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MEASURE THEORY AND INTEGRATION

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Preface

Measure Theory and Integration contains a detailed introduction to advanced real analysis. The work is expected to serve as an aid to advanced graduate and postgraduate students studying these topics as the measurable ideas dealt with here can be enjoyed with an elementary knowledge of advanced real analysis and probability theory. The proofs of theorems are separated from their formulations and are appended at the end of each chapter. This makes it look like a problem book and encourages the reader to think about each formulation.

In Chapter 1 we have introduced the concepts of sets theory, finite and infinite set, cardinal number and cardinality of a set, countable and uncountable sets, ordered pairs, ordinal numbers, Cantor's theorem and Continuum hypothesis.

Chapter 2 deals with the length of a set, measure, Borel set, $\Im_\sigma$-sets, ζ_σ-sets, Boolean ring, Boolean algebra, σ-Ring, Lebesgue measure, outer measure, exterior and interior measure of a set, measurable set, first fundamental theorem, Cantor's ternary set and non-measurable set.

Chapter 3 discusses the measurable function, Borel measurability, F. Riesz theorem, Egoroff's theorem and Lusin's theorem.

Chapter 4 deals with Riemann theory of integral, Lebesgue integral, first mean value theorem, Lebesgue bounded convergence theorem, Lebesgue dominated convergence theorem, Beppo-Levi's theorem and Fatou's lemma.

Chapter 5 deals with the function of bounded variation, Lipschitz condition, cover in the sense of Vitali, Vitali's Lemma, Lebesgue point and Lebesgue set and fundamental theorem of integral calculus.

Chapter 6 discusses the L^p, conjugate number, convergent sequence, Cauchy sequence, metric and normed space, L^p-space with properties, Riesz Holder inequalities, Minkowski's inequality and Schwarz's inequality.

Chapter 7 deals with product measures and signed measures, Fubini's theorem, Tonelli's theorem, Hahn decomposition theorem, Radon-Nikodym theorem and Lebesgue decomposition theorem.

Any comments from students and teachers are welcome. Their comments will naturally contribute to the improvement of the further edition of the book.

Authors

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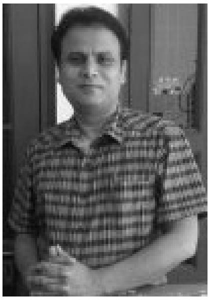

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CHAPTER

Countability of Sets

1.1 INTRODUCTION

Set theory has a great importance in the study of mathematics and computer sciences. German mathematician Georg Cantor (1845-1918) introduced the idea of set theory. The concept of set theory has a great contribution in analysis. In 1874, George Cantor discussed the term countable set. Countable sets have an important place in the branch of mathematics known as real and discrete mathematics.

In this chapter we shall discuss some introductory concepts of analysis such as Sets, Relations, Equivalence relation, Well ordering, Partial ordered relation, Zorn′ Lemma, Axiom of choice, Schröder-Bernstein′s equivalence theorem functions, Open and closed sets, Bolzano-Weierstrass theorem, Finite and infinite set, Cardinal number and cardinality of a set, Countable and uncountable sets, Ordered pairs, Ordinal numbers, Cantor′s theorem, Continuum hypothesis, Algebraic number and Transcendental number.

1.2 IMPORTANT TERMINOLOGY

Before discussing the countable and cardinal number of sets, we shall discuss certain necessary preliminaries.

1.2.1 Sets

A *set* is a well defined collection of objects. The objects in a set are known as members or elements or points. Suppose A is a set and a is an element of A, then we write $a \in A$ (a belongs to A). If a is not an element of A, then we write $a \notin A$ (a does not belongs to A). Let A be the set $A = \{1, 3, 5, 7, 9\}$. Here $1 \in A, 3 \in A, 5 \in A, 7 \in A, 9 \in A$ but $2 \notin A$. The form of presentation of above set A is known as *tabular method* or *roster method*. Also A can be written as $A = \{x \mid x \text{ is an odd positive integer and } x < 11\}$. It means that A is the set of all odd positive integers which are less than 11. This form of presentation of set A is known as set-builder method or rule method. For example, *the set consisting of all the letters in the word "DELHI" can be written as {D, E, L, H, I} or {x/x is a letter in the word "DELHI"}.*

A *multi set* is an unordered collection of objects in which an object can appear more than once. For example, $B = \{a, a, b, b, b, c\}$. Here a appears two times, b appears three times and c appear one time. A set is said to be *empty set* or *null set* or *void set* if it contains no element. It is denoted by ϕ or {}. For example, $P = \{x|x$ is a real number and $x^2 = -1\}$, $Q = \{x|x < x\}$ and $R = \{x|x \in I$ and $1 < x < 2\}$. Here P, Q and R are empty set. A set is said to be *singleton set* or *unit set* if it contains only one element. For example, $S = \{x \mid x$ is a positive integer and $x^2 = 4\}$ and $T = \{0\}$. Here S and T are singleton set. Let A and B be any two sets. If all the element of A belongs to B, then A is said to be *subset* of B. It is denoted as $A \subset B$, read as "A is a subset of B" or "A contained in B". For example, $A = \{a, b, c\}$. Then ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ are all subset of A. Let A and B be any two sets. Then A and B are said to be *disjoint sets* if they have no common elements. For example, $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$. Here A and B have no common elements. Therefore A and B are disjoint sets. Let A and B be any two sets. If all the elements of A belongs to B and all the elements of B belongs to A, (*i.e.*, $A \subseteq B$ and $B \subseteq A$) then A and B are said to be *equal set* and written as $A = B$. For example, $A = \{N, I, R, A, N, J, A, N\}$ and $B = \{N, I, R, A, J\}$. Here A and B are equal set, *i.e.*, $A = B$.

Let A and B be any two sets. Then A and B are said to be *comparable* if all the elements of A belongs to B or all the elements of B belongs to A (*i.e.*, $A \subseteq B$ or $B \subseteq A$). But if $A \not\subset B$ or $B \not\subset A$, then A and B are known as *non-comparable set*. For example, $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{1, 3, 5, 7\}$ and $C = \{2, 5, 6, 7, 9\}$. Here A and B are comparable set; A and C, and B and C are non-comparable set. A set under consideration in the problem is a fixed set in which includes each given set known as *universal set*. For the sets of numbers, the set of complex number (C) will be the universal set. It is denoted by U. If a set contains a number of sets as its elements then it is known as *set of sets* or *family of sets* or *class of sets*. For example, $A = \{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$ and $B = \{\{0\}, \{0, 1\}, \{0, 1, 2\}\}$. Here A and B are set of sets. Let A be any set. The *power set* of A is the set of all subsets of A. It is denoted by $P(A)$. For example, $A = \{a, b, c\}$ be the set. Then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The number of elements in a $P(A)$ is 2 raised to the cardinality of A i.e., Number of $P(A) = 2^{n(A)}$. If $A = \{a, b, c\}$, then number of $P(A) = 2^3 = 8$. Index set is a set whose elements are used as names. It is usually denoted by Λ. An *index set* may be finite or infinite. For example, $\{a, b, c, \ldots..\}$, $\{\alpha, \beta, \gamma, \ldots..\}$ are index sets.

Let U be the universal set. The *complement* of a set A with respect to U is the set of elements which belong to U but do not belong to A. It is denoted by $U - A$ or $\bar{A}$ or A' or A^c and is defined as $\bar{A} = \{x: x \in U \text{ and } x \notin A\}$. For example, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$. Then $\bar{A} = \{2, 4, 6, 8\}$. Another example, $U = \{x: x$ is a letter in English alphabet$\}$ and $A = \{x: x$ is a vowel$\}$. Then $\bar{A} = \{x: x$ is a consonant$\}$.

Let A and B be any two sets. The *union* of A and B is the set of all elements which belong to A or to B. It is denoted by $A \cup B = \{x: x \in A \text{ or } x \in B\}$. For example, $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$. Let A and B be any two sets. The *intersection* of A and B is the set of elements which belong to both A and B. It is denoted by $A \cap B = \{x: x \in A \text{ and } x \in B\}$. For example, $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$, then $A \cap B = \{2, 4\}$. Let A and B be any two sets. The *difference* of A and B is the set of elements which belong to A but do not belong to B. It is denoted by $A - B$ *or* $A \sim B$ or $A/B = \{x: x \in A \text{ and } x \notin B\}$. For example, $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 4, 5, 6, 7, 8\}$. Then $A - B = \{1, 2\}$ and $B - A = \{7, 8\}$. Let A and B be any two sets. The *symmetric difference* of A and B is the set of elements which belong to A or B but do not belong to A and B. It is denoted by $A \oplus B$ and defined as $A \oplus B = \{x: (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$ or $A \oplus B = (A - B) \cup (B - A)$. For example, $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7\}$. Then $A \oplus B = \{2, 4, 7\}$.

An *ordered pair* is represented by (a, b) in which a is first element and b is second element. Let (a, b) and (x, y) are two ordered pairs. Then we have $(a, b) = (x, y)$ if $a = x$ and $b = y$. Let A and B be any two sets. The *Cartesian products* of A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ *i.e.*, $A \times B = \{(a, b) : a \in A, b \in B\}$ and $B \times A = \{(b, a) : b \in B, a \in A\}$. For example, Let $A = \{a, b\}$, $B = \{1, 2, 3\}$. Then $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$ and $A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$.

In general, if $A_1, A_2, A_3, \ldots A_n$ are n sets then the product set of all these sets is $A_1 \times A_2 \times A_3 \times \ldots \times A_n = \{a_1, a_2, a_3 \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots., a_n \in A_n\}$.

Note:

1. If one of the two sets is infinite and the other is non-empty then the Cartesian product of two set is also infinite set.
2. Cartesian product of two set is not commutative.

1.2.2 Relations

In our day to life, a word used relation means something like as marriage and friendship, etc. “Is the mother of”, “is the father of”, “is the sister of”, “is the brother of”, “is the friend of”, are all relations over the set of men. Similarly, “is equal to”, “is less than”, “is greater than”, “is the divisor of” are relations on the set of numbers. In this book we study binary relations. A binary relation is the relation between two objects. For example, “is the son of” is a relation between two men a and b. Therefore the binary relation involves certain ordered pair (a, b) in which the first element a is related to the second element b. Let A and B be any two sets. *A relation R* from a set A to set B is a subset of $A \times B$ and defined as xRy if and only if $(x, y) \in R$, $x \in A$ and $y \in B$ or $xRy \Leftrightarrow (x, y) \in R$ and $x - y \Leftrightarrow (x, y) \notin R$, xRy reads "x is R-related to y". *For example:* Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Then $R = \{(a, 1), (a, 2), (b, 2), (c, 3)\}$ is a relation from A to B.

Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the *inverse relation* of R if and only if $R^{-1} = \{(y, x) : (x, y) \in R\}$. *For example:* Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Then $R = \{(1, 2), (1, 4), (2, 4), (3, 6)\}$ is a relation from A to B and $R^{-1} = \{(2, 1), (4, 1), (4, 2), (6, 3)\}$ is a inverse relation from B to A. Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an *identity relation* if $R = \{(a, a) : a \in A\}$. *For example:* Let $A = \{a, b, c, d\}$. Then $R = \{(a, a), (b, b), (c, c), (d, d)\}$ is an identity relation on A. Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as *universal relation* if $R = A \times A$ or $R = \{(a, a), (a, b), (a, b), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ is a universal relation on A. *For example:* Let $A = \{a, b\}$. Then $R = \{(a, a), (a, b), (b, a), (b, b)\}$ is a universal relation on A.

A relation R on a set A is known as *reflexive* relation if and only if aRa, $\forall a \in A$. A relation R on a set A is known as *symmetric* relation if and only if $aRb \Rightarrow bRa$ $\forall (a, b) \in R$. *A* relation R on a set A is known as *anti-symmetric* relation if and only if aRb, $bRa \Rightarrow a = b$ $\forall (a, b) \in R$. A relation R on a set A is known as *transitive* relation if and only if aRb, $bRc \Rightarrow aRc$, $(a, b, c \in A)$. *For example:* (i) In R, the relation "is equal to" is reflexive, symmetric and transitive. (ii) In R, the relation "less than" is anti-symmetric and transitive. (iii) The relation "is the friend of" on the set of all human beings is reflexive. (iv) the relation "less than", "greater than", "is the father of", "is the wife of" on the set of people are not reflexive. (v) The relation "a divides b" on set of natural numbers is anti-symmetric for a divides b and b divides a if and only if $a = b$. (vi) The relation "is the brother of" on any set of men is transitive for a is brother of b, b is brother of c then a is brother of c. (vii) The relation "is the father of" is not transitive.

1.2.3. Equivalence Relation

A relation R on a set A is known as an *equivalence relation* if and only if it is reflexive, symmetric and transitive. Equivalence relation is denoted by $\sim$. *For examples:* the relation "is the brother of". On any set of men, "is equal to" on the set of numbers are all equivalence relation. Let R be an equivalence relation on a set A. Let a be any arbitrary element of A. The set of all element $x \in A$ such that xRa constitute a subset of A (say $[a]$). Thus subset $[a]$ is known as *equivalence class* of a with respect to R, denoted as $[a] = \{x : x \in A \text{ and } xRa\}$.

A relation which is transitive but not an equivalence relation is known as an *order relation*. If R is an order relation on a set X, then xRy and $yRz \Rightarrow xRz$, $\forall\ x, y, z \in X$. A relation R on a set X is said to be a *partial order relation* if it is at the same time (i) Reflexive (ii) Anti-symmetric and (iii) Transitive. It is denoted by the symbol $\leq$. A set X together with a partial order relation defined on it, *i.e.*, $(X, \leq)$ is known as a *partial ordered set*. *For example,* the relation "x divides y" on the set of natural numbers is a partial order relation. The relation "sub-set of" on the set of all sub-sets of a set is a partial order relation.

1.2.4. Well Ordering, Partial Ordered Relation and Zorn Lemma

A partial ordered set is said to be *well ordered* and its ordering is known as a *well ordering* if every non-empty subset of it has a smallest element. *For example*, every finite set is well ordered but the set $\{\ldots, 5, 4, 3, 2, 1\}$ is not well ordered because it has no first element. *Well ordering theorem* state that every set can be well ordered. For example, the empty set is well ordered.

Let $(X, \leq)$ be a partially ordered set. An element p in X is said to be the largest element of X if $x \leq p$ for every x in X. An element p in X is said to be the maximal element of X if $p \leq x \Rightarrow p = x$ for some $x \in X$. Similarly an element $q \in X$ is said to be the smallest element of X if $q \leq x$ for every x in X. An element $q \in X$ is said to be the minimal element of X if $x \leq q \Rightarrow x = q$ for some x in X. Let $(X, \leq)$ be a partially ordered set and let A be any non-empty subset of X. An element $l \in X$ is said to be a lower bound of A if $l \leq x$ for every x in A. Similarly, an element $u \in x$ is said to be an upper bound of A if $x \leq u$ for every x in A. *For example,* Let N be a set of all natural numbers and let $A = \{5, 10, 15, 20, 25\}$. Obviously, 5 is a lower bound of A for $5 \leq x, \forall\ x \in A$. *Also 25 is an upper bound of A* for $25 \geq x, \forall\ x \in A$. Hence, 5 is the greatest lower bound and 25 is the least upper bound of A.

According to *Zorn's lemma*, if X is a partially ordered set such that every totally ordered set in X *has* an upper bound, then X contains a maximal element.

Note:

1. If R is a relation from A to A then R is known as relation on A.
2. A binary relation on a set A is a subset of $A \times A$.
3. Every relation has an inverse relation.
4. Let $A = \{1, 2, 3, 4\}$ and R be the relation > (is greater than). Then we have,
$$R = \{(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1)\}.$$

1.2.5 Axiom of Choice

Suppose $\{A_\alpha : \alpha \in \wedge\}$ is a non-empty family of non-empty subsets of a set P. A function $f: \{A_\alpha : \alpha \in \wedge\} \to P$ is called a *choice function,* if $f(A_\alpha) = a_\alpha \in A_\alpha$ for every $\alpha \in \wedge$.

The *axiom of choice* is equal to the following postulate know as Zermelo's postulate:

Consider $\{A_\alpha : \alpha \in \wedge\}$ is any non-empty family of disjoint non-empty sets. Then there exists a subset S of $\cup \{A_\alpha : \alpha \in \wedge\}$ such that the intersection of S and each set A_α consists of exactly one element.

1.2.6 Schröder-Bernstein's Equivalence Theorem

If A and B are two sets such that $A \leq B$ and $B \leq A$, then $A \sim B$. OR

If $A_1 \subset B \subset A$ and if $A \sim A_1$ then $A \sim B$.

Proof: Suppose if A is equivalent to A_1, then there exists a one-one onto function f from $A \to A_1$. Also it is given $B \subset A$, so f to B is also one-one. This means that the B is equivalent to a subset B_1 of A_1. Therefore the function $f: B \to B_1$ is one-one and onto, and so $B \sim B_1$. Continuing in this way we get the equivalent sets $A, A_1, A_2, \ldots$ and $B, B_1, B_2, \ldots$ such that

$$A \supset B \supset A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \ldots$$

Suppose $T = A \cap B \cap A_1 \cap B_1 \cap A_2 \cap B_2 \cap A_3 \ldots$

Thus we have $A = (A - B) \cup (B - A_1) \cup (A_1 - B_1) \cup (B_1 - A_2) \cup \ldots \cup T$

$B = (B - A_1) \cup (A_1 - B_1) \cup (B_1 - A_2) \cup \ldots \cup T$

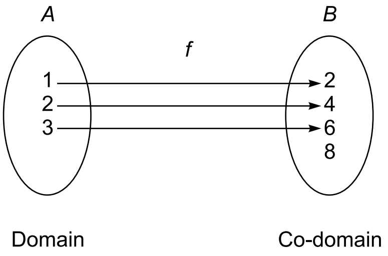
Now define a mapping $g: A \to B$ such that

$$\begin{aligned} g(A - B) &= A_1 - B_1 \\ g(A_1 - B_1) &= A_2 - B_2 \\ g(A_2 - B_2) &= A_3 - B_3 \\ \ldots\ & \ldots.. \\ \ldots\ & \ldots.. \\ g(B - A_1) &= B - A_1 \\ g(B_1 - A_2) &= B_1 - A_2 \\ \ldots\ & \ldots.. \\ \ldots\ & \ldots.. \\ g(T) &= T. \end{aligned}$$

Thus the mapping g is one-one and onto. Hence $A \sim B$.

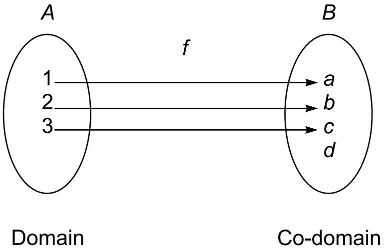
1.2.7 Functions

Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B. This mapping is denoted by $f : A \to B$ *or* $A \xrightarrow{f} B$. Here the set A is known as domain and the set B is known as co-domain of the function f. *For example,* Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$ and $f : A \to B$ is defined as



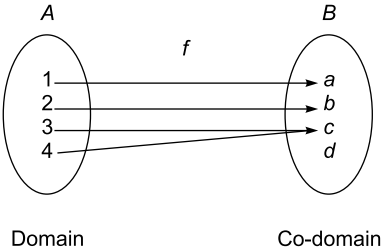
Here range is {2, 4, 6}. Range is a subset of co-domain.

A function f: $A \to B$ is called *one-one* if $x_1, x_2 \in A$, we have $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. *For example,* Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \to B$ is defined as



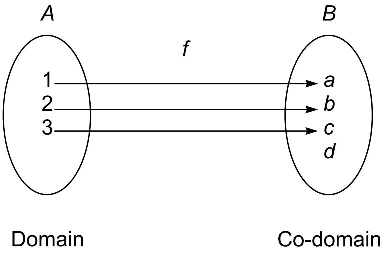
Here f is known as one-one function and range of f is $\{a, b, c\}$.

A function $f : A \to B$ is said to be *many-one* if at least one element of B has two or more than two pre-image in A. *For example,* Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $f : A \to B$ is defined as



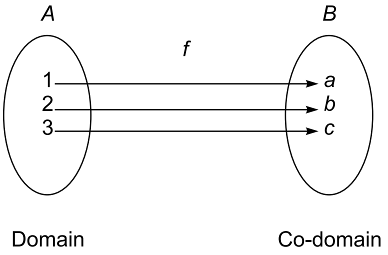
Here f is known as many-one function and range of f is $\{a, b, c\}$.

A function $f : A \to B$ is said to be *into* if there is at least one element of B, has no pre-image in A. *For example*, Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \to B$ is defined as



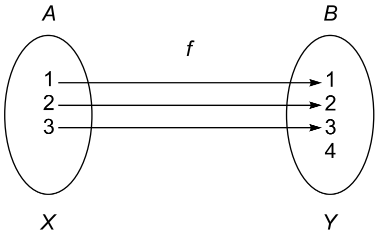
Here one element d of the set B has no pre-image in the set A. Then f is known as into function and range of f is $\{a, b, c\}$.

A function $f : A \to B$ is said to be *onto* if there is no element of B, which is not an image of some element of A. *For example*, Let A $\{1, 2, 3\}$, $B = \{a, b, c\}$ and $f : A \to B$ is defined as



Here f is known as onto function and range of f is $\{a, b, c\}$.

Let $f : X \to Y$ be a *one-one onto mapping* and $f(x) = y$, $\forall\, x \in X$, $\forall\, y \in Y$. Now we define a mapping $f^{-1} : y \to X$ such that $f^{-1}(y) = x$, $\forall\, x \in X$, $\forall\, y \in Y$, where f^{-1} is called the inverse of f. Here f is invertible mapping because inverse of f is exists. Let X be any subset of Y.



Then the mapping $f : X \to Y$ is said to be *inclusion mapping* if $f(x) = x$, $\forall\, x \in X$. $f : A \to B$ is defined as

Let $f : X \to X$ be a mapping. Then f is said to be *identity mapping* if $f(x) = x$, $\forall\, x \in X$. *A* mapping $f : X \to R$, where R is the set of real numbers, is known as *real valued mapping*. Let $f : X \to Y$ be a function. Then f is said to be *constant function* if $f(x) = a$, $\forall\, x \in X$ *i.e.*, a function $f : X \to Y$ is known as *constant function* if each element of X is mapped onto a single element of Y. The function $f : X \to Y$ is known as *zero function* if the image of each element of X under f is zero *i.e.*, $f(x) = 0$.

A mapping f is said to be *injective* (or *injection*) which is either one-one into or one-one onto. A mapping f is said to be *bijective* (or *bijection*) which is both one-one and onto. Let $f : X \to$ and $g : X \to Y$ be two mapping. Then the mapping f and g are said to be *equal mapping* if and only if $f(x) = g(x)\ \forall\ x \in X$. In case of equal mappings, the domains of mappings must be the same.

Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then a function $gof : X \to Z$ is defined as $gof = g[f(x)]$, $\forall x \in X$ is known as *composition of functions*. *For example,* Let $f(x) = x^2$, $g(x) = x + 3$, $\forall x \in R$. Here $gof = g[f(x)] = g(x^2) = x^2 + 3$ and $fog = f[g(x)] = f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$.

Theorem 1.1: Let $f : X \to Y$ be a one-one and onto mapping, then $f^{-1}: Y \to X$ is also a one-one and onto mapping.

Proof: Suppose $f : X \to Y$ is a one-one and onto mapping. To show that $f^{-1} : Y \to X$ is a one-one and onto mapping.

Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that

$$f(x_1) = y_1$$

and $f(x_2) = y_2$

If f^{-1} denotes the inverse of f, we have

$$f^{-1}(y_1) = x_1 \text{ and } f^{-1}(y_2) = x_2$$

Now we have

$$f^{-1}(y_1) = f^{-1}(y_2)$$

$$\Rightarrow \quad x_1 = x_2$$

$$\Rightarrow \quad f(x_1) = f(x_2) \qquad \text{(because } f \text{ is one-one)}$$

$$\Rightarrow \quad y_1 = y_2.$$

Therefore f^{-1} is a one-one mapping. Again f^{-1} is also an onto mapping for each element $x \in X$ is the inverse image of the element $y \in Y$, where $y = f(x)$.

Hence, the mapping $f^{-1} : Y \to X$ is always a bijective mapping.

Theorem 1.2: Let $f : X \to Y$ and, $A, B \subset X$ then

(i) $f[A \cup B] = f(A) \cup f(B)$

(ii) $f[A \cap B] \subset f(A) \cap f(B)$ But $f[A \cap B] \neq f(A) \cap f(B)$.

Proof:

(i) Suppose $y \in f[A \cup B]$

$$\Rightarrow \quad y = f(x) \text{ for some } x \in A \cup B$$

i.e., $\quad y = f(x)$ for some $x \in A$ or $x \in B$

Now $x \in A \Rightarrow y \in f(A)$

And $x \in B \Rightarrow y \in f(B)$

Hence, $y \in f[A \cup B]$

$\Rightarrow y \in f(A) \cup f(B)$

$$\therefore \quad f[(A \cup B] \subseteq f(A) \cup f(B) \tag{1.1}$$

Now, if $y \in f(A) \cup f(B)$

$\Rightarrow$ either $y \in f(A)$ or $y \in f(B)$

If $y \in f(A)$

$\Rightarrow$ there is an $x \in A$ such that $y = f(x)$

$\therefore \quad y \in f(A \cup B)$

If $y \in f(B)$

$\Rightarrow$ there is an $x \in B$ such that $y \in f(x)$

$\therefore \quad y \in f(A \cup B)$.

Hence, $y \in f(A) \cup f(B)$

$\Rightarrow \quad y \in f[A \cup B]$

$$\therefore \quad f(A) \cup f(B) \subseteq f[A \cup B] \tag{1.2}$$

Using Eqns. (1.1) and (1.2), we get

$f[A \cup B] = f(A) \cup f(B)$.

(ii) Suppose $y \in f[A \cap B]$

$\Rightarrow \quad y = f(x)$ for some $x \in A \cap B$

Since, $x \in A \Rightarrow f(x) \in f(A)$ and $x \in B$

$\Rightarrow \quad f(x) \in f(B)$

Hence, $x \in A \cap B$

$\Rightarrow \quad x \in A$ and $x \in B$

$\Rightarrow \quad f(x) \in f(A)$ and $f(x) \in f(B)$

$\Rightarrow \quad f(x) \in f(A) \cap f(B)$

$\therefore \quad f(A \cap B] \subset f(A) \cap f(B)$

Now to show that

$$f[A \cap B] \neq f(A) \cap f(B)$$

Consider a mapping $f : R \to R$ such that

$$f(x) = x^2$$

Let $A = [-1, 0]$ and $B = [0, 1]$

Then $A \cap B = \{0\}$ so that $f[A \cap B] = f(0) = \{0\}$

But $f(A) = [0, 1]$; $f(B) = [0, 1]$

$\therefore \quad f(A) \cap f(B) = [0, 1]$

Hence, $f(A \cap B] \neq f(A) \cap f(B)$.

Theorem 1.3: To show that composite of mappings is associative

Proof: Consider the functions $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$.

To show that $ho(gof) = (hog)of$

Suppose $x \in X$, then we have

$$\begin{aligned}[ho(gof)](x) &= h[(gof)(x)] \\ &= h[g(f(x)] \quad (\text{if } f(x) = y) \\ &= h[g(y)] \quad (\text{if } g(y) = z) \\ &= h(z) \qquad (1.3)\end{aligned}$$

Now we have

$$\begin{aligned}[(hog)of](x) &= (hog)[f(x)] \\ &= (hog)(y) \quad (\text{if } f(x) = y) \\ &= h[g(y)] \quad (\text{if } g(y) = z) \\ &= h(z) \qquad (1.4)\end{aligned}$$

Using Eqns. (1.3) and (1.4), we have

$$ho(gof) = (hog)of$$

Hence, the composite of mappings is associative.

Theorem 1.4: If $f : X \to Y$ and $g : Y \to Z$ be one-one and onto mappings then gof is invertible mapping and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: Consider $f : X \to Y$ and $g : Y \to Z$ both are the one-one onto mapping. First we show that *got* is invertible, *i.e.*, gof is one-one onto mapping because a one-one onto mapping is always invertible.

Let $x_1, x_2 \in X$, we have $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad g(f(x_1) = g(f(x_2))$

$\Rightarrow \quad f(x_1) = f(x_2)$, for g is one-one

$\Rightarrow \quad x_1 = x_2$, for f is one-one.

Hence, $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad x_1 = x_2$ which show that gof is one-one.

Now let $z \in Z$. Since g is one-one onto mapping and therefore exist one and only one element $y \in Y$

such that $g(y) = z$.

Again f is one-one onto, there exist a unique element $x \in X$ such that $f(x) = y$,

$$\begin{aligned}(gof)(x) &= g(f(x)) \\ &= g(y) \\ &= z\end{aligned}$$

i.e., $g\ of$ is an onto mapping.

Now we show that

$$(gof)^{-1} = f^{-1} og^{-1}.$$

Given that

$$f : X \to Y, \qquad g : Y \to Z$$

$$gof : X \to Z, \qquad (gof\,)^{-1} : Z \to X$$

$$\because \qquad f^{-1} : Y \to X, \qquad g^{-1} : Z \to Y$$

$$\therefore \qquad f^{-1}og^{-1} : Z \to X.$$

We have $(gof)(x) = z \;\Rightarrow\; x = (gof)^{-1} z$ ($\because$ *got* is one one-one onto)

$$f(x) = y \;\Rightarrow\; x = f^{-1}(y) \qquad (\because f \text{ is one-one onto})$$

$$g(y) = z \;\Rightarrow\; y = g^{-1}(z) \qquad (\because g \text{ is one-one onto})$$

$$\begin{aligned}(f^{-1}og^{-1})(z) &= f^{-1}(g^{-1})(z)\\ &= f^{-1}(y)\\ &= x\\ &= (gof\,)^{-1}(z)\end{aligned}$$

Hence, $\qquad (f^{-1}og^{-1})(z) = (gof\,)^{-1}(z)\ \forall z \in Z$

Here $f^{-1}og^{-1}$ and $(\mathrm{gof})^{-1}$ both the mapping have the same domain z. Using definition of equal mapping, we have

$$(gof)^{-1} = f^{-1}og^{-1}.$$

1.2.8 Open and Closed Sets

A subset of real numbers of the form $\{t \in R\text{: } a < t < b\}$ where $a, b \in R$ and $a < b$, is known as *open interval*. It is denoted as (a, b) or $[a, b]$. A subset of real numbers of the form $\{t \in R\text{: } a \le t \le b\}$ where $a, b \in R$ and $a < b$, is known as a *closed interval*. It is denoted as $[a, b]$. A subset of real numbers of the form $\{t \in R\text{: } a \le t < b\}$ where $a, b \in R$ and $a < b$, is known as a *closed-open interval*. It is denoted as $[a, b]$ or $[a, b]$. A subset X of R is said to be *U-open* (or open) set if (i) $X = \phi$ or (ii) if $X \ne \phi$ then for each $x \in X$ there exists an open interval I such that $x \in I$ and $I \in X$. *For example*, every open interval is an open set. *A* finite set is not an open set. Suppose x is a real number. A subset N of R containing x, is said to be a *neighbourhood* of x if N contains an open interval containing x. Hence, $N \in R$ is a neighbourhood of x if and only if there exists an open interval (a, b) such that $x \in (a, b) \in N$. *For example*, every open interval is a neighbourhood of each of its point. Let A be a subset of R. A point $x \in R$ is known as *limit point* of A if and only if every open set X containing x contains at least one point of A other than x, i.e., X is open, $x \in X \Rightarrow A \cap [X - \{x\}] \ne \phi$. The set of all limit point of a set A is said to be *derived* *set* of A. It is denoted by A'. A subset A of real number R is known as a *closed set* if and only if complement of A is an open set.

Let A be a subset of R. A point $x \in A$ is known as *adherent point* of A if and only if every neighbourhood of x contains at least one point of A. A point $x \in A$ is known as *isolated point* of A if an only if it is not an limit point of A. A set A is known as *discrete set* if each point A is an isolated point of A. Let A be a subset of R. A is known as *dense-in-itself* if and only if every point of A is a limit point of A. A subset A of R is known as *perfect set* if and only if $A = A'$. Let A be a subset of R. Then the set of all *adherent points* of a given subset A of R is known as closure of A. It is denoted by $\bar{A}$. Hence, $\bar{A} = A \cup A'$. Let A be a subset of R. A point $x \in R$ is known as *interior* point of A if A is a neighbourhood of x. A is the neighbourhood of x if and only if there exist an open interval (a, b) such that $x \in (a, b) \subset A$. Therefore, an interior point of a set A belongs to A. The set of all interior point of A is known as *interior* of A. It is denoted by A°. Let A be a subset of R. A point x is known as an exterior point of a subset A if x has a neighbourhood N such that $N \subset R - A$. Hence, if x is an exterior point of A then x is an interior point of $R - A$. The set of all exterior point of a set A of R is known as exterior of A and denoted by $e(A)$. Let A be a subset of R. A point x is known as a frontier point of A if every neighbourhood of x contains at least one point of A and at least one point of $R - A$. The set of all frontier points of A is known as *frontier* of A. It is denoted by $F(A)$. An exterior point of A which is also belongs to A, *i.e.*, $x \in A \cap e(A)$ is known as a boundary point of A.

1.2.9 Bolzano-Weierstrass Theorem

Every bounded infinite set of real numbers has at least one accumulation point.

Proof: Consider A is a bounded infinite set of real numbers R. To show that A contains at least one limit point of A. It is given that A is a bounded subset of R, so it may be contained in a closed interval $[a, b]$ *i.e.*, $A \subset [a, b] = I_1$ (say). Now suppose I_1 divided into two intervals, $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Since A is finite therefore I_1 if infinite and also one of the closed intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ is infinite.

Suppose $[a_1, b_1]$ be one of the above two intervals which contains infinite number of elements of A. Continuing this process, we get a sequence of closed intervals $I_1, I_2, I_3, \ldots$ such that $I_1 \supset I_2 \supset I_3 \supset \ldots$ and each interval I_n contains infinite number of points of A. Clearly the length of interval $I_n = \frac{a-b}{2^n}$. Thus $\lim|I_n| = 0$. Using the nested interval property of the real numbers there exist a point l in each interval I_n. Here we prove that l is a limit point of A. Suppose (p, q) be an open interval containing the point l. since $\lim|I_n| = 0$ there exists a positive integer r such that $|I_r| < r$ in $(l - p, q - l)$, then the interval I_r is a subset of the open interval (p, q), since I_r contains an infinite number of points of A. Hence, each open interval containing l contains points of A other than l and so l is a limit point of A.

1.3 FINITE AND INFINITE SETS

Finite sets are very important for the study of combinatory theory of counting. A set is said to be *finite set* if it contains finite number of elements, otherwise it is *infinite*. Let A be the set of all students of an engineering college, B is the set of vowels and N is the set of natural numbers. Here A and B are finite set and N is infinite set.

1.4 CARDINAL NUMBERS AND ITS ARITHMETIC WITH CARDINALITY OF A SET

The cardinal number of a finite set is the number of elements contains in it. For any set A the cardinal number is denoted as $\#(A)$. Hence, the cardinal number of the null set is zero and the cardinal number of a 100-elementic set is 100. Every cardinal number of a set is sometime called its power. If P is any finite set consisting of p element then its cardinal number is defined as p. The cardinal number of ϕ is defined as 0. A cardinal number corresponding to a finite set is called a *finite cardinal number*. A cardinal number corresponding to an infinite set is called a *transfinite cardinal number*.

Let P and Q be any two sets with cardinality p and q. Such that their intersection is empty *i.e.*, $|P| = p, |Q| = q, P \cap Q = \phi$. Then the sum of and is defined as $p + q = |P \cup Q|$.

For Example, If $A = \{1, 2\}, B = \{4, 5, 6, 7\}$, then $|A| = 2$, $|B| = 4$ and $A \cap B = \phi$ so that $|A \cup B| = |A| + |B| = 2 + 4 = 6$. Also $A \cup B = \{1, 2, 4, 5, 6, 7\}$ and $|A \cup B| = 6$. *For Example*, Let $A = \{a, b, c\}, B = \{a, b, q, r, s\}$. Then $A \cup B = \{a, b, c, p, q, r, s\}$, *i.e.*, $|A \cup B| = 7$.

Evidently $|A| = 3$, $|B| = 5$ and $A \cap B \neq \phi$. By definition, we have $|A \cup B| \neq |A| + |B| \Rightarrow 7 \neq 3 + 5$. This example show that $A \cap B = \phi$ is a necessary condition for defining the rule $|A \cup B| = |A| + |B|$.

The product of cardinal number p and q as $p \times q = |P \times Q|$. *For Example*, Let $P = \{a, b\}$, $Q = \{0, 1, 2, 3\}$. Then $|P| = 2, |Q| = 4, P \cap Q \neq Q$. By definition, we have $|P \times Q| = 2 \times 4 = 8$.

Now $|P \times Q| = |\{(a,0),(a,1),(a,2),(a,3),(b,0),(b,1),(b,2),(b,3)\}|$.

Let A be any finite set. The number of distinct elements contained in A is known as *cardinality* of the set A. It is denoted by $n(A)$ or $|A|$. *For Example,* Let A be the set $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then $n(A) = 7$. For a empty set, $n(\phi) = 0$.

Note: Every property of additional and multiplication of natural numbers is not true for cardinal numbers in general. *For example,* $a + a = a$ does not imply $a = 0$; $a \cdot a = a = 1.\ a$ does not imply $a = 1$. This proves that cancellation law is not true for the operations of addition and multiplication of cardinal numbers. In a finite set, the number of elements is a natural numbers and known as cardinality of a set.

1.5 COUNTABLE AND UNCOUNTABLE SETS

A set which is either finite or denumerable is called a *countable set*. An infinite set is said to be denumerable or enumerable if it equivalent to the set N, the set of all natural number. *For example*, Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then A is finite so that by definition A is countable. *A* set A is called on *uncountable set* if A is an infinite set and A is not cardinally equivalent to N. For example, R and C are uncountable sets. Here we state the following theorem without proof:

1. Every infinite set contains an enumerable set.
2. The open interval (0, 1) is not an enumerable.
3. The set of all irrational numbers is uncountable.

1.6 ORDERED SETS AND ORDINAL NUMBERS

A set X is said to be an *ordered set* if any order relation exist between every pair of distinct elements of set, i.e., for any two elements a and b such that (i) $a < b$ or $b < a$ if $a \neq b$ and (ii) $a < b$ or $b < c \Rightarrow a < c$. *For example,* the set of natural numbers and the set of integers are ordered sets, according to increasing or decreasing their magnitude.

Georg Cantor discussed the Ordinals numbers in 1883. Ordinal numbers are the extension of the natural numbers which is different from integers and cardinals numbers. Ordinal numbers can be added, multiplied, and exponentiated. An order type which is represented by well ordered set is known as *ordinal number*. The ordinal number of singleton set is 1. An ordinal number is a special type of well ordered set.

Note:

1. Subset of an ordered set are ordered set.
2. An empty set is an ordered set.
3. A singleton set is an ordered set.

1.7 CANTOR'S THEOREM

To prove that $|A| < |P(A)|$ for any set A.

Proof: Let A be any set.

To prove that $|A| < |P(A)|$

Write $B_1 = \{\{x\} : x \in A\}$

Then (i) $B_1 \subset P(A)$ and (ii) $B_1 \sim A$ under the map $\{x\} \to x$

(i) $\Rightarrow |B_1| \leq |P(A)|$ (1.5)

(ii) $\Rightarrow |B_1| = |A| \Rightarrow |A| \leq |P(A)|$,

on using (1.5) remains to prove that

$$|A| \neq |P(A)| \tag{1.6}$$

Suppose not, then $|A| = |P(A)|$ which $\Rightarrow A \sim P(A)$ so that $\exists$ one-one map

$$f : A \xrightarrow{\underline{onto}} P(A) \tag{1.7}$$

Take $B = \{x \in A : x \notin f(x)\}$ (1.8)

If $x \in A \Rightarrow \quad f(x) \in P(A)$ by Eqn. (1.7)

$\Rightarrow \quad f(x) \subset A$ since $P(A)$ is the family of all subsets of A.

Also $B \subset A$ according to Eqn. (1.8).

Thus we have $B \subset A$, $f(x) \subset A$ for any $x \in A$.

This implies there exists $\alpha \in A$ such that $f(\alpha) = B$.

$$\alpha \in A \quad \Rightarrow \quad \alpha \in B \quad \text{or} \quad \alpha \notin B.$$

Consider the case in which $\alpha \in B$ (1.9)

Now $\alpha \in A \quad \Rightarrow \quad \alpha \in B$

$\Rightarrow \quad \alpha \notin f(\beta)$ according to Eqn. (1.8)

$\Rightarrow \quad \alpha \notin B$ contrary to Eqn. (1.9) $[\because f(\alpha) = B]$

$\therefore$ The possibility $\alpha \in B$ is ruled out.

Consider the second possibility in which $\alpha \notin B$ (1.10)

Now $\alpha \in A \quad \Rightarrow \quad \alpha \notin B$

$\Rightarrow \quad \alpha \notin f(\alpha) \qquad \{\because f(\alpha) = B\}$

$\Rightarrow \quad \alpha \in B$ according to Eqn. (1.8) contrary to Eqn. (1.10).

$\therefore$ The possibility $\alpha \notin B$ is also ruled out it among to saying.

$\alpha \in A$ does not imply $\alpha \in B$ or $\alpha \notin B$. Again we get a contradiction.

It means that our initial assumption is wrong. This means that Eqn. (1.6) is true.

1.8 CONTINUUM HYPOTHESIS

It is guessed that there is no cardinal number between a and c. Hence c is supposed to be the second transfinite cardinal number. But, there are other cardinal number bigger than c. Such as card $P(R) > c$.

1.9 ALGEBRAIC NUMBERS

A real number is said to be an *algebraic number* if it is a solution of a polynomial equation $p(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n = 0, a_n \neq 0$ with integral coefficients. The set of al algebraic numbers is enumerable.

1.10 TRANSCENDENTAL NUMBERS

A real number which is not "algebraic" is called a *transcendental number*. The number e and π are the best know transcendental numbers. The number e and π were proved to be transcendental, respectively by Hermite in 1873 and by Lindemann in 1892.

Note:

1. The set of all transcendental numbers is non-enumerable.
2. Every transcendental number is irrational but the converse is not true.

Theorem 1.5: To prove that $|A \times B| = |B| + |B| + |B| \dots$ to $|A|$ terms.

Proof: Let A and B be any two sets. We have

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

$$= \bigcup_{x \in A} \{x, y\} : y \in B$$

Then $$|A \times B| = \left| \bigcup_{x \in A} \{(x, y) : y \in B\} \right| \tag{1.11}$$

For a fixed $x \in A$, consider the mapping

$$f : B \to \{(x, y) : y \in B\}$$

given by $f(y) = (x, y), \forall y \in B$

Evidently f is one-one onto. Then $B \sim \{(x, y) : y \in B\}$

This $\Rightarrow |B| = |(x, y) : y \in B|$

In this event Eqn. (1.11), show that

$|A \times B| = |B| + |B| + \dots$ to $|A|$ terms.

Verification of the Theorem

Let $A = \{1, 2\},\ B = \{1, 2, 3\}$.

Then $|A| = 2,\quad |B| = 3 \qquad \Rightarrow \qquad |A \times B| = 2 \cdot 3 = 6$

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

If we take $A_1 = \{(1, 1), (1, 2), (1, 3)\}$ and $B_1 = \{(2, 1), (2, 2), (2, 3)\}$

Then $A \times B = A_1 \cup B_1,\ \ A_1 \cap B_1 = \phi$.

Thus $|A \times B| = |A_1 \cup B_1|$ *i.e.*, $2 \cdot 3 = 3 + 3$.

Theorem 1.6: If α by any cardinal number, then $\alpha \leq |A| \leq \alpha \quad \Rightarrow \quad |A| = \alpha$.

Solution: Let α be any cardinal number.

Let $\alpha \leq |A| \leq \alpha$.

To prove that $|A| = \alpha$.

Let $\alpha = |B|$. Then we have to prove $|A| = |B|$ which is equivalent to proving that $A \sim B$.

$$\alpha \leq |A| \qquad \Rightarrow \qquad |B| \leq |A|$$

$\Rightarrow \quad B \sim$ to a subset of A

or $\quad B \sim A \qquad (1.12)$

So we have,

$$|A| \leq \alpha \qquad \Rightarrow \qquad |A| \leq |B|$$

$\Rightarrow \quad A \sim$ to a subset of B

or $\quad A \sim B \qquad (1.13)$

Using Eqns. (1.12) and (1.13), we have

$$|A| = |B| = \alpha.$$

Theorem 1.7: Every subset of a countable set is countable.

Proof: Let A be a countable set.

Then A is either finite or enumerable.

(i) When A is finite. Then every subset of A is also finite and so is countable.

(ii) When A is enumerable. Then A can be written as

$$A = \{x_1, x_2, \ldots\} \tag{1.14}$$

Let B be a subset of A

(i) If $B = \phi$ then evidently B is countable.

(ii) If $B \neq \phi$ then B is expressible as $B = \{x_{n1}, x_{n2}, \ldots\}$ where $x_{ni} \in A$.

Evidently A is enumerable by Eqn. (1.14).

Theorem 1.8: The set of all real numbers in the closed interval [0, 1] is not denumerable.

Proof: Let A denote the set of all real numbers in the closed interval [0, 1]. So we have

$$A = \{x \in R : 0 \le x \le 1\} = [0, 1].$$

To prove that A is uncountable. Suppose not. Then A is countable.

$\Rightarrow$every element of A must appear in the sequence $x_1, x_2, x_3 \dots, x_n \dots$ of distinct element, *i.e.,*

$$A = \{x_1, x_2, x_3, \dots\} \tag{1.15}$$

Write decimal expansion of those x_j' as follows:

$$x_1 = 0.x_{11}\ x_{12}\ x_{13}\ x_{14} \dots x_{1m} \dots$$

$$x_2 = 0.x_{21}\ x_{22}\ x_{23}\ x_{24} \dots x_{2m} \dots$$

$$x_m = 0.x_{m1}\ x_{m2}\ x_{m3}\ x_{m4} \dots x_{mn} \dots$$

Where $x_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \forall i$ and j and each decimal contains an infinite number of non-zero elements. We can write 1 and 1/2 in two ways as given below:

$$\begin{cases} 1 = 1.0000\dots \\ \dfrac{1}{2} = 0.5000\dots \end{cases}$$

and

$$\begin{Bmatrix} 1 = 0.99999\dots = 0.9 \\ \dfrac{1}{2} = 0.49999\dots = 0.49 \end{Bmatrix} \tag{1.16}$$

But in the present case we shall write the decimal representation of the elements A in the from (1.16).

Construct a real number $\xi = 0.\xi_1\xi_2\xi_3.........\xi_n$ such that

If $x_{11} = 5$, write $\xi_1 = 4$

And if $x_{11} \ne 5$, write $\xi_1 = 5$

In general if $x_{mm} = 5$, write $\xi_m = 4$

And if $x_{mm} \ne 5$, write $\xi_m = 5$

In either case, it is clear that $x_{mm} \ne \xi_m, \forall m$ (1.17)

Since, $\xi_m = 4$ or 5, $\forall m$

$$\therefore \quad \xi \in [0, 1] \quad i.e.,\ \xi \in A \tag{1.18}$$

$$x_m = \xi \quad \Rightarrow \quad 0.x_{m_1}, x_{m_2}x_{m_3} - = 0.\xi_1\xi_2\xi_3$$

$$\Rightarrow \quad x_{mm} = \xi_m \quad \forall m$$

Contrary to Eqn. (1.17), we have

$\because \quad x_m \neq \xi, \forall m$

In this event Eqn. (1.15) show that $\xi \notin A$ contrary Eqn. (1.18). Hence the required result follows.

Theorem 1.9:

(i) If A_i is countable infinite set then $\bigcup_{i=1}^{n} A_i$ is countable infinite and hence deduce that $na = a$.

(ii) If A_i is countable infinite set for $i = 1, 2, 3\ldots$ then $\bigcup_{i=1}^{n} A$ is countable infinite and hence deduce that $a + a + a + \ldots +$ to terms = a

or

Union of countable collection of countable set is countable

Solution:

(i) Let A_i be a countable infinite set. Let $A = \bigcup_{i=1}^{n} A_i$. To prove that A is countable infinite.

Let the elements of A_i be displayed as

$A_1 : a_{11}, a_{12}, a_{13}, \ldots, a_{1n}, \ldots$

$A_2 : a_{21}, a_{22}, a_{23}, \ldots, a_{2n}, \ldots$

$A_3 : a_{31}, a_{32}, a_{33}, \ldots, a_{3n}, \ldots$

Write $B = \{a_{11}, a_{12}, \ldots a_{1n}, \ldots; a_{21}, a_{22}, \ldots a_{n1}, a_{n2}, \ldots\}$

The set B considered as the set of distinct element is countably infinite

$$\therefore \quad |B| = a \tag{1.19}$$

If $A_i \cap A_j = \phi$ for $i \neq j$ then evidently $|A| = |B| = a$.

If $A_i \cap A_j \neq \phi$ for $i \neq j$, then A is equipollent with some subset of B.

$$\text{This} \Rightarrow |A| \leq |B| = a \text{ or } |A| \leq a \tag{1.20}$$

But $A_1 \subset A$ and $A_1 = a$

$$\therefore \quad a = |A_1| \leq |A| \; or \; a \leq |A| \tag{1.21}$$

Combining Eqns. (1.20) and (1.21), we get

$$a \leq |A| \leq a \qquad \text{which} \qquad \Rightarrow |A| = a.$$

$\therefore$ In either case $|A| = a$. This proves the required result.

Deduction. Further assume that $A_i \cap A_j = \phi$ for $i \neq j$ and $i, j = 1, 2, 3, \ldots n$. Then by what we have established it follows that $|A| = a$.

or $\left|\bigcup_{i=1}^{n} A_i\right| = a$ or $\sum_{i=1}^{n} |A_i| = a$

or $a = a = \cdots +$ to n terms $= a$

or $na = a$

(ii) Let A_j be a countable infinite set, and $A = \bigcup_{j=1}^{\infty} A_i$ to prove that A is countably infinite.

Let the elements of A_i be displayed as follows:

$$\begin{array}{l} A_1 : a_{11}, a_{12}, a_{13}, \ldots, a_{1n}, \ldots \\ A_2 : a_{21}, a_{22}, a_{23}, \ldots, a_{2n}, \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ A_n : a_{n1}, a_{n2}, a_{n3}, \ldots, a_{nn}, \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}$$

First we shall assume that $A_i \cap A_j = \phi$ for $i \neq j$.

Divide the elements of A into blocks such that m^{th} block contain m elements and a_{ij} will be in m^{th} block *iff* $i + j = m + 1$ within the m^{th} block the second suffix j of a_{ij} increases from 1 to m.

Thus A can be expressed as

$$A = \{a_{11}; a_{21}, a_{12}; a_{31}, a_{22}\ a_{13}; \ldots; a_{n1}, \ldots, a_{1n}, \ldots\}$$

Clearly A is expressible in the form of a sequence of distinct elements and hence denumerable

So that $|A| = a$.

Secondly we assume that $A_j \cap A_j \neq \phi$ for $i \neq j$

In this case A is equipollent with some subset of N so that

$$|A| \leq N \quad \text{or} \quad |A| \leq a \tag{1.22}$$

But $A_1 \subset A$ and $|A_1| = a$ so that

$$a = |A_1| \leq |A| \quad \text{or} \quad a \leq |A| \tag{1.23}$$

By Eqns. (1.22) and (1.23), we have $|A| = a$.

Deduction. Further assume that

$$A_i \cap A_j \neq \phi \quad \text{for} \quad i \neq j$$

Then from what has been done it follows that

$$|A| = a \left|\bigcup_{i=1}^{\infty} A_i\right| = a \quad \Rightarrow \quad \sum_{i=1}^{\infty} |A_i| = a$$

$\Rightarrow$ $a + a + \ldots +$ to a terms $= a$.

Theorem 1.10:

(i) If A_i is non-enumerable set $\forall i \in N$ then $\bigcup\limits_{i=1}^{\infty} A_i$ is non enumerable and hence deduce that $C + C + C + \ldots$ to a terms = C.

(ii) If A_i is non-enumerable set for $1 \le i \le n \bigcup\limits_{i=1}^{n} A_i$ is non-enumerable and hence deduce that $C + C + C + \ldots$ to n terms = C.

Solution:

(i) Let A_i be a non-enumerable set $\forall i \in N$ so that we can write $|A_i| = C,\ \forall i \in N.$

Let $A = \bigcup\limits_{i=1}^{n} A_i$. To prove that A is non-enumerable.

For proving this we must show that $|A| = C$. $|A_i| = C$ show that the set A_1 is numerically equivalent to the set of real numbers in the interval $\left[0, \frac{1}{2}\right)$, *i.e,.* to say, $A_1 \sim \left[0, \frac{1}{2}\right)$

Similarly, we have

$$A_2 \sim \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^2}\right)$$

$$A_3 \sim \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)$$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$$A_n \sim \left[\frac{1}{2} + \frac{1}{2^2} + \ldots \frac{1}{2^{n-1}}, \frac{1}{2} + \frac{1}{2^2} \ldots + \ldots \frac{1}{2^n}\right)$$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

i.e., $A_1 \sim \left[0, \frac{1}{2}\right), A_2 \sim \left[\frac{1}{2} +, \frac{3}{4}\right), A_3 \sim \left[\frac{3}{4}, \frac{7}{8}\right), \ldots, A_n \sim \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right)$

If we assume that $A_i \cap A_j = \phi$ for $i \neq j$ then above results taken together imply that $\bigcup\limits_{i=1}^{\infty} A_i \sim \bigcup\limits_{i=1}^{\infty} \left[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right)$

or $A \sim [0, 1)$ or $|A| = |[0, 1)| = C.$

If we assume that $A_i \cap A_j \neq \phi$ for $i \neq j$ then $\bigcup_{i=1}^{\infty} A_i$ is equivalent with some subset of $[0, 1)$.

$$\therefore \quad \left|\bigcup_{i=1}^{\infty} A_i\right| \le C \quad \text{or} \quad |A| \le C.$$

But $A_1 \subset A$ and $|A_1| = C$.

$$\therefore \quad C = |A_1| \le |A| \quad \text{or} \quad |A| \ge C$$

$$|A| \ge C, \quad |A| \le C \quad \Rightarrow \quad C \le |A| \le C \quad \Rightarrow \quad |A| = C.$$

$\therefore$ In either case $|A| = C$.

Deduction: Further assume that $A_i \cap A_j \neq \phi$ for $i \neq j$.
We have seen that

$$\left|\bigcup_{i=1}^{\infty} A_i\right| = C$$

This implies $\sum_{i=1}^{\infty} |A_i| = C$.

or $C + C + C + \ldots +$ to a terms = C.

(ii) Let A_i be a non-enumerable set for $1 \le i \le n$. Then we can write $|A_i| = C$ for $1 \le i \le n$. To prove that $\bigcup_{i=1}^{n} A_i$ is non-enumerable $|A_i| = C$ implies that A_i is cardinally equivalent with the set of real numbers in the semi-open interval $[a_1, a_2)$ where $a_1, a_2 \in R$ and $a_1 < a_2$.
This is expressed by writing $A_1 \sim [a_1, a_2)$.
Here we shall suppose that $a_r < a_{r+1}$,

Where $a_r, a_{r+1} \in R$ for $r = 1, 2, 3 \ldots, n$

Thus we have

$$A_1 \sim [a_1, a_2), \; A_2 \sim [a_2, a_3), \ldots A_n \sim [a_n, a_{n+1}).$$

If we assume that $A_i \cap A_j \neq \phi$ for $i \neq j$ then the above statements taken together imply that $\bigcup_{i=1}^{n} A_i \sim [a_1, a_{n+1})$

This implies $\left|\bigcup_{i=1}^{n} A_i\right| = |[a_1, a_{n+1})| = C$

$$\Rightarrow \quad \left|\bigcup_{i=1}^{n} A_i\right| = C$$

If we assume that $A_i \cap A_j \neq \phi$ for $i \neq j$ then $\bigcup_{i=1}^{n} A_i$ is equivalent with some subset of $[a_1, a_{n+1})$ so that $\left|\bigcup_{i=1}^{n} A_i\right| \leq C$ (1.24)

But $A_1 \subset \bigcup_{i=1}^{n} A_i$ and $|A_1| = C$

$\therefore \quad C = |A_1| \leq 1\left|\bigcup_{i=1}^{n} A_i\right|$

or $\quad C \leq \left|\bigcup_{i=1}^{n} A_i\right|$ (1.25)

By Eqns. (1.24) and (1.25), we get

$$\left|\bigcup_{i=1}^{n} A_i\right| = C$$

In this case $\left|\bigcup_{i=1}^{n} A_i\right| = C$

Deduction:

(i) Further assume that

$A_i \cap A_j \neq \phi$ for $i \neq j$

Then $\left|\bigcup_{i=1}^{n} A_i\right| = C$

$\Rightarrow \quad \left|\bigcup_{i=1}^{n} A_i\right| = C$

$\Rightarrow \quad C + C + \ldots + C + \ldots$ to n terms = C.

(ii) To prove that $a + a = a$ for every infinite cardinal a let A and B be two disjoint infinite sets such that $|A| = a = |B|$.
Then we have

$$A \cap B = \phi$$

$$\Rightarrow \quad |A \cup B| = |A| + |B|$$

$$\Rightarrow \quad a + a = a.$$

Note: If $|A| = n$ (finite cardinal number)
Then by Cantor's theorem $|A| < |P(A)|$

$\Rightarrow \quad n < 2^n$

$\because \quad |P(A)| = 2^{|A|} = 2^n.$

Theorem 1.11: Prove $a+\alpha=\alpha,$ α being any transfinite cardinal number.

or

If an enumerable set is add to an infinite set, the power of the infinite set is unaffected.

Proof: Let A be any infinite set with cardinality α so that $|A|=\alpha$.

Let $A \cap N=\phi$.

We have to prove that $|A \cup N|=|A|$.

If we show that $a+\alpha=\alpha,$ the result will follow:

A is an infinite set $\Rightarrow \exists \quad B \subset A$ such that $|B|=a$.

We have $A \cup N=(A-B) \cup B \cup N$

$$=(A-B) \cup (B \cup N)$$

or $$(A \cup N)=(A-B) \cup (B \cup N) \tag{1.26}$$

$B \cup N,$ being a finite union of countably infinite sets, is countably infinite.

$\because \quad B \cup N \sim N$ but $B \sim N$.

$\because \quad B$ is enumerable.

The relation $M \sim N$, where M and N are any two sets, is an equivalence relation in the family of sets.

$\because \quad B \sim N \Rightarrow N \sim B$ (by symmetry)

$B \cup N \sim N, \quad N \sim B \Rightarrow B \cup N \sim B$ (by transitivity)

Now $B \cup N \sim B$ and $A-B \sim A-B$ (by reflexivity)

Combining these two, we have,

$$(A-B) \cup (B \cup N) \sim (A-B) \cup B$$

Using Eqn. (1.26), we get

$$A \cup N \sim A$$

This implies $|A \cup N|=|A|$

Hence, $a+\alpha=\alpha$.

Theorem 1.12: If $\alpha,$ β and r are cardinal number and if $\alpha<\beta$ and $\beta<\gamma$ then $\alpha<\gamma$.

Proof: Let A, B and C be the sets having cardinal number $\alpha,$ β and γ respectively.

Since $\beta<\beta$, the set $A \sim B^{*}$, a proper subset of B (1.27)

Similarly, we have,

$\beta<\gamma$, the set $B \sim C^{*}$, a proper subset of C (1.28)

From Eqns. (1.27) and (1.28), we have

$$\alpha < \beta \quad \text{and} \quad \beta < \gamma$$

$$\Rightarrow \quad \alpha < \gamma$$

or $$A \sim B^* \quad \text{and} \quad B < C^*$$

$\Rightarrow A \sim C^*$, a subset of C^*

Hence $\beta < \gamma$.

Theorem 1.13: For any cardinal number $\alpha,\ \beta,\ \gamma$ prove that $\alpha^{\beta}\alpha^{\gamma} = \alpha^{\beta+\gamma}$

Proof: Let $\alpha = \overline{\overline{A}},\ \beta = \overline{\overline{B}}$ *and* $\gamma = \overline{\overline{C}}$, where $B \cap C = \phi$, then

$$\overline{\overline{B \cup C}} = \beta + \gamma$$

So, we have

$$\alpha^{\beta+\gamma} = (\overline{\overline{B \cup C}}),\ \ \alpha^{\beta} = (A^{\overline{\overline{B}}}) \text{ and } \alpha^{\gamma} = (A^{\overline{\overline{C}}})$$

Since $\alpha^{\beta} \cdot \alpha^{\gamma} = A^{\overline{\overline{B}}} \times A^{\overline{\overline{C}}}$.

We are left to prove that $A^{B\cup C} \sim A^{B} \times A^{C}$.

Suppose $f \in A^{B\cup C}$ corresponds to ordered pair of functions (f_{1B}, f_{1C}), where f_{1B} is the restriction of f to B and f_{1C} is the restriction of f to C. Evidently (f_{1B}, f_{1C}) belongs to $A^{B} \times A^{C}$. Now we define a mapping $\psi: A^{B\cup C} \to A^{B} \times A^{C}$ given by $\psi(f) = (f_{1B}, f_{1C})$ clearly ψ is one to one between $A^{B\cup C}$ and $A^{B} \times A^{C}$.

Hence, we have, $A^{B\cup C} \sim A^{B} \times A^{C}$.

Solved Problems

Problem 1.1: To prove that the set R is uncountable.

Proof: Firstly we shall show that $|R| = c$.

We know that cardinality of the set of real numbers in the open interval (a, b) is c, so that $|(a,b)| = c\ \ \forall a,b \in R$ such that $a < b$.

This implies $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = c$.

The set R is the set of all red numbers lying in the open interval $(-\infty, \infty)$, *i.e.*, $R(-\infty, \infty)$.

Now define a mapping

$f:\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$ by the formula

$f(x) = \tan x$.

Evidently f is well defined. Also it is easy to verify that f is one-one onto

$$\therefore \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim (-\infty, \infty)$$

This implies $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |(-\infty, \infty)|$.

i.e., $c = |R|$, in accordance.

This implies R is uncountable.

Problem 1.2: To prove that $N \times N$ is infinite countable.

Proof: Here we have to prove $|N \times N| = |N|$.

The element of $N \times N$ may be displayed as follows:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \ldots & (1,m) & \ldots \\ (2,1) & (2,2) & (2,3) & \ldots & (2,m) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (n,1) & (n,2) & (n,3) & \ldots & (n,m) & \ldots \end{matrix}$$

If we write,

$$A_i = \{(i, 1), (i, 2), (i, 3), \ldots\}$$

Then (i) $N \times N = \bigcup_{j=1}^{\infty} A_i$

(ii) $A_i \cap A_j = \phi \quad$ for $\quad i \neq j$

(iii) A_i is a countably infinite $\forall\, i$.

These facts lead to the conclusion that $\bigcup_{i=1}^{\infty} A_i$ is countably infinite.

i.e., $N \times N$ is countably infinite and hence

$$N \times N \sim N$$

This $|N \times N| = |N|$.

Problem 1.3: Prove $|P(A)| = 2^{|A|}$, if A is any finite set.

Solution: Let A by any finite set with cardinality n. $P(A)$ is the family of all subsets of A.

$$\begin{aligned} |P(A)| &= n_{C_0} + n_{C_1} + n_{C_2} + \ldots\ .\, n_{C_n} \\ &= (1+1)^n \\ &= 2^n \\ &= 2^{|A|} \end{aligned}$$

Hence, $\quad |P(A)| = 2^{|A|}$.

Problem 1.4: If α is any transfinite cardinal number, then $a \le \alpha$.

Solution: Let A be any infinite set with cardinality α, i.e. $|A| = \alpha$.

To prove that $a \le \alpha$.

A is infinite set $\Rightarrow \exists \quad B \subset A$ such that $|B| = a$.

$$B \subset A$$

$$\Rightarrow \qquad |B| \le |A|$$

$$\Rightarrow \qquad a \le \alpha.$$

Problem 1.5: Prove that the set of all rational is enumerable.

Solution: A rational number is a real number expressible in the from $\left(\dfrac{m}{n}\right)$ where m and n are integer and $n \ne 0$. We claim Q is countable.

Define a map $f : Q^+ \to N \times N$ by the formula $f\left(\dfrac{m}{n}\right) = (m, n)$.

where m and n are positive integers prime to each other. Evidently f is one = one. Q^+ is equipollent with some subset of $N \times N$.

Hence $|Q^+| \le |N \times N| = |N| = a$

or
$$|Q^+| \le a \tag{1.29}$$

Since $N \subset Q^+$, $\therefore \; |N| \le |Q^+|$, or $a \le |Q^+|$ $\qquad$ (1.30)

From Eqns. (1.29) and (1.30), we have

$a \le |Q^+| \le a,$

which is $\Rightarrow |Q^+| = a$, Q^+ is cardinally equivalent to Q^- under the map

$$\frac{p}{q} \to \frac{-p}{q}$$

$\therefore \; |Q^+| = |Q^-| = a$

Now we have $|Q| = |Q^+ \cup Q^- \cup \{0\}|$

$$= a + a + 1$$

$$= (a + a) + 1 \qquad \text{(associative law)}$$

$$= a + 1 = a$$

$\therefore \; |Q| = a$, which $\Rightarrow$ Q is enumerable and hence countable.

Problem 1.6: Prove that $a < c$.

Solution: We know $|N| = a, \quad |R| = c, \quad N \subset R$.

$N \subset R \Rightarrow |N| \le |R| \Rightarrow a \le c$.

N is not cardinally equivalent to R under any mapping

$\therefore \; a \ne c$.

Now $a \ne c, a \le c \quad \Rightarrow \quad a < c$.

Problem 1.7: Find the power of an aggregate of numbers given by $\frac{M}{2^m}$, M and m being positive and integral.

Proof: Let $A = \left\{\frac{M}{2^m} : M, \quad m \in N\right\}$

To determined the power of A.

Write $A_M = \left\{\frac{M}{2^m} : m \in N\right\}, \qquad \forall\, m \in N$

Elements of A_M can be displayed as follows:

$$\begin{array}{llllll}
A_1 : \frac{1}{2^1} & \frac{1}{2^2} & \frac{1}{2^3} & \cdots & \frac{1}{2^n} \\
A_2 : \frac{2}{2^1} & \frac{2}{2^2} & \frac{2}{2^3} & \cdots & \frac{2}{2^n} \\
\quad \cdots & \cdots & \cdots & \cdots & \cdots \\
A_n : \frac{n}{2^1} & \frac{n}{2^2} & \frac{n}{2^3} & \cdots & \frac{n}{2^n} \\
\quad \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Obviously,

(i) A_r is enumerable $\forall r \in N$

(ii) $A_r \cap A'_r = \phi$

(iii) $A = \bigcup_{r=1}^{\infty} A_r$

Being an enumerable union of enumerable sets, A is enumerable and hence its cardinal number is a, *i.e.*, the power of the given set is a.

RECAPITULATION

- A set is a well defined collection of objects. The objects in a set are known as members or elements or points.
- A multi set is an unordered collection of objects in which an object can appear more than once.
- A set is said to be empty set or null set or void set if it contains no element. It is denoted by ϕ or {}.
- Let A and B be any two sets. If all the element of A belongs to B, then A is said to be subset of B.
- If a set contains a number of sets as its elements then it is known as set of sets or family of sets or class of sets.

- Two sets A and B are said to be disjoint sets if they have no common elements.
- A set is said to be finite *set* if it contains finite number of elements, otherwise it is infinite.
- Let A be any set. The power set of A is the set of all subsets of A.
- Index set is a set whose elements are used as names.
- The difference of A and B is the set of elements which belong to A but do not belong to B.
- The symmetric difference of A and B is the set of elements which belong to A or B but do not belong to A and B.
- The Cartesian products of A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ *i.e.*, $A \times B = \{(a, b) : a \in A, b \in B\}$ and $B \times A = \{(b, a) : b \in B, a \in A\}$.
- Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the inverse relation of R if and only if $R^{-1} = \{(y, x) : (x, y) \in R\}$.
- Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an identity relation if $R = \{(a, a) : a \in A\}$.
- A relation R on a set A is known as reflexive relation if and only if $aRa, \forall a \in A$.
- A relation R on a set A is known as symmetric relation if and only if $aRb \Rightarrow bRa\ \forall (a, b) \in R$.
- A relation R on a set A is known as anti-symmetric relation if and only if $aRb, bRa \Rightarrow a = b\ \forall (a, b) \in R$.
- A relation R on a set A is known as transitive relation if and only if $aRb, bRc \Rightarrow aRc, (a, b, c \in A)$.
- A relation R on a set A is known as an equivalence relation if and only if it is reflexive, symmetric and transitive.
- A relation which is transitive but not an equivalence relation is known as an order relation. If R is an order relation on a set X, then xRy and $yRz \Rightarrow xRz, \forall\, x, y, z \in X$.
- A relation R on a set X is said to be a partial order relation if it is at the same time (i) Reflexive (ii) Anti-symmetric and (ii) Transitive.
- A set X together with a partial order relation defined on it, *i.e.*, $(X, \leq)$ is known as a partial ordered set.
- Let $(X, \leq)$ be a partially ordered set and let A be any non-empty subset of X. An element $l \in X$ is said to be a lower bound of A if $l \leq x$ for every x in A. Similarly, an element $u \in x$ is said to be an upper bound of A if $x \leq u$ for every x in A.
- If A and B are two sets such that $A \leq B$ and $B \leq A$, then $A \sim B$.
- Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B.

- *A* function $f : A \to B$ is called one-one if $x_1, x_2 \in A$, we have $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ or $x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$.
- A function $f : A \to B$ is said to be many-one if at least one element of B has two or more than two pre-image in A.
- A function $f : A \to B$ is said to be many-one if at least one element of B has two or more than two pre-image in A.
- A function $f : A \to B$ is said to be into if there is at least one element of B, has no pre-image in A.
- A function $f : A \to B$ is said to be onto if there is no element of B, which is not an image of some element of A.
- Let $f: X \to Y$ be a one-one onto mapping and $f(x) = y, \forall x \in X, \forall y \in Y$. Now we define a mapping $f^{-1} : y \to X$ such that $f^{-1}(y) = x, \forall x \in X, \forall y \in Y$, where f^{-1} is called the inverse of f.
- Let $f : X \to X$ be a mapping. Then f is said to be identity mapping if $f(x) = x, \forall x \in X$.
- A mapping $f : X \to R$, where R is the set of real numbers, is known as real valued mapping.
- Let $f: X \to Y$ be a function. Then f is said to be constant function if $f(x) = a, \forall x \in X$ *i.e.,* a function $f: X \to Y$ is known as constant function if each element of X is mapped onto a single element of Y.
- The function $f : X \to Y$ is known as zero function if the image of each element of X under f is zero *i.e.,* $f(x) = 0$.
- A mapping f is said to be injective (or injection) which is either one-one into or one-one onto.
- A mapping f is said to be bijective (or bijection) which is both one-one and onto.
- Let $f : X \to$ and $g : X \to Y$ be two mapping. Then the mapping f and g are said to be equal mapping if and only if $f(x) = g(x) \ \forall x \in X$.
- Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then a function $gof : X \to Z$ is defined as $gof = g[f(x)], \forall x \in X$ is known as composition of functions.
- Suppose x is a real number. *A* subset N of R containing x, is said to be a neighbourhood of x if N contains an open interval containing x.
- Let A be a subset of R. *A* point $x \in R$ is known as *limit point* of A if and only if every open set X containing x contains at least one point of A other than x, *i.e.,* X is open, $x \in X \Rightarrow A \cap [X - \{x\}] \neq \phi$.
- The set of all limit point of a set A is said to be *derived set* of A.
- A subset A of real number R is known as a *closed set* if and only if complement of A is an open set.
- Let A be a subset of R. *A* point $x \in A$ is known as *adherent point* of A if and only if every neighbourhood of x contains at least one point of A.

- A point $x \in A$ is known as *isolated point* of A if an only if it is not an limit point of A.
- A set A is known as *discrete set* if each point A is an isolated point of A.
- Let A be a subset of R. A is known as *dense-in-itself* if and only if every point of A is a limit point of A.
- A subset A of R is known as *perfect set* if and only if $A = A'$.
- Let A be a subset of R. Then the set of all *adherent points* of a given subset A of R is known as closure of A. It is denoted by $\bar{A}$.
- Let A be a subset of R. A point $x \in R$ is known as *interior* point of A if A is a neighbourhood of x. A is the neighbourhood of x if and only if there exist an open interval (a, b) such that $x \in (a, b) \subset A$.
- The set of all interior point of A is known as *interior* of A.
- Let A be a subset of R. A point x is known as an exterior point of a subset A if x has a neighbourhood N such that $N \subset R - A$.
- The set of all exterior point of a set A of R is known as exterior of A and denoted by $e(A)$.
- Let A be a subset of R. A point x is known as a frontier point of A if every neighbourhood of x contains at least one point of A and at least one point of $R - A$.
- The set of all frontier points of A is known as *frontier* of A.
- An exterior point of A which is also belongs to A, *i.e.*, $x \in A \cap e(A)$ is known as a boundary point of A.
- Every bounded infinite set of real numbers has at least one accumulation point.
- Every bounded infinite set of real numbers has at least one accumulation point.
- A set containing only a finite number of elements is known as finite set.
- The cardinal number of a finite set is the number of elements contains in it.
- A cardinal number corresponding to an infinite set is called a transfinite cardinal number.
- A set which is either finite or denumerable is called a countable set.
- Every subset of a countable set is countable.
- An order type which is represented by well ordered set is known as ordinal number.
- An ordinal number is a special type of well ordered set.
- A real number which is not "algebraic" is called a transcendental number.

EXERCISES

Multiple-choice Questions

1.1 Every subset of a countable set is
   (a) Countable
   (b) Uncountable
   (c) Superset
   (d) None of these.

1.2 If α, β and r are cardinal number and if $\alpha < \beta$ and $\beta < \gamma$ then
   (a) $\alpha < \gamma$
   (b) $\alpha > \gamma$
   (c) $\alpha = \gamma$
   (d) None of these.

1.3 A relation R on a set X is said to be a partial order relation if it is at the same time
   (a) Reflexive and symmetric
   (b) Anti-symmetric and transitive
   (c) Reflexive, anti-symmetric and transitive
   (d) None of these.

1.4 Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7\}$. Then $A \oplus B$ is equal to
   (a) $\{1, 2, 3, 4, 5, 7\}$
   (b) $\{3, 5\}$
   (c) $\{2, 4, 7\}$
   (d) None of these.

State True or False

1.1 A real number which is not "algebraic" is called a transcendental number.

1.2 The union of countable collection of countable set is countable.

1.3 A mapping $f : X \to R$, where R is the set of real numbers, is known as complex valued mapping.

1.4 A partial ordered set is said to be well ordered and its ordering is known as a well ordering if every non-empty subset of it has a greatest element.

Fill in the Blanks

1.1 The relation $A \sim B$ in the family of sets is an ____________ relation.

1.2 A set which is either finite or denumerable is called a __________.

1.3 The set of all rational is ____________.

1.4 A relation R on a set A is known as an ____________ if and only if it is reflexive, symmetric and transitive.

Exercises

1.1 List of elements of the following sets:
  (a) $\{x : x \in I, x^2 < 11\}$
  (b) $\{x : x \in N, x$ is even and $x < 17\}$
  (c) $\{x : x \in N, x$ is prime and $x < 21\}$
  (d) $\{x : x$ is a solution of $x^2 + 3x + 2 = 0\}$
  (e) $\{x : x \in I, x < 3\}$
  (f) $\{x : x \in N, x + 5 = 3\}$
  (g) $\{x : x$ is a month with exactly 30 days$\}$

1.2 Let $U = \{1, 2, 3, \ldots., 9, 10\}$ be the universal set and $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 7, 9\}$, $C = \{2, 5, 6, 8\}$. Find
  (a) A', B', C'
  (b) $A \cup B, B \cup C$, and $A \cup C$
  (c) $A \cap B, B \cap C, A \cap C$
  (d) $A - B, B - A, B - C, C - B, A - C$ and $C - A$.
  (e) $A \oplus B, B \oplus C$, and $A \oplus C$

1.3 Which of the sets are equal?
  (a) $\{x : x$ is a letter in the word 'wolf'$\}$
  (b) $\{x : x$ is a letter in the word 'follow'$\}$
  (c) The letters *f, l, o, w*.
  (d) The letters which appear in the word 'flow'.

1.4 Is a set A comparable with itself?

1.5 Find the power set of $\{1, 2\}$

1.6 Let $A = \{a, b, c\}$ and $B = \{c, d, e, f\}$. Find the $A - B$, $B - A$ and $A \oplus B$.

1.7 Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$

1.8 If $A = \{a, b, c\}$. Find all the subsets of A.

1.9 Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find $A \times B$ and $B \times A$.

1.10 Prove that $A \subset B$ and $C \subset D \Rightarrow (A \times C) \subset (B \times D)$

1.11 Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

1.12 Prove that $A \cup B = (A \oplus B) \oplus (A \cap B)$

1.13 Prove that $A \subseteq \phi \Rightarrow A = \phi$

1.14 Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

1.15 If $A = \{a, b, c, d\}$ and $R = \{(a, a), \{b, b), (c, c), (d, d)\}$. Prove that R is reflexive.

1.16 Give an example of a relation which is symmetric and transitive but not reflexive.

1.17 Give an example of a relation that is reflexive but neither symmetric nor transitive.

1.18 Give an example of a relation which is transitive but not reflexive or symmetric.

1.19 If the function $f : R \to R$ be defined by $f(x) = x^2$, find $f^{-1}(g)$ and $f^{-1}(-g)$.

1.20 If the function $f : R \to R$ be defined by $f(x) = x^2 - 1$ then find $f^{-1}(-2)$ and $f^{-1}\ \{8, 15\}$.

1.21 If $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2, y_3\}$. To show that $f = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ is an into mapping.

1.22 If $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$. To show that $f = \{\{x_1, y_1), \{x_2, y_2), \{x_3, y_2), \{x_4, y_3))$ is only mapping.

1.23 Define function with examples.

1.24 Give four type of function with examples.

1.25 Define identity, constant, zero, real valued and characteristic functions with examples.

1.26 Define injective and bijective mappings with examples.

1.27 Define composition of functions with examples.

1.28 If $f(x) = x^2$ and $g(x) = x + 5$, $x \in R$. Then find the composition fog and gof.

1.29 If $f : X \to Y$, $g : Y \to Z$ such that $f(x) = \log(1 + x)$, $g(x) = e^x$, then find $(gof)(x)$.

1.30 If $f : R \to R$ and $g : R \to R$ be functions such that $f(x) = x^2$ and $g(x) = x^3$, find $(gof)(x)$ and $(gof)(3)$.

1.31 If $f : R \to R$ and $g : R \to R$ be mappings such that $f(x) = \tan x$, $g(x) = x^3$, find $(fog)(x)$ and $(gof)(x)$.

1.32 Prove that the function $f : R \to R$ given by $f(x) = -\sin x$ is neither one-one nor onto.

1.33 Let $f : R \to R$ be a function on R defined by $f(x) = x^2 + 9x + 3$, find $f^{-1}(11)$.

1.34 If $f : R \to R$ is defined by $f(x) = x^2 + 1$. Find:

(a) $f^{-1}(-3)$

(b) $f^{-1}(17)$ and

(c) $f^{-1}\{10, 37\}$

1.35 Prove that if $f : X \to Y$ and $g : Y \to Z$ be two one-one onto functions then gof is also one-one onto function.

1.36 If $f : R \to R$ be defined by $f(x) = x^3 - 3$, where R is the set of all real numbers, then find

(a) 1/3

(b) $f(2)$ and

(c) $f(-g)$

1.37 If $f : R \to R$, $g : R \to R$ be mappings such that $f(x) = x^2 + 2$, $g(x) = 2x + 1$ then find $(fog)\,(x)$ and $(gof)\,(x)$.

1.38 Definite finite and infinite sets.

1.39 Define countable and uncountable sets.

1.40 Define order relation and partial order relation.

1.41 Define upper and lower bounds.

1.42 State the Zorn's Lemma and Cantor's theorem.

1.43 State the Schröder-Bernstein's theorem.

1.44 Prove that if P and Q are enumerable than $P \times Q$ is also enumerable.

1.45 Prove that the set of real numbers in the closed interval [0, 1] is uncountable.

1.46 Prove that the set of points in a plane, both of whose co-ordinates are rational, is enumerable.

1.47 Prove that the set of real numbers cannot be enumerable although the set of such of them as are rationals is enumerable.

1.48 Is the set {1, 2, 3 ...} a well ordered set? Give reason.

1.49 To show that the set $\{x \in R : 0 \leq x \geq 2\}$ is closed.

1.50 Let A and B be subsets of R, then

(a) $A \subset B \Rightarrow A' \subset B'$

(b) $(A \cup B)' = A' \cup B'$

(c) $(A \cap B)' \subset A' \cap B'$

1.51 Let A be any subset of R, then

(a) A° is an U-open subset of R.

(b) A° is the largest U-open set contained in A.

(c) A is U-open if an only if $A^\circ = A$.

1.52 If A denoted the derived set of A, then find the sets:

(a) $A \subset A'$

(b) $A' \subset A$

(c) $A = A'$

1.53 To show that a subset A of real numbers is closed if and only if all the limit points of A are contained in A.

1.54 To show that $A = A \cup A'$.

1.55 If A is a finite subset of R, then proves that the set A' is empty.

1.56 What do you understand by an algebraical number?

1.57 To show that the set of all algebraic number is countable.

1.58 Prove that the set of transcendental numbers is not countable.

ANSWERS

Multiple-choice Questions

1.1 (a)
1.2 (a)
1.3 (c)
1.4 (d)

State True or False

1.1 T
1.2 T
1.3 F
1.4 F

Fill in the Blanks

1.1 Equivalence
1.2 Countable set
1.3 Countable
1.4 Equivalence relation

Exercises

1.1 {a) $\{-3, -2, -1, 0, 1, 2, 3\}$
(b) $\{2, 4, 6, 8, 10, 12, 14, 16\}$
(c) $\{2, 3, 5, 7, 11, 13, 17, 19\}$
(d) $\{-1, -2\}$
(e) $\{\ldots -, -3, -1, 0, 1, 2\}$
(f) ϕ or { }
(g) {April, June, September, November}.

1.2 (a) $A' = \{5, 6, 7, 8, 9, 10\}$, $B' = \{1, 2, 5, 6, 8, 10\}$, $C' = \{1, 3, 4, 7, 9, 10\}$
(b) $A \cup B = \{1, 2, 3, 4, 7, 9\}$, $B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \cup C = \{1, 2, 3, 4, 5, 6, 8\}$
(c) $A \cap B = \{3, 4\}$, $B \cap \mathrm{C} = \varphi$ and $A \cap C = \{2\}$.
(d) $A - B = \{1, 2\}$, $B - A = \{7, 9\}$, $B - C = \{3, 4, 7, 9\}$, $C - B = \{2, 5, 6, 8\}$, $A - C = \{1, 3, 4\}$ and $C - A = \{5, 6, 8\}$.
(e) $A \oplus B = \{1, 2, 7, 9\}$, $B \oplus C = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \oplus C = \{1, 3, 4, 5, 6, 8\}$.

1.3 All the given sets are equal.

1.4 Yes

1.5 ϕ, {1}, {2}, {1, 2}.

1.6 $\{a, b\}$, $(d, e, f\}$ and $\{a, b, d, e, f\}$

1.8 A, ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b, c\}$

1.9 $A \times B = \{(1, 3\}, (1, 4), (2, 3), (2, 4)\}$
$B \times A = \{(3, 1), \{3, 2\}, (4, 1), (4, 2)\}$

1.16 $A = \{a, b, c\}$ and $R = \{(a, a), \{b, b), (a, b), \{b, a)\}$.

1.17 $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (b, c)\}$.

1.18 $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c)\}$

1.19 $\{3, -3\}$, ϕ

1.20 ϕ, $\{3, -3, 4, -4\}$.

1.28 $x^2 + 5$, $x^2 + 10x + 25$.

1.29 $1 + x$.

1.30 x^6, 3^6.

1.31 $\tan x^3$, $\tan^3 x$.

1.33 $\{-2, -7\}$.

1.34 (a) ϕ (b) $\{4, -4\}$ (c) $\{3, -3, 6, -6\}$

1.36 (a) $-80/27$ (b) 5 (c) -732.

1.37 $4x^2 + 12x + 10$ and $4x^2 + 4x + 3$, $2x^2 + 5$.

1.48 Yes, because it has first element.

1.52 (a) Let $A = (a, b)$, then $A' = [a, b]$ and $A \subset A'$.

(b) Let $A = \left\{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ then $A' = \{0\}$ and $A' \subset A$.

(c) Let $A = [1, 2]$ then $A' = [1, 2]$ and $A = A'$.

2

CHAPTER

Lebesgue Measure

2.1 INTRODUCTION

Measure theory and integration is the study of lengths, surface area, and volumes in the general spaces. Lebesgue measure has a great importance in the study of real and complex analysis, Fourier series and integrals. The development of measure in the late 19th century, measure is an extension of the length, area or volume. It is a part of real analysis and is used in many areas of mathematics like, for geometry, probability theory, dynamical systems, functional analysis, etc. In this chapter, we shall discuss the Length of a set, measure, Borel set, $\Im_\sigma$ -sets, ζ_σ-sets, Boolean ring, Boolean algebra, σ-ring, Lebesgue measure, Outer measure, exterior and interior measure of a set, Measurable set, First fundamental theorem, Cantor's ternary set and Non-measurable set.

2.2 IMPORTANT TERMINOLOGY

Before discussing the Lebesgue measure, we shall discuss certain necessary preliminaries.

2.2.1 Length of an Interval

We know that the length of an interval is defined to be the difference between two end points.

2.2.2 Measure of Interval, Rectangle and Parallelepiped

The length of any open interval H is defined as its length denoted by $m(H)$.

For example:

(i) $m(H) = m[(3,9)] = 9 - 3 = 6$.

(ii) $m[(a,b)] = b - a \quad \Rightarrow \quad m(H) > 0$.

Now let $[a, b]$ be the smallest closed interval containing a closed set F, then we have

$$\begin{aligned} m(F) &= b - a - m(F') \\ &= b - a - m(\phi) \qquad [\text{if } F' = \phi] \\ &= b - a - 0 \\ m([a, b]) &= b - a\,. \end{aligned}$$

For rectangle, the area of an open rectangle $R(a < x < b, c < y < d)$, *i.e.*, $(b - a)\,(d - c)$ is defined as the measure of R, thus $m(R) = (b - a)\,(d - c)$. The area of a closed rectangle $R(a \le x \le b, c \le y \le d)$, *i.e.*, $m(R) = (b - a)\,(d - c)$. The volume of an open parallelepiped $V(a < x < b, c < y < d, l < z < m)$ is defined as the measure of V, thus $m(V) = (b - a)\,(d - c)\,(m - l)$. Similarly, for closed parallelepiped, we have $m(V) = (b - a)\,(d - c)\,(m - l)$.

2.2.3 Borel Set

In this section, we study one of the most important σ-algebras, the Borel sets in R^n. The set which may be obtain from closed and open countable set by repeatedly applying union and intersection operation to them are called *Borel set*. A bonded Borel set is said to be Borel measurable. For example, the set of type $\Im_\sigma$ and ξ_σ are Borel set. A set which is a countable (finite or infinite) union of closed set is called an $\Im_\sigma$-set, *i.e.*, $E = \bigcup_{i=1}^{\infty}(f_i)$. A set which is a countable intersection of open set is ξ_σ-set, *i.e.*, $E = \bigcap_{i=1}^{\infty}[G_i]$.

Note:

1. The complement of a $\Im_\sigma$ set is a ξ_σ set and conversely.
2. Each of the classes $\Im_\sigma$ and ξ_σ of sets is wider than the classes of open and closed sets.

2.2.4 Boolean Ring

A non-empty class $\Re$ of sets is known as Boolean ring (or ring of sets) if $A, B \in \Re \Rightarrow A - B \in \Re$ and $A \cup B \in \Re$. Hence a Boolean ring is a non-empty family of sets which is closed under the formation of unions and differences. For example, the set ϕ containing the empty set only is the smallest possible ring of sets; the class of all finite unions of bounded closed-open intervals of real line.

Theorem 2.1: A non-empty collection of sets is a Boolean ring if it is closed under the formulation of intersections and symmetric differences.

Proof: Suppose $\mathfrak{R}$ is a non-empty collection of sets and $A, B \in \mathfrak{R}$.

It is given that the $A, B \in \mathfrak{R} \Rightarrow A \Delta B \in \mathfrak{R}$ and $A \cap B \in \mathfrak{R}$.

We have

$$\begin{aligned} A, A \cap \mathrm{B} \in \mathfrak{R} &\Rightarrow A \Delta\, (A \cap \mathrm{B}) \in \mathfrak{R} \\ &\Rightarrow A - B \in \mathfrak{R} \end{aligned} \tag{2.1}$$

Now we have

$$\begin{aligned} A \cap B, A \Delta B \in \mathfrak{R} &\Rightarrow (A \cap B) \Delta (A \Delta B) \in \mathfrak{R} \\ &\Rightarrow A \cup B \in \mathfrak{R} \end{aligned} \tag{2.2}$$

From Eqns. (2.1) and (2.2), we have

$$A, B \in \mathfrak{R} \Rightarrow A - B \in \mathfrak{R} \text{ and } A \cup B \in \mathfrak{R}.$$

Hence, $\mathfrak{R}$ is a Boolean ring.

Theorem 2.2: Intersection of two Boolean rings is a Boolean ring.

Proof: Suppose $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are any two Boolean rings. Suppose A and B are any two members of $\mathfrak{R}_1 \cap \mathfrak{R}_2$.

To show that the $\mathfrak{R}_1 \cap \mathfrak{R}_2$ is a Boolean ring, *i.e.,* $A - B \in \mathfrak{R}_1 \cap \mathfrak{R}_2$ and $A \cup B \in \mathfrak{R}_1 \cap \mathfrak{R}_2$.

Since A and B are any two members of $\mathfrak{R}_1 \cap \mathfrak{R}_2$, then we have

$$A \in \mathfrak{R}_1 \cap \mathfrak{R}_2 \Rightarrow A \in \mathfrak{R}_1 \text{ and } A \in \mathfrak{R}_2 \tag{2.3}$$

$$B \in \mathfrak{R}_1 \cap \mathfrak{R}_2 \Rightarrow B \in \mathfrak{R}_1 \text{ and } B \in \mathfrak{R}_2 \tag{2.4}$$

From Eqns. (2.3) and (2.4), we have

$$A \in \mathfrak{R}_1 \text{ and } B \in \mathfrak{R}_1$$

$$\Rightarrow A - B \in \mathfrak{R}_1 \text{ and } A \cup B \in \mathfrak{R}_1 \qquad \text{(Since } \mathfrak{R}_1 \text{ is a Boolean rings)} \tag{2.5}$$

$$A \in \mathfrak{R}_2 \text{ and } B \in \mathfrak{R}_2$$

$$\Rightarrow A - B \in \mathfrak{R}_2 \text{ and } A \cup B \in \mathfrak{R}_2 \qquad \text{(Since } \mathfrak{R}_2 \text{ is a Boolean rings)} \tag{2.6}$$

Using Eqns. (2.5) and (2.6), we have

$$A - B \in \mathfrak{R}_1 \cap \mathfrak{R}_2 \text{ and } A \cup B \in \mathfrak{R}_1 \cap \mathfrak{R}_2$$

Hence, $\mathfrak{R}_1 \cap \mathfrak{R}_2$ is a Boolean ring.

2.2.5 Algebra of Sets (Boolean Algebra)

A non-empty class $\mathfrak{R}$ of sets is known as algebra (or Boolean algebra) if

(i) $A, B \in \mathfrak{R} \Rightarrow A \cup B \in \mathfrak{R}$ and
(ii) $A \in \mathfrak{R} \Rightarrow A' \in \mathfrak{R}$.

Theorem 2.3: Every algebra is a ring.

Proof: Suppose $\mathfrak{R}$ is an algebra.

Using the definition of Boolean algebra, we have

$$A, B \in \mathfrak{R} \Rightarrow A \cup B \in \mathfrak{R} \tag{2.7}$$

and

$$A \in \mathfrak{R} \Rightarrow A' \in \mathfrak{R}. \tag{2.8}$$

Now we have

$A' \in \mathfrak{R}, B \in \mathfrak{R}$

$\Rightarrow \quad A' \cup B \in \mathfrak{R}$ {Using Eqn. (2.7)}

$\Rightarrow \quad (A' \cup B)' \in \mathfrak{R}$ {Using Eqn. (2.8)}

$\Rightarrow \quad A \cap B' \in \mathfrak{R}$

$\Rightarrow \quad A - B \in \mathfrak{R}$

Therefore $A, B \in \mathfrak{R} \Rightarrow A \cup B \in \mathfrak{R}$, $A - B \in \mathfrak{R}$, *i.e.*, $\mathfrak{R}$ is a Boolean ring. Hence every algebra is a Boolean ring.

2.2.6 σ-Ring

A non-empty class $\mathfrak{R}$ of sets is known as σ-ring (or ring of sets) if

(i) $A, B \in \mathfrak{R} \Rightarrow A - B \in \mathfrak{R}$ and

(ii) $A_i \in \mathfrak{R}$, $i = 1, 2, 3, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{R}$.

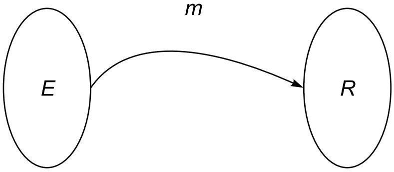
Hence a ring of sets is a σ-ring if it is closed under the formation of countable unions.

Note:

1. The member of a σ-ring $\mathfrak{R}$ are known as Borel sets.
2. Every σ-algebra is a ring, and every ring is a semi ring.

2.3 MEASURE (*m*)

Measure theory initially was proposed to provide an analysis and notions such as length, area and volume of subsets of Euclidean spaces. The approach to measure and integration is axiomatic, i.e., a measure is any function m defined on subsets which satisfy a certain list of properties. Suppose m is a function which assign to each set E in R a non-negative extended real number denoted by $m(E)$ called the measure of E and satisfying the following properties:



(i) $m(E)$ is defined for all sets $E \in P(R)$

(ii) $m(I) = \ell(I)$ for an interval I.

(iii) If $\{E_i\}$ is a sequence of disjoint sets, then $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$

This property is known as countable additivity.

(iv) $m(E+y) = m(E)$, where y is any fixed number. This property is known as translation invariance.

2.4 OUTER MEASURE

Let E be a set of real numbers. Then outer measure of set E is defined as $m^*(E) = \begin{cases} \inf \sum_i l(I_i) \\ 0, & if\ E = \phi \end{cases}$ where the infimum is taken over all countable collection $\{I_i\}$ of open interval such that $E \subset \bigcup \{I_i\}$.

Some important properties of outer Measure are as follows:

(i) $m^*(A) \geq 0$ for all sets A.

(ii) $m^*(\phi) = 0$.

(iii) *Inf* of $m^*(E)$ is the least length to cover the set E,

$$m^*(E) \leq \sum l(I_i).$$

(iv) If A and B are two sets with $A \subset B$ then

$$m^*(A) \leq m^*(B) \text{ \{monotonicity property\}}$$

(v) $m^*(A) = 0$ for every singleton set A.

(vi) The function m_o is translation invariant, *i.e.*, $m^*(A+x) = m^*(A)$ for every set A and for every $x \in R$.

(vii) For every $\in > 0$, there exist one countable family of open intervals such that $\sum l(I_i) < m^*(E) + \in$.

2.5 CARATHEODORY'S POSTULATES FOR OUTER MEASURE

An extended real valued set function m^* defined on a σ-ring $\Re$ said to be an outer measure if the following postulates are satisfied:

(i) $m^*(A_i) \geq 0$ for all sets $A_i \in \Re$.

(ii) $m^*(\phi) = 0$ iff $\phi \in \Re$.

(iii) If A, B $\in \Re$ and $A \subset B$ then we have $m^*(A) \leq m^*(B)$.

(iv) If A and B are two disjoint set of $\Re$, then we have $m^*(A) + m^*(B) = m^*(A \cup B)$.

(v) For every $E > 0$, there exist one countable family of open intervals such that $m^*(\cup\{A_i : i \in \wedge\}) \leq \sum_{i \in \wedge} m^*(A_i)$.

These above five postulates are called as Caratheodory's Postulates for Outer Measure.

2.6 MEASURABLE SET

A set E is said to be measurable with respect to the given outer measurable if for each set A, $m^*(A) = m^*(A \cap E) + m^*(A \cap E')$.

Note:

1. Any set E is said to be measurable with respect to the given outer measure function m^* *if* $m^*(A) = m^*(A \cap E) + m^*(A \cap E'), \forall A \in E$ or $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$.
2. If we say that the set E is measurable then we write $m^*(E) = m(E)$.
3. If E is a measurable set then E' is also measurable.
4. The set ϕ and R are measurable set.
5. Every countable set is measurable and its measure is zero.
6. The contour set C and all its subsets are measurable and each of them has measure zero.

2.7 EXTERIOR MEASURE OF A SET

It is defined by $m_e(A) = \text{Inf}\,\{m(G) : G \supset A.\, G \text{ is open}\} \Rightarrow m(G) \geq m_e(A)$. For any given $\in > 0$, there exists an open set $G \supset A$ such that $m(G) < m_e(A) + \in$.

Some important properties of exterior measure are as follows:

(i) $m_e(\phi) = 0$.

(ii) If $A \subset B$, then we have $m_e(A) \leq m_e(B)$.

(iii) If $A \subset (a, b)$ then we have $0 \leq m_e(A) \leq b - a$.

2.8 INTERIOR MEASURE OF A SET

It is defined by $m_i(A) = \sup\{m(F) : F \subset A, F \text{ is closed}\}$.

Some important properties of interior measure are as follows:

(i) $m_i(\phi) = 0$.

(ii) If $A \subset B$ then we have $m_i(A) \leq m_i(B)$.

(iii) If $A \subset (a, b)$ or $A \subset (b - a)$ then we have $m_i(A) = (b - a) - m_e(A')$.

Note:

1. A set A is said to be measurable set if $m_i(A) = m_e(A) = m(A)$.

2.9 MEASURABLE SPACE

Let E be a σ-algebra of subset of X and let m be a measure defined on E then (X, E, m) is known as a measurable space.

Note:

1. If (X, E, m) is a measurable space then
   (i) for $\boldsymbol{E}_1, E_2 \in E$ and $\boldsymbol{E}_1 \subset E_2 \Rightarrow m(E_1) \le m(E_2)$
   (ii) if $\boldsymbol{E}_1, E_2, E_3, \ldots$ Are the member of E then we have $m_o\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m(E_i)$.

Theorem 2.4: If $A \subset B$ then show that $m^*(A) \le m^*(B)$.

Proof: By definition, we have

$$m^*(B) = \sum l(I_p) \tag{2.9}$$

where $\{I_p\}$ is a countable family of open intervals such that

$$B \subset \bigcup \{I_p\} \tag{2.10}$$

It is given that $A \subset B \Rightarrow A \subset \bigcup \{I_p\}$

$$\Rightarrow \qquad m^*(A) \le \sum l\{I_p\}.$$

From Eqn. (2.9) and (2.10), we have

$$m^*(A) \le m^*(B).$$

Theorem 2.5: If A is countable then show that $m^*(A) = 0$.

Proof: Let $A = \{a_1, a_2, \ldots, a_n\}$ is a countable set.

Then we have $A = \bigcup (a_i)$ (*i.e.,* countable union of singleton set).

By definition, we have

$$m^*(A) \le m^* \bigcup \{a_i\} \le \sum m^*\{a_i\}$$

Since we know that the measure of singleton set is zero therefore we have

$$m^*(A) \le 0.$$

Also we know that length is never negative, *i.e.,* $m^*(A) = 0$.

Theorem 2.6: A set is outer measurable if A' is outer measurable.

Proof: Let a set A be outer measurable so that for any set T.

$$\begin{aligned} m^*(T) &= m^*(T \cap A) + m^*(T \cap A') \\ &= m^*(T \cap A') + m^*(T \cap A) \\ &= m^*(T \cap A') + m^*(T \cap (A')') \qquad \{(A')' = A\} \end{aligned}$$

$\Rightarrow$ A' is outer measurable

Conversely, Suppose the A' is outer measurable so that for any set T,

$$\begin{aligned} m^*(T) &= m^*(T \cap A') + m^*\{T \cap (A')'\} \\ &= m^*(T \cap A') + m^*(T \cap A) \\ &= m^*(T \cap A) + m^*(T \cap A') \end{aligned}$$

This implies A is outer measurable.

Theorem 2.7: The union of two outer measurable sets is outer measurable.

Proof: Let S_1 and S_2 be any two outer measurable sets and T any set.

To show that $S_1 \cup S_2$ is outer measurable, *i.e.*,

$$m^*(T) = m^*[T \cap (S_1 \cup S_2)] + m^*[T \cap (S_1 \cup S_2)']$$

For applying the criteria of measurable of S_1, we have

$$m^*(T) = m^*(T \cap S_1) + m^* = [T \cap S_1'] \tag{2.11}$$

Now taking $T \cap S_1'$, for the set T and applying the criteria of measurability of S_2, we get

$$\begin{aligned} m^*(T \cap S_1') &= m^*[(T \cap S_1') \cap S_2] + m^*[(T \cap S_1') \cap S_2'] \\ &= m^*(T \cap S_1' \cap S_2) + m^*(T \cap S_1' \cap S_2') \end{aligned}$$

Using $(S_1 \cap S_2)' = S_1' \cup S_2'$ and $S_1' \cap S_2' = (S_1' \cup S_2')$, we get

$$m^*(T \cap S_1') = m^*(T \cap S_1' \cap S_2) + m^*[T \cap (S_1 \cup S_2)'] \tag{2.12}$$

Again taking $T \cap (S_1 \cup S_2)$ for the set T and applying the criteria of measurability of S_1, we get

$$m^*[T \cap (S_1 \cup S_2)] = m^*[T \cap (S_1 \cup S_2) \cap S_1] + m^*[T \cap (S_1 \cup S_2) \cap S_1']$$

We know that $(S_1 \cup S_2) \cap S_1 = S_1$

$$\begin{aligned} \Rightarrow\ (S_1 \cup S_2) \cap S_1' &= (S_1 \cap S_1') \cup (S_2 \cap S_1') \\ &= \phi \cup (S_2 \cap S_1') \\ &= S_2 \cap S_1' \end{aligned}$$

We have $m^*[T \cap (S_1 \cup S_2)] = m^*[T \cap S_1] = m^*[T \cap S_2 \cap S_1']$ (2.13)

By Eqn. (2.12), we have

$$m^*(T \cap S_1' \cap S_2) = m^*(T \cap S_1') - m^*[T \cap (S_1 \cup S_2)']$$

Putting in Eqn. (2.13), we get

$$m^*[T \cap (S_1 \cup S_2)] = m^*(T \cap S_1) + m^*(T \cap S_1') - m^*[T \cap (S_1 \cup S_2)'] \tag{2.14}$$

From Eqns. (2.11) and (2.14), we have

$$m^*[T \cap (S_1 \cup S_2)] = m^*(T) - m^*[T \cap (S_1 \cup S_2)']$$

i.e., $m^*(T) = m^*[T \cap (S_1 \cup S_2)] + m^*[T \cap (S_1 \cup S_2)']$

This implies $S_1 \cup S_2$ is outer measurable.

Hence, the union of two outer measurable sets is outer measurable.

Theorem 2.8: The intersection of two outer measurable sets is outer measurable.

Proof: Let A_1 and A_2 be two any outer measurable sets.

To show that $A_1 \cap A_2$ is outer measurable.

Since, A_1 and A_2 are outer measurable sets $\Rightarrow A_1'$ and A_2' are outer measurable sets.

$\Rightarrow$ $A_1' \cup A_2'$ is an outer measurable set

(because we know that the union of two outer measurable set is outer measurable)

$\Rightarrow$ $(A_1' \cup A_2')'$ is an outer measurable set

$\Rightarrow$ $(A_1')' \cap (A_2')'$ is an outer measurable.

This implies $A_1 \cap A_2$ is outer measurable.

Hence, the intersection of two outer measurable sets is outer measurable.

Theorem 2.9: If S_1, S_2 are outer measurable sets and $S_2 \subset S_1$ then show that $S_1 - S_2$ is outer measurable set.

Proof: Suppose S_1, S_2 is outer measurable sets and $S_2 \subset S_1$.

To show that $S_1 - S_2$ is outer measurable.

Given S_1, S_2 are outer measurable sets. We know that if S_2' is outer measurable then S_2 is also outer measurable. This implies S_1, S_2' are outer measurable.

We also know that the intersection of two outer measurable sets is outer measurable.

$\Rightarrow$ $S_1 \cap S_2'$ is outer measurable.

$$\begin{aligned} S_1 \cap S_2' &= S_1 \cap (X \sim S_2) \\ &= S_1 \cap X - S_1 \cap S_2 \\ &= S_1 - S_2 \qquad \{S_2 \subset S_1\} \end{aligned}$$

$\therefore$ $S_1 - S_2$ is outer measurable set.

Hence, if S_1, S_2 are outer measurable sets and $S_2 \subset S_1$ then $S_1 - S_2$ is outer measurable set.

Theorem 2.10: If S_1 and S_2 are measurable sets and if $S_2 \cap S_1 = \phi$, then $m(S_1 \cup S_2) = m(S_1) + m(S_2)$.

Proof: Let S_1, S_2 be measurable sets and $S_2 \cap S_1 = \phi$.

This implies $m^*(S_1) = m(S_1), m^*(S_2) = m(S_2)$ and

$m^*(S_1 \cup S_2) = m(S_1 \cup S_2)$.

To show that $m(S_1 \cup S_2) = m(S_1) + m(S_2)$

Taking S_1, S_2 for the set T and applying the criteria of measurability of S_1, we get

$$\begin{aligned} m(S_1 \cup S_2) &= m[(S_1 \cup S_2 \cap S_1) + m_0((S_1 \cup S_2) \cap S_1')] \\ &= m(S_1) + m[(S_1 \cap S_1') \cup (S_2 \cap S_1')] \end{aligned}$$

We know that if $S_1 \cap S_2 = \phi$

$\Rightarrow\ S_2 \subset X - S_1$

$\Rightarrow\ S_2 \cap (X - S_1) = S_2$

$\Rightarrow\ S_2 \cap - S_1' = S_2$

Now we have $m^*(S_1 \cup S_2) = m^*(S_1) + m^*\left[\phi \cup (S_2 \cap S_1')\right]$

$$= m^*(S_1) + m^*(S_2 \cap X - S_1)$$

$$= m^*(S_1) + m^*(S_2)$$

According with initial hypothesis this takes the form, we have

$$m(S_1 \cup S_2) = m(S_1) + m(S_2).$$

Theorem 2.11: If the outer measure of a set is zero, then the set is outer measurable.

Proof: Let A be any set such that $m^*(A) = 0$

To show that A is outer measurable.

For any set T, we have $T = A \cap T + A' \cap T$

By Caratheodory postulates for outer measure.

$$m^*(T) = m^*(A \cap T + A' \cap T)$$

$$\leq m^*(A \cap T) + m^*(A' \cap T)$$

or $$m^*(T) \leq m^*(A \cap T) + m^*(T \cap A') \tag{2.15}$$

Since, $(T \cap A) = A$

$\Rightarrow\quad m^*(T \cap A) \leq m^*(A) = 0$

$\Rightarrow\quad m^*(T \cap A) \leq 0$

But $m^*(T \cap A) \geq 0$

$\therefore\ 0 \leq m^*(T \cap A) \leq 0$

Consequently, $m^*(T \cap A) = 0$ (2.16)

Again $T \cap A' \subset T$

$\Rightarrow\quad m^*(T \cap A') \leq m^*(T)$

Using Eqn. (2.16), we have

$m^*(T \cap A') + m^*(T \cap A) \leq m^*(T) + 0$

or $\quad m^*(T \cap A') + m^*(T \cap A) \leq m^*(T)$ (2.17)

By Eqns. (2.15) and (2.17), we get

$$m^*(T) = m^*(T \cap A) + m^*(T \cap A')$$

This implies A is outer measurable.

Hence, if the outer measure of a set is zero, then the set is outer measurable.

Theorem 2.12: Let A be a measurable set if B is any set then

$$m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B).$$

Proof: Let A is measurable set so that

$$m^*(T) = m^*(T \cap A) + m^*(T \cap A')$$

Taking one by one $T = B$ and $T = A \cup B$, we get

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A') \tag{2.18}$$

and $$m^*(A \cup B) = m^*[(A \cup B) \cap A] + m^*[(A \cup B) \cap A'] \tag{2.19}$$

We know that $(A \cup B) \cap A = A$ and

$$\begin{aligned}(A \cup B) \cap A' &= (A \cap A') \cup (B \cap A')\\ &= \phi \cup (B \cap A')\\ &= B \cap A'\end{aligned}$$

By Eqn. (2.19), we have

$$m^*(A \cup B) = m^*(A) + m^*(B \cap A') \tag{2.20}$$

Subtracting Eqn. (2.18) to (2.20), we get

$$m^*(B) - m^*(A \cup B) = m^*(B \cap A) - m^*(A)$$

$$\Rightarrow \qquad m^*(B) + m^*(A) = m^*(A \cup B) + m^*(B \cap A)$$

Hence, $m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B)$.

Theorem 2.13: The intersection of a finite number of measurable sets is measurable.

Proof: Let $E_1, E_2, \ldots, E_n$ be measurable sets.

To prove that $\bigcap_{k=1}^{n} E_k$ is measurable.

We know that union of finite number of measurable set is measurable.

Let all the set E_k be contained in an interval $[a, b]$. Then $E_k' =$ complement of E_k with respect to $[a, b]$.

Using De-Morgen law, we have

$$\bigcup_{k=1}^{n} E_k' = \left[\bigcap_{k=1}^{n} E_k\right]' \tag{2.21}$$

E_k is measurable.

$\Rightarrow$ its complement E_k' is measurable.

$\Rightarrow \bigcup_{k=1}^{n} E_k'$ is measurable

$\Rightarrow \left[\bigcap_{k=1}^{n} E_k\right]'$ is measurable [Using Eqn. (2.21)]

$\Rightarrow \bigcap_{k=1}^{n} E_k$ is measurable.

Theorem 2.14: The difference of two measurable sets is measurable.

Proof: Let E_1 and E_2 be measurable sets.

To show that $E_1 - E_2$ is a measurable set.

We have $E_1 - E_2 = E_1 \cap E_2'$.

Since E_2 is measurable.

$\Rightarrow \quad E_2'$ is measurable.

$\Rightarrow \quad E_1 \cap E_2'$ is measurable.

$\Rightarrow \quad E_1 - E_2$ is measurable.

Hence, the difference of two measurable sets is measurable.

Theorem 2.15: Prove that $m_i(A) \le m_e(A)$ for $A \subset R$.

Proof: Let H be a closed set such that $H \subset A$ (by definition of interior measure) and G is a open set such that $G \supset A$ (by definition of exterior measure).

$$\Rightarrow \qquad H \subset A \subset G$$

$$\Rightarrow \qquad H \subset G$$

$$\Rightarrow \qquad l(H) \subset l(G)$$

Taking infimum all over G such that

$$\Rightarrow \qquad l(H) \le \inf l(G) \text{ such that } A \subset G$$

$$\Rightarrow \qquad l(H) \le m_e(A) \qquad \text{(by definition)}$$

Now taking supremum all over H, we have

$$\text{Sup}\, l(H) \le m_e(A) \text{ such that } H \subset A$$

$$m_i(A) \le m_e(A) \qquad \text{(by definition)}$$

Hence, $m_i(A) \le m_e(A)$ for $A \subset R$.

Theorem 2.16. Prove that a set is said to be measurable if and only if its complement is measurable.

Proof: Let A be a given set which is contained in the interval $[a, b]$, then we have

$$m_i(A) = (b-a) - m_e(A') \tag{2.22}$$

Let A is measurable, *i.e.*, $\Rightarrow m_i(A) = m_e(A) = m(A)$ (2.23)

To show A' is measurable, *i.e.*, $\Rightarrow m_i(A') = m_e(A') = m(A')$ (2.24)

By Eqns. (2.22) and (2.23), we get

$$m(A) = (b-a) - m_e(A')$$

$$\Rightarrow \qquad m_e(A') = (b-a) - m(A) \tag{2.25}$$

Now put $A = A'$ in Eqn. (2.22), we get

$$\begin{aligned} m_i(A') &= (b-a) - m_e(A')' \\ &= (b-a) - m_e(A) \end{aligned}$$

By Eqn. (2.23), we have

$$m_i(A') = (b-a) - m(A) \tag{2.26}$$

From Eqns. (2.25) and (2.26), we have

$$m_e(A') = m_i(A') = m(A') \quad \Rightarrow \quad A \text{ is measurable.}$$

Conversely, let A' is measurable

i.e., $$m_e(A') = m_i(A') = m(A') \tag{2.27}$$

To show A is measurable, *i.e.,*

$$m_e(A) = m_i(A) = m(A)$$

By Eqn. (2.22), we have

$$\begin{aligned} m_i(A) &= (b-a) - m_e(A') \\ &= (b-a) - m(A') \quad \{\text{Using Eqn. (2.27)}\} \end{aligned} \tag{2.28}$$

Put $A = A'$ in Eqn. (2.22), we get

$$m_i(A') = (b-a) - m_e(A')'$$

or $\quad m_i(A') = (b-a) - m_e(A)$

$\Rightarrow \quad m_e(A) = (b-a) - m_i(A')$

$$\Rightarrow \quad m_e(A) = (b-a) - m(A') \qquad \{\text{Using Eqn. (2.27)}\} \tag{2.29}$$

From Eqns. (2.28) and (2.29), we get

$m_i(A) = m_e(A) = m(A) \quad \Rightarrow \quad A$ is measurable.

Hence, a set is said to be measurable if and only if its complement is measurable.

Theorem 2.17: If E_1 and E_2 are subset of $[a, b]$ then show that $m_e(E_1) + m_e(E_2) \geq m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$ **and** $m_i(E_1) + m_i(E_2) \leq m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$.

Proof: Suppose E_1 and E_2 be subset of $[a, b]$. For a given $\in > 0$, we can find open set G_1 and G_2 such that $G_1 \supset E_1$ and $G_2 \supset E_2$ (2.30)

By definition of exterior measure on a set G_1 and G_2, we have

$$m(G_1) < m_e(E_1) + \in/2 \tag{2.31}$$

$$m(G_2) < m_e(E_2) + \in/2 \tag{2.32}$$

Adding Eqns. (2.31) and (2.32), we get

$$m(G_1) < m(G_2) < m_e(E_1) + m_e(E_2) + \in \tag{2.33}$$

Since G_1 and G_2 are open set then $G_1 \cup G_2$ is also open set from (2.30), $G_1 \cup G_2$ and $G_1 \cap G_2$ set containing $E_1 \cup E_2$ and $E_1 \cap E_2$ respectively, then we know by definition

$$\left.\begin{aligned} m(G_1 \cup G_2) > m_e(E_1 \cup E_2) \\ m(G_1 \cap G_2) > m_e(E_1 \cap E_2) \end{aligned}\right\} \tag{2.34}$$

We know by a theorem,

$$m(G_1 \cup G_2) + m(G_1 \cap G_2) = m(G_1) + m(G_2) \tag{2.35}$$

By Eqn. (2.33) and (2.35), we have

$$m(G_1 \cup G_2) + m(G_1 \cap G_2) < m_e(E_1) + m_e(E_2) + \in \tag{2.36}$$

From Eqn. (2.34) and (2.36), we have

$$m_e(E_1) + m_e(E_2) + \in > m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$$

or $$m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2) \le m_e(E_1) + m_e(E_2) \tag{2.37}$$

We know that

$$m_i(A) = (b-a) - m_e(A')$$

$$\Rightarrow \quad m_e(A') = (b-a) - m_i(A) \tag{2.38}$$

Since, Eqn. (2.37) is true for every E_1, E_2 subset of $[a, b]$

$\Rightarrow m_e(E_1' \cup E_2') + m_e(E_1' \cap E_2') \le m_e(E_1') + m_c(E_2')$

$\Rightarrow m_e(E_1 \cap E_2)' + m_e(E_1 \cup E_2)' \le m_e(E_1') + m_c(E_2')$ {By De-Morgen law}

Using Eqn. (2.38), we get

$$(b-a) - m_i(E_1 \cap E_2) + (b-a) - m_i(E_1 \cup E_2)$$
$$\le (b-a) - m_i(E_1) + (b-a) - m_i(E_2)$$

$$\Rightarrow \quad -m_i(E_1 \cap E_2) - m_i(E_1 \cup E_2) \le -m_i(E_1) - m_i(E_2)$$

$$\Rightarrow \quad m_i(E_1) + m_i(E_2) \le m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2).$$

Hence, if E_1 and E_2 are subset of $\left[a, b\right]$ then

$m_e(E_1) + m_e(E_2) \ge m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$ and

$m_i(E_1) + m_i(E_2) \le m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$.

Theorem 2.18: If E_1 and E_2 are measurable set of a closed interval $[a, b]$ then prove that $m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$.

Proof: It is given E_1 and E_2 are measurable set, then we have

$$\left.\begin{aligned} m_e(E_1) = m_i(E_1) = m(E_1) \\ m_e(E_2) = m_i(E_2) = m(E_2) \end{aligned}\right\} \tag{2.39}$$

Also we know that union of two measurable set is measurable and intersection is also.

i.e., $E_1 \cup E_2$ and $E_1 \cap E_2$ is measurable

$$\Rightarrow \quad m_e(E_1 \cup E_2) = m_i(E_1 \cup E_2) = m(E_1 \cup E_2)$$
$$m_e(E_1 \cap E_2) = m_i(E_1 \cap E_2) = m(E_1 \cap E_2)$$

Also we know that

$$m_e(E_1) + m_e(E_2) \ge m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2) \tag{2.40}$$

and $$m_i(E_1) + m_i(E_2) \le m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2) \tag{2.41}$$

Using equation (2.39), we have

$$m(E_1) + m(E_2) \ge m(E_1 \cup E_2) + m_e(E_1 \cap E_2) \tag{2.42}$$

$$m(E_1) + m(E_2) \le m(E_1 \cup E_2) + m(E_1 \cap E_2) \tag{2.43}$$

From Eqns. (2.42) and (2.43), we get

$$m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$$

Theorem 2.19: Let $< E_i >$ be a sequence of measurable sets. Then prove that $m\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m(E_i)$.

Proof: Let $< E_n >$ be a sequence of sets in E and we have write $P_n = E_n - \bigcup_{i=1}^{n-1} E_i$.

Then $P_1 = E_1,\ P_2 = E_2 - E_1,\ P_3 = E_3 - (E_1 \cup E_2), \ldots$

This shows that $P_i \cap P; = \phi$ for $i \ne j$ (2.44)

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} P_i \tag{2.45}$$

$$P_n \subset E_n,\ \forall n \tag{2.46}$$

From Eqn. (2.44), we have $m\left(\bigcup_{i=1}^{\infty} P_i\right) = \sum_{i=1}^{\infty} m(P_i)$

$$\Rightarrow \qquad m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(P_i) \qquad \text{[using Eqn. (2.45)]}$$

$$\le \sum_{i=1}^{\infty} m(E_i) \qquad \text{[using Eqn. (2.46)]}$$

$$\Rightarrow \qquad m\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m(E_i).$$

Theorem 2.20: If $< E_n >$ is monotonic non-increasing sequence of measurable sets (outer measurable sets), then the limit set $E = \bigcap_{k=1}^{\infty} E_k$ is a measurable (outer measurable) set and for every T of finite outer measure, $m^*(T \cap E) = \lim_{n\to\infty} m^*(T \cap E_n)$.

Proof: Let $< E_n >$ is monotonic non-increasing sequence of measurable sets.

First we prove that $\lim_{n\to\infty} m_0(T \cap E_n)$ exists and

$$\lim_{n\to\infty} m^*(T \cap E_n) = m^*(T \cap E)$$

Let T be a set of finite measure.

Since, $E_1 \supset E_2 \supset E_3 \supset \ldots$

$$T \cap E_1 \supset T \cap E_2 \supset T \cap E_3 \supset \ldots$$

Therefore, we have

$$m^*(T \cap E_1) \ge m^*(T \cap E_2) \ge m^*(T \cap E_3) \ge \ldots$$

Consequently $< m_0(TE_n) : n \in N >$ is monotonic non-increasing sequence in brief we shall denote $A_1 \cap A_2 \cap A_3 \cap \ldots$ by $A_1\,A_2\,A_3 \ldots$ similarly the set $A \cup B \cup C$ is denoted by $A + B + C$. Also every member of $< m^*(TE_n), n \in N >$ is non-negative.

Hence, this sequence has a limit so that $\lim\limits_{n\to\infty} m^*(TE_n)$ exists.

$$E = \bigcap_{n=1}^{\infty} E_n \subset E_n \Rightarrow E \subset E_n \Rightarrow T \cap E \subset T \cap E_n$$

$$\Rightarrow \qquad TE \subset TE_n$$

$$\Rightarrow \qquad m^*(TE) \le m^*(TE_n)$$

$$\Rightarrow \qquad \lim_{n\to\infty} m^*(TE) \le \lim_{n\to\infty} m^*(TE_n)$$

Taking $\lim\limits_{n\to\infty} m^*(TE_n) = \lambda$, we get

$$m^*(TE) \le \lambda \tag{2.47}$$

The set T can be broken as

$$T = TE + TE_1' + E_1TE_2' + E_2TE_3' + \ldots + E_{n-1}\,TE_n' \tag{2.48}$$

Making use of Caratheodory postulates for outer measure, we have

$$m^*(T) \le m^*(TE) + m^*(TE_1') + m^*(E_1TE_2') + \ldots + m^*(E_{n-1}TE_n') \tag{2.49}$$

Since each E_n is outer measurable, so we have

$$m^*(T) = m^*(T\,E_n) + m^*(T\,E_n')$$
$$m^*(T) = m^*(T\,E_1) + m^*(T\,E_1') \tag{2.50}$$

Taking $E_{n-1}\,T$ for the set T and applying the condition of measurability of E_n, we get

$$m^*(E_{n-1}\,T) = m^*(E_{n-1}\,T\,E_n) + m^*(E_{n-1}\,T\,E_n')$$

$$\text{or} \qquad m^*(E_{n-1}\,T) = m^*(E_n\,T) + m^*(E_{n-1}\,TE_n')$$

$$\text{or} \qquad m^*(E_{n-1}T\,E_n') = m^*(TE_{n-1}) - m^*(TE_n) \tag{2.51}$$

By Eqns. (2.49), (2.50) and (2.51), we get

$$m^*(T) = m^*(TE) + \left[m^*(T) - m^*(TE_{1'})\right] + [m^*(TE_1) - m^*(TE_2)] + \ldots$$
$$= \left[m^*(TE_{n-1}) - m^*(TE_n)\right]$$
$$= m^*(TE) + m^*(T) - m^*(TE_n)$$

Taking limit as $n \to \infty$, we have

$$m^*(T) \le m^*(TE) + m^*(T) - \lambda$$
$$\lambda \le m^*(TE) \tag{2.52}$$

By Eqns. (2.47) and (2.52), we get

$$\lambda = m^*(TE)$$

i.e., $\lim\limits_{n\to\infty} m^*(TE_n) = m^*(TE)$

Now we show that $E = \bigcap_{n=i}^{\infty} E_n$ is measurable.

For any set $T \subset X$, we have

$$T = TE + TE' \tag{2.53}$$

From which we get

$$m^*(T) \leq m^*(TE) + m^*(TE') \tag{2.54}$$

By Eqn. (2.53), we get

$$TE' = T - TE$$

Putting in (2.48), we get

$$TE' = TE_1' + E_1TE_2' + E_2\,TE_3' + \ldots + E_{n-1}\,TE_n'$$

$$m^*(TE') \leq m^*(TE_1') + m^*(E_1\,TE_2') + \ldots + m^*(E_{n-1}\,TE_n')$$

Using Eqns. (2.50) and (2.51), we have

$$\begin{aligned} m^*(TE') &\leq [m^*(T) - m^*(TE_1)] + [m^*(TE_1) - m^*(TE_2)] \\ &\quad + [m^*(TE_2) - m^*(TE_3)] + \ldots + [m^*(TE_{n-1}) - m^*(TE_n)] \\ &= m^*(T) - m^*(TE_n) \end{aligned}$$

Making $n \to \infty$, we get

$$m^*(TE') \leq m^*(T) - \lim_{n\to\infty} m^*(TE_n)$$

or $m^*(TE') \geq m^*(TE') + \lim_{n\to\infty} m^*(TE_n)$

But $\lim m^*(TE_n) = m^*(TE)$

$$\Rightarrow \quad m^*(T) \geq m^*(TE') + m^*(TE) \tag{2.55}$$

By Eqns. (2.54) and (2.55), we get

$$m^*(T) = m^*(TE') + m^*(TE)$$

Hence, E is outer measurable.

Theorem 2.21: If $< E_n >$ be a sequence of disjoint measurable sets then for any set A,

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof: Let $< E_n >$ be a sequence of outer measurable set and $A = \bigcup_{i=1}^{\infty} E_i$, $B_n = \bigcup_{i=1}^{n} E_i$.

Then we have

$$A = \bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i = B_n$$

$$\Rightarrow \qquad m^*(T \cap B_n) = \sum_{r=1}^{n} m^*(T \cap A_r)$$

$$\Rightarrow \qquad m^*(T\cap B_n)=\sum_{r=1}^{n} m^*(T\cap E_r), \text{ where } A=\bigcup_{i=1}^{\infty} E_i,\ B_n=\bigcup_{i=1}^{n} E_i$$

Now we have

$$m^*\left[T\cap\left(\bigcup_{i=1}^{n} E_i\right)\right]=\sum_{i=1}^{n} m^*(T\cap E_i)$$

Replacing T by A, we get

$$m^*\left[A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right]=\sum_{i=1}^{n} m^*(A\cap E_i).$$

Theorem 2.22: Prove that every Borel set is Lebesgue measurable.

Proof: Since we know that open and closed sets are measurable therefore countable union and intersection of measurable set is measurable.

This implies Borel sets are measurable.

Another Proof:

Let a set E be Borel measurable so that E is a bounded Borel set.

To show that E is Lebesgue measurable.

E is expressible as $E=\left[\bigcup_{k} G_k\right]\cap\left[\bigcap_{k} F_k\right]$

Where G_k is an open set and F_k is a closed set.

E is bounded $\Rightarrow$ all G_k and all F_k are bounded.

Also G_k is open and F_k is closed $\Rightarrow G_k$ and F_k both are Lebesgue measurable sets. This implies $\bigcup_{k} G_k, \bigcap_{k} F_k$ are Lebesgue measurable sets.

For a countable union or intersection of measurable sets is measurable.

$\Rightarrow\left[\bigcup_{k} G_k\right], \cap\left[\bigcap_{k} F_k\right]$ is also Lebesgue measurable by same reasoning.

$\Rightarrow E$ is Lebesgue measurable.

Hence, every Borel set is Lebesgue measurable.

2.10 FIRST FUNDAMENTAL THEOREM

If E_1, E_2, … are disjoint measurable sets and $E = E_1 + E_2 + \ldots$ then E is measurable and $m(E)=\sum_{k=1}^{\infty} m(E_k)$.

Proof: Let E_1, E_2, … be disjoint measurable [a, b] then

$$m_e(E_k) = m_i(E_k) = m(E_k), \forall k$$

It is given that $E_i \cap E_j = \phi$ for $i \neq j$

Also $E = E_1 + E_2 + \ldots + = \sum_{k=1}^{\infty} E_k$

Suppose all the sets is contained in an [a, b], then we have

$$m_e(E) \leq \sum_{k=1}^{\infty} m_e(E_k) = \sum_{k=1}^{\infty} m(E_k)$$

or

$$m_e(E) \leq \sum_{k=1}^{\infty} m(E_k) \tag{2.56}$$

Let $S_n = \sum_{r=1}^{n} E_k$, then $m(S_n) = \sum_{r=1}^{n} m(E_r)$ for $E_r \cap E_s = \phi$ for $r \neq s$.

Then $S_n \subset \sum_{r=1}^{n} E_r = E$

i.e., $S_n \subset E$ so that $E' \subset S'_n$

$E' \subset S'_n \Rightarrow m_e(E') \leq m_e(S'_n) = m(S'_n)$

$\Rightarrow \quad m_e(E') \leq m(S'_n)$.

Since S_n is measurable $\Rightarrow S'_n$ is measurable.

$$\begin{aligned}\Rightarrow \quad m_e(E') &\leq m(S'_n) \\ &= (b-a) - m(S_n) \\ &= (b-a) - \sum_{r=1}^{n} m(E_r)\end{aligned}$$

$$\Rightarrow \quad m_e(E') \leq (b-a) - \sum_{r=1}^{n} m(E_r)$$

$$\Rightarrow \quad \sum_{r=1}^{n} m(E_r) \leq (b-a) - m_e(E') = m_i(E)$$

$$\Rightarrow \quad m_i(E) \geq \sum_{r=1}^{n} m(E_r).$$

Making $n \to \infty$, we obtain $m_i(E) \geq \sum_{r=1}^{n} m(E_r)$

By Eqn. (2.56), we have

$$m_i(E) \geq \sum_{r=1}^{n} m(E_r) \geq m_e(E) \tag{2.57}$$

$$\Rightarrow \quad m_i(E) \geq m_e(E).$$

But $m_i(E) \leq m_e(E)$ is always true.

$\therefore \quad m_e(E) \geq m_i(E)$.

This implies E is measurable

and $m(E) \geq \sum_{k=1}^{\infty} m(E_k) \geq m(E)$ [Using Eqn. (2.57)]

Hence, $m(E) \geq \sum_{k=1}^{\infty} m(E_k)$.

2.11 CANTOR'S TERNARY SET

This section we discuss a compact (and hence, a Borel) uncountable set of real numbers with measure zero. This set is known as the Cantor set after Georg Cantor (1845–1918) who constructed the set in 1883. Let the closed interval $I = [0, 1]$ and suppose H_1. Now divide H_1 in three equal parts and remove the middle part, *i.e.*, the open interval (1/3, 2/3). Now suppose H_2 be the remaining closed set, *i.e.*, $H_2 = (0, 1/3) \cup (2/3, 1)$. Repeating this process infinitely, we get a closed set $H = \bigcap_n \{H_n\}$. This set is called the Cantor's ternary set. The cantor's ternary set H contains the end points of the closed interval which make up each set H_n.

Note:

1. The Cantor's ternary set is measurable and its measure is zero.
2. The Cantor's set F is uncountable.
3. The Cantor's ternary set is a perfect set.

2.12 NON-MEASURABLE SET

First we define the equivalence relation on the set of real numbers. Consider two real number p and q is called to equivalent if and only if $p - q$ is a rational number. This equivalence relation is partition the whole set into disjoint equivalence classes such that any two elements of one class differ by a rational number while any two elements of different classes differ by an irrational number. Using the axiom of choice that there exist a set S such that S contains exactly one element from each equivalence class. Since the set of all rational numbers in [0, 1) is enumerable, the members can be arranged in the form of a sequence. Suppose $\{x_i\}_{i=0}^{\infty}$ be an enumeration of the rational numbers in [0, 1), where $x_0 = 0$. Consider $S_i = S \otimes x_i$, where $S_0 = S$ and if $r \in S_i \cap S_j$ then we have $r = p_i + x_i = p_j + x_j$, where $x_i, x_j \in$ S. But $p_i - p_j = x_i - x_j$ is a rational number and so $p_i \sim p_j$. Since S contains only one element from each class $p_i \sim p_j \Rightarrow i = j$. It follows that if $i \neq j$ then S_i and S_j are disjoint. It means that $\{S_i\}$ is a sequence of pairwise disjoint sets. Again, since each real number $p \in [0, 1)$ belongs to some equivalence class and so is equivalent to some element in S. But if p differs from an element in S by a rational number x_i then $p \in S_i$.

Therefore, we have $\cup\{S_i\}=[0,1)$. Since each S_i is translation modulo 1 of S, every S_i will be measurable if S is measurable and then $m\{S_i\}=m(S)$. But then we have $m[0,1)=\sum_{i=1}^{\infty} m(S_i)=\sum_{i=1}^{\infty} m(S)$. Now the right hand side is either zero {when $m(S)$ = 0} or ∞ {m (S) is positive}. This contradicts the fact that m [0, 1) = 1. Hence S is non-measurable.

RECAPITULATION

- The set which may be obtain from closed and open countable set by repeatedly applying union and intersection operation to them are called Borel set.
- A non-empty class $\mathfrak{R}$ of sets is known as Boolean ring (or ring of sets) if $A, B \in \mathfrak{R} \Rightarrow A - B \in \mathfrak{R}$ and $A \cup B \in \mathfrak{R}$.
- A non-empty class $\mathfrak{R}$ of sets is known as algebra (or Boolean algebra) if
- (i) $A, B \in \mathfrak{R} \Rightarrow A \cup B \in \mathfrak{R}$ and (ii) $A \in \mathfrak{R} \Rightarrow A' \in \mathfrak{R}$.
- A non-empty class $\mathfrak{R}$ of sets is known as σ-ring (or ring of sets) if
- (i) $A, B \in \mathfrak{R} \Rightarrow A\text{-}B \in \mathfrak{R}$ and (ii) $A_i \in \mathfrak{R}$, i = 1, 2, 3, …, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{R}$.
- The outer measure or the Lebesgue outer measure of an arbitrary set E is given by $m^*(E)=\begin{cases}\inf \sum_i l(I_i) \\ 0, \quad \text{if } E=\phi\end{cases}$ where the infimum is taken over all countable collection $\{I_i\}$ of open interval such that $E \subset \bigcup\{I_i\}$.
- A set E is said to be measurable with respect to the given outer measurable if for each set A, $m^*(A)=m^*(A\cap E)+m^*(A\cap E')$.
- Every countable set is measurable and its measure is zero.
- The union of two outer measurable set is outer measurable. The intersection of two outer measurable set is outer measurable.
- If S_1 and S_2 are measurable sets and if $S_2 \cap S_1=\phi$, then $m(S_1 \cup S_2)=m(S_1)+m(S_2)$.
- The exterior measure of a set $m_e(A)=\text{Inf}\,\{m(G): G\supset A. G \text{ is open}\}$ $\Rightarrow m(G) \geq m_e(A)$. For any given $\in > 0$, there exists an open set $G \supset A$ such that $m(G) < m_e(A)+\in$.
- The interior measure of a set $m_i(A)=\sup\,\{m(F): F\subset A, F \text{ is closed}\}$.
- If $A \subset B$, then $\Rightarrow m_e(A)\leq m_e(B)$ and If $A\subset B$ then $m_i(A)\leq m_i(B)$.
- If E_1 and E_2 are subset of $[a,b]$ then
  $$m_e(E_1)+m_e(E_2)\geq m_e(E_1\cup E_2)+m_e(E_1\cap E_2)$$
  and $m_i(E_1)+m_i(E_2)\leq m_i(E_1\cup E_2)+m_i(E_1\cap E_2)$.
- The union of two measurable sets is measurable.

- The difference of two measurable sets is measurable.
- **First Fundamental Theorem**: If E_1, E_2, … are disjoint measurable sets and $E = E_1 + E_2 + \dots$ then E is measurable and $m(E) = \sum_{k=1}^{\infty} m(E_k)$.

EXERCISES

Multiple-choice Questions

2.1 If A and B are two sets with $A \subset B$ then
  (a) $m^*(A) \leq m^*(B)$
  (b) $m^*(A) \geq m^*(B)$
  (c) $m^*(A) = m^*(B)$
  (d) None of these.

2.2 If S_1 and S_2 are measurable sets and if $S_2 \cap S_1 = \phi$, then $m(S_1 \cup S_2) =$
  (a) $m(S_1) + m(S_2)$
  (b) $m(S_1) - m(S_2)$
  (c) $m(S_1) \cdot m(S_2)$
  (d) None of these.

2.3 Which statement is wrong?
  (a) If $A \subset B$ then $m^*(A) \leq m^*(B)$
  (b) If $A \subset B$, then $m_e(A) \leq m_e(B)$
  (c) If $A \subset B$ then $m_i(A) \leq m_i(B)$
  (d) None of these.

2.4 If A and B are any two subsets then, $m^*(A - B) =$
  (a) $m^*(A) \cdot m^*(B)$
  (b) $m^*(A) + m^*(B)$
  (c) $m^*(A) - m^*(B)$
  (d) None of these.

2.5 If E_1 and E_2 are subset of $[a, b]$ then
  (a) $m_e(E_1) + m_e(E_2) \geq m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$
  (b) $m_i(E_1) + m_i(E_2) \leq m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$
  (c) $m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$
  (d) All above.

2.6 The Cantor's ternary set is measurable and its measure is equal to
  (a) 0
  (b) 1
  (c) ∞
  (d) None of these.

State True or False

2.1 The measure of singleton set is zero.
2.2 Every countable set is measurable and its measure is zero.
2.3 The intersection of two outer measurable set is not outer measurable.
2.4 A set is said to be measurable if and only if its complement is measurable.
2.5 Borel measurable set is not Lebesgue measurable.

Fill in the Blanks

2.1 If E is a measurable set then ______ is also measurable.
2.2 The union of two outer measurable set is ______ measurable.
2.3 Let A be a measurable set and if B is any set then

$$m^*(A \cup B) + m^*(A \cap B) \_\_\_\_\_ \; m^*(A) + m^*(B).$$

2.4 If two sets are measurable then there intersection is ______ measurable.
2.5 The Cantor's ternary set is a ______ set.

Exercises

2.1 What is Measure?
2.2 What do you mean by Lebesgue measure?
2.3 Discuss about the Exterior and Interior measure of a set.
2.4 State and prove first fundamental theorem.
2.5 What is Measurable set?
2.6 Prove that $m^*(A \cup B) \le m^*(A) + m^*(B)$.
2.7 Prove that a singleton subset is a measurable set with a measure zero.
2.8 Prove that the difference of two measurable sets is measurable.
2.9 Prove that the outer measure of an interval is its length.
2.10 If A and B are any two subsets then show that $m^*(A - B) = m^*(A) - m^*(B)$.
2.11 Prove that the symmetric difference of two measurable sets is also measurable.
2.12 Prove that
   (i) $m_e(E_1) + m_e(E_2) \ge m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$
   (ii) $m_i(E_1) + m_i(E_2) \le m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$.
2.13 Construct a set which is not measurable in the sense of Lebesgue.
2.14 Define Lebesgue measure of a set. Construct a non-dense perfect set in the interval (0, 1) whose measure is 1/2.
2.15 Write a short note on Cantor's ternary set.
2.16 To show that a Borel measurable set is Lebesgue measurable but not conversely.

2.17 Prove that the intersection of any collection of Boolean rings is again a Boolean ring.
2.18 Prove that the Lebesgue exterior measure is a Caratheodory outer measure.
2.19 Prove that the intersection of two rings is a ring.
2.20 Prove that open sets and closed sets are measurable.
2.21 Prove that a singleton set is measurable.
2.22 State Caratheodory's criterian for measurability and prove that the union of two measurable sets is measurable.
2.23 Write a short note on non-measurable set.

ANSWERS

Multiple-choice Questions

2.1 (a)	2.2 (a)	2.3 (d)	2.4 (c)
2.5 (d)	2.6 (a)		

State True or False

2.1 T	2.2 T	2.3 F	2.4 T	2.5 F

Fill in the Blanks

2.1 E'	2.2 outer
2.3 =	2.4 also
2.5 perfect.	

3

CHAPTER

Measurable Functions

3.1 INTRODUCTION

The concept of a measurable function on an abstract measurable space is the almost identical with that for real valued functions. However, measure-theoretic ideas are essential for a deep understanding of the probability, because probability is itself a measure. A measurable function is one whose pre-images are measurable sets and functions in the Lebesgue world are not "measured" but integrated. In this chapter we shall discuss the step function, Real valued function, Characteristic function, Simple function, Equivalent functions, Sequences of functions, Measurable function, Measurable function as a random variable, Littlewood's three principles, Borel measurability, Point-wise convergence, Convergence in measure, Uniform convergence, Convergence in mean, F. Riesz theorem, Egoroff's theorem and Lusin's theorem.

3.2 IMPORTANT TERMINOLOGY

Before discussing the measurable functions, we shall discuss certain necessary preliminaries.

3.2.1 Step Function

A real valued function ϕ defined over on a closed interval $[a, b]$ is called a step function if there exists is partition $\{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ such that function takes one and only one value of each interval.

3.2.2 Real Valued Function, Characteristic Function and Simple Function

A function f is said to be a real valued function if domain is a family of sets and its co-domain is a set of real numbers. For examples, trigonometric functions, exponential functions and polynomials functions are real valued functions.

Let U be the universal set and X be a subset of U. Then the real valued function $\phi_X : u \to \{0, 1\}$ is known as *characteristic function* of X and it is defined by $\phi_X = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$

An extended real valued function f is known as simple function if it is measurable and its range is finite. A characteristic function is a simple function.

3.2.3 Equivalent Functions

Let the two functions f_1 and f_2 are defined on the same set X, then they are said to be equivalent function if $m(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$.

3.2.4 Sequences of Functions

If $\{f_n(x)\}$ is a sequence of sets, then the supremum of $\{f_n(x)\}$ is the set $\bigcup_{n=1}^{\infty} \{f_n(x)\}$ and the supremum limit of is denoted by $\overline{\lim} f_n(x)$ and defined by $\lim_{n\to\infty} \sup \{f_n(x)\} = \overline{\lim} f_n(x) = \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{f_m(x)\}$.

If $\{f_n(x)\}$ is a sequence of sets, then the infimum of $\{f_n(x)\}$ is the set $\bigcap_{n=1}^{\infty} \{f_n(x)\}$ and the infimum limit of is denoted by $\underline{\lim} f_n(x)$ and defined by $\lim_{n\to\infty} \inf \{f_n(x)\} = \underline{\lim} f_n(x) = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{f_m(x)\}$. If the supremum and infimum limits are equal, *i.e.*, $\lim f_n(x) = \overline{\lim} f_n(x) = \underline{\lim} f_n(x)$, then we say that the limit of sequence is exist.

3.3 MEASURABLE FUNCTION

In this section we study the concept of measurability. We shall see that measurable functions are basically very strong or durable continuous-like functions. The class of measurable function has an important role in Lebesgue theory of integers. It takes a place comparable to that of class of function which are bounded and continuous in Riemann theory of integer and function of bounded variation. Let f is extended real valued function whose domain is measurable then following statement are equivalent:

(i) For each real number a collection of all x set $\{x : f(x) < a\}$ or $E(f(x) < a) = \{x \in E : f(x) < a\}$

(ii) $E(f(x) \geq a) = \{x \in E : f(x) \geq a\}$
(iii) $E(f(x) > a) = \{x \in E : f(x) > a\}$
(iv) $E(f(x) \leq a) = \{x \in E : f(x) \leq a\}$

Note:

1. Any continuous function of the sets on which function is defined is measurable.
2. Continuous function with measurable domain is measurable.

3.4 MEASURABLE FUNCTION AS A RANDOM VARIABLE

Suppose f is a function from a measurable space (Ω, F) into the real numbers R. We state that the function is measurable if for each Borel set $B_1 \in B$, the set $\{\omega; f(\omega) \in B_1\} \in F$. A random variable X is a measurable function from a probability space (Ω, F, P) into the real numbers R.

3.5 LITTLEWOOD'S THREE PRINCIPLES

J. Littlewood give the three principles for the theory of functions of real variables, are follows:

1. Every (measurable) set is nearly a finite union of intervals.
2. Every (measurable) function is nearly continuous.
3. Every convergent sequence of (measurable) functions is nearly uniformly continuous.

Theorem 3.1: Prove that the conditions involved in the definition for a measure function are equivalent to each other.

Proof: Let f be an extended real valued function defined over a measurable set E. Then f is Lebesgue measurable function if and only if any one of the following is measurable:

(i) $E(f > a)$ (ii) $E(f \geq a)$ (iii) $E(f < a)$ (iv) $E(f \leq a)$

(i) Let $\{x : f(x) \geq a\}$ is measurable.
To show $\{x : f(x) > a\}$ is measurable.

Now we have $E(f > a) = \{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq a + \frac{1}{n}\right\}$

We know that the union of measurable is measurable. Hence $\{x : f(x) > a\} = E(f > a)$ is measurable.

(ii) Let $\{x : f(x) > a\}$ is measurable.
To show $\{x : f(x) \geq a\}$ is measurable.

Now we have $E(f \geq a) = \{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}$

We know that the intersection of measurable is measurable. Hence $\{x : f(x) > a\} = E(f \geq a)$ is measurable.

(iii) Let $\{x : f(x) \geq a\}$ is measurable.
To show $\{x : f(x) < a\}$ is measurable.
Now we have $E(f < a) = \{x : f(x) < a\} = \{x : f(x) \geq a\}'$
We know that complement of measurable set is measurable. Hence $\{x : f(x) < a\} = E(f < a)$ is measurable.

(iv) Let $\{x : f(x) > a\}$ is measurable.
To show $\{x : f(x) \leq a\}$ is measurable.
Now we have $E(f \leq a) = \{x : f(x) \leq a\} = \{x : f(x) > a\}'$
We know that complement of measurable set is measurable. Hence $\{x : f(x) \leq a\} = E(f \leq a)$ is measurable.

3.6 BOREL MEASURABILITY

A function *f* is known as Borel measurable if for each *a*, the set $\{x : f(x) > a\}$ is a Borel set.

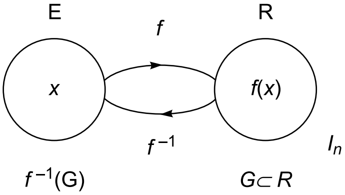
Note:

1. The product of two Borel measurable functions is a Borel measurable function.

Theorem 3.2: Let *f* be a function defined over a measurable set *E* then show that *f* is measurable if and only if for any open set $G \subset R$, $f^{-1}(G)$ is measurable.

Proof: Let *f* be defined over a measurable set *E* and $G \subset R$, where *R* is the set of real number and *G* and an open set.

To show that $f^{-1}(G)$ is measurable.



Since we know that any subset *G* of *R* can be expressed as a countable union of disjoint open intervals, therefore we have $G = \bigcup_{n=1}^{\infty}\{I_n\}$, where I_n is open interval then

$$\begin{aligned} f^{-1}(G) &= \cup\{x \in E : f(x) \in I_n\} = \cup\{x \in E : f(x) \in (a_n, b_n)\} \\ &= \cup\{x \in E : a_n < f(x) < b_n\} \\ &= \cup\,[\{x \in E : f(x) > a_n\} \cap \{x \in E : f(x) < b_n\}] \\ &= \cup\{E(f > a_n) \cap E(f < b_n)\} \end{aligned}$$

Since $E(f > a_n)$ and $E(f < b_n)$ is measurable.
$= E(f > a_n) \cap E(f < b_n)$ is measurable.
$\Rightarrow f^{-1}(G)$ is measurable.
Conversely, let $f^{-1}(G)$ is measurable.
To show f is also measurable.
Let $G = (a, \infty)$, where $a > 0$.

$$\begin{aligned} \text{Then } f^{-1}(G) &= \{x \in E : f(x) \in (a, \infty)\} \\ &= \{x \in E : a < f(x) < \infty\} \\ &= \{x \in E : f(x) > a\} \\ &= E(f > a) \end{aligned}$$

Give $f^{-1}(G)$ is measurable. Hence $E(f > a)$ is measurable.
Thus f is measurable on E.

Theorem 3.3: Prove that a constant function over a measurable set E is measurable.

Proof: Let $f(x) = c$. Then we have $E(f > a) = \begin{Bmatrix} E & \text{if } a < c \\ \phi & \text{if } a \geq c \end{Bmatrix}$. Since, E is measurable and ϕ is measurable. Hence, f is measurable function over a measurable set E. Hence, a constant function over a measurable set E is measurable.

Theorem 3.4: Every bounded open set and bounded closed set are measurable.

Proof: Let G be a bounded open set. From G by means of a finite number of disjoint closed interval. Then the sum of the lengths of their intervals is equal to the length (measure) of G. Therefore we have $m_e(G) = m(G)$. Then G can be obtained in an open set consisting of G itself and contains a closed set such that difference of measures of closed set and open set is arbitrarily small. Consequently $m_e(G) = m(G) = m_i(G) \Rightarrow$ G is measurable.

Since complement of a measurable set is measurable. Also complement of open set is closed set. This implies G' is measurable and G' is closed.

Theorem 3.5: Let f and g are real valued measurable function over a measurable set E show that $f \cup g$, $f \cap g$ are measurable function.

Proof: Let c is any real number. It is given f and g are measurable functions.

$\Rightarrow$ $E(f > c)$ and $E(g > c)$ are measurable.

To show that $E\{(f \cup g) > c\}$ is measurable.

$\Rightarrow E\{(f \cup g) > c\} = E(f > c) \cup E(g > c)$

Since $E(f > c)$ and $E(g > c)$ are measurable $\Rightarrow$ $E(f > c) \cup E(g > c)$ is measurable.

Hence, $f \cup g$ is measurable over a measurable set.

Now to show that $f \cap g$ is measurable.

$$\Rightarrow E[(f \cap g) > c] = E(f > c) \cap E(g > c)$$

Since $E(f > c)$ and $E(g > c)$ are measurable $\Rightarrow$ $E(f > c) \cap E(g > c)$ is measurable.

Hence, $f \cap g$ is measurable over a measurable set.

Theorem 3.6: If f and g are measurable function defined over a measurable set E then the set $E(f > g)$ is measurable.

Proof: Let f and g are measurable function over E.

$\Rightarrow$ $\exists$ a rational number r such that $f > r > g$. Therefore we have

$$E(f > g) = \cup [E(f > r) \cap E(g < r)]$$

Since it is given f and g are measurable function $\Rightarrow$ $E(f > r)$ and $E(g < r)$ are measurable set. We know that intersection of measurable set is measurable. Therefore $E(f > r) \cap E(g < r)$ is measurable. Also we know that union of measurable functions (sets) is measurable. Therefore $\cup \{E(f > r) \cap (g < r)\}$ is measurable.

Hence, $E(f > g)$ is measurable.

Theorem 3.7: Let f be a sequence function defined over a measurable set E_k, $\forall k \in N$ and $E = \bigcup_{k=1}^{\infty} E_k$. Then f is measurable over E.

Proof: Let a be any real number. Since it is given $E = \bigcup_{k=1}^{\infty} E_k$, *i.e.*, enumerable union of measurable set.

$\Rightarrow$ E is measurable.

$\Rightarrow$ $\{x \in E : f(x) > a\}$

$$\Rightarrow \quad E(f > a) = \bigcup_{k=1}^{\infty} E_k\ (f > a)$$

$$\text{or} \quad \bigcup_{k=1}^{\infty} [E_k(f > a)] = E_1(f > a) \cup E_2(f > a) \cup \ldots$$

Since each function f defined on each measurable set $E_1, E_2, \ldots, E_k$ and we know that union of measurable set is measurable. Therefore we have

$E_k(f > a)$ is measurable $\Rightarrow$ $E(f > a)$ is measurable.

Hence, f is measurable over E.

Theorem 3.8: Prove that if f is measurable over a measurable set E then f is also measurable over any subset E.

Proof: Let A be subset of E, where E is measurable set. To show that f is also measurable over a set A.

We have $A(f > a) = \{E(f > a) \cap A\}$.

Given $E(f > a)$ and A is measurable $\Rightarrow$ $E(f > a) \cap A$ is measurable.

Hence, f is measurable over a set A.

Theorem 3.9: Let f be a measurable function defined over a measurable set E and Let c be any real number. Show that cf, $f + c$, $|f|$, f^2 and $1/f$ are measurable where $f \neq 0$.

Proof:

(i) To show cf is measurable function over measurable set E.

$$\text{We have } E(cf > a) = \begin{cases} E\left(f > \dfrac{a}{c}\right), \text{ if } c \text{ is positive real number.} \\ E\left(f < \dfrac{a}{c}\right), \text{ if } c \text{ is negative real number.} \end{cases}$$

Since a, c are real number $\Rightarrow$ a/c is a real number.

Since the both R.H.S. is measurable therefore cf is measurable.

If $c = 0 \;\Rightarrow\; cf = 0, \forall x \in E \;\Rightarrow\; cf$ is a constant function.

We know that constant function is always measurable.

Hence, cf is measurable.

(ii) To show that $f + c$ is a measurable function over E.

We have $E(f + c > a) = \{E(f > a - c)\}$

Since a, c are real number $\Rightarrow$ $a - c$ is a real number.

Therefore $E(f + c > a)$ is measurable.

Hence, $f + c$ is measurable function over E.

(iii) To show that $|f|$ is measurable function over the measurable set E.

$$E(|f| < a) = \begin{cases} E(f < a) \cup E(f > -a) & \text{if } a \geq 0 \\ \phi & \text{if } a < 0 \end{cases}$$

By definition we know that $E(f < a), E(f > -a)$ and ϕ are measurable.

Therefore $E(f < a) \cup E(f > a)$ is measurable.

Thus $E(|f| < a)$ is measurable, *i.e.*, $|f|$ is measurable function over a measurable over E.

(iv) To prove that f^2 is measurable function over the measurable set E.

We have $E(f^2 < a) = E(|f| < \sqrt{a})$ if $a \geq 0$ (3.1)

But $E(|f| < \sqrt{a}) = [E(f < \sqrt{a}) \cup E(f > -\sqrt{a})]$, where a is any real number.

From Eqn. (3.1), we have $E(f^2 < a) = E(f < \sqrt{a}) \cup E(f > -\sqrt{a})$

By definition of measurable $\Rightarrow$ $E(f < \sqrt{a})$ and $E(f > \sqrt{a})$ is measurable.

$\Rightarrow$ $E(f < \sqrt{a}) \cup E(f > -\sqrt{a})$ is measurable.

$\Rightarrow$ $E(f^2 < a)$ is measurable.

Hence, f^2 is measurable function over a measurable set E.

(v) To show that $\frac{1}{f}$ is measurable function over the measurable set E.

Let f is vanish nowhere on E that $f(x) \neq 0, \forall x \in E$. Therefore $1/f$ exist.
Now we observe for any real number a, we have

$$E\left(\frac{1}{f} > a\right) = \begin{cases} E(f > 0) & \text{if } a = 0 \\ E(f > 0) \cap E\left(f < \frac{1}{a}\right) & \text{if } a > 0 \\ E(f < 0) \cap E\left(f < \frac{1}{a}\right) \cup E(f > 0) & \text{if } a < 0 \end{cases}$$

By definition of measurable function, we have $E(f > 0)$ is measurable and $E\left(f < \frac{1}{a}\right)$ is measurable $\Rightarrow E(f > 0) \cap E\left(f < \frac{1}{a}\right)$ is also measurable.

Now again $E(f < 0)$ is measurable and $E\left(f < \frac{1}{a}\right)$ is measurable this implies $E(f < 0) \cap E\left(f < \frac{1}{a}\right) \cup E(f > 0)$ is also measurable, *i.e.,* we know that union and intersection of two measurable set is measurable $\Rightarrow$ $E\left(\frac{1}{f} > 0\right)$ is measurable.

Hence, $\frac{1}{f}$ is measurable function over the measurable set E.

Theorem 3.10: Let f and g are measurable function defined over a measurable set E such that $f + g, f - g, fg$ and f/g (where g vanishes nowhere on the set E) are measurable function over E.

Proof: Let f and g are measurable function defined over a measurable set E.

First we will prove that $E(f > g)$ is measurable.

(i) Now to show that $f+g$ is measurable.
First we will prove $c-f$ is measurable.
We know that it is given that g is measurable $\Rightarrow$ cg is measurable.

$\Rightarrow$ $a+cg$ is measurable {Let a be any real number}
$\Rightarrow$ $a-g$ is measurable {put $c = -1$}

Now we have $E(f+g>a)=E(f>a-g)$
We know that f and $a-g$ are both measurable functions over E.
Thus $E(f+g>a)$ is a measurable set.
Hence, $f+g$ is measurable function over E.

(ii) Given g is measurable $\Rightarrow$ $-g$ is measurable.

$\Rightarrow$ $c(-g)$ is measurable.
$\Rightarrow$ $a+c(-g)$ is measurable.
$\Rightarrow$ $a-(-g)$ is measurable {put $c = -1$}

Now we have $E[f+(-g)>a]=E[f>a-(-g)]$
We know that f and $a-(-g)$ are both measurable functions over E
Thus $E[f+(-g)]=E[f-g]$ is a measurable
Hence, $f-g$ is measurable function over E.

(iii) To show that fg is measurable function over E.
First we prove $f+g$ and $f-g$ are measurable functions over E.
Since f and g are measurable function over $E \Rightarrow f+g$ is measurable function over E.

Also $(f+g)^2$ is measurable function over E.

Again if $f-g$ is measurable function over E $\Rightarrow$ $(f-g)^2$ is measurable function over E.
We know that difference of two measurable functions is also measurable.

$\Rightarrow$ $(f+g)^2-(f-g)^2$ is measurable.
$\Rightarrow$ $f^2+g^2+2fg-f^2-g^2+2fg$ is measurable.
$\Rightarrow$ $4\,fg$ is measurable.

We know that for any real number c,
A function cf is also measurable.

$\Rightarrow$ $c(4\,fg)$ is measurable over E. {put $c = 1/4$}

$\Rightarrow$ $\frac{1}{4}(4\,fg)$ is measurable over E.

$\Rightarrow$ fg is measurable over measurable set E.

(iv) Let g vanish nowhere on E so that $g(x)\neq 0\ \forall x\in E$. Therefore $1/g$ exists.
Give f and g are measurable function over E.

$\Rightarrow f+g,\ f-g,\ fg$ are measurable function over E.

Now to show that $(1/g)$ is measurable over E
Let a be any real number, we have

$$E\left(\frac{1}{g} > a\right) = \begin{bmatrix} E(g>0) & \text{if } a=0 \\ E(g>0)\cap E\left(g<\frac{1}{a}\right) & \text{if } a>0 \\ [E(g>0)]\cup[E(g<0)]\cap\left[E\left(g<\frac{1}{a}\right)\right] & \text{if } a<0 \end{bmatrix}$$

Since $E(g>0)$ and $E\left(g<\frac{1}{a}\right)$ are measurable function over E. We know that finite intersection of measurable is also measurable function. Thus $E(g>0)\cap E\left(g<\frac{1}{a}\right)$ is also measurable.

Now $E(g<0)$ is measurable. We know that finite union and intersection of measurable function is measurable function.

$\Rightarrow$ $[E(g>0)]\cup\left[\left\{E(g<0)\cap E\left(g<\frac{1}{a}\right)\right\}\right]$ is measurable.

$\Rightarrow$ $E\left(\frac{1}{g}>a\right)$ is measurable in every case.

$\Rightarrow$ $1/g$ is measurable over E.

Now to prove f/g is measurable over E.

We know $f/g = f\cdot\left(\frac{1}{g}\right)$

Since we know that the product of two measurable functions is measurable function.
Hence f/g is measurable function over E.

Theorem 3.11: A function f is measurable if and only if the set $\{x = f(x) < r\}$ is measurable for every rational number r.

Proof: Suppose a function f is measurable over E.
To show $\{x : f(x) < r\}$ is measurable.
Let r be a rational number. Let c be any real number.
Since f is measurable $\Rightarrow$ $E(f<c)$ is measurable $\forall c\in R$

$\Rightarrow$ $E(f<c)$ is measurable $c\in Q$ for $Q\subset R$.

$\Rightarrow \quad E(f < r)$ is measurable, $\forall r \in Q$.

$\Rightarrow \quad \{x : f(x) < r\} = \{x \in E : f(x) < r\}$ is measurable.

Conversely, suppose $\{x : f(x) < r\}$ is measurable.

We have $E(f < c) = \{x \in E : f(x) < c\} = \cup\{x \in E = f(x) < r < c\}$,

$= \cup\{x \in E : f(x) < r, r \in Q\}$

= enumerable union of measurable set.

= Measurable set.

Hence, f is measurable function over E.

Theorem 3.12: Prove that a continuous function f defined over a measurable set E is measurable.

Proof: Let f is the continuous function defined over a measurable set E.

To show that f is measurable function over the set A.

Let a be any real number then we have

$$A = E(f \geq a) = \{x \in E : f(x) \geq a\} \tag{3.2}$$

To show that f is measurable function then prove A is closed because we know that every closed set is measurable. Let x_0 be the limit point of A.

To show that $x_0 \in A \Rightarrow \exists$ a sequence $\{x_n\}$ of A such that

$$\lim x_n = x_0 \qquad \text{(by definition)} \tag{3.3}$$

Since it is given f is continuous, so we have

$$\begin{aligned} x_n \to x_0 &\Rightarrow f(x_n) \to f(x) \\ &\Rightarrow x_n \in A \Rightarrow f(x_n) \geq a \qquad \text{\{From Eqn. (3.2)\}} \\ &\Rightarrow \lim_{n\to\infty} f(x_n) \geq a \\ &\Rightarrow f\left[\lim_{n\to\infty} x_n\right] \geq a \\ &\Rightarrow f(x_0) \geq a \qquad \text{\{From Eqn. (3.3)\}} \\ &\Rightarrow x_0 \in A \end{aligned}$$

Thus A is closed. Since we know that every closed set is measurable. Hence, f is measurable function over the measurable set E.

Theorem 3.13: Let $< f_n >$ be a sequence of measurable function defined over a measurable set E show that $\sup\{f_1, f_2, \dots f_n\}$, $\inf\{f_1, f_2, \dots f_n\}$, $\overline{\lim}\, f_n$, $\underline{\lim}\, f_n$ are measurable over E. Hence, show that $\lim f_n$ is measurable over E if $\lim f_n$ exists.

Proof: Let $< f_n >$ be a sequence of measurable functions defined over a measurable set E.

Step I: To prove that $\sup\{f_r : 1 \leq r \leq n\}$ and $\inf\{f_r : 1 \leq r \leq n\}$ are measurable over E.

Define $M(x)=\sup\{f_r(x):1\le r\le n\}=\bigcup\limits_{r=1}^{n} f_r(x)$

and $M(x)=\inf\{f_r(x):1\le r\le n\}$

$$=-\sup\{f_r(x):1\le r\le n\}=\bigcup_{r=1}^{n} f_r(x)$$

If we show that $M(x)$ is measurable over E, the result will follows. For proving this, it is enough to prove that $E(M>a)$ is a measurable set over E, a being any real number.

Since f_n is a measurable function over $E \Rightarrow E(f_n>a)$ is a measurable set $\forall\, n$.

We have $E(M>a)=\bigcup\limits_{r=1}^{n} E[f_r>a]$

= a finite union of measurable set
= a measurable set.

This prove that $E(M>a)$ is measurable set.

Step II: To show that $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ are measurable function over the set E.

Define $M_k(x)=\sup\limits_{n\ge k}\{f_n(x)\}$

and $m_k(x)=\inf\limits_{n\ge k}\{f_n(x)\}$

Then $\overline{\lim}\, f_n(x)=\inf\limits_{k\ge 1}\{M_k(x)\}$

and $\underline{\lim}\, f_n(x)=\sup\limits_{k\ge 1}\{m_k(x)\}$.

By case (i) $M_k(x), m_k(x)$ both are measurable defined over $E, \forall k\in N$ and again by case (i).

$\inf\limits_{k\ge 1}\{M_k(x)\}, \sup\limits_{k\ge 1}\{M_k(x)\}$ are measurable over E.

i.e., $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ both are measurable over E.

Step III: Let $\lim f_n$ exist.

To show that $\lim f_n$ is measurable over E

By case (ii), $\overline{\lim}\, f_n \cdot \underline{\lim}\, f_n$ both are measurable over E.

By hypothesis, we have $\lim f_n = \overline{\lim}\, f_n = \underline{\lim}\, f_n$.

From what has been done, it follows that $\lim f_n$ is measurable over E.

Theorem 3.14: If $< f_n >$ is a sequence of measurable functions then show that $\lim f_n$ is measurable.

Proof: Let $< f_n >$ be a sequence of measurable functions.

To show that $\lim f_n$ is measurable.

$\therefore$ $< f_n >$ is a monotonic sequence and hence it is either monotonic increasing or monotonic decreasing.

If $< f_n >$ is a monotonic increasing sequence, then we have

$$\lim f_n = \sup_{n \geq 1} \{f_n\}$$

If $< f_n >$ is monotonic decreasing sequence, then we have

$$\lim f_n = \inf_{n \geq 1} \{f_n\}$$

If prove that $\sup\{f_1, f_2, \ldots f_n\}$ and $\inf\{f_1, f_2, \ldots f_n\}$ are measurable then the problem will be solved.

Let $g(x) = \sup\{f_1, f_2, \ldots, f_n\}$

and $f_n(x) = \inf\{f_1, f_2, \ldots, f_n\}$.

Then $E(g > a) = \bigcup_{n=1}^{\infty} [E(f > a)]$

= an enumerable union of measurable sets.

Hence, $E(g > a)$ is measurable and so g is measurable.

Now we have

$$h(x) = -\sup\{-f_n(x) : n = 1, 2, \ldots\}$$

$\Rightarrow$ $h(x)$ is also measurable as g is measurable.

Hence, the limit of a convergent sequence of measurable function is measurable.

3.7 POINT-WISE CONVERGENCE

Let $\{f_n\}$ is the sequence of measurable function on a measurable set E then this sequence $\{f_n\}$ is said to converge point-wise on a set E if $\exists$ a measurable function f on E such that $f_n(x) \to f_n, \forall x \in E$, where $n \to \infty$.

3.8 CONVERGENCE IN MEASURE

Let $\{f_n\}$ is a sequence of measurable functions defined over a measurable set E then $\{f_n\}$ is said to converge in measure to a function f if the following conditions are satisfied:

(i) f is a measurable function on a set E such that $f(x) < \infty, \forall x \in E$.

(ii) $m\left\{E\left(|f_n - f| \geq \in\right)\right\} = 0, \quad \forall \in > 0$.

3.9 UNIFORM CONVERGENCE

Let $\{f_n\}$ is a sequence of measurable functions defined over a measurable set E then $\{f_n\}$ is said to converge uniformly almost everywhere to a function f if there exists a set E_0 such that (i) $m(E_0) = 0$ and (ii) $\{f_n\}$ converges uniformly to a function f on the set $E - E_0$.

Note:

1. A real valued function f is said to be continuous at the point x_0 if given $\in > 0 \; \exists \, \delta > 0$ such that $|f(x) - f(x_0) < \in|$ wherever $|x - x_0| < \delta$. If a function is continuous at each point for which it is defined, then it is called a continuous functions.
2. The main difference between continuity and uniformly continuity, in continuity $f(x)$ depend upon $\in$ and δ but in case of uniformly continuity $f(x)$ depend upon only $\in$.

3.10 CONVERGENCE IN MEAN

A sequence $\{f_n\}$ of a Lebesgue integrable function is said to converge in mean to a function f if $\lim\limits_{n \to \infty} \int_E |f_n - f| \, dx = 0$.

3.11 F. RIESZ THEOREM

If $\{f_n\}$ be a sequence of measurable functions which converges in measure to f, then there exists a sub-sequence $\{f_{n_k}\}$ which also converges to f almost everywhere.

Proof: Let $\{\delta_n\}$ be a monotonic decreasing sequence of positive terms such that $\lim \delta_n = 0$. Suppose $\sum \sigma_i$ is a convergent series of positive terms. Consider we choose a monotonic sequence $\{\sigma_i : i \in N\}$ of positive integers such that $m\left(\left\{x \in A : |f_n(x) - f(x)| \geq \delta_k\right\}\right) < \sigma_k$ for each $k \in N$.

Since the sequence $\{f_n\}$ converges is measure to f

$$\Rightarrow \lim_{n \to \infty} m\left(\left\{x \in A : |f_n(x) - f(x)| \geq \delta_k\right\}\right) = 0 .$$

Now we have to show that the sequence $\{f_{n_k}\}$ also converges is measure to f almost everywhere on A, *i.e.,* $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ almost everywhere on A.

Let $\bigcup_{k=i}^{\infty}\left(\left\{x\in A : \left|f_{n_k}(x) - f(x)\right| \ge \delta_k\right\}\right) = A_i$ and $\bigcap_{i=1}^{\infty}\{A_i\} = B$

Then obviously, $\{A_i\}$ is a monotonic decreasing sequence of measurable sets, *i.e.,* $A_1 \supset A_2 \supset A_3 \supset \dots$.

Thus $\lim_{n\to\infty} m(A_n) = m(B)$ and also $m(A_n) < \sum_{k=n}^{\infty} \sigma_k$

Hence, $\lim_{n\to\infty} m(A_n) = 0 = m(B)$.

Let y be any element of (A – B). Then $y \notin B$ implies that $y \notin A_{n_0}$ for some positive integer n_0 such that

$$y \notin \left\{x\in A : \left|f_{n_k}(x) - f(x)\right| \ge \delta_k\right\} \quad \forall\, k \ge n_0$$

$$\Rightarrow \qquad \left|f_{n_k}(y) - f(y)\right| \ge \delta_k \quad \forall\, k \ge n_0$$

$$\Rightarrow \qquad \lim_{k\to\infty} f_{n_k}(y) = f(y) \quad \forall\, y \in A - B$$

$$\Rightarrow \qquad \lim_{k\to\infty} f_{n_k}(y) = f \text{ almost everywhere on } A.$$

3.12 EGOROFF'S THEOREM

If a sequence of measurable functions is convergent almost everywhere on a measurable set A, of finite measure then for every $\in > 0$ there exists a measurable set $B \subset A$ such that

(i) $m(B) < \in_j$ and (ii) the sequence is uniformly convergent on (A – B).

Proof: Let $\{f_n\}$ be a sequence of measurable functions defined on A. Suppose this sequence converges almost everywhere on A to a finite measurable function f. Then the function f is measurable. For if B is the set on which $\lim_{n\to\infty} f_n(x) = f(x)$ then $m(A - B) = 0$. Hence, f is measurable on B. Since on a set of measure zero every function is measurable, f is measurable on (A – B). It follows that f is measurable on A also. If we set $E_n^m = \bigcap_{i=n}^{\infty}\left\{x : \left|f_i(x) - f(x)\right| < 1/m\right\}$ then the set E_n^m, for fixed m and n are the sets of those points x for which $\left|f_i(x) - f(x)\right| < 1/m$ for all $i \ge n$. Let $E^m = \bigcup_{n=1}^{\infty} E_n^m$

Then obviously, for fixed m, we have $E_1^m \supset E_2^m \supset E_3^m \supset \ldots$. Since the sequence $\{f_n\}$ converges to f on A, we have $\lim\limits_{n\to\infty} (E_n^m) \supset A$ for every m = 1, 2, 3, ...; for given $\in > 0$. Thus $\lim\limits_{n\to\infty} m(A - E_n^m) = 0$ and so for given $\in > 0$ there exists a positive integer n_0, dependent on m and $\in$, such that $m(A - E_n^m) < \in/2^m, \in > 0$. Let $B = \bigcup\limits_{m=1}^{\infty} [A - E_{n_0}^m]$. Then B is a subset of A and being the union of measurable sets, B is a measurable set.

Also
$$m(B) = m\left(\bigcup_{m=1}^{\infty} [A - E_{n_0}^m]\right)$$

$$\leq \sum_{m=1}^{\infty} (A - E_{n_0}^m) < \in$$

Again we have

$$A - B = A - m \bigcup_{m=1}^{\infty} [A - E_{n_0}^m]$$

$$= A \cap m \bigcup_{m=1}^{\infty} E_{n_0}^m \qquad \{\text{Using De-Morgen law}\}$$

Thus for every $n \geq n_0$ and $x \in A - B \quad \Rightarrow \quad x \in E_n^m$

Therefore, $|f_i(x) - f(x)| < 1/m$ for every $x \in A - B$.

Hence, $\{f_n\}$ is uniformly converges on A – B.

3.13 LUSIN'S THEOREM

Let f be a measurable function defined on a finite interval $[a, b]$. For every pair $\delta, \varepsilon > 0$ there exists a continuous function Ψ on $[a, b]$ such that $m\left(\{x : |f(x) - \psi(x)| \geq \delta\}\right) < \in$.

Proof: Let the function f is finite almost everywhere on $[a, b]$.

First we shall prove that when f is bounded. If f is bounded then there exists a number λ such that $|f(x)| < \lambda$ for $\forall x \in [a, b]$

If δ, ε are arbitrary positive real numbers then for their fixed values let m be a positive integer such that $\lambda/m < \delta$.

Now we have

$$A_r = \left(x : \left\{\frac{r-1}{m}\lambda \leq f(x) < \frac{r}{m}\lambda\right\}\right), \quad r = 1-m, 2-m, 3-m, \ldots, m$$

and $A_m = \left(x : \left\{\dfrac{m-1}{m}\lambda \leq f(x) < \lambda\right\}\right)$

Then $A_i \cap A_j = \phi$ if $i \neq j$ for $i, j = 1-m, 2-m, 3-m, \ldots, m$

and $\underset{r=1-m}{\overset{m}{\cup}} A_r = [a, b]$.

Let B_i be a closed interval contained in A_i such that

$$m(A_r) = m(B_r) + \frac{\in}{2M} \text{ and } B = \underset{r=1-m}{\overset{m}{\cup}} B_r$$

Then we have $m([a,b]) - m(B) < \in$

Now we define a function ϕ on B given by

$$\phi(x) = r\lambda/m \ \ \forall\ x \in B_r, \quad r = 1-m, 2-m, 3-m, \ldots, m$$

Evidently, ϕ is constant on each closed set B_r. Also $B_r \cap B_{r'} = \phi$ if $r \neq r'$.
Hence, ϕ is continuous on B.
Now we have

$$\begin{aligned} |\phi(x)| &= \left|\frac{r\lambda}{m}\right| \\ &= \frac{r\lambda}{m} \\ &\leq \lambda. \end{aligned}$$

Then we have

$$\begin{aligned} |f(x) - \phi(x)| &= \left|\lambda - \frac{r\lambda}{m}\right| \\ &= \frac{\lambda}{m}|m - r| \\ &\leq \frac{\lambda}{m} \\ &< \delta \qquad \forall x \in B. \end{aligned}$$

We know that if B is a closed set contained in $[a, b]$ and ϕ is defined and is continuous on B then there exists a function Ψ on $[a, b]$ such that

(i) Ψ is continuous,

(ii) $\Psi(x) = \phi(x)$, $\forall x \in B$ and

(iii) $\max|\psi(x)| = \max|\phi(x)|$.

Now we can find a function Ψ defined on $[a, b]$ which has the above the three properties.

Furthermore $\{x : |f(x) - \psi(x)| \geq \delta\} \subset [a, b] - B$.

Thus we have $m(\{x : |f(x) - \psi(x)| \geq \delta\}) \leq m([a,b]) < \in$.

This prove that the theorem for a bounded function f. Now if f is unbounded, then there exists a bounded function g such that $m(\{x : |g(x) - \psi(x)| \geq \delta\}) < \in/2$.

It follows that

$$m(\{x : |f(x) - \psi(x)| \geq \delta\}) \leq m(\{x : f(x) \neq g(x) \geq \delta\}) + m(\{x : |g(x) - \psi(x)| \geq \delta\})$$
$$< \in/2 + \in/2 = \in.$$

Solved Problem

Problem 3.1: Show that the function f defined on R by

$$f(x) = \begin{bmatrix} x+5 & x < -1 \\ 2 & -1 \leq x < 0 \\ x^2 & x \geq 0 \end{bmatrix} \text{ is a measurable function.}$$

Solution: Let a be any real number. Then $a < 0$ and $R(f \leq a)$ $\Rightarrow x + 5 \leq a \;\Rightarrow\; x \leq a - 5$

$\therefore\; R(f \leq a) = [-\infty, a-5]$, we get

$$R(f \leq a) = \begin{cases} (-\infty, a-5) & \text{if } a < 0 \\ (-\infty, -5) \cup \{0\} & \text{if } a = 0 \\ (-\infty, a-5) \cup [0, 5a] & \text{if } 0 < a < 2 \\ (-\infty, a-5) \cup [-1, 5a] & \text{if } 2 \leq a \leq 4 \\ (-\infty, -5a) & \text{if } 4 \leq a \end{cases}$$

Since each set on R.H.S. is measurable.

$\therefore R(f \leq a)$ is measurable for every real value of a. Hence, f is measurable.

RECAPITULATION

- A real valued function ϕ defined over on a closed interval $[a, b]$ is called a step function if there exists is partition $\{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ such that function takes one and only one value of each interval.
- A function f is said to be a real valued function if domain is a family of sets and its co-domain is a set of real numbers.
- Let U be the universal set and X be a subset of U. Then the real valued function $\phi_X : u \rightarrow \{0, 1\}$ is known as *characteristic function* of X and it is defined by $\phi_X = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$.

- An extended real valued function f is known as simple function if it is measurable and its range is finite. A characteristic function is a simple function.
- Let the two functions f_1 and f_2 are defined on the same set X, then they are said to be equivalent function if $m(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$.
- If the supremum and infimum limits are equal, *i.e.,* $\lim f_n(x) = \overline{\lim}\, f_n(x) = \underline{\lim}\, f_n(x)$, then we say that the limit of sequence is exist.
- A function f is known as Borel measurable if for each a, the set $\{x : f(x) > a\}$ is a Borel set.
- Let $\{f_n\}$ is a sequence of measurable functions defined over a measurable set E then $\{f_n\}$ is said to converge uniformly almost everywhere to a function f if $\exists$ a set E_0 such that (i) $m(E_0) = 0$ (ii) $\{f_n\}$ converges uniformly to a function f on the set $E - E_0$.
- Let f be an extended real valued function defined over a measurable set E. Then f is Lebesgue measurable function if and only if one of the following is measurable:

  (i) $E(f > a)$ (ii) $E(f \geq a)$ (iii) $E(f < a)$ (iv) $E(f \leq a)$

EXERCISES

Multiple-choice Questions

3.1 Let f and g are measurable function defined over a measurable set E such that the following function is measurable over E

(a) $f + g$ and $f - g$
(b) fg
(c) f/g (where g vanishes nowhere on the set E)
(d) All above.

3.2 Let f be an extended real valued function defined over a measurable set E. Then f is Lebesgue measurable function if the following is measurable:

(a) $E(f > a)$ or $E(f < a)$
(b) $E(f \geq a)$ or $E(f \leq a)$
(c) $E(f = a)$
(d) Any one of all above.

3.3 Two functions f_1 and f_2 are defined on the same set X, then they are said to be equivalent function if

(a) $m(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$
(b) $m(\{x \in X : f_1(x) = f_2(x)\}) = 0$
(c) $m(\{x \in X : f_1(x) \neq f_2(x)\}) \neq 0$
(d) None of these.

State True or False

3.1 A constant function over a measurable set E is measurable.
3.2 Let f and g are real valued measurable function over a measurable set E than $f \cup g$, $f \cap g$ are measurable function.
3.3 If f is measurable over a measurable set E then f is not measurable over any subset E.
3.4 The limit of a convergent sequence of measurable function is measurable.

Fill in the Blanks

3.1 Every bounded open set ______ bounded closed set is measurable.
3.2 If f and g are measurable function defined over a measurable set E then the set $E(f > g)$ is ______
3.3 A continuous function defined in a ______ internal is measurable.

Exercises

3.1 What is real valued function?
3.2 What do you mean by measurable function? Show that the characteristic function of a measurable set is a measurable function.
3.3 Discuss about the Step Function.
3.4 Prove that the limit of a convergent sequence of measurable function is measurable.
3.5 Prove that if the functions f and g are finite and measurable then $f + g$, $f - g$ and fg are measurable.
3.6 Prove that the sum, difference and product of two simple functions is simple.
3.7 If f is a Borel measurable function and A is a Borel set then proves that $f^{-1}(A)$ is a Borel set.
3.8 Explain the Borel measurability.
3.9 If f and g are two measurable functions and c be a constant then prove that the function cf and $f + g$ are measure.
3.10 Prove that a real function which is continuous in an open interval is measurable.
3.11 State and Prove F. Riesz Theorem.
3.12 State and Prove Egoroff's Theorem.
3.13 State and Prove Lusin's Theorem.
3.14 Show that a continuous function defined in a closed internal is measurable.
3.15 To prove that the limit of a convergent sequence of measurable function is measurable.

ANSWERS

Multiple-choice Questions

3.1 (d) 3.2 (d) 3.3 (a)

State True or False

3.1 T 3.2 T 3.3 F 3.4 T

Fill in the Blanks

3.1 and 3.2 measurable
3.3 closed

4

CHAPTER

Lebesgue Integral

4.1 INTRODUCTION

Lebesgue integration theory is one of the most attainments of modern mathematics. It has a great importance in real analysis and in many other fields of the mathematical sciences, and is named after the Henri Lebesgue who introduced this integral in 1904. The term "Lebesgue integration" introduced by Lebesgue for the specific case of integration of a function defined on a sub-domain of the real line with respect to Lebesgue measure. In Riemann integral usually proved that if the sequence of integrable functions is uniformly convergent, then its limit function is integrable and the integral of this limit function is equal to the limit of the integrals. Otherwise, a sequence of functions integrable in the Riemann sense that is not uniformly convergent may have a limit function that is not integrable in the Riemann sense, even if the functions are uniformly bounded. In the theory of the Lebesgue integral, the theorem on the integration of sequences permits of expression in a much stronger form, simpler and more suitable in applications, and furthermore it has a fundamental theoretical significance. Lebesgue integration is the superior to Riemann integration. It does not require the functions to be bounded with bounded support, it ignores "local nonsense" on sets of measure 0, and it is better behaved with respect to limits. This chapter discusses the definite Riemann theory of integral, Lebesgue integral, First mean value theorem, Lebesgue bounded convergence theorem, Lebesgue dominated convergence theorem, Beppo Levi's theorem and Fatou's Lemma.

4.2 IMPORTANT TERMINOLOGY

Before discussing the Lebesgue integrals in details, we shall discuss certain necessary preliminaries.

4.2.1 Partition

By a partition of [a, b] we mean a finite set of points $x_0, x_1, x_2, \ldots x_n$, where $a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$. The partition P consists of n + 1 points clearly any number of partitions of [a, b] can be considered $[x_0, x_1], [x_1, x_2] \ldots [x_{i-1}, x_i] \ldots [x_{n-1}, x_n]$ are the sub-interval of [a, b]. We shall use the same symbol Δx_i to denote the i^{th} sub interval $[x_{i-1}, x_i]$ as also its length $x_i - x_{i-1}$. Thus $\Delta x = x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$.

4.4.2 Refinement of a Portion

Let P is a measurable partition of [a, b] and p_1 is any other measurable partition of [a, b] such that every component of partition p_1 is contained in same components of P then P is called Refinement of p_1. Let P is the subset of [a, b] and f is a bounded function on [a, b] then $M(f : E) = \sup\{f(x) : x \in E\}$. Similarly, we have $m(f : E) = \inf\{f(x) : x \in E\}$.

4.2.3 Definition (Darboux's Condition of Integrability)

When the two integrals are equal, *i.e.*, $\int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx$. We say the f is Riemann integrable (or simply integrable) over [a, b] and the common value of these integrals is called the Riemann integral of f over [a, b]. This implies f is integrable over [a, b] we express by writing $f \in R[a, b]$ or R simply, $m, (b - a) \le \int_a^b f\,dx \le M(b - a), \quad b \ge a.$

Obviously, $w(f, P) = U(f, P) - L(f, P) = \sum (M_r - m_r)\,S_r$
Hence, $U(L, P) \le U(f, P)$.

Note:

(1) If f_1 and f_2 are two bounded and integrable function on [a, b] then $f = f_1 + f_2$ is also integrable on $[a, b]$ and $\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$.

(2) If f_1 and f_2 are two bounded and integrable function on [a, b] then $f = f_1 - f_2$ is also integrable on $[a, b]$ and $\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$.

(3) (i) If a bounded function f is integrable on [a, b] then it is also integrable on $[a, b]$ and $[c, b]$, where c is a point of $[a, b]$. (ii) Conversely if f is bounded and integrable on $[a, c]$, $[c, b]$ then it is also integrable on $[a, b]$. (iii) Also in either case $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$, $a \le c \le b$.

(4) The oscillation of a bounded function f on an interval [a, b] is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ of numbers.

(5) If f_1, f_2 are two bounded and integrable function on $[a, b]$ and there exists a number $\lambda > 0$ such that $|f_2(x) \geq \lambda|$, for all x in $[a, b]$ then f_1/f_2 is bounded and integrable on $[a, b]$.

(6) If f is integrable on $[a, b]$ then f^2 is also integrable on $[a, b]$.

(7) Every continuous function is integrable.

(8) If a function f is monotonic on $[a, b]$ then it is integrable on $[a, b]$.

(9) A bounded function f, having a finite number of point of discontinuity on $[a, b]$ is integrable on $[a, b]$.

(10) A bounded function f is integrable on $[a, b]$ if the set of its points of discontinuity has only a finite number of limit points.

4.3 RIEMANN THEORY OF INTEGRAL

The classical definition of an integral, given first by Cauchy and later developed by Riemann, runs as follows:

Let F be a bounded real valued function defined on the interval $[a, b]$ and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition (or Subdivision) of $[a, b]$, where f is bounded on each subinterval corresponding to each partition P.

Let M_i, m_i be the bounds (supremum and infimum) of f in Δx_i.

From the two sums, we have

$$S(P) = U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

$$= M_1 \Delta x_1 + M_2 \Delta x_2 + \ldots + M_n \Delta x_n$$

and $S(P) = L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$

$$= m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots + m_n \Delta x_n$$

respectively called the upper and the lower (Darboux) sums of f corresponding to the partition P.

where $M_i = \sup\{f(x) : \ x \in]x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : \ x \in]x_{i-1}, x_i]\}$ for $i = 1, 2, \ldots, n$.

If M, m are the bounds of f in $[a, b]$, we have $m \leq m_i \leq M_i \leq M$

$\Rightarrow \ m\Delta x_i \leq m_i \Delta x_i \leq M_i \Delta x_i \leq M \Delta x_i$

Putting $i = 1, 2, \ldots n$ and adding all the inequalities, we get

$$m(x_n - x_0) \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq M(x_n - x_0)$$

$$\Rightarrow \qquad m(b-a) \leq L(P, f) \leq U(p, f) \leq M(b-a), \quad b \geq a \qquad (4.1)$$

Now each partition gives rise to a pair of sums the upper and the lower sums by considering all portions of $[a, b]$, we get a set U of upper sums and a set L of lower sums Eqn. (4.1) show that the both of these sets are bounded and so each set has the supremum and the infimum. The infimum of the set of upper sums is called the upper integral and the supremum of the set of lower sums is called the lower integral of *f* over [*a*, *b*]. Thus the upper Riemann integral of *f* over [*a*, *b*] is defined by

$$R\int_a^{\bar{b}} f(x)\,dx = \inf S(P) \quad \text{or} \quad \inf U$$
$$\text{or} \quad \inf \{U(P, f) : P \text{ is a partition of } [a, b]\}$$

and the lower – Riemann integral of *f* over [*a*, *b*] is defined by

$$R\int_{-a}^{b} f(x)\,dx = \sup(P) \quad \text{or} \quad \sup L$$
$$\text{or} \quad \sup \{L(P, f) : P \text{ is a partition of } [a, b]\}$$

Thus the two intervals may or may not be equal if $R\int_a^{\bar{b}} f(x)\,dx =$ $R\int_a^b f(x)dx$, then we say that the Riemann integral of *f* – over [*a*, *b*] exists and denote it by $R\int_a^b f(x)\,dx$. It may be noted that in order that the function *f* be Riemann integrable it is necessary for it to be bounded, *i.e.*, *f* is Riemann integrable if $R\int_a^{\bar{b}} f(x)\,dx = R\int_{\underline{a}}^{b} f(x)\,dx = R\int_a^b f(x)\,dx.$

Note:

1. For the shape of convenience, whenever the scope for confusion is not there we shall omit the limits of integration and write simply

$$\overline{\int} f(x)\,dx, \quad \underline{\int} f\,dx, \quad \int f\,dx.$$

4.4 LEBESGUE INTEGRAL

Let *f* is a bounded function defined on [*a*, *b*] then sup of lower sum, *i.e.*, $\sup L(f : P) = \int_{\underline{a}}^{b} f$, where sup is taken over all measurable partition of a [*a*, *b*].

Similarly, we have $\inf U(f : P) = \int_a^{\bar{b}} f$, where in *f* is taken over all measurable partition of [*a*, *b*].

If $\int_{\underline{a}}^{b} f = \int_a^{\bar{b}} f$ then it is called Lebesgue integral and it is denoted as $\int_a^b f$.

Note:

1. $\int_{\underline{a}}^{b} f \le \int_a^{\bar{b}} f$.
2. We abbreviate Riemann integrable as R-integrable and Lebesgue integrable as L-integrable.

Theorem 4.1: If f is a bounded function defined on $[a, b]$ and f is a Riemann–integrable on $[a, b]$ then to show that f is also Lebesgue integral on $[a, b]$, i.e., $L\int_a^b f = R\int_a^b f$.

Proof: Let P is the measurable function of $[a, b]$ in the sense of Lebesgue and σ is subdivision of $[a, b]$ in the sense of Riemann, *i.e.*, $P = \{S_1, S_2, \ldots, S_n\}$ and $\sigma = \{x_1, x_2, \ldots, x_n\}$.

Since the set of number of $U(f:P)$ is a super set of the set of number of $U(f:\sigma)$

$$\Rightarrow \qquad U(f:P) \le U(f:\sigma)$$

$$\Rightarrow \qquad \inf U(f:P) \le \inf U(f:\sigma)$$

By definition, we have $L\int_a^{\overline{b}} f \le R\int_a^{\overline{b}} f$ (4.2)

Similarly, we have $L(f:P) \ge U(f:\sigma)$

$$\Rightarrow \qquad \sup L(f:P) \ge \sup U(f:\sigma)$$

By definition, we have $L\int_{\underline{a}}^b f \ge R\int_{\underline{a}}^b f$ (4.3)

We know that $\int_{\underline{a}}^b f \le \int_a^{\overline{b}} f$ or $L(f:P) \le U(f:P)$

By Eqns. (4.2) and (4.3), we have

$$R\int_{\underline{a}}^b f \le L\int_{\underline{a}}^b f \le L\int_a^{\overline{b}} f \le R\int_a^{\overline{b}} f \tag{4.4}$$

It is given that

$$R\int_{\underline{a}}^b f = R\int_a^{\overline{b}} f = R\int_a^b f \tag{4.5}$$

By Eqns. (4.4) and (4.5), we have

$$R\int_{\underline{a}}^b f = L\int_{\underline{a}}^b f = L\int_a^{\overline{b}} f = R\int_a^{\overline{b}} f$$

$$\Rightarrow \qquad L\int_{\underline{a}}^b f = L\int_a^{\overline{b}} f = L\int_a^b f$$

Hence f, which is Riemann integral, is Lebesgue integral.

4.5 FIRST MEAN VALUE THEOREM

If f is a bounded measurable real valued function such that $f(x) \in [a, b]$ over a measurable set E then $a \cdot m(E) \le \int_E f(x)\,dx \le b \cdot m(E)$.

Proof: Since $a \le f(x) \le b \quad \Rightarrow \quad \left(a - \frac{1}{n}\right) \le f(x) \le \left(b + \frac{1}{n}\right)$

when $n \to \infty, \forall x \in E$

$\Rightarrow \quad \alpha \le f(x) \le \beta$

Now $[\alpha, \beta]$ divide in finite points, $\alpha = y_0 \le y_1 \le \ldots \le y_n = \beta$

$$E_0 = \{x \in E : y_0 \le f(x) \le y_1\}$$

and $$E_r = \{x \in E : y_r < f(x) < y_{n-1}, r = 1, 2, \ldots\}$$

$$\Rightarrow \qquad E = \bigcup_{r=0}^{n-1} E_r \qquad \text{where } E_r \cap E_s = \phi, r \ne s$$

Since *f* is a measurable function over a set *E*.

$$\Rightarrow \qquad m(E) = \sum_{r=0}^{n-1} m(E_r) \tag{4.6}$$

Let P be the measurable partition $P = [E_0, E_1, E_2, \ldots, E_n]$
Since $\alpha \le y_r \le \beta$
Multiply by $m(E_r)$, we have $\alpha\, m(E_r) \le y_r\, m(E_r) \le \beta\, m(E_r)$

$$\Rightarrow \qquad \alpha \sum m(E_r) \le \sum y_r\, m(E_r) \le \beta \sum m(E_r)$$

By Eqn. (4.6), we have

$$\alpha \cdot m(E) \le \sum y_r\, m(E_r) \le \beta \cdot m(E) \tag{4.7}$$

If we take max. $y_{r+1} - y_r \to 0$ and define $\int_E f = \sum y_r\, m(E_r)$

By Eqn. (4.7), we have $\alpha\, m(E) \le \int_E f(x)dx \le \beta \cdot m(E)$

$$\Rightarrow \qquad \left(a - \frac{1}{n}\right) m(E) \le \int_E f(x)dx \le \left(b + \frac{1}{n}\right) m(E)$$

Taking $n \to \infty$, we have $a \cdot m(E) \le \int_E f(x)dx \le b\, m(E)$.

Now we shall see four theorems about how the Lebesgue integral behaves with respect to limit operations. The properties revealed in these theorems are what distinguish the Lebesgue integral from competitor integrals.

4.6 LEBESGUE BOUNDED CONVERGENCE THEOREM

Let $\{f_n\}$ is a sequence of bounded measurable functions defined on a set E of a finite measure if $\exists$ a real number M such that

$|f_n(x)| \le M \ \forall n \in N, \forall x \in E.$ Let $\{f_n\}$ converges in measure to a measurable function f on the set E then show that

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx$$

Proof: Since f_n is bounded and measurable function on a set E

$\Rightarrow$ it is integrable on a set E.

Also it is given $\{f_n\}$ converges in measure

$$\Rightarrow \quad m\big[E|f_n - f| \ge \delta\big] = 0, \quad \text{where } \delta > 0 \tag{4.8}$$

$$\Rightarrow \quad |f_n| \le M, \quad \forall x \in N$$

$$\Rightarrow \quad |f_n(x)| \le M, \quad \forall x \in E$$

$\Rightarrow$ $f(x)$ is bounded and measurable on a set E therefore it is integrable on E.

Let $\alpha = 0$. Now we make two set

$$E_n = E\big(|f_n - f| \ge \lambda\big)$$

and

$$E_n' = E\big(|f_n - f| < \lambda\big)$$

$$\Rightarrow \quad E_n \cap E_n' = \phi \text{ and } E = E_n \cup E_n'$$

From Eqn. (4.8), we have $m(E_n) = 0$ (4.9)

Then we come to the property by countable additive property of the integral

$$\int_E |f_n - f| = \int_{E_n} |f_n - f| + \int_{E_n'} |f_n - f| \tag{4.10}$$

$$|f_n - f| < \lambda \quad \forall x \in E_n'$$

Then by mean value theorem, we have

$$\int_{E_n'} |f_n - f| < \lambda\, m(E_n')$$

$$\Rightarrow \quad \int_{E_n'} |f_n - f| < \lambda\, m(E) \tag{4.11}$$

Choose λ in such a way such that

$$\lambda\, m(E) < \in/2$$

$$\Rightarrow \quad \int_E |f_n - f| < \in/2 \tag{4.12}$$

Also $|f_n - f| \le |f_n| + f \le M + M$ (because each function is bounded)

$$\Rightarrow \quad |f_n - f| \le 2M$$

By mean value theorem on the set E_n

$$\int_E |f_n - f| < 2M\, m(E_n)$$

From Eqn. (4.9), we have

$$\Rightarrow \qquad |m(E_n) - 0| < \in/4M$$

$$\Rightarrow \qquad \int_E |f_n - f| < 2M \times \frac{\in}{4M}$$

$$\Rightarrow \qquad \int_E |f_n - f| \le \in/2 \tag{4.13}$$

From Eqns. (4.10), (4.12) and (4.13), we have

$$\Rightarrow \qquad \int_E |f_n - f| = \in/2 + \in/2$$

$$\Rightarrow \qquad \left|\int_E (f_n - f)\right| \le \int_E |f_n - f| < \in$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E |f_n - f| = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E f_n(x)\,dx - \int_E f(x)\,dx = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx$$

4.7 LEBESGUE DOMINATED CONVERGENCE THEOREM

Let $\{f_n\}$ is a sequence of measurable functions defined over a measurable set E such that $|f_n(x)| < \psi(x) \quad \forall x \in E, \quad \forall n \in N$
where ψ is integrable over E and the sequence $\{f_n\}$ converges in measure to a measurable function f on E then prove that $\int_E f(x)\,dx = \lim\limits_{n\to\infty} \int_E f_n(x)\,dx$

Proof: It is given $\{f_n\}$ be a sequence of measurable function over a measurable set E. Let ψ be integrable over E such that

$$|f_n(x)| < \psi(x) \quad \forall x \in E \quad \text{and } n \in N \tag{4.14}$$

$$\Rightarrow \qquad f_n(x) \text{ is bounded } \forall n$$

Also $f_n(x)$ is given to be measurable. Therefore $\{f_n\}$ is a sequence of bounded measurable function over $E \Rightarrow \{f_n\}$ is Lebesgue integrable over E.

Also it is given $\{f_n\}$ is converge in measure to a measurable function f over E such that

$$\lim_{n\to\infty} m\Big[E\big(|f_n - f| \ge E\big)\Big] = 0, \forall \in > 0 \tag{4.15}$$

Now to show that $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx$

From Eqns. (4.14) and (4.15), we have $|f(x)| < \psi(x)$ (4.16)

$\Rightarrow$ $\psi(x)$ is integrable over E.

Using Eqn. (4.16), we have $f(x)$ is integrable over $E \Rightarrow (f_n - f)$ is integrable over E.

Let $\delta > 0$ be an arbitrary and suppose that

$$E_n = E(|f_n - f| \geq \delta) \quad \text{and} \quad E'_n = E(|f_n - f| < \delta)$$

Then $E = E_n \cup E'_n, E_n \cap E'_n = \phi$

$$\Rightarrow \qquad \lim_{n\to\infty} m(E_n) = 0 \tag{4.17}$$

By countable additively property of the integral

$$\int_E |f_n - f|dx = \int_{E_n} |f_n - f|dx + \int_{E'_n} |f_n - f|dx \tag{4.18}$$

We have $|f_n - f| < \delta,\ \forall x \in E'_n$

Using first mean value theorem, we get

$$\int_{E'_n} |f_n - f|dx < \delta \cdot m(E'_n) \leq \delta m(E)$$

i.e.,

$$\int_{E'_n} |f_n - f|dx < \delta \cdot m(E) \tag{4.19}$$

Take $\in > 0$ and choose δ such that $\delta \cdot m(E) < \in/2$

Eqn. (4.19) becomes $\int_{E'_n} |f_n - f|dx < \in/2$ (4.20)

Now δ is fixed, so we have

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &< \psi + \psi = 2\psi \end{aligned}$$

$$\Rightarrow \qquad |f_n - f| < 2\psi, \quad \forall x \in E$$

On integrating, we get

$$\int_{E_n} |f_n - f|dx < 2\int_{E_n} \psi(x)dx \tag{4.21}$$

$$\because \qquad |f_n| < \psi, \quad \forall x \in E$$

Also it is given $\psi(x) \geq 0 \quad \Rightarrow \quad \left|\int_{E_n} \psi(x)dx\right| = \int_{E_n} \psi(x)dx$

From Eqn. (4.21), we have

$$\int_{E_n} |f_n - f|dx < 2\left|\int_{E_n} \psi(x)dx\right| \tag{4.22}$$

Using Eqn. (4.18), given $\eta > 0, \exists\, n_0 \in N$ such that

$$\Rightarrow\ m(E_n) < \eta$$

$$\therefore\ m(E_n) \geq 0 \quad \Rightarrow \quad \left|\int_{E_n} \psi(x)\,dx\right| < \in/4 \tag{4.23}$$

Because we know that by absolute continuity of the integral which is as if f be integrable over a measurable set E and let $\{E_n\}$ be a sequence of subset of E such that

$$\lim_{n\to\infty} m(E_n) = 0 \text{ then } \lim_{n\to\infty} \int_{E_n} f(x)\,dx = 0$$

or $$\left|\int_{E_n} f(x)\,dx\right| < \in,\ n \geq n_0 \text{ whenever } m(E_n) < \delta$$

Now from Eqns. (4.22) and (4.23), we have

$$\int_{E_n} |f_n - f|\,dx < \in/2 \tag{4.24}$$

From Eqn. (4.18), using Eqns. (4.20) and (4.24), we have

$$\int_E |f_n - f|\,dx < \in/2 + \in/2 = \in$$

or $$\int_E |f_n - f|\,dx < \in \text{ for } n \geq n_0$$

$$\left|\int (f_n - f)\,dx\right| \leq \int_E |f_n - f|\,dx < \in$$

or $$\left|\int (f_n - f)\,dx\right| < \in,\ \forall n \geq n_0$$

$\Rightarrow$ $$\lim_{n\to\infty} \int_E (f_n - f)\,dx = 0$$

or $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx - \int_E f(x)\,dx = 0.$

Hence, $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$

4.8 LEBESGUE MONOTONIC CONVERGENCE THEOREM *OR* BEPPO LEVI'S THEOREM

Let $\{f_n\}$ be a non-decreasing sequence of integrable functions defined over a measurable set E and $\lim\limits_{n\to\infty} f_n = f(x)$ is integrable over E then $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim\limits_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx.$

Proof: Let $\{f_n\}$ be a non-decreasing sequence of integrable functions defined over a measurable set E.

$\Rightarrow$ $$f_1 \leq f_2 \leq f_3 \leq \ldots \leq f_n \leq \ldots$$

$\Rightarrow$ $$f_n - f_1 \geq 0, \quad \forall n \in N$$

Let $f_n - f_1 = \psi_n \quad \Rightarrow \quad \psi_n \geq 0 \quad \forall n \in N$

$\Rightarrow$ $\{\psi_n\}$ is a sequence of non-negative value function.

Also $\{f_n - f_1\}$ is a sequence of integrable function $\Rightarrow$ $\{\psi_n\}$ is the sequence of integrable function.

Case 1: Let ψ_n is a bounded measurable function then we know by Lebesgue bounded convergence theorem $\lim\limits_{n\to\infty} \int_E f_n(x) = \int_E f(x)$.

Case 2: If $\{\psi_n\}$ is unbounded $\Rightarrow$ $\{\psi_n\}$ is bounded measurable function. Then by bounded convergence theorem, we have

$$\lim_{n\to\infty} \int_E (\psi_n(x))_m \; dx = \int_E \lim_{n\to\infty} (\psi_n(x))_m$$

Let $n \to \infty$ then $(\psi_n(x))_m = \psi_n(x)$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E \psi_n(x)\,dx = \int_E \lim_{n\to\infty} \psi_n(x)$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E (f_n - f_1)\,dx = \int \lim_{n\to\infty} (f_n - f_1)$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E f_n(x)\,dx - \int_E f_1\,dx = \lim_{n\to\infty} f_n(x)\,dx - \int f_1\,dx$$

Hence $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int \lim\limits_{n\to\infty} f_n(x)\,dx$

Another Proof: Let $< f_n >$ be a non-decreasing sequence of integrable defined over a measurable set E.

Let $\lim\limits_{n\to\infty} f_n$ be integrable over E. To prove that

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx.$$

Since $< f_n >$ a non decreasing sequence and hence $f_1 \leq f_2 \leq f_3 \leq \ldots.$

This implies $f_1 \leq f_n, \forall n \quad \Rightarrow \quad f_n - f_1 \geq 0 \quad \Rightarrow \quad \psi_n \geq 0, \quad \forall n$ on taking $\psi_n = f_n - f_1$

Moreover $< f_n >$ is a sequence of integrable functions implies that $< \psi_n >$ is a sequence of integrable function. Finally $< \psi_n >$ is a sequence of non-negative integrable function. Applying this to the Lebesgue bounded convergence theorem, we have $\lim\limits_{n\to\infty} \int_E \psi_n\,dx = \int_E \lim\limits_{n\to\infty} \psi_n\,dx$

$$\text{or} \qquad \lim_{n\to\infty} \int_E (f_n - f_1)\,dx = \int_E \lim_{n\to\infty} (f_n - f_1)\,dx$$

$$\text{or} \qquad \lim_{n\to\infty} \int_E f_n\,dx - \int_E f_1\,dx = \int_E \lim_{n\to\infty} f_n\,dx - \int_E f_1\,dx$$

$$\text{or} \qquad \lim_{n\to\infty} \int_E f_n\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx$$

Hence, theorem is also known as Lebesgue's monotonic convergence theorem.

4.9 FATOU'S LEMMA

Let $\{f_n\}$ be a sequence of non-negative integrable functions defined over a measurable set E such that (i) $\liminf\limits_{n\to\infty} f_n = f$ almost everywhere on E (ii) $\liminf\limits_{n\to\infty} \int f_n(x)dx < \infty$. Then $\int_E f(x)dx \le \liminf\limits_{n\to\infty} \int_E f_n(x)dx$

Proof: Let $\{f_n\}$ be a sequence of non-negative integrable function defined over a measurable set E such that

(i) $f = \liminf\limits_{n\to\infty} f_n$ almost everywhere on E (4.25)

(ii) $\liminf\limits_{n\to\infty} \int_E f_n(x)dx < \infty$ (4.26)

To show that $\int_E f(x)dx \le \liminf\limits_{n\to\infty} \int_E f_n(x)dx$

Define $g_k(x) = \inf\limits_{n \ge k}\{f_n(x)\} = \inf\{f_n(x) : n \ge k\}$ (4.27)

$\Rightarrow g_n(x) \le f_n(x) \quad \forall n \in N \Rightarrow \int_E g_n(x)dx \le \int_E f_n(x)dx$

Consequently, $\lim\limits_{n\to\infty} \int_E g_n(x)dx \le \liminf\limits_{n\to\infty} \int_E f_n(x)dx$ (4.28)

Since $\{g_n\}$ is an increasing sequence of non-negative integrable functions and hence by Lebesgue monotonic converges theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \int_E g_n(x) &= \int_E \lim_{n\to\infty} g_n(x)dx \\ &= \int_E \liminf_{n\to\infty} f_n(x)dx && \text{[from Eqn. (4.27)]} \\ &= \int_E f(x)dx && \text{[from Eqn. (4.25)]}\end{aligned}$$

i.e., $\lim\limits_{n\to\infty} \int_E g_n(x)dx = \int_E f(x)dx$

Using Eqn. (4.25), we get

$$\int_E f(x)dx \le \liminf_{n\to\infty} \int_E f_n(x)dx$$

Theorem 4.2: Let $\{f_n\}$ is a sequence of integrable functions which converges in mean to a function f then show that $\{f_n\}$ converges to f in measure.

Proof: Let $\{f_n\}$ is a sequence of integrable function which converges in mean to a function f then $E_n = E\left(|f_n - f| \ge \delta\right)$

$$\Rightarrow \qquad |f_n - f| \geq \delta, \quad \forall x \in E_n$$

Then by first mean value theorem, we have

$$\int_{E_n} |f_n - f| \geq \delta\, m(E) \tag{4.29}$$

Also it is given $\{f_n\}$ converges to f in mean to function f if

$$\lim_{n\to\infty} \int_{E_n} |f_n - f|\, dx = 0 \tag{4.30}$$

From Eqns. (4.29) and (4.30), we have

$$\lim_{n\to\infty} \delta \cdot m(E_n) = 0 \qquad \text{since } \delta > 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} \cdot\, m(E_n) = 0$$

Hence, $\{f_n\}$ converges to f in measure.

Theorem 4.3: Suppose f is measurable on a measurable set E. To prove that f is integrable if $|f|$ is integrable that $\left|\int_E f\right| \leq \int_E |f|$.

Proof:

(I) Let f be a measurable function on a measurable set E. Suppose f is Lebesgue integrable over E. Then its positive and negative parts f_+ and f_- are also Lebesgue integrable over E.

But $|f| = f_+ + f_-$, this means that $|f|$ is Lebesgue integrable over E.

(II) Let f be a measurable function on a measurable set E such that $|f|$ is Lebesgue integrable over E. Then $\int_E |f| < \infty$

Since $0 \leq f_+(x) \leq |f(x)|, \quad \forall x \in E$

Hence, $\int_E f_+(x)\, dx \leq \int_E |f(x)|\, dx < \infty$

$$\int_E f_+(x)\, dx < \infty$$

$\Rightarrow$ f_+ is Lebesgue integrable.

Similarly, we can prove that f_- is Lebesgue integrable. But $|f| = f_+ + f_-$

Hence, $|f|$ is Lebesgue integrable on E.

(III) Let $f(x) \geq 0$ on E_1 and $f(x) < 0$ on E_2.

Then $E_1 \cap E_2 = \phi$ and $E = E_1 \cup E_2$.

By countable property of the integrable

$$\int_E |f|\,dx = \int_{E_1} |f|\,dx + \int_{E_2} |f|\,dx$$

and $\int_E f\,dx = \int_{E_1} f\,dx + \int_{E_2} f\,dx \;=\; \int_{E1} |f|\,dx - \int_{E_2} |f|\,dx$

$$\Rightarrow \quad \left|\int_E f\,dx\right| \le \left|\int_{E_1} |f|\,dx\right| + \left|\int_{E_2} |f|\,dx\right|$$

$$= \int_{E_1} |f| + \int_{E_2} |f|$$

$$= \int_E |f|$$

Hence, $\left|\int_E f\right| \le \int_E |f|.$

Solved Problems

Problem 4.1: Shown example that a function which Lebesgue integrable is not necessary Riemann integrable.

or

Let the given function $h(x) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$

Prove that this function is a Lebesgue integrable.

Solution: Let A is the set of all points of irrational number in [0, 1] and B is the set of all points of rational number in [0, 1].

$$\Rightarrow \quad A \cup B = [0, 1] \text{ and } A \cap B = \phi.$$

We know that measure of null set is null.

i.e., $m(A \cap B) = 0$ because $A \cap B = \phi$.

Then $P = \{A, B\}$

Since the value of h is identically 1 on the set A and zero on the set B

$$\Rightarrow \quad M[h : A] = 1$$

$$\Rightarrow \quad m[h : A] = 1$$

Similarly, $M[h : B] = 0 = m[h : B]$

Hence, $U(h : P) = \sum_{r=1}^{2} M[h : S_r] \times m(S_r) \qquad S_r = \{A, B\}$

$$= M[h : A] \times m[A] + M[h : B] \times m(B)$$

$$= m(A) = 1$$

Similarly,

$$
\begin{aligned}
L[h:P] &= \sum_{r=1}^{2} m[h:S_r] \times m(S_r) \\
&= m[h:A] \times m[A] + m[h:B] \times m(B) \\
&= 1 \times m(A) + 0 \times m(B) \\
&= m(A) = 1
\end{aligned}
$$

Hence, $U(h:P) = 1 = L(h:P)$, *i.e.*, h is Lebesgue integrable.

$\Rightarrow \qquad h \in L[0,1]$

Converse, let $h(x)$ is discontinuous and point of discontinuous for a measure set since

$\Rightarrow \quad M_r = 1, m_r = 0$ like this.

$$
\begin{aligned}
\text{Riemann lower sum } &= \sum m_r\, S_r \\
&= \sum 0.1 = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Riemann upper sum } &= \sum M_r\, S_r \\
&= \sum 1.1 = 1
\end{aligned}
$$

Sup (lower sum) $= R\int_{\underline{a}}^{b} f = \int_{\underline{0}}^{1} h = 0$

Inf (upper sum) $= R\int_{a}^{\bar{b}} f = R\int_{0}^{\bar{1}} h = 1$

$\Rightarrow \qquad R\int_{\underline{0}}^{1} h \neq R\int_{0}^{\bar{1}} h \neq R\int_{0}^{1} h$

$\Rightarrow \qquad h \notin, R[0,1]$

Hence, h is not Riemann integrable.

Problem 4.2: If the function f is constant on the measurable set E say $f(x) = e$ then $\int_E f(x)\,dx = e \cdot m(E)$.

Solution: Let f be real valued function defined over a measurable set E such that

$$f(x) = e \,(\text{constant}) \;\; \forall x \in E$$

To prove that $\int_E f(x)\,dx = e \cdot m(E)$

We have $f(x) = e \quad \forall x \in E$

$\Rightarrow \quad e \leq f(x) \leq e \quad \forall x \in E$

Using first mean value theorem

$$e\,m(E) \le \int_E f(x)\,dx \le e \cdot m(E)$$

$\Rightarrow$
$$\int_E f(x)\,dx = e \cdot m(E)$$

Problem 4.3: Using Lebesgue dominated convergence theorem to evaluate the following integral: $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx$, where $f_n(x) = \dfrac{n^{3/2}}{1+n^2x^2}$, $0 \le x \le 1$, $n = 1, 2, 3, \ldots$.

Solution: Given that $f_n(x) = \dfrac{n^{3/2}}{1+n^2x^2}$, $0 \le x \le 1, \quad n = 1, 2, 3, \ldots$

It can be written as $f_n(x) = \dfrac{1}{x}\dfrac{n^{3/2}x^2}{1+n^2x^2} \le \dfrac{1}{n} = \psi(x)$ (say)

$\Rightarrow$
$$f_n(x) \le \psi(x).$$

Also $\psi(x)$ is integrable in $[a, b]$. Hence by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \frac{n^{3/2}x}{1+n^2x^2}\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \frac{n^{3/2}x}{n^2\left(\dfrac{1}{n^2}n^2\right)}\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \frac{x}{n^{1/2}\left(\dfrac{1}{n^2}+x^2\right)}\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \frac{x}{\sqrt{n}\left(\dfrac{1}{n^2}n^2\right)}\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}\right)\left(\frac{x}{\dfrac{1}{n^2}+x^2}\right)dx$$

$$= \int_0^1 0 \cdot \left(\frac{x}{0+x^2}\right)dx$$

$$= \int_0^1 0 \cdot \left(\frac{1}{x}\right) dx$$

$$= \int_0^1 0 \cdot dx$$

$$= 0$$

Hence, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$

Problem 4.4: Show that the theorem of bounded convergence is applicable to $f_n(x) = \dfrac{nx}{1+n^2x^2}$, for $0 \le x \le 1$.

Solution: Given that $f_n(x) = \dfrac{nx}{1+n^2x^2}$, for $0 \le x \le 1$

It can be written as $f_n(x) = \dfrac{nx}{1+n^2x^2} = \dfrac{\frac{nx}{nx}}{\frac{1}{nx} + \frac{n^2x^2}{nx}} = \dfrac{1}{\frac{1}{nx} + nx}$

$$f_n(x) = \frac{1}{\left(\frac{1}{\sqrt{nx}} - \sqrt{nx}\right)^2 + 2} \le \frac{1}{2} = M \text{ (say)}$$

Then $|f_n(x)| \le M$. Hence, by Lebesgue convergence theorem, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx \tag{4.31}$$

L.H.S. of Eqn. (4.31), we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx$$

Put $1 + n^2x^2 = t \quad \Rightarrow \quad 2n^2x\,dx = dt \quad \Rightarrow \quad dx = dt/2n^2x$

$$\therefore \quad \lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \lim_{n\to\infty} \int_1^{1+n^2} \frac{nx}{t}\,\frac{dt}{2n^2x}$$

$$= \lim_{n\to\infty} \int_1^{1+n^2} \frac{1}{2n}\,\frac{dt}{t}$$

$$= \lim_{n\to\infty} \frac{1}{2n} \int_1^{1+n^2} \frac{dt}{t}$$

$$= \lim_{n\to\infty} \left[\frac{1}{2n} (\log t)_1^{1+n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2n}(\log(1+n^2)-\log(1))\right]$$

$$= \lim_{n\to\infty}\frac{1}{2n}\log(1+n^2) \qquad \left[\therefore \text{ form } \frac{\infty}{\infty}\right]$$

$$= 0$$

By using L. Hospital rule, we have $\{\because \log\infty = \infty, \log -\infty = 0\}$
Now we have

$$\int_0^1 \lim_{n\to\infty}\frac{nx}{1+n^2x^2}\,dx = \int_0^1 \lim_{n\to\infty}\frac{nx}{n^2x^2\left(1+\frac{1}{n^2x^2}\right)}\,dx$$

$$= \int_0^1 \lim_{n\to\infty}\frac{1}{nx\left(1+\frac{1}{n^2x^2}\right)}\,dx$$

$$= \int_0^1 0\cdot dx$$

$$= 0$$

Thus, $\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^2}\,dx = 0 = \int_0^1 \lim_{n\to\infty}\frac{nx}{1+n^2x^2}\,dx$

Hence, the theorem of bounded convergence is applicable to $f_n(x)$.

Problem 4.5: Show that if $\alpha > 1$, then show that

$$\int_0^1 \frac{x\sin(x)}{1+(nx)^\alpha}\,dx = 0(n^{-1}), \text{ if } \quad n\to\infty.$$

Solution: Suppose the sequence $< f_n(x) >$ defined by

$$f_n(x) = \frac{x\sin(x)}{1(nx)^\alpha} \quad \text{for} \quad n = 1, 2, 3$$

Then $|f_n(x)| \le 1$ as $|\sin x| \le 1$ and $\alpha. > 1$
Take $g(x) = 1, \ \forall x$
Then $|f_n(x)| \le g(x), \ \forall x$
Using dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^1 n\cdot f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} n\,f_n(x)\,dx$$

$$= \int_0^1 \lim_{n\to\infty}\frac{nx\sin(x)}{1+(nx)^\alpha}\,dx$$

$$= \int_0^1 \lim_{n\to\infty}\frac{x\sin(x)}{n^{\alpha-1}\left(\frac{1}{n^\alpha}+x^\alpha\right)}\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \frac{x\sin(x)}{\infty\cdot(0+x^{\alpha})}\cdot dx$$

$$= \int_0^1 0\cdot dx = 0 \quad \text{if} \quad \alpha > 1$$

or $$\lim_{n\to\infty}\int_0^1 n\,f_n(x)\,dx = 0$$

or $$\int_0^1 f_n(x)\,dx = 0\cdot(n^{-1})$$

Hence, $$\int_0^1 \frac{x\sin(x)}{1+(nx)^{\alpha}}\,dx = 0(n^{-1})$$

Problem 4.6: If $\alpha > 0$, prove that $\lim\limits_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n x^{\alpha-1}dx = \int_0^{\infty} e^{-x}x^{\alpha-1}dx,$ where the integral are taken in the Lebesgue sense.

Solution: Let $f_n(x) = \left(1-\frac{x}{n}\right)^n x^{\alpha-1}, \quad \alpha > 0$

Then $f_n(x) \le \psi_n(x), \forall n,$ where $\psi(x) = e^{-x}x^{\alpha-1}$
Also $\psi(x) \in L(0,\infty) \Rightarrow \psi(x)$ is Lebesgue integrable.
The by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^n f_n(x)\,dx = \int_0^{\infty} \lim_{n\to\infty} f_n(x)\,dx$$

$$\Rightarrow \quad \lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n x^{\alpha-1}\,dx = \int_0^{\infty}\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n x^{\alpha-1}\,dx$$

$$= \int_0^{\infty} e^{-x}\,x^{\alpha-1}\,dx \qquad \left[\therefore \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}\right]$$

Hence, $$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n x^{\alpha-1}\,dx = \int_0^{\infty} e^{-x}x^{\alpha-1}dx$$

Problem 4.7: Prove that the function $\frac{\sin x}{x}$ is not Lebesgue integrable over $[0,\infty]$.

Solution: To show that the function $\frac{\sin x}{x}$ is not L-integrable if we show that $\frac{\sin x}{x}$ is not summable over $[0,\infty]$ we can conclude the result. For this we must show that $\int_0^{\infty}\frac{|\sin x|}{x} = \infty.$

Consider the integral $\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}, \qquad x = y + (r-1)\pi$

$$= \sum_{r=1}^{n} \int_0^{\pi} \frac{|\sin\{y+(r-1)\pi\}|}{y+(r-1)\pi} dy$$

$$= \sum_{r=1}^{n} \int_0^{\pi} \frac{\sin y \, dy}{y+(r-1)\pi}$$

$$\geq \sum_{r=1}^{n} \frac{1}{r\pi} \int_0^{\pi} |\sin y| dy = \sum_{r=1}^{n} \frac{2}{r\pi}$$

Now we have $\therefore \lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x}, dy \geq \frac{2}{\pi} \sum_{r-1}^{n} \frac{1}{r} = \frac{2}{\pi} \times \infty = \infty$

i.e., $\int_0^{\infty} \frac{|\sin x|}{x} dx = \infty.$

Hence, the function $\frac{\sin x}{x}$ is not Lebesgue integrable over $[0, \infty]$.

RECAPITULATION

- Henri Lebesgue introduced the Lebesgue integral in 1904.
- Every continuous function is integrable.
- If f is a bounded measurable real valued function such that $f(x) \in [a, b]$ over a measurable set E then $a \cdot m(E) \leq \int_E f(x)dx \leq b \cdot m(E)$.
- If f is a bounded function defined on [a, b] and f is a Riemann–integrable on [a, b] then $L\int_a^b f = R\int_a^b f$.
- **Lebesgue bounded convergence theorem:** Let $\{f_n\}$ is a sequence of bounded measurable functions defined on a set E of a finite measure if $\exists$ a real number M such that $|f_n(x)| \leq M \quad \forall n \in N, \quad \forall x \in E$. Let $\{f_n\}$ converges in measure to a measurable function f on the set E then $\lim_{n\to\infty} \int_E f_n(x)dx = \int_E \lim_{n\to\infty} f_n(x)dx = \int_E f(x)dx$
- **Lebesgue dominated convergence theorem:** Let $\{f_n\}$ is a sequence of measurable functions defined over a measurable set E such that $|f_n(x)| < \psi(x) \quad \forall x \in E, \quad \forall n \in N$ where ψ is integrable over E and the sequence $\{f_n\}$ converges in measure to a measurable function f on E then $\int_E f(x)dx = \lim_{n\to\infty} \int_E f_n(x)dx$

- **Lebesgue monotonic convergence theorem** ***or*** **Beppo Levi's theorem:** Let $\{f_n\}$ be a non-decreasing sequence of integrable functions defined over a measurable set E and $\lim\limits_{n\to\infty} f_n = f(x)$ is integrable over E then $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim\limits_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx.$
- **Fatou's Lemma:** Let $\{f_n\}$ be a sequence of non-negative integrable functions defined over a measurable set E such that (i) $\liminf\limits_{n\to\infty} f_n = f$ almost everywhere on E (ii) $\liminf\limits_{n\to\infty} \int f_n(x)\,dx < \infty.$ Then $\int_E f(x)\,dx \le \liminf\limits_{n\to\infty} \int_E f_n(x)\,dx$
- If the function f is constant on the measurable set E say $f(x) = C$ then $\int_E f(x)\,dx = C \cdot m(E).$

EXERCISES

Multiple-choice Questions

4.1 If f is a bounded measurable real valued function such that $f(x) \in [a, b]$ over a measurable set E then

(a) $a \cdot m(E) \le \int_E f(x)\,dx \le b \cdot m(E)$
(b) $a \cdot m(E) < \int_E f(x)\,dx < b \cdot m(E)$
(c) $a \cdot m(E) \ge \int_E f(x)\,dx \ge b \cdot m(E)$
(d) None of these.

4.2 If f is a bounded function defined on $[a, b]$ and f is a Riemann–integrable on $[a, b]$ then

(a) $L\int_a^b f = R\int_a^b f$
(b) $L\int_a^b f > R\int_a^b f$
(c) $L\int_a^b f < R\int_a^b f$
(d) None of these.

4.3 Let $< f_n >$ be a non-decreasing sequence of integrable functions defined over a measurable set E. Let $\lim\limits_{n\to\infty} f_n$ be integrable over E. Then $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim\limits_{n\to\infty} f_n(x)\,dx.$ This is known as

(a) Lebesgue convergence theorem
(b) Beppo Levi's theorem

(c) First fundamental theorem

(d) None of these.

State True or False

4.1 Let $\{f_n\}$ be a sequence of non-negative integrable functions defined over a measurable set E such that (i) $\liminf_{n\to\infty} f_n = f$ almost every where on E (ii) $\liminf_{n\to\infty} \int f_n(x)\,dx < \infty$. Then $\int_E f(x)\,dx \le \liminf_{n\to\infty} \int_E f_n(x)\,dx$

4.2 If the function f is constant on the measurable set E say $f(x) = C$ then $\int_E f(x)\,dx = C/m(E)$.

4.3 Every continuous function is integrable.

Fill in the Blanks

4.1 Henri Lebesgue introduced the Lebesgue integral in _______

4.2 Let $\{f_n\}$ is a sequence of bounded measurable functions defined on a set E of a finite measure if $\exists$ a real number M such that $|f_n(x)| \le M \quad \forall n \in N, \ \forall x \in E$. Let $\{f_n\}$ converges in measure to a measurable function f on the set E then _______

4.3 If f_1 and f_2 are two bounded and integrable function on $[a, b]$ then $f = f_1 + f_2$ is also integrable on $[a, b]$ and $\int_a^b f\,dx$ _____ $\int_a^b f_1\,dx + \int_a^b f_2\,dx$.

Exercises

4.1 What do you mean by Lebesgue integral?

4.2 Explain Riemann theory of integral.

4.3 State and prove Lebesgue bounded convergence theorem.

4.4 State and prove Lebesgue dominated convergence theorem.

4.5 State and prove Lebesgue monotonic convergence theorem or Beppo Levi's theorem.

4.6 State and prove First mean value theorem.

4.7 State and prove Fatou's Lemma.

4.8 If f, g are non-negative measurable function defined on E, then $\int_E (f+g) = \int_E f + \int_E g$.

4.9 If $f, g \in L\,[a, b]$ and if $f(x) \le g(x)$ almost everywhere on $[a, b]$, then show that $\int_a^b f \le \int_a^b g$.

4.10 If $\int_E f(x)\,dx = 0$ for every measurable subset E of a measurable set A, show that $f(x) = 0$ almost everywhere on A.

4.11 Show that a function which is Lebesgue integrable is not necessarily to Riemann integrable.

4.12 Let f be a function defined on the interval $[a, b]$ such that

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Is f integrable in Riemann sense? Is f integrable in the Lebesgue sense?

4.13 Give an example of a function which is not integrable in the sense of lebesgue.

ANSWERS

Multiple-choice Questions

4.1 (a) 4.2 (a) 4.3 (b)

State True or False

4.1 T 4.2 F 4.3 T

Fill in the Blanks

4.1 1904

4.2 $\lim\limits_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim\limits_{n\to\infty} f_n(x)\,dx = \int_E f(x)\,dx$

4.3 =

5

CHAPTER

Differentiation and Integration

5.1 INTRODUCTION

In this chapter we shall discuss the Lebesgue theorem on differentiability of monotonic functions to a larger class of functions (class of functions of bounded variation), integration and differentiation functions of the finite variation. Functions of bounded variation are important not only in differentiation but also the in the study of function analysis, Fourier series, complex analysis, etc.

The chapter begins with a preliminary discussion of continuous functions, absolute continuous function, differentiable, monotonic function, function of bounded variation, Lipschitz condition, cover in the sense of Vitali, Vitali's Lemma, Lebesgue point and Lebesgue set and Fundamental Theorem of Integral Calculus.

5.2 IMPORTANT TERMINOLOGY

Before discussing the differentiation and integration, we shall discuss certain necessary preliminaries.

5.2.1 Continuous Function

A real valued function $f(x)$ is said to be continuous at x_0 if given $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \in$ whenever $|x - x_0| < \delta$.

5.2.2 Absolute Continuous Function

Let f be a finite real valued function defined over a closed interval $[a, b]$ such that given $\in > 0, \exists\, \delta > 0$ such that $\left|\sum_{r=1}^{n}\left[f(b_r) - f(a_r)\right]\right| < \in$ whenever

$\sum\limits_{r=1} (b_r - a_r) < \delta$, where $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ without altering the sense of definition, we replace the condition $\left|\sum\limits_{r=1}^{n} [f(b_r) - f(a_r)]\right| < \in$ by the stronger condition $\sum\limits_{r=1}^{n} |f(b_r) - f(a_r)| < \in$. Then the function $f(x)$ is said to be absolutely continuous in the interval $[a, b]$.

5.2.3 Differentiable

If $f(x)$ be a continuous function, then $F(x) = \int_a^x f(t)\,dt + F(a)$ is said to be differentiable, where $F(a)$ being any finite constant.

5.2.4 Monotonic Functions

Let f be a real valued function defined on $[a, b]$. Then f is monotonic increasing on $[a, b]$, if $x < y \Rightarrow f(x) \le f(y)$. Also f is monotonic decreasing on $[a, b]$, if $x > y \Rightarrow f(x) \ge f(y)$.

Increasing and decreasing function are known as monotonic functions.

5.2.5 Function of Bounded Variation

A real valued function $f : [a, b] \to R$ is said to be of bounded variation if there is a constant c such that $\sum\limits_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le c$ hold for any position $a = x_0 < x_1 < x_2 \ldots < x_n = b$. The total variation $\overset{b}{\underset{a}{V}}$ of f on $[a, b]$ is

$$\overset{b}{\underset{a}{V}}(f) = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : a = x_0 < x_1 < x_2 \ldots < x_n = b\right\}$$

where the supremum is taken over all partitions of $[a, b]$.

Note:

1. A real valued function f is said to be of bounded variation in a closed interval $[a, b]$ if and only if it can be expressed as $f(x) = \phi(x) - \psi(x), \forall x \in [a, b]$ where $\phi(x)$ and $\psi(x)$ are monotonic functions.
2. If f is a monotonic function on $[a, b]$, then f is bounded variation and $\overset{b}{\underset{a}{V}}(f) = |f(b) - f(a)|$.
3. If P_1 and P_2 are two partition and $P_1 \subset P_2 \Rightarrow \overset{b}{\underset{a}{V}}(f : P_1) \le \overset{b}{\underset{a}{V}}(f : P_2)$
4. If $a < b < c$ then $\overset{b}{\underset{a}{V}} f < \overset{c}{\underset{a}{V}} f$.

5.2.6 Lipschitz Condition

A function *f* is said to satisfy Lipschitz condition if there exists a positive constant *M* such that $|f(x) - f(y)| \leq M|x - y|$.

5.2.7 Cover in the Sense of Vitali

Let *E* be a set and *C* be the collection of intervals. The set *E* is known as *covered by C in the sense of Vitali* if for every $\in > 0$ and any $x \in E$ there exists an interval *I* in *C* such that $x \in I$ and $m(I) < \in$.

5.2.8 Vitali's Lemma

If *E* is a set of finite measure and *C* is a collection of intervals which *cover E in the sense of Vitali*, then for a given $\in > 0$ there exists a finite, pairwise-disjoint subclass $\{C_1, C_2, \ldots, C_{n_0}\}$ of *C* such that $m\left(E - \overset{n_0}{\underset{n=1}{\cup}} C_n\right) < \in$.

5.3 LEBESGUE POINT AND LEBESGUE SET

If $\lim_{n \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0$, then x is said to be a Lebesgue point of the function $f(t)$. The set of all lebesgue point in [*a*, *b*] of *f* is called the lebesgue set of the function *f*.

Theorem 5.1: A monotonic function on [*a*, *b*] has finite variation on [*a*, *b*].

or

A monotonic function defined on closed interval [*a*, *b*] is of bounded variation.

Proof: Let *f* be an increasing function defined on the interval [*a*, *b*] so that

$$f(x_r) \leq f(x_{r+1}) \text{ for } x_r < x_{r+1}$$

$$\Rightarrow \quad f(x_{r+1}) - f(x_r) \geq 0 \tag{5.1}$$

Now divide the closed interval [*a*, *b*] by means of points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

From Eqn. (1), we have $|f(x_{r+1}) - f(x_r)| \geq 0$

$$\text{or} \quad |f(x_{r+1}) - f(x_r)| = f(x_{r+1}) - f(x_r) \tag{5.2}$$

$\because \; x_{r+1} > x_r$

Now we have $V = \sum_{r=0}^{n-1} |f(x_{r+1}) - f(x_r)|$

$$= \sum_{r=0}^{n-1} \{f(x_{r+1}) - f(x_r)\}$$

$$= f(x_n) - f(x_0)$$

$$V = f(b) - f(a) = \text{finite number}$$

Since f is monotonic $\Rightarrow f(b), f(a)$ are finite number.

$\Rightarrow V$ is a finite number independent of mode of subdivision.

Hence f is bounded variation or V is finite or total variation is constant.

5.4 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $f(x)$ is continuous and if $f(x) = \int_a^b f(t)\,dt + F(a)$. Then the theorem states that $F'(x) = f(x)$ i.e., to say differentiation and integration are reverse processes.

Proof: Let $f(x)$ is continuous and $f(x) = \int_a^b f(t)\,dt + F(a)$ (5.3)

To show that $F'(x) = f(x)$.

We prove this theorem by using elementary theory of integral calculus.

From Eqn. (5.3), we have

$$F(x) = \int_a^x f(t)\,dt + F(a) \text{ and } F(x+h) = \int_a^{x+h} f(t)\,dt + F(a)$$

Now we have

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt + F(a) - \int_a^x f(t)\,dt - F(a) \\ &= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x) \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\int_x^{x+h} dt \qquad \left[\therefore \int_x^{x+h} dt = 1\right] \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt - \int_x^{x+h} f(x)\,dt \\ &= \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt \end{aligned}$$

Now we have

$$\begin{aligned} \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \frac{1}{|h|}\left|\int_x^{x+h} [f(t) - f(x)]\,dt\right| \\ &\le \frac{1}{|h|}\int_x^{x+h} |f(t) - f(x)|\,dt \end{aligned} \tag{5.4}$$

Since $f(x)$ is continuous and therefore given $\in > 0,$ there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \in \quad \text{whenever } |t - x| < \delta$$

From Eqn. (3), we have

$$\left| \frac{f(x+h) - F(x)}{h} - f(x) \right| < \frac{1}{|h|} \int_x^{x+h} \in dt, \qquad |h| < \delta$$

or $$\left| \frac{f(x+h) - F(x)}{h} - f(x) \right| < \in, \quad |h| < \delta$$

Making $h \to 0$ and consequently $\in \to 0$, we have

$\lim\limits_{n \to 0} \dfrac{f(x+h) - F(x)}{h} - f(x) = 0$ is almost everywhere.

or $\lim\limits_{n \to 0} \dfrac{f(x+h) - F(x)}{h} = f(x)$

or $\dfrac{d \cdot F(x)}{dx} = f(x)$

Hence, $F'(x) = f(x)$ is almost everywhere.

Theorem 5.2: Prove that every absolutely continuous function is of bounded variation.

Proof: Since f is absolutely continuous function on a closed interval $[a, b]$ $\Rightarrow \exists, \delta > 0$ such that

$$\sum_{r=1}^{n} |f(b_r) - f(a_r)| < \in \tag{5.5}$$

Whenever $$\sum_{r=1}^{n} (b_r - a_r) < \delta$$

For all numbers $a_1, b_1, a_2, b_2, \ldots, a_n\, b_n$

Where $a = a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n = b$

Consider another subdivision of a close interval $[a, b]$ [or refinement of b by adjoining same addition point of p (partition) in such a way such that all the intervals can be divided into r parts of total length $< \delta$.

Let r subinterval $[c_0, c_1], [c_1, c_2], \ldots, [c_{r-1}, c_r]$ such that

$$a = c_0 < c_1, \ldots < c_r = b$$

and $c_{r+1} - c_r < \delta.$

From Eqn. (5.5), we have

$$\Rightarrow \qquad \sum_i |f(x_{i+1}) - f(x_i)| < \in \text{ where } x_{i+1}, x_i \in [c_r, c_{r+1}]$$

The total variation $\overset{c_{r+1}}{\underset{c_r}{V}} f < \in$

$$\Rightarrow \qquad \overset{b}{\underset{a}{V}}(f) = \overset{c_1}{\underset{c_0}{V}}(f) + \overset{c_2}{\underset{c_1}{V}}(f) + \ldots + \overset{c_r}{\underset{c_{r-1}}{V}}(f)$$

$$< \in + \in + \ldots + \in$$

$$= r \in$$

i.e., $\overset{b}{\underset{a}{V}}(f) < r \cdot \in$ (which is finite)

Hence, f is of bounded variation.

Theorem 5.3: Let f and g be integrable functions over $[a, b]$. Suppose $F(x) = \int_a^x f(t)\,dt + F(a),\ G(x) = \int_a^x g(t)\,dt + G(a), \quad \forall x \in [a, b]$.

Then $\int_a^b F(t)g(t)\,dt + \int_a^b f(t)G(t)\,dt = F(b)G(b) - F(a)G(a)$.

Proof: Since $F(x)$ and $G(x)$ are integrable functions $\Rightarrow F, G$ are absolutely continuous function on a closed interval $[a, b]$ then we know that by a theorem that F, G are also absolutely continuous.

Also we know that by a theorem if F is absolutely continuous on $[a, b]$ and F' is integrable on a closed interval $[a, b]$ then $F'(x) = f(x)$ almost everywhere *or* F is an indefinite integrable if its all derivative then $\int_a^b F(t)\,dt = F(x) - F(a)$

Also $\int_a^b (FG)' = (FG)(b) - (FG)(a)$ (5.6)

$$= F(b)\,G(b) - f(a)\,G(a)$$

Let $F(x) = \int_a^x f(t)\,dt + F(a)$

Then $F'(x) = f$ almost everywhere and

$G'(x) = g$ almost everywhere in $[a, b]$ (5.7)

Also $(FG)' = FG' + F'G$

$= Fg + fG$ almost everywhere {from Eqn. (5.7) in $[a, b]$}

From Eqns. (5.6) and (5.7), we have

$$\int_a^b F(t)g(t)\,dt + \int_a^b f(t)G(t)\,dt = F(b)G(b) - F(a)G(a)$$

Theorem 5.4: If f is bounded variation on $[a, b]$ then $V = P + N$ and $P - N = f(b) - f(a)$

where *V, P, N* respectively denote total, positive and negative variation of f on $[a, b]$.

Proof: Consider $V = \sum_{r=1}^{n-1} |f(x_{r+1}) - f(x_r)|$

Here the closed interval $[a, b]$ is divided by means of points $a = x_0 < x_1, < x_2 < \ldots < x_n = b$.

Let p be the sum of those differences $f(x_{r+1}) - f(x_r)$ which are positive $-n$ that the sum of those differences which are negative. Evidently,

$$v = p + n,\ f(b) - f(a) = p - n$$

From which we get

$$v + f(b) - f(a) = 2p$$

and $$v - f(b) + f(a) = 2n$$

i.e., $$v = 2p + f(a) - f(b) \tag{5.8}$$

and $$v = 2n + f(b) - f(a) \tag{5.9}$$

Now set $P = \sup p,\ N = \sup n,\ V = \sup v$ where we take the supremum over all possible.

Subdivision of [a, b] taking supremum in Eqns. (5.8) and (5.9), we have

$$V = 2P + f(a) - f(b) \tag{5.10}$$

$$V = 2N + f(b) - f(a) \tag{5.11}$$

Adding Eqns. (5.10) and (5.11), we get

$$2V = 2P + 2N$$

or $$V = P + N$$

Now subtracting Eqn. (5.10) to Eqn. (5.11), we get

$$0 = 2(P - N) + 2[f(a) - f(b)]$$

Hence, $f(b) - f(a) = P - N$

Theorem 5.5: Every point of continuity of a summable function $f(t)$ is a Lebesgue point of $f(t)$.

Proof: Let $f(t)$ be integrable (summable) over the closed interval $[a, b]$. Also let $f(t)$ be continuous at $t = x_0$.

To prove that x_0 is Lebesgue point of the function $f(t)$

$\Rightarrow$ $f(t)$ is integrable over $[a, b]$ if $\int_a^b f(t)\,dt =$ a finite quantity

Since $f(t)$ is continuous at $t = x_0 \Rightarrow$ given $\in > 0\ \exists\ \delta > 0$ such that $|f(t) - f(x_0)| < \in$ whenever $|t - x_0| < \delta$

From which are get

$$\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt < \in \int_{x_0}^{x_0+h} dt = \in h$$

or $$\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt < \in \tag{5.12}$$

Making $\in \to 0,\ h \to 0$ we get

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(t)|\,dt \le 0 \tag{5.13}$$

But $\lim_{h\to 0}\left|\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt\right| \le \lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt \le 0$

[Using Eqn. (5.13)]

or $$\lim_{h\to 0}\left|\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt\right| \le 0 \tag{5.14}$$

But $$\lim_{h\to 0}\left|\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt\right| \ge 0 \tag{5.15}$$

Using Eqns. (5.14) and (5.15), we have

$$\lim_{h\to 0}\left|\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt\right| = 0$$

i.e., $$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt = 0$$

Hence, x_0 is a Lebesgue point of $f(t)$.

Theorem 5.6: If x be a Lebesgue point of the function f, then prove that the indefinite integral $F(x)=\int_a^x f(t)\,dt+F(a)$ is differentiable at the point x and $F'(x)=f(x)$.

Proof: Let x be a Lebesgue point of the function $f(t)$ so that

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt = 0 \tag{5.16}$$

Also let $$F(x)=f(a)+\int_a^x f(t)\,dt \tag{5.17}$$

To prove that $F'(x)=f(x)$

Since $$F(x)=\lim_{n\to 0}\frac{F(x+h)-F(x)}{h}$$

Hence, if we show that $\lim_{n\to 0}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=0$ the result will follow.

From Eqn. (5.16), we have

$$F(x+h)-F(x)=\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt$$

$$=\int_a^{x+h} f(t)\,dt+\int_x^a f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

$$\therefore \qquad \frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \tag{5.18}$$

Also $\dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt = \dfrac{1}{h} f(x) \displaystyle\int_x^{x+h} dt$

$$= \frac{1}{h} f(x)\,[t]_x^{x+h}$$

$$= f(x)$$

or $\dfrac{1}{h}\displaystyle\int_a^{x+h} f(x)\,dt = f(x)$

$$\text{or} \quad f(x) = \frac{1}{h}\int_a^{x+h} f(x)\,dt \tag{5.19}$$

Subtracting Eqns. (5.19) from (5.18), we have

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} [f(t)-f(x)]\,dt\right|$$

$$\leq \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt$$

Making $h \to 0$ and using Eqn. (5.16), we get

$$\lim_{n\to 0}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq 0 \tag{5.20}$$

$$\text{But} \quad \lim_{n\to 0}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \geq 0 \tag{5.21}$$

Since modulus of any quantity is always non-negative. Combining Eqns. (5.20) and (5.21), we have

$$\lim_{n\to 0}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = 0$$

$$\text{or} \quad \lim_{n\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$$

Hence, $F'(x) = f(x)$.

RECAPITULATION

- A real valued function $f(x)$ is said to be continuous at x_0 if given $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \in$ whenever $|x - x_0| < \delta$.
- If $f(x)$ be a continuous function, then $F(x) = \int_a^x f(t)\,dt + F(a)$ is said to be differentiable, where $F(a)$ being any finite constant.
- Let f be a real valued function defined on $[a, b]$. Then f is monotonic increasing on $[a, b]$, if $x < y \Rightarrow f(x) \le f(y)$. Also f is monotonic decreasing on $[a, b]$, if $x > y \Rightarrow f(x) \ge f(y)$.
- A real valued function $f : [a, b] \to R$ is said to be of bounded variation if there is a constant c such that $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le c$ hold for any position $a = x_0 < x_1 < x_2 \ldots < x_n = b$.
- The total variation $\overset{b}{\underset{a}{V}}$ of f on $[a, b]$ is

  $$\overset{b}{\underset{a}{V}}(f) = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : a = x_0 < x_1 < x_2 \ldots < x_n = b\right\}$$

  where the supremum is taken over all partitions of $[a, b]$.
- If f is a monotonic function on $[a, b]$, then f is bounded variation and $\overset{b}{\underset{a}{V}}(f) = |f(b) - f(a)|$.
- If P_1 and P_2 are two partition and $P_1 \subset P_2 \Rightarrow \overset{b}{\underset{a}{V}}(f : P_1) \le \overset{b}{\underset{a}{V}}(f : P_2)$.
- A function f is said to satisfy Lipschitz condition if there exists a positive constant M such that $|f(x) - f(y)| \le M|x - y|$.
- Let E be a set and C be the collection of intervals. The set E is known as *covered by C in the sense of Vitali* if for every $\in > 0$ and any $x \in E$ there exists an interval I in C such that $x \in I$ and $m(I) < \in$.
- If E is a set of finite measure and C is a collection of intervals which *cover E in the sense of Vitali*, then for a given $\in > 0$ there exists a finite, pairwise-disjoint subclass $\{C_1, C_2, \ldots, C_{n_0}\}$ of C such that

  $$m\left(E - \bigcup_{n=1}^{n_0} C_n\right) < \in.$$

- If $\lim_{n \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\,dt = 0$, then x is said to be a Lebesgue point of the function $f(t)$. The set of all Lebesgue point in $[a, b]$ of f is called the Lebesgue set of the function f.

- **Fundamental theorem of integral calculus:** If $f(x)$ is continuous and if $f(x)=\int_a^b f(t)\,dt+F(a)$. Then the theorem states that $F'(x)=f(x)$ *i.e.,* to say differentiation and integration are reverse processes.

EXERCISES

Multiple-choice Questions

5.1 Let *f* be a real valued function defined on $[a, b]$. Then *f* is monotonic increasing on $[a, b]$, if $x<y$ then

(a) $f(x)\le f(y)$
(b) $f(x)\ge f(y)$
(c) $f(x)>f(y)$
(d) None of these.

5.2 The total variation $\overset{b}{\underset{a}{V}}$ of *f* on $[a, b]$ is

(a) $\overset{b}{\underset{a}{V}}(f)=\inf\left\{\sum_{k=1}^{n}\left|f(x_k)-f(x_{k-1})\right| : a=x_0<x_1<x_2\ldots<x_n=b\right\}$

(b) $\overset{b}{\underset{a}{V}}(f)=\sup\left\{\sum_{k=1}^{n}\left|f(x_k)-f(x_{k-1})\right| : a=x_0<x_1<x_2\ldots<x_n=b\right\}$

(c) $\overset{b}{\underset{a}{V}}(f)=\sup\left\{\sum_{k=1}^{n}\left|f(x_k)+f(x_{k-1})\right| : a=x_0<x_1<x_2\ldots<x_n=b\right\}$

(d) None of these.

5.3 A function *f* is said to satisfy Lipschitz condition if there exists a positive constant *M* such that

(a) $\left|f(x)-f(y)\right|\le M\left|x-y\right|$

(b) $\left|f(x)-f(y)\right|\ge M\left|x-y\right|$

(c) $\left|f(x)-f(y)\right|=M\left|x-y\right|$

(d) None of these.

True or False

5.1 Let *f* be a real valued function defined on $[a, b]$. Then *f* is monotonic decreasing on $[a, b]$, if $x>y \Rightarrow f(x)\ge f(y)$.

5.2 If P_1 and P_2 are two partition and $P_1\subset P_2 \Rightarrow \overset{b}{\underset{a}{V}}(f:P_1)\ge \overset{b}{\underset{a}{V}}(f:P_2)$

5.3 Every absolutely continuous function is of bounded variation.

Fill in the Blanks

5.1 If $f(x)$ be a continuous function, then $F(x)=\int_a^x f(t)\,dt+F(a)$ is said to be ________, where $F(a)$ being any finite constant

5.2 Let f be a real valued function defined on $[a, b]$. Then f is monotonic ________ on $[a, b]$, if $x<y \quad \Rightarrow \quad f(x)\le f(y)$.

5.3 A real valued function $f:[a,b]\to R$ is said to be of ________ if there is a constant c such that $\sum_{k=1}^{n}\left|f(x_k)-f(x_{k-1})\right|\le c$ hold for any position $a=x_0<x_1<x_2\ldots<x_n=b$.

5.4 If f is a monotonic function on $[a, b]$, then f is bounded variation and $\overset{b}{\underset{a}{V}}(f)=$ ________

Exercises

5.1 What is an Absolute continuous function?

5.2 What do you mean by Monotonic functions?

5.3 Explain the Lebesgue point and Lebesgue set.

5.4 Write the Lipschitz condition.

5.5 State and prove Fundamental theorem of integral calculus.

5.6 To show that a monotonic function defined on closed interval $[a, b]$ is of bounded variation.

5.7 Prove that if f is absolutely function, then $f'(x)$ exists almost everywhere.

5.8 Prove that a monotonic function is differentiable almost everywhere.

5.9 Let x be a Lebesgue point of a function $f(t)$. Then show that the indefinite integral $F(x)=f(a)+\int_a^x f(t)\,dt$ is differentiable at each point x and $F'(x)=f(x)$.

5.10 Show that the indefinite integral is an absolutely continuous function.

ANSWERS

Multiple-choice Questions

5.1 (a) 5.2 (b) 5.3 (a)

State True or False

5.1 T 5.2 F 5.3 T

Fill in the Blanks

5.1 differentiable

5.2 increasing

5.3 bounded variation

5.4 $\left|f(b)-f(a)\right|$

6

CHAPTER

L^p Spaces

6.1 INTRODUCTION

This chapter discussed the spaces of p-th power of integrable functions. The class of measurable functions $f(x)$ whose p-th power $|f(x)|_p$ $(p > 0)$ is integrable on the interval (a, b) will be denoted by $L^p(a, b)$. If the interval of integration is known, we use briefly the symbol L^p. It is said that the sequence of functions f_n of the class $L^p(a, b)$ is mean convergent of order p to the function f if the distance between the functions f_n and f tends to zero, or if the sequence $\{f_n\}$ tends to f, in the metric of the space under consideration. By the uniqueness of the limit function, the uniqueness of the function $f(x)$ can be understood as a point in the space L^p functions differing only on a set of measure 0 and are considered as the same point of the space. This chapter is devoted to the study of L^p, conjugate number, convergent and Cauchy sequence. Metric and Normed space, L^p-space with properties, Riesz Holder Inequalities, Minkowski's Inequality and Schwarz's Inequality.

6.2 IMPORTANT TERMINOLOGY

Before discussing the L^p-space, we shall discuss the certain necessary preliminaries.

6.2.1 Conjugate Number

Let $p, q \in R$ such that (i) $p > 1$ (ii) $\frac{1}{p}+\frac{1}{q}=1$

Then q is known as conjugate of p.

Note: If $p = 2$ then $q = 2$ $\Rightarrow$ 2 is self conjugate number.

6.2.2 Convex Set and Convex Function

Suppose S is a subset of a complex vector space X, then S is called a convex set if $x, y \in S \Rightarrow (1 - \lambda)\, x + \lambda y \in S$, $\forall\ \lambda \in [0, 1]$. Consider $f : X \to R$ is a real valued function defined on a complex vector space X, then f is known as convex function if $f((1 - \lambda)\, x + \lambda y) \le (1 - \lambda)\, f(x) + \lambda f(y)$, $\forall x, y \in X$, $\lambda \in [0, 1]$. In particular, if the equality holds if and only if $x = y$, then f is said to be strictly convex. For example: exponential, (– log) and $p \to p^{\alpha}$ with $\alpha \ge 1$, etc.

6.2.3 Metric Space

Let X be a non-empty set and $d(x, y)$, $\forall (x, y) \in X$ is a distance function. A real valued function

d: $X \times X \to R$ which satisfies the following axioms:

(i) $d(x, y) \ge 0$, $\forall\ x, y \in X$.

(ii) $d(x, y) = d(y, x)$, $\forall x, y \in X$. (Symmetric property)

(iii) $d(x, y) \le d(x, z), + d(z, y)$, $\forall x, y, z \in X$. (Triangular in equality)

(iv) If $x = y \quad \Rightarrow \quad d(x, y) = 0$.

(v) If $d(x, y) = 0 \quad \Rightarrow \quad x \ne y$.

Then d is said to be metric on X and the pair (X, d) is called a metric space. The real number $d(x, y)$ is called the distance of x to y. The first axiom means that the distance between any two points x and y of X is a non-negative real number. The second axiom means that the distance does not depend on the order of the points x and y. The third axiom means that in the triangle, the sum of the length of two sides is greater than the length of the third side and equal sign shows that three points are in a straight line. The fourth axiom means that if the two points x and y are the same then the distance between x and y is equal to zero. The fifth axiom means that if the distance between two points x and y is equal to zero then the points x and y are the same.

6.2.4 Norm and Norm of an Element of L^p -space

The size of an element x is a real number denoted by $\|x\|$ and is called norm (which is distance $d(x, 0)$) if satisfies the following properties.

(i) $\|x\| \ge 0$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|kx\| = |k| \cdot \|x\| \qquad \{\therefore \|-x\| = \|x\|\}$

(iv) $\|x + y\| \le \|x\| + \|y\|$

Now we define a metric d for a set X with the help of norm as follows:

$$d(x, y) = \|x - y\| \qquad \forall\ x, y \in X$$

This metric is known as metric induced by the norm. Let f and g be two real bounded functions defined on the closed interval [0, 1]. Define the norms of f and g by

$$\|f\| = \int_0^1 |f(x)|\,dx \text{ and } \|g\| = \int_0^1 |g(x)|\,dx$$

The induced metric is defined by

$$d(f, g) = \| f - g \| = \int_0^1 |f(x) - g(x)|\,dx$$

The norm of any $f \in L^p[a, b]$ is defined by $\|f\|_p = \left[\int_a^b |f|^p\right]^{1/p}$.

If, $p = 1$ then $\|f\|_1 = \|f\| = \int_a^b |f|\,dx$, where (a, b) is finite or infinite interval.

Let $f(x), g(x) \in L^p\,[a, b]$ be arbitrary, we define $d(f, g) = \|f - g\|_p$.

Then d is called distance function *or* metric on $L^p\,[a, b]$, $d(f, g)$ denoted the distance between the functions $f(x)$ and $g(x)$.

6.2.5 Convergent Sequence

Let $< f_n >$ be a sequence of function belonging to a L^p-space. The sequence is said to be a converge in mean with index p if given $\in > 0$, there exists $n_0 \in N$ such that $m, n \geq n_0 \Rightarrow \|f_m(x) - f_n(x)\|_p < \in$ or if $\int |f_n(x) - f_m(x)|^p dx \to 0$ as $m \to \infty$ and $n \to \infty$.

6.2.6 Cauchy Sequence

Let $< f_n >$ be a sequence of function belonging to a L^p-space. This sequence is said to be a fundamental sequence or Cauchy sequence if given $\in > 0, \exists\ n_0 \in N$ such that $m, n \geq n_0 \Rightarrow \|f_m - f_n\|_p < \in$.

Note: Every convergent sequence in an L^p-space is a Cauchy sequence.

6.2.7 Completeness of L^p-space

An L^p-space is said to be complete if every Cauchy sequence in the space is convergent at some point of the space, *i.e.*, for every Cauchy sequence $< f_n >$ in the space, there is an element f in the space such that $f_n \to f$. A complete normal linear space is called Banach space.

6.2.8 Distance Function on $L^p[a, b]$

Let $f, g \in L^p[a, b]$ then distance function d on $L^p[a, b]$ is defined as $d(f, g) = \|f - g\|_p$.

6.3 L^p -SPACE

The L^p spaces are the spaces of p-power integrable functions and form an important class of examples of Banach spaces. Let E is measurable subset of R and $1 \leq p \leq \infty$. A measurable function x on E is said to be p-integrable on E if the function $|x|^p$ is integrable on E.

In particular x is 1-integrable on E if and only if it is integrable on E. A 2-integrable function is also called square integrable function.

The set $L^p[E]$ of all equivalence class of p-integrable function on E then we have $\|x\|_p = \left(\int_E |x|^p\right)^{1/p}$, where $L^p[E]$ is the linear space and $\|.\|_p$ is called the norm of L^p -space.

or

The family of all measurable function $f(x)$ is L^p -space if $|f|^p$ is Lebesgue integrable over $[a, b]$ for, *i.e.*, $\int_a^b |f|^p \, dx < \infty, (p > 0)$ and is denoted by L^p -space.

Following are the properties of L^p-space:

(i) $\alpha \in R$ and $f \in L^p[a, b] \;\Rightarrow\; \alpha f \in L^p[a, b]$.

(ii) $f, g \in L^p[a, b] \;\Rightarrow\; f + g \in L^p[a, b]$ where $p \geq 1$.

Theorem 6.1: If $f \in L^p[a, b]$ and $g \leq f$ then $g \in L^p[a, b]$.

Proof: Let α is any positive real number

$$\Rightarrow \quad \{x \in [a, b] : g(x) > \alpha\}$$
$$= \{x \in [a, b] : \alpha < g(x) \leq f(x)\}$$

It is given $g(x) \leq f(x)$

And also $f \in L^p[a, b]$

$\Rightarrow$ f is measurable $[a, b]$

$\Rightarrow$ $\{x \in [a, b] : f(x) \leq \alpha\}$ is measurable

$\Rightarrow$ $\{x \in [a, b] : g(x) \leq \alpha\}$ is measurable $\{g \leq f\}$

$\Rightarrow$ $g(x)$ is measurable function over $[a, b]$

Also we have

$$g(x) \leq f(x), \quad \forall x \in [a, b]$$

$$\Rightarrow \quad \int_a^b |g|^p\, dx \le \int_a^b |f|^p\, dx < \infty$$

$$\int_a^b |g|^p\, dx < \infty \qquad \left\{\because |f|^p \in L^p[a,b]\right\}$$

$$\Rightarrow \quad |g|^p \in L^p\,[a, b]$$

Hence, $g \in L^p\,[a, b]$

Theorem 6.2: If $f \in L^p\,[a, b], p > 1$ then $f \in L[a, b]$.

Proof: Let $f \in L^p\,[a, b]$.

To show $f \in L[a, b]$

Let
$$\left.\begin{aligned} A_1 &= \{x \in [a, b] : |f(x)| \ge 1\} \\ A_2 &= \{x \in [a, b] : |f(x)| < 1\} \end{aligned}\right\} \tag{6.1}$$

$$\Rightarrow \quad [a, b] = A_1 \cup A_2 \text{ and } A_1 \cup A_2 = \phi$$

Using countable additive property of the integral, we have

$$\int_a^b |f| = \int_{A_1} |f| + \int_{A_2} |f| \tag{6.2}$$

Since $f(x) \ge 1, \forall x \in A_1$ [from Eqn. (6.1)]

$$\Rightarrow \quad |f| \le |f|^p \text{ on } A \text{, where } p > 1$$

$$\Rightarrow \quad \int_{A_1} |f| \le \int_{A_1} |f|^p < \infty \text{ (given)}$$

$$\Rightarrow \quad \int_{A_1} |f| < \infty \tag{6.3}$$

On A_2, we have

$$|f(x)| < 1 \text{ from Eqn. (6.1) } \forall x \in A_2$$

Then by first mean value theorem, we have

$$\int_{A_2} |f| \le m(A_2) \tag{6.4}$$

From Eqns. (6.2), (6.3) and (6.4), we have

$$\Rightarrow \quad \int_a^b |f| = \text{integrable and infinite } < \infty$$

$$\Rightarrow \quad f \in L[a, b]$$

Another Proof: Let $f \in L^p\,[a, b], p > 1$.

To prove that $f \in L(a \cdot b)$ *i.e.,* to show that

(i) f is measurable over [a, b]

(ii) $|f|$ is integrable over (a, b)

By hypothesis

(iii) f is measurable over (a, b)

(iv) $|f|^p$ is integrable over (a, b)

Let $E = (a, b)$

Clearly $(\text{iii}) \Rightarrow (\text{i})$

Let $A = E(|f| \geq 1)$, $B = E(|f| < 1)$

Then $E = A \cup B, \quad A \cap B = \phi$

By countable additively property of the integral, we have

$$\int_E |f| dx = \int_A |f| dx + \int_B |f| dx \tag{6.5}$$

$$\Rightarrow \qquad |f(x) < 1|, \quad \forall x \in B$$

Using first mean value theorem, we have

$\int_B |f(x)| < m(B) = a$ finite quantity

This prove that $|f(x)|$ is integrable over *B*. (6.6)

$\because \quad f(x) \geq 1 \quad \forall x \in A$

$\therefore \quad |f| \leq |f|^p$ on the set A, for $\phi > 1$.

Integrating, we get

$$\int_A |A| dx \leq \int_A |f|^p dx < \infty \qquad \{\text{according to (iv)}\}$$

i.e., $\int_A |f| dx < \infty$

$\Rightarrow \quad |f|$ is integrable over A (6.7)

From Eqns. (6.5), (6.6) and (6.7) $\Rightarrow$ $|f|$ is integrable over *(a, b)*

Hence, $f \in L(a, b)$.

Theorem 6.3: If $f \in L^p\,[a, b], g \in L^p\,[a, b]$, then $f + g \in L^p\,[a, b]$.

Proof: Given that $f, g \in L^p\,[a, b] \Rightarrow f, g$ are measurable over (a, b).

$\Rightarrow f + g$ is measurable over (a, b).

Let

$$\left.\begin{aligned} A_1 &= \{x \in [a, b] : |f(x)| \geq |g(x)|\} \\ A_2 &= \{x \in [a, b] : |f(x)| < |g(x)|\} \end{aligned}\right\} \tag{6.8}$$

$\Rightarrow [a, b] = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$.

Then by countable additive property, we have

$$\int_a^b |f+g|^p = \int_{A_1} |f+g|^p + \int_{A_2} |f+g|^p$$

Again we have

$$|f+g|^p \le (|f|+|g|)^p$$

$$\le (|g|+|g|)^p \text{ on } A_1 \text{ and } \le (|f|+|f|)^p \text{ on } A_2$$

Now $|f+g|^p \le 2^p |g|^p$ on A_1

$\le 2^p |f|^p$ on A_2

Integrating both sides, we get

$$\int_{A_1} |f+g|^p \le 2^p \int_{A_1} |g|^p \text{ on } A_1 \tag{6.9}$$

$$\int_{A_2} |f+g|^p \le 2^p \int_{A_2} |f|^p \text{ on } A_2 \tag{6.10}$$

Given f, g are $L^p[a, b]$

$$\Rightarrow \qquad \int_{A_2} |f|^p, \int_{A_1} |g|^p < \infty$$

$$\Rightarrow \qquad \int_{A_1} |f+g|^p < \infty, \int_{A_2} |f+g|^p < \infty$$

From Eqns. (6.9) and (6.10), we have

$$f+g \in L^p[a, b]$$

Note: Given f, g are integrable. Since right hand sides of both relations are integrable then left hand sides are also integrable. Hence, $f+g \in L^p[a, b]$.

Another Proof: Let $f, g \in L^p[a, b]$ so that

(i) f, g over measurable over (a, b)

(ii) $|f|^p, |g|^p$ are integrable over (a, b)

i.e., $\int_a^b |f|^p \, dx < \infty, \int_a^b |g|^p \, dx < \infty$

To prove that $f+g \in L^p(a, b)$ it suffices to prove that

(i) $f+g$ is measurable over (a, b)

(ii) $|f+g|^p$ is integrable over (a, b)

i.e., $\int_a^b |f+g|^p \, dx < \infty$

Evidently (i) $\Rightarrow$ (iii),

Let $E=(a,b)$

and $A=E(|f|\le|g|), B=E(|f|>|g|)$

By countable additive property of the integral, we have

$$\int_E |f+g|^p\,dx=\int_A |f+g|^p\,dx+\int_B |f+g|^p\,dx \tag{6.11}$$

$$\begin{aligned}|f+g|^p &\le (|f|+|g|)^p\\ &\le (|f|+|g|)^p\\ &=2^p|g|^p \text{ on the set } A\end{aligned}$$

Integrating, we get

$$\begin{aligned}\int_A |f+g|^p\,dx &\le 2^p\int_A |g|^p\,dx\\ &<2^p\times\infty \quad \text{according to (ii)}\end{aligned}$$

i.e.,
$$\int_A |f+g|^p\,dx<\infty \tag{6.12}$$

Again
$$\begin{aligned}|f+g|^p &\le (|f|+|g|)^p\\ &<2^p|f|^p \text{ on the set } B\end{aligned}$$

Consequently

$$\begin{aligned}\int_B |f+g|^p\,dx &<2^p\int_B |f|^p\,dx\\ &<2^p\times\infty=\infty \text{ by (ii)}\end{aligned}$$

i.e.,
$$\int_B |f+g|^p\,dx<\infty \tag{6.13}$$

From Eqns. (6.11), (6.12) and (6.13), we have

$$\int_E |f+g|^p<\infty$$

or
$$\int_a^b |f+g|^p\,dx<\infty$$

Hence, $f+g\in L^p(a,b)$.

Lemma: Let *A* and *B* are two positive real number and $0<\lambda<1$ then $A^\lambda B^{1-\lambda}\le\lambda A+(1-\lambda)B$ with equality when *A* = *B*.

Proof:

Case 1: If either A is zero or B is zero lemma is obvious.
Case 2: Let $A>0$, $B>0$

Let $\phi(x)=x^\lambda-\lambda x,\quad x\ge 0,\quad 0<\lambda<1 \qquad (6.14)$

Now the function $\phi(x)$ is maximum at $x = 1$, *i.e.,* $\phi(1)$ maximum.

$\Rightarrow \phi(x) \leq \phi(1)$

From Eqn. (6.14), we have

$\Rightarrow x^{\lambda} - \lambda x \leq 1^{\lambda} - \lambda$

Let $x = A/B$

$$\Rightarrow \qquad \left(\frac{A}{B}\right)^{\lambda} - \lambda\left(\frac{A}{B}\right) \leq 1 - \lambda$$

$$\Rightarrow \qquad A^{\lambda} B^{-\lambda} - \lambda \frac{A}{B} \leq 1 - \lambda$$

$$\Rightarrow \qquad A^{\lambda} B^{1-\lambda} \leq \lambda A + B(1-\lambda) \tag{6.15}$$

Hence, if $x = 1$ and if $A = B$

$$\begin{aligned} A^{\lambda-\lambda^2} &\leq \lambda A + A(1-\lambda) \\ &\leq \lambda A + A - \lambda A \\ &\leq A \end{aligned}$$

The equality sign will hold good then $A = B$

6.4 RIESZ HOLDER INEQUALITIES

Let p, q are conjugate numbers and $f \in L^p[a, b], g \in L^p[a, b]$, then show that (i) $fg \in L(a, b)$ (ii) $\|fg\| \leq \|f\|_p \|g\|_q \Rightarrow \int |fg| \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$ with equality only when $\beta|f|^p = \beta|g|^q$.

Proof:

Case 1: When $p = 1$, $q = \infty$ the result is obvious.

Case 2: Let $1 < p < \infty$, $1 < q < \infty$ $\quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$$\text{Then } \lambda = \frac{1}{p}, p > 1 \;\Rightarrow\; \lambda < 1 \tag{6.16}$$

Also $\frac{1}{q} = 1 - \lambda$

Putting these value in Eqn. (6.15), we have

$$A^{1/p} B^{1/q} \leq \frac{A}{p} + \frac{B}{q} \tag{6.17}$$

It one of the $f(x)$ are zero

$\Rightarrow$ result is trivial

$$\Rightarrow \qquad \int_a^b |f|^p \text{ and } \int_a^b |g|^q > 0$$

$$\Rightarrow \qquad \|f\|_p > 0, \quad \|g\|_q > 0$$

Let $F(x) = f(x)/\|f\|_p$, $G(x) = g(x)/\|g\|_q$, $A^{1/p} = |F(x)|$ and $B^{1/q} = |G(x)|$.

From Eqn. (6.17), we have

$$|F(x)G(x)| \leq \frac{|\mathrm{F}(x)|^p}{p} + \frac{|G(x)|^q}{q}$$

Integrating both sides between a and b, we get

$$\begin{aligned}
\int_a^b |F(x)G(x)| dx &\leq \frac{1}{p}\int_a^b |F(x)|^p dx + \frac{1}{q}\int_a^b |G(x)|^q dx \\
&\leq \frac{1}{p}\int_a^b \frac{|f(x)|^p}{\|f\|^p} dx + \frac{1}{q}\int_a^b \frac{|g(x)|^q}{\|g\|^q} dx \\
&\leq \frac{1}{p}\frac{\int_a^b |f(x)|^p dx}{\int_a^b |f|^p dx} + \frac{1}{q}\frac{\int_a^b |g(x)|^q dx}{\int_a^b |g|^q dx} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

$$\Rightarrow \qquad \int_a^b |F(x)G(x)| \leq 1$$

Putting the value of $F(x)$ and $G(x)$, we have

$$\Rightarrow \qquad \frac{\int_a^b |f(x)g(x)| dx}{\|f\|_p \|g\|_q} \leq 1$$

$$\Rightarrow \qquad \int_a^b |f(x)g(x)| dx \leq \|f\|_p \|g\|_q$$

$$\Rightarrow \qquad \int_a^b |fg| \leq \|f\|_p \|g\|_q$$

$$\text{or} \qquad \|fg\| \leq \|f\|_p \|g\|_q \tag{6.18}$$

It is given $f \in L^p(a,b)$, $g \in L^q(a,b)$
$\Rightarrow f$ and g are integrable.

$$\Rightarrow \qquad \int_a^b |f|^p < \infty \Rightarrow \|f\|_p < \infty \tag{6.19}$$

$$\text{or} \qquad \int_a^b |g|^q < \infty \Rightarrow \|g\|_q < \infty \tag{6.20}$$

Using Eqns. (6.18), (6.19) and (6.20), we have

Eqn. (6.18) $\Rightarrow \|g\|, < \infty$

$$\Rightarrow \qquad f g \cdot \in L(a,b)$$

Also this result will be true that $A = B$

$$\Rightarrow \qquad |F(x)|^p = |G(x)|^q$$

$$\Rightarrow \qquad \frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q} \qquad \text{[Norm is scalar quantity]}$$

$$\Rightarrow \qquad \|g\|_q^q\,|f|^p = \|f\|_p^p\,|q|^q$$

or there exists two non zero constant $\propto$ and β such that

$$\alpha|f|^p = \beta|g|^q.$$

6.5 MINKOWSKI'S INEQUALITY

If $f, g \in L^p(a,b),\ \ p \geq 1$ then (i) $f+g \in L^p(a,b)$ (ii) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Proof:

Case 1: If $p = 1$ or ∞ then the result is trival.

Case 2: When $p > 1$, if $f, g \in L^p(a,b)$

Then $f+g \in L^p(a,b)$ (6.21)

Also we know that $\frac{1}{p}+\frac{1}{q} = 1$ where q is conjugate to p since $f, g \in L^p(a,b)$.

$$\Rightarrow \qquad f+g \in L^p(a,b)$$

$$\Rightarrow \qquad (f+g)^{p/q} \in L^p(a,b)$$

Applying holder inequality for f and $(f+g)^{p/q}$, we have

$$\int_a^b |f|\cdot|f+g|^{p/q}\,dx \leq \left(\int_a^b |f|^p\,dx\right)^{1/p}\left(\int_a^b |f+g|^{p/q}\,dx\right)^{1/q}$$

$$\Rightarrow \int_a^b |f|\cdot|f+g|^{p/q}\,dx \leq \left(\int_a^b |f|^p\,dx\right)^{1/p}\left(\int_a^b |f+g|^{p}\,dx\right)^{1/q} \qquad (6.22)$$

Since $g \in L^p(a,b)$ and therefore interchanging f and g in the last inequality, we get

$$\int_a^b |g||f+g|^{p/q}\,dx \leq \left(\int_a^b |g|^p\,dx\right)^{1/p}\left(\int_a^b |f+g|^{p}\,dx\right)^{1/q} \qquad (6.23)$$

Adding Eqn. (6.22) and (6.23), we get

$$\int_a^b |f||f+g|^{p/q}\,dx + \int_a^b |g|\cdot|f+g|^{p/q}\,dx \le$$

$$\left[\left(\int_a^b |f|^p\,dx\right)^{1/p} + \left(\int_a^b |g|^p\,dx\right)^{1/p}\right]\left(\int_a^b |f+g|^p\,dx\right)^{1/q} \tag{6.24}$$

We know that

$$\frac{1}{p}+\frac{1}{q}=1 \quad\Rightarrow\quad p+q=pq \quad\Rightarrow\quad 1+\frac{p}{q}=p \tag{6.25}$$

We know that $|f+g|^p = |f+g|\cdot|f+g|^{p-1} \quad \left(p-1=\dfrac{p}{q}\right)$

$$= |f+g||f+g|^{p/q} \qquad \text{[Using Eqn. (6.25)]}$$

$$\le (|f|+|g|)\,|f+g|^{p/q}$$

$$\le |f|\cdot|f+g|^{p/q} + |g|\cdot|fg|^{p/q}$$

Integrating both sides from a to b, we get

$$\int_a^b |f+g|^p\,dx \le \int_a^b |f||f+g|^{p/q}\,dx + \int_a^b |g||f+g|^{p/q}\,dx \tag{6.26}$$

From Eqns. (6.24) and (6.26), we get

$$\int_a^b |f+g|^p\,dx \le \left[\left(\int_a^b |f|^p\,dx\right)^{1/p} + \left(\int_a^b |g|^p\,dx\right)^{1/p}\right]\left(\int_a^b |f+g|^p\,dx\right)^{1/q}$$

Dividing by $\left(\int_a^b |f+g|^p\,dx\right)^{1/p}$ and observing that $1-\dfrac{1}{q}=\dfrac{1}{p}$, we get

$$\frac{\int_a^b |f+g|^p\,dx}{\left(\int_a^b |f+g|^p\,dx\right)^{1/q}} \le \left[\left(\int_a^b |f|^p\,dx\right)^{1/p} + \left(\int_a^b |g|^p\,dx\right)^{1/p}\right]$$

$$\Rightarrow \qquad \left(\int_a^b |f+g|^p\,dx\right)^{1-1/q} \le \left[\left(\int_a^b |f|^p\,dx\right)^{1/p} + \left(\int_a^b |g|^p\,dx\right)^{1/p}\right]$$

$$\Rightarrow \qquad \left(\int_a^b |f+g|^p\,dx\right)^{1/p} \le \left(\int_a^b |f|^p\,dx\right)^{1/p} + \left(\int_a^b |g|^p\,dx\right)^{1/p}$$

$$\Rightarrow \qquad \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Let $\|f+g\|_p$ is non zero finite.

If $\|f+g\|_p = 0 \quad \Rightarrow \quad$ also result trivial.

If $\|f+g\|_p = \infty$ then either $\|f\|_p = \infty$ or $\|g\|_p = \infty$.

6.6 SCHWARZ'S INEQUALITY

Let f and g be Lebesgue integrable, *i.e.*, $f, g \in L^2(a,b)$, then $fg \in L(a,b)$ and $\|fg\| \le \|f\|_2 \|g\|_2$.

Proof: It is a particular case of Holder inequality.

Let $x \in (a,b)$ be arbitrary then $\left[|f(x)| - |g(x)|\right]^2 \ge 0$

$\Rightarrow 2|f(x)|\,g(x)|| \le |f(x)|^2 + |g(x)|^2$

On integrating, we get

$$2\int_a^b |f(x)g(x)|\,dx \le \int_a^b |f(x)|^2\,dx + \int_a^b |g(x)|^2\,dx \tag{6.27}$$

Since $f, g \in L^2(a,b) \Rightarrow f \cdot g$ are integrable over (a, b)

$$\text{And } \int_a^b |f(x)|^2\,dx \le \infty, \int_a^b |g(x)|^2\,dx < \infty \tag{6.28}$$

Using Eqn. (6.27), we have

$$\Rightarrow \qquad \int_a^b |f(x)g(x)|\,dx < \infty$$

$$\Rightarrow \qquad fg \in L(a,b)$$

We know that if $|f(x)| \ge 0 \Rightarrow \int_a^b |f(x)|\,dx \ge 0$

Let $\alpha \in R$ is arbitrary then $\left(\alpha|f(x)| + |g(x)|\right)^2 \ge 0$

$$\Rightarrow \qquad \int_a^b \left(\alpha|f(x)| + |g(x)|\right)^2 \ge 0$$

$$\Rightarrow \qquad \alpha^2 \int_a^b |f(x)|^2\,dx + 2\alpha \int_a^b |f(x)g(x)|\,dx + \int_a^b |g(x)|^2\,dx \ge 0$$

Put $A = \int_a^b |f|^2\,dx,\ B = 2\int_a^b |fg|\,dx,\ C = \int_a^b |g|^2\,dx$

$$\Rightarrow \qquad \alpha^2 A + \alpha B + C \ge 0 \tag{6.29}$$

Case 1: If $A = 0 \Rightarrow f(x) = 0$ are in (a, b) then L.H.S and R.H.S are zero result is trivial.

Case 2: If $A \ne 0$. Let $\alpha = \dfrac{B}{2A}$.

Using Eqn. (6.29), we have

$$\Rightarrow \qquad A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C \geq 0$$

$$\Rightarrow \qquad B^2 \leq 4AC \tag{6.30}$$

Putting the value of *A*, *B*, *C* in Eqn. (6.30), we have

$$4\left[\int_a^b |fg|\,dx\right]^2 \leq 4\left[\int_a^b |f|^2\,dx\right]\left[\int_a^b |g|^2\,dx\right]$$

$$\text{or} \qquad \int_a^b |fg|\,dx \leq \left(\int_a^b |f|^2\,dx\right)^{1/2}\left(\int_a^b |g|^2\,dx\right)^{1/2}$$

$$\Rightarrow \qquad \|fg\| \leq \|f\|_2\|g\|_2$$

Note: $$\left|\int_a^b fg\right| \leq \int_a^b |fg|\,dx \leq \left(\int_a^b f^2\,dx\right)^{1/2}\left(\int_a^b g^2\,dx\right)^{1/2}$$

Theorem 6.4: Show that (L^p, d) is a metric space.

Proof: To show that d is a metric, we will prove the following properties:

(i) $d(f, g) \geq 0$
(ii) $d(f, g) = 0 \Leftrightarrow f = g$
(iii) $d(f, g) = d(g, f)$
(iv) $d(f, g) \leq d(f, h) + d(h, g)\ \forall f, g, h \in L^p,\ p \geq 1$.

From (i), we have Let $f, g \in L^p[a, b]$

$$\because \qquad \vdots\ \|f-g\|_p = \left(\int_a^b |f-g|^p\,dx\right)^{1/p} \geq 0$$

$$\Rightarrow \qquad d(f, g) = \|f-g\| \geq 0$$

(ii) $d(f, g) = 0 \Leftrightarrow \|f-g\|p = 0$

$$\Leftrightarrow \quad \int_a^b |f-g|^p\,dx = 0$$

$$\Leftrightarrow \quad |f-g|^p = 0$$

$$\Leftrightarrow \quad f = g$$

(iii) $|f-g| = |g-f|$

$$\Rightarrow \quad |f-g|^p = |g-f|^p$$

$$\Rightarrow \quad \left(\int_a^b |f-g|^p\,dx\right)^{1/p} = \left(\int_a^b |g-f|^p\,dx\right)^{1/p}$$

$$\Rightarrow \quad \|f-g\|_p = \|g-f\|_p$$
$$\Rightarrow \quad d(f, g) = d(g, f)$$

(iv) Let $f, g, h \in L^p [a, b]$

Then $d(f, g) = \|f-g\|_p$

$$= \|f-h+(h-g)\|_p$$
$$\leq \|f-h\|_p + \|h-g\|_p \qquad \text{[using Minkowski property]}$$
$$\leq d(f, h) + d(h, g)$$

Hence, (L^p, d) is a metric space.

Theorem 6.5: An L^p space is a normed linear space.

Proof: Let $f, g \in L^p [a, b]$ and $\subset \in R$

(i) $f, g \in L^p [a, b] \quad \Rightarrow \quad f+g \in L^p [a, b]$

(ii) $f \in L^p [a, b], c \in R \quad \Rightarrow \quad cf \in L^p [a, b]$

Since $\because$ $f \in L^p \Rightarrow f$ is measurable over $[a, b]$

$\Rightarrow$ cf is measurable over $[a, b]$

$\Rightarrow$ L^p is a linear space.

(iii) $|f(x)| \geq 0 \quad \forall\, x \in [a, b]$

$$\Rightarrow \left[\int_a^b \left|f(x)^p\right|\right]^{1/p} \geq 0 \Rightarrow \|f\|_p \geq 0$$

(iv) Now $\|f\|_p \geq 0 \quad \Leftrightarrow \quad f = 0.$

(v) $$\|cf\|_p = \left[\int_a^b |cf|^b \, dx\right]^{1/p} = |c| \left(\int_a^b |f|^p \, dx\right)^{1/p} = |c| \|f\|_p$$

(vi) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ (using Minkowski properly)

Hence, L^p is a normed linear space.

Solved Problems

Problem 6.1: If f and g are square integrable in the Lebesgue sense then prove that $f + g$ is also square integrable in the Lebesgue sense and $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

Solution: Let f and g are square integrable in the Lebesgue sense so that $f, g \in L^2 [a, b]$.

To prove $(f+g) \in L^2 (a, b)$ and $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

By Schwarz's inequality, we have

$$f,\ g \in L^2(a,b) \Rightarrow fg \in L(a,b) \tag{6.31}$$

Also $$f,\ g \in L^2(a,b) \Rightarrow f^2 g^2 \in L(a,b) \tag{6.32}$$

From Eqns. (6.31) and (6.32), we have

$$\Rightarrow \quad f^2 + g^2 + 2fg \in L(a,b) \quad \Rightarrow \quad (f+g)^2 \in L(a,b)$$

$$\Rightarrow \quad f + g \in L^2(a,b)$$

We have

$$\begin{aligned} |f+g|^2 &\le (|f|+|g|)^2 \\ &\le |f|^2 + |g|^2 + 2|f|\cdot|g| \\ &\le |f|^2 + |g|^2 + 2|fg| \end{aligned}$$

Taking limit a to b and integrating both sides, we get

$$\int_a^b |f+g|^2 \le \int_a^b |f|^2 + \int_a^b |g|^2 + 2\int_a^b |fg|$$

By Schwarz's inequality, we have

$$\begin{aligned} \int_a^b |f+g|^2 &\le \int_a^b |f|^2 + \int_a^b |g|^2 + 2\left[\int_a^b |f|^2\right]^{1/2}\left[\int_a^b |g|^2\right]^{1/2} \\ &\le \left[\left(\int_a^b |f|^2\right)^{1/2} + \left(\int_a^b |g|^2\right)^{1/2}\right]^{1/2} \end{aligned}$$

Taking positive square root, we have

$$\left(\int_a^b |f+g|^2\right)^{1/2} \le \left(\int_a^b |f|^2\right)^{1/2} + \left(\int_a^b |g|^2\right)^{1/2}$$

Hence, $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$.

Problem 6.2: If $f \in L^2(0,1)$ show that $\left|\int_0^1 f(x)\,dx\right| \le \left[\int_0^1 f(x)\,dx\right]^{1/2}$.

Solution: If $f,\ g \in L^2(0,1)$ then by Schwarz's inequality, we get

$$\|f+g\| \le \|f\|_2 \|g\|_2$$

or $$\int_0^1 |fg|\,dx \le \left(\int_0^1 |f|^2\,dx\right)^{1/2}\left(\int_0^1 |g|^2\,dx\right)^{1/2}$$

Taking $g(x) = 1, \forall x$, we get

$$\int_0^1 |f|\,dx \le \left[\int_0^1 |f|^2\,dx\right]^{1/2} \tag{6.33}$$

or $$\left|\int_0^1 |f|\,dx\right| \le \int_0^1 |f|\,dx \le \left(\int_0^1 |f|^2\,dx\right)^{1/2}$$ [Using Eqn. (6.33)]

Hence, $$\left|\int_0^1 |f|\,dx\right| \le \left(\int_0^1 |f|^2\,dx\right)^{1/2}.$$

RECAPITULATION

- The norm of any $f \in L^p[a, b]$ is defined by $\|f\|_p = \left[\int_a^b |f|^p\right]^{1/p}$.
- Let $< f_n >$ be a sequence of function belonging to a L^p-space. This sequence is said to be a Cauchy sequence if given $\in > 0$, $\exists\ n_0 \in N$ such that $m, n \ge n_0 \Rightarrow \|f_m - f_n\|_p < \in$.
- A complete normal linear space is called Banach space.
- Let $f, g \in L^p[a, b]$ then distance function d on $L^p[a, b]$ is defined as $d(f, g) = \|f - g\|_p$.
- The family of all measurable function $f(x)$ is L^p-space if $|f|^p$ is Lebesgue integrable over $[a, b]$ for $p > 0$ *i.e.*, $\int_a^b |f|^p\,dx < \infty, (p > 0)$ and is denoted by L^p-space.
- Let A and B are two positive real number and $0 < \lambda < 1$ then $A^\lambda B^{1-\lambda} \le \lambda A + (1-\lambda)B$ with equality when $A = B$.
- Let p, q are conjugate number and $f \in L^p[a, b], g \in L^p[a, b]$, then show that (i) $fg \in L(a, b)$ (ii) $\|fg\| \le \|f\|_p \|g\|_q \Rightarrow \int |fg| \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$ with equality only when $\beta|f|^p = \beta|g|^q$.
- If $f, g \in L^p(a, b), p \ge 1$ then (i) $f + g \in L^p(a, b)$ (ii) $\|f + g\|_p \le \|f\|_p + \|g\|_p$
- Let f and g be Lebesgue integrable, *i.e.*, $f \cdot g \in L^2(a, b)$, then $fg \in L(a, b)$ and $\|fg\| \le \|f\|_2 \|g\|_2$.

EXERCISES

Multiple-choice Questions

6.1 Let p and q be any two real numbers, such that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then q is

(a) called the conjugate of p.

(b) equal to p.

(c) less then p.
(d) None of these.

6.2 Let A and B are two positive real numbers and $0 < \lambda < 1$ then $A^{\lambda} B^{1-\lambda} \leq \lambda A + (1-\lambda)B$ with equality when $A = B$.

(a) $A^{\lambda} B^{1-\lambda} \leq \lambda A + (1-\lambda)B$ with equality when $A = B$.
(b) $A^{\lambda} B^{1-\lambda} \geq \lambda A + (1-\lambda)B$ with equality when $A = B$.
(c) $A^{\lambda} B^{1-\lambda} \neq \lambda A + (1-\lambda)B$ with equality when $A = B$.
(d) None of these.

6.3 If $f, g \in L^p(a,b), p \geq 1$ then (i) $f+g \in L^p(a,b)$ (ii) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. This is known as

(a) Riesz Holder Inequality
(b) Minkowski's inequality
(c) Schwarz's inequality
(d) None of these.

True or False

6.1 The number 2 is self conjugate number.
6.2 If $f \in L^p[a,b]$, $g \in L^p[a,b]$, then $f+g \in L^p[a,b]$.
6.3 If $f \in L^p[a,b], p < 1$ then $f \in L[a,b]$.
6.4 If $f \in L^p[a,b]$ and $g \geq f$ then $g \in L^p[a,b]$.

Fill in the Blanks

6.1 An L^p-space is said to be _______ if every Cauchy sequence in the space is convergent at some point of the space A.
6.2 Let f and g be Lebesgue integrable, *i.e.*, $f \cdot g \in L^2(a,b)$, then $fg \in L(a,b)$ and $\|fg\| \ldots \|f\|_2 \|g\|_2$.
6.3 Let A and B are two positive real numbers and $0 < \lambda < 1$ then $A^{\lambda} B^{1-\lambda} \leq \lambda A + (1-\lambda)B$ with equality when A _______ B.
6.4 An L^p space is a _______ linear space.

Exercises

6.1 Write a short note on Norm.
6.2 What do you mean by L^p-space?
6.3 Explain the convergent and Cauchy sequence.
6.4 State and prove Riesz Holder inequality
6.5 State and prove Minkowski's inequality
6.6 State and prove Schwarz's inequality.
6.7 Define the space $L^2[a, b]$. Prove that it is a complete metric space.

6.8 If $\{f_n\}$ is a Cauchy sequence in L^p-space then prove that it has a limit.

6.9 Prove that a convergent sequence $\{f_n\}$ in an L^p-space has a limit in L^p-space.

6.10 To show that L^p-space are complete.

ANSWERS

Multiple-choice Questions

6.1 (a)	6.2 (a)	6.3 (b)

State True or False

6.1 T	6.2 T	6.3 F	6.4 F

Fill in the Blanks

6.1 complete	6.2 $\leq$
6.3 =	6.4 normed

7

CHAPTER

Product and Signed Measures

7.1 INTRODUCTION

The purpose of this chapter is to study the products of two measure spaces: product measures and signed measures. The limit function of a sequence of measurable functions that converges almost everywhere is a measurable function. In the preceding chapters that a measure is a non-negative function, if a measure is allowed to take both positive and negative values, then this idea leads to the consideration of signed measure. Fubini's theorem and Tonelli's theorem are also being discussed in details. Fubini's theorem has a great importance in calculation of certain explicit integrals. Hahn Decomposition theorem, Radon-Nikodym theorem and Lebesgue Decomposition theorem are also be discussed in details.

7.2 IMPORTANT TERMINOLOGY

Before discussing the product and signed measure, we shall discuss certain necessary preliminaries:

7.2.1 Cartesian Product of Sets

Let X and Y be any two sets. The *Cartesian products* of X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$, *i.e.*, $X \times Y = \{(x, y): x \in X, y \in Y\}$ and $Y \times X = \{(y, x): y \in Y, x \in X\}$.

7.2.2 Rectangular

If A is a subset of X and B is a subset of Y then $A \times B \subset X \times Y$ is known as a rectangular and A and B is called its sides. A rectangle is empty if and only if one of its sides is empty.

If (X, M) and (Y, N) be any two measurable spaces then $(X \times Y, M \times N)$ is the Cartesian product of the given spaces. If $A \in M$ and $B \in N$ then the rectangle $A \times B$ is known as measurable.

7.2.3 Sections

Let (X, M) and (Y, N) be two measurable spaces and E be any subset of the Cartesian product $X \times Y$ and $x \in X, y \in Y$ then the sets $E_x = \{y : (x, y) \in E\}$ and $E_y = \{x : (x, y) \in E\}$ are called the section of E determined by x and y respectively. Since this section is determined by an element x and therefore E_x is called X-section and similarly E_y is called Y-section. Every section of a measurable set is measurable. So E_x and E_y are measurable sets.

Let f be measurable function on E. We write $f_x(y) = f(x, y)$ and $f_y(x) = f(x, y)$. Then $f_y(x)$ and $f_x(y)$ are functions over the sets E_y and E_x respectively. f_x and f_y are known as X and Y-section of f respectively. Since every section of a measurable function is measurable and so f_x and f_y are measurable function.

7.2.4 Double Integral

If g is an integrable function on $X \times Y$ then the integral denoted by $\int g(x, y)\, d\pi(x, y)$ or $\int g(x, y)\, d(A \times B)(x, y)$ is known as double integral of g, where $\pi = \mu \times \lambda$.

7.3 PRODUCT MEASURE

Consider (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces and $E \subset X \times y$. A set function π, defined for every set E in $M \times N$ by $\pi(E) = \int \lambda(E_x) d\ \mu(x) = \int \mu(E_y)\, d\lambda(y)$ is a σ-finite measure such that $\pi(A \times B) = \mu(A) \cdot \lambda(B)$ where $A \in M, B \in N$.

Here π is a measure function which is referred to as product of μ and λ, *i.e.*, $\pi = \mu \times \lambda$.

Theorem 7.1: Let E be a measurable subset of $X \times Y$. Then E has a measure zero if and only if almost every X-section (or Y-section) has measure zero.

Proof: Consider (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces and $E \subset X \times Y$. If the function π, defined by $\pi = \mu \times \lambda$ than

$$\pi(E) = \int \lambda(E_x) d\,\mu(x) = \int \mu(E_y)\, d\,\lambda(y) \tag{7.1}$$

Suppose if E has a measure zero, *i.e.*, $\pi(E) = 0$, than by equation (7.1), integrals are finite in (7.1). Therefore non-negative integrals must be zero almost everywhere $\lambda(E_x) = 0$ and $\mu(E_y) = 0$. Thus the X-section or Y-section has measure zero in almost everywhere.

Conversely, if the X-section or Y-section has measure zero in almost everywhere, then by equation (7.1), we get $\pi(E) = 0$ *i.e.*, E has a measure zero.

7.4 TONELLI'S THEOREM

Suppose (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces. If g is a non-negative measurable function on $X \times Y$ then $\int g \, d(\mu \times \lambda) = \int g \, d\mu \, d\lambda = \int g \, d\lambda \, d\mu$.

Proof: Consider (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces and if g is a non-negative measurable function on $X \times Y$ then there exists an increasing sequence $\{g_n\}$ of non-negative simple function such that the sequence $\{g_n\}$ converges to g everywhere.

Using the property of integrals, we have $\lim\limits_{n \to \infty} \int g_n \, d\pi = \int g \, d\pi = 0$. If $f_n(x) = \int g_n(x, y) \, d\lambda(y)$, then using the properties of sequence $\{g_n\}$, follows that the $\{f_n\}$ is also a n increasing sequence of non-negative measurable functions which converges to $f(x)$ for every x where $f(x) = \int g(x, y) \, d\lambda(y)$.

Therefore f is a non-negative measurable function and $\lim\limits_{n \to \infty} \int f_n \, d\pi = \int f \, d\pi = 0$. Thus $\int g \, d(\mu \times \lambda) = \int g \, d\lambda \, d\mu$. Similarly, we also show that $\int g \, d(\mu \times \lambda) = \int g \, d\mu \, d\lambda$ where $\pi = \mu \times \lambda$.

7.5 FUBINI'S THEOREM

Suppose (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces. If g is an integrable function on $X \times Y$ then almost every section of g is integrable. If g_1 and g_2 are defined by $g_1(x) = \int g(x, y) d\lambda(y)$ and $g_2(x) = \int g(x, y) d\mu(x)$ then g_1 and g_2 are integrable and $\int g \, d(\mu \times \lambda) = \int g_1 \, d\mu = \int g_2 \, d\lambda$.

Proof: Consider (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces and g is integrable function on $X \times Y$ if and only if positive and negative parts of g are integrable.

Using Tonelli's theorem, if g is a non-negative measurable function than it is integrable. Therefore g_1 and g_2 are finite valued almost everywhere so the sections of g have required properties. Hence, $\int g \, d(\mu \times \lambda) = \int g_1 \, d\mu = \int g_2 \, d\lambda$.

Note: Other form of Fubini's theorem, consider $E = [a, b] \times [c, d]$. Let f be Lebesgue integrable over E. Then we have

$$\int_E \int f(x, y) dx \, dy = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

and

$$\int_E \int f(x, y) dx \, dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy .$$

Problem 7.1: *Using Fubini's theorem to show that*

$$\int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)}\, dx \right\} dy \neq \int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)}\, dy \right\} dx .$$

Solution: Suppose $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)}$, where $(x, y) \in [0, 1]$.

We have $\displaystyle\int_0^1 f(x, y)\, dy = \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)}\, dy$

$$= \int_0^1 \frac{\partial}{\partial y} \left(\frac{y}{(x^2 + y^2)} \right) dy$$

$$= \left[\frac{y}{(x^2 + y^2)} \right]_{y=0}^{1}$$

$$= \frac{1}{1 + x^2}.$$

Now we have

$$\int_0^1 \left\{ \int_0^1 f(x, y)\, dy \right\} dx = \int_0^1 \frac{dx}{1 + x^2}$$

$$= [\tan^{-1} x]_{x=0}^{1}$$

$$= \frac{\pi}{4} \tag{7.2}$$

Again we have

$$\int_0^1 f(x, y)\, dx = \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)}\, dx$$

$$= \int_0^1 \frac{\partial}{\partial x} \left(\frac{-x}{x^2 + y^2} \right) dx$$

$$= \left[\frac{-x}{x^2 + y^2} \right]_{x=0}^{1}$$

$$= -\frac{1}{1 + y^2}$$

Now we have

$$\int_0^1 \left\{\int_0^1 f(x, y)dx\right\} dy = \int_0^1 - \frac{dy}{1+y^2}$$

$$= [-\tan^{-1} y]_{y=0}^1$$

$$= -\frac{\pi}{4} \tag{7.3}$$

Therefore, from equations (7.2) and (7.3), we have

$$\int_0^1 \left\{\int_0^1 f(x, y)dx\right\} dy \neq \int_0^1 \left\{\int_0^1 f(x, y)dx\right\} dx$$

7.6 SIGNED MEASURE

Consider A is a σ-algebra of sub-sets of X. An extended real valued set function v such that $v: A \to [-\infty, \infty]$ is said to be a signed measure on a measurable space (X, A), if the following conditions are satisfied:

(i) v takes almost one of the values $-\infty$ or $+\infty$.
(ii) $v(\phi) = 0$.
(iii) v is countable additive, *i.e.*, $v\left(\bigcup_{i=1}^{\infty} A_{1i}\right) = \sum_{i=1}^{\infty} v(A_{1i})$ for any sequence $\{A_{1i}\}$ of disjoint measurable sets.

Note:

1. A signed measure v is said to be finite if $v(A_1) < \infty, \forall A_1 \in A$.

7.7 POSITIVE, NEGATIVE, NULL SETS AND A SET OF MEASURE ZERO

A subset E of X is said to be positive relative (or negative relative) to a signed measure ν defined on a measurable space (X, A) if the following conditions are satisfied:

(i) E is measurable and
(ii) For all $A_1 \subset E$ such that A_1 is measurable implies $v(A_1) \geq 0$ {or $(\nu(A_1) \leq 0)$}.

A set $E \subset X$ is said to be a null set relative to a signed measure v if E is both positive and negative. It is noted that every measurable subset of a positive set is a positive set.

A measurable set E is a set of measure zero if and only if every measurable subset of it has ν measure zero. The measure of every null set is zero. A set of measure zero may be a union of two measurable sets whose measures are not zero but are negative of each other.

7.8 SINGULAR MEASURES, JORDAN DECOMPOSITION AND ABSOLUTELY CONTINUOUS MEASURABLE FUNCTION

Two measures v_1 and v_2 are said to be mutually singular $(v_1 \perp v_2)$ if there exist a measurable set $A \subset X$ such that $v_1(A) = 0 = v_2(X - A)$.

Let v be a signed measure on a measurable space (X, A). As a result of Jordan decomposition of v, we represent a pair (v_1, v_2) of mutually singular measures v_1 and v_2 satisfying the condition $v = v_1 - v_2$. For illustration, a pair of measures (v^+, v^-) cleared as above, is a Jordan decomposition of v.

Suppose v_1 and v_2 are two measure functions defined on a space (X, A). The measure v_1 is said to be an absolutely continuous with respect to v_2 if and only if $v_2(A_1) = 0, \forall A_1 \in A \quad \Rightarrow \quad v_1(A_1) = 0$. It is denoted by $v_2 << v_1$.

Theorem 7.2: A union of any countable collection of positive subsets of X is positive.

Proof: Consider $\{A_n\}$ is a sequence of positive sets in X. Suppose $A = \bigcup_{i=1}^{\infty} A_i$ and B is any measurable subset of A. Suppose we have written $B_n = B \cap A'_n \cap A'_{n-1} \cap \ldots \cap A'_1, \forall n \in N$. Here dashes denote complement of respective sets with respect to X,. i.e., $A'_n = X - A_n$.

We recognize that if a set is measurable then its complement is also measurable and also a countable intersection of measurable sets is measurable. These details lead to the finish that B_n is a measurable subset of the positive set A_n and thus $v(B_n) \geq 0$, using the explanation of the positive set. From the construction of B_n it is clear that sets B_n are disjoint. Also $B_n = \bigcup_{n=1}^{\infty} B_n$. Hence, $v(B) = \bigcup_{n=1}^{\infty} v(B_n) \geq 0$. Therefore we have exposed that (i) A is measurable set. For A_n is a positive set $\Rightarrow$ A_n is a measurable set $\Rightarrow$ $\bigcup_{n=1}^{\infty} A_n$ is measurable $\Rightarrow$ A is measurable. (ii) $\forall B \subset A$ such that B is measurable set $\Rightarrow v(B) \geq 0$.

Using the definition of positive set, this confirms that A is a positive set.

Here we state the following theorem without proof:

1. Every subset of a negative set is negative set and a countable union of negative sets is a negative set.

2. If E is a measurable set with finite negative measure, *i.e.,* if $-\infty < v(E) < 0$, then E contains a negative set A with the property $v(A) > 0$.
3. If E is a measurable set of finite positive measure, *i.e.,* $0 < v(E) < \infty$, then E contains a positive set A with $v(A) > 0$.

7.9 HAHN DECOMPOSITION THEOREM

Let v be a signed measure on measurable space (X, A). Then there exists a positive set P and a negative set Q such that $P \cap Q = \phi, P \cup Q = X$.

Proof: Consider A is an σ-algebra of subsets of X and v is a signed measure on a measurable space (X, A). Since v consider almost one of the values ∞ and $+\infty$. Without failure of simplification we can understand that v does not take $-\infty$. Consider the family B of all negative subsets of X and let

$$\lambda = \inf \{v(E) : E \in B\} \tag{7.4}$$

Then there exist a sequence $\{E_n\}$ in B such that $\lim\limits_{n \to \infty} v(E_n) = \lambda$.

B is a family of negative sets $\Rightarrow$ $\{E_n\}$ is a sequence of negative sets

$\Rightarrow$ $\bigcup\limits_{i=1}^{\infty} E_i$ is a negative set,

$\Rightarrow$ Q is a negative set on taking $Q = \bigcup\limits_{i=1}^{\infty} E_i$.

Thus Q is negative subset of X. Then, according to equation (7.4), $v(Q) \geq \lambda$. Now we assume the subset $Q - E_n$ of Q.

$\therefore\ Q = (Q - E_n) \cup E_n$

$\therefore\ v(Q) = v(Q - E_n) + v(E_n) \leq v(E_n)$

or $v(Q) \leq v(E_n),\ \forall n \in N$ where $E_n \in B,\ \forall n \in N$.

In outlook of equation (7.4), these two particulars prove that $v(Q) \leq \lambda$.
Therefore we have exposed that $v(Q) \leq \lambda$ and $v(Q) \geq \lambda$.
This implies $v(Q) = \lambda \Rightarrow \lambda > -\infty$.
Now we have to prove that $P = X - Q$ is a positive subset of X.

Consider the opposing. Then P is not positive and therefore P is negative. By the definition, for every measurable set $E \subset P$, $v(E) < 0$. Now we have E is a measurable subset of X with negative measure. Making use of result:

If E is a measurable set of finite negative measure, *i.e.,* $-\infty < v(E) < 0$, then E contains a set A with $v(A) < 0$, we obtain a negative set $A \subset E$ such that $v(A) < 0$.

Since A and Q both are disjoint negative subsets of X and their union $A \cup Q$ is also negative.

We know that the every subset of a negative set is negative set and a countable union of negative sets is a negative set.

Consequently, we have $v(A \cup Q) \geq \lambda$, by asset of (7.4)

But $\lambda \leq (A \cup Q) = v(A) + v(Q) = v(A) + \lambda$.

or $\lambda \leq v(A) + \lambda$

or $v(A) \geq 0$.

Contrary to the fact that $v(A) < 0$.

Hence, our assumption is wrong, i.e., "*P* – is not positive" is wrong. So *P* is positive. Thus $P = X - Q$ is positive and Q is negative.

$\therefore\ X = P \cup Q, \quad P \cap Q = \phi.$

A decomposition of a measurable space *X* into two subsets *P* and *Q* such that $X = P \cup Q, P \cap Q = \phi,$ where *P* and *Q* are positive and negative sets respectively relative to the signed measure *v*, is known as Hahn decomposition for the signed measure *v*. Also *P* and *Q* are called positive and negative components of *X*.

7.10 RADON-NIKODYM THEOREM

Let (X, A, μ) be a σ-finite measure space. Let v be a measure defined on A such that v is absolutely continuous with respect to μ. Then there exists a non-negative measurable function *f* such that $v(E) = \int_E f\, d\mu,\ \forall E \in A.$ The function *f* is unique in the sense that if *g* is any measurable function with this property, then $g(x) = f(x)$ almost everywhere in *X* with respect to μ.

Proof: Suppose (X, A, μ) is a σ-finite measure space. To set up the existence of the function *f*, suppose that μ is finite. Since μ is finite this implies $v - \alpha\mu$ is a signed measure on A for each rational number α. Suppose *T* denote the set of all non-negative rational numbers. For every $t \in T$, consider (P_i, Q_i) be a Hahn decomposition for the signed measure $v - t\mu$. If $\alpha \geq t$, then Q_t is negative for $v - \alpha\mu$. For if *E* is a measurable subset of Q_t then $v(E) - \alpha\mu(E) \leq v(E) - t$. For *E* is measurable subset of $Q_t, \alpha \geq t \Rightarrow v(E) - \alpha\mu(E) \leq v(E) - t\mu(E) \leq 0,$ Since we have $-\alpha \leq t$ $\Rightarrow v(E) - \alpha\mu(E) \leq 0 \Rightarrow (v - \alpha\mu)(E) \leq 0$. Similarly P_t is positive for each signed measure $v - \alpha t$ if $\alpha \leq t$. Now we have write $P = \cap\{P_t : t \in T\}$. Being an intersection of countable collection of measurable sets, *P* is measurable. Suppose *E* is any measurable subset of *P*. Then we have

$$E \subset P = \cap P_i \quad \Rightarrow \quad E \subset \cap P_t \subset P_t, \quad \forall t \in T \quad \Rightarrow \quad E \subset P_t, \forall t \in T$$
$$\Rightarrow \quad (v - t\mu) E \geq 0.$$

For P_t is positive, this implies $v(E) - t\mu(E) \geq 0$

$$\Rightarrow v(E) \geq t\mu(E) \quad \Rightarrow \quad t\mu(E) \leq v(E), \forall t \in T$$

$\Rightarrow\ \mu(E)=0$ or $v(E)=\infty$.

Thus $\mu(E)>0 \Rightarrow v(E)=\infty$.

Now define a non-negative function $f: X \to [-\infty,\infty]$ by

$$f(x)=\inf\{t\in T : x\in Q_t\},\quad \forall x\in X$$

Here we assume the normal convention that the infimum of any empty collection of real numbers is ∞. So we have $f(x)=0, \forall x\in P$. To show that *f* is measurable. Suppose $r\in R$ and $q\in Q$ are arbitrary. Then we have

$S_q=\cup(Q_t:\forall t<q)$ = Union of measurable set = measurable set

Now using the definition of *f* states that

$$\{x\in X: f(x)\le r\}=\cap\{S_q : q\in Q \text{ such that } q>r\}$$

= intersection of measurable sets = measurable set.

Thus $\{x\in X: f(x)\le r\}$ is measurable and so *f* is measurable.

Next our intent is to demonstrate that $v(E)=\int_E f\,d\mu,\ \ \forall E\in A$. Consider $\mu(E\cap P)=0$. If $\mu(E\cap P)>0$, then $v(E\cap P)=0$. Also $\int_E f\,d\mu \ge \int_{E\cap P} d\mu=\infty$. This implies $\int_E f\,d\mu=\infty$. [For $f(x)=0\ \forall x\in P$, hence $f: E\to[-\infty,\infty]$ is not integrable]. Hence, $v(E)=\int_E f\,d\mu$.

This establishes the equality in case $\mu(E\cap P)>0$. Absolute continuity of *v* with respect to μ implies that $v(E\cap P)=0$. Choose a positive integer *n* and set $E_k=\left\{x\in E: \dfrac{k-1}{n}<f(x)\le\dfrac{k}{n}\right\}$. For each rational number $q<\dfrac{k}{n}$, we have $E_k\subset\{x\in X: f(x)\le n\}\subset S_q=\cap\{Q_t:\forall t<q\}$. Also Q_t is a negative set for the signed measure $v-q\mu$ and E_k is a subset of Q_t. So E_k is a negative set. This implies

$$(v-q\mu)(E_k)\le 0 \quad\Rightarrow\quad v(E_k)\le q\mu(E_k) \quad\Rightarrow\quad v(E_k)\le\frac{k}{n}\mu(E_k) \tag{7.5}$$

Alternatively, since $E_k\subset\left\{x\in X: f(x)>\dfrac{k-1}{n}\right\}\subset P_{(k-1)/n}$

and hence $\left[v-\left(\dfrac{k-1}{n}\right)\mu\right](E_k)\ge 0$. For $P_{(k-1)/n}$ is positive,

$$\text{This gives } v(E_k)\ge\left(\frac{k-1}{n}\right)\mu(E_k) \tag{7.6}$$

From equations (7.5) and (7.6), we have

$$\left(\frac{k-1}{n}\right)\mu\,(E_k) \le v(E_k) \le \frac{k}{n}\,\mu(E_k) \tag{7.7}$$

As E is a disjoint union of sets $E \cap P$ and E_k. Thus $\mu(E \cap P) = 0 = v(E \cap P)$. It pursue from the countable additive property of integral that

$$\int_E f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu, \quad v(E) = \sum_{k=0}^{\infty} v(E_k)$$

Using definition of E_k, we have $\frac{k-1}{n} \le f(x) \le \frac{k}{n}$ on E_k.

Using the first mean value theorem, we have

$$\left(\frac{k-1}{n}\right)\mu(E_k) \le \int_{E_k} f(x)\,d\mu \le \frac{k}{n}\,\mu(E_k) \tag{7.8}$$

From equations (7.7) and (7.8), we have

$$\left(\frac{k-1}{n}\right)\mu(E_k) \le \int_{E_k} f\,d\mu - v(E_k) \le \frac{k}{n}\,\mu(E_k)$$

Summing this over k, we have

$$-\frac{1}{n}\,\mu(E_k) \le \int_E f(x)\,d\mu - v(E) \le \frac{k}{n}\,\mu(E)$$

Since n is arbitrary and hence making $n \to \infty$, we have

$$0 \le \int_E f(x)\,d\mu - v(E) \le 0,$$

or
$$\int_E f(x)\,d\mu = v(E) \tag{7.9}$$

Hence we complete the existence proof.

Now to prove that the almost uniqueness of the function $f : X \to [-\infty, \infty\}$.

Consider $g : X \to [-\infty, \infty]$ is any measurable function satisfying the condition $v(E) = \int_E g\,d\mu\ \forall E \in A$. For each $n \in N$, suppose $A_n = \{x \in X : f(x) - g(x) \ge 1/n\} \in A$ and $B_n = \{x \in X : g(x) - f(x) \ge 1/n\} \in A$. Using definition of A_n, $f(x) - g(x) \ge 1/n$ on A_n. Using first mean value theorem, we get

$$\int_{A_n} (f-g)\,d\mu \ge \frac{1}{n}\,\mu(A_n)$$

or
$$\int_{A_n} f\,d\mu - \int_{A_n} g\,d\mu\,\frac{1}{n}\,\mu(A_n)$$

or
$$v(A_n) - v(A_n) \ge (1/n)\,\mu(A_n) \qquad \text{[Using equation (7.9)]}$$

or $(1/n)\,\mu(A_n) \le 0$ or $\mu(A_n) \le 0$

But $\mu(A_n) \ge 0$ is always true.

$\therefore$ $\mu(A_n) = 0$. Similarly we also have $\mu(B_n) = 0$.

Suppose $C = \{x \in X : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} (A_n \cup B_n)$.

Then we have $\mu(C) = \sum_n \mu(A_n) + \sum_n \mu(B_n) = \sum_n 0 + \sum_n 0 = 0$.

Then we have $\mu(C) = 0$.

This implies $f = g$ almost everywhere on X with respect to μ.

Precisely in a comparable way the theorem can be proved in case μ is σ-infinite.

7.11 LEBESGUE DECOMPOSITION THEOREM

Suppose (X, A, μ) is a σ-finite measure space and v a σ-finite measure defined on A. Then there exists two uniquely determined measures v_0 and v_1 such that $v = v_0 + v_1$, $v_0 \perp \mu$, $v_1 << \mu$.

Proof: Suppose (X, A, μ) is a σ-finite measure space and assume $\lambda = \mu + v$. Since μ and v are σ-finite implies that λ is σ-finite. Obviously $\mu \le \lambda$ and $v << \lambda$, where $\mu << \lambda$ place for "μ is absolutely continuous with respect to λ". Using Radon-Nikodym theorem, we find the non-negative functions $f,\ g : X \to [-\infty, \infty]$ such that $\mu(E) = \int_E f\, d\lambda,\quad v(E) = \int_E g\, d\lambda, \forall E \in A$.

Consider $A = \{x \in X : f(x) > 0],\ B = \{x \in X : f(x) = 0\}$.

Then we have $X = A \cup B,\ A \cap B = \phi$. Also we have $\mu(B) = \int_B f\, d\lambda = 0$.

Now, define two functions $v_0, v_1 : A \to [\infty, \infty]$ by involving that

$$v_0(E) = v(E \cap B),\ v_1(E) = v(E \cap B), \quad \forall E \in A$$

Then v_0 and v_1 are measures on A and satisfy the condition $v = v_0 + v_1$.

$$v_0(A) = v(A \cap B) = v(\phi) = 0, \text{or } v_0(A) = 0$$

Thus $\mu(B) = 0 = v_0(A) = v_0(X - B)$

i.e. $\mu(B) = 0 = v_0(X - B)$

This implies v_0 is mutually singular to $\mu \Rightarrow v_0 \perp \mu$.

Now to prove that $v_1 \le \mu$. For this assume $E \in A$ is arbitrary such that $\mu(E) = 0$.

Then $\int_E f\, d\lambda = \mu(E) = 0$ or $\int_E f\, d\lambda = 0$. Also $f(x) \ge 0, \quad \forall x \in E$.

This implies $f = 0$ almost everywhere on E relative to λ.

Since $f > 0$ on $A \cap E$ and thus $v_1(E) = v(E \cap A)$ (by definition of v_1)

$$\leq \lambda(E \cap A) = 0$$

$\therefore$ $v_1(E) \leq 0$. But $v_1(E) \geq 0$.

Combining these two results, $v_1(E) = 0$.

Thus we have prove that $\mu(E) = 0 \;\Rightarrow\; v_1(E) = 0$. This implies $v_1 << \mu$.

Now we have to prove that the uniqueness of v_0 and v_1.

To proving that v_0 and v_1 are unique, consider v_0' and v_1' are measures such that $v = v_0' + v_1'$ and has the same property as that of the measure v_0 and v_1 respectively. Then we have $v = v_0 + v_1$ and $v = v_0' + v_1'$ are the two Lebesgue decomposition of v. So we have $v_0 - v_0' = v_1' - v_1$. Again $v_0' - v_1$ is absolutely continuous and $v_0 - v_0'$ is singular relative to v. We have $v = v_0', v_1 = v_1'$. Hence v_0 and v_1 are unique.

Solved Problem

Problem 7.1: Let A and B be two measurable sets and v a signed measure such that $A \subset B$ and $|v(B)| < \infty$. Then prove that $|v(A)| < \infty$.

Solution: It is given that A and B are two measurable set. So we assume $B = (B - A) + A, \quad (B - A) \cap A = \phi$. Thus $v(B) = v(B - A) + v(A)$.

If $v(B - A)$ and $v(A)$ both are ∞, then $v(B)$ is ∞. If $v(B - A)$ or $v(A)$ is ∞, then again $v(B)$ is ∞. Given that $|v(B)| < \infty$, i.e., $v(B)$ is finite. Hence, $v(B)$ will be finite if both $v(B - A)$ or $v(A)$ both are finite. Hence, $|v(A)| = \infty$.

RECAPITULATION

- The *Cartesian products* of X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$, *i.e.*, $X \times Y = \{(x, y): x \in X, y \in Y\}$.
- If A is a subset of X and B is a subset of Y then $A \times B \subset X \times Y$ is known as a rectangular and A and B is called its sides.
- Let (X, M) and (Y, N) be two measurable spaces and E be any subset of the Cartesian product $X \times Y$ and $x \in X, y \in Y$ then the sets $E_x = \{y : (x, y) \in E\}$ and $E_y = \{x : (x, y) \in E\}$ are called the section of E determined by x and y respectively.
- Every section of a measurable function is measurable.
- Consider (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces and $E \subset X \times y$. A set function π, defined for every set E in $M \times N$ by $\pi(E) = \int \lambda(E_x) d\mu(x) = \int \mu(E_y)\, d\lambda(y)$ is a σ-finite measure such that $\pi(A \times B) = \mu(A) \cdot \lambda(B)$ where $A \in M, B \in N$.

- Let E be a measurable subset of $X \times Y$. Then E has a measure zero if and only if almost every X-section (or Y-section) has measure zero.
- Suppose (X, M, μ) and (Y, N, λ) are two σ-finite measure spaces. If g is a non-negative measurable function on $X \times Y$ then $\int g\, d(\mu \times \lambda) = \int g\, d\mu\, d\lambda = \int g\, d\lambda\, d\mu.$
- Suppose (X, M, μ) and (Y, N, λ) are two $\sigma -$ finite measure spaces. If g is an integrable function on $X \times Y$ then almost every section of g is integrable. If g_1 and g_2 are defined by $g_1(x) = \int g(x, y)\, d\lambda(y)$ and $g_2(x) = \int g(x, y)\, d\mu(x)$ then g_1 and g_2 are integrable and $\int g\, d(\mu \times \lambda) = \int g_1\, d\mu = \int g_2\, d\lambda.$
- A signed measure v is said to be finite if $v(A_1) < \infty, \forall A_1 \in A.$
- A measurable set E is a set of measure zero if and only if every measurable subset of it has v measure zero. The measure of every null set is zero.
- A set of measure zero may be a union of two measurable sets whose measures are not zero but are negative of each other.
- Two measures v_1 and v_2 are said to be mutually singular $(v_1 \perp v_2)$ if there exist a measurable set $A \subset X$ such that $v_1(A) = 0 = v_2(X - A).$
- Countable union of positive sets is positive and the countable union of negative sets is negative.
- Every subset of a negative set is negative set and a countable union of negative sets is a negative set.
- If E is a measurable set with finite negative measure, *i.e.,* if $-\infty < v(E) < 0$, then E contains a negative set A with the property $v(A) < 0$.
- If E is a measurable set of finite positive measure, *i.e.,* $0 < v(E) < \infty$, then E contains a positive set A with $v(A) > 0$.
- Let v be a signed measure on measurable space (X, A). Then there exists a positive set P and a negative set such that $P \cap Q = \phi, P \cup Q = X.$
- Let (X, A, μ) be a σ-finite measure space. Let v be a measure defined on A such that v is absolutely continuous with respect to μ. Then there exists a non-negative measurable function f such that $v(E) = \int_E f\, d\mu, \quad \forall E \in A$. The function f is unique in the sense that if g is any measurable function with this property, then $g(x) = f(x)$ almost everywhere in X with respect to μ.
- Suppose (X, A, μ) is a σ-finite measure space and v a σ-finite measure defined on A. Then there exists two uniquely determined measures v_0 and v_1 such that $v = v_0 + v_1, v_0 \perp \mu, v_1 << \mu.$

EXERCISES

Multiple-choice Questions

7.1 The measure of every null set is
  (a) 1
  (b) 0
  (c) ∞
  (d) None of these.

7.2 Countable union of positive sets is
  (a) Positive
  (b) Negative
  (c) may be both (a) and (b)
  (d) None of these.

7.3 Let v be a signed measure on measurable space (X, A). Then there exists a positive set P and a negative set Q such that $P \cap Q = \phi, P \cup Q = X.$ This is known as
  (a) Lebesgue decomposition theorem
  (b) Hahn decomposition theorem
  (c) Randon-Nikodym theorem
  (d) None of these.

State True or False

7.1 If f is not integrable then Fubini's theorem may be fail.

7.2 Signed measure is not a measure in universal.

7.3 Every subset of a negative set is negative set and a countable union of negative sets is a negative set.

7.4 Countable union of negative sets is positive.

Fill in the Blanks

7.1 Every measurable subset of a positive set is a _______

7.2 If v is σ-finite then it converse is _______ finite.

7.3 The decomposition $v = v_0 + v_1$ is called _______ of σ-finite measure v with respect to σ-finite measure μ.

Exercises

7.1 What a short note on sections.

7.2 Define the product measure.

7.3 State and prove Tonelli's theorem.

7.4 State and prove Fubini's theorem.

7.5 State and prove Hahn decomposition theorem and the Jordan decomposition theorem for signed measures.

7.6 State and prove Lebesgue decomposition theorem.

7.7 State and prove Radon-Nikodym theorem.

7.8 State Fubini's theorem. Discuss its applicability to the function

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)}$$

$$x^2 + y^2 \neq 0\,, x \in [0, 1], y \in [0, 1].$$

7.9 Using Fubini's theorem explain the

$$\int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)}\, dx \right\} dy \neq \int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy \right\} dx$$

ANSWERS

Multiple-choice Questions

7.1 (b) 7.2 (a) 7.3 (b)

State True or False

7.1 T 7.2 T 7.3 T 7.4 F

Fill in the Blanks

7.1 positive set 7.2 also

7.3 Lebesgue decomposition

Appendix: Sets

Some Important Theorems on Set Theory

In this section we discussed some important theorems on set theory, which are following:

Theorem 1: Let A, B and C be any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$
(ii) $A \cup (B \cup C) = (A \cup B) \cap (A \cup C)$
(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(iv) $A - (B \cup C) = (A - B) \cap (A - C)$
(v) $A - (B \cap C) = (A - B) \cup (A - C)$
(vi) $(A - C) \cap (B - C) = (A \cap B) - C$
(vii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof:

(i) Let x be any element of $A \cup (B \cup C)$. Then we have

$$
\begin{aligned}
& x \in A \cup (B \cup C) \\
\Leftrightarrow \quad & x \in A \text{ or } x \in (B \cup C) \\
\Leftrightarrow \quad & x \in A \text{ or } (x \in B \text{ or } x \in C) \\
\Leftrightarrow \quad & x \in A \text{ or } x \in B \text{ or } x \in C \\
\Leftrightarrow \quad & (x \in A \text{ or } x \in B) \text{ or } x \in C \\
\Leftrightarrow \quad & x \in (A \cup B) \text{ or } x \in C \\
\Leftrightarrow \quad & x \in [(A \cup B) \cup C]
\end{aligned}
$$

Hence, $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) Let x be any element of $A \cup (B \cap C)$. Then we have

$$
\begin{aligned}
& x \in [A \cup (B \cap C)] \\
\Leftrightarrow \quad & x \in A \text{ or } x \in B \cap C \\
\Leftrightarrow \quad & x \in A \text{ or } (x \in B \text{ and } x \in C)
\end{aligned}
$$

$$\begin{aligned}
\Leftrightarrow \quad & (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
\Leftrightarrow \quad & (x \in A \cup B) \text{ and } (x \in A \cup C) \\
\Leftrightarrow \quad & x \in [(A \cup B) \cap (A \cup C)]
\end{aligned}$$

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(iii) Let x be any element of $A \cap (B \cup C)$. Then we have

$$\begin{aligned}
& x \in A \cap (B \cup C) \\
\Leftrightarrow \quad & x \in A \text{ and } x \in B \cup C \\
\Leftrightarrow \quad & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Leftrightarrow \quad & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
\Leftrightarrow \quad & x \in (A \cap B) \text{ or } x \in (A \cap C) \\
\Leftrightarrow \quad & x \in [(A \cap B) \cup (A \cap C)]
\end{aligned}$$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(iv) Let x be any element of $A - (B \cup C)$. Then we have

$$\begin{aligned}
& x \in A - (B \cup C) \\
\Leftrightarrow \quad & x \in A,\ x \notin (B \cup C) \\
\Leftrightarrow \quad & x \in A,\ (x \in B \text{ or } \in C) \\
\Leftrightarrow \quad & (x \in A,\ x \notin B) \text{ and } (x \in A,\ x \notin C) \\
\Leftrightarrow \quad & x \in (A - B) \text{ and } x \in (A - C) \\
\Leftrightarrow \quad & x \in [(A - B) \cap (A - C)]
\end{aligned}$$

Thus, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$

and $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

(v) Let x be any element of $A - (B \cap C)$. Then we have

$$\begin{aligned}
& x \in A - (B \cap C) \\
\Leftrightarrow \quad & x \in A,\ x \notin (B \cap C) \\
\Leftrightarrow \quad & x \in A,\ x \text{ belongs either to } B' \text{ or to } C' \\
\Leftrightarrow \quad & x \in A,\ x \in B' \text{ or } x \in A,\ x \in C' \\
\Leftrightarrow \quad & x \in A - B \text{ or } x \in A - C \\
\Leftrightarrow \quad & x \in (A - B) \cup x\ (A - C)
\end{aligned}$$

Thus, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

and $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$.

(vi) Let x be any element of $(A - C) \cap (B - C)$. Then we have

$$\begin{aligned}
& x \in [(A - C) \cap (B - C)] \\
\Leftrightarrow \quad & x \in A - C \text{ and } x \in B - C \\
\Leftrightarrow \quad & x \in A,\ x \notin C \text{ and } x \in B,\ x \notin C \\
\Leftrightarrow \quad & x \in A, \text{ and } x \in B,\ x \notin C \\
\Leftrightarrow \quad & x \in (A \cap B),\ x \notin C \\
\Leftrightarrow \quad & x \in (A \cap B) - C
\end{aligned}$$

Hence, $(A - C) \cap (B - C) = (A \cap B) - C$.

(vii) First we have to prove

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$$

Let (a, b) be any element of $A \times (B \cup C)$. Then we have

$$\begin{aligned}
& (a, b) \in A \times (B \cup C) \\
\Rightarrow \quad & a \in A \text{ and } b \in B \cup C \\
\Rightarrow \quad & a \in A \text{ and } (b \in B \text{ or } b \in C) \\
\Rightarrow \quad & (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C) \\
\Rightarrow \quad & (a, b) \in A \times B \text{ or } (a, b) \in A \times C \\
\Rightarrow \quad & (a, b) \in (A \times B) \cup (A \times C)
\end{aligned}$$

Thus, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
Now we have to prove

$$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$$

Let (x, y) be any element of $(A \times B) \cup (A \times C)$. Then we have

$$\begin{aligned}
& (x, y) \in (A \times B) \cup (A \times C) \\
\Rightarrow \quad & (x, y) \in A \times B \text{ or } (x, y) \in A \times C \\
\Rightarrow \quad & (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \\
\Rightarrow \quad & x \in A \text{ and } (y \in B \text{ or } y \in C) \\
\Rightarrow \quad & (x, y) \in A \times (B \cup C)
\end{aligned}$$

Thus, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Theorem 2: Let A and B be any two sets, then

(i) $A - B = A \cap B'$

(ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(iii) $B - A \subseteq A'$

(iv) $B - A' = B \cap A$

(v) $\mathrm{A} - B \subset A$

(vi) $(A - B) \cap B = \phi$.

(vii) $(A \cup B)' = A' \cap B'$

(viii) $(A \cap B)' = A' \cup B'$

Proof:

(i) We have

$$\begin{aligned} A - B &= \{x : x \in A, x \notin B\} \\ &= \{x : x \in A, x \in B'\} \\ &= \{x : x \in (A \cap B')\} \end{aligned}$$

Hence, $A - B = A \cap B'$

(ii) Let x be any element of $(A - B) \cup (B - A)$. Then we have

$$x \in (A - B) \cup (B - A)$$

$\Leftrightarrow$ $x \in (A - B)$ or $x \in (B - A)$

$\Leftrightarrow$ $(x \in A \text{ } or \text{ } x \notin B)$ or $(x \in B, x \notin A)$

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ but x does not belong to both A and B

$\Leftrightarrow$ $x \in (A \cup B)$ but $x \notin (A \cap B)$

$\Leftrightarrow$ $x \in (A \cup B) - (A \cap B)$

Thus $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

and $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

Hence, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

(iii) Let x be any element of $B - A$. Then we have

$x \in B - A$

$\Rightarrow$ $x \in B, x \notin A$

$\Rightarrow$ $x \in B, x \in A'$

i.e., each element of $B - A$ belongs to A'.

Hence, $B - A \subseteq A'$.

(iv) Let x be any element of $B - A'$. Then we have

$$\begin{aligned} B - A' &= \{x : x \in B, x \notin A'\} \\ &= \{x : x \in B, x \in A\} \\ &= \{x : x \in B \cap A\} \\ &= B \cap A \end{aligned}$$

Hence, $B - A' = B \cap A$.

(v) Using definition of difference, we know that the all the elements of $A - B$ are the elements of A. So, we have $A - B \subset A$.
Hence, $A - B \subset A$.

(vi) Let x be any element of $(A - B) \cap B$. Then we have

$$\begin{aligned}(A - B) \cap B &= \{x : x \in A - B \text{ and } x \in B\} \\ &= \{x : x \in A,\, x \notin B \text{ and } x \in B\}\end{aligned}$$

Here, there is no element in A which belongs to B and also does not belong to B.
Hence, $(A - B) \cap B = \phi$.

(vii) Let x be any element of $(A \cup B)'$. Then we have

$$\begin{aligned} & x \in (A \cup B)' \\ \Leftrightarrow \quad & x \notin A \cup B \\ \Leftrightarrow \quad & x \notin A \text{ and } x \notin B \\ \Leftrightarrow \quad & x \in A' \text{ and } x \in B' \\ \Leftrightarrow \quad & x \in (A' \cap B')\end{aligned}$$

Thus, $(A \cup B)' \subseteq (A' \cap B')$
and $(A' \cap B') \subseteq (A \cup B)'$
Hence, $(A \cup B)' = A' \cap B'$.

(viii) Let x be any element of $(A \cap B)'$. Then we have

$$\begin{aligned} & x \in (A \cap B)' \\ \Leftrightarrow \quad & x \notin (A \cap B) \\ \Leftrightarrow \quad & x \notin A \text{ or } x \notin B \\ \Leftrightarrow \quad & x \in A' \text{or } x \in B' \\ \Leftrightarrow \quad & x \in (A' \cup B')\end{aligned}$$

Thus, $(A \cap B)' \subseteq (A' \cup B')$ and $(A' \cup B') \subseteq (A \cap B)'$

Hence, $(A \cap B)' = A' \cup B'$

Theorem 3: Let A and B be any two sets such that $A \subseteq B$. Then

(i) $A \cap B = A$ (ii) $A \cup B = B$ (iii) $B' \subseteq A'$

Proof:

(i) It is given that $A \subseteq B$. Then we have,
$A \cap B \subseteq A$
Now if, $x \in A$

$$\begin{aligned} \Rightarrow \quad & x \in B \\ \Rightarrow \quad & x \in A,\, x \in B \\ \Rightarrow \quad & x \in A \cap B \\ \therefore \quad & A \subseteq A \cap B\end{aligned}$$

Hence, $A \cap B = A$

(ii) We have, $B \subseteq A \cup B$

Now we have,

$$\begin{aligned} & x \in A \cup B \\ \Rightarrow\quad & x \in A \text{ or } x \in B \\ \Rightarrow\quad & x \in B \qquad \{\because A \subset B\} \\ \therefore\quad & A \cup B \subseteq B \end{aligned}$$

Hence, $A \cup B = B$.

(iii) Let x be any element of B'. Then we have

$$\begin{aligned} & x \in B' \\ \Rightarrow\quad & x \notin B \\ \Rightarrow\quad & x \notin A \qquad \{\because A \subseteq B\} \\ \Rightarrow\quad & x \in A' \end{aligned}$$

Hence, $B' \subset A'$.

Theorem 4: Let A and B be any two sets, then

(i) $(A \cup B) \cap B' = A$ *iff* $A \cap B = \phi$

(ii) $A - B = A$ *iff* $A \cap B = \phi$

(iii) $A \oplus A = \phi$

(iv) $A \oplus \phi = A$

(v) $A \oplus B = \phi$ *iff* $A = B$.

Proof:

(i) Using distributive law, we have

$$\begin{aligned} (A \cup B) \cap B' &= (A \cap B') \cup (B \cap B') \\ &= (A \cap B') \cup \phi \qquad \{\because B \cap B' = \phi\} \\ &= A \cap B' \end{aligned}$$

Now we have to prove, $A \cap B = \phi$ *iff* $A \cap B' = A$

Let $A \cap B' = A$

$\Rightarrow \quad A \subseteq B'$

$\Rightarrow \quad A \cap B = \phi$

Again we have

$A \cap B = \phi$

$\Rightarrow \quad A \subseteq B'$

$\Rightarrow \quad A \cap B' = A$

Hence, $(A \cup B) \cap B' = A$ *iff* $A \cap B = \phi$.

(ii) We have

$$\begin{aligned} A - B &= \{x : x \in A, x \notin B\} \\ &= \{x : x \in A, x \in B'\} \\ &= \{x : x \in A \cap B'\} \end{aligned}$$

Thus, $A - B = A \cap B'$

Now if, $A - B = A$ then we have

$$A = A \cap B'$$
$\Rightarrow$ $A \subseteq B'$
$\Rightarrow$ all elements of A belong to B'
$\Rightarrow$ none element of A belong to B
$\Rightarrow$ $A \cap B = \phi$.

Conversely, if $A \cap B = \phi$.
Then we have

$$A - B' = \phi$$
$\Rightarrow$ $A \subseteq B'$
Now we have,

$$A \cap B' = A$$
$\Rightarrow$ $A - B = A$.

(iii) We have
$$A \oplus A = (A - A) \cup (A - A)$$
$$= \phi \cup \phi$$
$$= A$$

(iv) We have
$$A \oplus \phi = (A - \phi) \cup (\phi - A)$$
$$= A \cup \phi$$
$$= A$$

(v) We have
$$A \oplus B = (A - B) \cup (B - A)$$
Let $A \oplus B = \phi$, then we have
$\Leftrightarrow$ $(A - B) \cup (B - A) = \phi$
$\Leftrightarrow$ $A - B = \phi$ and $B - A = \phi$
$\Leftrightarrow$ $A = B$
Hence, $A \oplus B = \phi$ *iff* $A = B$.

Theorem 5: If $A \subset B$, show that $A \times A \subset (A \times B) \cap (B \times A)$.

Proof: Let (a, b) be any two element of $A \times A$. Then we have

$a \in A$ and $b \in A$
Since $A \subset B$
$\Rightarrow$ $a \in B, b \in B$
Now we have $a \in A, b \in B$
$\Rightarrow$ $(a, b) \in A \times B$
and $a \in B, b \in A$
$\Rightarrow$ $(a, b) \in B \times A$

Therefore (a, b) belongs to $A \times B$ and $B \times A$ both.

i.e., $(a, b) \in (A \times B) \cap (B \times A)$

Hence, $A \times A \subset (A \times B) \cap (B \times A)$.

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Index

MEASURE THEORY AND INTEGRATION

Integration is one of the two cornerstones of analysis. In the fundamental work of Lebesgue, integration is presented in terms of measure theory. This introductory text starts with the historical development of the notion of the set theory and integral theory. From here, the reader is naturally led to the consideration of the Lebesgue Integral, where abstract integration is developed via the measure theory. The important topics like the Outer Measure, Cantor's Ternary Set, Measurable Function, the Lebesgue Integral, Fundamental Theorem of Calculus, L^p-spaces, Fubini's Theorem, the Radon-Nikodym Theorem, and so on are discussed. The text is written in an informal style to make the subject matter easily comprehensible. Concepts have been developed with the help of motivating examples, probing questions, followed by exercises. The book is suitable both as a textbook for an introductory course on the topic or for self-study. The core material is interspersed with examples, theorems, recapitulations, multiple-choice questions, true/false questions and fill-in-the-blank questions after relevant discussions of the topics.

Salient Features

- Contains a detailed account of some of the popular measures like Lebesgue, Borel, Jordan.
- Integration as a measuring technique is appropriately discussed.
- All the theorems are accompanied by detailed, step-by-step proofs.
- Each chapter ends with exercise problems, objective questions and summary.
- Contains a useful appendix on important theorems on Set Theory.

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