frac{2 \cos \theta}{r^2} \cos(\omega t - \beta r) \tilde{a}_r - \frac{\beta}{r} \sin \theta \sin(\omega t - \beta r) \tilde{a}_\phi\end{aligned}$$

$$\tilde{H} = \frac{2 \cos \theta}{\omega r^2} \sin(\omega t - \beta r) \tilde{a}_r + \frac{\beta}{\omega r} \sin \theta \cos(\omega t - \beta r) \tilde{a}_\phi.$$

$$\beta = \frac{\omega}{c} = \frac{6 \times 10^7}{3 \times 10^8} = \underline{\underline{0.2 \text{ rad/m}}}.$$

$$\begin{aligned}\tilde{H} &= \frac{10^{-7}}{3r^2} \cos \theta \sin(6 \times 10^7 t - 0.2r) \tilde{a}_r \\ &\quad + \frac{10^{-8}}{3r} \sin \theta \cos(6 \times 10^7 t - 0.2r) \tilde{a}_\phi.\end{aligned}$$

P.E. 9.8

$$\omega = \frac{3}{\sqrt{\mu \epsilon}} = \frac{3c}{\sqrt{\epsilon_r \mu_r}} = \frac{9 \times 10^8}{\sqrt{10}} = \underline{\underline{2.846 \times 10^8 \text{ rad/s}}}.$$

$$\begin{aligned}\tilde{E} &= \frac{1}{\epsilon} \int \nabla \times \tilde{H} dt = -\frac{6}{\omega \epsilon} \cos(\omega t - 3y) \tilde{a}_x \\ &= \frac{-6}{\frac{9 \times 10^8}{\sqrt{10}} \cdot \frac{10^{-9}}{36} (5)} \cos(\omega t - 3y) \tilde{a}_x\end{aligned}$$

$$\underline{\underline{\tilde{E} = -476.8 \cos(2.846 \times 10^8 t - 3y) \tilde{a}_x \text{ V/m}}}.$$

Prob. 9.1

$$\begin{aligned}
 V &= -\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s} = -\frac{\partial \vec{B}}{\partial t} \cdot \vec{s} \\
 &= 3770 \sin 377t \quad \pi (0.2)^2 \times 10^{-3} \\
 &= \underline{\underline{0.4738 \sin 377t \text{ V}}}
 \end{aligned}$$

Prob. 9.2

$$\begin{aligned}
 V_{\text{emf}} &= -\frac{\partial \lambda}{\partial t} = -N \frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s} = -NBS \frac{d\phi}{dt} \\
 &= -NBS\omega = -50 \times 0.06 \times 0.3 \times 0.4 \\
 &= \underline{\underline{-54 \text{ V}}}
 \end{aligned}$$

Prob 9.3

$$\begin{aligned}
 \text{(a)} \quad I &= \frac{V}{R} = \frac{1}{R} \frac{\partial \Psi}{\partial t} = \frac{1}{R} S \frac{\partial B}{\partial t} \\
 &= \underline{\underline{0.3 \times 0.5 \times 0.05 = 0.75 \text{ mA}}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \Psi &= \int \vec{B} \cdot d\vec{s} = \int_{y=0}^{10} \int_{x=0}^{0.3} 5 \sin (120\pi t - 2y) \times 10^{-3} dx dy \\
 &= \frac{2.5}{2} \cos (120\pi t - 2y) \Big|_0^{0.3} \text{ mWb} \\
 &= 1.25 [\cos (120\pi t - 0.6) - \cos 120\pi t] \text{ mWb} \\
 I &= \frac{1}{R} \frac{\partial \Psi}{\partial t} = \frac{1}{10} \times 1.25 (-120\pi) \times 10^{-3} [\sin (120\pi t - 0.6) - \sin 120\pi t]
 \end{aligned}$$

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$$\begin{aligned}
 I &= -15\pi [\sin(120\pi t - 0.6) - \sin 120\pi t] \text{ mA} \\
 &= -15\pi(2) \cos(\omega t - 0.3) \sin(-0.3) \\
 &= \underline{\underline{27.85 \cos(120\pi t - 0.3) \text{ mA}}}
 \end{aligned}$$

Prob. 9.4

$$\vec{B} = \frac{\mu_0 I}{2\pi y} (-\hat{a}_x)$$

$$\Psi = \int \vec{B} \cdot d\vec{s} = \frac{\mu_0 I}{2\pi} \int_{z=0}^a \int_{y=p}^{p+a} \frac{dz dy}{y} = \frac{\mu_0 I a}{2\pi} \ln \frac{p+a}{p}$$

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t} = -\frac{\partial \Psi}{\partial p} \cdot \frac{\partial p}{\partial t} = -\frac{\mu_0 I a}{2\pi} \mu_0 \frac{d}{dp} [\ln(p+a) - \ln p]$$

$$= -\frac{\mu_0 I a \mu_0}{2\pi} \left[\frac{1}{p+a} - \frac{1}{p} \right] = \underline{\underline{\frac{\mu_0 a^2 I \mu_0}{2\pi p(p+a)}}}$$

Prob 9.5

This is similar to Prob. 9.4. Assume loop is of width z .

$$\Psi = \frac{\mu_0 I z}{2\pi} \ln \frac{p+a}{p}$$

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t} = -\frac{d\Psi}{dz} \cdot \frac{dz}{dt} = -\frac{\mu_0 I}{2\pi} \ln \frac{p+a}{p} \cdot u$$

$$= -\frac{4\pi \times 10^{-7}}{2\pi} \times 15 \times 3 \ln \frac{60}{20} = -9.888 \mu\text{V}$$

Thus the induced emf = 9.888 mV, point A at higher potential.

Prob 9.6

$$V_{\text{emf}} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

where $\vec{B} = B_0 \cos \omega t \vec{a}_x$, $\vec{u} = u_0 \cos \omega t \vec{a}_y$, $d\vec{l} = dz \vec{a}_z$.

$$V_{\text{emf}} = \int_{z=0}^L \int_{y=-a}^y B_0 \omega \sin \omega t \, dy \, dz - \int_0^L B_0 u_0 \cos^2 \omega t \, dz$$

$$= B_0 \omega L (y+a) \sin \omega t - B_0 u_0 L \cos^2 \omega t$$

Alternatively,

$$\psi = \int \vec{B} \cdot d\vec{s} = \int_{z=0}^L \int_{y=-a}^y B_0 \cos \omega t \vec{a}_x \cdot dy \, dz \vec{a}_x$$

$$= B_0 (y+a) L \cos \omega t$$

$$V_{\text{emf}} = - \frac{\partial \psi}{\partial t} = B_0 (y+a) L \omega \sin \omega t - B_0 \frac{dy}{dt} L \cos \omega t$$

But $\frac{dy}{dt} = u = u_0 \cos \omega t \rightarrow y = \frac{u_0}{\omega} \sin \omega t$

$$V_{\text{emf}} = B_0 \omega L (y+a) \sin \omega t - B_0 u_0 L \cos^2 \omega t$$

$$= B_0 u_0 L \sin^2 \omega t + B_0 \omega a L \sin \omega t - B_0 u_0 L \cos^2 \omega t$$

$$= -B_0 u_0 L \cos 2\omega t + B_0 \omega a L \sin \omega t$$

$$= 6 \times 10^{-3} \times 5 [10 \times (0 \sin 10t - 2 \cos 20t)]$$

$$V_{emf} = \underline{3 \sin 10t - 0.06 \cos 20t \text{ V.}}$$

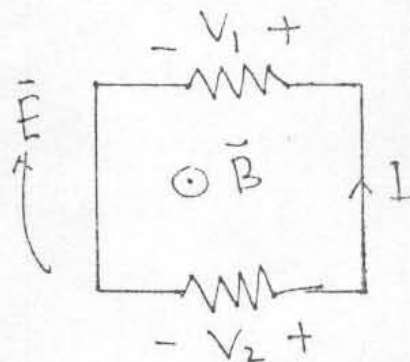
Prob. 9.7

$$\begin{aligned} V_{emf} &= \int (\vec{u} \times \vec{B}) \cdot d\vec{l} = uBl \\ &= \frac{80 \times 10^3 \times 70 \times 10^{-6} \times 2.5}{3600} \\ &= \underline{3.89 \text{ mV.}} \end{aligned}$$

Prob. 9.8

$$\begin{aligned} \oint \vec{E} \cdot d\vec{l} &= -\frac{d}{dt} \int \vec{B} \cdot d\vec{S} \\ &= -L(R_1 + R_2) \end{aligned}$$

$$\frac{dB}{dt} \cdot S = L(R_1 + R_2) \quad - (1)$$



$$\text{Also, } \oint \vec{E} \cdot d\vec{l} = V_1 - V_2 = -\frac{dB}{dt} \cdot S \quad - (2)$$

$$\text{Hence, } V_1 = IR_1 = -\frac{SR_1}{R_1 + R_2} \frac{dB}{dt}$$

$$V_2 = -IR_2 = -\frac{SR_2}{R_1 + R_2} \frac{dB}{dt}$$

$$V_1 = \frac{10 \times 10^{-4} \times 10}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{0.0628 \sin 150\pi t \text{ V}}$$

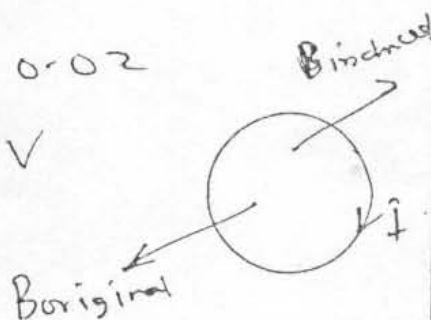
$$V_2 = \frac{-10 \times 10^{-4} \times 5}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{-0.0314 \sin 150\pi t \text{ V}}$$

Prob. 9.9

$$d\psi = 0.63 - 0.45 = 0.18, \quad dt = 0.02$$

$$V_{\text{emf}} = N \frac{d\psi}{dt} = 10 \left(\frac{0.18}{0.02} \right) = 90 \text{ V}$$

$$I = \frac{V_{\text{emf}}}{R} = \frac{90}{15} = \underline{\underline{6 \text{ A}}}$$



Using Lenz's law, the direction of the induced current is counterclockwise

Prob. 9.10

$$R = R_c + R_s = \frac{1}{\mu_0 S} \left(\frac{L_c}{N} + \frac{L_s}{1} \right)$$

$$= \frac{1}{4\pi \times 10^{-7} \times 4 \times 10^{-4}} \left[6 \times 10^{-3} + \frac{(40 - 0.6) \cdot 10^{-2}}{1000} \right]$$

$$= 12.72 \times 10^6$$

$$i = \frac{\mathcal{E}}{R} = \frac{N_1 I_1}{R}, \quad V_2 = -N_2 \frac{\partial \psi}{\partial t} = -\frac{N_1 N_2}{R} \frac{dI_1}{dt}$$

$$V_2 = \frac{500(300) 1200\pi \sin 120\pi t}{12.72 \times 10^6}$$

$$= \underline{\underline{44.46 \sin 120\pi t \text{ V.}}}$$

Prob. 9.11

$$V = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}, \text{ where } \vec{u} = \rho \omega \vec{a}_\phi, \vec{B} = B_0 \vec{a}_z$$

$$V = \int_{R_1}^{R_2} \rho \omega B_0 d\rho = \frac{\omega B_0}{2} (R_2^2 - R_1^2)$$

$$V = \frac{60 \times 5 \cdot 10^{-3}}{2} (100 - 4) \cdot 10^{-4} = \underline{\underline{4.32 \text{ mV}}}$$

Prob. 9.12

$$\begin{aligned} J_{ds} = j\omega D_s &\rightarrow |J_{ds}|_{\max} = \omega \epsilon E_s = \omega \epsilon \frac{V_s}{d} \\ &= \frac{10^{-9}}{36\pi} \times \frac{2\pi \times 20 \times 10^6 \times 50}{0.2 \times 10^{-3}} \\ &= \underline{\underline{277.8 \text{ A/m}^2}} \end{aligned}$$

$$I_{ds} = J_{ds} \cdot S = \frac{1000}{36} \times 2.8 \times 10^{-4} = \underline{\underline{77.78 \text{ mA}}}$$

Prob. 9.13

$$J = \sigma E = \underline{\underline{4 \cos 10^7 t \text{ nA/m}^2}}$$

$$\begin{aligned} J_d = \frac{\partial D}{\partial t} &= \epsilon_0 \epsilon_0 \frac{\partial E}{\partial t} = -20 \times 10^7 \times 10^{-6} \times \frac{10^{-9}}{36\pi} \times 5 \sin 10^7 t \\ &= \underline{\underline{-8.842 \sin 10^7 t \text{ nA/m}^2}} \end{aligned}$$

Prob. 9.14

$$J_c = \frac{J_c}{S} = \sigma E \rightarrow E = \frac{J_c}{\sigma S}$$

$$J_d = j\omega \epsilon E \rightarrow |J_d| = \omega \epsilon E = \frac{\omega \epsilon J_c}{\sigma S}$$

$$|J_d| = \frac{10^9 \times 4.6 \times 10^{-9} / 36\pi \times 0.2 \times 10^{-3}}{2.5 \times 10^6 \times 10 \times 10^{-4}} \text{ A/m} = \underline{\underline{3.254 \text{ nA/m}^2}}$$

Prob. 9.15

If $|\vec{J}_d| = |\vec{J}|$, then $\epsilon \omega |\vec{E}| = \sigma |\vec{E}| \quad \text{or}$
 $2\pi f \epsilon_0 \epsilon_r = \sigma \rightarrow f = \frac{\sigma}{2\pi \epsilon_0 \epsilon_r} = \frac{4 \times 10^{-3}}{2\pi \times \frac{10^{-9}}{36\pi} \times 80}$
 $= \underline{\underline{900 \text{ kHz}}}$

Prob. 9.16

(a) $\nabla \cdot \vec{E}_s = \rho_s / \epsilon, \quad \nabla \cdot \vec{H}_s = 0$
 $\nabla \times \vec{E}_s = j\omega \mu \vec{H}_s, \quad \nabla \times \vec{H}_s = (\sigma - j\omega \epsilon) \vec{E}_s$

(b) $\nabla \cdot \vec{D} = \rho_v \rightarrow \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v \quad - (1)$

$\nabla \cdot \vec{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad - (2)$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad - (3)$

$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad - (4)$

$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad - (5)$

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \quad - (6)$

$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \quad - (7)$

$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \quad - (8)$

Prob. 9.19

If $\rho_v = 0$, $\nabla \cdot \vec{E} = 0$ — (1)

$$\vec{H} = -\frac{1}{\mu_0} \int \nabla \times \vec{E} \, dt$$

But $\nabla \times \vec{E} = -30k \sin 2x \cos(kz - \omega t) \vec{a}_x$
 $+ 60 \cos 2x \sin(kz - \omega t) \vec{a}_z$

$$\vec{H} = \frac{-30k}{\mu_0 \omega} \sin 2x \sin(kz - \omega t) \vec{a}_x$$

$$- \frac{60}{\mu_0 \omega} \cos 2x \cos(kz - \omega t) \vec{a}_z \quad \text{A/m.}$$

$\nabla \cdot \vec{H} = 0$ — (2)

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \underline{\underline{-30\omega\epsilon_0 \sin 2x \cos(kz - \omega t) \vec{a}_y \quad \text{A/m}^2}}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \vec{a}_y$$

$$= \frac{-30\epsilon_0}{\omega} \sin 2x \cos(kz - \omega t) \left[\frac{k^2}{\mu_0 \epsilon_0} + \frac{4}{\mu_0 \epsilon_0} \right] \vec{a}_y$$

$$= -30\epsilon_0 \omega \sin 2x \cos(kz - \omega t) \vec{a}_y = \vec{J}_d \quad \text{--- (3)}$$

Since $\frac{k^2}{\mu_0 \epsilon_0} + \frac{4}{\mu_0 \epsilon_0} = \omega^2$. Thus the four Maxwell's equations are satisfied by $\vec{E}$ showing that $\vec{E}$ is a genuine EM field.

Prob. 9.20

$$\epsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = -\frac{\partial H_z}{\partial x} \vec{a}_y$$

$$= 0.6\beta \sin \beta x \cos \omega t \vec{a}_y$$

$$\vec{E} = \frac{1}{\epsilon} \int \nabla \times \vec{H} dt = \frac{0.6\beta}{\omega \epsilon} \sin \beta x \sin \omega t \vec{a}_y$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial x} \vec{a}_z$$

$$= \frac{0.6\beta^2}{\omega \epsilon} \cos \beta x \sin \omega t \vec{a}_z$$

$$\vec{H} = -\frac{1}{\mu} \int \nabla \times \vec{E} dt = \frac{0.6\beta^2}{\omega^2 \mu \epsilon} \cos \beta x \cos \omega t \vec{a}_z$$

$$\text{Thus } \beta = \omega \sqrt{\mu \epsilon} = \frac{\omega}{c} \sqrt{\mu_r \epsilon_r} = \frac{10^8 (2.25)}{3 \times 10^8}$$

$$= \underline{\underline{0.8333 \text{ rad/m.}}}$$

$$E_0 = \frac{0.6\beta}{\omega \epsilon} = \frac{0.6 \omega \sqrt{\mu \epsilon}}{\omega} = 0.6 \sqrt{\frac{\mu_r}{\epsilon_r}} = 0.6 (377) \sqrt{\frac{\mu_r}{\epsilon_r}}$$

$$= \frac{0.6 \times 377}{2.25} = 100.5$$

$$\vec{E} = \underline{\underline{100.5 \sin \beta x \sin \omega t \vec{a}_y \text{ V/m.}}}$$

Prob. 9.21

from Maxwell's equations,

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad - (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad - (2)$$

Dotting both sides of (2) with $\vec{E}$ gives

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad - (3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Applying this on the left-hand side of (3) by letting $\vec{A} \equiv \vec{B}$ and $\vec{B} \equiv \vec{E}$, we get

$$\vec{H} \cdot (\nabla \times \vec{E}) + \nabla \cdot (\vec{H} \times \vec{E}) = \vec{E} \cdot \vec{J} + \frac{1}{2} \frac{\partial (\vec{D} \cdot \vec{E})}{\partial t} \quad - (4)$$

from (1),

$$\vec{H} \cdot (\nabla \times \vec{E}) = \vec{H} \cdot \left(- \frac{\partial \vec{B}}{\partial t} \right) = - \frac{1}{2} \frac{\partial (\vec{B} \cdot \vec{H})}{\partial t}$$

Substituting this in (4) gives

$$- \frac{1}{2} \frac{\partial (\vec{B} \cdot \vec{H})}{\partial t} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \frac{1}{2} \frac{\partial (\vec{D} \cdot \vec{E})}{\partial t}$$

Rearranging terms and then taking the volume integral of both sides

$$\int_V \nabla \cdot (\vec{E} \times \vec{H}) dV = - \frac{\partial}{\partial t} \frac{1}{2} \int_V (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) dV - \int_V \vec{J} \cdot \vec{E} dV$$

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = - \frac{\partial W}{\partial t} - \int_V \vec{J} \cdot \vec{E} dV$$

$$\frac{\partial W}{\partial t} = - \oint (\vec{E} \times \vec{H}) \cdot d\vec{s} - \int \vec{E} \cdot \vec{J} dv$$

as required.

Prob. 9.22

In air, $\sigma=0$, $\epsilon=\epsilon_0$, $\mu=\mu_0$

$$\nabla \times \vec{H} = \vec{J}_d = -\frac{\partial H_0}{\partial z} \vec{a}_\phi$$

$$\vec{J}_d = \frac{10}{3\rho} \cos(10^8 t - \frac{z}{3}) \vec{a}_\phi \text{ A/m}^2.$$

$$\begin{aligned} \vec{J}_d = \frac{\partial \vec{D}}{\partial t} &\rightarrow \vec{E} = \frac{1}{\epsilon_0} \int \vec{J}_d dt \\ &= \frac{10}{3\rho} \cdot \frac{1}{10^8 \epsilon_0} \sin(10^8 t - \frac{z}{3}) \vec{a}_\phi \\ &= \frac{1200\pi}{\rho} \sin(10^8 t - \frac{z}{3}) \vec{a}_\phi \text{ V/m.} \end{aligned}$$

Prob 9.23

$$\begin{aligned} \nabla \times \vec{E} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) \vec{a}_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 t e^{-\rho-t}) \vec{a}_z \\ &= (2-\rho) t e^{-\rho-t} \vec{a}_z \end{aligned}$$

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \nabla \times \vec{E} \rightarrow \vec{B} = - \int (\nabla \times \vec{E}) dt \\ &= \int \underbrace{(\rho-2)}_v \underbrace{t e^{-\rho-t}}_{d\psi} dt \vec{a}_z \end{aligned}$$

Integrating by parts yields

$$\vec{B} = [-(p-2)t e^{-p-t} + \int (p-2) e^{-p-t} dt] \vec{a}_z$$

$$= \underline{(2-p)(1+t) e^{-p-t} \vec{a}_z \text{ Wb/m}^2}.$$

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial \rho} \vec{a}_\phi$$

$$= -\frac{1}{\mu_0} (1+t) (-1-2t+p) e^{-p-t} \vec{a}_\phi$$

$$\vec{J} = \frac{(1+t)(3-p) 10^7 e^{-p-t}}{4\pi} \vec{a}_\phi \text{ A/m}^2$$

Prob. 9.24

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A} \quad (1)$$

$$\nabla \cdot \vec{A} = \frac{\mu}{4\pi} e^{j\omega(t-r/c)} \cos \theta \left(\frac{1}{r^2} + \frac{j\omega}{cr} \right)$$

$$\nabla(\nabla \cdot \vec{A}) = \frac{\mu}{4\pi} e^{j\omega(t-r/c)} \left[\cos \theta \left(\frac{2j\omega}{cr^2} - \frac{\omega^2}{c^2 r} + \frac{2}{r^3} \right) \vec{a}_r \right.$$

$$\left. + \sin \theta \left(\frac{1}{r^3} + \frac{j\omega}{cr^2} \right) \vec{a}_\theta \right] \quad (2)$$

$$\nabla \times \vec{A} = \frac{\mu}{4\pi} e^{j\omega(t-r/c)} \sin \theta \left(\frac{j\omega}{cr} + \frac{1}{r^2} \right) \vec{a}_\phi$$

$$\nabla \times \nabla \times \vec{A} = \frac{\mu}{4\pi} e^{j\omega(t-r/c)} \left[\cos \theta \left(\frac{2j\omega}{cr^2} + \frac{2}{r^3} \right) \vec{a}_r \right.$$

$$\left. + \sin \theta \left(-\frac{\omega^2}{c^2 r} + \frac{j\omega}{cr^2} + \frac{1}{r^3} \right) \vec{a}_\theta \right] \quad (3)$$

Substituting (2) and (3) into (1) give

$$\nabla^2 \vec{A} = -\frac{\mu^2}{4\pi c^2 r} (\cos\theta \vec{a}_r - \sin\theta \vec{a}_\theta) e^{j\omega(t-r/c)} \quad (4)$$

$$\mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{\mu^2}{c^2} \vec{A} = -\frac{\mu^2}{4\pi c^2 r} (\cos\theta \vec{a}_r - \sin\theta \vec{a}_\theta) e^{j\omega(t-r/c)} \quad (5)$$

Thus

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

But $\nabla \cdot \vec{A} = -\mu\epsilon \frac{\partial V}{\partial t} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$

$$\begin{aligned} \text{or } V &= -c^2 \int \nabla \cdot \vec{A} dt \\ &= \frac{\mu_0 \cos\theta}{4\pi c} e^{j\omega(t-r/c)} \left(\frac{1}{cr} + \frac{1}{j\omega r^2} \right) \end{aligned}$$

Prob. 9.25

$$\nabla^2 (\nabla \cdot \vec{A}) = \nabla \cdot \nabla^2 \vec{A}$$

But $\nabla \cdot \vec{A} = -\mu \vec{J}$ and $\nabla \cdot \vec{A} = -\mu\epsilon \frac{\partial V}{\partial t}$

$$-\mu\epsilon \frac{\partial}{\partial t} \nabla^2 V = -\mu \nabla \cdot \vec{J}$$

Also $\nabla^2 V = -\frac{\rho_v}{\epsilon}$, $-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J}$

which is the continuity equation.

Prob. 9.26

$$(a) \frac{3-j2}{4-j} - \frac{2}{1+j} \neq \frac{(2+j)^{\frac{1}{2}}}{(-3+j5)^*} = (0.8235 - j0.2941)$$

$$-(1-j) + (-0.1791 + j0.1837)$$

$$= \underline{\underline{-0.3554 + j0.8896}}$$

$$(b) -10 \angle -30^\circ + (20 \angle 120^\circ)^* = -8.66 - j5 - 10 - j17.32$$

$$= \underline{\underline{-18.66 - j22.32}}$$

$$(c) 10 \ln(3-j4) = 10 \ln 5 + 10 \ln e^{-j53.1^\circ}$$

$$= \underline{\underline{16.09 + j(20n - 2.95)\pi}}$$

where $n = \text{integer}$.

Prob. 9.27

$$(a) \frac{2+j3}{j(-10+j2)} = \frac{3.606 \angle 56.31^\circ}{10.198 \angle 258.69^\circ} = \underline{\underline{0.354 \angle -202.38^\circ}}$$

$$(b) \frac{8 \angle 30^\circ - 5 \angle 60^\circ}{1+j} = \frac{4.4282 - j0.33}{1+j} = \frac{4.44 \angle -4.26^\circ}{1.414 \angle 45^\circ}$$

$$= \underline{\underline{3.14 \angle -49.26^\circ}}$$

$$(c) \frac{(1+j3)(4-j2)}{(1+j3) + (4-j2)} = \frac{10+j10}{5+j} = \underline{\underline{2.77 \angle 33.7^\circ}}$$

$$(d) \left[\frac{(15-j7)(3+j2)^*}{(4+j6)^* (3 \angle 70^\circ)} \right]^* = \left[\frac{16.555 \angle -25.02^\circ}{7.211 \angle -56.31^\circ} \cdot \frac{3.606 \angle -33.69^\circ}{3 \angle 70^\circ} \right]^*$$

$$= (2.7596 \angle -92.4^\circ)^* = \underline{\underline{2.76 \angle 72.4^\circ}}.$$

$$\begin{aligned} (e) \left[(-10 \angle 30^\circ) (4 e^{j\pi/4}) \right]^{1/2} &= \left[40 \angle 210^\circ + 45^\circ \right]^{1/2} \\ &= \underline{\underline{6.325 \angle 127.5^\circ}}. \end{aligned}$$

Prob. 9.28

$$(a) \vec{A} = \text{Re} \left[5 e^{j\omega t} e^{-j2z} e^{-j\pi/4} \vec{a}_x \right] = \text{Re} \left[\vec{A}_s e^{j\omega t} \right]$$

$$\rightarrow \underline{\underline{\vec{A}_s = -j e^{-j2z} \vec{a}_x}}.$$

$$(b) \vec{B} = \text{Re} \left[15 e^{-2y} e^{j\omega t} e^{-jy} \vec{a}_z \right]$$

$$\underline{\underline{\vec{B}_s = 15 e^{-2y} e^{-jy} \vec{a}_z = 15 e^{-y(2+j)} \vec{a}_z}}.$$

$$(c) \vec{C} = 5 \cos \omega t \vec{a}_y - 8 \cos (\omega t - x - \pi/2) \vec{a}_z$$

$$\underline{\underline{\vec{C}_s = 5 \vec{a}_y - 8 e^{-jx} e^{-j\pi/2} \vec{a}_z = 5 \vec{a}_y + j 8 e^{-jx} \vec{a}_z}}.$$

$$(d) \vec{D} = \text{Re} \left[\frac{10}{\rho} e^{j\omega t} e^{-j3z} \vec{a}_\phi \right]$$

$$\underline{\underline{\vec{D}_s = \frac{10}{\rho} e^{-j3z} \vec{a}_\phi}}.$$

Prob. 9.29

$$(a) \vec{A} = \text{Re} \left[\vec{A}_s e^{j\omega t} \right] = \text{Re} \left[5 e^{j90^\circ} e^{-j20^\circ} e^{j\omega t} \vec{a}_x - 5 x e^{j\alpha} e^{j\omega t} \vec{a}_y \right]$$

$$\text{where } \alpha = \tan^{-1} \frac{4}{3} = 53.13^\circ$$

$$\underline{\underline{\vec{A} = 5 \cos (\omega t + 70^\circ) \vec{a}_x - 5x \cos (\omega t + 53.13^\circ) \vec{a}_y}}.$$

$$\begin{aligned}
 (b) \quad \vec{B} &= \text{Re} \left[10 e^{-jkz} e^{j\omega t} \vec{a}_x + 5 e^{j1/2} e^{jkz + j1/4} e^{j\omega t} \vec{a}_y \right] \\
 &= 10 \cos(\omega t - kz) \vec{a}_x + 5 \cos(\omega t + kz + 1/2 + 1/4) \vec{a}_y \\
 &= \underline{10 \cos(\omega t - kz) \vec{a}_x - 5 \sin(\omega t + kz + 3/4) \vec{a}_y}.
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad c &= \text{Re} [C e^{j\omega t}] \\
 &= \text{Re} \left[2 e^{-j90^\circ} e^{-j3x} e^{j\omega t} \sin x + e^{3x - j4x + j\omega t} \right] \\
 &= 2 \sin x \cos(\omega t - 3x - 90^\circ) + e^{3x} \cos(\omega t - 4x) \\
 &= \underline{2 \sin x \sin(\omega t - 3x) + e^{3x} \cos(\omega t - 4x)}.
 \end{aligned}$$

Prob. 9.30

for time factor $e^{-j\omega t}$, replace every j by $-j$ and obtain

$$\begin{aligned}
 \vec{B}_s &= \nabla \times \vec{A}_s \\
 \vec{E}_s &= -\nabla V_s - j\omega \vec{A}_s \\
 \nabla \cdot \vec{A}_s &= -j\omega \mu \epsilon V_s \\
 \nabla^2 V_s + \omega^2 \mu \epsilon V_s &= -\rho_s / \epsilon \\
 \nabla^2 \vec{A}_s + \omega^2 \mu \epsilon \vec{A}_s &= -\mu \vec{J}_s.
 \end{aligned}$$

Prob. 9.31

$$\vec{E} = \text{Re} [\vec{E}_s e^{j\omega t}] = \underline{20 \sin(k_x x) \sin(k_y y) \cos \omega t \vec{a}_z} \text{ V/m.}$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \frac{\partial E_z}{\partial y} \vec{a}_x - \frac{\partial E_z}{\partial x} \vec{a}_y$$

$$= 20k_y \sin(k_x x) \cos(k_y y) \cos \omega t \hat{a}_x \\ - 20k_x \cos(k_x x) \sin(k_y y) \cos \omega t \hat{a}_y$$

$$\vec{B} = - \int \nabla \times \vec{E} dt$$

$$= -\frac{20}{\omega} \left[k_y \sin(k_x x) \cos(k_y y) \sin \omega t \hat{a}_x \right. \\ \left. - k_x \cos(k_x x) \sin(k_y y) \sin \omega t \hat{a}_y \right] \text{ Wb/m}^2.$$

CHAPTER 10

P.E. 10.1

$$(a) T = \frac{2\pi}{\omega} = \frac{2\pi}{2 \times 10^8} = 10\pi \text{ ns} = \underline{\underline{31.42 \text{ ns}}}.$$

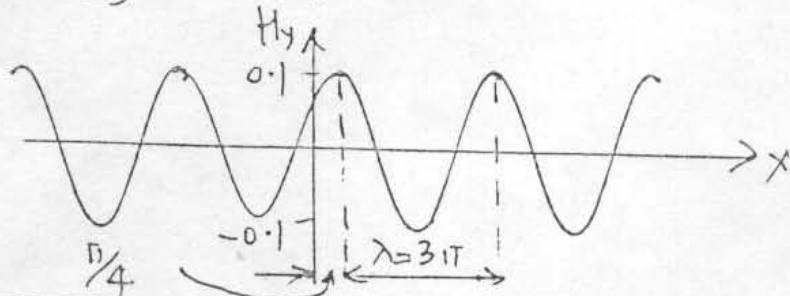
$$\lambda = uT = 3 \times 10^8 \times \frac{\pi}{10^8} = \underline{\underline{9.425 \text{ m}}}.$$

$$k = \beta = \frac{2\pi}{\lambda} = \frac{2}{3} = \underline{\underline{0.667 \text{ rad/m}}}$$

$$(b) t_1 = \frac{T}{8} = \underline{\underline{3.927 \text{ ns}}}.$$

$$(c) \vec{H}(t=t_1) = 0.1 \cos \left(2 \times 10^8 \cdot \frac{\pi}{8 \times 10^8} - \frac{2}{3}x \right) \hat{a}_y \\ = 0.1 \cos \left(\frac{2}{3}x - \frac{\pi}{4} \right) \hat{a}_y$$

as sketched below.



i.e. $\alpha^2 + \beta^2 = \omega\mu \sqrt{(\sigma^2 + \omega^2\epsilon^2)} \quad - (1)$

$\text{Re } \gamma^2 = \alpha^2 - \beta^2 = -\omega^2\mu\epsilon$

or $\beta^2 - \alpha^2 = \omega^2\mu\epsilon \quad - (2)$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right]}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]}$$

(b) from Eq. (10.29), $\vec{E}_s(z) = E_0 e^{-\gamma z} \vec{a}_x$

$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\mu \vec{H}_s \longrightarrow \vec{H}_s = \frac{j}{\omega\mu} \nabla \times \vec{E}_s \\ &= \frac{j}{\omega\mu} (-\gamma E_0 e^{-\gamma z} \vec{a}_y) \end{aligned}$$

But $\vec{H}_0 = H_0 e^{-\gamma z} \vec{a}_y$, hence $H_0 = \frac{E_0}{\eta} = -\frac{j\gamma E_0}{\omega\mu}$

$\eta = \frac{j\omega\mu}{\gamma}$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}, \quad \tan 2\theta_\eta = \left(\frac{\omega\epsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\epsilon}$$

Prob. 10.3

(a) Along the positive x direction.

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \underline{1.0472 \text{ m}}$.

(c) $w = 10^8$, $\beta = 6$, $u = \frac{w}{\beta} = \frac{10^8}{6} = \underline{1.667 \times 10^7 \text{ m/s}}$.

Prob. 10.4

$$(\nabla^2 - \gamma^2)(E_{xs}, E_{ys}, E_{zs}) = (0, 0, 0)$$

i.e. $\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} - \gamma^2 E_{xs} = 0$

$$\frac{\partial^2 E_{ys}}{\partial x^2} + \frac{\partial^2 E_{ys}}{\partial y^2} + \frac{\partial^2 E_{ys}}{\partial z^2} - \gamma^2 E_{ys} = 0$$

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} - \gamma^2 E_{zs} = 0.$$

Prob. 10.5

(a) Let $u = \frac{S}{wE} = \text{loss tangent}$

$$\beta = w \sqrt{\frac{\mu E}{2} [\sqrt{1+u^2} + 1]} = \frac{w}{c} \sqrt{\frac{5 \times 10^2}{2}} \sqrt{\sqrt{1+u^2} + 1}$$

$$10 = \frac{w}{c} \sqrt{5} \sqrt{\sqrt{1+u^2} + 1}$$

$$\text{or } \sqrt{\sqrt{1+u^2} + 1} = \frac{10c}{w\sqrt{5}} = \frac{10 \times 3 \times 10^8}{2\pi \times 5 \times 10^6 \sqrt{5}} = \frac{300}{\pi\sqrt{5}}$$

$$\sqrt{1+u^2} = 1823 \rightarrow u = \frac{S}{wE} = \underline{1823}$$

$$(b) \sigma = \omega \epsilon u = 2\pi \times 5 \times 10^6 \times 1823 \times \frac{10^{-9}}{36\pi}$$

$$= \underline{1.013 \text{ V/m.}}$$

$$(c) \epsilon_c = \epsilon' - j\epsilon'' = \epsilon - j\frac{\sigma}{\omega} = \frac{2 \times 10^{-9}}{36\pi} - j \frac{1.013}{2\pi \times 5 \times 10^6}$$

$$= \underline{1.768 \times 10^{-11} - j3.224 \times 10^{-8} \text{ F/m.}}$$

$$(d) \frac{\alpha}{\beta} = \frac{\sqrt{\sqrt{1+u^2}-1}}{\sqrt{\sqrt{1+u^2}+1}} = \sqrt{\frac{1822}{1824}}$$

$$\alpha = \underline{9.995 \text{ Np/m.}}$$

$$(e) |\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1+u^2}} = \frac{120\pi \sqrt{5/2}}{\sqrt{1823}} = 13.96 \Omega$$

$$\tan 2\theta_\eta = u = 1823 \rightarrow \theta_\eta = 44.98^\circ$$

$$\underline{\eta = 13.96 \angle 44.98^\circ \Omega}$$

Prob. 10.6

$$(a) \frac{\sigma}{\omega \epsilon} = \tan 2\theta_\eta = \tan 60^\circ = \underline{1.732}$$

$$(b) |\eta| = 240 = \frac{120\pi}{\sqrt{\epsilon_r}} = \frac{120\pi}{\sqrt{2\epsilon_r}}$$

$$\epsilon_r = \frac{\pi}{8} = 1.234$$

$$(c) \epsilon_c = \epsilon (1 - j\frac{\sigma}{\omega \epsilon}) = 1.234 \times \frac{10^{-9}}{36\pi} (1 - j1.732)$$

$$\epsilon_c = (1.091 - j1.89) \times 10^{-11} \text{ f/m.}$$

$$\begin{aligned} (d) \alpha &= \frac{W}{c} \sqrt{\frac{\mu_r \epsilon_r}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]} \\ &= \frac{2\pi \times 10^6}{3 \times 10^8} \sqrt{\frac{1}{2} \cdot \frac{\pi^2}{8} \left[\sqrt{1+3} - 1 \right]} = \frac{\pi^2}{600} \\ &= \underline{0.0164 \text{ Np/m.}} \end{aligned}$$

Prob. 10.7

$$\tan 2\theta_\eta = \tan 2 \times 35.26^\circ = 2.827 = \frac{\sigma}{\omega \epsilon}$$

$$\text{loss tangent} = \underline{2.827}$$

$$\text{loss angle} = \theta = 2\theta_\eta = \underline{70.52^\circ}$$

$$\frac{\alpha}{\beta} = \frac{\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1}{\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} + 1} = \frac{\sqrt{9} - 1}{\sqrt{9} + 1} = \frac{1}{2}$$

$$\beta = \alpha \sqrt{2} = 0.1 \sqrt{2}, \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.1\sqrt{2}} = \underline{44.43 \text{ m.}}$$

Prob. 10.8

$$\frac{\sigma}{\omega \epsilon} = \frac{4}{2\pi \times 10^5 \times 81 \times 10^{-9} / 36\pi} = \frac{80,000}{9} \gg 1$$

$$\alpha = \beta = \sqrt{\frac{W \mu \sigma}{2}} = \sqrt{\frac{2\pi \times 10^5 \times 4\pi \times 10^{-7} \times 4}{2}} = 0.4\pi$$

$$(a) u = \frac{W}{\beta} = \frac{2\pi \times 10^5}{0.4\pi} = \underline{5 \times 10^5 \text{ m/s.}}$$

$$(b) \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.4\pi} = \underline{5 \text{ m.}}$$

(c) $\delta = \frac{1}{\alpha} = \frac{1}{0.411} = \underline{\underline{0.796}}$.

Prob. 10.9

(a) Along $\underline{\underline{\vec{a}_z}}$.

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1} = \underline{\underline{6.283 \text{ m}}}$.

$T = \frac{2\pi}{\omega} = \frac{2\pi}{3 \times 10^8} = \underline{\underline{20.94 \text{ ms}}}$.

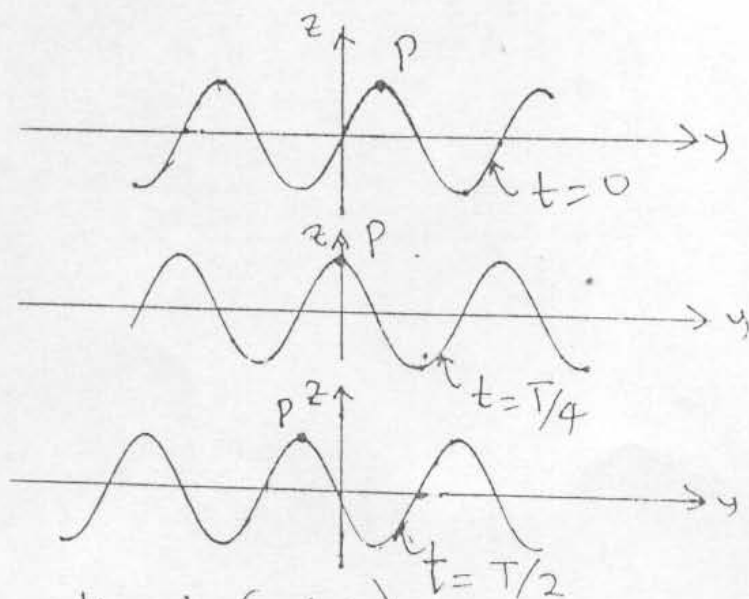
$u = c = \underline{\underline{3 \times 10^8 \text{ m/s}}}$.

(c) At $t=0$, $\vec{E} = 5 \sin y \vec{a}_z \text{ V/m}$

At $t = T/4$, $\vec{E} = 5 \sin(\omega \cdot \frac{2\pi}{4\omega} + y) \vec{a}_z = 5 \cos y \vec{a}_z$

At $t = T/2$, $\vec{E} = 5 \sin(y + \omega \cdot \frac{2\pi}{2\omega}) \vec{a}_z = -5 \sin y \vec{a}_z$

The wave is sketched below.



(d) Let $\vec{H} = H_0 \sin(\omega t + y) \vec{a}_H$,

$H_0 = \frac{E_0}{|\eta|} = \frac{5}{377} = 0.01326$, $\vec{a}_z \times \vec{a}_H = -\vec{a}_y \rightarrow \vec{a}_H = -\vec{a}_x$

$$\vec{H} = -13.26 \sin(3 \times 10^8 t + y) \vec{a}_x \text{ mA/m.}$$

Prob. 10.10

$$(a) \beta = \frac{\omega}{c} = \frac{2\pi \times 10^6}{3 \times 10^8} = \frac{2\pi}{300} = 0.02094 \text{ rad/m.}$$

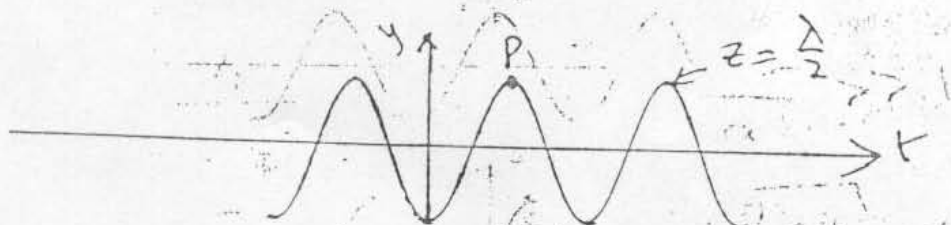
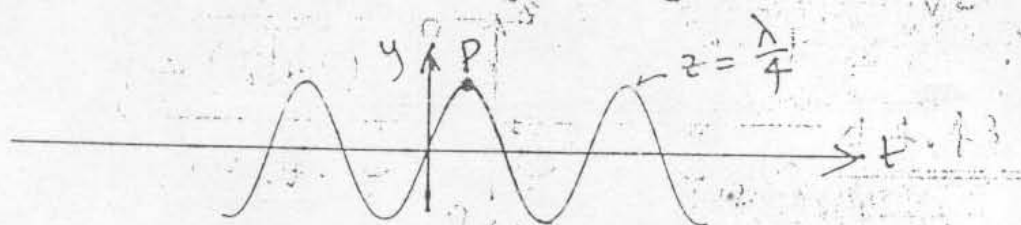
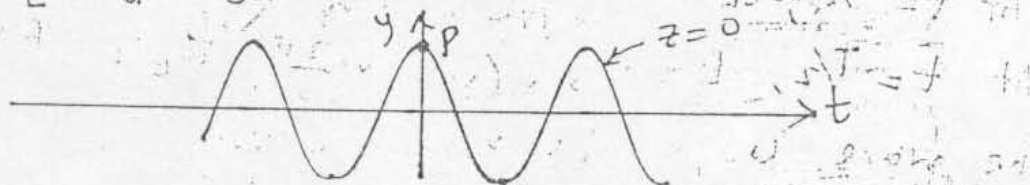
$$\lambda = \frac{2\pi}{\beta} = 300 \text{ m.}$$

$$(b) \text{ When } z=0, E_y = 10 \cos \omega t$$

$$z = \lambda/4, E_y = 10 \cos(\omega t - \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4}) = 10 \sin \omega t$$

$$z = \lambda/2, E_y = 10 \cos(\omega t - \pi) = -10 \cos \omega t$$

Thus E is as sketched below.



$$(c) \vec{a}_E \times \vec{a}_H = \vec{a}_k \rightarrow \vec{a}_y \times \vec{a}_H = \vec{a}_z \rightarrow \vec{a}_H = \vec{a}_x$$

$$\vec{H} = \frac{10}{120\pi} \cos(2\pi \times 10^6 t - 2\pi z/300) \vec{a}_x$$

$$= 26.53 \cos(2\pi \times 10^6 t - 0.02094 z) \vec{a}_x \text{ mA/m.}$$

Prob. 10.11

$$\frac{\sigma}{\omega \epsilon} = \frac{5 \times 10^{-9}}{2\pi \times 10^9 \times 9 \times 10^{-9} \times \frac{36\pi}{36\pi}} = 10^{-8} \approx 0$$

$$\alpha \approx 0, \quad \beta \approx \omega \sqrt{\mu \epsilon} = \frac{2\pi f}{c} \sqrt{\mu \epsilon} = \frac{2\pi \times 10^9 \sqrt{9}}{3 \times 10^8} = 20\pi$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = \underline{\underline{0.1 \text{ m}}}$$

It is a lossless medium since $\frac{\sigma}{\omega \epsilon} \approx 0$.

Prob. 10.12

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{4\pi \times 10^6}{3 \times 10^8} = \frac{4\pi}{300} = \underline{\underline{0.04189 \text{ rad/m}}}$$

$$\vec{a}_k = \underline{\underline{\vec{a}_z}}$$

$$E_0 = \eta_0 H_0 = 120\pi (0.3) = 36\pi$$

$$-\vec{a}_E = \vec{a}_k \times \vec{a}_H = \vec{a}_z \times (-\vec{a}_x) \rightarrow \vec{a}_E = \vec{a}_y$$

$$\vec{E} = (36\pi \vec{a}_y) \sin(4\pi \times 10^8 t - kx) \rightarrow \vec{A} = \underline{\underline{36\pi \vec{a}_y}}$$

Prob. 10.13

If $\vec{A}$ is a uniform vector and $\phi(r)$ is a scalar,

$$\nabla \times (\phi \vec{A}) = \nabla \phi \times \vec{A} + \phi (\nabla \times \vec{A}) = \nabla \phi \times \vec{A}$$

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z \right) \times \vec{E}_0 e^{j(k_x x + k_y y + k_z z - \omega t)}$$

$$= j(k_x \bar{a}_x + k_y \bar{a}_y + k_z \bar{a}_z) e^{j(\dots)} \times \bar{E}_0$$

$$= j \vec{k} \times \bar{E}_0 e^{j(\dots)} = j \vec{k} \times \bar{E}$$

Also $-\frac{\partial \vec{B}}{\partial t} = j\omega\mu\vec{H}$. Hence $\nabla \times \bar{E} = -\frac{\partial \vec{B}}{\partial t}$ becomes

$$\vec{k} \times \bar{E} = \omega\mu\vec{H}$$

from this, $\bar{a}_E \times \bar{a}_E = \bar{a}_H$.

Prob. 10.14

$$\nabla \cdot \bar{E} = \left(\frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z \right) \cdot \bar{E}_0 e^{j(k_x x + k_y y + k_z z - \omega t)}$$

$$= j(k_x \bar{a}_x + k_y \bar{a}_y + k_z \bar{a}_z) e^{j(\dots)} \cdot \bar{E}_0$$

$$= j \vec{k} \cdot \bar{E}_0 e^{j(\dots)} = j \vec{k} \cdot \bar{E} = 0 \rightarrow \vec{k} \cdot \bar{E} = 0$$

Similarly, $\nabla \cdot \bar{H} = j \vec{k} \cdot \bar{H} = 0 \rightarrow \vec{k} \cdot \bar{H} = 0$

It has been shown in Prob. 10.13 that

$$\nabla \times \bar{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \vec{k} \times \bar{E} = \mu\omega\vec{H}$$

Similarly, $\nabla \times \bar{H} = \frac{\partial \vec{D}}{\partial t} \rightarrow \vec{k} \times \bar{H} = -\epsilon\omega\bar{E}$

from $\vec{k} \times \bar{E} = \mu\omega\vec{H}$, $\bar{a}_E \times \bar{a}_E = \bar{a}_H$

and from $\vec{k} \times \bar{H} = -\epsilon\omega\bar{E}$, $\bar{a}_H \times \bar{a}_H = -\bar{a}_E$.

Prob. 10.15

$$(a) \beta = \frac{\omega}{c} \sqrt{\epsilon_r} \rightarrow \sqrt{\epsilon_r} = \frac{\beta c}{\omega} = \frac{5 \times 3 \times 10^8}{2\pi \times 10^8} = \frac{15}{2\pi}$$

$$\epsilon_r = \underline{\underline{5.6993}}$$

$$(b) \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{5} = \underline{\underline{1.2566 \text{ m}}}$$

$$u = \frac{c}{\sqrt{\mu_r \epsilon_r}} = \frac{3 \times 10^8}{\frac{15}{2\pi}} = \underline{\underline{1.257 \times 10^8 \text{ m/s}}}$$

$$(c) \eta = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{120\pi}{15/(2\pi)} = 16\pi^2 = \underline{\underline{157.91 \Omega}}$$

$$(d) \vec{a}_E \times \vec{a}_H = \vec{a}_z \rightarrow \vec{a}_E \times \vec{a}_z = \vec{a}_x \rightarrow \vec{a}_E = \underline{\underline{\vec{a}_y}}$$

$$(e) \vec{E} = 30 \times 10^{-3} (157.91) \sin(\omega t - \beta x) \vec{a}_E$$

$$= \underline{\underline{4.737 \sin(2\pi \times 10^8 t - 5x) \vec{a}_y \text{ V/m}}}$$

$$(f) \vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \vec{\nabla} \times \vec{H} = \underline{\underline{0.15 \cos(2\pi \times 10^8 t - 5x) \vec{a}_y \text{ A/m}}}$$

Prob. 10.16

$$\beta = \omega \sqrt{\mu \epsilon} = \frac{\omega}{c} \sqrt{\mu_r \epsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8} (10) = \frac{2\pi}{3}$$

$$= \underline{\underline{2.0943 \text{ rad/m}}}$$

$$\vec{H} = -\frac{1}{\mu} \int \vec{\nabla} \times \vec{E} dt$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x) & E_z(x) \end{vmatrix} = -\frac{\partial E_z}{\partial x} \vec{a}_x + \frac{\partial E_y}{\partial x} \vec{a}_z$$

$$= -10\beta \sin(\omega t - \beta x) (\vec{a}_y - \vec{a}_z)$$

$$\begin{aligned}\vec{H} &= \frac{-10\beta}{\omega\mu} \cos(\omega t - \beta x) (\vec{a}_y - \vec{a}_z) \\ &= \frac{10 \times \frac{2\pi}{3}}{2\pi \times 10^7 \times 50 \times 4\pi \times 10^{-7}} \cos(\omega t - \beta x) (-\vec{a}_y + \vec{a}_z) \\ \vec{H} &= \underline{5.305 \cos(2\pi \times 10^7 t - 2.0943x) (-\vec{a}_y + \vec{a}_z) \text{ mA/m.}}\end{aligned}$$

Prob. 10.17

$$(a) \frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 10^6 \times \frac{10^{-9}}{36\pi} \times 80} = \underline{900}$$

$$(b) \frac{\sigma}{\omega\epsilon} = 18000 \times \frac{10^{-3}}{80} = \underline{0.225}$$

$$(c) \frac{\sigma}{\omega\epsilon} = 18000 \times \frac{10^{-3}}{10} = \underline{1.8}$$

$$(d) \frac{\sigma}{\omega\epsilon} = 18000 \times \frac{10^{-5}}{10} = \underline{0.06}$$

Prob. 10.18

$$\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$\begin{aligned}\alpha f &= \frac{1}{\delta^2 \pi \mu \sigma} = \frac{1}{4 \times 10^{-6} \times \pi \times 4\pi \times 10^7 \times 6.1 \times 10^7} \\ &= \underline{1.038 \text{ kHz}}\end{aligned}$$

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta = 4\pi = \underline{12.566 \text{ mm}}$$

$$u = \frac{w}{\beta} = \frac{2}{\delta_{\text{no } \delta}} = \frac{2}{2 \times 10^{-3} \times 4\pi \times 10^{-7} \times 6.1 \times 10^7}$$

$$= \underline{13.04 \text{ m/s.}}$$

Prob. 10.19

$$\alpha = w \sqrt{\frac{\mu \epsilon}{2}} \left[\sqrt{1 + (\delta/w\epsilon)^2} - 1 \right]$$

$$= \frac{2\pi f}{c} \sqrt{\frac{\mu \epsilon_r}{2}} \left[\sqrt{1.0049} - 1 \right]$$

$$= \frac{2\pi \times 6 \times 10^6}{3 \times 10^8} \sqrt{\frac{4}{2} \times 2.447 \times 10^{-3}} = 8.791 \times 10^{-3}$$

$$\delta = \frac{1}{\alpha} = \underline{113.75 \text{ m.}}$$

$$\beta = \frac{4\pi}{100} \sqrt{\frac{4}{2} \left[\sqrt{1.0049} + 1 \right]} = 0.2515$$

$$u = \frac{w}{\beta} = \frac{2\pi \times 6 \times 10^6}{0.2515} = \underline{1.5 \times 10^8 \text{ m/s.}}$$

Prob. 10.20

$$(a) 0.4 E_0 = E_0 e^{-\alpha z} \rightarrow \frac{1}{0.4} = e^{2\alpha}$$

$$\alpha = \frac{1}{2} \ln \frac{1}{0.4} = 0.4581$$

$$\delta = \frac{1}{\alpha} = \underline{2.183 \text{ m.}}$$

$$(b) \lambda = 2\pi/\beta = 2\pi/1.6$$

$$u = f\lambda = 10^7 \times \frac{2\pi}{1.6} = \underline{3.927 \times 10^7 \text{ m/s.}}$$

Prob. 10.21

$$(a) R_{dc} = \frac{L}{\sigma S} = \frac{1}{\sigma \pi a^2} = \frac{600}{5.8 \times 10^7 \times \pi \times (1.25 \times 10^{-6})^2}$$

$$= \underline{2.287 \Omega}$$

$$(b) R_{ac} = \frac{L}{\sigma 2 \pi a \delta}$$

At 100 MHz, $\delta = 6.6 \times 10^{-3}$ mm for copper (see Table 10.2).

$$R_{ac} = \frac{600}{5.8 \times 10^7 \times 2 \pi \times 1.2 \times 6.6 \times 10^{-3} \times 10^{-6}}$$

$$= \underline{207.88 \Omega}$$

$$(c) \frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} \xrightarrow{\delta = \frac{a}{2}} \delta = \frac{a}{2} = \frac{66.1 \times 10^{-3}}{\sqrt{f}}$$

$$\sqrt{f} = \frac{66.1 \times 2 \times 10^{-3}}{1.2} = \frac{66.1 \times 2}{1.2}$$

$$\rightarrow f = \underline{12.137 \text{ kHz}}$$

Prob. 10.22

$$W = 10^6 \text{ W} = 2 \pi f \rightarrow F = 0.5 \times 10^6$$

$$\delta = \frac{1}{\sqrt{\pi f G}} = \left(\frac{1}{\pi \times 0.5 \times 10^6 \times 3.5 \times 10^7 \times 4 \pi \times 10^{-7}} \right)^{1/2}$$

$$= \underline{0.1203 \text{ mm}}$$

$$R_{ac} = \frac{L}{\delta \delta w}$$

since δ is very small, $w = 2\pi \rho_{outer}$

$$R_{ac} = \frac{L}{\delta \delta 2\pi \rho_{outer}} = \frac{40}{3.5 \times 10^7 \times 0.1203 \times 2\pi \times 12 \times 10^{-6}} \\ = \underline{\underline{0.126 \mu}}$$

Prob. 10.23

$$P_{ave} = \frac{E_0^2}{2\eta_0} = \frac{1}{2}\eta_0 H_0^2 = \frac{1}{2}(120\pi)(0.1)^2 = \underline{\underline{1.885 \text{ W/m}^2}}$$

$\vec{P}$ is along $-\vec{a}_y$.

Prob. 10.24

$$(a) \vec{E} = \text{Re}(\vec{E}_s e^{j\omega t}) = \underline{\underline{40 \cos(\omega t - 8y + 20^\circ) \vec{a}_z}}$$

$$(b) \eta = 377 \sqrt{\frac{\mu_r}{\epsilon_r}} = 377 \sqrt{2.25} = 565.5 \mu$$

$$\vec{a}_E \times \vec{a}_H = \vec{a}_E \rightarrow \vec{a}_z \times \vec{a}_H = \vec{a}_y \rightarrow \vec{a}_H = \vec{a}_x$$

$$\vec{H} = \frac{40}{565.5} \cos(\omega t - 8y + 20^\circ) \\ = \underline{\underline{0.0707 \cos(\omega t - 8y + 20^\circ) \vec{a}_x \text{ A/m}}}$$

$$(c) \lambda = \frac{2\pi}{\beta} = \frac{\pi}{4} = \underline{\underline{0.7854 \text{ m}}}, \quad \eta = \underline{\underline{565.5 \mu}}$$

$$(d) \vec{P}_{ave} = \frac{E_0^2}{2\eta} \vec{a}_y = \frac{(40)^2}{2(565.5)} \vec{a}_y = \underline{\underline{1.415 \vec{a}_y \text{ W/m}^2}}$$

Prob. 10.25

$$(a) \vec{P} = \vec{E} \times \vec{H} = \frac{E_0^2}{\eta_0} \cos^2(\omega t - \beta z) \vec{a}_z = \frac{100}{120\pi} \cos^2(\omega t - \beta z) \vec{a}_z$$

$$= 0.2652 \cos^2(2\pi \times 10^6 t - \beta z) \vec{a}_z \text{ W/m}^2.$$

$$(b) \vec{P} = \vec{E} \times \vec{H} = \frac{E_0^2}{|\eta|} e^{-2z/3} \sin(\omega t - \beta z) \sin(\omega t - \beta z - \theta_n) \vec{a}_z$$

$$= \frac{(0.5)^2}{177.72} e^{-2z/3} \sin(\omega t - \beta z) \sin(\omega t - \beta z - \theta_n) \vec{a}_z$$

$$= 1.41 e^{-2z/3} \sin(10^8 t - \beta z) \sin(10^8 t - \beta z - 13.63^\circ) \vec{a}_z \text{ mW/m}^2.$$

Prob 10.26

Let $\vec{E}_s = \vec{E}_r + j\vec{E}_i$ and $\vec{H}_s = \vec{H}_r + j\vec{H}_i$

$$\vec{E} = \text{Re}[\vec{E}_s e^{j\omega t}] = \vec{E}_r \cos \omega t - \vec{E}_i \sin \omega t$$

similarly, $\vec{H} = \vec{H}_r \cos \omega t - \vec{H}_i \sin \omega t$

$$\vec{P} = \vec{E} \times \vec{H} = \vec{E}_r \times \vec{H}_r \cos^2 \omega t + \vec{E}_i \times \vec{H}_i \sin^2 \omega t$$

$$- \frac{1}{2} (\vec{E}_r \times \vec{H}_i + \vec{E}_i \times \vec{H}_r) \sin 2\omega t$$

$$\vec{P}_{\text{ave}} = \frac{1}{T} \int_0^T \vec{P} dt = \frac{1}{T} \int_0^T \cos^2 \omega t dt (\vec{E}_r \times \vec{H}_r)$$

$$+ \frac{1}{T} \int_0^T \sin^2 \omega t dt (\vec{E}_i \times \vec{H}_i) - \frac{1}{2T} \int_0^T \sin 2\omega t dt (\vec{E}_i \times \vec{H}_i + \vec{E}_i \times \vec{H}_r)$$

$\downarrow 0$

$$= \frac{1}{2} (\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i)$$

$$= \frac{1}{2} \operatorname{Re} [(\vec{E}_r + j\vec{E}_i) \times (\vec{H}_r - j\vec{H}_i)]$$

$$\vec{P}_{\text{ave}} = \frac{1}{2} \operatorname{Re} (\vec{E}_s \times \vec{H}_s) \quad \text{as required.}$$

Prob. 10.27

$$(a) \beta = \frac{\omega}{c} = \frac{10^7}{3 \times 10^8} = \underline{\underline{0.0333 \text{ rad/m.}}}$$

$$E_0 = \eta_0 H_0 = 377, \quad -\vec{a}_E = \vec{a}_K \times \vec{a}_H = \vec{a}_r \times \vec{a}_\phi \rightarrow \vec{a}_E = -\vec{a}_\phi$$

$$\vec{E} = \underline{\underline{\frac{-377 \sin \theta}{r} \cos(10^7 t - 0.0333 r) \vec{a}_\phi \text{ V/m.}}}$$

$$(c) P_t = \int \vec{P}_{\text{ave}} \cdot d\vec{s} = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{2\pi} \frac{60\pi \sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi$$

$$= 60\pi (2\pi) \int_0^{\pi/3} (1 - \cos^2 \theta) d(-\cos \theta)$$

$$= 120\pi^2 \left(\frac{5}{24} \right) = \underline{\underline{246.74 \text{ W.}}}$$

$$(b) \vec{P} = \vec{E} \times \vec{H} = \frac{377 \sin^2 \theta}{r^2} \cos^2(10^7 t - \beta r) \vec{a}_r$$

$$= \underline{\underline{\frac{188.5 \sin^2 \theta}{r^2} \vec{a}_r \text{ W/m}^2}}$$

Prob. 10.28

$$\tilde{H}_i(o) + \tilde{H}_r(o) = \tilde{H}_t(o) \rightarrow H_{io} + H_{ro} = H_{to} \quad (1)$$

$$\tilde{E}_i(o) + \tilde{E}_r(o) = \tilde{E}_t(o) \rightarrow \eta_1 (H_{io} - H_{ro}) = \eta_2 H_{to} \quad (2)$$

$$(1) \times \eta_2 - (2) \Rightarrow H_{io} (\eta_2 - \eta_1) + H_{ro} (\eta_2 + \eta_1) = 0$$

$$\text{or} \quad -\frac{H_{ro}}{H_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma$$

$$(1) \times \eta_1 + (2) \Rightarrow 2\eta_1 H_{io} = (\eta_1 + \eta_2) H_{to}$$

$$\text{or} \quad \frac{H_{to}}{H_{io}} = \frac{2\eta_1}{\eta_1 + \eta_2} = \frac{\eta_1}{\eta_2} \tau$$

Prob 10.29

$$P_{iave} = \frac{E_{io}^2}{2\eta_1}, \quad P_{rave} = \frac{E_{ro}^2}{2\eta_1}, \quad P_{tave} = \frac{E_{to}^2}{2\eta_2}$$

$$R = \frac{P_{rave}}{P_{iave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \right)^2$$

$$T = \frac{P_{tave}}{P_{iave}} = \frac{\eta_1}{\eta_2} \frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2} \tau^2 = \frac{4\eta_1 \eta_2}{(\eta_1 + \eta_2)^2}$$

$$R + T = \frac{\eta_2^2 - 2\eta_1 \eta_2 + \eta_1^2 + 4\eta_1 \eta_2}{(\eta_1 + \eta_2)^2} = 1$$

Alternatively,

$$P_{i,ave} = P_{r,ave} + P_{t,ave}$$

$$1 = \frac{P_{r,ave}}{P_{i,ave}} + \frac{P_{t,ave}}{P_{i,ave}} = R + T.$$

Prob. 10.30

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \frac{\eta_0}{2}, \quad \eta_2 = \eta_0$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{3}, \quad \tau = \frac{4}{3}$$

$$E_{or} = \Gamma E_{io} = \left(\frac{1}{3}\right)(5) = \frac{5}{3}, \quad E_{ot} = \tau E_{io} = \frac{20}{3}$$

$$\beta = \frac{\omega}{c} \sqrt{\mu_r \epsilon_1} = \frac{10^8}{3 \times 10^8} \sqrt{4} = \frac{2}{3}$$

$$(a) E_r = \frac{5}{3} \cos(10^8 t - \frac{2}{3} y) \bar{a}_z$$

$$\begin{aligned} \vec{E}_1 &= \vec{E}_i + \vec{E}_r \\ &= \underline{\underline{5 \cos(10^8 t + \frac{2}{3} y) \bar{a}_z + \frac{5}{3} \cos(10^8 t - \frac{2}{3} y) \bar{a}_z \text{ V/m.}}} \end{aligned}$$

$$\begin{aligned} (b) \vec{P}_{ave1} &= \frac{E_{io}^2}{2\eta_1} (-\bar{a}_y) + \frac{E_{ro}^2}{2\eta_1} (\bar{a}_y) \\ &= \frac{25}{2(60\pi)} \left(1 - \frac{1}{9}\right) (-\bar{a}_y) = \underline{\underline{-0.0589 \bar{a}_y \text{ W/m}^2.}} \end{aligned}$$

$$(c) \vec{P}_{ave2} = \frac{E_{to}^2}{2\eta_2} (-\bar{a}_y) = \frac{400}{9(2)(120\pi)} (-\bar{a}_y) = \underline{\underline{-0.0589 \bar{a}_y \text{ W/m}^2.}}$$

Prob. 10.31

$$(a) \beta = 1 = \frac{w}{u} = \frac{w}{c} \sqrt{\mu_r \epsilon_r} \rightarrow w = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

$$w = \frac{3 \times 10^8}{\sqrt{3 \times 12}} = \underline{\underline{0.5 \times 10^8 \text{ rad/s}}}$$

$$(b) \eta_1 = \eta_0, \eta_2 = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_0 \sqrt{\frac{3}{12}} = \frac{\eta_0}{2}$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}, \tau = 1 + \Gamma = \frac{2}{3}$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = \underline{\underline{2}}$$

$$(c) \text{ Let } \vec{H}_r = H_{0r} \cos(\omega t + z) \vec{a}_H$$

where $\vec{E}_r = -\frac{1}{3}(30) \cos(\omega t + z) \vec{a}_y = -10 \cos(\omega t + z) \vec{a}_y$

$$H_{0r} = \frac{10}{\eta_0} = \frac{10}{120\pi}$$

$$\vec{a}_E \times \vec{a}_H = \vec{a}_K \rightarrow -\vec{a}_y \times \vec{a}_H = -\vec{a}_z \rightarrow \vec{a}_H = -\vec{a}_x$$

$$\vec{H}_r = \frac{-10}{120\pi} \cos(0.5 \times 10^8 t + z) \vec{a}_x \text{ A/m}$$

$$= \underline{\underline{-26.53 \cos(0.5 \times 10^8 t + z) \vec{a}_x \text{ mA/m}}}$$

Prob. 10.32

$$(a) \vec{a}_E \times \vec{a}_H = \vec{a}_K \rightarrow \vec{a}_E \times \vec{a}_z = \vec{a}_x \rightarrow \vec{a}_E = -\vec{a}_y$$

ie. polarization is along the y-axis.

$$(b) \beta = \omega \sqrt{\mu \epsilon} = \frac{2\pi f}{c} \sqrt{\mu_r \epsilon_r} = \frac{2\pi \times 30 \times 10^6}{3 \times 10^8} \sqrt{4 \times 9}$$

$$\underline{\underline{\beta = 3.77 \text{ rad/m.}}}$$

$$(c) \vec{J}_d = \nabla \times \vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(x,t) \end{vmatrix} = - \frac{\partial H_z}{\partial x} \vec{a}_y$$

$$= -10\beta \cos(\omega t + \beta x) \vec{a}_y, \quad \beta = \frac{6\pi}{5}$$

$$\underline{\underline{= -37.6 \cos(\omega t + \beta x) \vec{a}_y \text{ mA/m.}}}$$

$$(d) \eta_2 = \eta_0, \quad \eta_1 = \eta_0 \sqrt{\frac{4}{9}} = \frac{2}{3} \eta_0$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1 - \frac{2}{3}}{1 + \frac{2}{3}} = \frac{1}{5}, \quad \tau = 1 + \Gamma = \frac{6}{5}$$

$$\vec{E}_i = 10\eta_1 \sin(\omega t + \beta x) \vec{a}_E \text{ mV/m}, \quad \vec{a}_E = -\vec{a}_y$$

$$\vec{E}_r = \Gamma 10\eta_1 \sin(\omega t - \beta x) (-\vec{a}_y) \text{ mV/m}$$

$$\vec{a}_E \times \vec{a}_H = \vec{a}_z \rightarrow -\vec{a}_y \times \vec{a}_H = +\vec{a}_z \rightarrow \vec{a}_H = -\vec{a}_z$$

$$\vec{H}_r = \Gamma 10 \sin(\omega t - \beta x) (-\vec{a}_z) \text{ mA/m}$$

$$\underline{\underline{= -2 \sin(\omega t - \beta x) \vec{a}_z \text{ mA/m.}}}$$

$$\vec{E}_t = \tau 10\eta_1 \sin(\omega t + \beta x) (-\vec{a}_y) \text{ mV/m}$$

$$\vec{a}_E \times \vec{a}_H = \vec{a}_z \rightarrow -\vec{a}_y \times \vec{a}_H = -\vec{a}_z \rightarrow \vec{a}_H = \vec{a}_z$$

$$\vec{H}_t = \frac{6}{5} (10) \frac{\eta_1}{\eta_2} \sin(\omega t + \beta x) \tilde{a}_z \text{ mA/m}$$

$$= \underline{\underline{8 \sin(\omega t + \beta x) \tilde{a}_z \text{ mA/m}}}$$

$$(e) \vec{P}_{1, \text{ave}} = \frac{E_{10}^2}{2\eta_1} (-\tilde{a}_x) + \frac{E_{10}^2}{2\eta_1} \tilde{a}_x = -\frac{E_{10}^2}{2\eta_1} (1 - \Gamma^2) \tilde{a}_x$$

$$= -\frac{\eta_1^2 H_{10}^2}{2\eta_1} (1 - \Gamma^2) \tilde{a}_x = -\frac{1}{3} \eta_0 100 (1 - \frac{1}{25}) \tilde{a}_x$$

$$= -32 \eta_0 \tilde{a}_x \mu\text{W/m}^2 = \underline{\underline{-0.012064 \tilde{a}_x \text{ W/m}^2}}$$

$$\vec{P}_{2, \text{ave}} = \frac{E_{0t}^2}{2\eta_2} (-\tilde{a}_x), \quad E_{0t} = \tau E_{0i} = \tau \eta_1 H_{10}$$

$$= -\frac{\tau^2 \eta_1^2 H_{10}^2}{2\eta_2} \tilde{a}_x = 32 \eta_0 (-\tilde{a}_x) \mu\text{W/m}^2$$

$$= \underline{\underline{-0.012064 \tilde{a}_x \text{ W/m}^2}}$$

Prob. 10.33

$$(a) \text{ In air, } \beta_1 = 1, \quad \lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = \underline{\underline{6.283 \text{ m}}}$$

$$\omega = \beta_1 c = \underline{\underline{3 \times 10^8 \text{ rad/s}}}$$

In the dielectric medium, ω is the same

$$\omega = \underline{\underline{3 \times 10^8 \text{ rad/s}}}$$

$$\beta_2 = \frac{\omega}{c} \sqrt{\epsilon_{r2}} = \beta_1 \sqrt{\epsilon_{r2}} = 1\sqrt{3}$$

$$\lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276 \text{ m.}}}$$

$$(b) H_0 = \frac{E_0}{\eta_0} = \frac{10}{120\pi} = 0.0265$$

$$\vec{a}_H = \vec{a}_E \times \vec{a}_E = \vec{a}_z \times \vec{a}_y = \vec{a}_x$$

$$\vec{H}_i = \underline{\underline{-26.5 \cos(\omega t - z) \vec{a}_x \text{ mA/m}}}$$

$$(c) \eta_1 = \eta_0, \quad \eta_2 = \frac{\eta_0}{\sqrt{3}}$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{\sqrt{3}} - 1}{\frac{1}{\sqrt{3}} + 1} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = \underline{\underline{-0.268}}, \quad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

$$(d) E_{to} = \tau E_{io} = 7.32, \quad E_{ro} = \Gamma E_{io} = -2.68$$

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r = \underline{\underline{10 \cos(\omega t - z) \vec{a}_y - 2.68 \cos(\omega t + z) \vec{a}_y \text{ V/m.}}}$$

$$\vec{E}_2 = \vec{E}_r = \underline{\underline{7.32 \cos(\omega t - z) \vec{a}_y \text{ V/m.}}}$$

$$\vec{P}_{1 \text{ ave}} = \frac{1}{2\eta_1} \vec{a}_z (\vec{E}_{io}^2 - \vec{E}_{ro}^2) = \frac{1}{2(120\pi)} \vec{a}_z (10^2 - 2.68^2) \\ = \underline{\underline{0.1231 \vec{a}_z \text{ W/m}^2.}}$$

$$\vec{P}_{2 \text{ ave}} = \frac{E_{to}^2}{2\eta_2} \vec{a}_z = \frac{\sqrt{3}}{2 \times 120\pi} (7.32)^2 \vec{a}_z = \underline{\underline{0.1231 \vec{a}_z \text{ W/m}^2.}}$$

Prob. 10.34

$$(a) \omega = \beta c = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \text{ rad/s.}}}$$

$$(b) \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{3} = \underline{\underline{2.094 \text{ m}}}$$

$$(c) \quad \frac{\sigma}{\omega \epsilon} = \frac{4}{9 \times 10^8 \times 80 \times 10^{-9} \times \frac{36\pi}{36\pi}} = 2\pi = \underline{\underline{6.288}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega \epsilon} = 6.288 \rightarrow \theta_\eta = 40.47^\circ$$

$$|\eta_2| = \frac{\sqrt{\mu_2 \epsilon_2}}{4 \sqrt{1 + \left(\frac{\sigma_2}{\omega \epsilon_2}\right)^2}} = \frac{377 \sqrt{80}}{4 \sqrt{1 + 4\pi^2}} = 16.71$$

$$\eta_2 = \underline{\underline{16.71 \angle 40.47^\circ \Omega}}$$

$$(d) \quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{16.71 \angle 40.47^\circ - 377}{16.71 \angle 40.47^\circ + 377} = 0.935 \angle 176.7^\circ$$

$$E_{or} = \tau E_{oi} = 9.35 \angle 176.7^\circ$$

$$\vec{E}_r = \underline{\underline{9.35 \sin(\omega t - 3z + 176.7^\circ) \hat{a}_x \text{ V/m}}}$$

$$\alpha_2 = \frac{\omega}{c} \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left[\sqrt{1 + \left(\frac{\sigma_2}{\epsilon_2 \omega}\right)^2} - 1 \right]$$

$$= \frac{9 \times 10^8}{3 \times 10^8} \sqrt{40} \sqrt{\sqrt{1 + 4\pi^2} - 1} = 43.94 \text{ Np/m}$$

$$\beta_2 = \frac{9 \times 10^8}{3 \times 10^8} \sqrt{40} \sqrt{\sqrt{1 + 4\pi^2} + 1} = 51.48 \text{ rad/m}$$

$$\bar{\Gamma} = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{33.42 \angle 40.47^\circ}{377 + 16.71 \angle 40.47^\circ} = 0.0857 \angle 38.89^\circ$$

$$E_{ot} = \tau E_o = 0.857 \underline{[38.59]}$$

$$\underline{E_t = 0.857 e^{43.94z} \sin(9 \times 10^8 t + 51.48z + 38.89^\circ) \text{ V/m.}}$$

Prob. 10.35

Both media are lossless, hence Γ is real.

$$\Gamma = \frac{s-1}{s+1} = \frac{1.2}{3.2} = \frac{3}{8}, \quad \tau = 1 + \Gamma = \frac{11}{8}$$

$$\text{from Prob. 10.29, } P_t = \tau P_i = \frac{\eta_1}{\eta_2} \tau^2 P_i.$$

$$\text{But } \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1-x}{1+x} = \frac{3}{8}, \text{ where } x = \frac{\eta_1}{\eta_2}.$$

$$3x + 3 = 8 - 8x \rightarrow x = \frac{5}{11} = \frac{\eta_1}{\eta_2}$$

$$P_t = \frac{5}{11} \left(\frac{11}{8} \right)^2 (3) = \underline{\underline{2.578 \text{ W/m}^2}}.$$

Prob. 10.36

$$\eta_1 = \eta_0, \quad \eta_2 = \sqrt{\frac{\mu}{\epsilon}} = 3\eta_0.$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{2}, \quad \tau = 1 + \Gamma = \frac{3}{2}, \quad E_{to} = \tau E_{io} = \frac{3}{2} (2) = 2$$

$$P_{\text{ave}} = \frac{E_{to}^2}{2\eta_2} = \frac{\frac{1}{2} (3)^2}{3(120\pi)} = \frac{1}{80\pi} = \underline{\underline{3.98 \text{ mW/m}^2}}.$$

Prob. 10.37

$$\eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$$

$$\tan 2\theta_{\eta_2} = \frac{\epsilon_2}{\mu\epsilon_2} = \frac{0.02}{10^9 \times 10^{-9} \times 36\pi} = 2.262 \rightarrow \theta_{\eta_2} = 33.08^\circ$$

$$|\eta_2| = \frac{377}{4\sqrt{1 + (2.262)^2}} = 239.72$$

$$\eta_2 = 239.72 \angle 33.08^\circ$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{239.72 \angle 33.08^\circ - 377}{239.72 \angle 33.08^\circ + 377}$$

$$= 0.371 \angle 130.61^\circ$$

$$E_{r0} = \Gamma E_{i0}, E_{rs} = 0.371 \angle 130.61^\circ \left(40 \bar{e}^{j\beta_1 z} \bar{a}_x + 30 \bar{e}^{-j(\beta_1 z - 90^\circ)} \bar{a}_y \right)$$

$$\vec{E}_r = 14.84 \cos(10^9 t + 3.33z + 130.61^\circ) \bar{a}_x$$

$$+ 11.13 \sin(10^9 t + 3.33z + 130.61^\circ) \bar{a}_y \text{ V/m.}$$

$$a_H = \bar{a}_r \times \bar{a}_z \rightarrow \bar{a}_{H1} = -\bar{a}_z \times \bar{a}_x = -\bar{a}_y$$

$$\bar{a}_{H2} = -\bar{a}_z \times \bar{a}_y = \bar{a}_x$$

$$\vec{H}_r = \frac{-14.84}{377} \cos(\omega t + 3.33z + 130.61^\circ) \bar{a}_y$$

$$+ \frac{11.13}{377} \sin(\omega t + 3.33z + 130.61^\circ) \bar{a}_x$$

$$\vec{H}_r = 29.52 \sin(10^9 t + 3.33z + 130.61^\circ) \bar{a}_x$$

$$\underline{-39.36 \cos(10^9 t + 3.33z + 130.61^\circ) \vec{a}_y \text{ mA/m.}}$$

$$\Gamma = 1 + \Gamma = 1 + 0.371 \angle 130.61^\circ = 0.81 \angle 20.3^\circ,$$

$$E_{t0} = \tau E_{i0},$$

$$\vec{E}_{ts} = 0.81 \angle 20.3^\circ \left[40 e^{-j\beta_2 z} \vec{a}_x + 30 e^{-j(\beta_2 z - 90^\circ)} \vec{a}_y \right]$$

$$\text{But } \beta_2 = 10^9 \sqrt{\frac{\mu_0 \epsilon_0}{2} \left[\sqrt{1 + (2.262)^2} + 1 \right]} = 4.393$$

$$\vec{E}_t = 32.4 \cos(10^9 t - 4.393z + 20.3^\circ) \vec{a}_x \\ + 24.3 \sin(10^9 t - 4.393z + 20.3^\circ) \vec{a}_y \text{ V/m.}$$

$$\text{Since } \vec{a}_{H1} = \vec{a}_z \times \vec{a}_k = \vec{a}_y, \vec{a}_{H2} = \vec{a}_z \times \vec{a}_y = -\vec{a}_x$$

$$\text{and } \eta_2 = 239.72 \angle 33.08^\circ,$$

$$\vec{H}_t = \frac{32.4}{239.72} \cos(10^9 t - 4.393z + 20.3^\circ - 33.08^\circ) \vec{a}_y$$

$$- \frac{24.3}{239.72} \sin(10^9 t - 4.393z + 20.3^\circ - 33.08^\circ) \vec{a}_x$$

$$\vec{H}_t = -101.6 \sin(10^9 t - 4.393z - 12.78^\circ) \vec{a}_x \\ + 135.5 \cos(10^9 t - 4.393z - 12.78^\circ) \vec{a}_y \text{ mA/m.}$$

Prob. 10.38

$$(a) \frac{\lambda}{2} = 4\text{m} \text{ or } \lambda = 8\text{m}, f = \frac{c}{\lambda} = \frac{3 \times 10^8}{8} = 37.5 \text{ MHz.}$$

$$(b) \Gamma = \frac{s-1}{s+1} = \frac{1}{3}, \theta_r = 0, \Gamma = 0.3333.$$

Prob. 10.39

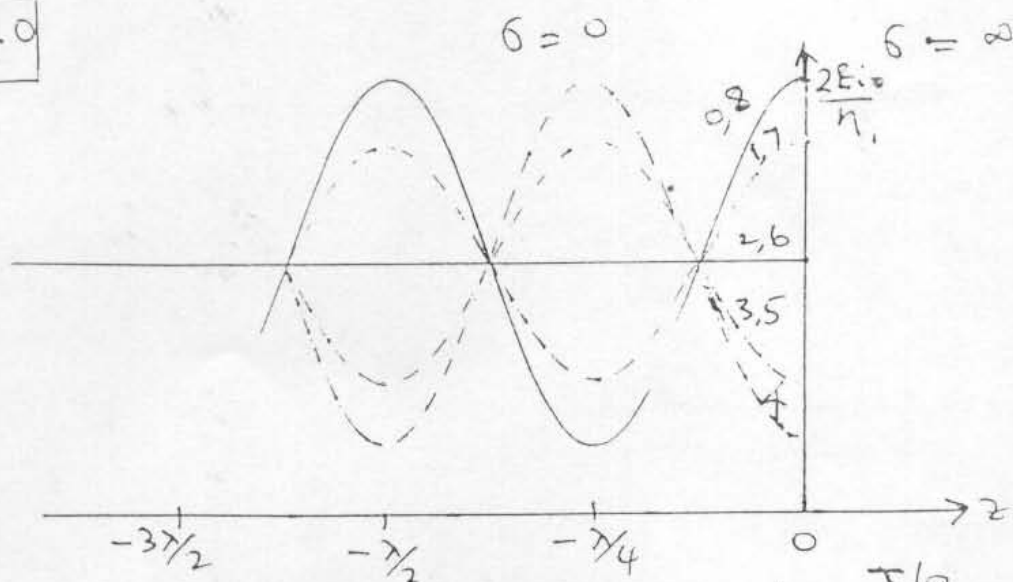
$$\begin{aligned}\vec{E} &= \vec{E}_i + \vec{E}_r \\ &= -40 [\cos(\omega t + \beta z) - \cos(\omega t - \beta z)] \vec{a}_x \\ &\quad - 30 [\sin(\omega t + \beta z) - \sin(\omega t - \beta z)] \vec{a}_y \text{ V/m} \\ &= 80 \sin \omega t \sin \beta z \vec{a}_x - 60 \cos \omega t \sin \beta z \vec{a}_y \text{ V/m}\end{aligned}$$

which is a standing wave

$$\begin{aligned}\vec{H} &= \vec{H}_i + \vec{H}_r \\ &= \frac{1}{3\pi} [\cos(\omega t + \beta z) + \cos(\omega t - \beta z)] \vec{a}_y \\ &\quad - \frac{1}{4\pi} [\sin(\omega t + \beta z) + \sin(\omega t - \beta z)] \vec{a}_x \text{ A/m} \\ &= \frac{2}{3\pi} \cos \omega t \cos \beta z \vec{a}_y - \frac{2}{4\pi} \sin \omega t \sin \beta z \vec{a}_x \text{ A/m}\end{aligned}$$

which is a standing wave.

Prob. 10.40



Curve 0 is at $t=0$, curve 1 is at $t=T/8$,
curve 2 is at $t=T/4$, curve 3 is at $t=3T/8$, etc.

Prob. 10.41

Since $\mu_0 = \mu_1 = \mu_2$,

$$\sin \theta_{t1} = \sin \theta_i \sqrt{\frac{\epsilon_0}{\epsilon_1}} = \frac{\sin 45^\circ}{\sqrt{4.5}} = 0.3333 \rightarrow \theta_{t1} = \underline{19.47^\circ}$$

$$\sin \theta_{t2} = \sin \theta_{t1} \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{1}{3} \sqrt{\frac{2.25}{4.5}} = 0.2357$$

$$\rightarrow \theta_{t2} = \underline{13.63^\circ}$$

Prob. 10.42

$$\vec{E}_s = \frac{20}{2j} (e^{jk_x x} - e^{-jk_x x}) \frac{(e^{jk_y y} + e^{-jk_y y})}{2} \vec{a}_z$$

$$= -j5 [e^{j(k_x x + k_y y)} + e^{j(k_x x - k_y y)} - e^{-j(k_x x - k_y y)} - e^{-j(k_x x + k_y y)}] \vec{a}_z$$

which consists of four plane waves.

$$\nabla \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s \rightarrow \vec{H}_s = \frac{j}{\omega\mu_0} \nabla \times \vec{E}_s$$

$$= \frac{j}{\omega\mu_0} \left(\frac{\partial E_z}{\partial y} \vec{a}_x - \frac{\partial E_z}{\partial x} \vec{a}_y \right)$$

$$\vec{H}_s = \frac{-j20}{\omega\mu_0} [k_y \sin(k_x x) \sin(k_y y) \vec{a}_x + k_x \cos(k_x x) \cos(k_y y) \vec{a}_y]$$

Prob. 10.43

If $\mu_1 = \mu_2 = \mu_0$, $n_1 = \frac{n_0}{\sqrt{\epsilon_{r1}}}$, $n_2 = \frac{n_0}{\sqrt{\epsilon_{r2}}}$

$$\Gamma_{11} = \frac{\frac{1}{\sqrt{E_{r2}}} \cos \theta_t - \frac{1}{\sqrt{E_{r1}}} \cos \theta_i}{\frac{1}{\sqrt{E_{r2}}} \cos \theta_t + \frac{1}{\sqrt{E_{r1}}} \cos \theta_i}$$

$$\sqrt{E_{r1}} \sin \theta_i = \sqrt{E_{r2}} \sin \theta_t \longrightarrow \frac{\sqrt{E_{r2}}}{\sqrt{E_{r1}}} = \frac{\sin \theta_i}{\sin \theta_t}$$

$$\Gamma_{11} = \frac{\cos \theta_t - \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i}{\cos \theta_t + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i} = \frac{\sin \theta_t \cos \theta_t - \sin \theta_i \cos \theta_i}{\sin \theta_t \cos \theta_t + \sin \theta_i \cos \theta_i}$$

Dividing both numerator and denominator by $\cos \theta_i \cos \theta_t$ gives

$$\begin{aligned} \Gamma_{11} &= \frac{\tan \theta_t - \tan \theta_i}{\tan \theta_t + \tan \theta_i} = \frac{\frac{\tan \theta_t - \tan \theta_i}{1 + \tan \theta_i \tan \theta_t}}{\frac{\tan \theta_t - \tan \theta_i}{1 + \tan \theta_i \tan \theta_t}} \\ &= \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{\Gamma}_{11} &= \frac{\frac{2}{\sqrt{E_{r2}}} \cos \theta_i}{\frac{1}{\sqrt{E_{r2}}} \cos \theta_t + \frac{1}{\sqrt{E_{r1}}} \cos \theta_i} = \frac{2 \cos \theta_i}{\cos \theta_t + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{\sin \theta_t \cos \theta_t (\sin^2 \theta_i + \cos^2 \theta_i) + \sin \theta_i \cos \theta_i (\sin^2 \theta_t + \cos^2 \theta_i)} \end{aligned}$$

$$\begin{aligned} T_{11} &= \frac{2 \cos \theta_i \sin \theta_t}{(\sin \theta_i \cos \theta_t + \sin \theta_t \cos \theta_i) (\cos \theta_i \cos \theta_t + \sin \theta_i \sin \theta_t)} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i) \cos(\theta_t - \theta_i)} \end{aligned}$$

$$\begin{aligned} \Gamma_{\perp} &= \frac{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i - \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t} = \frac{\cos \theta_i - \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t}}{\cos \theta_i + \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t}} \\ &= \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \end{aligned}$$

$$\begin{aligned} T_{\perp} &= \frac{\frac{2}{\sqrt{\epsilon_{r2}}} \cos \theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t} = \frac{2 \cos \theta_i}{\cos \theta_i + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_t} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i)} \end{aligned}$$

Prob. 10.44

$$\vec{H} = H_0 \vec{a}_H = \frac{E_0}{\eta} \vec{a}_E \times \vec{a}_E = \frac{50}{120\pi} \vec{a}_E \times \vec{a}_E$$

where $\vec{a}_E = \sin 45^\circ \vec{a}_x + \cos 45^\circ \vec{a}_z$

$$\vec{a}_E \times \vec{a}_E = -\cos 45^\circ \vec{a}_x + \sin 45^\circ \vec{a}_z$$

$$\vec{H} = \frac{50}{120\pi} (-\cos 45^\circ \vec{a}_x + \sin 45^\circ \vec{a}_z) \cos(\omega t - \dots)$$

$$\vec{H} = \underline{93.78 (-\vec{a}_x + \vec{a}_z) \cos(\omega t - \beta_1 x \sin 45^\circ - \beta_1 z \cos 45^\circ)} \text{ mV/m}$$

$$\tilde{T}_\perp = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \frac{2}{1 + \frac{\eta_1}{\eta_2} \frac{\cos \theta_t}{\cos \theta_i}},$$

$$\cos \theta_i = \vec{a}_n \cdot \vec{a}_k = \cos 45^\circ \rightarrow \theta_i = 45^\circ$$

$$k_i \sin \theta_i = k_t \sin \theta_t \rightarrow \omega \sqrt{\mu_0 \epsilon_0} \sin 45^\circ = \omega \sqrt{\mu_0 \epsilon_0 (2.25)} \sin \theta_t$$

$$\sin \theta_t = \frac{\sin 45^\circ}{1.5} \rightarrow \theta_t = 28.125^\circ, \frac{\eta_1}{\eta_2} = 1.5$$

$$\tilde{T}_\perp = \frac{2}{1 + 1.5 \frac{\cos 28.125^\circ}{\cos 45^\circ}} = \underline{0.6967}.$$

Prob. 10.45

$$(a) \tan \theta_i = \frac{k_{ix}}{k_{iz}} = \frac{1}{\sqrt{8}} \rightarrow \theta_i = \theta_r = \underline{19.47^\circ}.$$

$$\sin \theta_t = \sin \theta_i \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{1}{3} (3) = 1 \rightarrow \theta_t = \underline{90^\circ}.$$

$$(b) \beta_1 = \frac{\omega}{c} \sqrt{\epsilon_{r1}} = \frac{10^9}{3 \times 10^8} \times 3 = 10 = k \sqrt{1+8} = 3k$$

$$\rightarrow k = \frac{10}{3} = \underline{3.333}.$$

$$(c) \lambda = \frac{2\pi}{\beta}, \quad \lambda_1 = \frac{2\pi}{\beta_1} = \frac{2\pi}{10} = \underline{0.6283 \text{ m}}.$$

$$\beta_2 = \frac{\omega}{c} = \frac{10}{3}, \quad \lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi \times 3}{10} = \underline{1.885 \text{ m}}.$$

$$(d) \vec{E}_i = \eta_1 \vec{a}_r \times \vec{H}_i = 40\pi (\vec{a}_x + \sqrt{3} \vec{a}_z) \times 0.2 \cos(10^9 t - \vec{k} \cdot \vec{r}) \text{ V/m}$$

$$= \underline{\underline{(-213.3 \vec{a}_x + 75.4 \vec{a}_z) \cos(10^9 t - kx - k\sqrt{3}z) \text{ V/m}}}$$

$$(e) \tau_{11} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_t - \theta_i)}$$

$$= \frac{2 \cos(9.47^\circ) \sin 90^\circ}{\cos 19.47^\circ \sin 19.47^\circ} = 6$$

$$\Gamma_{11} = -\frac{\cot 19.47^\circ}{\cot 19.47^\circ} = -1$$

$$\text{Let } \vec{E}_t = -E_{t0} (\cos \theta_t \vec{a}_x - \sin \theta_t \vec{a}_z) \cos(10^9 t - \beta_2 x \sin \theta_t - \beta_2 z \cos \theta_t)$$

$$\text{where } \vec{E}_i = -E_{i0} (\cos \theta_i \vec{a}_x - \sin \theta_i \vec{a}_z) \cos(10^9 t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i)$$

$$\sin \theta_t = 1, \cos \theta_t = 0, \beta_2 \sin \theta_t = \frac{10}{3}$$

$$E_{t0} \sin \theta_t = \tau_{11} E_{i0} = 6 (24\pi) (3) (1) = 1357.2$$

Hence

$$\vec{E}_t = \underline{\underline{1357 \cos(10^9 t - 3.333x) \vec{a}_z \text{ V/m}}}$$

Since $\Gamma = -1$, $\theta_r = \theta_i$

$$\vec{E}_r = \underline{\underline{(213.3 \vec{a}_x + 75.4 \vec{a}_z) \cos(10^9 t - kx + k\sqrt{3}z) \text{ V/m}}}$$

$$(f) \tan \theta_{B11} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sqrt{\frac{\epsilon_0}{9\epsilon_0}} = \frac{1}{3} \rightarrow \theta_{B11} = \underline{\underline{18.43^\circ}}$$

Prob. 10.46

Total transmission occurs at Brewster angle. Since $\mu_1 = \mu_2 = \mu_0$, this can only occur for parallel polarization.

$$\eta_1 = 1, \quad \eta_2 = \sqrt{\epsilon_2} = 2$$

$$\theta_{B11} = \theta_i = \tan^{-1} \frac{\eta_2}{\eta_1} = \tan^{-1} \frac{2}{1} \rightarrow \theta_i = 63.43^\circ$$

$$\text{from Snell's law, } \sin \theta_t = \frac{\eta_1}{\eta_2} \sin \theta_i = \frac{\sin 63.43^\circ}{2}$$

$$\rightarrow \theta_t = \underline{\underline{26.56^\circ}}$$

Prob. 10.47

$$B_1 = \sqrt{3^2 + 4^2} = 5 = \frac{\omega}{c} \rightarrow \omega = \beta_1 c = \underline{\underline{15 \times 10^8 \text{ rad/s}}}$$

$$\text{Let } \vec{E}_r = (E_{0x}, E_{0y}, E_{0z}) \sin(\omega t + 3x + 4y). \text{ In order for } \nabla \cdot \vec{E}_r = 0, \quad 3E_{0x} + 4E_{0y} = 0 \quad \text{---(1)}$$

$$\text{Also at } y=0, \quad \vec{E}_1 \tan = E_2 \tan = 0$$

$$\vec{E}_1 \tan = 0 \rightarrow 8\hat{a}_x + 5\hat{a}_z + E_{0x}\hat{a}_x + E_{0z}\hat{a}_z = 0$$

$$\text{Equating components, } E_{0x} = -8, \quad E_{0z} = -5$$

$$\text{from (1), } 4E_{0y} = -3E_{0x} = 24 \rightarrow E_{0y} = 6$$

$$\text{Hence } \vec{E}_r = \underline{\underline{(-8\hat{a}_x + 6\hat{a}_y - 5\hat{a}_z) \sin(15 \times 10^8 t + 3x + 4y) \text{ V/m}}}$$

CHAPTER 11

P.E. 11.1

Since z_0 is real and $\alpha \neq 0$, this is a distortionless line.

$$z_0 = \sqrt{\frac{R}{G}} \quad (1)$$

$$\text{or } \frac{L}{R} = \frac{C}{G} \quad (2)$$

$$\alpha = \sqrt{RG} \quad (3)$$

$$\beta = \omega L \sqrt{\frac{G}{R}} = \frac{\omega L}{z_0} \quad (4)$$

$$(1) \times (3) \rightarrow R = \alpha z_0 = 0.04 \times 80 = \underline{\underline{3.2 \, \Omega/m}}$$

$$(3) \div (1) \rightarrow G = \frac{\alpha}{z_0} = \frac{0.04}{80} = \underline{\underline{5 \times 10^{-4} \, S/m}}$$

$$L = \frac{\beta z_0}{\omega} = \frac{1.5 \times 80}{2\pi \times 5 \times 10^8} = \underline{\underline{38.2 \, nH/m}}$$

$$C = \frac{LG}{R} = \frac{12}{\pi} \cdot 10^{-8} \times \frac{0.04}{80} \times \frac{1}{0.04 \times 80} = \underline{\underline{5.97 \, pF/m}}$$

P.E. 11.2

$$(a) \, z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{0.03 + j2\pi \times 10^3 \times 0.1 \times 10^{-3}}{0 + j2\pi \times 10^3 \times 0.02 \times 10^{-6}}} \\ = 70.73 - j1.688 = \underline{\underline{70.75 \angle -1.367^\circ \, \Omega}}$$

$$(b) \, \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(0.03 + j0.2\pi)(j0.4 \times 10^{-4}\pi)} \\ = \underline{\underline{2.121 \times 10^{-4} + j8.888 \times 10^{-3} \, /m}}$$

$$(c) u = \frac{W}{P} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5 \text{ m/s}}}$$

P.E. 11.3

$$(a) Z_o = Z_L \rightarrow Z_{in} = Z_o = \underline{\underline{30 + j60 \Omega}}$$

$$(b) V_{in} = V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{V_g}{2} = \underline{\underline{7.5 \angle 0^\circ \text{ V}_{rms}}}$$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15 \angle 0^\circ}{2(30 + j60)}$$

$$= \underline{\underline{0.05 \angle -63.43^\circ \text{ A}}}$$

$$(c) \text{ Since } Z_o = Z_L, \Gamma = 0 \rightarrow V_o^- = 0, V_o^+ = V_o.$$

The load voltage is

$$V_L = V_o(z=l) = V_o^+ e^{-\gamma l}$$

$$e^{-\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5 \angle 0^\circ}{5 \angle -48^\circ} = 1.5 \angle 48^\circ$$

$$e^{\alpha l} e^{j\beta l} = 1.5 \angle 48^\circ$$

$$e^{\alpha l} = 1.5 \rightarrow \alpha = \frac{1}{l} \ln 1.5 = \frac{1}{4} \ln 1.5 = 0.0101$$

$$e^{j\beta l} = e^{j48^\circ} \rightarrow \beta = \frac{1}{l} \frac{48^\circ}{180} \pi \text{ rad} = 0.02094$$

$$\gamma = \underline{\underline{0.0101 + j0.02094 \text{ /m}}}$$