Textbook of
Analytical Geometry of Two Dimensions

Second Edition

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CONTENTS

CHAPTER I

PRELIMINARY

1. INTRODUCTION

The purpose of this chapter is to explain the basic concepts of the subject like coordinate systems, ratio formula and the idea of locus which are necessary for the development of the subject in the subsequent chapters. Assuming that the students have already studied an elementary course at the school level. The details are omitted, it is only for the sake of completeness that these concepts have been discussed here.

2. CARTESIAN COORDINATES

The lines $X'OX$ and $Y'OY$ are called the x-axis and y-axis, respectively. These two lines are also referred to coordinate axes or simply the axes. The axes divide the plane into four parts XOY, YOX', $X'OY'$ and $Y'OX$ which are called respectively the

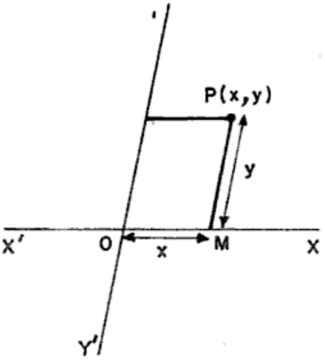


Fig. 1.1

first, second, third and fourth quadrants. The point O is called the origin. Here too we have the same rule for the sign convention as in trigonometry i.e., the lengths measured along OX and OY are positive while the lengths measured along OX' and OY' are negative.

Let P be any arbitrary point in the plane. Draw a line from P parallel to y-axis meeting the x-axis in the point M. Let $OM=x$ and $MP=y$. Then x and y are, respectively, called the x-coordinate and y-coordinate of the point P. They are also called the **abscissa** and **ordinate** of P. The pair (x, y) is referred to the cartesian coordinates or simply coordinates of the point P with reference to the coordinate axes $X'OX$ and $Y'OY$ and is usually denoted by $P(x, y)$.

Note. When the axes $X'OX$ and $Y'OY$ are perpendicular to each other, the coordinate axes are called rectangular otherwise they are called oblique axes. Throughout we shall be dealing with rectangular coordinate axes unless specified otherwise and therefore by coordinate axes or simply axes we shall mean rectangular coordinate axes. However, for those students who are interested, we are giving some details about the oblique axes in the Appendix.

Remark. With reference to an arbitrary but fixed system of coordinate axes, for a given point P in the plane, there correspond unique coordinates (x, y), and conversely for a given ordered pair of real numbers (x, y), there corresponds a unique point in the plane with (x, y) as its coordinates. Thus there is one-to-one correspondence between the set of all points in the plane and the set of all ordered pairs of real numbers, of course, with reference to a fixed system of coordinate axes.

2.1 Distance between two points. *To express the distance between two points in terms of their coordinates.*

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two given points.

Draw PM, QN parallel to y-axis and PL parallel to x-axis as shown in the Fig. 1.2.

Then $\qquad OM=x_1,\ MP=y_1,$

$\qquad\qquad ON=x_2,\ NQ=y_2.$

Now $\qquad PL=MN=ON-OM=x_2-x_1,$

and $\qquad LQ=NQ-NL=NQ-MP=y_2-y_1.$

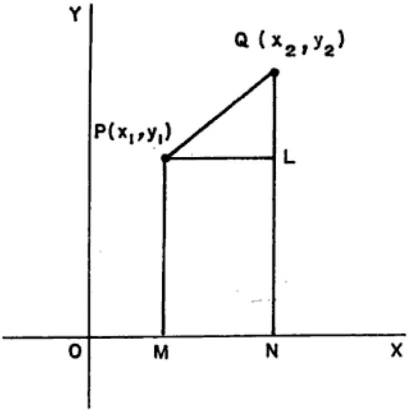


Fig. 1.2

Therefore, from right angled triangle PLQ, we have

$$PQ^2 = PL^2 + LQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

i.e. $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Thus, we have obtained the distance formula between two points P and Q when both of these are lying in the first quadrant. But this formula will be found to be true for all positions of P and Q in the plane when due regard is paid to the signs of the coordinates (x_1, y_1) and (x_2, y_2).

2.2 The ratio formula. *To find the coordinates of a point which divides the straight line joining two given points in a given ratio.*

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two given points. Let the point $R(x, y)$ divide PQ internally in the ratio $m_1 : m_2$.

Draw PM, RL, QN parallel to y-axis and $PL'N'$ parallel to x-axis. Then

$$PL' = ML = OL - OM = x - x_1,$$

and $$L'N' = LN = ON - OL = x_2 - x.$$

By geometry, we have

$$\frac{PL'}{L'N'} = \frac{PR}{RQ} = \frac{m_1}{m_2}$$

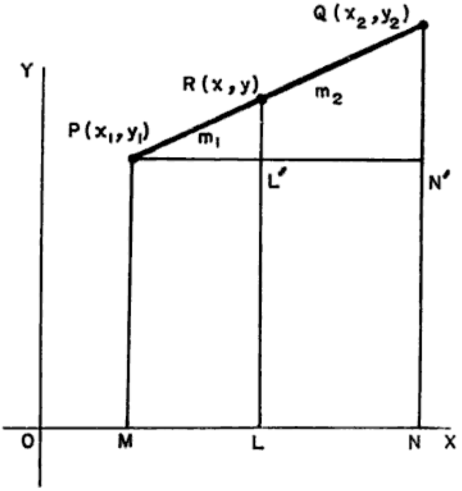


Fig. 1.3

$$\Rightarrow \qquad \frac{x-x_1}{x_2-x}=\frac{m_1}{m_2}$$

$$\Rightarrow \qquad x=\frac{m_1x_2+m_2x_1}{m_1+m_2}.$$

Similarly, $$y=\frac{m_1y_2+m_2y_1}{m_1+m_2}.$$

Corollary 1. The coordinates of the middle point of the line joining the points (x_1, y_1) and (x_2, y_2) are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Corollary 2. If R divides PQ externally in the ratio $m_1 : m_2$, then the coordinates of R are

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}\right).$$

Since in this case the distances PL' and $L'N'$ are measured in the opposite sense, we have

$$\frac{PL'}{L'N'}=-\frac{m_1}{m_2},$$

which gives $$x=\frac{m_1x_2-m_2x_1}{m_1-m_2},\ m_1\neq m_2.$$

Similarly, $$y=\frac{m_1y_2-m_2y_1}{m_1-m_2},\ m_1\neq m_2.$$

Remark. The above results can be extended to other quadrants provided the coordinates of the points are taken with proper signs.

Ex. 1. Find the coordinates of the points which divide the line joining the points (2, −8) and (−5, 6) internally and externally in the ratio 3 : 4.

Ex. 2. Show that the medians of a triangle are concurrent.

2.3. Area of a triangle. *To express the area of a triangle in terms of the coordinates of its vertices.*

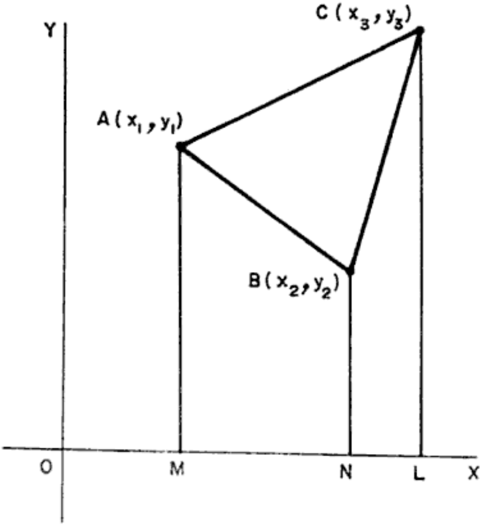


Fig. 1.4

Let ABC be the triangle. Let the coordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively.

Draw the lines AM, BN and CL parallel to y-axis.

Then

$$\begin{aligned}\triangle ABC &= \text{trapezium } AMLC - \text{trapezium } AMNB - \text{trapezium } BNLC\\ &= \tfrac{1}{2}(MA+LC)\cdot ML - \tfrac{1}{2}(MA+NB)\cdot MN - \tfrac{1}{2}(NB+LC)\cdot NL\\ &= \tfrac{1}{2}(y_1+y_3)(x_3-x_1) - \tfrac{1}{2}(y_1+y_2)(x_2-x_1) - \tfrac{1}{2}(y_2+y_3)(x_3-x_2)\\ &= \tfrac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)\}.\end{aligned}$$

Thus *the area of the triangle whose vertices are* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *is*

$$\tfrac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)\}.$$

Note. The area of the triangle can also be expressed in the determinant form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}.$$

Remark. The area of the triangle ABC is positive if the vertices A, B and C are taken in the counter-clockwise order but negative if the vertices are taken in clockwise order. However, in most of the cases only the absolute area is required.

Corollary. Three points are collinear if and only if the area of the triangle so formed by taking the points as vertices is zero.

Ex. Find the value of λ so that the points $(\lambda, 2)$, $(-2, 1)$ and $(-3, -2)$ are collinear.

EXERCISES

1. Find the distance between the following pair of points:
    - (*i*) $(5, 1)$ and $(6, 0)$.
    - (*ii*) $(2, 8)$ and $(2, -3)$.
    - (*iii*) $(1, 2)$ and $(3, 4)$.
    - (*iv*) $(-1, -3)$ and $(5, -6)$.
    - (*v*) $(2, -2)$ and $(1, -5)$.

2. Find the lengths of the sides of the triangle whose vertices are (5, 1), (−3, 7) and (8, 5) and prove that one of the angles is a right angle.
3. Prove that the triangle with vertices at the points (0, 3), (−2, 1) and (−1, 4) is a right angled triangle.
4. Show that the points (a, a), $(-a, -a)$ and $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle.
5. Show that the quadrilateral formed by joining the points (1, 0), (6, 1), (5, 6) and (0, 5) is a square.
6. Show that the four points (1, −4), (1, 0), (3, −2) and (−1, −2) form the vertices of a square and calculate the length of a diagonal.
7. Find the coordinates of the point which divides the straight line joining the given points in a given ratio:
   - (*i*) (−7, 3), (2, −4); ratio 4:5.
   - (*ii*) (0, 0), (7, 5); ratio 5:2.
   - (*iii*) (5, 0), (−1, 2); ratio −4:3.
   - (*iv*) (5, 2), (−1, 1); ratio −1:2.
   - (*v*) (−3, 9), (5, −7); ratio 5:3.
8. Find the coordinates of the point which divides the line joining the points (−3, 4) and (5, 6) internally in the ratio 3:2.
9. Find the coordinates of the point which divides the line joining the points (−1, −5) and (1, −2) externally in the ratio 4:3.
10. Find the centroid of the triangle whose vertices are (−4, 6), (2, −2) and (2, 5).
11. In what ratio does the point (−1, −1) divide the join of (−5, −3) and (5, 2)?
12. Find the areas of the triangles with the following vertices:
   - (*i*) (0, 0), (12, 0), (0, 5).
   - (*ii*) (−2, 3), (4, 3), (1, 1).
   - (*iii*) $(0, 0)$, (a, b), (b, a).
   - (*iv*) $(0, 0)$, $(2a \sin \alpha, 2a \cos \alpha)$, $(\sin \beta, \cos \beta)$.
13. Find the area of the quadrilateral whose vertices taken in order are (1, 2), (6, 2), (5, 3) and (3, 4).
14. If A, B, C and D are the points (3, 1), (7, −3), (8, −1) and (19, −3), respectively, prove that the areas of the triangles ABC and ADC are equal in magnitude but opposite in sign.

15. Find the coordinates of the middle points of the sides of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) and show that the area of the triangle formed by joining these points is one-fourth of that of the original triangle.
16. Show that the points (0, 4), (3, 2) and (6, 0) are collinear.
17. For what value of λ the points $(0, \lambda)$, $(-2, 1)$ and $(-3, -2)$ are collinear?

3. POLAR COORDINATES

There are various types of coordinate systems. The cartesian system, in particular, the rectangular cartesian system with which we have been dealing with is probably the most important. In that system a point is located by its distances from two perpendicular lines. We shall introduce in this section a coordinate system in which the coordinates of a point in a plane are its distance from a fixed point and its direction from a fixed line. The coordinates given in this way are called polar coordinates. The proper choice of a coordinate system depends on the nature of the problem at hand. For some problems either the rectangular cartesian or the polar system may be satisfactory; usually, however, one of the two is preferable. In some situations it is advantageous to use both systems, shifting from one to the other.

In polar system of coordinates, we consider a fixed point *O*, called the **origin** (or **pole**) and a fixed straight line *OX*, called the

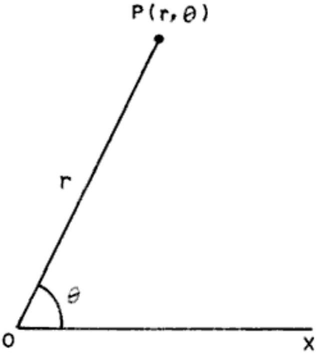


Fig. 1.5

polar axis (or **initial line**). Consider a point P in the plane and let $OP=r$ and $\angle XOP=\theta$. The length r is called the **radius vector** of the point P and the angle XOP the **vectorial angle.** The ordered pair (r, θ) is called the **polar coordinates** of the point P.

Note. The radius vector is positive if it is measured from the pole along the line bounding the vectorial angle and is negative in the opposite direction.

If PO is produced to P' so that $OP'=OP$ in magnitude, and if (r, θ) be the coordinates of P, then coordinates of P' will be either $(-r, \theta)$ or $(r, \theta+\pi)$.

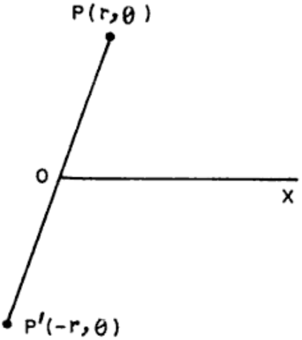


Fig. 1.6

It is easy to see that

$$(r, \theta), (r, \theta+2\pi), (r, \theta+4\pi), \ldots\ldots;$$
$$(r, \theta-2\pi), (r, \theta-4\pi), \ldots\ldots;$$
$$(-r, \theta+\pi), (-r, \theta+3\pi), \ldots\ldots;$$

and $$(-r, \theta-\pi), (-r, \theta-3\pi), \ldots\ldots;$$

all represent the coordinates of the same point P. Thus in polar coordinates, unlike in cartesian system; the representation of a point is not unique.

3.1 Distance between two points. *To find the distance between two points whose polar coordinates are given.*

Let the coordinates of two points P and Q be (r_1, θ_1) and (r_2, θ_2), respectively.

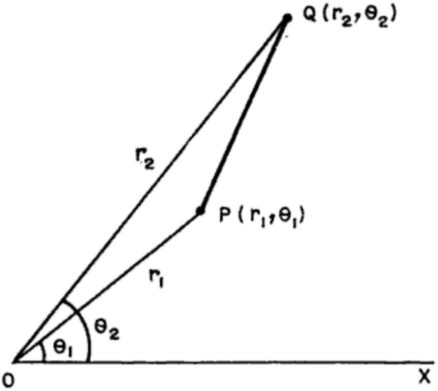


Fig. 1.7

Then $$OP=r_1,\ OQ=r_2,$$
$$\angle XOP=\theta_1,\ \angle XOQ=\theta_2.$$

By trigonometry, we have

$$PQ^2=OP^2+OQ^2-2OP\cdot OQ\cos POQ$$
$$=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)$$

since $$\angle POQ=\angle XOQ-\angle XOP=\theta_2-\theta_1.$$

Hence *the distance between the points* $P(r_1, \theta_1)$ *and* $Q(r_2, \theta_2)$ *is*

$$\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}.$$

3.2 Area of a triangle. *To find the area of a triangle the polar coordinates of whose vertices are given.*

Let ABC be the triangle. Let the polar coordinates of A, B and C be (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3), respectively.

Then $\triangle ABC=\triangle OAB+\triangle OBC-\triangle OAC$

$$=\tfrac{1}{2}\,OA\cdot OB\sin AOB+\tfrac{1}{2}\,OB\cdot OC\sin BOC-\tfrac{1}{2}\,OA\cdot OC\sin AOC$$

$$=\tfrac{1}{2}\,r_1r_2\sin(\theta_2-\theta_1)+\tfrac{1}{2}\,r_2r_3\sin(\theta_3-\theta_2)-\tfrac{1}{2}\,r_3r_1\sin(\theta_3-\theta_1)$$

$$=\tfrac{1}{2}\,\{r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)\}.$$

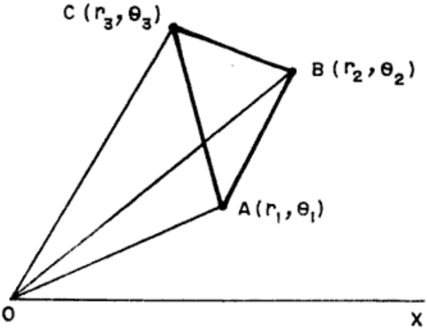


Fig. 1.8

Hence *the area of the triangle ABC is*

$$\tfrac{1}{2}\{r_1 r_2 \sin(\theta_2-\theta_1)+r_2 r_3 \sin(\theta_3-\theta_2)+r_3 r_1 \sin(\theta_1-\theta_3)\}.$$

4. TRANSFORMATION

To change the cartesian coordinates of a point into polar coordinates and vice versa.

Let P be any point with cartesian coordinates (x, y) and polar coordinates (r, θ).

Draw PM parallel to y-axis.

Then $$OM=x,\ MP=y,$$
$$OP=r,\ \angle XOP=\theta.$$

Therefore,

$$x=OP\cos\theta=r\cos\theta$$

and $$y=OP\sin\theta=r\sin\theta.$$

Also $$r=OP=\sqrt{x^2+y^2}$$

and $$\tan\theta=\frac{MP}{OM}=\frac{y}{x}.$$

From these transforms it is easy to change any equation from cartesian form to polar form and vice versa.

Ex. 1. Transform $x^2+y^2-2x+2y=0$ into polar form.

Ex. 2. Transform $r=4a\cos\theta$ into cartesian form.

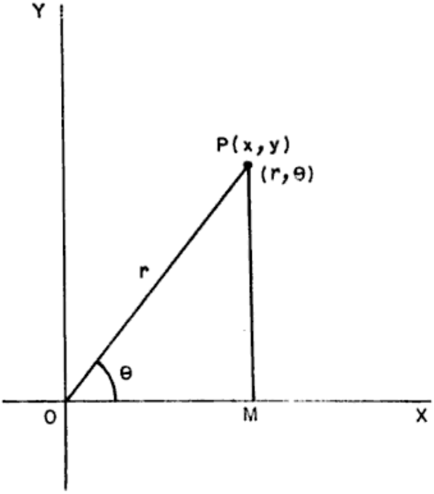


Fig. 1.9

EXERCISES

1. Find the lengths of the straight lines joining the pair of points whose polar coordinates are:

   (*i*) (3, 60°) and (5, 150°).
   (*ii*) (−6, 30°) and (4, 90°).
   (*iii*) (2, 40°) and (4, 100°).
   (*iv*) (2*a*, 30°) and (4*a*, 120°).

2. Prove that the points (0, 0°), (3, 90°) and (3, 30°) form an equilateral triangle.
3. Find the areas of the triangles with vertices

   (*i*) (2, 60°), (3, 90°), (4, 120°).
   (*ii*) (−6, 30°), (4, 90°), (5, 150°).
   (*iii*) (2*a*, 30°), (4*a*, 60°), (6*a*, 90°).

4. Transform the following equations to the corresponding polar coordinate equations:

   (*i*) $3x+y=0$.

(*ii*) $x^2 + y^2 = 16.$

(*iii*) $y^2 = 4\,ax.$

(*iv*) $(x^2 + y^2)^2 = 2\,a^2xy.$

5. Transform the following equations to the corresponding cartesian coordinate equations:

(*i*) $r^2 = a^2 \cos 2\theta.$

(*ii*) $r = 8 \cos \theta.$

(*iii*) $r = \dfrac{4}{1 + \cos \theta}.$

5. LOCUS

The path traced by a moving point under certain geometrical conditions is called the **locus** *of that point.* For instance, the locus of the point *P* which moves such that it remains at a constant distance 2 from the fixed point *A* is a circle, Fig. 1.10.

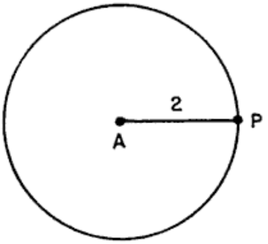


Fig. 1.10

The sciences of algebra and geometry correlated by means of the coordinate systems lead to a study of coordinate geometry. In plane coordinate geometry, the underlying feature is the correspondence between an equation in x and y, the coordinates of a variable point, and a geometric figure of the locus (path) of the variable point. In regard to this there are two problems to be studied. One is that we are given some geometrical facts which limits the positions of the moving point P and asked to find what relation is then satisfied by x and y, the coordinates of P. Conversely, an equation connecting x and y is given, and the problem is to find out on what geometrical locus P lies.

5.1 Equation of the locus. *An equation of a locus is a relation between x and y which is satisfied by the coordinates of all points of the locus and by no others.*

Illustration 1. $P(x, y)$ is equidistant from the point $A(2, 0)$ and y-axis. Find the equation of the locus of P.

Here $PA = \sqrt{(x-2)^2 + y^2}$.

It is given that

$$PA = \text{distance of } (x, y) \text{ from } y\text{-axis}$$

i.e. $$\sqrt{(x-2)^2 + y^2} = x$$

$$\Rightarrow \quad (x-2)^2 + y^2 = x^2$$

$$\Rightarrow \quad y^2 = 4(x-1)$$

This is the required equation of the locus of P.

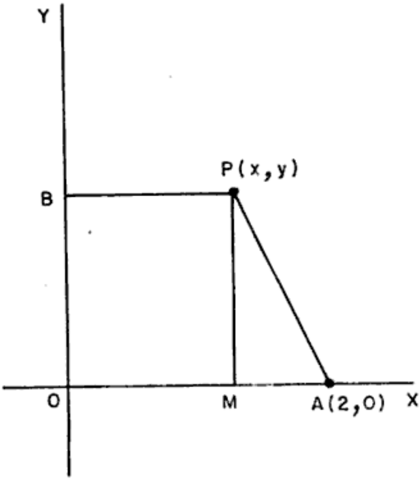


Fig. 1.11

Illustration 2. $P(x, y)$ is equidistant from the points $A(2, 3)$ and $B(3, -1)$. Find the equation of the locus of P.

Here $PA = \sqrt{(x-2)^2 + (y-3)^2}$

$PB = \sqrt{(x-3)^2 + (y+1)^2}$

Given that $PA = PB$.

$$\therefore \quad (x-2)^2 + (y-3)^2 = (x-3)^2 + (y+1)^2$$

$$\Rightarrow \quad 2x - 8y + 3 = 0.$$

This is the required equation of the locus of P.

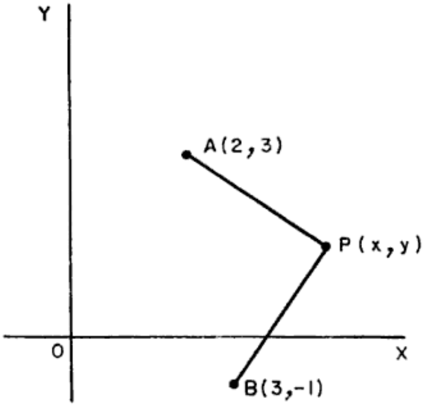


Fig. 1.12

5.2 Locus of the equation. *The locus (or graph) of an equation in x and y consists of all points whose coordinates satisfy the given equation.*

Illustration 1. Plot the locus of the equation $4x+6y-3=0$.

Solving for y in terms of x, we get

$$y = \frac{3-4x}{6}.$$

By giving different values to x compute the corresponding values of y.

x	0	3	6	9	12
y	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{7}{2}$	$-\frac{11}{2}$	$-\frac{15}{2}$

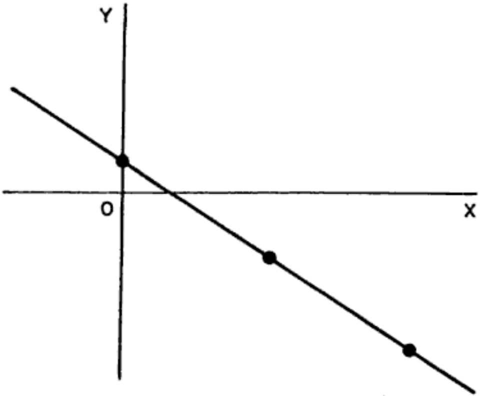


Fig. 1.13

Illustration 2. Plot the locus of the equation

$$x^2+y^2-4x-5=0.$$

Solving for y in terms of x, we get

$$y=\pm\sqrt{5+4x-x^2}.$$

x	0	1	2	3	4	5
y	$\pm\sqrt{5}$	$\pm 2\sqrt{2}$	± 3	$\pm 2\sqrt{2}$	$\pm\sqrt{5}$	0

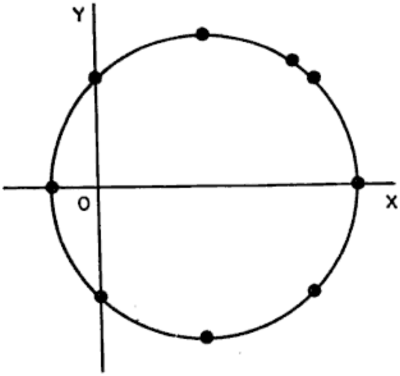


Fig. 1.14

EXERCISES

1. A point P moves so that its distances from the two points (3, 4) and (5, −2) are equal to one another. Find the equation of the locus of P.
2. Prove that the locus of a point which is equidistant from the point $(a+b, b-a)$ and $(a-b, b+a)$ is $bx=ay$.
3. The sum of the squares of the distance of a moving point from the two fixed points (2. 0) and (−2, 0) is equal to 16. Find the equation of its locus.
4. A point P moves so that its distance from the fixed point (0, 2) is equal to its distance from the x-axis. Prove that the equation of the locus is $x^2=4(y-1)$.
5. Find the equation of the locus of a point which is at a distance 3 from the point (3, −1).
6. A point moves so that its distance from the x-axis is half of its distance from the origin. Find the equation of its locus.
7. Find the equation of the locus of a point whose distance from (−1, 1) is equal to twice its distance from the x-axis.
8. Given $A(a, b)$ and $B(3a, 3b)$, show that if $P(x, y)$ is a point such that $PA=PB$, then $ax+by=2(a^2+b^2)$.

9. Find the locus of a point P which moves such that its distance from (0, 3) is equal to the ordinate of P.
10. Find the locus of a point P which moves such that the difference of the distance from P to $(-5, 0)$ and $(5, 0)$ is numerically equal to 8.

CHAPTER II

THE STRAIGHT LINE

1. INTRODUCTION

The simplest locus of a point in a plane is a straight line. In this chaper, we obtain the equations of a straight line, in various forms, and obtain a number of basic properties of the straight lines. Throughout the book we shall be calling a straight line simply a line.

2. PARTICULAR LINES

Before we come to a general discussion of the equations of a line, we examine one or two particular cases.

To find the equation of a line parallel to one of the coordinate axes.

Let AB be a line parallel to y-axis. Clearly AB is the locus of a point which moves such that its distance from y-axis remains fixed. Let $P(h, k)$ be the moving point. Then, by definition, P is

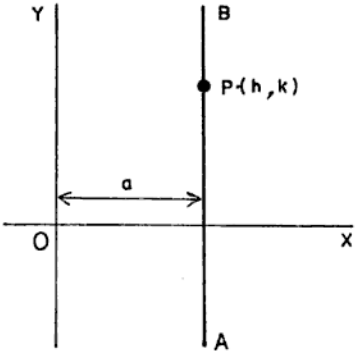


Fig. 2.1

at a distance a (say) from y-axis. Therefore,

$$h=a.$$

Thus, the locus of (h, k) is $x=a$, which is the required equation of the line AB.

Similarly, the equation of the line parallel to x-axis which it at a distance b is

$$y=b.$$

Corollary. The equation of x-axis is $y=0$ and that of y-axis is $x=0$.

3. SLOPE OF A LINE

If any two points on a line are taken, their join makes a constant angle with a fixed direction, and the angle so formed is independent of the choice of the two points on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle α which a line makes with the positive direction of the x-axis. Then, the quantity $\tan\alpha$ is defined to be the **slope** of the line and is denoted by 'm'.

Note. The slope of a line is also sometimes referred to **gradient** of the line.

3.1 *To find the slope of the line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.*

Draw PM, QN parallel to y-axis and PL parallel to x-axis. Let the line joining P and Q make an angle α with x-axis.

Then $$\tan\alpha=\frac{LQ}{PL}.$$

But $$LQ=NQ-NL=NQ-MP=y_2-y_1,$$

and $$PL=MN=ON-OM=x_2-x_1,$$

$\therefore$ $$\tan\alpha=\frac{y_2-y_1}{x_2-x_1},$$

which is the required slope of the line PQ.

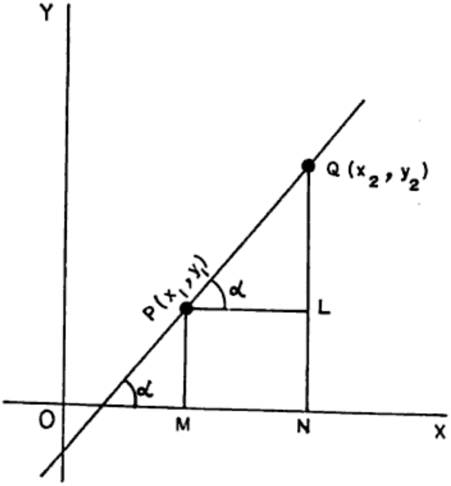


Fig. 2.2

Ex. Find the slope of the line joining the points (1, −4) and (5, −3).

3.2 Parallel and perpendicular lines. Two lines are said to be **parallel** if they make the same angle with x-axis and hence they have the same slope. Conversely, if two lines have the same slope, they make the same angle with x-axis and hence they are parallel. Thus, if m and m' be the slopes of any two parallel lines, then

$$m = m'.$$

Note. The lines parallel to x-axis have **zero slope** whereas the lines parallel to y-axis have the **slope** ∞.

Further consider any two perpendicular lines with slopes m, m' and making angles α, β with x-axis, respectively. Then

$$\tan \alpha = m, \tan \beta = m'.$$

From Fig. 2.3, it is clear that

$$\beta = \frac{\pi}{2} + \alpha.$$

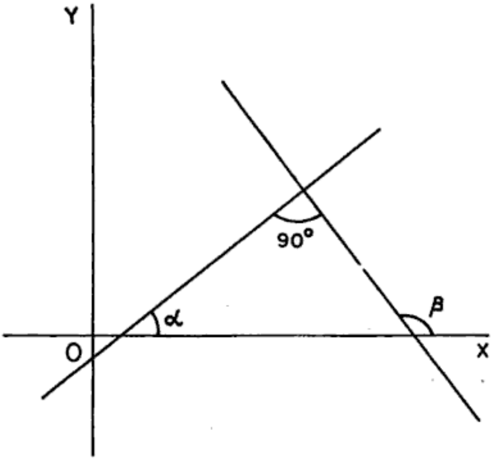


Fig. 2.3

$$\therefore \qquad \tan\beta=\tan\left(\frac{\pi}{2}+\alpha\right)=-\cot\alpha=-\frac{1}{\tan\alpha}$$

$$\Rightarrow \qquad \tan\alpha\tan\beta=-1$$

$$\Rightarrow \qquad mm'=-1.$$

Conversely, if $mm'=-1$, then the lines with slopes m, m' are perpendicular to each other.

4 VARIOUS FORMS OF THE EQUATION OF A LINE

4.1 Slope intercept form. *To find the equation of a line which cuts off a given intercept on y-axis and is inclined to a given angle to the x-axis.*

Let AB be the line having slope m and making intercept c on y-axis. Let α be the angle which the line AB makes with the positive direction of x-axis so that $m=\tan\alpha$.

Let $P(x, y)$ be any point on the line AB. Draw PM parallel to y-axis and CN parallel to x-axis.

Then $\qquad CN=OM=x$

and $\qquad NP=MP-MN=MP-OC=y-c$

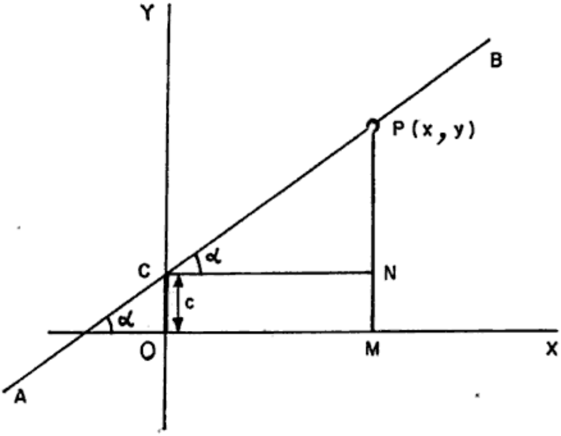


Fig. 2.4

In ΔPCN, $\angle PNC = 90°$. Therefore

$$\frac{NP}{CN} = \tan\alpha$$

$$\Rightarrow \qquad \frac{y-c}{x} = \tan\alpha$$

$$\Rightarrow \qquad y = x\tan\alpha + c.$$

Hence *the equation of the line having slope m and making an intercept c on y-axis is*

$$y = mx + c$$

Corollary. The equation of the line passing through the origin is $y = mx$.

4.2 Intercept form. *To find the equation of a line in terms of the intercept which it makes on the axes.*

Let AB be the line which cuts off intercepts a and b, respectively, on x-axis and y-axis so that

$$OA = a \text{ and } OB = b.$$

Let $P(x, y)$ be any point on AB. Draw PM parallel to y-axis and PN parallel to x-axis. Then, from similar triangles BNP and

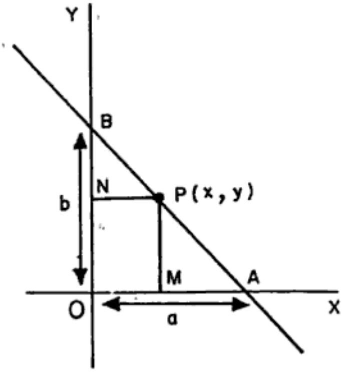


Fig. 2.5

PMA, we have

$$\frac{NB}{MP}=\frac{NP}{MA}$$

$$\Rightarrow \quad \frac{OB-ON}{ON}=\frac{OM}{OA-OM}$$

$$\Rightarrow \quad \frac{b-y}{y}=\frac{x}{a-x}$$

$$\Rightarrow \quad bx+ay=ab.$$

Hence *the equation of the line making intercepts a and b on the axes is*

$$\frac{x}{a}+\frac{y}{b}=1.$$

4.3 Normal form. *To find the equation of a line in terms of the length of the perpendicular on it from the origin and the angle which that perpendicular makes with* **x***-axis.*

Let AB be the line whose perpendicular distance from the origin is p (say). Draw ON perpendicular to AB. Then $ON=p$. Let ON make an angle α with the positive direction of x-axis.

Now $OA=p \sec \alpha$ and $OB=p \operatorname{cosec} \alpha$.

Clearly OA and OB are the intercepts of AB on the axes. Hence, by §4.2, we have

$$\frac{x}{p \sec \alpha}+\frac{y}{p \operatorname{cosec} \alpha}=1$$

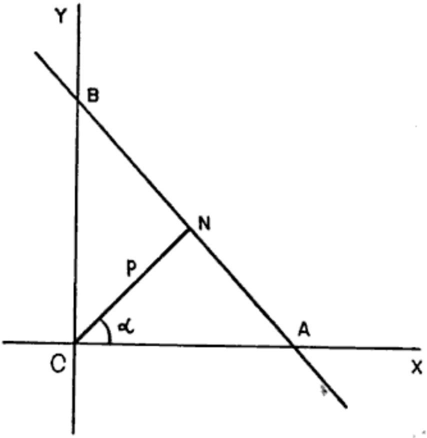


Fig. 2.6

$\Rightarrow \quad x \cos \alpha + y \sin \alpha = p.$

This is the required equation of the line AB.

Note. Normal form is also referred to '**perpendicular form**'.

4.4 Point-slope form. *To find the equation of a line which passes through a given point and makes an angle with x-axis.*

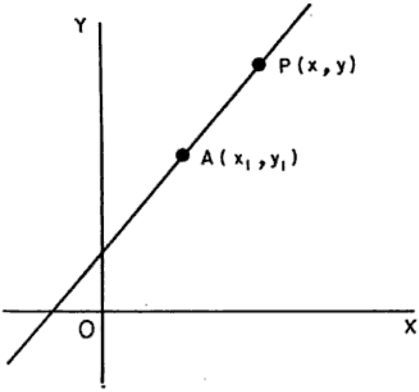


Fig. 2.7

Let m be the slope of the line passing through $A(x_1, y_1)$. Let $P(x, y)$ be any point on the line

Then $$m=\frac{y-y_1}{x-x_1}.$$

Hence *the equation of the line passing through the point* (x_1, y_1) *and having slope m is*

$$y-y_1=m(x-x_1).$$

Ex. Find the equation of the line passing through $(-2, 3)$ and having slope 2.

1.5 Two-point form. *To find the equation of a line which passes through two given points.*

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points. Since the points A and B both lie on the line, the slope m of the line is given by

$$m=\frac{y_2-y_1}{x_2-x_1}.$$

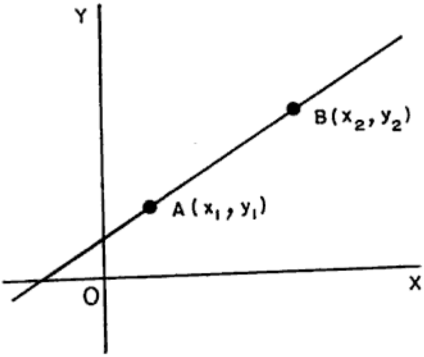


Fig. 2.8

The equation of the line through the point $A(x_1, y_1)$ having slope m is

$$y-y_1=m(x-x_1).$$

Hence, *the line through the points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is*

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$

Corollary. The equation of the line joining the origin to the point (x_1, y_1) is $y=\frac{y_1}{x_1}x$ *i.e.*, $xy_1-x_1y=0$.

Ex. Find the equation of the line passing through the points (1, 2) and (3, −4).

4.6 Parametric form. In coordinate geometry, very oftenly, we are interested to express the coordinates of a variable point on a line or on a curve in terms of a single variable called the **parameter,** viz., in the form

$$\left.\begin{aligned} x&=\phi(t) \\ y&=\psi(t) \end{aligned}\right\}. \tag{1}$$

Here t is the parameter. In this regard we must have the following properties:

(*i*) Given a point on the locus there corresponds a unique value of the parameter.

(*ii*) Given a value of the parameter, there corresponds a unique point on the locus.

Thus there is one-to-one correspondence between the values of the parameter and the points on the locus. Moreover, if we eliminate the parameter between the two relations in (1), the resultant is the equation of the line or the curve, as the case may be.

To find the equation of a line in the parametric form which passes through a given point and intercepts an angle with x-axis.

Let $P(x, y)$ be any point on the line passing through the point $A(x_1, y_1)$ and inclined at an angle θ with the positive direction of x axis.

Let $AP=r$. If we allow r to vary with any positive or negative values, P will take any position on the line; and conversely, if P is given to be any point on the line, the unique value of r can be found which, in fact, is the distance of P from A. Thus, it follows that r serves as a parameter of P.

Now, to find the coordinates of P in terms of the parameter r, let us draw AL and PM parallel to y-axis and AN parallel to x-axis. Then, it follows that

$$\begin{cases} OM=OL+LM=OL+AN \\ MP=MN+NP=LA+NP \end{cases}$$

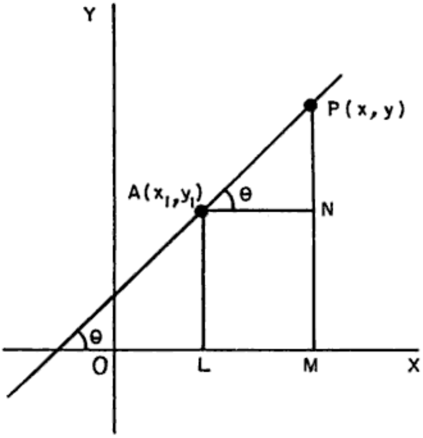


Fig. 2.9

$$\Rightarrow \qquad \begin{cases} x=x_1+r\cos\theta \\ y=y_1+r\sin\theta \end{cases}$$

Hence, *the equation of the line, in the parametric form, through* $A(x_1, y_1)$ *and inclined at any angle* θ *is*

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r. \tag{2}$$

Remark. The parametric form (2) of the equation of a line is very useful. It is noted that the coordinates of any point P on the line are immediately expressed if we know the algebraic distance of P from the given point $A(x_1, y_1)$ measured along the line which, in fact, are

$$x=x_1+r\cos\theta,\ y=y_1+r\sin\theta.$$

EXERCISES

1. In each of the following cases, find the equation of the line through the point P with slope m:
   - (*i*) P is (4, 1), $m=-\frac{1}{2}$.
   - (*ii*) P is (1, 5), $m=3$.

(*iii*) P is $(-3, -2)$, $m=\frac{1}{5}$.

(*iv*) P is $(4, -2)$, $m=-1$.

2. Find the equation of the line which passes through the point $(3, -1)$ and makes equal intercepts on the axes.
3. In each of the following cases, find the equation of the line through the points:

   (*i*) $(-4, 2)$ and $(1, -3)$.

   (*ii*) $(-1, 5)$ and $(2, 3)$.

   (*iii*) (a, b) and (b, a).

   (*iv*) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

   (*v*) $(a \cos \theta, b \sin \theta)$ and $(a \cos \phi, b \sin \phi)$.
4. Find the equation of the line which has intercept a on x-axis and intercept b on y-axis in the following:

   (*i*) $a=3, b=4$.

   (*ii*) $a=5, b=-3$.

   (*iii*) $a=-3, b=4$.

   (*iv*) $a=8, b=-2$.

   (*v*) $a=\frac{2}{3}$, $b=-\frac{5}{8}$.
5. Prove that the line through the points $(4, 3)$ and $(2, 5)$ cuts off equal intercepts on the axes.
6. Find the equation of the line through the point $(3, 6)$ which makes an angle $\tan^{-1} 3$ with x-axis.
7. Find the equation of the line joining the points $(3, 2)$ and $(-1, -5)$. Prove that the point $\left(1, \frac{7}{2}\right)$ lies on it, and find where it meets the axes.
8. What is the slope of the line $3x+4y=1$? Find the equations of the lines through the point $(3, -1)$ which are parallel and perpendicular to it.
9. Find the equations of the medians of the triangle whose vertices are $(4, 3)$, $(-2, 5)$ and $(2, -1)$.
10. Find the equation of the line through the point $(2, -1)$ perpendicular to $4x-3y=6$.
11. Find the equation of the line with slope 3 which meets the x-axis in the same point as the line $x+4y=8$.

5. GENERAL EQUATION OF FIRST DEGREE

From the various forms of the equations of a line obtained so far in §4, we observe that the equation of a line is a first degree (or linear) equation in x and y, the coordinates of the moving point on the line. It is a natural temptation to examine the converse problem, viz., whether a first degree equation in x and y always represents a line.

5.1 *To prove that every equation of first degree in x and y represents a line.*

The general equation of first degree in x and y is

$$ax+by+c=0, \tag{1}$$

where a, b, c are constants.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points lying on the locus of (1). Therefore,

$$ax_1+by_1+c=0 \tag{2}$$

$$ax_2+by_2+c=0 \tag{3}$$

and $$ax_3+by_3+c=0. \tag{4}$$

On eliminating a, b, c from equations (2), (3) and (4), we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0. \tag{5}$$

Equation (5) shows that the area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is zero [§2.3, Chapter I]. Hence the three points lying on the locus of (1) are collinear which shows that equation (1) represents a line.

5.2 Reduction to standard forms. *To reduce the general equation of first degree in standard forms of the line.*

Let $$ax+by+c=0 \tag{1}$$

be the general equation of first degree in x and y. From §5.1, equation (1) represents a line whose various forms have been obtained as follows:

(I) Slope-intercept form. Equation (1) can be written as

$$y=-\frac{a}{b}x-\frac{c}{b},\ b\neq0,$$

which is of the form $y=mx+k$, where

$$m=-\frac{a}{b} \text{ and } k=-\frac{c}{b}.$$

This is the **Slope-intercept form** of the line.

Note. The slope of the line $ax+by+c=0$ is given by

$$m=-\frac{a}{b}=-\frac{\text{coefficient of } x}{\text{coefficient of } y}.$$

(II) Intercept form. On dividing equation (1) by $-c(c\neq0)$, we get

$$\frac{a}{-c}x+\frac{b}{-c}y-1=0$$

$$\Rightarrow \qquad \frac{x}{-\frac{c}{a}}+\frac{y}{-\frac{c}{b}}=1,$$

which is the **intercept form** of the line. Clearly $-\frac{c}{a}$ and $-\frac{c}{b}$ are the intercepts on the axes.

Note. If $a\to0$ or $b\to0$ and $c\neq0$, then the line (1) is parallel to one of the coordinate axes as the case may be. If both $a\to0$ and $b\to0$, the line (1) is parallel neither to x-axis nor to y-axis and in this case $c=0$ which represents a line lying at infinity.

(III) Normal form. Equation (1) can also be written as

$$\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y=-\frac{c}{\sqrt{a^2+b^2}}.$$

This equation is of the form

$$x\cos\alpha+y\sin\alpha=p,$$

which is the **normal form**, where

$$\cos\alpha=\frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha=\frac{b}{\sqrt{a^2+b^2}} \text{ and } p=-\frac{c}{\sqrt{a^2+b^2}}.$$

Ex. Reduce the equation $2x+3y+4=0$ to various standard forms.

5.3 Intersection of two lines. *To find the coordinates of the point of intersection of two given lines.*

Let $$a_1x+b_1y+c_1=0 \tag{1}$$
and $$a_2x+b_2y+c_2=0 \tag{2}$$
be any two given lines. Let (x_1, y_1) be the point of intersection of (1) and (2). Then the point (x_1, y_1) lies on both (1) and (2).

Therefore, we get
$$a_1x_1+b_1y_1+c_1=0 \tag{3}$$
and $$a_2x_1+b_2y_1+c_2=0. \tag{4}$$

On solving (3) and (4), we get
$$x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Thus the point of intersection is
$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right).$$

Note. When $a_1b_2-a_2b_1=0$, the coordinates of the point of intersection become infinite *i.e.* the point of intersection lies at infinity. This means that the two lines are parallel and do not intersect provided the lines are distinct.

Working rule. In order to obtain the point of intersection of two lines we simply solve the equations and obtain the values of x and y which will give the coordinates of the point of intersection of the lines.

Ex. Find the coordinates of the point of intersection of the lines
$$2x+3y-7=0$$
and $$3x+4y+6=0.$$

EXERCISES

1. Reduce the following equations into standard forms:
   - (*i*) $x+y+4=0$.
   - (*ii*) $2x+5y+13=0$.
   - (*iii*) $4x-3y=4$.
   - (*iv*) $7x-2y=8$.
   - (*v*) $3x+4y=1$.

2. Find the coordinates of the points of intersection of the following pair of lines:
   (*i*) $4x+y-6=0$, $5x-2y+12=0$.
   (*ii*) $2x-y+1=0$, $x+3y-10=0$.
   (*iii*) $2x+3y+4=0$, $4x+3y+2=0$.
3. Find the equation of the line passing through the point (2, 4) and the point of intersection of the lines

$$2x-y=4$$

and

$$3x+2y=13.$$

6. AGNLE BETWEEN TWO LINES

To find the angle between two lines whose equations are given.

Let the equations of the lines be

$$y=mx+c$$

and

$$y=m'x+c'$$

making angles θ and θ′, respectively with *x*-axis. Therefore

$$m=\tan\theta \text{ and } m'=\tan\theta'.$$

Let ϕ be the angle between the lines. Then, from the Fig. 2.10, we have

$$\phi=\theta-\theta'.$$

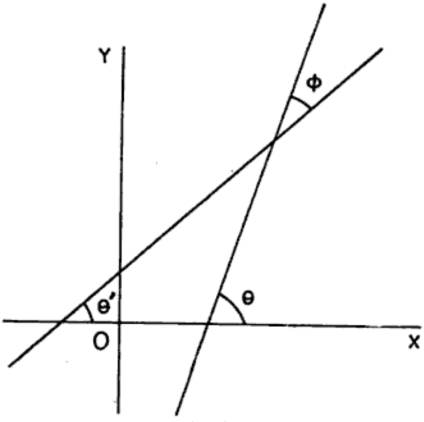


Fig. 2.10

$$\therefore \quad \tan\phi=\tan(\theta-\theta')$$

$$=\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}$$

$$=\frac{m-m'}{1+mm'}.$$

Thus the required angle is

$$\phi=\tan^{-1}\left(\frac{m-m'}{1+mm'}\right).$$

Remark. Sometimes the value of $\tan\phi$ may be negative which is,in fact,due to the factor $m-m'$ and it corresponds to the obtuse angle between the two lines. However, in numerical problems we oftenly consider the value of $\tan\phi$ to be positive. Thus, consider the acute angle only.

Corollary 1. The lines $y=mx+c$ and $y=m'x+c'$ are parallel if $m=m'$.

Corollary 2. The lines $y=mx+c$ and $y=m'x+c'$ are perpendicular if $mm'=-1$.

Ex. 1. Prove that the angle between the lines

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0 \text{ is}$$

$$\tan^{-1}\left(\frac{a_2b_1-a_1b_2}{a_1a_2-b_1b_2}\right).$$

Ex. 2. The angle between the lines in which the origin lies, is the supplement of the angle between their perpendiculars.

(**Hint:** Consider the equations of the lines in normal form).

EXERCISES

1. Find the angle between the following pair of lines:
   (*i*) $4x-3y+1=0, \quad x-y+1=0.$
   (*ii*) $4x-3y=1, \quad x-5y=4.$
   (*iii*) $5x-y=9, \quad x+6y=8.$
   (*iv*) $3x+y=7, \quad x+2y+9=0.$
2. Find the equation of the line through the point $(-2, 3)$ which makes an angle of 45° with the line $3x+y+5=0$.
3. Find all the angles of the triangle whose sides are

$$2x-3y-1=0,\ 4x+3y-5=0,\ x+y+2=0.$$

7. PERPENDICULAR DISTANCE OF A POINT FROM A LINE

To find the perpendicular distance of a given point from a given line.

Let us first consider the equation of the line AB in the normal form

$$x \cos \alpha + y \sin \alpha - p = 0. \tag{1}$$

Let $P(x_1, y_1)$ be the given point from which perpendicular distance of the line AB is to be calculated.

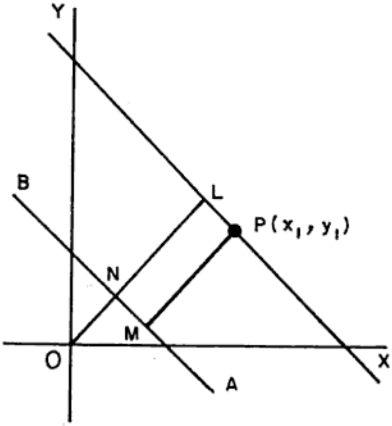


Fig. 2.11

Draw PM perpendicular from P to the line AB. Also draw a line parallel to AB passing through (x_1, y_1). Its equation is given by

$$x \cos \alpha + y \sin \alpha - p_1 = 0. \tag{2}$$

But it passes through (x_1, y_1).

$$\therefore \qquad x_1 \cos \alpha + y_1 \sin \alpha - p_1 = 0. \tag{3}$$

Draw OL perpendicular from O to the line (2) which meets AB in the point N. From Fig. 2.11, we have

$$OL = p_1, \quad ON = p.$$

Now $\qquad MP = NL = OL - ON = p_1 - p.$

Thus *the perpendicular distance of the point* $P(x_1, y_1)$ *from* AB *is*

$$x_1 \cos \alpha + y_1 \sin \alpha - p.$$

Consider now the general equation of the line

$$ax+by+c=0. \tag{4}$$

This equation should be written so that c is negative. This can be written in the normal form as

$$\frac{a}{\sqrt{a^2+b^2}}\,x+\frac{b}{\sqrt{a^2+b^2}}\,y+\frac{c}{\sqrt{a^2+b^2}}=0.$$

Hence *the perpendicular distance of the point* $P(x_1, y_1)$ *from the line* (4) *is*

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \qquad \square$$

The formula for the perpendicular distance can also be derived without reducing the given equation of the line to the normal form as below:

Let
$$ax+by+c=0 \tag{1}$$
be the given equation of the line AB and $P(x_1, y_1)$ be the given point.

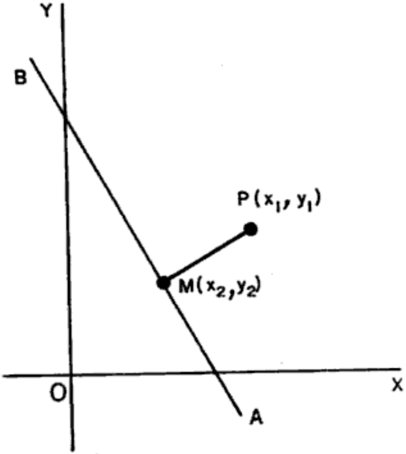


Fig. 2.12

Draw PM perpendicular from P to AB. Let the coordinates of the foot of the perpendicular M be (x_2, y_2) and let $MP=p$. The equation of the line passing through $P(x_1, y_1)$ and perpendicular to the line (1) *i.e.* equation of MP is

$$b(x-x_1)-a(y-y_1)=0. \tag{2}$$

But this passes through (x_2, y_2).

$$\therefore \qquad -b(x_1-x_2)+a(y_1-y_2)=0. \tag{3}$$

Since line (1) also passes through (x_2, y_2), we have

$$ax_2+by_2+c=0. \tag{4}$$

Equation (4) can also be written as

$$a(x_1-x_2)+b(y_1-y_2)=ax_1+by_1+c. \tag{5}$$

On squaring and adding (3) and (5), we get

$$(a^2+b^2)\{(x_1-x_2)^2+(y_1-y_2)^2\}=(ax_1+by_1+c)^2.$$

Hence

$$p=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$=\pm\frac{(ax_1+by_1+c)}{\sqrt{a^2+b^2}}.$$

The sign should be chosen so that the value of p is positive.

Working rule. In order to obtain the perpendicular distance of a point from a line, substitute the coordinates of the point in the expression of the equation and divide it by the quantity which is the square root of the sum of the squares of the coefficients of x and y in the equation of the line.

EXERCISES

1. Calculate the perpendicular distapces of
   (*i*) the point $(1, -2)$ from the line $3x-4y=2$;
   (*ii*) the point $(-3, -4)$ from the line $12x-5y+82=0$;
   (*iii*) the point $(0, 0)$ from the line $4x-3y-12=0$.
2. Find the distance between the line $4x-3y-4=0$ and a line parallel to it passing through the point $(3, -2)$.
3. Calculate perpendicular distance of the point $(3, -1)$ from the line $\frac{x}{2}-\frac{y}{3}=-1$.
4. If p is the length of the perpendicular from the origin to the line $\frac{x}{a}+\frac{y}{b}=1$, prove that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

5. If p and p' are the perpendiculars from the origin to the lines whose equations are $x \sec \theta + y \operatorname{cosec} \theta = a$ and $x \cos \theta - y \sin \theta = a \cos 2\theta$, prove that $4p^2 + p'^2 = a^2$.
6. Prove that the product of the perpendiculars from the two points $(\pm\sqrt{a^2-b^2}, 0)$ to the line $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$ is b^2.

8. POSITIVE AND NEGATIVE SIDES OF A LINE

We know that the line $X'OX (y=0)$ divides the plane into two parts, one above the line $X'OX$ where $y>0$ and one below the line $X'OX$ where $y<0$. The boundary between the two regions is the line $y=0$. In general, any line $ax+by+c=0$ also divides the plane into two parts. It is natural curiousity to know if the expression $L \equiv ax+by+c$ has the same sign, positive or negative, for all the points on one side of the line.

To prove that the expression $L \equiv ax+by+c$ has the same sign for all the points lying on one side of the line $L=0$.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane and let their join meet the line $L=0$ in R, where

$$\frac{QR}{PR} = \frac{\lambda}{\mu}.$$

Therefore the coordinates of the point R are

$$\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right).$$

Since R lies on the line $L=0$, we have

$$a\left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}\right) + b\left(\frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right) + c = 0$$

$$\Rightarrow \qquad \frac{\lambda}{\mu} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign, the ratio $\frac{\lambda}{\mu}$ is negative and so R divides PQ externally [see Fig. 2.13(*a*)]. Hence, P and Q lie on the same side of the line $L=0$. If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, the ratio $\frac{\lambda}{\mu}$ is positive and so R divides PQ internally [see Fig. 2.13(*b*)]. Hence, P and Q lie on the opposite sides of the line $L=0$.

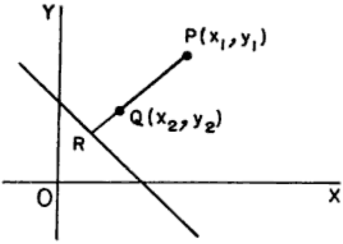


Fig. 2.13(*a*)

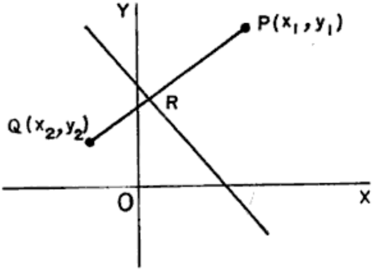


Fig. 2.13(b)

Thus *the expression* $L \equiv ax+by+c$ *has the same sign for all the points lying on one side of the line* $L=0$.

Corollary. The point (x_1, y_1) and the origin are on the same side of the line $ax+by+c=0$ if and only if ax_1+by_1+c and c have the same sign.

Remark. It may be noted that the sign of ax_1+by_1+c has in itself no particular significance for deciding the *positive* or *negative side** of the line $ax+by+c=0$ since $ax+by+c=0$ and $-ax-by-c=0$ represent the same line and as such the positive and negative regions of the plane would be interchanged. Therefore, we can

---

*A side of the line $ax+by+c=0$ is said to be the positive or the negative side according as the expression $ax+by+c$ has the positive or negative sign for the points on that side.

not attach the term positive or negative infallibly to the two sides of the line.

9. BISECTORS OF ANGLES

The **bisector** *of the acute angle between the two lines is the locus of a point which moves so that the perpendicular distances of the point from the lines are equal in magnitude.*

To find the equations of the lines which bisect the angles between the two given lines.

Let $$a_1x+b_1y+c_1=0 \qquad (1)$$

and $$a_2x+b_2y+c_2=0 \qquad (2)$$

be any two lines intersecting at R. Let $P(x', y')$ be any point on either of the bisectors. Then perpendicular distances of the lines (1) and (2) from $P(x', y')$ must be equal in magnitude *i.e.*

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Hence the point (x', y') lies on one or the other of the lines

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}. \qquad (3)$$

The two lines given by (3) are therefore the required bisectors.

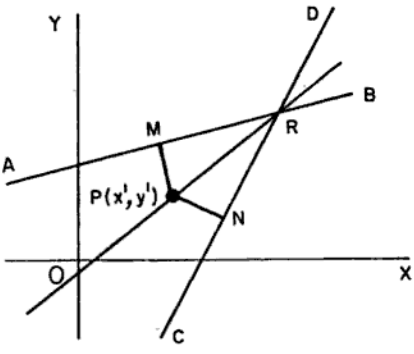


Fig. 2.14

Note. We can distinguish between the two bisectors given by (3) with the help of origin. Let us assume, without any loss of generality, that c_1 and c_2 both be positive for if it is not so it could be made on multiplying the equation by -1. Let (x', y') be any point on a bisector. Now, we have the following different cases:

Case I. When both the expressions $a_1x'+b_1y'+c_1$ and $a_2x'+b_2y'+c_2$ are positive. Then the point (x', y') and the origin lie on the same side of the lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0.$$

Thus the point (x', y') lies on the bisector of the angle which contains the origin. We call it **internal bisector** and its equation is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Case II. When the expressions $a_1x'+b_1y'+c_1$ and $a_2x'+b_2y'+c_2$ have opposite signs. Then the point (x', y') and the origin lie on the same side of one line and on opposite side of the other. Thus, the point (x', y') lies on the bisector of the angle which does not contain the origin. We call it **external bisector** and its equation is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Remark. The internal and external bisectors of the angles between the two lines are perpendicular to each other.

Working rule. In order to obtain the equations of the bisectors of the angles between the two lines, we use the above formula with a remark that the equations of the lines should be so written that the constant terms in the equations have the same sign.

Example. Find the equations of the bisectors of the angles between the pair of lines

$$12x+5y-4=0, \qquad 3x+4y+7=0.$$

Solution. First of all we write the equations of the lines so that the constant terms are positive *i.e.*

$$-12x-5y+4=0, \qquad 3x+4y+7=0.$$

The equation of the bisector of the angle between the lines which contain the origin is

$$\frac{-12x-5y+4}{\sqrt{(-12)^2+(-5)^2}}=\frac{3x+4y+7}{\sqrt{3^2+4^2}}$$

$$\Rightarrow \qquad 99x+77y+71=0.$$

The equation of the other bisector is

$$\frac{-12x-5y+4}{\sqrt{(-12)^2+(-5)^2}}=-\frac{3x+4y+7}{\sqrt{3^2+4^2}}$$

$$\Rightarrow \qquad 7x-9y-37=0.$$

EXERCISES

1. Show that the point (1, 2) is on the negative side of

$$3x-4y-1=0$$

and on the positive side of

$$4x+2y-2=0$$

2. Prove that the points (3, −2) and (1, 2) are on opposite sides of the line $2x+3y-5=0$.
3. Find the equations of the bisectors of the angles between the pair of lines:

   (*i*) $2x+3y=1$, $2x-y+3=0$;

   (*ii*) $3x-4y=5$, $4x-3y=3$.

4. Find the equation of the bisector of that angle between the lines $4x-3y+1=0$ and $12x+5y+13=0$ in which the origin lies.

10. SYSTEMS OF LINES

We have expressed equations of the lines in various forms. Among these are the equations

$$y=mx+c, \frac{x}{a}+\frac{y}{b}=1 \text{ and } x\cos\alpha+y\sin\alpha=p.$$

Each of these equations has two contants which have geometrical significance. The constants of the first equation are m and c. When definite values are assigned to these letters, a line is completely determined. Other values of these, of course, determine other lines. Thus the quantities m and c are fixed for any particular line but change from line to line. These letters are called **parameters.** In the second equation a and b are the parameters whereas in the third equation α and p are the parameters.

A linear equation with only one parameter is obtained if the other parameter is replaced by a fixed value. The resulting equation represents all lines with a particular property if the remaining parameter is allowed to vary. Each value assumed by the parameter yields an equation which represents a definite line.

The collection of lines defined by a linear equation with one parameter is called a **family,** *or* **system of lines.** For example, if $m=2$, the slope-intercept equation becomes

$$y=2x+c.$$

This equation represents the family of parallel lines each of which has slope 2, where c is a parameter. There are, of course, infinitely many lines in the family.

10.1 System of lines parallel to a given line

Let the given line be

$$ax+by+c=0. \tag{1}$$

Then the system of lines parallel to the line (1) is

$$ax+by+\lambda=0,$$

where λ is a parameter, since the slopes of the parallel lines are equal.

10.2 System of lines perpendicular to a given line

Let the given line be

$$ax+by+c=0. \tag{1}$$

Then the equation

$$bx-ay+\lambda=0, \tag{2}$$

for any value of λ, represents a line perpendicular to the line (1) since the product of the slopes of the perpendicular lines is -1. If λ is a parameter, then the equation (2) represents a family of lines each of which is perpendicular to the line (1).

10.3 System of lines through a given point. Let the given point be $A(x_1, y_1)$. Then a system of lines through A is

$$y-y_1=m(x-x_1),$$

where m, the slope of the line, is a parameter. For different values of m, we have a different line of the system through the point $A(x_1, y_1)$.

A system of lines passing through a given point is called a **pencil** *of lines and the given point is called the* **vertex** *of the pencil.*

10.4 System of lines through the intersection of two lines. *To find the system of lines passing through the point of intersection of two given lines.*

Consider the two distinct non-parallel lines

$$L_1 \equiv a_1x+b_1y+c_1=0, \tag{1}$$

and

$$L_2 \equiv a_2x+b_2y+c_2=0. \tag{2}$$

Then the equation

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0, \tag{3}$$

where λ is parameter, represents a system of lines, through the point of intersection of the lines (1) and (2). To verify this statement we observe that:

(*i*) Equation (3) being linear in x and y, for any value of the parameter λ, represents a line.

(*ii*) If (x_1, y_1) is the point of intersection of the given lines (1) and (2), then

$$a_1x_1+b_1y_1+c_1=0$$

and

$$a_2x_1+b_2y_1+c_2=0$$

and thus equation (3) is also satisfied by the coordinates (x_1, y_1). Therefore the line (3) passes through the point (x_1, y_1). Hence *the equation*

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$$

represents a system of lines through the point of intersection of the given lines.

Note 1. The system of lines (3) represents a pencil of lines with vertex as the point of intersection of the given lines (1) and (2). We refer to the given lines (1) and (2) as the **base lines** of the pencil. Clearly any two lines of the pencil can be chosen as base lines.

Note 2. If the lines (1) and (2) are parallel, equation (3) represents all lines parallel to (1) or (2) since all they have the same slope $-\frac{a_1}{b_1}$. In numerical examples, it is enough to choose

$a_1x+b_1y+c_1+\lambda=0$ to represent all such lines which, in fact, is a system of lines parallel to $a_1x+b_1y+c_1=0$.

10.5 Concurrent lines. *To find the condition for the given three lines to be concurrent.*

Let the three lines be

$$a_1x+b_1y+c_1=0 \tag{1}$$

$$a_2x+b_2y+c_2=0 \tag{2}$$

and

$$a_3x+b_3y+c_3=0. \tag{3}$$

These lines, no two of which are assumed to be parallel, will be concurrent if any one of them (say) line (3) is a member of the pencil determined by the other lines (1) and (2) here; in other words, if line (3) can be expressed in the form of

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$$

for some suitable value of λ. This is the samething as to find the values of two constants λ and μ such that

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)+\mu(a_3x+b_3y+c_3)=0$$

is satisfied.

Alternative test for concurrency

Let the lines represented by above equations are concurrent and (x_1, y_1) be the common point of intersection. Then

$$a_1x_1+b_1y_1+c_1=0 \tag{4}$$

$$a_2x_1+b_2y_1+c_2=0 \tag{5}$$

and

$$a_3x_1+b_3y_1+c_3=0. \tag{6}$$

On eliminating x_1 and y_1 from (4), (5) and (6), we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0,$$

which is the required condition for concurrency of the lines.

EXERCISES

1. Find the equation of the line which is perpendicular to $3x+8y-5=0$ and which passes tnrough the point $(2, -1)$.
2. Prove that the three lines $5x+3y-7=0$, $3x-4y-10=0$ and $x+2y=0$ are concurrent.
3. Verify that the lines $4x+7y-9=0$, $5x-8y+15=0$ and $9x-y+6=0$ are concurrent.
4. Prove that the line through the point $(-4, 6)$ concurrent with the lines $3x-2y+3=0$ and $5x+6y-2=0$ passes through the origin.

11. EQUATION OF A LINE IN POLAR COORDINATES

11.1 *To find the general equation of a line in the polar coordinates.*

Let O be the pole and OX the initial line. Let $P(r, \theta)$ be any point on the line. Draw perpendicular ON to the line.

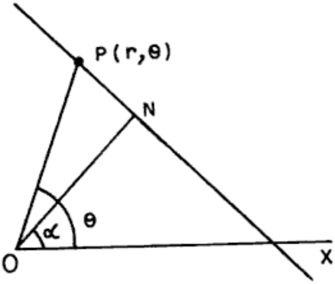


Fig. 2.15

Let $ON=p$ and $\angle XON=\alpha$.

Now in right angled triangle ONP,

$$ON=OP \cos NOP$$

i.e.
$$p=r\cos(\theta-\alpha).$$

This is the required equation of the line in polar coordinates.

Note. Transforming the equation to cartesian coordinates, this equation becomes the normal form of the line.

11.2 Polar equation of the line joining two Points. *To find the polar equation of the line joining two given points.*

Let O be the pole and OX the initial line. Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be the given points. Let $R(r, \theta)$ be any point on the line joining the points P and Q.

Then area of $\triangle POQ$ = area of $\triangle POR$ + area of $\triangle ROQ$

i.e. $\frac{1}{2}OP.OQ \sin POQ = \frac{1}{2}OP.OR \sin POR + \frac{1}{2}OR.OQ \sin ROQ$

i.e. $\frac{1}{2}r_1r_2 \sin(\theta_2-\theta_1) = \frac{1}{2}rr_1 \sin(\theta-\theta_1) + \frac{1}{2}rr_2 \sin(\theta_2-\theta)$

$$\Rightarrow \quad \frac{\sin(\theta_2-\theta_1)}{r} = \frac{\sin(\theta-\theta_1)}{r_2} + \frac{\sin(\theta_2-\theta)}{r_1}.$$

This is the required equation of the line PQ.

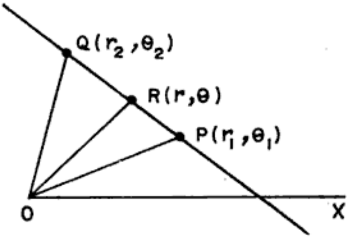


Fig. 2.16

Corollary. The euqation of the line joining the point (r_1, θ_1) to the pole is $\theta = \theta_1$.

This equation can be obtained easily by taking $r_2=0$ and $\theta_2=0$ in the above equation.

EXERCISES

1. Find the polar coordinates of the foot of the perpendicular from the pole on the line joining two points (r_1, θ_1) and (r_2, θ_2).
2. Prove that the equation to any line passing through the point of intersection of the lines

$$\frac{l}{r} = a\cos\theta + b\sin\theta \text{ and } \frac{l'}{r} = a'\cos\theta + b'\sin\theta \text{ is}$$

$$\frac{l+\lambda l'}{r}=(a+\lambda a')\cos\theta+(b+\lambda b')\sin\theta.$$

Hence obtain the equation of the line passing through the pole and the point of intersection of the two given lines.

3. A variable line cuts off from n given concurrent lines intercepts, the sum of the reciprocals of which is constant. Show that it always passes through a fixed point.

MISCELLANEOUS EXERCISES

1. If the line $\lambda x+6y+5=0$ is parallel to the line $4x-3y+2=0$, what is the value of λ?
2. Find the perpendicular distance from the origin of the line $\frac{x}{3}+\frac{y}{4}=1$.
3. Find the equation of the line through (2, 3) parallel to $5x-12y=3$.
4. Write down the equation of the lines through $(-2, 3)$, respectively, parallel and perpendicular to $3x-5y=1$.
5. Find the lines through (4, 5) which cut the axes so that the intercepts are equal in magnitude.
6. The vertices of a triangle are $(0, -2)$, $(2, 0)$ and $(3, -3)$. Prove that it is isoscles and find the equation of the median which bisects the shortest side. Verify that this line is perpendicular to the side it bisects.
7. Show that the line joining (2, 3) to $(3, -2)$ is perpendicular to the line joining $(-5, 0)$ to $(0, 1)$.
8. Show that the lines $x\cos\alpha+y\sin\alpha=p$ and $x\cos\beta+y\sin\beta=p$ intersect at
$$\left(p\cos\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2},\ p\sin\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2}\right).$$
9. Find the coordinates of the circumcentre of the triangle formed by the lines
$$3x-y-5=0,\ x+2y-4=0 \text{ and } 5x+3y+1=0.$$
10. Prove that the area of the triangle formed by the three lines $y=m_1x+c_1$, $y=m_2x+c_2$ and $y=m_3x+c_3$ is
$$\frac{1}{2}\left[\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}+\frac{(c_1-c_2)^2}{m_1-m_2}\right].$$

11. Prove that the medians of a triangle are concurrent.
12. Find λ such that the lines $x+y+5=0$, $5x-y+7=0$ and $\lambda x+4y+6=0$ are concurrent.
13. Three vertices of a parallelogram $ABCD$ are $A(1, 0)$, $B(-1, 0)$ and $C(2, 3)$. Find the coordinates of D.
14. Find the equation of the line joining $(1, 1)$ to the point of intersection of the lines $2x+4y-5=0$ and $4x+2y-5=0$.
15. Prove that the locus of a point which moves so that the sum of the perpendiculars let fall from it upon two given lines is constant, is a line.

CHAPTER III

CHANGE OF AXES

1. INTRODUCTION

In coordinate geometry, the coordinates of a point, the equation of a locus etc. are always with reference to some system of coordinate axes.

Sometimes while solving the problems in coordinate geometry, it is desirable as well as convenient to consider a figure under discussion and to change its equation, either temporarily or permanently, to a different system of axes such that an important set of points assume the coordinates of simple form. In general, we come across with the following three types of change of axes.

(*i*) **Translation of axes** (Change of origin only, the new axes being parallel to the old ones).

(*ii*) **Rotation of axes** (Change of directions of the axes without changing the origin).

(*iii*) **General transformation** (Translation and rotation of the axes together).

2. TRANSLATION OF AXES

To change the origin of a system of coordinate axes without changing the directions of the axes.

Let OX and OY be the original axes and (x, y) be the coordinates of a point P referred to these axes.

Let O' (α, β) be the new origin and $O'X'$ and $O'Y'$ the coordinate axes parallel to the original axes. Let (x', y') be the coordinates of the point P with respect to new axes.

Draw PM and $O'N$ perpendiculars to OX, where PM meets $O'X'$ in M'.

Then $ON=\alpha,\ NO'=\beta,\ OM=x,\ MP=y.$

Also $O'M'=x'$ and $M'P=y'$.

Now $x=OM=ON+NM=ON+O'M'=\alpha+x'$

and $y=MP=MM'+M'P=NO'+M'P=\beta+y'.$

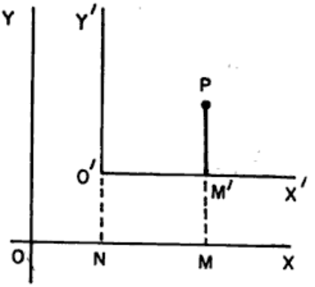


Fig. 3.1

Hence *the new coordinates* (x', y') *of the point P are connected with the old one* (x, y) *by means of the relations*

$$x=x'+\alpha,\ y=y'+\beta.$$

Note. The pair of equations

$$x=x'+\alpha,\ y=y'+\beta$$

giving the relationship between the two sets of coordinate axes is called the translation of axes.

Ex. Transform to parallel axes through the point (3, 5) the equation

$$x^2+y^2-6x-10y-2=0.$$

3. ROTATION OF AXES

To change the directions of the axes without changing the origin.

Let OX and OY be the original axes and OX', OY' the new axes which are obtained by rotating the original axes through an angle θ. Rotation is taken in anti-clockwise direction.

Let (x, y) be the coordinates of a point P with respect to original axes and (x', y') the coordinates of P with respect to new axes.

Draw PM and PN perpendiculars to OX and OX' and also NN' and NM' perpendiculars to OX and PM, respectively.

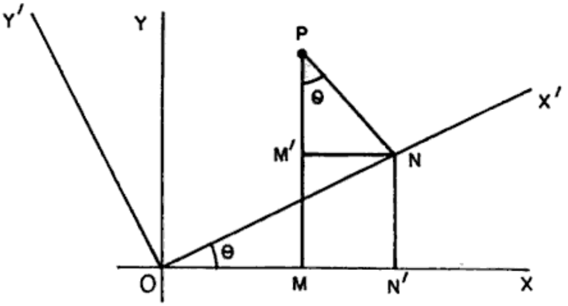
Then $\qquad OM=x,\ MP=y,\ ON=x',\ NP=y'$

and $\qquad \angle M'PN=\theta.$

Now, we have

$$x=OM=ON'-MN'=ON'-M'N=ON\cos\theta-NP\sin\theta$$
$$=x'\cos\theta-y'\sin\theta$$

and $$y = MP = MM' + M'P = N'N + M'P = ON \sin\theta + NP \cos\theta$$
$$= x' \sin\theta + y' \cos\theta.$$



Hence the relations connecting the two systems of axes are given by

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned}\right\}. \qquad (1)$$

Note. An equivalent form of the relations (1) is

$$\left.\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ y' &= -x \sin\theta + y \cos\theta \end{aligned}\right\}.$$

Ex. What does the equation

$$4x^2 + 2\sqrt{3}xy + 2y^2 = 1$$

become when the axes are turned through an angle of 30°?

4. GENERAL TRANSFORMATION

To change the origin and directions of the axes together.

Let OX and OY be the original axes. Shifting the origin to $O'(\alpha, \beta)$ so that $O'X'$ and $O'Y'$ are parallel to the original axes. Further rotate $O'X'$ and $O'Y'$ through an angle θ so that we get the new axes as $O'X''$ and $O'Y''$. Let (x, y) be the coordinates of any point P with respect to the axes OX and OY; (x', y') the coordinates of P with respect to $O'X'$ and $O'Y'$ and (x'', y'') the coordinates of P with respect to $O'X''$ and $O'Y''$.

Then $OM = x$, $MP = y$, $O'M' = x'$, $M'P = y'$,

$O'N'' = x''$, $N''P = y''$.

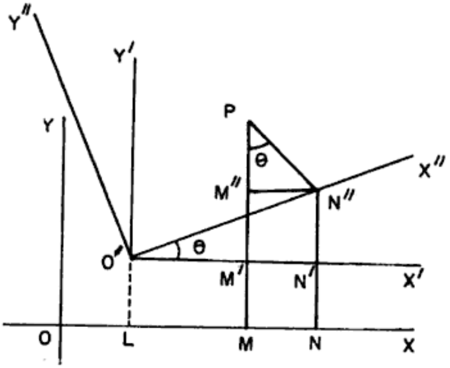


Fig. 3.3

Now, we have

$$\begin{aligned} x &= OM = ON - MN = OL + LN - MN \\ &= OL + O'N' - M''N'' \\ &= OL + O'N'' \cos\theta - N''P \sin\theta \\ &= \alpha + x'' \cos\theta - y'' \sin\theta \end{aligned}$$

and

$$\begin{aligned} y &= MP = MM' + M'M'' + M''P \\ &= O'L + N'N'' + M''P \\ &= \beta + O'N'' \sin\theta + N''P \cos\theta \\ &= \beta + x'' \sin\theta + y'' \cos\theta. \end{aligned}$$

Hence the new coordinates (x'', y'') of the point P are connected by the old one $(x. y)$ by means of the relations

$$\left.\begin{aligned} x &= \alpha + x'' \cos\theta - y'' \sin\theta \\ y &= \beta + x'' \sin\theta + y'' \cos\theta \end{aligned}\right\}.$$

Ex. Transform the equation

$$x^2 - 4xy + y^2 + 8x + 2y - 5 = 0$$

to new axes through (2, 3) rotated through an angle of 45°.

5. INVARIANTS

5.1 *If (x, y) be the coordinates of a point referred to axes OX, OY and (x', y') be the coordinates of the same point referred to* *axes OX', OY' and if $ax^2+2hxy+by^2$ in which a, b, h are independent of x and y become $a'x'^2+2h'x'y'+b'y'^2$, then*

$$a+b=a'+b' \text{ and } ab-h^2=a'b'-h'^2.$$

We are given that

$$ax^2+2hxy+by^2=a'x'^2+2h'x'y'+b'y'^2. \tag{1}$$

From § 3, we have

$$x=x'\cos\theta-y'\sin\theta$$

and $$y=x'\sin\theta+y'\cos\theta.$$

Now
$$\begin{aligned}ax^2+2hxy+by^2&=a(x'\cos\theta-y'\sin\theta)^2\\&\quad+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)\\&\quad+b(x'\sin\theta+y'\cos\theta)^2\\&=(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)\,x'^2\\&\quad+\{-2a\sin\theta\cos\theta+2h(\cos^2\theta-\sin^2\theta)\\&\qquad\qquad+2b\sin\theta\cos\theta\}\,x'y'\\&\quad+(a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta)\,y'^2\end{aligned} \tag{2}$$

On comparing (1) and (2), we get

$$a'=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta \tag{3}$$

$$b'=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta \tag{4}$$

and $$h'=-a\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta)+b\sin\theta\cos\theta. \tag{5}$$

On adding (3) and (4), we get

$$a'+b'=a+b.$$

Equations (3), (4) and (5) respectively, can also be written as

$$2a'=(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta \tag{6}$$

$$2b'=(a+b)-\{(a-b)\cos 2\theta+2h\sin 2\theta\} \tag{7}$$

and $$2h'=-(a-b)\sin 2\theta+2h\cos 2\theta. \tag{8}$$

Subtracting square of (8) from the product of (6) and (7), we get

$$\begin{aligned}4a'b'-4h'^2&=(a+b)^2-\{(a-b)\cos 2\theta+2h\sin 2\theta\}^2\\&\qquad-\{-(a-b)\sin 2\theta+2h\cos 2\theta\}^2\\&=(a+b)^2-\{(a-b)^2+4h^2\}\\&=4ab-4h^2.\end{aligned}$$

$$\Rightarrow\ a'b'-h'^2=ab-h^2 \quad \square$$

Note. The quantities $a+b$ and $ab-h^2$ are called **invariants.** These invariants are of great importance in the development of the theory of the general equation of second degree.

Ex. Prove that if

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1$$

represent the same conic and the axes are rectangular, then

$$(a-b)^2+4h^2=(a'-b')^2+4h'^2.$$

5.2 Removal of the xy term.

To find the angle through which the axes be rotated so that the expression $ax^2+2hxy+by^2$ may become of the form $a'x'^2+b'y'^2$.

Let the axes be rotated through an angle θ. Then by § 3, we have

$$x=x' \cos \theta-y' \sin \theta$$

and
$$y=x' \sin \theta+y' \cos \theta.$$

$$\begin{aligned}\therefore \quad ax^2+2hxy+by^2=&(a \cos^2 \theta+2h \sin \theta \cos \theta+b \sin^2 \theta)\, x'^2\\ &+\{-2(a-b) \sin \theta \cos \theta+2h(\cos^2 \theta-\sin^2 \theta)\}\, x'y'\\ &+(a \sin^2 \theta-2h \sin \theta \cos \theta+b \cos^2 \theta)\, y'^2. \qquad (1)\end{aligned}$$

Hence the expression $ax^2+2hxy+by^2$ will be of the form $a'x'^2+b'y'^2$ if coefficient of $x'y'$ in right side of (1) is zero,

i.e., if $-2(a-b) \sin \theta \cos \theta+2h(\cos^2 \theta-\sin^2 \theta)=0$

i.e., if $-(a-b) \sin 2\theta+2h \cos 2\theta=0$

$$\Rightarrow \quad \tan 2\theta=\frac{2h}{a-b}.$$

Hence the required angle is given by

$$\theta=\frac{1}{2} \tan^{-1} \frac{2h}{a-b} \quad \square$$

Remark. In the special case where $ab-h^2=0$, the removal of the xy term in $ax^2+2hxy+by^2$ by means of invariants, we note that

$$a'b'=ab-h^2=0$$

$$\Rightarrow \quad a'=0 \text{ or } b'=0.$$

Thus, in this special case the removal of the xy term in

$$ax^2+2hxy+by^2$$

by turning the axes through an angle θ given by $\tan 2\theta=\dfrac{2h}{a-b}$ makes either $a'=0$ or $b'=0$.

EXERCISES

1. Simplify the following equations by changing to a new origin in each case:

   (*i*) $x^2+y^2+2x-4y+1=0$; new origin $(-1, 2)$.

   (*ii*) $x^2+2y^2-6x+16y+39=0$; new origin $(3, -4)$.

2. Write down the equations for a rotation of axes through an angle $\frac{\pi}{4}$. Hence, prove that the curve $2xy=a^2$ can be transformed to $x^2-y^2=a^2$.

3. Transform the equation

$$x^2-6xy+9y^2+4x+8y+15=0$$

   to new axes through $(-2, -1)$ rotated through an angle $\tan^{-1}\frac{1}{2}$.

4. Transform the equation

$$3x^2-24xy+10y^2+6x+52y=0$$

   to new axes through $(3, 1)$ rotated through an angle $\tan^{-1}\frac{3}{4}$.

5. If (x, y) and (x', y') be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux+vy$, where u and v are independent of x and y, becomes $u'x'+v'y'$, then

$$u^2+v^2=u'^2+v'^2.$$

6. Show that the equation $x\cos\alpha+y\sin\alpha=p$, when the axes are turned through an angle α, becomes $x=p$. Interpret this fact.

7. What does the equation

$$(x-a)^2+(y-b)^2=c^2$$

   become when it is transferred to parallel axes through (*i*) the point $(a-c, b)$ (*ii*) the point $(a, b-c)$?

8. What does the equation

$$(a-b)(x^2+y^2)-2abx=0$$

become if the origin be moved to the point $\left(\frac{ab}{a-b}, 0\right)$?

9. Prove that the transformation of rectangular axes which converts

$\frac{X^2}{p}+\frac{Y^2}{q}$ into $ax^2+2hxy+by^2$ will convert $\frac{X^2}{p-\lambda}+\frac{Y^2}{p-\lambda}$

into $\frac{ax^2+2hxy+by^2-\lambda(ab-h^2)(x^2+y^2)}{1-(a+b)\lambda+(ab-h^2)\lambda^2}$.

CHAPTER IV

PAIR OF LINES

1. INTRODUCTION

Consider the equation

$$(lx+my+n)(l'x+m'y+n')=0. \tag{1}$$

This can be satisfied by the points whose coordinates satisfy

$$lx+my+n=0 \text{ or } l'x+m'y+n'=0. \tag{2}$$

Thus all the points which satisfy equation (1) lie on one or the other of the equations in (2); and conversely, any point whose coordinates satisfy any of the two equations in (2) satisfies the equation (1). The equations in (2) being linear, the equation (1), therefore, represents a pair of lines

If we multiply the two factors in (1) together, then it takes the form

$$ll'x^2+(lm'+l'm)xy+mm'y^2+(ln'+l'n)x+(mn'+m'n)y+nn'=0.$$

This is a second degree equation in x and y. Thus, we note that a pair of lines is given by a second degree equation in x and y.

Conversely, one is interested to know if the most general second degree equation in x and y *i.e.*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{3}$$

represents a pair of lines. It is not possible, in general. However, it would only be possible when the left hand expression in equation (3) can be decomposed into two real linear factors for which some relation among the coefficients a, b, c, f, g, h is required to be satisfied. Before, we deal with this problem in a later section, we come to a particular case when both the lines pass through the origin. In this case the equation (1) takes the form

$$(lx+my)(l'x+m'y)=0$$

i.e. $$ll'x^2+(lm'+l'm)xy+mm'y^2=0.$$

This is a homogeneous* equation of degree two in x and y. The general expression of this form (homogeneous form) is

$$ax^2+2hxy+by^2=0.$$

We shall now prove that this always represents a pair of lines through the origin, unlike the case for the most general second degree equation in x and y.

2. HOMOGENEOUS EQUATION OF SECOND DEGREE

2.1 *To prove that the general homogeneous equation of second degree represents a pair of lines passing through the origin.*

The general homogeneous equation of second degree is

$$ax^2+2hxy+by^2=0. \tag{1}$$

This can be written as

$$(ax+hy)^2-(h^2-ab)y^2=0, \qquad a\neq0$$

$$\Rightarrow \quad \{ax+(h+\sqrt{h^2-ab})y\}\{ax+(h-\sqrt{h^2-ab})y\}=0.$$

So equation (1) represents the lines

$$ax+(h+\sqrt{h^2-ab})y=0$$

and $$ax+(h-\sqrt{h^2-ab})y=0,$$

through the origin. The lines are real and distinct, coincident or imaginary according as $h^2>=\text{or}<ab$, respectively.

If $a=0$, the equation (1) becomes

$$2hxy+by^2=0,$$

which represents the lines $y=0$ and $2hx+by=0$.

Remark. Most of the times, the problems regarding the pair of lines through the origin require the technique of introducing separate equations for the lines. This we shall be tackling by considering that $y=mx$ and $y=m'x$ are the lines represented by the given equation

$$ax^2+2hxy+by^2=0$$

so that $$ax^2+2hxy+by^2\equiv b(y-mx)(y-m'x)$$

---

*An equation of the form

$$a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots+a_ny^n=0,$$

in which the sum of the powers of x and y in every term is the same (say n), is called a homogeneous equation (of degree n).

$$\Rightarrow \qquad \begin{cases} m+m'=-\dfrac{2h}{b} \\[2ex] mm'=\dfrac{a}{b}. \end{cases}$$

In the problems, we shall be using the equations $y=mx$ and $y=m'x$, and finally eliminate m and m' by the use of the relations involving $m+m'$ and mm'.

Illustration. Find the equation of the pair of lines through the origin which represents the lines perpendicular to the pair of lines $ax^2+2hxy+by^2=0$.

Solution. Let $y=mx$ and $y=m'x$ be the lines represented by

$$ax^2+2hxy+by^2=0.$$

Then $$m+m'=-\frac{2h}{b}, \quad mm'=\frac{a}{b}.$$

Now, the lines through the origin and perpendicular to the given lines are $y=-\frac{1}{m}x$ and $y=-\frac{1}{m'}x$. Thus the required equation of the pair of lines is

$$(x+my)(x+m'y)=0$$

i.e. $$x^2+(m+m')xy+mm'y^2=0.$$

Substituting the values of $m+m'$ and mm', we get

$$bx^2-2hxy+ay^2=0.$$

2.2 Angle between the lines $\mathbf{ax^2+2hxy+by^2=0}$. *To find the angle between the lines given by the equation*

$$ax^2+2hxy+by^2=0.$$

Let the lines represented by the given equation of the pair of lines be

$$y=mx \text{ and } y=m'x.$$

Then $$ax^2+2hxy+by^2\equiv b(y-mx)(y-m'x)$$

$$\Rightarrow \qquad m+m'=-\frac{2h}{b}, \quad mm'=\frac{a}{b}.$$

Let ϕ be the angle between the lines. Then

$$\tan\phi=\frac{m-m'}{1+mm'}$$

$$=\frac{\pm\sqrt{(m+m')^2-4mm'}}{1+mm'}$$

$$=\frac{\pm\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}$$

$$=\frac{\pm 2\sqrt{h^2-ab}}{a+b}.$$

The sign positive or negative is to be taken according as the angle ϕ between the lines is an acute or obtuse angle.

Remarks. (*i*) It is important, in the formula for the angle, that $h^2-ab \geqslant 0$ for otherwise the expression for $\tan\phi$ would have no meaning.

(*ii*) If $h^2-ab=0$, $\phi=0$ and the two lines given by

$$ax^2+2hxy+by^2=0$$

would coincide.

(*iii*) The two lines given by $ax^2+2hxy+by^2=0$ are perpendicular if and only if $a+b=0$, *i.e.* the sum of the coefficients of x^2 and y^2 is zero.

Ex. Find the angle between the lines $x^2+3xy+2y^2=0$.

2.3 Bisectors of the angles between the lines $ax^2+2hxy+by^2=0$.

To find the equation of the pair of lines bisecting the angles between the lines $ax^2+2hxy+bv^2=0$.

Let $y=mx$ and $y=m'x$ be the two lines represented by

$$ax^2+2hxy+by^2=0. \qquad (1)$$

Then, as before, we have

$$m+m'=-\frac{2h}{b},\ mm'=\frac{a}{b}.$$

First method. Let θ and θ' be the angles which the two lines make with the positive direction of x-axis so that

$$m=\tan\theta \text{ and } m'=\tan\theta'.$$

Let $y=x\tan\phi$ be the equation of a bisector. Then

$$2\phi=\theta+\theta' \text{ or } \theta+\theta'+\pi$$

so that

$$\tan 2\phi=\tan(\theta+\theta')$$

$$\Rightarrow \quad \frac{2\tan\phi}{1-\tan^2\phi}=\frac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'}$$

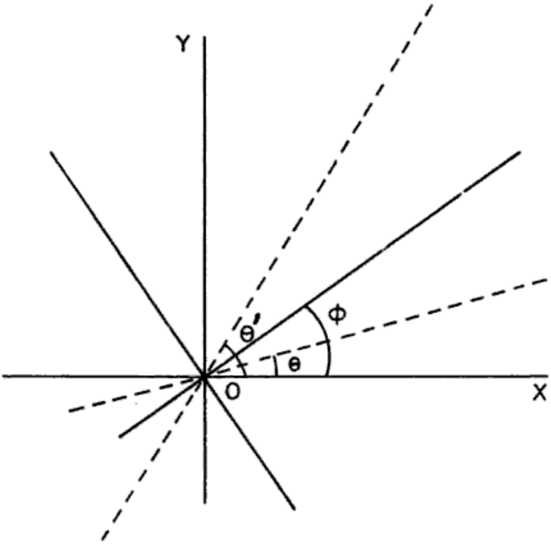


Fig. 4.1

$$\Rightarrow \qquad \frac{2\frac{y}{x}}{1-\frac{y^2}{x^2}}=\frac{m+m'}{1-mm'}$$

$$=\frac{2h}{a-b}.$$

Hence *the equation of the bisectors of the angles between the lines* $ax^2+2hxy+by^2=0$ *is given by*

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Second method. The pair of bisectors of the angles betwee the lines represented by (1) is given by [Art. 9 Chapter II]

$$\frac{mx-y}{\sqrt{m^2+1}}=\pm\frac{m'x-y}{\sqrt{m'^2+1}}$$

$$\Rightarrow \qquad (m'^2+1)(mx-y)^2=(m^2+1)(m'x-y)^2$$

$$\Rightarrow \quad (m^2-m'^2)(x^2-y^2)-2(m-m')(1-mm')\,xy=0$$

$$\Rightarrow \quad (m+m')(x^2-y^2)-2(1-mm')\,xy=0$$

$$\Rightarrow \quad -\frac{2h}{b}(x^2-y^2)-2\left(1-\frac{a}{b}\right)xy=0$$

$$\Rightarrow \quad \frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Ex. Find the equation of the bisectors of the angles between the lines represented by

$$6x^2-13xy+5y^2=3.$$

2.4 Examples

Example 1. Show that the two lines represented by

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0 \qquad \ldots(1)$$

make angles α, β with x-axis such that

$$\tan\alpha-\tan\beta=2.$$

Solution. Let $y=mx$ and $y=m'x$ be the lines given by (1). Then

$$m+m'=\frac{2\tan\theta}{\sin^2\theta}=\frac{2}{\sin\theta\cos\theta}$$

and
$$mm'=\frac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}=\sec^2\theta+\cot^2\theta.$$

$$\therefore\ \tan\alpha-\tan\beta=m-m'$$
$$=\sqrt{(m+m')^2-4mm'}$$
$$=\sqrt{\frac{4}{\sin^2\theta\cos^2\theta}-4(\sec^2\theta+\cot^2\theta)}$$
$$=2.$$

Example 2. Find the condition that one of the two lines

$$ax^2+2hxy+by^2=0$$

may be perpendicular to one of the lines given by

$$a'x^2+2h'xy+b'y^2=0.$$

Solution. Let $y=mx$ be one of the lines of

$$ax^2+2hxy+by^2=0.$$

Then $$a+2hm+bm^2=0. \tag{1}$$

Any line perpendicular to $y=mx$ passing through the origin is $x=-my$. Clearly $x=-my$ will satisfy

$$a'x^2+2h'xy+b'y^2=0$$

if $$a'm^2-2h'm+b'=0. \tag{2}$$

Now, from (1) and (2), we get

$$\frac{m^2}{2hb'+2h'a}=\frac{m}{aa'-bb'}=\frac{1}{-2h'b-2ha'}$$

$$\Rightarrow \quad \left\{ \begin{aligned} m&=\frac{aa'-bb'}{-2(h'b+ha')} \\ m^2&=\frac{hb'+h'a}{-(h'b+ha')} \end{aligned} \right.$$

$$\Rightarrow \quad \frac{hb'+h'a}{-(h'b+ha')}=\frac{(aa'-bb')^2}{4(h'b+ha')^2}$$

$$\Rightarrow \quad 4(ah'+hb')(ha'+bh')+(aa'-bb')^2=0.$$

Example 3 Prove that the lines represented by

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$$

have the same pair of bisectors for all values of λ. Interpret the case for $\lambda=-(a+b)$.

Solution. The equation representing the bisectors of the given pair of lines is

$$\frac{x^2-y^2}{(a+\lambda)-(b+\lambda)}=\frac{xy}{h}$$

i.e. $$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Since the equation of the bisectors is free from λ, the given equation has the same pair of bisectors for all values of λ.

Consider now the case when $\lambda=-(a+b)$.

The given equation reduces to

$$ax^2+2hxy+by^2-(a+b)(x^2+y^2)=0$$

i.e. $$bx^2-2hxy+ay^2=0.$$

This represents a pair of lines which are perpendicular to the lines given by

$$ax^2+2hxy+by^2=0 \quad \text{(See Illustration in § 2.1).}$$

Example 4. Let the pairs of the lines

$$x^2-2pxy-y^2=0 \text{ and } x^2-2qxy-y^2=0$$

be such that each pair bisects the angles between the other pair. Prove that $pq=-1$.

Solution. The bisectors of the angles between the lines $x^2-2pxy-y^2=0$ are

$$\frac{x^2-y^2}{1-(-1)}=\frac{xy}{-p}$$

i.e. $$px^2+2xy-py^2=0.$$

Since these bisectors are given to be represented by

$$x^2-2qxy-y^2=0,$$

we note that

$$\frac{p}{1}=\frac{2}{-2q}=\frac{-p}{-1}$$

$\Rightarrow$ $$pq=-1.$$

EXERCISES

1. Form the equations which represent the following pair of lines:
   (*i*) $3x-y=0$, $x+3y=0$;
   (*ii*) $y=0$, $4y=x$.
2. Show that each of the following equations represents a pair of lines, and find the angle between each pair:
   (*i*) $x^2-4y^2=0$.
   (*ii*) $x^2+2xy \cot 2\alpha-y^2=0$.
   (*iii*) $x^2-2xy \sec \alpha+y^2=0$.
   (*iv*) $2x^2-7xy+3y^2=0$.
3. Find the value of λ for which the two lines $3x^2-8xy+\lambda y^2=0$ are perpendicular to one another
4. Prove that, if the lines $ax^2+2hxy+by^2=0$ are perpendicular, then so are the lines

$$bx^2+2kxy+ay^2=0.$$

5. Find the condition that one of the lines

$$ax^2+2hxy+by^2=0,$$

may coincide with one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

6. Prove that the two lines

$$(x^2+y^2)(\cos^2\theta\sin^2\alpha+\sin^2\theta)=(x\tan\alpha-y\sin\theta)^2$$

include an angle 2α.

7. Find the equations of the bisectors of the angles between the pair of lines:

   (*i*) $3x^2+8xy+4y^2=0$;

   (*ii*) $15x^2-xy-6y^2=0$;

   (*iii*) $12x^2-7xy-y^2=0$.

8. Show that the angle between one of the lines given by

$$ax^2+2hxy+by^2=0,$$

and one of the lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0,$$

is equal to the angle between the other two lines of the systems.

9. Find the condition on m such that the bisectors of the angles between the lines $m^2x^2+2\ (m+1)\ xy-y^2=0$ are the axes of coordinates.

10. If the bisectors of the angles between the lines

$$ax^2+2hxy+by^2=0,$$

coincide with the bisectors of the angles between the lines

$$\alpha x^2+2\lambda xy+\beta y^2=0,$$

prove that

$$h(\alpha-\beta)=\lambda(a-b).$$

11. Find the condition that one of the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$ is the line $lx+my+n=0$.

3. GENERAL SECOND DEGREE EQUATION

3.1 Condition that the most general second degree equations in x and y represent a pair of lines. *To find the necessary condition that the general equation of the second degree*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (1)$$

may represent a pair of lines.

First method. Equation (1) may represent a pair of lines if the expression in the left hand side of it can be decomposed into two real linear factors in x and y. Suppose $a\neq 0$. Multiply equation (1) by a and arranging it in descending powers of x, we get

$$a^2x^2+2ax(hy+g)=-aby^2-2afy-ac.$$

Completing the perfect square on the left hand side, we have

$$(ax+hy+g)^2=(h^2-ab)\,y^2+2(gh-af)\,y+g^2-ac$$

$$\Rightarrow \qquad ax+hy+g=\pm\sqrt{(h^2-ab)y^2+2(gh-af)y+g^2-ac} \qquad (2)$$

Clearly equation (2) represents two real linear equations if the quantity inside the square root forms a perfect square for which

$$4(gh-af)^2=4(h^2-ab)(g^2-ac).$$

This, after simplification, gives the required condition as

$$abc+2fgh-af^2-bg^2-ch^2=0. \qquad (3)$$

If $a=0$ and $b\neq 0$, multiply equation (1) by b and arrange the terms in descending powers of y. Proceeding similarly as in the previous case, we get the required condition.

In case $a=0$ and $b=0$, we must have $h\neq 0$ for otherwise the equation (1) will reduce to a linear equation. The equation (1) reduces to

$$hxy+gx+fy+\tfrac{1}{2}c=0$$

$$\Rightarrow \qquad (hx+f)(hy+g)=fg-\tfrac{1}{2}ch.$$

Therefore, the condition for this equation to represent a pair of lines is that

$$fg-\tfrac{1}{2}ch=0,$$

which is the same as (3) for the case $a=0$ and $b=0$.

Second Method. Let equation (1) represent a pair of lines intersecting at the point (α, β). Shifting the origin to the point (α, β) by the transformations

$$x=X+\alpha, \quad y=Y+\beta,$$

equation (1) reduces to

$$a(X+\alpha)^2+2h\,(X+\alpha)(Y+\beta)+b(Y+\beta)^2+2g(X+\alpha) \\ +2f(Y+\beta)+c=0$$

$$\Rightarrow \quad aX^2+2hXY+bY^2+2(a\alpha+h\beta+g)X+ \\ 2(h\alpha+b\beta+f)Y+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0. \qquad (4)$$

Equation (4) represents a pair of lines passing through the new origin. Therefore, it must be a second degree homogeneous equation in X and Y, and consequently the terms of the first degree in X, Y and the constant terms must disappear from equation (4) so that we are left only with

$$aX^2+2hXY+bY^2=0.$$

Thus it is required that the following conditions are satisfied

$$a\alpha+h\beta+g=0 \qquad (5)$$

$$h\alpha+b\beta+f=0 \qquad (6)$$

and $a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0.$ (7)

Multiplying equation (5) by α, equation (6) by β and adding, and then subtracting this sum from equation (7), we get

$$g\alpha+f\beta+c=0. \qquad (8)$$

Now eliminating α, β from the equations (5), (6) and (8), we obtain

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

On solving the determinant, we get the condition

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Third Method. Suppose equation (1) represents a pair of lines. Let the lines be

$$lx+my+n=0 \quad \text{and} \quad l'x+m'y+n'=0.$$

Then

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv(lx+my+n)(l'x+m'y+n').$$

Comparing the coefficients of various terms on both the sides, we get

$$ll'=a,\ mm'=b,\ nn'=c,$$
$$lm'+l'm=2h,\quad mn'+m'n=2f,\quad ln'+l'n=2g.$$

By multiplication, we get

$$\begin{aligned}8fgh&=(lm'+l'm)(mn'+m'n)(ln'+l'n)\\&=2ll'mm'nn'+ll'(m^2n'^2+m'^2n^2)\\&\qquad+mm'(l^2n'^2+l'^2n^2)+nn'(l^2m'^2+l'^2m^2)\\&=2abc+a(4f^2-2bc)+b(4g^2-2ac)+c(4h^2-2ab)\end{aligned}$$

$$\Rightarrow\qquad abc+2fgh-af^2-bg^2-ch^2=0.$$

Note. The above conditon is also sufficient.

Corollary. If the general equation of second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of lines, the coordinates (α, β) of the point where they intersect are given by

$$\alpha=\frac{hf-bg}{ab-h^2},\ \beta=\frac{gh-af}{ab-h^2}.$$

[**Hint.** The values of α and β are determined by solving the equations $lx+my+n=0$ and $l'x+m'y+n'=0$].

Note. If $h^2=ab$, then the point of intersection is at infinity and as such the two lines would be parallel.

Ex. Show that the equation

$$x^2-4xy+y^2+6x-3=0$$

represents a pair of lines.

3.2 *If the general equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$

represents a pair of lines, then the equation

$$ax^2+2hxy+by^2=0 \tag{2}$$

represents a pair of lines parallel to them through the origin.

Let $y=mx+c$ and $y=m'x+c'$ be the lines represented by (1) so that

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv b(y-mx-c)(y-m'x-c').$$

Then

$$ax^2+2hxy+by^2\equiv b(y-mx)(y-m'x).$$

Thus equation (2) represents the lines $y=mx$ and $y=m'x$ through the origin, which are parallel to $y=mx+c$ and $y=m'x+c'$, respectively.

Corollary. If equation (1) represents a pair of lines, then

(*i*) the angle between these lines is given by

$$\tan\phi=\frac{2\sqrt{h^2-ab}}{a+b},$$

(*ii*) the two lines are parallel if $h^2=ab$,

(*iii*) the two lines are perpendicular if and only if

$$a+b=0.$$

Ex. Calculate the angele between the lines

$$x^2+8xy+y^2+16x+4y+4=0.$$

3.3 Example. Find the value of λ so that the equation

$$2x^2+xy-y^2-11x-5y+\lambda=0,$$

may represent a pair of lines

Solution. Here $a=2$, $h=\frac{1}{2}$, $b=-1$,

$$g=-\frac{11}{2},\ f=-\frac{5}{2},\ c=\lambda.$$

The given equation represents a pair of lines if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

i.e. $$-2\lambda+\frac{55}{4}-\frac{50}{4}+\frac{121}{4}-\frac{\lambda}{4}=0.$$

Hence $$\lambda=14.$$

EXERCISES

1. Show that each of the following equations represents a pair of lines

   (*i*) $x^2+8xy+y^2+16x+4y+4=0$;

   (*ii*) $x^2+6xy+9y^2+4x+12y-5=0$;

   (*iii*) $x^2-2xy+y^2-4x+4y+4=0$;

   (*iv*) $2x^2+3xy-2y^2+5x-10y-12=0$;

   (*v*) $x^2+xy-6y^2+7x+31y-18=0$.

2. Find the value of λ so that the following equations may represent a pair of lines

   (*i*) $12x^2-10xy+2y^2+11x-5y+\lambda=0$;

   (*ii*) $x^2+2\lambda xy+y^2+6x+2y+9=0$;

   (*iii*) $12x^2+36xy+\lambda y^2+6x+6y+3=0$;

   (*iv*) $\lambda x^2+2xy+\lambda y^2+4x+4y+3=0$.

3. Find the angle between the pair of lines

   (*i*) $x^2-xy-6y^2-7x+31y-18=0$;

   (*ii*) $4x^2+5xy+y^2+3y-4=0$;

   (*iii*) $x^2-xy-6y^2-3x+14y-4=0$.

4. For what value of λ does the equation

$$12x^2+7xy+\lambda y^2+13x-y+3=0,$$

   represent a pair of lines, and what is then the angle between them?

4. PAIR OF LINES JOINING THE ORIGIN TO THE POINTS OF INTERSECTION OF A CURVE AND A LINE

4.1 *To find the equation of the pair of lines joining the origin to the points of intersection of the curve*

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \qquad (1)$$

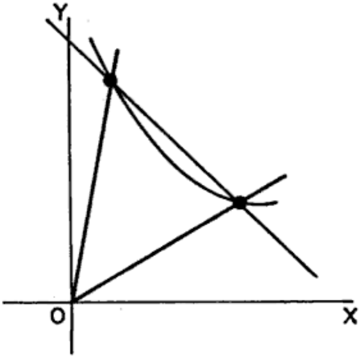
and the line

$$lx+my+n=0. \qquad (2)$$

Make the equations (1) and (2) homogeneous by introducing a third variable z in the following way:

$$ax^2+2hxy+by^2+2gxz+2fyz+cz^2=0 \tag{3}$$

$$lx+my+nz=0. \tag{4}$$



Now eliminating z from (3) and (4), we get

$$ax^2+2hxy+by^2-2gx\left(\frac{lx+my}{n}\right)$$
$$-2fy\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2=0$$

$$\Rightarrow \quad n^2(ax^2+2hxy+by^2)-2n(gx+fy)(lx+my)$$
$$+c(lx+my)^2=0. \tag{5}$$

This is a homogenous equation of degree two in x and y and so it represents a pair of lines through the origin. Moreover, we note that if (x_1, y_1) is a point of intersection of (1) and (2), then

$$\begin{cases} ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \\ lx_1+my_1+n=0 \end{cases}$$

and so the point (x_1, y_1) also satisfies equation (5). Thus, equation (5) represents a pair of lines through the origin and the points of intersection of (1) and (2).

Working Rule. The equation of the pair of lines joining the origin to the points of intersection of a curve and a line is obtained by making the equation of the curve homogeneous with the help of the equation of the line.

4.2 Examples

Example 1. Find the equation to the pair of lines obtained by joining the origin to the points of intersection of the line $y=mx+c$ and the circle $x^2+y^2=a^2$, and prove that they are at right angles if $2c^2=a^2(1+m^2)$.

Solution. Make the equation of the circle and the equation of the line homogeneous by introducing a third variable z as follows:

$$\left.\begin{aligned} x^2+y^2&=a^2z^2 \\ y&=mx+cz \end{aligned}\right\}.$$

Eliminating z from the above pair of equations, we get

$$x^2+y^2=a^2\left(\frac{y-mx}{c}\right)^2$$

$$\Rightarrow \qquad (c^2-a^2m^2)x^2+2a^2m\,xy+(c^2-a^2)y^2=0. \qquad (1)$$

This is the equation representing a pair of lines through the origin which pass through the points of intersection of the given circle and the line.

Further, the lines given by (1) will be at right angles to each other if

$$\text{coeff. of } x^2+\text{coeff. of } y^2=0$$

i.e. if
$$c^2-a^2m^2+c^2-a^2=0$$

$$\Rightarrow \qquad 2c^2=a^2(1+m^2).$$

Example 2. Prove that the lines joining the origin to the points of intersection of the two curves

$$ax^2+2hxy+by^2+2gx=0$$

and
$$a'x^2+2h'xy+b'y^2+2g'x=0$$

will be at right angles to one another, if

$$g'(a+b)=g(a'+b').$$

Solution. Any curve through the points of intersection of the given curves is

$$ax^2+2hxy+by^2+2gx+\lambda(a'x^2+2h'xy+b'y^2+2g'x)=0$$

i.e.
$$(a+\lambda a')x^2+2(h+\lambda h')xy+(b+\lambda b')y^2+2(g+\lambda g')x=0. \qquad (1)$$

In order that this equation should represent a pair of lines passing through the origin, it must be homogeneous equation of second degree, for which the coefficient of x must vanish *i.e.*

$$2(g+\lambda g')=0$$

$$\Rightarrow \qquad \lambda=-\frac{g}{g'}. \tag{2}$$

The two lines represented by (1) will he at right angles if

$$\text{Coeff. of } x^2+\text{Coeff. of } y^2=0$$

i.e.

$$a+\lambda a'+b+\lambda b'=0$$

$$\Rightarrow a+b+\lambda(a'+b')=0$$

$$\Rightarrow (a+b)-\frac{g}{g'}(a'+b')=0 \quad \text{(On using (2))}$$

$$\Rightarrow g'(a+b)=g(a'+b').$$

Example 3. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines, prove that the equation to the pair of the lines passing through the points where these meet the axes is

$$ax^2-2hxy+by^2+2gx+2fy+c+\frac{4fg}{c}xy=0.$$

Solution. If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$

represents a pair of lines, then

$$abc+2fgh-af^2-bg^2-ch^2=0. \tag{2}$$

The joint equation of the axes is

$$xy=0. \tag{3}$$

Any curve through the points of intersection of (1) and (3) is

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda xy=0$$

i.e.

$$ax^2+(2h+\lambda)xy+by^2+2gx+2fy+c=0. \tag{4}$$

This equation will represent a pair of lines if

$$abc+2fg\left(\frac{2h+\lambda}{2}\right)-af^2-bg^2-c\left(\frac{2h+\lambda}{2}\right)^2=0$$

$$\Rightarrow abc+2fgh-af^2-bg^2-ch^2+fg\lambda-ch\lambda-\frac{c\lambda^2}{4}=0$$

$$\Rightarrow fg\lambda - ch\lambda - \frac{c\lambda^2}{4} = 0 \qquad \text{(On using (2))}$$

$$\Rightarrow \lambda = \frac{4(fg-ch)}{c}.$$

Substituting this value of λ in (4), we get

$$ax^2 + \left\{2h + \frac{4(fg-ch)}{c}\right\}xy + by^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0.$$

EXERCISES

1. Prove that the lines joining the origin to the points common to $3x^2+5xy-3y^2+2x+3y=0$ and $3x-2y=1$ are at right angles.
2. Show that the angle between the lines joining the origin to the intersections of the line $y=3x+2$ with the curve

   $$x^2+2xy+3y^2+4x+8y-11=0,$$

   is $\tan^{-1}\frac{2\sqrt{2}}{3}$.
3. Show that all chords of the curve $3x^2-y^2-2x+4y=0$ which subtend a right angle at the origin pass through a fixed point.
4. Find the angle between the lines joining the origin to the points of intersection of the line $x-3y+2=0$ and the curve

   $$y^2-17xy+16y-12=0.$$
5. Show that the lines joining the origin to the points common to $x^2+hxy-y^2+gx+fy=0$ and $fx-gy=\lambda$ are at right angles for all values of λ.
6. Prove that the lines joining the origin to the points of intersection of

   $$ax^2+2hxy+by^2+2gx+2fy+c=0$$

   and $lx+my+n=0$ are perpendicular to each other if

   $$n^2(a+b)-2n(gl+fm)+c(l^2+m^2)=0.$$
7. If the circle $x^2+y^2+2gx+2fy+c=0$ intercepts on the line

$lx+my=1$ a length which subtends a right angle at the origin, prove that

$$c(l^2+m^2)+2(gl+fm+1)=0.$$

MISCELLANEOUS EXERCISES

1. The distance of a point (x_1, y_1) from each of the two lines which pass through the origin of coordinates is δ; prove that the two lines are given by

$$(x_1y-xy_1)^2=\delta^2(x^2+y^2).$$

2. Show that the product of the perpendiculars from the point (x', y') to the lines $ax^2+2hxy+by^2=0$ is

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

3. If p_1, p_2 be the perpendiculars from (x', y') to the lines

$$ax^2+2hxy+by^2=0,$$

prove that

$$(p_1^2+p_2^2)\{(a-b)^2+4h^2\}=2(a-b)(ax'^2-by'^2)+$$
$$4h(a+b)xy+4h^2(x'^2+y'^2).$$

4. Prove that the pair of lines $ax^2+2hxy+by^2=0$ is equally inclined to the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0.$$

5. Show that the lines given by $ax+by+c=0$ and

$$(ax+by)^2-3(ay-bx)^2=0$$

form an equilateral triangle.

6. For what values of λ is the conic

$$\lambda(x^2+y^2+2gx+2fy+c)+2xy=0$$

a pair of lines? If two of these pairs consist of parallel lines, what condition must be satisfied by g, f and c?

7. Prove that the lines joining the origin to the points of intersection of the line $kx+hy=2hk$, with the curve

$(x-h)^2+(y-k)^2=c^2$ are at right angles if $h^2+k^2=c^2$.

8. Find the equation of the pair of lines joining the origin to the points of intersection of the line $ax+by=a+2b$ and the locus $x^2-xy+y-1=0$.

9. Find the condition of a chord $ax+by+c=0$ of the curve $x^2-3y^2-4x+2y=0$ if it subtends a right angle at the origin.

10. Show that the area of the triangle formed by the lines

$$ax^2+2hxy+by^2=0 \text{ and } lx+my+n=0 \text{ is}$$

$$\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}.$$

11. Show that the lines

$$(a^2-3b^2)x^2+8ab\,xy+(b^2-3a^2)y^2=0$$

and $ax+by+c=0$ form an equilateral triangle whose area is

$$\frac{c^2}{(a^2+b^2)\sqrt{3}}$$

12. Prove that the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel lines if

$$h^2=ab \text{ and } bg^2=af^2.$$

Prove also that the distance between them is

$$2\sqrt{\frac{g^2-ac}{a(a+b)}}.$$

13. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines, prove that

(*i*) the lines are equidistant from the origin, if

$$f^4-g^4=c(bf^2-ag^2);$$

(*ii*) the product of the perpendiculars from the origin on these lines is

$$\frac{c}{\sqrt{(a-b)^2+4h^2}};$$

(*iii*) the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2};$$

(*iv*) the area of the triangle formed by these lines and the x-axis is

$$\frac{g^2-ac}{a\sqrt{h^2-ab}}.$$

14. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines intersecting in (α, β), show that

(*i*) the equation of the pair of lines bisecting the angles between them is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h};$$

(*ii*) the area of the triangle formed by their bisectors and the x-axis is

$$\frac{\sqrt{(a-b)^2+4h^2}}{2h}\cdot\frac{ca-g^2}{ab-h^2}.$$

15. The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two lines at right angles to each other. Prove that the square of the distance of their point of intersection from the origin is

$$\frac{f^2+g^2}{b^2+h^2}.$$

16. Prove that the locus of a point which moves such that the distance between the feet of the perpendiculars from it on the lines $ax^2+2hxy+by^2=0$ is a constant $2k$ is

$$(x^2+y^2)(h^2-ab)=k^2\{(a-b)^2+4h^2\}.$$

17. Show that the orthocentre of the triangle formed by the lines

$$ax^2+2hxy+by^2=0 \quad \text{and} \quad lx+my=1$$

is a point (x', y') such that

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$

18. If $$ax^2+2hxy+by^2+2gx+2fy+c=0$$
and $$ax^2+2hxy+by^2-2gx-2fy+c=0,$$
each represents a pair of lines, prove that the area of the parallelogram enclosed by them is

$$\frac{2c}{\sqrt{h^2-ab}}.$$

19. If the lines $ax^2+2hxy+by^2=0$ be two sides of a parallelogram and the line $lx+my=1$ be one of its diagonals, prove that the equation of the other diagonal is

$$y(bl-hm)=x(am-hl).$$

20. If two of the lines given by

$$ax^3+3bx^2y+3cxy^2+dy^3=0$$

are at right angles, prove that

$$a^2+3ac+3bd+d^2=0.$$

21. Prove that the equation

$$a(x^4+y^4)-4bxy(x^2-y^2)+6cx^2y^2=0$$

represents two pairs of lines which are at right angles, and if $2b^2=a^2+3ac$, the two pairs will coincide.

CHAPTER V

THE CIRCLE

1. DEFINITION OF THE CIRCLE

A **circle** *is the locus of a point which moves such that it remains at a fixed distance from a fixed point.*

The fixed point is called the **centre** of the circle and the fixed distance the **radius** of the circle.

1.1 Equation of a circle. *To find the equation of a circle with given centre and radius.*

Let $C(h, k)$ be the centre and r the radius of the circle.

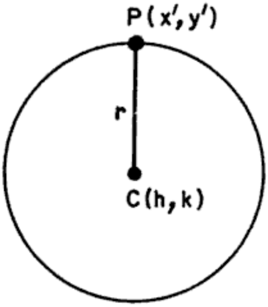


Fig. 5.1

Let $P(x', y')$ be any point on the locus (circle). Then, by definition

$$CP=r.$$

But $$CP=\sqrt{(x'-h)^2+(y'-k)^2}.$$

$$\therefore \quad \sqrt{(x'-h)^2+(y'-k)^2}=r$$

$$\Rightarrow \quad (x'-h)^2+(y'-k)^2=r^2$$

Hence *the equation of the circle with centre* (h, k) *and radius* r *is*

$$(x-h)^2+(y-k)^2=r^2.$$

1.2 The general form of the equation. The equation of the circle is

$$(x-h)^2+(y-k)^2=r^2 \tag{1}$$

which on expanding takes the form of

$$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0.$$

We note the following three things about this equation:

(*i*) This is a second degree equation in x and y;

(*ii*) The coefficients of x^2 and y^2 are equal;

(*iii*) There is no term containing the product xy.

Conversely, we consider the most general equation satisfying these three conditions:

$$ax^2+ay^2+2gx+2fy+c=0, \qquad (a\neq 0). \tag{2}$$

To prove that equation (2) *represents a circle.*

Equation (2) may be written as

$$\left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\frac{g^2+f^2-ac}{a^2}.$$

By comparing it with equation (1) we find that it represents a circle with centre at the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius given by

$$r=\frac{\sqrt{g^2+f^2-ac}}{a}.$$

Hence equation (2) always represents a circle.

There will be no loss of generality if we choose $a=1$ so that the general equation of the cirle takes the form

$$x^2+y^2+2gx+2fy+c=0. \tag{3}$$

This circle has the centre $(-g, -f)$ and radius equal to

$$\sqrt{g^2+f^2-c}.$$

Remarks. (*i*) If $g^2+f^2>c$, equation (3) represents a circle with centre $(-g, -f)$ and real radius.

(*ii*) If $g^2+f^2=c$, then equation (3) represents a circle whose centre is $(-g, -f)$ and radius is zero *i.e.* the circle coincides with the centre. This circle is called the **point circle.**

(*iii*) If $g^2+f^2<c$, then equation (3) represents a circle with centre $(-g, -f)$ and radius imaginary. In this case there are no real points on the circle and we call it a **virtual circle.**

Corollary. The equation of the circle centred at the origin and of radius a is

$$x^2+y^2=a^2.$$

1.3 Circle with given diameter.

To find the equation of the circle described on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter.

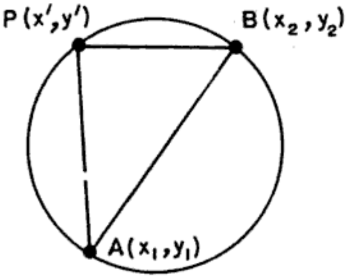


Fig. 5.2

Let $P(x', y')$ be any point on the circle. Then

$$m = \text{slope of the line } AP = \frac{y'-y_1}{x'-x_1},$$

$$m' = \text{slope of the line } PB = \frac{y'-y_2}{x'-x_2}.$$

Since $\angle APB = 90°$, $mm' = -1$, we have

$$\left(\frac{y'-y_1}{x'-x_1}\right)\left(\frac{y'-y_2}{x'-x_2}\right) = -1$$

$$\Rightarrow \quad (x'-x_1)(x'-x_2)+(y'-y_1)(y'-y_2)=0.$$

Hence the required equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Ex. Find the equation to the circle of which the points (8, −2) and (2, −6) are the end points of a diameter.

1.4 Parametric form of the circle. *To find the parametric equations of a circle.*

Consider the equation of the circle

$$x^2+y^2=a^2. \tag{1}$$

Let $P(x, y)$ be any point on it. Let $\angle XOP=\theta$. Then the coordinates (x, y) of P can be expressed as

$$\left.\begin{array}{l} x=a\cos\theta \\ y=a\sin\theta \end{array}\right\} \tag{2}$$

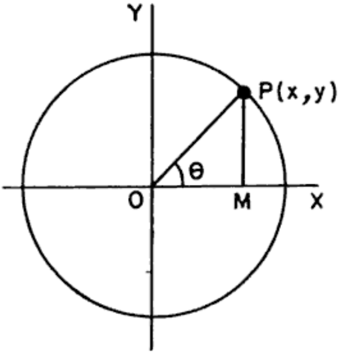


Fig. 5.3

Conversely, the point $(a\cos\theta, a\sin\theta)$ lies on the circle for all values of θ, $0 \leqslant \theta \leqslant 2\pi$. Hence equation (2) can be regarded as a parametric form of the circle given by equation (1), where θ is a parameter.

More generally, if we consider the circle with centre (h, k) and radius r, then its parametric form is

$$x=h+r\cos\theta$$

$$y=k+r\sin\theta.$$

1.5 Circle and a point. A circle divides the plane into two regions. A point may be either outside or inside the circle, if it is not on the circle itself which is the boundary between the two regions. We now proceed to find an algebraic condition for a point to be in a particular region determined by the circle.

To prove that the point (x_1, y_1) *lies outside, on or inside the circle* $x^2+y^2=a^2$ *according as* $x_1^2+y_1^2-a^2 >=$ *or* <0.

The distance between the centre of the circle and the point (x_1, y_1) is $>=$or $<$ the radius of the circle *i.e.* $\sqrt{x_1^2+y_1^2} >=$or $<a$. Hence the point (x_1, y_1) lies outside, on or inside the circle $x^2+y^2=a^2$ according as

$$x_1^2+y_1^2-a^2 >= \text{or} <0.$$

Ex. Prove that the point (x_1, y_1) lies outside, on or inside the circle $x^2+y^2+2gx+2fy+c=0$ according as

$$x_1^2+y_1^2+2gx_1+2fy_1+c >= \text{or} <0.$$

1.6 Examples

Example 1. Whatever be the value of α, prove that the locus of the intersection of the lines

$$x \cos \alpha+y \sin \alpha=a \text{ and } x \sin \alpha-y \cos \alpha=b$$

is a circle.

Solution. The locus of the point of intersection of the given lines is obtained by eliminating α between them. For this squaring and adding the given equations, we get

$$(x \cos \alpha+y \sin \alpha)^2+(x \sin \alpha-y \cos \alpha)^2=a^2+b^2$$

$$\Rightarrow \qquad x^2+y^2=a^2+b^2,$$

which is a circle.

Example 2. Prove that the circle on the chord $lx+my+n=0$ of the circle $x^2+y^2=a^2$ as a diameter is

$$(l^2+m^2)(x^2+y^2-a^2)+2n(lx+my+n)=0.$$

Solution. Any circle through the intersection of the given circle and the chord is

$$x^2+y^2-a^2+\lambda(lx+my+n)=0. \qquad (1)$$

The centre of the circle (1) is $(-\frac{1}{2}\lambda l, -\frac{1}{2}\lambda m)$. The chord $lx+my+n=0$ will be a diameter of circle (1) if its centre lies on the chord *i.e.*

$$l(-\tfrac{1}{2}\lambda l)+m(-\tfrac{1}{2}\lambda m)+n=0$$

$$\Rightarrow \qquad \lambda=\frac{2n}{l^2+m^2}.$$

Substituting this value of λ in (1), we get

$$(l^2+m^2)(x^2+y^2-a^2)+2n(lx+my+n)=0.$$

EXERCISES

1. Find the equation to the circle:
   (*i*) whose radius is 4 and centre is (1, 2).
   (*ii*) whose radius is 3 and centre is $(-3, -4)$.
   (*iii*) whose radius is $\sqrt{3}$ and centre is $(\frac{2}{3}, \frac{1}{3})$.
2. Find the coordinates of the centre and the radius of the circle whose equation is:
   (*i*) $x^2+y^2+4x+8y-41=0$.
   (*ii*) $x^2+y^2-4x-16=0$.
   (*iii*) $3x^2+3y^2+6x-5y=0$.
3. Find the equation to the circle whose diameter is the line joining the points:
   (*i*) (5, 1) and (7, 5).
   (*ii*) $(-4, -2)$ and (1, 1).
   (*iii*) $(-2, 1)$ and $(4, -3)$.
4. Find the equation to the circle which passes through the points:
   (*i*) (0, 0), $(5, -7)$, $(1, -3)$.
   (*ii*) (0, 0), $(a, 0)$, $(0, b)$.
   (*iii*) (3, 0), (4, 2), (0, 1).
   (*iv*) $(-1, 1)$, $(-2, 1)$, (4, 3).
5. Find the equation to the circle which passes through the points $(-1, 2)$, $(-4, 3)$ and has its centre on the line $4x-3y=5$.
6. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 4 and 5 from the axes.
7. Find the equation to the circle
   (*i*) which touches the axes at a distance 4 from the origin.
   (*ii*) which passes through (5, 3) and has its centre at $(-3, 1)$.
8. Find the equation to the circle circumscribing the triangle formed by the lines

$$x+y=6,\ 2x+y=4 \text{ and } x+2y=5.$$

9. Find the value of f for which the circle

$$x^2+y^2-4x+2fy+13=0.$$

has radius 4.

10. Prove that the locus of the point of intersection of the lines drawn through the points $(a, 0)$ and $(-a, 0)$ which include a constant angle θ is the circle

$$x^2+y^2-a^2\pm 2ay\cot\theta=0.$$

Hence prove that the angles in the same segment of a circle are equal.

2. TANGENT AND NORMAL

Definition of tangent

Let P and Q be any two points on a curve such that P and Q are near to one another. Join the secant PQ. If $Q \to P$, then *the secant PQ is called the tangent at P to the curve.* In other words *the tangent at P is the limiting position of the secant PQ through P as Q tends to coincide with P.* Consequently, the tangent PT cuts the curve in two coincident points at P.

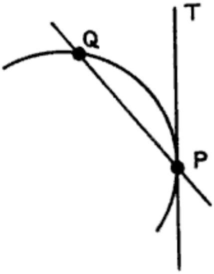


Fig. 5.4

Definition of normal

A line through the point P perpendicular to the tangent at P is called the normal to the curve at P.

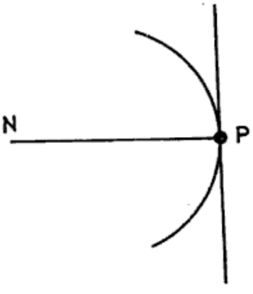


Fig. 5.5

2.1 Equation of the tangent at a point. *To find the equation of the tangent to a circle at a point on it.*

Let $$x^2+y^2=a^2 \tag{1}$$

be the equation of the circle and $P(x_1, y_1)$ be the given point on it.

Since (x_1, y_1) lies on (1), we have

$$x_1^2+y_1^2=a^2. \tag{2}$$

Take $Q(x_2, y_2)$ any other point on (1). Then

$$x_2^2+y_2^2=a^2. \tag{3}$$

The equation of the secant PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \tag{4}$$

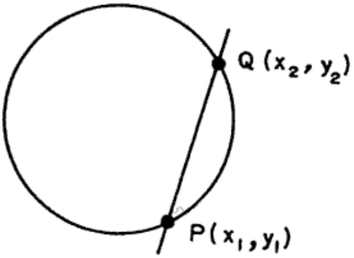


Fig. 5.6

As $Q \to P$, by definition, secant PQ becomes the tangent at P and therefore on taking the limit as $x_2 \to x_1$ and $y_2 \to y_1$, equation (4) would represent the equation of the tangent to (1) at $P(x_1, y_1)$. Since, on taking the limit, the value of $\frac{y_2-y_1}{x_2-x_1}$ is of the form $\frac{0}{0}$ which is an indeterminate form, we proceed to obtain the same as follows.

Subtracting (2) from (3), we get

$$x_2^2-x_1^2+y_2^2-y_1^2=0$$

$$\Rightarrow \qquad \frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1}.$$

Thus, as $Q \to P$, $x_2 \to x_1$ and $y_2 \to y_1$, and we get

$$\frac{y_2-y_1}{x_2-x_1}\to-\frac{2x_1}{2y_1}=-\frac{x_1}{y_1},$$

and hence as $Q\to P$, equation (4) becomes

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

$$\Rightarrow \qquad xx_1+yy_1=x_1^2+y_1^2.$$

Therefore, on using (2), we get

$$xx_1+yy_1=a^2.$$

Hence *the equation of the tangent to the circle* $x^2+y^2=a^2$ *at the point* (x_1, y_1) *is*

$$xx_1+yy_1=a^2 \qquad \square$$

More generally the equation of the tangent to the circle

$$x^2+y^2+2gx+2fy+c=0$$

at the point (x_1, y_1) on it is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Working rule. The equation of the tangent at the point (x_1, y_1) to any circle can be obtained by writing xx_1 for x^2, yy_1 for y^2, $x+x_1$ for $2x$ and $y+y_1$ for $2y$.

Ex. Find the equation of the tangent to the circle

$$x^2+y^2-4y-1=0$$

at the point (2, 1).

2.2 Intersection of a circle and a line

Let $$x^2+y^2=a^2 \tag{1}$$

and $$y=mx+c \tag{2}$$

be the equations of a circle and a line, respectively.

For the points of intersection we solve (1) and (2) simultaneously and we get

$$x^2+(mx+c)^2=a^2$$

i.e. $$(1+m^2)x^2+2mcx+c^2-a^2=0. \tag{3}$$

Since equation (3) is quadratic in x it gives two values of x which may be real and distinct, coincident or imaginary. Accordingly there are two points of intersection of the circle (1) and the line (2). Hence a line always cuts a circle in two points which may be real and distinct, coincident or imaginary.

2.3 Condition of tangency. *To find the condition when the line* $y=mx+c$ *becomes a tangent to the circle* $x^2+y^2=a^2$.

First Method. It is clear from equation (3) in § 2.2 that the line $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$ if the roots of the equation (3) in § 2.2 are equal for which

$$m^2c^2=(1+m^2)(c^2-a^2)$$

i.e.

$$c=a\sqrt{1+m^2},$$

which is the required condition of tangency.

Second Method. Equation of the tangent to the given circle $x^2+y^2=a^2$ at any point (x_1, y_1) is

$$xx_1+yy_1=a^2.$$

The line $y=mx+c$ is a tangent to the given circle if

$$xx_1+yy_1-a^2\equiv mx-y+c$$

$$\Rightarrow \quad \frac{x_1}{m}=\frac{y_1}{-1}=\frac{-a^2}{c}$$

$$\Rightarrow \quad x_1=\frac{-a^2m}{c},\ y_1=\frac{a^2}{c}.$$

But (x_1, y_1) lies on the given circle.

$$\therefore \quad \left(-\frac{a^2m}{c}\right)^2+\left(\frac{a^2}{c}\right)^2=a^2$$

$$\Rightarrow \quad c=a\sqrt{1+m^2},$$

which is the required condition.

Third Method. The line $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$ if the perpendicular distance of the line from the centre of the circle is equal to the radius of the circle *i.e.* if

$$\frac{c}{\sqrt{1+m^2}}=a$$

$$\Rightarrow \quad c=a\sqrt{1+m^2}.$$

Remarks. (*i*) The line $y=mx+a\sqrt{1+m^2}$ is always a tangent to the circle $x^2+y^2=a^2$ whatever the value of m may be. In fact, it is a family of tangents to the circle $x^2+y^2=a^2$.

(*ii*) The line $y=mx+a\sqrt{1+m^2}$ is called the equation of the tangent in **'m' form** or in **slope form** to the circle $x^2+y^2=a^2$.

(*iii*) The point of contact of the tangent $y=mx+a\sqrt{1+m^2}$ to the circle $x^2+y^2=a^2$ is

$$\left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right).$$

Ex. Find the condition that the line $x\cos\alpha+y\sin\alpha=p$ may touch the circle $x^2+y^2=a^2$.

2.4 Equation of normal. *To find the equation of the normal to the circle* $x^2+y^2=a^2$ *at any point* (x_1, y_1) *on it.*

The equation of the tangent at the point (x_1, y_1) is

$$xx_1+yy_1=a^2.$$

Slope of the tangent $=-\dfrac{x_1}{y_1}$.

$\therefore$ Slope of the normal $=\dfrac{y_1}{x_1}$.

Hence the equation of the normal at (x_1, y_1) to the circle $x^2+y^2=a^2$ is

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$

i.e. $xy_1-x_1y=0$ □

Following similar lines, one can easily show that the equation of the normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x_1, y_1) is

$$x(y_1+f)-y(x_1+g)-fx_1+gy_1=0.$$

Ex. Prove that the normal at any point to a circle passes through the centre of the circle.

2.5 Examples

Example 1. Find the equation to the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is

$$\alpha x+\beta y=0.$$

Solution. Let r be the radius of the circle. Then equation of the circle with centre (α, β) and radius r is

$$(x-\alpha)^2+(y-\beta)^2=r^2. \tag{1}$$

It will pass through the origin if

$$\alpha^2+\beta^2=r^2. \tag{2}$$

Now eliminating r from (1) and (2), we get

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$$

$$\Rightarrow \qquad x^2+y^2-2\alpha x-2\beta y=0. \tag{3}$$

This is the required equation of the circle.

Further, equation of the tangent to the circle (3) at the origin is

$$x.0+y.0-\alpha(x+0)-\beta(y+0)=0$$

i.e. $$\alpha x+\beta y=0.$$

Example 2. Find the condition that the line $lx+my+n=0$ may be normal to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

Solution. The line will be a normal to the circle if it passes through the centre $(-g, -f)$ of the circle *i.e.*

$$l(-g)+m(-f)+n=0$$

$$\Rightarrow \qquad lg+mf=n.$$

This is the required condition.

EXERCISES

1. Find the equation of the tangent to the circle $x^2+y^2=13$ at the point (2, 3).
2. Prove that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches the axes.
3. Prove that the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$ if

$$(c-f^2)l^2+2fglm+(c-g^2)m^2-2n(gl+fm)+n^2=0.$$

4. Prove that the tangent to a circle at any point on it is perpendicular to the radius at that point.
5. Find the equations of the two tangents to the circle $x^2+y^2=4$ which make an angle of 60° with the x-axis.
6. Prove that the line $2x+y=4$ is a tangent to the circle
$$x^2+y^2+6x-10y+29=0.$$
7. Find the condition that the line $lx+my+n=0$ may touch the circle $(x-h)^2+(y-k)^2=r^2$.
8. A line moves so that the sum of the perpendiculars drawn to it from the points $(a, 0)$, $(-a, 0)$ is constant; show that it always touches a circle.
9. Prove that the line $x+y=2$ touches the circles $x^2+y^2=2$ and $x^2+y^2+3x+3y-8=0$ at the same point.
10. Find the vertices of the triangle formed by the tangents at A, B and C to the circumcircle of the triangle whose vertices are $A(2, 3)$, $B(-2, 1)$ and $C(-3, -2)$.
11. For what value of λ does the line $4x+\lambda y+7=0$ touch the circle $x^2+y^2-6x+4y-12=0$.
12. Find the equation of the normal to the circle $x^2+y^2=5$ at the point $(-1, 2)$.
13. Find the condition that the line $lx+my+n=0$ may be normal to the circle $x^2+y^2=a^2$.
14. Find the equation of the normal to the circle $x^2+y^2-2ax=0$ at the point $(a(1+\cos\alpha), a\sin\alpha)$.

3. TANGENTS FROM A POINT

3.1 *To prove that from any point there can be drawn two tangents, real or imaginary, to a circle.*

Let the equation of the circle be
$$x^2+y^2=a^2. \tag{1}$$
Let $P(x_1, y_1)$ be the given point.

Any tangent to the circle (1) is
$$y=mx+a\sqrt{1+m^2},$$
where m is the slope and can take any value. Since the tangent passes through the given point $P(x_1, y_1)$, we have

$$y_1=mx_1+a\sqrt{1+m^2}$$

$$\Rightarrow \qquad (y_1-mx_1)^2=a^2(1+m^2)$$

$$\Rightarrow \qquad (x_1^2-a^2)m^2-2x_1y_1m+y_1^2-a^2=0. \qquad (2)$$

This is a quadratic equation in m giving two values of m (real and distinct, coincident or imaginary) corresponding to which there will be two tangents passing through the point $P(x_1, y_1)$. The two tangents will be real and distinct, coincident or imaginary according as the two roots of equation (2) are so. But the roots of equation (2) are real and distinct, coincident or imaginary according as

$$4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)>=\text{ or }<0$$

i.e. according as

$$x_1^2+y_1^2-a^2>=\text{ or }<0.$$

But this is the condition for the point (x_1, y_1) to be outside, on or inside the circle. Hence from a point there can be drawn two tangents to a circle and these tangents will be real and distinct, coincident or imaginary according as the point lies outside, on or inside the circle.

Note. This result can also be obtained by considering the general equation of the circle.

3.2 Pair of tangents from a given point. *To find the equation of the pair of tangents drawn from an external point $A(x_1, y_1)$ to the circle $x^2+y^2=a^2$.*

First Method. Let $P(x', y')$ be any point on the locus (*i.e.* on either of the tangents). Then the equation of the line AP is

$$y-y_1=\frac{y'-y_1}{x'-x_1}(x-x_1)$$

i.e.
$$x(y'-y_1)-y(x'-x_1)+x'y_1-y'x_1=0. \qquad (1)$$

Since AP is a tangent which touches the circle at Q, the perpendicular distance of the line (1) from C is a, the radius of the circle *i.e.*

$$\frac{x'y_1-y'x_1}{\sqrt{(x'-x_1)^2+(y'-y_1)^2}}=a$$

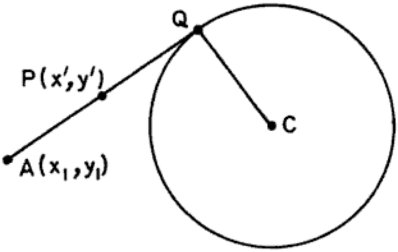


Fig. 5.7

$$\Rightarrow \qquad (x'y_1-y'x_1)^2=a^2\{(x'-x_1)^2+(y'-y_1)^2\}.$$

Hence the locus of (x', y') is

$$(xy_1-yx_1)^2=a^2\{(x-x_1)^2+(y-y_1)^2\}.$$

After simplification and rearrangement of the terms this can be written as

$$(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2.$$

This is the required equation of the pair of tangents.

Hence, using the notations:

$$S\equiv x^2+y^2-a^2$$

$$S_1\equiv x_1^2+y_1^2-a^2$$

$$T\equiv xx_1+yy_1-a^2,$$

we find that the equation of the pair of tangents drawn from the point (x_1, y_1) to the circle $x^2+y^2=a^2$ is

$$T^2=SS_1.$$

Second Method. The equations of a line through $A(x_1, y_1)$ making an angle θ with x-axis are

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \qquad (1)$$

r being the algebraic distance of the point $P(x, y)$ from $A(x_1, y_1)$ measured along the line. Therefore, the coordinates of a point on the line which is at a distance r from $A(x_1, y_1)$ are given by

$$x=x_1+r\cos\theta$$

$$y=y_1+r\sin\theta.$$

If this point be on the circle, we must have

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2$$

i.e. $$(\cos^2\theta+\sin^2\theta)\,r^2+2(x_1\cos\theta+y_1\sin\theta)\,r+x_1^2+y_1^2-a^2=0.$$

But this is a quadratic equation in r giving two values of r corresponding to which there are two points common to the circle and the line (1).

Now, if the line (1) be a tangent to the circle, both the values of r given by the quadratic equation must be equal. Therefore, we must have

$$(x_1\cos\theta+y_1\sin\theta)^2-(\cos^2\theta+\sin^2\theta)(x_1^2+y_1^2-a^2)=0. \qquad (1)$$

This equation gives us the value of θ *i.e.* the direction of the line (1) so that it may become the tangent to the circle. To obtain the actual equation of the tangent (*s*), eliminate θ between (1) and (2).

Thus the equation of the tangents is

$$\left\{x_1\left(\frac{x-x_1}{r}\right)+y_1\left(\frac{y-y_1}{r}\right)\right\}^2$$

$$=\left\{\left(\frac{x-x_1}{r}\right)^2+\left(\frac{y-y_1}{r}\right)^2\right\}\{x_1^2+y_1^2-a^2\}$$

$$\Rightarrow\quad (xx_1+yy_1-x_1^2-y_1^2)^2$$

$$=\{x^2+y^2+x_1^2+y_1^2-2(xx_1+yy_1)\}\{x_1^2+y_1^2-a^2\}$$

$$\Rightarrow\quad (xx_1+yy_1-a^2)^2=(x^2+y^2-a^2)(x_1^2+y_1^2-a^2).$$

Using the symbols S, S_1 and T as earlier, we find that the equation of the pair of tangents is

$$T^2=SS_1.$$

Note. The equation of the pair of tangents drawn from the point (x_1, y_1) to the circle given by the general equation

$$x^2+y^2+2gx+2fy+c=0$$

is $$T^2=SS_1, \text{ where}$$

$$S\equiv x^2+y^2+2gx+2fy+c$$

$$S_1\equiv x_1^2+y_1^2+2gx_1+2fy_1+c$$

$$T\equiv xx_1+yy_1+g(x+x_1)+f(y+y_1)+c.$$

3.3 Length of the tangent. *To find the length of the tangent drawn from an external point* $A(x_1, y_1)$ *to the circle*

$$x^2+y^2+2gx+2fy+c=0.$$

Let the tangent drawn from the point $A(x_1, y_1)$ to the given circle touch it at the point T.

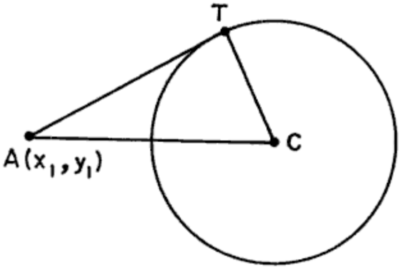


Fig. 5.8

Then CT=radius of the circle

$$=\sqrt{g^2+f^2-c}$$

and $AC=\sqrt{(x_1+g)^2+(y_1+f)^2}$.

Now in right angled triangle ACT, we have

$$AT^2=AC^2-CT^2$$
$$=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c.$$

Hence *the length of the tangent from the point* (x_1, y_1) *to the circle* $x^2+y^2+2gx+2fy+c=0$ *is*

$$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}.$$

Working rule. After the equation of the circle being written in the form so that each of the coefficients of x^2 and y^2 is unity and the right side is zero, write x_1 for x and y_1 for y in the left side of the equation of the circle and then take the square root. This gives the length of the tangent from the point (x_1, y_1) to the circle.

Ex. Find the length of the tangnnt from the point (—3, 4) to the circle

$$x^2+y^2-4x-6y-3=0.$$

3.4 Example. The angle between two tangents to the circle $x^2+y^2=a^2$ is constant and equal to α. Prove that the locus of their points of intersection is given by

$$4a^2(x^2+y^2-a^2)=(x^2+y^2-2a^2)^2 \tan^2 \alpha.$$

What happens to this locus if $\alpha=\pi/2$?

Solution. Let (h, k) be the point of intersection of the two tangents to the given circle. Then, the equation of the pair of tangents is

$$(x^2+y^2-a^2)(h^2+k^2-a^2)=(xh+yk-a^2)^2$$

$$\Rightarrow \quad x^2(k^2-a^2)-2hkxy+y^2(h^2-a^2)+\text{first degree terms}$$
$$+\text{constant terms}=0.$$

Since α is the angle between the tangents, we have

$$\tan \alpha=\frac{2\sqrt{h^2k^2-(k^2-a^2)(h^2-a^2)}}{k^2-a^2+h^2-a^2}$$

$$\Rightarrow \quad 4a^2(h^2+k^2-a^2)=(h^2+k^2-2a^2) \tan^2 \alpha.$$

Hence the locus of (h, k) is

$$4a^2(x^2+y^2-a^2)=(x^2+y^2-2a^2) \tan^2 \alpha.$$

When $\alpha=\pi/2$, the locus becomes

$$x^2+y^2=2a^2.$$

EXERCISES

1. Prove that the pair of tangents from the point $(1, 2)$ to the circle $x^2+y^2-4x+2y=0$ are mutually perpendicular.
2. Find the equation of the pair of tangents from the origin to the circle $x^2+y^2+2gx+2fy+c=0$.
3. From any point on the circle $x^2+y^2+2gx+2fy+c=0$ tangents are drawn to the circle

$$x^2+y^2+2gx+2fy+c \sin^2 \alpha+(g^2+f^2) \cos^2 \alpha=0.$$

Prove that the angle between them is 2α.

4. Find the length of the tangent drawn from the point $(a+b, a-b)$ to the circle

$$x^2+y^2+2bx-3b^2=0.$$

5. Prove that the locus of a point, which moves so that the tangents from it to the two circles

$$x^2+y^2-5x-3=0, \; 3x^2+3y^2+2x+4y-6=0$$

are equal, is a line.

4. CHORD OF CONTACT

The line through the points of contact of the tangents drawn from a point (x_1, y_1) *to a circle (or a conic in general) is called the* **chord of contact** *of tangents from the point* (x_1, y_1) *to the circle (or conic).*

4.1 Equation of the chord of contact. *To find the equation of the chord of contact of tangents drawn from the point* (x_1, y_1) *to the circle* $x^2+y^2=a^2$.

Let $A(x', y')$ and $B(x'', x'')$ be the points of contacts of the tangents drawn from the point $P(x_1, y_1)$ to the circle

$$x^2+y^2=a^2. \tag{1}$$

Then, the equations of the tangents at the points A and B to the circle (1) are

$$x'x+y'y=a^2$$

and

$$x''x+y''y=a^2.$$

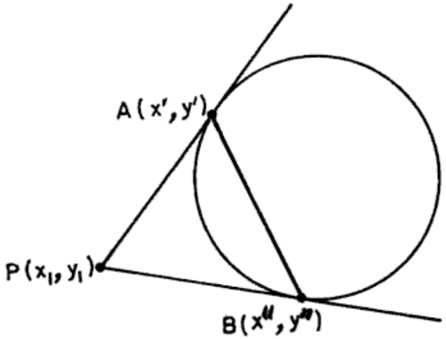


Fig. 5.9

But each of these tangents passes through the point $P(x_1, y_1)$. Therefore

$$x' x_1+y' y_1=a^2$$

and

$$x''x_1+y''y_1=a^2.$$

It is clear from these relations that (x', y') and (x'', y'') both satisfy the equation

$$xx_1 + yy_1 = a^2.$$

This is the required equation of the chord of contact of the tangents drawn from the point $P(x_1, y_1)$.

Remarks. (*i*) We note that the equation of the chord of contact of tangents drawn from a point (x_1, y_1) to a circle is of the same form as that the equation of the tangent at the point (x_1, y_1) to the circle. The reason is simple since in case the point $P(x_1, y_1)$ lies on the circle, the points A and B both coincide with P, and in this limiting position the chord of contact becomes the tangent to the circle at the point $P(x_1, y_1)$.

(*ii*) If the point $P(x_1, y_1)$ lies outside the circle, the situation is quite simple and clear geometrically also.

(*iii*) If the point $P(x_1, y_1)$ is inside the circle, the tangents drawn from the point $P(x_1, y_1)$ to the circle will be imaginary. But the chord of contact still exists and is real since x_1 and y_1 are real. Thus, there is a real line joining the imaginary points of contact of the two imaginary tangents drawn from the point inside the circle. This is consistent with the result that a line always meets a circle in two points, real or imaginary.

Thus the chord of contact of the tangents drawn from a point (x_1, y_1) to a circle always exists and is real whatever the position of the point P may be, outside, on or inside the circle.

4.2 Example. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$. Prove that the area of the triangle formed by them and the line joining their points of contact is

$$\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}.$$

Solution. Let A be the point (h, k) and PQ the chord of contact of the tangents drawn from $A(h, k)$. Then the equation of PQ is

$$xh+yk=a^2. \tag{1}$$

Let AM be the perpendicular drawn from A on PQ. Then

AM=perpendicular distance of (1) from $A(k, k)$

$$=\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}.$$

Also AP=length of the tangent$=\sqrt{h^2+k^2-a^2}$.

Now in right angled triangle APM,

$$PM^2=AP^2-AM^2$$
$$=(h^2+k^2-a^2)-\frac{(h^2+k^2-a^2)^2}{h^2+k^2}=\frac{a^2(h^2+k^2-a^2)}{h^2+k^2}$$

Hence the area of the triangle APQ is

$$2(\tfrac{1}{2}\,PM.AM)$$
$$=\frac{a\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}\cdot\frac{(h^2+k^2-a^2)}{\sqrt{h^2+k^2}}$$
$$=\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}.$$

EXERCISES

1. Find the equation of the chord of contact of the point (4, 3) with respect to the circle $x^2+y^2-3x+12y+6=0$.
2. Find the equation of the chord of contact of the point $(a, -b)$ with respect to the circle $x^2+y^2+2ax-2by+a^2-b^2=0$.
3. Prove that the chord of contact of the point (1, -2) with respect to the circles $x^2+y^2+6y+5=0$ and $x^2+y^2+2x+8y+5=0$ coincide.
4. Find the condition that the chord of contact of tangents from the point (x_1, y_1) to the circle $x^2+y^2=a^2$ should subtend a right angle at the centre.
5. Find the locus of a point the chord of contact of the tangents from which subtends a right angle at the centre of the circle $x^2+y^2=a^2$.

5. POLE AND POLAR

The **Polar** *of a point with respect to a circle (or a conic in general) is the locus of the points of intersection of the tangents drawn at the end points of chords through that point and the point is said to be the* **pole** *of the polar.*

Let P be a point outside or inside the circle. Take three chords PA_1B_1, PA_2B_2 and PA_3B_3. Let the tangents at the ends of these chords meet at points T_1, T_2 and T_3, respectively. Then the path of the points T's is, in fact, the polar of the point P. One may refer to Fig. 5.10(a) when the point P is outside the circle and to Fig. 5.10(b) when the point P is inside the circle.

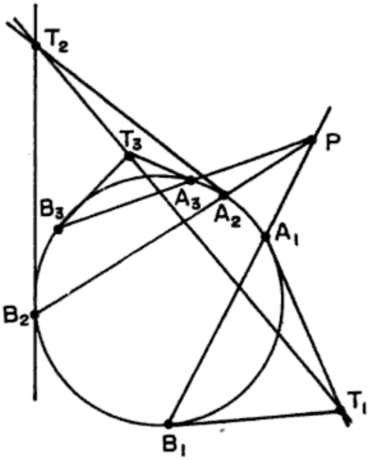


Fig. 5.10(*a*)

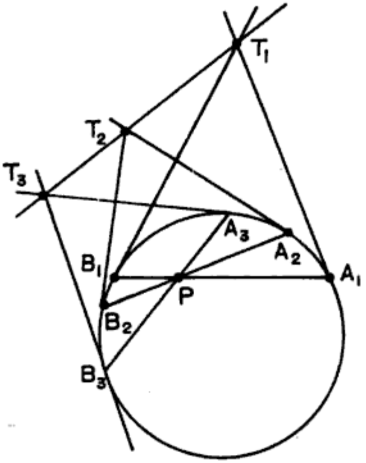


Fig. 5.10(*b*)

5.1 Equation of the polar. *The find the equation of the polar of he point* (x_1, y_1) *wiih respect to the circle* $x^2+y^2=a^2$.

Let $P(x', y')$ be any point on the locus *i.e.* the point of intersection of the tangents drawn at the extremities of a chord through the point (x_1, y_1).

The equation of the chord of contact of tangents from (x', y') is

$$xx'+yy'=a^2.$$

But it passes tnrough the point (x_1, y_1).

$$\therefore \qquad x_1x'+y_1y'=a^2.$$

Hence the locus of (x', y') is

$$xx_1+yy_1=a^2.$$

Note. The terms pole and polar are correlating and these terms here are nothing to do with those occurring in polar coordinates.

5.2 Pole of a line. *To find the pole of a line* $lx+my+n=0$ *with respect to the circle* $x^2+y^2+a^2$.

Let (x_1, y_1) be the pole of the line $lx+my+n=0$ with respect to the circle $x^2+y^2=a^2$.

Then this line must be identical with the polar of (x_1, y_1) with respect to the cicle $x^2+y^2=a^2$ *i.e.* with

$$xx_1+yy_1=a^2.$$

$$\therefore \qquad \frac{x_1}{l}=\frac{y_1}{m}=\frac{-a^2}{n}$$

$$\Rightarrow \qquad \begin{cases} x_1=-\dfrac{a^2l}{n} \\ y_1=-\dfrac{a^2m}{n}. \end{cases}$$

Hence the required pole is $\left(-\dfrac{a^2l}{n}, -\dfrac{a^2m}{n}\right)$.

Ex. Find the pole of the line $x+2y-1=0$ with respect to the circle $x^2+y^2=3$.

5.3 Propositions on polars.

Proposition I. *The polar is perpendicular to the line joining the centre to the pole.*

Let $$x^2+y^2=a^2$$
be the equation of the circle and $P(x_1, y_1)$ the pole.

Then equation of the polar is

$$xx_1+yy_1=a^2.$$

Now m_1=slope of the polar $=-\dfrac{x_1}{y_1}$

m_2=slope of the line joining the centre to the pole (x_1, y_1)

$=\dfrac{y_1}{x_1}$

We note that $m_1m_2=-1$. Hence the proposition follows.

Proposition II. *The pole and the point of intersection of the polar with the join of the centre and the pole are inverse points* with respect to the circle* $x^2+y^2=a^2$.

Let C be the centre of the circle and Q the point of intersection of the polar with CP, where $P(x_1, y_1)$ is the pole. Then, by Proposition *I*, we note that

CQ=perpendicular distance of C from the polar of P

$$=\frac{a^2}{\sqrt{x_1^2+y_1^2}}$$

and $$CP=\sqrt{x_1^2+y_1^2}\ .$$

$\therefore\quad CP.CQ=a^2=(\text{radius})^2.$

Hence the points P and Q are the inverse points with respect to the circle $x^2+y^2=a^2$.

Proposition III. *If the polar of a point P passes through the point Q, then polar of Q passes through P.*

Let (x_1, y_1) and (x_2, y_2), respectively, be the coordinates of P and Q. Since the polar of the point P passes through the point Q, we have

$$x_1x_2+y_1y_2=a^2.$$

But this is the condition also that the point (x_1, y_1) should lie on the line

$$xx_2+yy_2=a^2,$$

---

*Two points P and Q are said to be inverse points with respect to a circle if $CP.CQ=(\text{radins})^2$, C being the centre of the circle.

which, in fact, is the polar of the point Q. This proves the proposition.

Two points such that each lies on the polar of the other are called **conjugate points.**

Proposition IV. *If the pole of a line L_1 lies on another line L_2, then the pole of L_2, lies on L_1.*

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the poles of the lines L_1 and L_2, respectively. In other words, L_1 and L_2 are the polars of P and Q, respectively. Since the polar of Q passes through P, by Proposition III, the polar of P(*i.e.* the line L_1) must pass through Q. Hence the proposition follows.

Two lines such that each contains the pole of the other are called **conjugate lines.**

Proposition V. *If the polars of any two points P and Q meet in R, then the polar of R is the line PQ.*

Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points P and Q, respectively. Then the polars of P and Q with respect to the circle $x^2+y^2=a^2$ are

$$\left.\begin{array}{ll} & xx_1+yy_1=a^2 \\ \text{and} & xx_2+yy_2=a^2 \end{array}\right\}.$$

By solving the simultaneous equations the coordinate of R are given by

$$\left(\frac{a^2(y_2-y_1)}{x_1y_2-x_2y_1}, \frac{-a^2(x_2-x_1)}{x_1y_2-x_2y_1}\right).$$

Now the polar of R is

$$\frac{xa^2(y_2-y_1)}{x_1y_2-x_2y_1}-\frac{ya^2(x_2-x_1)}{x_1y_2-x_2y_1}=a^2$$

$$\Rightarrow \qquad y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1),$$

which, in fact, is the equation of the line PQ.

5.4 Examples

Example 1. Prove that the locus of a point whose polar with respect to the circle $x^2+y^2=a^2$ touches the circle $(x-c)^2+(y-d)^2=b^2$ is $b^2(x^2+y^2)=(a^2-cx-dy)^2$.

Solution. Let (h, k) be a point on the locus. Then its polar with respect to the circle $x^2+y^2=a^2$ is

$$xh+yk=a^2.$$

This line will touch the circle $(x-c)^2+(y-d)^2=b^2$ if the perpendicular distance of it from the centre of the circle is equal to the radius *i.e.* if

$$\frac{a^2-ch-dk}{\sqrt{h^2+k^2}}=b$$

$$\Rightarrow \qquad b^2(h^2+k^2)=(a^2-ch-dk)^2.$$

Hence the locus of (h, k) is

$$b^2(x^2+y^2)=(a^2-cx-dy)^2.$$

Example 2. Prove that the lines

$$l_1x+m_1y+n_1=0 \text{ and } l_2x+m_2y+n_2=0$$

are conjugate with respect to the circle

$$x^2+y^2=a^2 \text{ if } (l_1l_2+m_1m_2)a^2=n_1n_2.$$

Solution. Let (x_1, y_1) be the pole of the line $l_1x+m_1y+n_1=0$. Then its polar with respect to the given circle is

$$xx_1+yy_1=a^2$$

But $l_1x+m_1y+n_1=0$ and $xx_1+yy_1-a^2=0$ are identical. Therefore, we have

$$\frac{x_1}{l_1}=\frac{y_1}{m_1}=\frac{-a^2}{n_1},$$

which gives $x_1=-\dfrac{a^2l_1}{n_1}$ and $y_1=-\dfrac{a^2m_1}{n_1}$. The given lines are conjugate if the point (x_1, y_1), pole of $l_1x+m_1y+n_1=0$, lies on $l_2x+m_2y+n_2=0$ *i.e.* if

$$l_2\left(-\frac{a^2l_1}{n_1}\right)+m_2\left(-\frac{a^2m_1}{n_1}\right)+n_2=0$$

i.e. if $\qquad (l_1l_2+m_1m_2)\,a^2=n_1n_2.$

EXERCISES

1. Find the polar of the point $(3, -1)$ with respect to the circle

$$x^2+y^2=4.$$

2. Find the polar of the point $(1, -1)$ with respect to the circle

$$x^2+y^2+4x+8y-5=0.$$

3. Find the pole of the line $2x+3y-6=0$ with respect to the circle

$$x^2+y^2=5.$$

4. Find the pole of the line $3x-2y-5=0$ with respect to the circle

$$x^2+y^2+2x-4y-4=0.$$

5. Find the locus of a point whose polar with respect to the circle $x^2+y^2=a^2$ touches the circle

$$(x-a)^2+y^2=a^2.$$

6. Prove that the polar of a given point with respect to anyone of the circles $x^2+y^2-2kx+c^2=0$, where k is variable, always passes through a fixed point, whatever the value of k be.
7. Prove that the polar of the point (p, q) with respect to the circle $x^2+y^2=a^2$ touches $(x-c)^2+(y-d)^2=b^2$, if

$$b^2(p^2+q^2)=(a^2-cp-dq)^2.$$

8. Find the locus of a point P which is such that its polar with respect to one circle touches a second circle.

6. CHORD WITH GIVEN MIDDLE POINT

6.1 *To find the equation of the chord of the circle $x^2+y^2=a^2$ in terms of its middle point (x_1, y_1).*

First Method. The equations of a line through the point (x_1, y_1) are

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \tag{1}$$

θ being its inclination with the x-axis and r the algebraic distance of (x, y) from (x_1, y_1) measured along the line. Then any point on the line has the coordinates $(x_1+r\cos\theta, y_1+r\sin\theta)$ and this point will be common to the circle if

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2$$

i.e. $(\cos^2\theta+\sin^2\theta)\,r^2+2(x_1\cos\theta+y_1\sin\theta)\,r+x_1^2+y_1^2-a^2=0.$ (2)

This equation being quadratic in r, gives two values of r. Since (x_1, y_1) is the middle point of the chord, the two values of r given by equation (2) must be equal in magnitude and opposite in sign *i.e.* the sum of the roots of equation (2) must be zero.

$$\therefore \quad x_1\cos\theta+y_1\sin\theta=0. \tag{3}$$

This equation gives us the suitable value of θ so that the line (1) may be the chord of the circle having (x_1, y_1) as the middle point.

Therefore, the equation of the chord is

$$x_1\left(\frac{x-x_1}{r}\right)+y_1\left(\frac{y-y_1}{r}\right)=0$$

i.e. $$xx_1+yy_1=x_1^2+y_1^2.$$

Hence *the equation of the chord of the circle* $x^2+y^2=a^2$ *in terms of its middle point is*

$$xx_1+yy_1=x_1^2+y_1^2.$$

Using the notations of § 3.2, the equation of the chord can be written as

$$T=S_1.$$

Second Method. The equation of any line through the point $M(x_1, y_1)$ is

$$y-y_1=m(x-x_1),$$

where m is the slope of the line.

The point (x_1, y_1) being the middle point of the chord PQ, CM is perpendicular to PQ, where C is the centre of the circle.

$$\therefore \qquad m=-\frac{1}{\text{Slope of } CM}=-\frac{x_1}{y_1}.$$

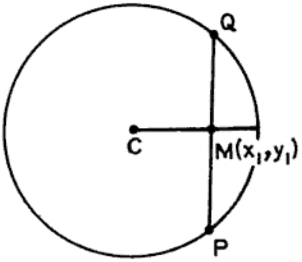


Fig. 5.11

Hence the equation of the chord of the circle $x^2+y^2=a^2$ in terms of its middle point (x_1, y_1) is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

i.e. $$xx_1+yy_1=x_1^2+y_1^2.$$

Note. The second method is applicable only for the circle whereas the first method will be applicable alike for other conics like parabola, ellipse and hyperbola also.

Ex. Find the equation of the chord of the circle

$$x^2+y^2+2gx+2fy+c=0$$

in terms of its middle point.

6.2 Example. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which pass through the fixed point (h, k).

Solution. Let (x', y') be the middle point of the chord of the given circle. Then its equation is

$$xx'+yy'=x'^2+y'^2.$$

But this passes through (h, k).

$$\therefore \qquad hx'+ky'=x'^2+y'^2.$$

Hence the locus of (x', y') is

$$x^2+y^2-hx-ky=0.$$

EXERCISES

1. Find the equation of the chord of the circle

   $$x^2+y^2-4x+3y-1=0$$

   whose middle point is $(-2, 1)$.
2. Find the middle point of the chord of the circle

   $$x^2+y^2=a^2$$

   lying along the line $lx+my+n=0$.
3. Find the locus of the middle points of chords of the circle

   $$x^2+y^2=a^2$$

   which subtend a right angle at the point $(c, 0)$.
4. Tangents are drawn to a circle from a point which always lies on a given line. Prove that the locus of the middle point of the chord of contact is another circle.
5. Find the locus of the middle points of the chords of a circle which are of constant length.

7. POLAR EQUATION OF A CIRCLE

To find the polar equation of a circle in terms of the given radius and the coordinates of the centre.

Let $C(r_1, \theta_1)$ be the cenrre and a the radius of the circle. Let $P(r, \theta)$ be any point on the circle.

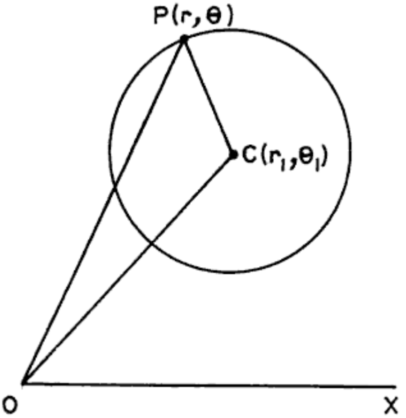


Fig. 5.12

Then from triangle OCP, we have

$$CP^2 = OC^2 + OP^2 - 2OC.OP \cos COP$$

But $CP = a$, $OC = r_1$, $OP = r$ and $\angle COP = \theta - \theta_1$.

$\therefore$ $a^2 = r_1^2 + r^2 - 2rr_1 \cos(\theta - \theta_1)$,

which is the polar equation of the circle.

Corollary 1. The equation of the circle passing through the pole is

$$r = 2a \cos(\theta - \theta_1).$$

Corollary 2. The equation of the circle passing through the pole and having the centre on the initial line is

$$r = 2a \cos\theta.$$

EXERCISES

1. Find the equation to the circle described on the line joining the points (r_1, θ_1) and (r_2, θ_2) as diameter.
2. Find the coordinates of the centre of the circle

$$r=A \cos \theta+B \sin \theta.$$

3. Find the polar equation of a circle, the initial line being a tangent.
4. Find the equation to the chord joining the points on the circle $r=2a \cos \theta$ whose vectorial angles are θ_1 and θ_2, and deduce the equation to the tangent at the point θ_1.
5. Find the condition that the line

$$\frac{1}{r}=A \cos \theta+B \sin \theta$$

may touch the circle $r=2a \cos \theta$.

6. Prove that for all values of α, the line $r \cos(\theta-\theta_1)=a+r_1 \cos \theta_1$ is a tangent to the circle

$$a^2=r_1^2+r^2-2rr_1 \cos \theta.$$

MISCELLANEOUS EXERCISES

1. Find the equation of the circle which passes through three points (0, 1), (1, 0) and (2, 1).
2. Find the centre and the radius of the circle

$$2(x^2+y^2)+6x-7y=0.$$

3. Show that the locus of a point such that the sum of the squares of its distances from two fixed points is constant, is a circle.
4. Show that the locus of a point such that the ratio of its distances from two given points is constant, is a circle.
5. Find the locus of a point which moves so that the square of the tangent drawn from it to the circle $x^2+y^2=a^2$ is equal to c times its distance from the line $lx+my+n=0$.
6. Find the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle.

7. Prove that the locus of a point which moves so that the chords of contact of the tangents from the point to two fixed circles are perpendicular, is a circle.
8. Prove that the locus of a point which moves so that the sum of the squres of its distances from the the three vertices of a triangle is constant, is a circle whose centre is at the centroid of the triangle.
9. Find the equation of the circle inscribed in the triangle the equations of whose sides are $x=1$, $2y=5$ and $3x-4y=5$.
10. Prove that the circle whose centre is (3, 5) and which touches the y-axis is $x^2+y^2-6x-10y+25=0$. Find the equation of the other tangent from the origin, and the coordinates of the point of contact.
11. Find the coordinates of the poles of the lines $3x-11y-13=0$, $8x+y-2=0$ and $3x+2y+1=0$ with respect to the circle $x^2+y^2-4x+3=0$, and show that they are collinear.
12. Find the length of the common chord of the circles whose equations are

$$(x-a)^2+y^2=a^2 \text{ and } x^2+(y-b)^2=b^2,$$

and prove that the equation to the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

13. Find the locus of a point which moves so that its polars with respect to two fixed circles are mutually orthogonal.
14. Find the locus of a point of intersection of the tangent to a given circle and the perpendicular let fall on this tangent from a fixed point on the circle.
15. Prove that if the length of the tangent from (h, k) to the circle $x^2+y^2=6$ be twice the length of the tangent from (h, k) to the circle $x^2+y^2+3x+3y=0$, then $h^2+k^2+4h+4k+2=0$.
16. A point moves so that the square of its distance from a fixed point varies as its perpendicular distance from a fixed line. Prove that it describes a circle.
17. A point moves so that the sum of the square of its distances from the four sides of a square is constant. Prove that the locus of the point is a circle.
18. Prove that the locus of the mid-points of the chords of the

circle $x^2+y^2+2gx+c=0$ which pass through the origin is the circle $x^2+y^2+gx=0$.

19. The line $y=mx+c$ cuts off a chord of length $2d$ from the circle $x^2+y^2=a^2$. Prove that $c^2=(a^2-d^2)(1+m^2)$.

20. Prove that the circle on the chord $x\cos\alpha+y\sin\alpha=p$ of the circle $x^2+y^2=a^2$ as diameter is

$$x^2+y^2-a^2-2p(x\cos\alpha+y\sin\alpha-p)=0.$$

21. Find the locus of a point the polars of which with respect to two given circles make a given angle with one another.

22. Find the locus of the foot of the perpendicular let fall from the origin upon any chord of the circle

$$x^2+y^2+2gx+2fy+c=0$$

which subtends a right angle at the origin.

Find also the locus of the middle points of these chords.

23. A tangent is drawn to the circle $(x-a)^2+y^2=b^2$ and a perpendicular tangent to the circle $(x+a)^2+y^2=c^2$. Find the locus of their point of intersection, and prove that the bisector of the angle between them always touches one or other of two fixed circles.

24. Prove that the distances of two points from the centre of a circle are proportional to the perpendicular drawn from one point on the polar of the other.

25. Find the locus of the poles of the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ with respect to the circles which touch the coordinate axes.

26. Prove that the locus of the middle points of a system of parallel chords of a circle is a line passing through the centre.

27. O is a fixed point and P any point on a given circle; OP is joined and on it a point Q is taken so that $OP.OQ=k^2$, a constant quantity. Prove that the locus of Q is a circle which becomes a line when O lies on the original circle.

28. The distances from the origin of the centres of three circles $x^2+y^2-2\lambda x=c^2$ (where c is a constant and λ a variable) are in geometrical progression. Prove that the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are also in geometrical progression.

29. From any point on one given circle tangents are drawn to

another given circle. Prove that the locus of the middle point of the chord of contact is a third circle.

30. Prove that the equation of a line meeting the circle $x^2+y^2=a^2$ in two points at equal distances d from a point (x_1, y_1) on the circumference is

$$xx_1+yy_1-a^2+\tfrac{1}{2}d^2=0.$$

Apply this to find the equation of the tangent at (x_1, y_1).

CHAPTER VI

SYSTEMS OF CIRCLES

1. INTERSECTION OF TWO CIRCLES

Let the equations of the two circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0. \tag{2}$$

Now consider the equation $S_1+\lambda S_2=0$, where λ is an arbitrary constant. The equation takes the form of

$$(1+\lambda)x^2+(1+\lambda)y^2+2(g_1+\lambda g_2)x+2(f_1+\lambda f_2)y+c_1+\lambda c_2=0. \tag{3}$$

The coordinates (x, y) which satisfy (1) and (2), also satisfy (3). Therefore, equation (3) represents, in general, a curve (circle or line) through the points of intersection of (1) and (2).

Case I. $\lambda=-1$. Equation (3) becomes a linear equation in x and y, and as such represents a line through the common points of the circles (1) and (2).

Note. There is only one line through the points of intersection of the two circles. As such there are two points of intersection, real or imaginary, of two circles. When the two points are real and distinct, the line $S_1-S_2=0$ is the common chord of the two circles whereas in case the two points are coincident, the line $S_1-S_2=0$ is the common tangent of the circles $S_1=0$ and $S_2=0$. Finally, when these circles do not meet in real points, still the equation $S_1-S_2=0$ represents a real line which of course passes through the imaginary common points of the circles $S_1=0$ and $S_2=0$. In this case this line may represent some other locus.

Case II $\lambda\neq-1$. Equation (3) represents circles through the common points of the circles $S_1=0$ and $S_2=0$. In this case it gives a family of such circles since for each value of $\lambda(\neq-1)$ there is a member of the family.

1.1 Angle of intersection of two circles.

Let there be any two circles intersecting at points P and Q. It is well known from the elementary geometry that the angles between the tangents to the two circles at point P is the same as at Q.

The **angle of intersection** *of any two circles is the common value of the angle between the two tangents at their point of intersection.*

Two circles are said to cut **orthogonally** *if their angle of intersection is a right angle.*

1.2 Condition for orthogonal intersection of two circles.

To find a necessary and sufficient condition for the circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \tag{2}$$

to cut orthogonally.

Let A and B be the centres of the two circles, and P the point of intersection. Suppose the two circles cut orthogonally. Then the tangents at P to the two circles are at right angles. But the tangent to a circle is always at right angles to its radius.

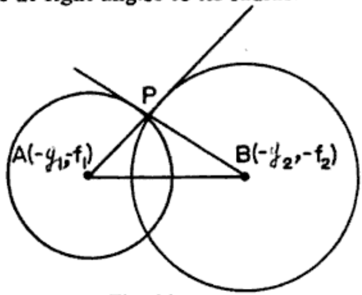


Fig. 6.1

Therefore, $\angle BPA$ is a right angle and so

$$AB^2=AP^2+PB^2.$$

But

$$AB^2=(g_1-g_2)^2+(f_1-f_2)^2$$

$$AP^2=g_1^2+f_1^2-c_1$$

and

$$PB^2=g_2^2+f_2^2-c_2.$$

$$\therefore \quad (g_1-g_2)^2+(f_1-f_2)^2=g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2$$

$$\Rightarrow \quad 2g_1g_2+2f_1f_2=c_1+c_2.$$

This is the required necessary condition for the circles (1) and (2) to cut orthogonally. This condition can also be proved to be sufficient by working the same steps backwards.

Ex. Prove that the circles $2x^2+2y^2-7x+5=0$ and $x^2+y^2-6x+4y+8=0$ cut orthogonally.

1.3 Exumples

Example 1. Find the equation of the circle which cuts orthogonally each of the circles

$$x^2+y^2+2x+17y+4=0,$$

$$x^2+y^2+7x+6y+11=0,$$

$$x^2+y^2-x+22y+3=0.$$

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0. \tag{1}$$

It will cut orthogonally the given circle if

$$2g+17f=c+4, \tag{2}$$

$$7g+6f=c+11, \tag{3}$$

$$-g+22f=c+3. \tag{4}$$

Subtracting (2) from (3) and (4) from (2), we get

$$5g-11f=7 \text{ and } 3g-5f=1,$$

which give $g=-3$ and $f=-2$.

Substituting these values of g and f in (2), we get $c=-44$.

Hence the required equation of the circle is

$$x^2+y^2-6x-4y-44=0.$$

Example 2. Find the equations of the circles which intersect the circles $x^2+y^2-6y+1=0$ and $x^2+y^2-4y+1=0$ orthogonally and touch the line $3x+4y+5=0$.

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0. \tag{1}$$

It will cut the given circles orthogonally if

$$-6f=c+1 \text{ and } -4f=c+1,$$

which give $f=0$ and $c=-1$. Therefore, the circle (1) becomes

$$x^2+y^2+2gx-1=0. \tag{2}$$

This circle will touch the line $3x+4y+5=0$ if the distance of the line from the centre of the circle=the radius of the circle, *i.e.*

$$\frac{-3g+5}{\sqrt{3^2+4^2}}=\sqrt{g^2-1}$$

$$\Rightarrow \qquad g=0 \text{ or } -\frac{15}{8}.$$

Substituting these values of g in (2), the required equations of the circles become

$$x^2+y^2=1;\ 4(x^2+y^2)-15x-4=0.$$

Example 3. Prove that the two circles which pass through the two points $(0, a)$ and $(0, -a)$ and touch the line $y=mx+c$ will cut orthogonally if $c^2=a^2(2+m^2)$.

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+k=0. \tag{1}$$

It will pass through $(0, a)$ and $(0, -a)$ if

$$a^2+2fa+k=0$$

and
$$a^2-2fa+k=0.$$

On solving these equations, we get

$$f=0 \text{ and } k=-a^2.$$

Therefore, the equation (1) becomes

$$x^2+y^2+2gx-a^2=0. \tag{2}$$

This circle will touch the line $y=mx+c$ if the distance of the line from the centre of the circle equal to the radius of the circle *i.e.*

$$\frac{-gm+c}{\sqrt{m^2+1}}=\sqrt{g^2+a^2}$$

$$\Rightarrow \qquad g^2+2gcm+a^2(1+m^2)-c^2=0.$$

Let g_1, g_2 be the roots of this equation. Then

$$g_1+g_2=-2cm \text{ and } g_1g_2=a^2(1+m^2)-c^2;$$

and the equations of the two circles are

$$x^2+y^2+2g_1x-a^2=0,$$

$$x^2+y^2+2g_2x-a^2=0.$$

These circles will cut orthogonally if

$$2g_1g_2=-2a^2$$

$$\Rightarrow \quad a^2(1+m^2)-c^2=-a^2$$

$$\Rightarrow \quad c^2=a^2(2+m^2).$$

EXERCISES

1. Prove that the circles

$$x^2+y^2-8x-2y+16=0$$

and $$3x^2+3y^2-14x+23y-15=0$$

cut orthogonally.

2. Find the equation of the circle which passes through the origin and cuts orthogonally each of the circles

$$x^2+y^2-6x+8=0 \text{ and } x^2+y^2-2x-2y-7=0.$$

3. Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles

$$x^2+y^2-8x-2y+16=0$$

and $$x^2+y^2-4x-4y-1=0.$$

4. Find the equation of the circle which passes through (3, 0), cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$ and touches y-axis.

5. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2-2x+3y-7=0,$$

$$x^2+y^2+5x-5y+9=0,$$

and $$x^2+y^2+7x-9y+29=0.$$

6. Find the equation of the circle which cuts orthogonally each of the three circles

$$x^2+y^2=a^2,$$

$$(x-c)^2+y^2=a^2,$$

$$x^2+(y-b)^2=a^2.$$

7. Find the equation to the circle passing through the points

(1, 0), (4, 0) and (0, 2). Prove that this circle cuts orthogonally the circle $x^2+y^2=4$.

8. Prove that the the two circles which pass through two points $(a, 0)$ and $(-a, 0)$ and touch the line $lx+my+n=0$ will cut orthogonally if $a^2(2l^2+m^2)=n^2$.
9. Find the locus of the centre of the circle which cuts two given circles orthogonally.
10. Prove that any circle which passes through the point $(b, 0)$ and cuts the circle $x^2+y^2=a^2$ orthogonally also passes through the point $\left(\frac{-a^2}{b}, 0\right)$.
11. Prove that any circle which passes through the point (α, β) and cuts the circle $x^2+y^2=c^2$ orthogonally also passes through the point $\left(\frac{c^2\alpha}{\alpha^2+\beta^2}, \frac{c^2\beta}{\alpha^2+\beta^2}\right)$.
12. A circle has its centre at the point (1, 2) and passes through the point (0, 3). Find its equation and also that of the circle whose centre is at the point (4, 3) and which cuts the first circle at right angles.
13. A and B are the centres of the circles $x^2+y^2+4x-2y+4=0$, $x^2+y^2-2x+6y+1=0$. Find the equation of the circle whose centre lies on AB and which cuts both the circles orthogonally.
14. If the equations of two circles with radii a, a' are $S=0$, $S'=0$; prove that the circles $\frac{S}{a}\pm\frac{S'}{a'}=0$ will intersect orthogonally.
15. AB is a diameter of a circle. Prove that the polar of A with respect to any circle which cuts the first circle orthogonally passes through B.

2. RADICAL AXIS

Radical axis *of two circles is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal.*

2.1 Equation of the radical axis. *To find the equation of the radical axis of two circles.*

Let the equations of the two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0. \tag{2}$$

Let $P(x', y')$ be the moving point on the locus from which the lengths of the tangents drawn to the circles (1) and (2) are equal *i.e.*

$$PQ = PR$$

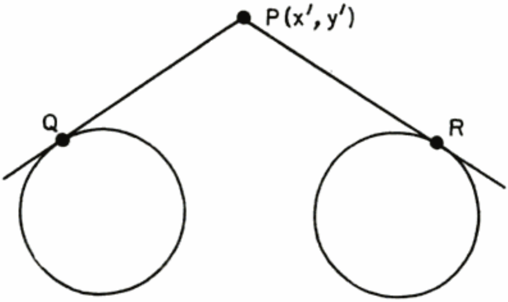


Fig. 6.2

But $\qquad PQ$=length of the tangent to the circle (1) from $P(x', y')$

$$=\sqrt{x'^2+y'^2+2g_1x'+2f_1y'+c_1},$$

and $\qquad PR$=length of the tangent to the circle (2) from $P(x', y')$

$$=\sqrt{x'^2+y'^2+2g_2x'+2f_2y'+c_2}.$$

$$\therefore \quad \sqrt{x'^2+y'^2+2g_1x'+2f_1y'+c_1}=\sqrt{x'^2+y'^2+2g_2x'+2f_2y'+c_2}$$

$$\Rightarrow \quad 2(g_1-g_2)x'+2(f_1-f_2)y'+c_1-c_2=0.$$

Hence the locus of (x', y') is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0.$$

This is the required equation of the radical axis of the circles (1) and (2).

Remark. The equation of the radical axis being linear in x and y, represents a line.

Working rule. In order to obtain the radical axis of two circles, reduce the equations of the circles to the standard forms in such a way that the coefficients of x^2 and y^2 are unity in the equations of the circles and then subtract one equation from the other. Thus the radical axis of the two circles

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0,$$
$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0$$

is

$$S_1-S_2=0.$$

Note 1. Radical axis is a line through the common points, real or imaginary, of the two circles (compare the equation of the radical axis with that of the line through the points of intersection of the two circles as obtained in § 1). As such, the concept of the radical axis seems to be wider than that of the common chord or the common tangent of the two circles. Fig. 6.3 (*a*) shows the case of two intersecting circles and the radical axis coincides with the common chord of the circles. Fig. 6.3 (*b*) and Fig. 6.3 (*c*) explain

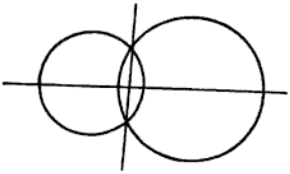


Fig. 6.3 (*a*)

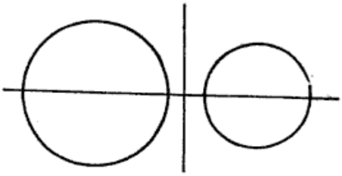


Fig. 6.3 (*b*)

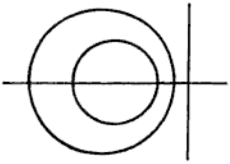


Fig. 6.3 (*c*)

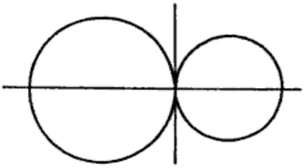


Fig. 6.3 (*d*)

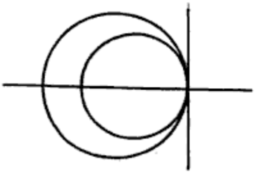


Fig. 6.3 (*e*)

the cases of non-intersecting circles whereas Fig. 6.3(*d*) and Fig. 6.3(*e*) point out the intermediate case of two touching circles when the radical axis coincides with the common tangent.

Note 2. In case of the circles having a common chord, Fig. 6.3 (*a*), the points which lie inside both the circles *i.e.* the points on the common chord do not strictly speaking belong to the locus (*i.e.* radical axis) since real tangents cannot be drawn from them to the circles. As such the lengths of the tangents from these points to the circles have no meaning. This type of anomaly may be avoided by introducing the concept of the power of a point (x_1, y_1) with respect to the circle $x^2+y^2+2gx+2fy+c=0$ to be the expression $x_1^2+y_1^2+2gx_1+2fy_1+c$ irrespective of the fact that this expression may be positive or negative, and then defining the radical axis as the locus of a point whose powers with respect to the two circles are equal.

2.2 Propositions on radical axis

Proposition I. *The radical axis of two circles is perpendicular to the line joining their centres.*

Let the equations of the two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0. \tag{2}$$

The centres of these circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$ respectively. Therefore, the slope of the line joining them is

$$m_1=\frac{f_1-f_2}{g_1-g_2}.$$

Radical axis of the circles (1) and (2) is

$$2(g_1-g_2)\,x+2(f_1-f_2)\,y+c_1-c_2=0.$$

Its slope is

$$m_2=-\frac{g_1-g_2}{f_1-f_2}.$$

Now

$$m_1m_2=\left(\frac{f_1-f_2}{g_1-g_2}\right)\left(-\frac{g_1-g_2}{f_1-f_2}\right)=-1.$$

Hence the proposition follows.

Proposition II. *The radical axes of three circles taken in pairs meet in a point.*

Let the equations of the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0$$

and $$x^2+y^2+2g_3x+2f_3y+c_3=0.$$

The equations of the radical axes of these circles taken in pairs are

$$2(g_1-g_2)\,x+2\,(f_1-f_2)\,y+c_1-c_2=0 \tag{1}$$

$$2(g_2-g_3)\,x+2\,(f_2-f_3)\,y+c_2-c_3=0 \tag{2}$$

and $$2(g_3-g_1)\,x+2\,(f_3-f_1)\,y+c_3-c_1=0. \tag{3}$$

On adding (1) and (2), we get

$$2(g_1-g_3)\,x+2\,(f_1-f_3)\,y+c_1-c_3=0.$$

This is the same as equation (3) which shows that the coordinates of the point of intersection of the lines (1) and (2) also satisfy equation (3). As such, all the three lines meet in a point.

The point of concurrence of the three radical axes of three circles taken in pairs is called the **radical centre** *of the three circles.*

Note. To find the radical centre of any given three circles, it is enough to find the point of intersection of any two of the three radical axes.

Proposition III. *The difference of the squares of the lengths of the tangents to two circles from any point in their plane varies as the distance of the point from their radical axis.*

Let equations of the circle be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0. \tag{2}$$

Let $P(x_1, y_1)$ be any point in the plane of the circles (1) and (2). The equation of the radical axis of these circles is

$$2(g_1-g_2)\,x+2\,(f_1-f_2)\,y+c_1-c_2=0.$$

The perpendicular distance from the point $P(x_1, y_1)$ to the radical axis is given by

$$d=\frac{2\,(g_1-g_2)\,x_1+2\,(f_1-f_2)\,y_1+c_1-c_2}{\sqrt{4\,(g_1-g_2)^2+4\,(f_1-f_2)^2}}.$$

Now the difference of the squares of the lengths of the tangents drawn from the point $P(x_1, y_1)$ to the circles (1) and (2) is

$$\begin{aligned}(x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1)-(x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2)\\ =2(g_1-g_2)\,x_1+2(f_1-f_2)\,y_1+c_1-c_2\\ =d\sqrt{4(g_1-g_2)^2+4(f_1-f_2)^2},\end{aligned}$$

which is proportional to d.

Proposition IV. *If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.*

Let the equations of the two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \tag{1}$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0. \tag{2}$$

The radical axis of the circles (1) and (2) is

$$2(g_1-g_2)\,x+2(f_1-f_2)\,y+c_1-c_2=0. \tag{3}$$

Let the equation of the third circle be

$$x^2+y^2+2gx+2fy+c=0. \tag{4}$$

Since the circles (1) and (2) cut the circle (4) orthogonally, we have

$$2gg_1+2ff_1=c+c_1 \tag{5}$$

and

$$2gg_2+2ff_2=c+c_2. \tag{6}$$

Subtracting (6) from (5), we get

$$2g(g_1-g_2)+2f(f_1-f_2)=c_1-c_2. \tag{7}$$

The radical axis given by (3) will pass through the centre $(-g, -f)$ of circle (4) if

$$-2(g_1-g_2)\,g-2(f_1-f_2)\,f+c_1-c_2=0,$$

which is true by virtue of (7).

Hence the proposition follows.

2.3 Examples

Example 1. Find the radical centre of the circles

$$x^2+y^2+\ x+2y+3=0, \tag{1}$$

$$x^2+y^2+2x+4y+5=0, \tag{2}$$

$$x^2+y^2-7x-8y-9=0. \tag{3}$$

Solution. The radical axis of the circles (1) and (2) is

$$x+2y+2=0, \tag{4}$$

and the radical axis of the circles (1) and (3) is

$$4x+5y+6=0. \tag{5}$$

On solving (4) and (5), we get $x=y=-\frac{2}{3}$. Hence the required radical centre is $(-\frac{2}{3}, -\frac{2}{3})$.

Example 2. Find the equation of the circle whose diameter is the common chord of the circles

$$x^2+y^2+2x+3y+1=0$$

and $$x^2+y^2+4x+3y+2=0.$$

Solution. The equation of the common chord (radical axis) of the given circles is

$$2x+1=0.$$

Now any circle through the extremities of this chord is

$$x^2+y^2+2x+3y+1+\lambda(2x+1)=0$$

i.e. $$x^2+y^2+2(1+\lambda)x+3y+1+\lambda=0. \tag{1}$$

The common chord $2x+1=0$ will be the diameter of circle (1) if its centre $(-1-\lambda, -\frac{3}{2})$ lie on the chord.

i.e. $$2(-1-\lambda)+1=0$$

$$\Rightarrow \quad \lambda=-\tfrac{1}{2}.$$

Substituting the value of λ in (1), the required equation of the circle becomes

$$x^2+y^2+x+3y+\tfrac{1}{2}=0.$$

Example 3. If the circles

$$x^2+y^2+2gx+2fy=0$$

and $$x^2+y^2+2g'x+2f'y=0$$

touch each other, prove that $f'g=fg'$.

Solution. The radical axis of the given circles is

$$(g-g')x+(f-f')y=0.$$

The two circles will touch if the radical axis become the tangent to any of the circles, *i.e.*

$$\frac{(g-g')(-g)+(f-f')(-f)}{\sqrt{(g-g')^2+(f-f')^2}}=\sqrt{g^2+f^2}$$

$$\Rightarrow \qquad f'g=fg',$$

Aliter. Since the given circles pass through the origin, in case the circles touch each other, the origin will be the point of contact and the tangents at the origin will coincide. But the tangents at the origin are given by

$$gx+fy=0 \text{ and } g'x+f'y=0.$$

Hence, we get

$$\frac{g}{g'}=\frac{f}{f'}$$

$$\Rightarrow \qquad f'g=fg'.$$

Example 4. Prove that if the four points of intersection of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+a'x+b'y+c'=0$ by the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, respectively, are concyclic, then

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0.$$

Solution. Let the four points of intersection lie on the circle

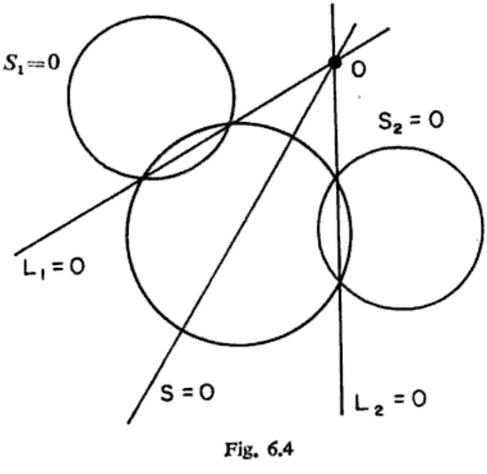


Fig. 6.4

$$S = x^2+y^2+2gx+2fy+d=0,$$

and let the given circles and the lines be named

$$S_1=0,\ S_2=0,\ L_1=0,\ L_2=0, \text{ respectively.}$$

The radical axis of $S_1=0$ and $S=0$ is

$$L_1 \equiv Ax+By+C=0. \tag{1}$$

The radical axis of $S_2=0$ and $S=0$ is

$$L_2 \equiv A'x+B'y+C'=0. \tag{2}$$

The radical axis of $S_1=0$ and $S_2=0$ is

$$S_1-S_2 \equiv (a-a')x+(b-b')y+c-c'=0. \tag{3}$$

But the lines (1), (2) and (3) are concurrent.

$$\therefore \qquad \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Hence the result.

EXERCISES

1. Find the radical axis of each of the following pairs of circles:

   (*i*) $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$.

   (*ii*) $x^2+y^2-3x-4y+5=0$ and $3x^2+3y^2-7x+8y+11=0$.

2. Prove that the three circles

$$x^2+y^2+3x+6y+12=0,$$
$$x^2+y^2+2x+8y+16=0,$$
and $$x^2+y^2+12y+24=0,$$

   have a common radical axis.

3. Find the radical centre of the three circles in each of the following:

   (*i*) $$x^2+y^2=9,$$
   $$x^2+y^2-2x-2y-5=0,$$

and $x^2+y^2+4x+6y-19=0.$

(*ii*) $x^2+y^2-2x+6y=0,$

$x^2+y^2-4x-2y+6=0,$

and $x^2+y^2-12x+2y+30=0.$

(*iii*) $x^2+y^2+4x+7=0,$

$x^2+y^2+y=0,$

and $2x^2+2y^2+3x+5y+9=0.$

4. Prove that the locus of the points such that the difference of the squares of the lengths of the tangents from them to two given circles is constant, is a line parallel to their radical axis.

5. Find the radical axis of the circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0.$$

Prove that the first of these circles will bisect the circumference of the other if

$$2g_2(g_1-g_2)+2f_2(f_1-f_2)=c_1-c_2.$$

6. Prove that the circles

$$x^2+y^2+2ax+c=0 \text{ and } x^2+y^2+2by+c=0$$

touch if $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c}$.

3. COAXAL CIRCLES

Let $S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$

and $S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0$

be the equations of any two circles.

Now consider the equation

$$S_1+\lambda S_2=0, \qquad (1)$$

where λ is a constant.

The value $\lambda=-1$ corresponds to the radical axis $S_1-S_2=0$ of the two circles $S_1=0$ and $S_2=0$ whereas any other value of λ yields a circle. Thus equation (1) represents a system of circles, except when $\lambda=-1$. Consider now two distinct values λ_1 and λ_2 of λ such that none of them is -1. Then

$$S_1+\lambda_1 S_2=0 \quad \text{and} \quad S_1+\lambda_2 S_2=0$$

are the equations of the two circles of the family (1), and their radical axis is given by

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0$$

$$\Rightarrow \qquad (\lambda_2-\lambda_1)(S_1-S_2)=0$$

$$\Rightarrow \qquad S_1-S_2=0 \qquad (\because \ \lambda_1\neq\lambda_2).$$

Thus, we note that any two distinct circles of the family (1) have the radical axis $S_1-S_2=0$ which is independent of λ.

A system of circles is said to be **coaxal** *if every pair of circles of the system has the same radical axis.*

Thus $S_1+\lambda S_2=0$ represents a coaxal system of circles.

To prove that the centres of all circles of a coaxal system are collinear and lie on a line perpendicular to the common radical axis.

We note that the centre of the coaxal system $S_1+\lambda S_2=0$ is

$$\left(-\frac{g_1+\lambda g_2}{1+\lambda}, -\frac{f_1+\lambda f_2}{1+\lambda}\right).$$

This is a point which lies on the line joining the centres of the circles $S_1=0$ and $S_2=0$. Thus, the centres of all the circles in the system are collinear. It is trivial to see that the line joining the centres $(-g_1, -f_1)$ and $(-g_2, -f_2)$ is perpendicular to the common radical axis $S_1-S_2=0$.

3.1 Standard form of the equation to a coaxal system.

To find the equation of a coaxal system of circles in its standard form.

Let us choose the line of centres as the x-axis and the common radical axis (perpendicular to the line of centres) as the y-axis.

Since the centre of each circle lies on the x-axis, the y-coordinate of it is zero. As such any two circles from the system may be represented by the equations

$$\left.\begin{aligned} x^2+y^2+2gx+c=0 \\ x^2+y^2+2g'x+c'=0 \end{aligned}\right\}. \tag{1}$$

The radical axis of these circles is

$$2(g-g')\,x+c-c'=0.$$

But the radical axis is $x=0$

$$\therefore \qquad c'=c.$$

Hence the equation of any circle in the system takes the form

$$x^2+y^2+2gx+c=0,$$

where the constant term c is fixed for the whole system, and the coefficient g is a parameter which varies from one circle to another in the family.

3.2 Intersection of circles of a coaxal system

Let
$$x^2+y^2+2gx+c=0$$

be the equation of a coaxal system of circles with centres on x-axis and common radical axis as y-axis. The points of intersection of the system with its radical axis are given by (taking $x=0$)

$$y^2+c=0$$

$$\Rightarrow \qquad y=\pm\sqrt{-c}.$$

Thus, $(0, \sqrt{-c})$ and $(0, -\sqrt{-c})$ are the coordinates of the points where the radical axis meets a circle in the system. Since these coordinates are independent of g (the parameter of the system), every circle of the system meets the radical axis in the same two points.

The points $(0, \sqrt{-c})$ *and* $(0, -\sqrt{-c})$ *are called the* **common points** *of the co-axal system.*

Now, there are three cases:

Case I. When c is negative, the common points are real and different [see Fig. 6.5 (*a*)]. In this case the system is said to be of **intersecting species.**

Case II. When $c=0$, the common points are coincident [see Fig. 6.5 (*b*)] and hence every circle of the system touches y-axis (the radical axis) at the origin. In other words any two circles of the system touch at the origin.

Case III. When c is positive, the common points are imaginary [see Fig. 6.5 (*c*)] and the system is said to be of **non-intersecting species.**

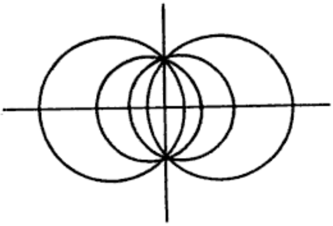


Fig. 6.5 (*a*)

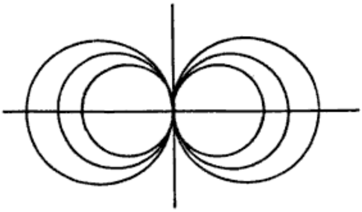


Fig. 6.5 (*b*)

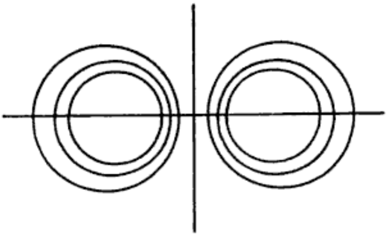


Fig. 6.5 (*c*)

3.3 Limiting points of a coaxal system.

The equation

$$x^2+y^2+2gx+c=0$$

gives any circle of a coaxal system which can be written as

$$(x+g)^2+y^2=(\sqrt{g^2-c})^2.$$

This equation represents a circle whose centre is the point $(-g, 0)$ and radius $=\sqrt{g^2-c}$. If the radius of the circle is zero *i.e.*

$g^2-c=0$, the circle becomes a point circle. But in this case

$$g=\pm\sqrt{c}.$$

Thus, at the particular points $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$, we have point circles which are in the system.

The points $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$ *are called the* **limiting points** *of the coaxal system* $x^2+y^2+2gx+c=0$.

We note that the limiting points are real or imaginary according as c is positive or negative.

Note 1. The limiting points of a coaxal system are real or imaginary according as the common points are imaginary or real. Consequently, if the limiting points are real, the system is in the non-intersecting species whereas if the limiting points are imaginary, the system is in the intersecting species. Hence, we see that, except in the second case of § 3.2 which is a compromise between the two, there are either limiting points or common points but not both simultaneously.

Note 2. The circle coaxal with two given circles $S_1=0$ and $S_2=0$ can be written as $S_1+\lambda S_2=0$ or $S_1+\lambda L=0$, where $L=0$ is the equation of the radical axis of the two given circles.

3.4 Conjugate system of coaxal circles. A system of circles through the limiting points, real or imaginary, of a coaxal system is of particular interest.

To find the system of circles through the limiting points $(\pm\sqrt{c}, 0)$ *of the coaxal system* $x^2+y^2+2gx+c=0$.

Let the circle given by the equation

$$x^2+y^2+2g'x+2f'y+c'=0$$

pass through the limiting points $(\pm\sqrt{c}, 0)$. Then

$$\left.\begin{aligned} c+2g'\sqrt{c}+c'=0 \\ c-2g'\sqrt{c}+c'=0 \end{aligned}\right\}$$

$$\Rightarrow \qquad c'=-c \text{ and } g'=0.$$

Hence the family of circles through the limiting points $(\pm\sqrt{c}, 0)$ is given by the equation

$$x^2+y^2+2fy-c=0,$$

where f (written for f') is the parameter of the famliy.

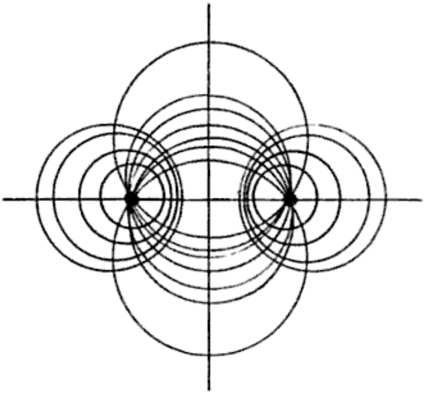


Fig. 6.6

Remark. The system of circles $x^2+y^2+2fy-c=0$ represents another coaxal system of circles having y-axis as the line of centres and x-axis as the common radical axis. The limiting points of the system are $(0, \pm\sqrt{-c})$ and the common points are $(\pm\sqrt{c}, 0)$.

We note several things relating the two coaxal systems

$$x^2+y^2+2gx+c=0$$

and

$$x^2+y^2+2fy-c=0$$

which are as follows:

(*i*) The line of centres of either system is the common radical axis of the other system.

(*ii*) The limiting points of the either system are the common points of the other.

(*iii*) If either system is of the intersecting species, the other must be of the non-intersecting species.

(*iv*) Every member of one system cuts orthogonally every member of the other.

To prove that any circle which cuts any two given circles of the coaxal system

$$x^2+y^2+2gx+c=0$$

orthogonally is a circle of the system

$$x^2+y^2+2fy-c=0.$$

Let the given two circles of the coaxal system be represented by the equations

$$x^2+y^2+2g_1x+c=0 \tag{1}$$

and

$$x^2+y^2+2g_2x+c=0. \tag{2}$$

Let

$$x^2+y^2+2g'x+2f'y+c'=0 \tag{3}$$

be a circle which cuts the circles (1) and (2) orthogonally. Then, we have

$$\begin{cases}2g_1g'-c-c'=0\\2g_2g'-c-c'=0\end{cases}$$

$$\Rightarrow \qquad g'=0,\ c'=-c.$$

Thus circle (3) must be of the form

$$x^2+y^2+2f'y-c=0,$$

which is a member of the family

$$x^2+y^2+2fy-c=0.$$

Hence, any circle orthogonal to the coaxal system

$$x^2+y^2+2gx+c=0,$$

is in the family $x^2+y^2+2fy-c=0$; and conversely.

Two systems of circles such that any circle of one system cuts any circle of the other system orthogonally are said to be **conjugate systems** *(or* **orthogonal systems***) of circles.*

Note. Through each point of the plane there is just one member of each system of the conjugate systems.

Remark. The orthogonal property of two systems of curves appears in everyday life, for example, in an electric or magnetic field, lines of force and equipotentials form orthogonal systems.

3.5 Examples

Example 1. Find the radical axis of the circles

$$x^2+y^2+4x-3=0 \quad \text{and} \quad x^2+y^2+6x-8y+7=0.$$

Find the equation to the circle coaxal with these two circles and passing through the point (1, 1).

Solution. The radical axis of the given circles is

$$(x^2+y^2+4x-3)-(x^2+y^2+6x-8y+7)=0$$

i.e. $$x-4y+5=0.$$

The equation to the circle coaxal with the given circles is

$$x^2+y^2+4x-3+\lambda(x-4y+5)=0.$$

But this circle passes through (1, 1).

$$\therefore \qquad \lambda=-\frac{3}{2}.$$

Hence the required equation of the circle is

$$x^2+y^2+4x-3-\tfrac{3}{2}(x-4y+5)=0$$

i.e. $$2x^2+2y^2+5x+12y-21=0.$$

Example 2. Find the limiting points of the coaxal system of circles determined by

$$x^2+y^2-6x-4y+3=0 \quad \text{and} \quad x^2+y^2+10x+4y-1=0.$$

Solution. The equation to the coaxal system containing the given circles is

$$x^2+y^2-6x-4y+3+\lambda(x^2+y^2+10x+4y-1)=0$$

$$\Rightarrow \quad x^2+y^2-\frac{6-10\lambda}{1+\lambda}x-\frac{4-4\lambda}{1+\lambda}y+\frac{3-\lambda}{1+\lambda}=0. \qquad (1)$$

Centre of (1) is $\left(\frac{3-5\lambda}{1+\lambda}, \frac{2-2\lambda}{1+\lambda}\right)$ and the radius of (1) is

$$\sqrt{\left(\frac{3-5\lambda}{1+\lambda}\right)^2+\left(\frac{2-2\lambda}{1+\lambda}\right)^2-\frac{3-\lambda}{1+\lambda}}.$$

For limiting points of the coaxal system (1), radius of (1) should be zero *i.e.*

$$\left(\frac{3-5\lambda}{1+\lambda}\right)^2+\left(\frac{2-2\lambda}{1+\lambda}\right)^2-\frac{3-\lambda}{1+\lambda}=0$$

$$\Rightarrow \qquad 3\lambda^2-4\lambda+1=0$$

$$\Rightarrow \qquad \lambda=\tfrac{1}{3}, 1.$$

Putting these values of λ in the coordinates of the centre of (1), we get

(1, 1) and (−1, 0),

which are the required limiting points.

Example 3. Find the equation to the circle which passes through the origin and belongs to the coaxal system of which the limiting points are (1, 2) and (4, 3).

Solution. The equations to the point circles are

$$\left.\begin{aligned}(x-1)^2+(y-2)^2=0\\(x-4)^2+(y-3)^2=0\end{aligned}\right\}. \qquad (1)$$

The equation to the circle coaxal with these circles is

$$(x-1)^2+(y-2)^2+\lambda\{(x-4)^2+(y-3)^2\}=0. \qquad (2)$$

But this circle passes through the origin.

$$1+4+\lambda(16+9)=0$$

$$\Rightarrow \qquad \lambda=-\tfrac{1}{5}.$$

Putting this value of λ in equation (2), we get

$$(x-1)^2+(y-2)^2-\tfrac{1}{5}\{(x-4)^2+(y-3)^2\}=0$$

$$\Rightarrow \qquad 2x^2+2y^2-x-7y=0.$$

This is the required equation of the circle.

EXERCISES

1. Find the coordinates of the limiting points of each of the pairs of circles:
   (*i*) $x^2+y^2+2x+5=0$ and $x^2+y^2+2y+5=0$.
   (*ii*) $x^2+y^2+2x+4y+7=0$ and $x^2+y^2+4x+2y+5=0$.
   (*iii*) $x^2+y^2+2x-6y=0$ and $2x^2+2y^2-10y+5=0$.
   (*iv*) $x^2+y^2-6x+12y+5=0$ and $3x^2+3y^2+10x-20y+15=0$.
2. The point (2, 1) is a limiting point of a coaxal system of the circles of which $x^2+y^2-6x-4y+3=0$ is a member. Find the equation of the radical axis and the coordinates of the other limiting point.
3. A system of coaxal circles is defined by one of the limiting points $(-1, 2)$ and the circle $x^2+y^2+18x+4y-35=0$. Find the coordinates of the second limiting point and also the equation of the other circle of the system which has the same radius as the given circle.

4. Find the general equation of the system of circles any pair of which have the same radical axis as the circles

$$x^2+y^2+x-5y-3=0 \text{ and } x^2+y^2+3x+4y+6=0.$$

Prove that the equation of the member of the system which passes through the origin is

$$3x^2+3y^2+5x-6y=0.$$

5. Find the coordinates of the limiting points of the system of circles coaxal with the circles $x^2+y^2-6x-6y+4=0$ and $x^2+y^2-2x-4y+3=0$. Find also the equations of the circles of this coaxal system which touch the line

$$x+y-5=0.$$

6. Prove that the limiting points of the coaxal system of circles

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$$

are real if $(c-c')^2>4(f^2g^2-f^2c-g^2c')$.

7. Find the equation to the circle which is coaxal with the circles

$$x^2+y^2+3x-4y+5=0$$

and $$x^2+y^2-5x+2y-1=0$$

and which passes through the point (3, 1). Find also the radical axis of the system.

8. Find the equation to the circle which belongs to the coaxal system of which the limiting points are (1, −2), (2, 3) and which passes through the origin.

9. Find the equation of the circle which belongs to the coaxal system of which the limiting points are (1, −1), (2, 0) and which passes through the origin.

10. Prove that the polar of one limiting point of a coaxal system with respect to any circle of the system passes through the other limiting point.

11. The circle $x^2+y^2+4x-6y+3=0$ is one of the circles of a coaxal system having as radical axis the line $2x-4y+1=0$. Find the circle of the system which touches the line

$$x+3y-2=0.$$

12. Prove that the locus of points, the lengths of the tangents drawn from them are in a given ratio is a circle coaxal with the given circles.

13. Find the limiting points of the coaxal system given by

$$x^2+y^2+2\lambda(x+y-4)-6=0.$$

Find the equations of the two circles through these points which have radius 3.

14. Prove that the polar of a limiting point of a coaxal system with respect to any circle of the system is the same for all circles of the system.

15. The equation to a circle of a given coaxal system is

$$x^2+y^2+2gx+2fy+c=0$$

and the origin is a limiting point of this system. Prove that the equation to the orthogonal system is

$$(x^2+y^2)(g+\mu f)+c(x+\mu y)=0.$$

MISCELLANEOUS EXERCISES

1. Show that the condition for the two circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0$$

should touch is

$$(2g_1g_2+2f_1f_2-c_1-c_2)^2=4(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2).$$

2. Find the equation to the circle through the points of intersection of the circles $x^2+y^2=1$ and $x^2+y^2+2x+4y+1=0$, which touches the line $x+2y+5=0$.

3. Find the radical axis of the circles $x^2+y^2+6x+2y+1=0$ and $x^2+y^2-6x-2y+1=0$. Also, find the equation to the circle coaxal with these two circles and passing through the point (1, 1).

4. The circle $x^2+y^2+2x+4y+1=0$ belongs to a coaxal system and a limiting point is $(0, -2)$. Find the equation of the radical axis and coordinates of the other limiting point.

5. Prove that the circles $x^2+y^2-4y-5=0$

and $$x^2+y^2-6x-12y+29=0$$

intersect each other at right angles. Find the equation of their common chord and that of the tangent to the first circle at each point of intersection.

6. Prove that the circles $x^2+y^2+2x-8y+8=0$

and $x^2+y^2+10x-2y+22=0$

touch each other. Find the coordinates of the point of contact P and the equation of the common tangent at that point. Find the equation to the circle which touches each of these circles at P and passes through the point (3, 7).

7. If two circles cut orthogonally, prove that the polar of any point P on the first circle with respect to the second passes through the other end of the diameter of the first circle which goes through P.

Hence prove that the equation of the circle which cuts the three circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0$$
$$x^2+y^2+2g_3x+2f_3y+c_3=0$$

orthogonally is

$$\begin{vmatrix} x+g_1 & y+f_1 & g_1x+f_1y+c_1 \\ x+g_2 & y+f_2 & g_2x+f_2y+c_2 \\ x+g_3 & y+f_3 & g_3x+f_3y+c_3 \end{vmatrix}=0.$$

8. Prove that the common tangent to two circles of a coaxal system subtends a right angle at either limiting point of the system.

9. Prove that the equation to the circle cutting orthogonally the circles

$$(x-a)^2+(y-b)^2=b^2,$$
$$(x-b)^2+(y-a)^2=a^2,$$

and $$(x-a-b-c)^2+y^2=ab+c^2,$$

is $x^2+y^2-2(a+b)\,x-(a+b)\,y+a^2+3ab+b^2=0.$

10. Prove that the polar lines of a fixed point P with respect to the circles of a given coaxal system pass through a fixed point Q.

11. Find the limiting points of the system of circles

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$$

and show that the square of the distance between them is

$$\frac{(c-c')^2-4f^2g^2+4f^2c+4g^2c'}{f^2+g^2}.$$

12. Prove that if a circle cuts two of a coaxal system of circles at right angles, it will cut them all at right angles.

13. Prove that the equations of two given circles can always be put in the form

$$x^2+y^2+ax+b=0, \quad x^2+y^2+a'x+b=0$$

and that one of the circles will be within the other if aa' and b are both positive.

14. If A, B, C be the centres of three coaxal circles and t_1, t_2, t_3 be the tangents to them from any point, show that

$$BC\cdot t_1^2+CA\cdot t_2^2+AB\cdot t_3^2=0.$$

15. Prove that the locus of the centre of a circle which cuts three given circles at the same angle is a line.

16. If polars of a point P with respect to two given circles meet in Q, prove that the radical axis of the circles bisects PQ.

17. Prove that as λ varies the circles

$$x^2+y^2+2ax+2by+c+\lambda(ax-by+1)=0$$

form a coaxal system, and find the equations of the radical axis and the line of centres. Find the equations of the circles which are orthogonal to all the circles of the given system.

18. Find the equations of the two circles of radius $\frac{1}{2}$ each of which is orthogonal to both of the circles

$$x^2+y^2+2x=0, \; x^2+y^2-2x-2y=0.$$

19. A coaxal system is defined by the circles

$$x^2+y^2+2ax+2by+c=0 \quad (a^2+b^2>c),$$

and $$(a^2+b^2)(x^2+y^2)+2acx+2bcy+c^2=0.$$

Prove that the origin is one of the limiting points and find the equation of the orthogonal coaxal system of circles.

20. The polars of a point P with respect to two fixed circles meet in the point Q. Prove that the circle on PQ as diameter passes through two fixed points and cuts both the given circles at right angles.

21. Prove that the tangents drawn from any point of a fixed circle of a coaxal system to two other fixed circles of the system are in a constant ratio.

22. Prove that the circles with respect to which the given line

$$lx+my+n=0$$

is polar of the origin form a coaxal system with radical axis

$$2lx+2my+n=0.$$

Find the coordinates of the limiting points of this system and the equation of the system of circles which cuts the system orthogonally.

23. Prove that the limiting points of a system of coaxal circles are inverse points with respect to every circle of the system.

24. Prove that the limiting points of the system

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$$

subtend a right angle at the origin if

$$\frac{c}{g^2}+\frac{k}{f^2}=2.$$

CHAPTER VII

THE PARABOLA

1. INTRODUCTION

The parabola, ellipse and hyperbola are cases of curves called conic sections, or simply conics. The name is derived from the fact that they may be obtained as sections made by a plane with a double right circular cone, and they were first studied in this way. In this book, however, we shall discuss them as loci by means of equations.

In the present chapter we shall study the conic section—parabola. This curve may already have been met in other branches of mathematics. A projectile, for example, a ball or bullet, travels in a path which is approximately a parabola. The paths of some comets are nearly parabolic. Cables of some suspension bridges hang in the form of a parabola. The surface generated by revolving a parabola about its axis is called a paraboloid of revolution. A reflecting surface in this form has the property that light emanating at the focus is reflected in the direction of the axis. This kind of surface is used in headlights, in some telescopes and in devices to reflect sound waves. A comparatively recent application of parabolic metal surfaces is found in radar and other microwave equipments. The surfaces reflect radio waves in the same way that light is reflected and are used in directing outgoing beams and also in receiving incoming beams.

2. PARABOLA

The locus of a point which moves such that it is equidistant from a fixed point and a fixed line is called a **parabola.**

The fixed point is called the **focus** and the fixed line the **directrix.**

Ex. Find the equation of the parabola whose focus is $(-1, 1)$ and directrix is $x+3y+3=0$.

The line passing through the focus and perpendicular to the directrix is called the **axis** of the parabola.

2.1 Standard equation of a parabola. *To find the equation of a parabola in its standard form.*

Let S be the focus and ZK the directrix. Draw SZ perpendicular from the focus S to the directrix ZK.

Take A as the middle point of ZS. Then, by definition, ZS is the axis and the point A lies on the parabola. The point A is called the vertex of the parabola.

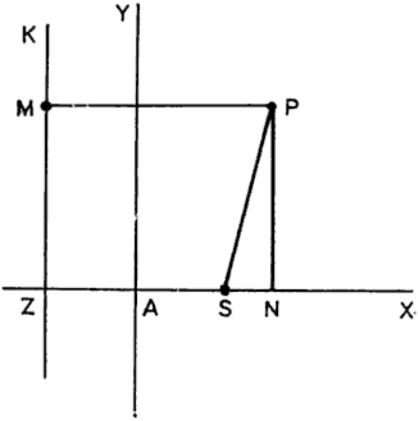


Fig. 7.1

Take the point A as origin of coordinates, the x-axis along the line AS and y-axis along the perpendicular to AS at A, as shown in Fig. 7.1.

Let $AS=a$. Then the coordinates of the focus S are $(a, 0)$ and the equation of the directrix is $x=-a$. Let $P(h, k)$ be a moving point on the locus (parabola). Then, by definition of the parabola

$$SP=MP.$$

But $$SP=\sqrt{(h-a)^2+k^2}$$

and $$MP=ZN=ZA+AN=a+h.$$

$\therefore$ $$(h-a)^2+k^2=(a+h)^2$$

$\Rightarrow$ $$k^2=4ah.$$

Hence the locus of (h, k) is

$$y^2=4ax.$$

2.2 General equation of a parabola. *To find the equation of the parabola whose focus is* (α, β) *and equation of the directrix is* $ax+by+c=0$.

Let $P(h, k)$ be a moving point on the locus (parabola). Then, by definition

$$SP=MP.$$

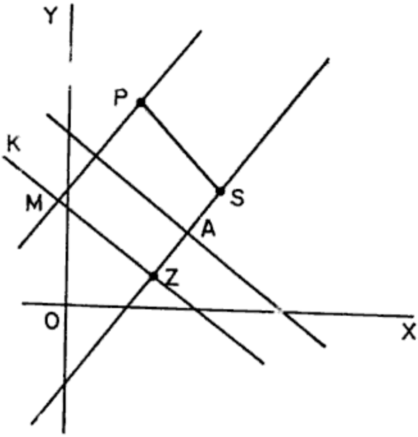


Fig. 7.2

But $$SP=\sqrt{(h-\alpha)^2+(k-\beta)^2}$$

and MP=perpendicular distance of $ax+by+c=0$ from (h, k)

$$=\frac{ah+bk+c}{\sqrt{a^2+b^2}}.$$

$$\therefore \quad (h-\alpha)^2+(k-\beta)^2=\frac{(ah+bk+c)^2}{a^2+b^2}.$$

Hence the locus of (h, k) is

$$(x-\alpha)^2+(y-\beta)^2=\frac{(ax+by+c)^2}{a^2+b^2}.$$

This is the required equation of the parabola

2.3 Tracing of a parabola. *To trace the parabola $y^2=4ax$, where a is positive.*

We have the following observations regarding this curve:

(*i*) No part of the curve lies on the left of y-axis since when x is negative, the corresponding values of y are imaginary.

(*ii*) The curve is symmetrical about x-axis since for each positive value of x, there are two equal and opposite values of y.

(*iii*) x-axis meets the curve only at the point A.

(*iv*) y increases as x increases and y tends to infinity as x tends to infinity. Hence the curve extends upto infinity on the right of y-axis.

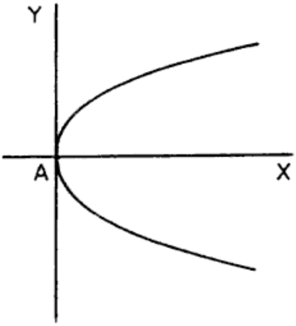


Fig. 7.3

2.4 Standard forms of parabola. (*i*) The equation of the parabola with vertex at the origin and focus at $(a, 0)$ is

$$y^2=4ax.$$

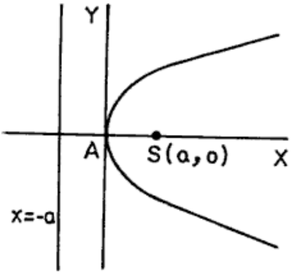


Fig. 7.4 (*a*)

(*ii*) The equation of the parabola with vertex at the origin and focus at $(-a, 0)$ is

$$y^2=-4ax.$$

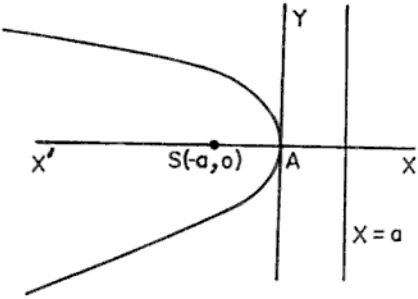


Fig. 7.4 (*b*)

(*iii*) The equation of the parabola with vertex at the origin and focus at $(0, b)$ is

$$x^2=4by.$$

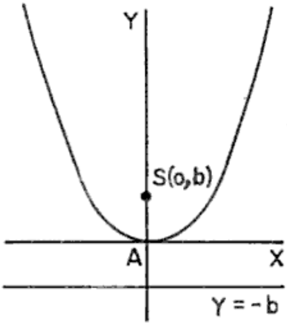


Fig. 7.4 (*c*)

(*iv*) The equation of the parabola with vertex at the origin and focus at $(0, -b)$ is

$$x^2=-4by.$$

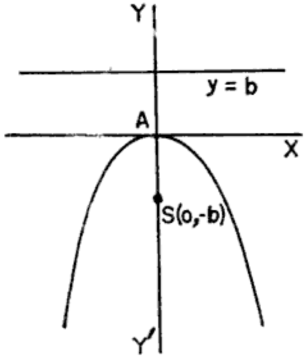


Fig. 7.4 (*d*)

Remarks. (*i*) The parabola $y^2=4ax$ opens to the right if $a>0$ [see Fig. 7.4 (*a*)] and opens to the left if $a<0$ [see Fig. 7.4 (*b*)].

(*ii*) The parabola $x^2=4by$ opens to the upward if $b>0$ [see Fig. 7.4 (*c*)] and opens to the downward if $b<0$ [see Fig. 7.4 (*d*)].

2.5 Latus rectum and focal distance. *The chord through focus parallel to the directrix is called the* **latus rectum.**

The distance of any point on the parabola from its focus is called the **focal distance** *of the point.*

Let the equation of the parabola be

$$y^2=4ax.$$

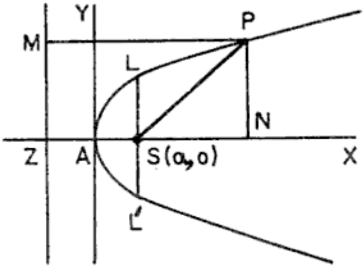


Fig. 7.5

Then latus rectum $= LSL'$

$$= 2LS$$

$$= 4a, \qquad \because \ LS = 2a.$$

Focal distance of any point $P(x, y)$ on the parabola

$$= SP$$

$$= MP = ZN = ZA + AN = a + x.$$

2.6 Parabola and a point. *To prove that the point* (x_1, y_1) *lies outside, on or inside* the parabola* $y^2 = 4ax$ *according as the expression* $y_1^2 - 4ax_1 >= $ *or* < 0.

Let the point (x_1, y_1) be outside the given parabola.

Draw PM perpendicular to x-axis and let it meet the parabola in the point N. Let ordinate of the point N be y_2. Then coordinates of the point N are (x_1, y_2).

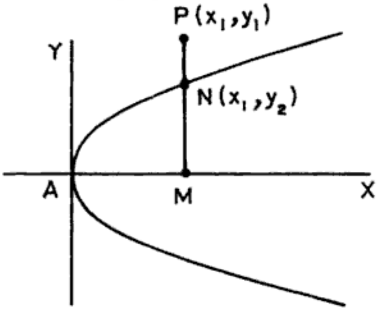


Fig. 7.6

The point $P(x_1, y_1)$ lies outside the parabola if

$$MP > MN$$

i.e. if $$MP^2 > MN^2.$$

But $MP^2 = y_1^2$ and $MN^2 = y_2^2 = 4ax_1$.

Thus the point (x_1, y_1) lies outside the parabola if $y_1^2 - 4ax_1 > 0$.

Similarly, we can prove that the point (x_1, y_1) lies inside the parabola if $y_1^2 - 4ax_1 < 0$ whereas the point (x_1, y_1) lies on the parabola if $y_1^2 - 4ax_1 = 0$.

---

*By the points inside a parabola, we mean the points lying in the region, determined by the parabola which contains the focus.

Hence *the point* (x_1, y_1) *lies outside, on or inside the parabola* $y^2=4ax$ *according as* $y_1^2-4ax_1 >=$ *or* <0.

2.7 Example. PQ is a double ordinate of the parabola $y^2=4ax$. Find the locus of its points of trisection.

Solution. Let (x_1, y_1) be the coordinates of P. Then the coordinates of Q are $(x_1, -y_1)$. Let M and N be the points of trisection. Then the coordinates of these points are, respectively, $(x_1, \frac{1}{3}y_1)$ and $(x_1, -\frac{1}{3}y_1)$.

Let (h, k) be the coordinates of either points of trisection. Then

$$h=x_1 \quad \text{and} \quad k=\pm\tfrac{1}{3}y_1$$

$$\Rightarrow \qquad x_1=h \quad \text{and} \quad y_1=\pm 3k.$$

But (x_1, y_1) lies on the parabola. Therefore

$$9k^2=4ah.$$

Hence the locus of (h, k) is $y^2=\dfrac{4a}{9}x$ which is a parabola.

EXERCISES

1. Find the equation of the parabola whose
   (*i*) focus is $(5, 2)$ and directrix is $x-1=0$;
   (*ii*) focus is $(2, 1)$ and directrix is $3x+4y=0$;
   (*iii*) focus is $(1, 2)$ and directrix is $x+y-2=0$;
   (*iv*) focus is $(4, -3)$ and directrix is $3x-4y=6$.
2. Find the equation of the parabola whose
   (*i*) vertex is $(0, 0)$ and focus is $(0, -2)$;
   (*ii*) vertex is $(3, 3)$ and focus is $(-3, 3)$;
   (*iii*) vertex is $(5, -2)$ and focus is $(-2, -2)$.
3. Find the equation of the parabola in each problem:
   (*i*) vertex $(-1, -2)$, axis vertical; passes through $(3, 6)$.
   (*ii*) axis vertical, passes through $(0, 0)$, $(3, 0)$ and $(-1, 4)$.
   (*iii*) axis horizontal, passes through $(0, 4)$, $(0, -1)$ and $(6, 1)$.
4. Find the vertex, focus, directrix and length of the latus rectum of the following parabolas:
   (*i*) $5x^2+24y=0$.
   (*ii*) $y^2+2x-4y+3=0$.

(*iii*) $x^2-4x-5y-1=0.$

(*iv*) $y^2+4x+2y-8=0.$

5. A double ordinate of the parabola $y^2=4ax$ is of length $8a$. Prove that the lines from the vertex to its two ends are at right angles.
6. A variable circle is described to pass through $(a, 0)$ and touch the line $x+y=0$. Prove that the locus of the centre of the circle is a parabola.
7. If a circle be drawn so as always to touch a given line and also a given circle, prove that the locus of its centre is a parabola.

3. TANGENT AND NORMAL

3.1 Equation of tangent at a point. *To find the equation of the tangent to a parabola at a point on it.*

Let
$$y^2=4ax \tag{1}$$
be the equation of the parabola and $P(x_1, y_1)$ be the given point on it.

Since (x_1, y_1) lies on (1), we have
$$y_1^2=4ax_1. \tag{2}$$

Take $Q(x_2, y_2)$ any other point on (1). Then
$$y_2^2=4ax_2. \tag{3}$$

The equation of the secant PQ is
$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \tag{4}$$

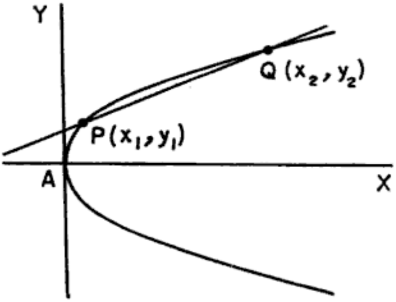


Fig. 7.7

As $Q \to P$, by definition, secant PQ becomes the tangent at P and therefore on taking the limit as $x_2 \to x_1$ and $y_2 \to y_1$, equation (4) would represent the equation of the tangent to (1) at $P(x_1, y_1)$. Since on taking the limit, the value of $\frac{y_2-y_1}{x_2-x_1}$ is of the form $\frac{0}{0}$ which is an indeterminate form, we proceed to obtain the same as follows.

Subtracting (2) from (3), we get

$$y_2^2-y_1^2=4a\,(x_2-x_1)$$

$$\Rightarrow \qquad \frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_2+y_1}.$$

Thus as $Q \to P$, $x_2 \to x_1$ and $y_2 \to y_1$, we get

$$\frac{y_2-y_1}{x_2-x_1} \to \frac{4a}{2y_1}=\frac{2a}{y_1}$$

and hence as $Q \to P$, equation (4) becomes

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

$$\Rightarrow \qquad yy_1-y_1^2=2ax-2ax_1.$$

On using (2), we get

$$yy_1-4ax_1=2ax-2ax_1.$$

Hence *the equation of the tangent to the parabola* $y^2=4ax$ *at the point* (x_1, y_1) is

$$yy_1=2a\,(x+x_1).$$

Working rule. The equation of the tangent to the parabola $y^2=4ax$ at the point (x_1, y_1) is obtained by writing yy_1 for y^2 and $x+x_1$ for $2x$ in the equation of the parabola.

Corollary. The tangent to the parabola $y^2=4ax$ at the vertex is the y-axis.

3.2 Intersection of a parabola and a line

Let $\qquad y^2=4ax \qquad (1)$

and $\qquad y=mx+c \qquad (2)$

be the equations of a parabola and a line, respectively. For the points of intersection, we solve (1) and (2) simultaneously and we get

$$(mx+c)^2=4ax$$

$$\Rightarrow \qquad m^2x^2+2(mc-2a)\,x+c^2=0. \tag{3}$$

Since equation (3) is quadratic in x it gives two values of x which may be real and distinct, coincident or imaginary. Accordingly there are two points of intersection of the parabola (1) and the line (2).

Hence a line always cuts a parabola in two points which may be real and distinct, coincident or imaginary.

Note. From equation (3), we have

$$x=\frac{-2(mc-2a)\pm\sqrt{4(mc-2a)^2-4m^2c^2}}{2m^2}$$

$$\Rightarrow \qquad x=\frac{2a-mc\pm\sqrt{4a^2-4amc}}{m^2}.$$

If $m=0$, x becomes infinite *i.e.* if the line is parallel to x-axis then it meets the parabola at infinity. Further, if we take $m=0$ in equation (3), then one root of equation (3) is real and finite. Thus, we conclude that *a line parallel to the axis of the parabola meets the curve in two points of which one is finite and the other infinite.*

3.3 Condition of tangency. *To find the condition when the line $y=mx+c$ becomes tangent to the parabola $y^2=4ax$.*

First Method. It is clear, from equation (3) in §3.2, that the line $y=mx+c$ is tangent to the parabola $y^2=4ax$ if the roots of the equation (3) in §3.2 are equal for which

$$4(mc-2a)^2=4m^2c^2$$

i.e.

$$c=\frac{a}{m}.$$

This is the required condition.

Second Method. The equation of the tangent to the given parabola $y^2=4ax$ at any point (x_1, y_1) is

$$yy_1=2a\,(x+x_1).$$

The line $y=mx+c$ is tangent to the given parabola if

$$2ax-yy_1+2ax_1\equiv mx-y+c$$

$$\Rightarrow \qquad \frac{2a}{m}=\frac{-y_1}{-1}=\frac{2ax_1}{c}$$

$$\Rightarrow \qquad x_1=\frac{c}{m},\ y_1=\frac{2a}{m}.$$

But (x_1, y_1) lies on the given parabola.

$$\therefore \qquad \frac{4a^2}{m^2}=4a\ \frac{c}{m}$$

$$\Rightarrow \qquad c=\frac{a}{m}.$$

This is the required condition.

Remarks. (*i*) The line $y=mx+\dfrac{a}{m}$ is always a tangent to the parabola $y^2=4ax$ whatever the value of m may be. In fact, it is a family of tangents to the parabola $y^2=4ax$, where m stands for a parameter.

(*ii*) The point of contact of the tangent $y=mx+\dfrac{a}{m}$ to the parabola $y^2=4ax$ is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

(*iii*) The line $y=mx+\dfrac{a}{m}$ is called the equation of the tangent in 'm' form or in slope form to the parabola $y^2=4ax$.

Ex. Show that the equation of the tangent to the parabola $y^2=4ax$ which makes an angle θ with its axis is

$$y=x\tan\theta+a\cot\theta.$$

3.4 Equation of the normal. *To find the equation of the normal to a parabola at a given point on it.*

Let $$y^2=4ax \tag{1}$$

be the equation of the parabola and $P(x_1, y_1)$ be the given point on it.

The equation of the tangent at the point $P(x_1, y_1)$ is

$$yy_1=2a(x+x_1).$$

Slope of the tangent $=\dfrac{2a}{y_1}$.

$\therefore$ Slope of the normal $=-\dfrac{y_1}{2a}$.

Hence the equation of the normal at the point $P(x_1, y_1)$ to the parabola $y^2=4ax$ is

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

i.e. $$xy_1+2ay=(2a+x_1)y_1 \quad \square$$

If we denote the slope of the normal by m *i.e.* $m=-\frac{y_1}{2a}$. Then

$$y_1=-2am$$

and $$x_1=\frac{y_1^2}{4a}=am^2.$$

Hence the equation of the normal in terms of m is

$$y+2am=m(x-am^2)$$

i.e. $$y=mx-2am-am^3.$$

Note. The line $y=mx-2am-am^3$ is called the equation of the normal in **'m' form** or in **slope form.** In fact, this represents a system of normals to the parabola $y^2=4ax$, where m stands for the parameter.

Remark. The coordinates of the foot of the normal $y=mx-2am-am^3$ to the parabola $y^2=4ax$ are $(am^2, -2am)$

3.5 *To prove that the three normals can be drawn to a parabola from an external point and that the algebraic sum of the ordinates of the feet of these normals is zero.*

The equation of the normal at the point $(am^2, -2am)$ to the parabola $y^2=4ax$ is

$$y=mx-2am-am^3. \tag{1}$$

Let it pass through a fixed point (h, k).

Then $$k=mh-2am-am^3$$

i.e. $$am^3+(2a-h)\,m+k=0. \tag{2}$$

Equation (2) is cubic in m, therefore it has three roots m_1, m_2 and m_3 (say). Hence through any point, three normals can be drawn to a parabola of which one at least must be real.

Further, by theory of equations, we have

$$m_1+m_2+m_3=0$$

$$\Rightarrow \qquad -2am_1-2am_2-2am_3=0$$

$$\Rightarrow \qquad y_1+y_2+y_3=0,$$

where y_1, y_2, y_3 are the ordinates of the feet of the normals.

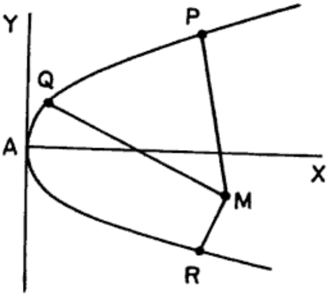


Fig. 7.8

The feet of the normals drawn from a point are called **co-normal points.**

In Fig. 7.8 three normals are drawn from the point M. The feet of the normals are P, Q and R. Thus P, Q and R are co-normal points.

3.6 Circle through co-normal points. *To find the equation to the circle through the three points the normals at which meet in a point.*

Let the normals at the points P, Q, R to the parabola $y^2=4ax$ meet in the point (h, k). Then we have

$$am^3+(2a-h)\,m+k=0. \tag{1}$$

Taking $-2am=y$, the ordinates of P, Q, R are the roots of the equation

$$y^3+4a\,(2a-h)\,y-8a^2k=0. \tag{2}$$

Let equation of the circle passing through P, Q, R be

$$x^2+y^2+2gx+2fy+c=0. \tag{3}$$

Multiply equation (3) by $16a^2$ and put y^2 for $4ax$. Then the ordinates of the points of intersection of the circle and the parabola are the roots of the equation

$$y^4+8a\,(2a+g)y^2+32a^2\,fy+16a^2c=0. \tag{4}$$

If y_1, y_2, y_3, y_4 be the roots of the equation (4), then by theory of equations, we have

$$y_1+y_2+y_3+y_4=0.$$

But from equation (2), we have

$$y_1+y_2+y_3=0,$$

where y_1, y_2 and y_3 are the ordinates of P, Q and R, respectively. Therefore $y_4=0$ which shows that the circle passing through P, Q and R passes through the vertex of the parabola, for all values of h and k. Thus $c=0$.

Now from equation (4) the ordinates of P, Q and R are the roots of the equation

$$y^3+8a\,(2a+g)\,y+32a^2f=0. \tag{5}$$

On comparing (2) and (5), we get

$$\begin{cases} 2(2a+g)=2a-h \\ \quad -8a^2k=32a^2f \end{cases}$$

$$\Rightarrow \qquad \begin{cases} 2g=-(2a+h) \\ 2f=-\dfrac{k}{2}. \end{cases}$$

Hence *the equation of the circle passing through the points the normals at which meet in the point* (h, k) *is*

$$x^2+y^2-(2a+h)x-\tfrac{1}{2}ky=0.$$

3.7 Examples

Example 1. Two lines are at right angles to one another, and one of them touches the parabola $y^2=4a(x+a)$ and the other to $y^2=4a'(x+a')$. Prove that the locus of the point of intersection of the lines is $x+a+a'=0$.

Solution. Any tangents to the given parabolas are

$$y=m(x+a)+\frac{a}{m} \tag{1}$$

and
$$y=m'(x+a')+\frac{a'}{m'}. \tag{2}$$

Since the tangents are at right angles, we have

$$mm' = -1. \tag{3}$$

Substituting the value of m' from (3) in (2), we get

$$y = -\frac{1}{m}(x+a') - a'm. \tag{4}$$

Subtracting (4) from (1), we obtain

$$0 = \left(m+\frac{1}{m}\right)x + \left(m+\frac{1}{m}\right)a + \left(m+\frac{1}{m}\right)a'$$

$$\Rightarrow \qquad x+a+a'=0.$$

Example 2. Find the locus of the foot of the perpendicular drawn from the vertex on a tangent to the parabola $y^2=4ax$.

Solution. Any tangent to the given parabola is

$$y = mx + \frac{a}{m}. \tag{1}$$

The equation of the perpendicular drawn from the vertex (0, 0) to the tangent (1) is

$$y = -\frac{1}{m}x. \tag{2}$$

The locus of the foot of the perpendicular is obtained by eliminating m between (1) and (2). Hence the required locus is

$$y = -\frac{x}{y}\cdot x + \left(-\frac{y}{x}\right)a$$

$$\Rightarrow \qquad (x+a)y^2 + x^3 = 0.$$

Example 3. If the tangent to the parabola $y^2=4ax$ meets the axis in T and the tangent at the vertex A in Y, and the rectangle $TAYQ$ is completed. Prove that the locus of Q is the parabola

$$y^2+ax=0.$$

Solution. Any tangent to the parabola $y^2=4ax$ is

$$y = mx + \frac{a}{m}.$$

This tangent meets the axis in $T\left(-\frac{a}{m^2}, 0\right)$ and the tangent at the vertex in $Y\left(0, \frac{a}{m}\right)$.

Let (h, k) be the coordinates of Q. Then

$$h=-\frac{a}{m^2} \text{ and } k=\frac{a}{m}.$$

Eliminating m from these equations, we get

$$k^2+ah=0.$$

Hence the locus of (h, k) is $y^2+ax=0$.

Example 4. Prove that the locus of the middle point of the portion of a normal to the parabola $y^2=4ax$ intercepted between the curve and the axis is another parabola. Find the vertex and the latus rectum of this parabola.

Solution. The equation of the normal at $(am^2, -2am)$ to the given parabola is

$$y=mx-2am-am^3.$$

This normal meets the axis in the point $(2a+am^2, 0)$. Let (x', y') be the middle point of the given portion of the normal. Then

$$x'=\frac{2a+am^2+am^2}{2}=a+am^2$$

and
$$y'=\frac{0+(-2am)}{2}=-am.$$

Eliminating m between these equations, we get

$$y'^2=a(x'-a).$$

Hence the locus of (x', y') is

$$y^2=a(x-a).$$

This equation represents a parabola whose vertex and latus rectum are, respectively, $(a, 0)$ and a.

Example 5. The normals to the parabola $y^2=4ax$ from a point P meet the axis in A, B, C. If B is the middle point of AC, prove that the locus of P is

$$27ay^2=2(x-2a)^3.$$

Solution. Any normal to the given parabola is

$$y=mx-2am-am^3.$$

Let it pass through the point $P(h, k)$. Then

$$k = mh - 2am - am^3$$

$$\Rightarrow \qquad am^3 + (2a-h)m + k = 0.$$

Let m_1, m_2, m_3 be the roots of the equation. Then, by the theory of equations, we have

$$m_1 + m_2 + m_3 = 0, \tag{1}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a} \tag{2}$$

and

$$m_1m_2m_3 = -\frac{k}{a}. \tag{3}$$

The normals through (h, k) are

$$y = m_1x - 2am_1 - am_1^3,$$

$$y = m_2x - 2am_2 - am_2^3,$$

and

$$y = m_3x - 2am_3 - am_3^3.$$

These normals meet the axis in $A(2a+am_1^2, 0)$, $B(2a+am_2^2, 0)$ and $C(2a+am_3^2, 0)$, respectively. Since B is the middle point of AC, we have

$$2a + am_2^2 = \frac{2a + am_1^2 + 2a + am_3^2}{2}$$

$$\Rightarrow \qquad 2m_2^2 = m_1^2 + m_3^2$$

$$\Rightarrow \qquad 2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3$$

$$\Rightarrow \qquad 2m_2^2 = m_2^2 - 2m_1m_3$$

$$\Rightarrow \qquad m_1m_3 = -\frac{m_2^2}{2}. \tag{4}$$

Now from (2), we have

$$(m_1 + m_3)m_2 + m_3m_1 = \frac{2a-h}{a}$$

$$\Rightarrow \qquad (-m_2)m_2 - \frac{m_2^2}{2} = \frac{2a-h}{a}$$

$$\Rightarrow \qquad m_2^2 = \frac{2}{3}\left(\frac{h-2a}{a}\right). \tag{5}$$

Further, from (3) and (4), we get

$$m_2^3 = \frac{2k}{a}. \tag{6}$$

Eliminating m_2 between (5) and (6), we get

$$m_2^6=\left(\frac{2k}{a}\right)^2=\left\{\frac{2}{3}\left(\frac{h-2a}{a}\right)\right\}^3=m_2^6$$

$$\Rightarrow \qquad 27ak^2=2(h-2a)^3.$$

Hence the locus of (h, k) is

$$27ay^2=2(x-2a)^3.$$

Example 6. Prove that the locus of the points such that two of the three normals to the parabola $y^2=4ax$ from them coincide is

$$27ay^2=4(x-2a)^3.$$

Solution. Any normal to the given parabola is

$$y=mx-2am-am^3.$$

Let it pass through the point (h, k). Then

$$k=mh-2am-am^3$$

$$\Rightarrow \qquad am^3+(2a-h)\, m+k=0.$$

Let m_1, m_2, m_3 be the roots of the equation. Then, by theory of equations, we have

$$m_1+m_2+m_3=0 \tag{1}$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-h}{a} \tag{2}$$

and

$$m_1m_2m_3=\frac{-k}{a}. \tag{3}$$

Since any two normals coincide, we can take $m_1=m_2$. Now equations (1), (2) and (3) give

$$m_3=-2m_1,\ m_1^2=\frac{h-2a}{3a},\ m_1^3=\frac{k}{2a}.$$

Thus

$$m_1^6=\frac{(h-2a)^3}{27a^3}=\frac{k^2}{4a^2}$$

$$\Rightarrow \qquad 27ak^2=4(h-2a)^3.$$

Hence the locus of (h, k) is

$$27ay^2=4\,(x-2a)^3.$$

EXERCISES

1. Find the equations of the tangents and the normals to the parabola $y^2=4ax$ at the ends of its latus rectum.
2. Prove that the line $lx+my+n=0$ touches the parabola $y^2=4ax$ if $ln=am^2$.
3. Prove that the line $y=mx+c$ touches the parabola $y^2=4a(x+a)$ if $c=ma+\frac{a}{m}\cdot$
4. Prove that the tangent to the parabola $y^2=4ax$ at the point (x_1, y_1) is perpendicular to the tangent at the point
$$\left(\frac{a^2}{x_1}, -\frac{4a^2}{y_1}\right).$$
5. Prove that the normal chord at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.
6. Two equal parabolas have the same vertex and their axes are at right angles. Prove that the common tangent touches each at the end of a latus rectum.
7. Prove that the chord of the parabola $y^2=4ax$ whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$ is a normal to the curve, and that its length is $6\sqrt{3}a$.
8. Prove that the chord of the parabola $y^2=4ax$ which is normal at the point whose abscissa is $2a$ subtends a right angle at the vertex.
9. PNP' is a double ordinate of the parabola $y^2=4ax$. Prove that the locus of the point of intersection of the normal at P and the line through P' parallel to the axis is equal parabola
$$y^2=4a(x-4a).$$
10. The normal at any point P meets the axis in G and the tangent at the vertex in G'. If A is the vertex and the rectangle $AGQG'$ is completed, prove that the equation of the locus of Q is
$$x^3=2ax^2+ay^2.$$
11. Find the locus of the foot of the perpendicular drawn from a fixed point on any tangent to a parabola.
12. Prove that a normal chord of a parabola which subtends a right angle at the vertex makes an angle θ with the axis, where $\tan\theta=\sqrt{2}$.

13. From the point where any normal to the parabola $y^2=4ax$ meets the axis is drawn a line perpendicular to this normal. Prove that this line always touches an equal parabola.

14. If three normals from a point to the parabola $y^2=4ax$ cut the axis in points whose distances from the vertex are in arithmetical progression. Prove that the point lies on the curve

$$27ay^2=2(x-2a)^3.$$

15. Find the locus of a point which is such that two of the normals drawn from it to the parabola $y^2=4ax$ are at right angles.

16. Find the locus of a point P when three normals drawn from it are such that one bisects the angle between the other two.

4. TANGENTS FROM A POINT

4.1 *To prove that from any point there can be drawn two tangents, real or imaginary, to a parabola.*

Let the equation of the parabola be

$$y^2=4ax. \tag{1}$$

Let $P(x_1, y_1)$ be the given point. Any tangent to the parabola (1) is

$$y=mx+\frac{a}{m},$$

where m is the slope and can take any value. Since tangent passes through the given point $P(x_1, y_1)$, we have

$$y_1=mx_1+\frac{a}{m}$$

$$\Rightarrow \qquad m^2x_1-my_1+a=0. \tag{2}$$

This is a quadratic equation in m, giving two values of m (real and distinct, coincident or imaginary) corresponding to which there will be two tangents passing through the point $P(x_1, y_1)$. The two tangents will be real and distinct, coincident or imaginary according as the roots of the equation (2) are so. In fact the roots of the equation (2) are real and distinct, coincident or imaginary according as $y_1^2-4ax_1 >=$ or <0.

But this is the condition for the point (x_1, y_1) to be outside, on or inside the parabola.

Hence from a point there can be drawn two tangents to a parabola and these tangents will be real and distinct, coincident or imaginary according as the point lies outside, on or inside the parabola.

4.2 Pair of tangents from a given point. *To find the equation of the pair of tangents drawn from an external point* $P(x_1, y_1)$ *to the parabola* $y^2=4ax$.

Let $Q(x, y)$ be any point on the locus (*i.e.* on either of the tangents). Then equations of the line PQ are

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \tag{1}$$

θ being its inclination from the axis of the parabola and r the algebraic distance of the point $Q(x, y)$ from $P(x_1, y_1)$ measured along the line. Therefore a point on the line and at a distance r from $P(x_1, y_1)$ is given by

$$x=x_1+r\cos\theta$$

$$y=y_1+r\sin\theta.$$

If this point be on the parabola, we must have

$$(y_1+r\sin\theta)^2=4a(x_1+r\cos\theta)$$

i.e. $$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1^2-4ax_1=0.$$

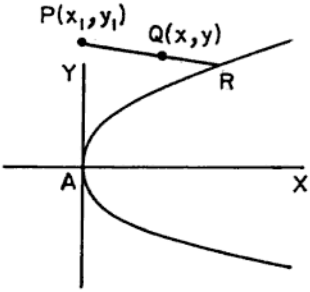


Fig. 7.9

This is a quadratic equation in r giving two values of r corresponding to which there are two points common to the parabola and the line (1).

Now if the line (1) be a tangent to the parabola, both the values of r given by the quadratic equation must be equal. Therefore, we must have

$$(y_1 \sin \theta - 2a \cos \theta)^2 - (y_1^2 - 4ax_1) \sin^2 \theta = 0. \tag{2}$$

This equation gives us the value of θ *i.e.* the direction of the line (1) so that it may become the tangent to the parabola. To obtain the actual equation of the tangent (*s*), eliminate θ between (1) and (2). Thus the equation of the tangent is

$$\left\{y_1\left(\frac{y-y_1}{r}\right) - 2a\left(\frac{x-x_1}{r}\right)\right\}^2 = \left(\frac{y-y_1}{r}\right)^2 (y_1^2 - 4ax_1)$$

$$\Rightarrow \quad (yy_1 - 2ax - y_1^2 + 2ax_1)^2 = (y^2 - 2yy_1 + y_1^2)(y_1^2 - 4ax_1)$$

$$\Rightarrow \quad (yy_1 - 2ax - 2ax_1)^2 = (y^2 - 4ax)(y_1^2 - 4ax_1).$$

Hence the equation of the pair of tangents is

$$T^2 = SS_1,$$

where $S \equiv y^2 - 4ax$

$S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2ax - 2ax_1$.

Ex. Find the equation of the tangents drawn from the point $(-2, 0)$ to the parabola $y^2 = 8x$ and prove that they are perpendicular.

4.3 Example. The angle between two tangents to the parabola $y^2 = 4ax$ is constant and equal to α. Prove that the locus of their point of intersection is given by

$$y^2 - 4ax = (a+x)^2 \tan^2 \alpha.$$

What happens to this locus if $\alpha = \frac{\pi}{2}$?

Solution. Let (h, k) be the point of intersection of the two tangents to the given parabola. Then the equation of the pair of tangents is

$$(y^2 - 4ax)(k^2 - 4ah) = (ky - 2ax - 2ah)^2$$

$\Rightarrow$ $4a^2x^2 - 4akxy + 4ahy^2 +$ first degree terms $+$ constant terms $= 0$.

Since α is the angle between the tangents, we have

$$\tan \alpha = \frac{2\sqrt{4a^2k^2-16a^3h}}{4a^2+4ah}$$

$$\Rightarrow \qquad k^2-4ah=(a+h)^2 \tan^2 \alpha.$$

Hence the locus of (h, k) is

$$y^2-4ax=(a+x)^2 \tan^2 \alpha.$$

When $\alpha=\frac{\pi}{2}$, the locus becomes

$$x+a=0,$$

which is the equation of the directrix.

EXERCISES

1. Find the equation of the locus of the point of intersection of two tangents to a parabola which make an angle of 45° with one another.
2. Prove that the locus of the point of intersection of two tangents to a parabola at points on the curve whose ordinates are in a constant ratio is a parabola.
3. The two tangents from a point P to the parabola $y^2=4ax$ make angles α, β with x-axis. Find the locus of P when

   (*i*) $\tan \alpha+\tan \beta$ is constant.

   (*ii*) $\tan^2 \alpha+\tan^2 \beta$ is constant.
4. Prove that the two tangents which make equal angles, respectively with the axis and the directrix but are not at right angles, intersect on the latus rectum.
5. A pair of tangents are drawn which are equally inclined to a line whose inclination to the axis is α. Prove that the locus of their point of intersection is the line

$$y=(x-a) \tan 2\alpha.$$

5. CHORD OF CONTACT

5.1 Equation of the chord of contact. *To find the equation of the chord of contact of the tangents drawn from the point* (x_1, y_1) *to the parabola* $y^2=4ax$.

Let $Q(x', y')$ and $R(x'', y'')$ be the points of contacts of the tangents drawn from the point $P(x_1, y_1)$ to the parabola

$$y^2=4ax. \tag{1}$$

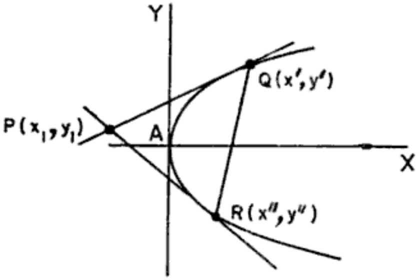


Fig. 7.10

Then, the equations of the tangents at the points Q and R to the parabola (1) are, respectively

$$y'y=2a(x'+x)$$

and
$$y''y=2a(x''+x).$$

But each of these tangents passes through the point $P(x_1, y_1)$. Therefore

$$y'y_1=2a(x'+x_1)$$

and
$$y''y_1=2a(x''+x_1).$$

It is clear from these relations that the points (x', y') and (x'', y'') both satisfy the equation

$$yy_1=2a(x+x_1).$$

This is the required equation of the line which passes through the points $Q(x', y')$ and $R(x'', y'')$, the points of contacts of the tangents drawn from the point $P(x_1, y_1)$ to the parabola (1).

Remarks. (*i*) If the point $P(x_1, y_1)$ lies inside the parabola, the two tangents will be imaginary, but the equation of the chord of contact is real since x_1 and y_1 are real.

(*ii*) If the point $P(x_1, y_1)$ lies on the parabola, then the chord of contact becomes the tangent at that point.

Working rule. The equation of the chord of contact of the tangents drawn from an external point (x_1, y_1) to the parabola $y^2=4ax$ is obtained by substituting yy_1 for y^2 and $x+x_1$ for $2x$ in the equation of the parabola.

5.2 Example. Prove that if tangents be drawn to the parabola $y^2=4ax$ from a point on the line $x+4a=0$, their chord of contact will subtend a right angle at the vertex.

Solution. Any point on the line $x+4a=0$ is $(-4a, k)$. The equation of the chord of contact of the tangents drawn from the point $(-4a, k)$ to the given parabola is

$$ky=2a(x-4a).$$

The equation of the lines joining the vertex to the points of intersection of the parabola and the chord of contact is

$$y^2=4ax\left(\frac{2ax-ky}{8a^2}\right)$$

$$\Rightarrow \quad 2ax^2-kxy-2ay^2=0.$$

Since coefficient of x^2+coefficient of $y^2=0$, the chord of contact subtends a right angle at the vertex.

EXERCISES

1. Tangents are drawn from the point (α, β) to the parabola $y^2=4ax$. Prove that the length of their chord of contact is

$$\frac{(\beta^2-4a\alpha)^{1/2}\,(\beta^2+4a^2)^{1/2}}{a}.$$

2. Tangents are drawn at the ends of a normal chord of the parabola $y^2=4ax$. Prove that the locus of their point of intersection is $(x+2a)\,y^2+4a^3=0$.
3. If two tangents to a parabola make equal angles with a fixed line, prove that the chord of contact must pass through a fixed point.
4. A variable chord PQ of the parabola passes through the fixed point R. Prove that the tangents at P, Q meet on a fixed line, and show that this line is the chord of contact of the tangents drawn from the point R.

6. POLE AND POLAR

6.1 Equation of the polar. *To find the equation of the polar of the point* (x_1, y_1) *with respect to the parabola* $y^2=4ax$.

Let $P(x', y')$ be any point on the locus *i.e.* the point of intersection of the tangents drawn at the extremities of a chord through the point (x_1, y_1).

The equation of the chord of contact of the tangents from the point (x', y') is

$$yy'=2a\,(x+x').$$

But it passes through the point (x_1, y_1).

$$\therefore \qquad y_1y'=2a(x_1+x').$$

Hence the locus of (x', y') is

$$yy_1=2a(x+x_1).$$

6.2 Pole of a line. *To find the pole of the line* $lx+my+n=0$ *with respect to the parabola* $y^2=4ax$.

Let (x_1, y_1) be the pole of the line $lx+my+n=0$ with respect to the parabola $y^2=4ax$. Then this line must be identical with the polar of (x_1, y_1) with respect to the parabola $y^2=4ax$ *i.e.* with

$$yy_1=2a(x+x_1).$$

$$\therefore \qquad \frac{2a}{l}=\frac{-y_1}{m}=\frac{2ax_1}{n}$$

$$\Rightarrow \qquad x_1=\frac{n}{l},\ y_1=-\frac{2am}{l}.$$

Hence the required pole is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

Ex. Prove that the polar of the focus of a parabola is the directrix.

6.3 Examples

Example 1. Prove that the locus of poles of normal chords of the parabola $y^2=4ax$ is

$$(x+2a)\,y^2+4a^3=0.$$

Solution. Any normal chord of the given parabola is

$$y=mx-2am-am^3. \tag{1}$$

Let (h, k) be the pole of (1). Then its polar with respect to the given parabola is

$$ky=2a(x+h). \tag{2}$$

But equations (1) and (2) represent the same line. Therefore, on comparing (1) and (2), we get

$$\frac{m}{2a}=\frac{1}{k}=\frac{-2am-am^3}{2ah}$$

$$\Rightarrow \qquad m=\frac{2a}{k} \text{ and } -2m-m^3=\frac{2h}{k}.$$

$$\therefore \qquad -\frac{4a}{k}-\frac{8a^3}{k^3}=\frac{2h}{k}.$$

$$\Rightarrow \qquad -2ak^2-4a^3=hk^2.$$

Hence locus of (h, k) is

$$(x+2a)y^2+4a^3=0.$$

Example 2. Prove that the locus of the poles of tangents to the parabola $y^2=4ax$ with respect to the circle $x^2+y^2-2ax=0$ is the circle $x^2+y^2-ax=0$.

Solution. Any tangent to the given parabola is

$$y=mx+\frac{a}{m}. \tag{1}$$

Let (h, k) be the pole of (1). Then its polar with respect to the circle $x^2+y^2-2ax=0$ is

$$hx+ky-a(x+h)=0$$

i.e.

$$(h-a)x+ky-ah=0. \tag{2}$$

But equations (1) and (2) represent the same line. Therefore, on comparing (1) and (2), we get

$$\frac{h-a}{m}=\frac{k}{-1}=\frac{-ah}{\frac{a}{m}}$$

$$\Rightarrow \qquad m=-\left(\frac{h-a}{k}\right) \text{ and } m=\frac{k}{h}.$$

$$\therefore \qquad -\left(\frac{h-a}{k}\right)=\frac{k}{h}$$

$$\Rightarrow \qquad -h^2+ah=k^2.$$

Hence the locus of (h, k) is

$$x^2+y^2-ax=0.$$

EXERCISES

1. Prove that the locus of the poles of chords of the parabola $y^2=4ax$ which subtend a constant angle α at the vertex is the curve
$$(x+4a)^2=4(y^2-4ax)\cot^2\alpha.$$
Hence or otherwise prove that if the constant angle is a right angle the locus is a line perpendicular to the axis of the parabola.
2. Prove that the polar of $(-a, 2a)$ with respect to the circle $x^2+y^2-2ax-3a^2=0$ touches the parabola $y^2=4ax$.
3. Prove that the polar of any point on the circle
$$x^2+y^2-2ax-3a^2=0,$$
with respect to the circle
$$x^2+y^2+2ax-3a^2=0,$$
will touch the parabola $y^2=-4ax$.
4. Prove that the locus of the poles of tangents to the parabola $y^2=4ax$ with respect to the parabola $y^2=4bx$ is the porabola
$$y^2=\frac{4b^2}{a}x.$$
5. A point P is such that the line drawn through it perpendicular to its polar with respect to the parabola $y^2=4ax$ touches the parabola $x^2=4by$. Prove that the locus of P is the line $2ax+by+4a^2=0$.
6. The middle point of a chord of a parabola is on a fixed line perpendicular to the axis of the parabola. Prove that the locus of the pole of the chord is another parabola.
7. Prove that the locus of the poles of the chords which subtend a right angle at the fixed point (h, k) is
$$ax^2-hy^2+(4a^2+2ah)x-2aky+a(h^2+k^2)=0.$$

7. CHORD WITH GIVEN MIDDLE POINT

7.1 *To find the equation of a chord of the parabola* $y^2=4ax$ *in terms of its middle point* (x_1, y_1).

First Method

The equations of a line through the point (x_1, y_1) are

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \tag{1}$$

θ being its inclination with the axis of the parabola and r the algebraic distance of (x, y) from (x_1, y_1) measured along the line. Then any point on the line has the coordinates $(x_1+r\cos\theta, y_1+r\sin\theta)$ and this point will be common to the parabola if

$$(y_1+r\sin\theta)^2=4a(x_1+r\cos\theta)$$

i.e.
$$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1^2-4ax_1=0. \tag{2}$$

This equation being quadratic in r, gives two values of r. Since (x_1, y_1) is the middle point of the chord, the two values of r given by the equation (2) must be equal in magnitude and opposite in sign *i.e.* the sum of the roots of equation (2) must be zero *i.e.*

$$y_1\sin\theta-2a\cos\theta=0. \tag{3}$$

This equation gives us the suitable value of θ so that the line (1) may be the chord of the parabola having (x_1, y_1) as the middle point. Therefore the equation of the chord is

$$y_1\left(\frac{y-y_1}{r}\right)-2a\left(\frac{x-x_1}{r}\right)=0$$

i.e.
$$yy_1-2ax=y_1^2-2ax_1.$$

Second Method

Let $P(x', y')$ and $Q(x'', y'')$ be the end points of the chord whose middle point is (x_1, y_1).

Then $x_1=\dfrac{x'+x''}{2}$ and $y_1=\dfrac{y'+y''}{2}$.

Slope of PQ is $\dfrac{y''-y'}{x''-x'}$. Hence its equation is

$$y-y_1=\frac{y''-y'}{x''-x'}(x-x_1). \tag{1}$$

Since $P(x', y')$ and $Q(x'', y'')$ lie on the given parabola, we have

$$y'^2=4ax' \tag{2}$$

and $$y''^2=4ax''. \tag{3}$$

Subtracting (2) from (3), we get

$$y''^2-y'^2=4a(x''-x')$$

$$\Rightarrow \qquad \frac{y''-y'}{x''-x'}=\frac{4a}{y''+y'}=\frac{4a}{2y_1}.$$

Therefore the equation (1) becomes

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

i.e. $$yy_1-2ax=y_1^2-2ax_1.$$

Hence *the equation of the chord of the parabola* $y^2=4ax$ *in terms of its middle point* (x_1, y_1) *is*

$$yy_1-2ax=y_1^2-2ax_1.$$

This equation of the chord can be written as

$$T=S_1,$$

where $$T\equiv yy_1-2a(x+x_1)$$

and $$S_1\equiv y_1^2-4ax_1.$$

7.2 *To find the locus of the middle points of a system of parallel chords of a parabola.*

Let the equation of the parabola be

$$y^2=4ax$$

and m the slope of the chords in the given system of parallel chords.

Let (x_1, y_1) be any point on the locus *i.e.* the middle point of one of the chords parallel to the line $y=mx$. Then the equation of the chord is

$$yy_1-2ax=y_1^2-2ax_1.$$

$$\therefore \qquad m=\frac{2a}{y_1}.$$

$$\Rightarrow \qquad y_1=\frac{2a}{m}.$$

Hence *the locus of the middle points of a system of parallel chords having slope m is*

$$y=\frac{2a}{m}.$$

This is the equation of a line parallel to x-axis *i.e.* the axis of the parabola $y^2=4ax$.

7.3 Examples

Example 1. Find the locus of the middle points of chords of the parabola $y^2=4ax$ which pass through the fixed point (h, k).

Solution. Let (x', y') be the middle point of the chord of the given parabola. Then its equation is

$$yy'-2ax=y'^2-2ax'.$$

This will pass through the point (h, k) if

$$ky'-2ah=y'^2-2ax'.$$

Hence the locus of (x', y') is

$$y^2-ky=2a(x-h).$$

This equation represents a parabola.

Example 2. Prove that the locus of the middle points of normal chords of the parabola $y^2=4ax$ is

$$\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a.$$

Solution. Any normal chord of the given parabola is

$$y=mx-2am-am^3. \tag{1}$$

Let (x', y') be the middle point of (1). Then equation of the chord with middle point (x', y') is

$$yy'-2ax=y'^2-2ax'. \tag{2}$$

But equations (1) and (2) represent the same line. Therefore, on comparing (1) and (2), we get

$$\frac{m}{2a}=\frac{1}{y'}=\frac{-2am-am^3}{y'^2-2ax'}$$

$$\Rightarrow \quad m=\frac{2a}{y'} \quad \text{and} \quad -2am-am^3=\frac{y'^2-2ax'}{y'}.$$

$$\therefore \qquad \frac{4a^2}{y'}-\frac{8a^4}{y'^3}=\frac{y'^2-2ax'}{y'}$$

$$\Rightarrow \qquad \frac{y'^2}{2a}+\frac{4a^3}{y'^2}=x'-2a.$$

Hence locus of (x', y') is

$$\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a.$$

EXERCISES

1. Find the locus of the middle points of chords of the parabola $y^2=4ax$ which
   - (*i*) pass through vertex (0, 0).
   - (*ii*) pass through a fixed point $(a, 0)$.
   - (*iii*) subtend a right angle at the vertex.
   - (*iv*) subtend a constant angle α at the vertex.
2. Two parabolas have a common focus and their axes in opposite directions. Prove that the locus of the middle points of chords of either which touch the other is another parabola.
3. A tangent to the parabola $y^2=4bx$ meets the parabola $y^2=4ax$ in P and Q. Prove that the equation to the locus of the mid-point of PQ is $y^2(2a-b)=4a^2x$.
4. The line $y=m(x+a)$ meets the parabola in two points P and Q. Find the coordinates of the mid-point of PQ in terms of a and m, and prove that the locus of this point is the parabola $y^2=2a(x+a)$.
5. Prove that the locus of the middle point of a variable chord of the parabola $y^2=4ax$ such that the focal distances of its extremities are in the ratio 2 : 1 is

$$9(y^2-2ax)^2=4a^2(2x-a)(4x+a).$$

8. PARAMETRIC COORDINATES

Sometimes it is convenient to express the coordinates of a point on the parabola in terms of a single variable. If we substitute $y=2at$ in the equation of the parabola $y^2=4ax$, we obtain $x=at^2$. That is, as t varies the point $(at^2, 2at)$ always lies on the parabola

$y^2=4ax$. Accordingly, the parabola can be represented by the equations

$$x=at^2,\ y=2at.$$

These equations are called **parametric equations** of the parabola. The point $(at^2, 2at)$ is referred to the point 't' of the curve.

For the parabola $y^2=4ax$, one can easily have the following:

(*i*) The equation of the chord joining the points t_1 and t_2 is

$$(t_1+t_2)\,y=2\,(x+at_1t_2).$$

(*ii*) The equation of the tangent at the point t is

$$ty=x+at^2.$$

(*iii*) The equation of the normal at the point t is

$$tx+y=2at+at^3.$$

8.1 Propositions on the parabola

Proposition I. *The tangent at any point of a parabola bisects the angle between the focal chord through the point and the perpendicular on the directrix from the point.*

Let $P(at^2, 2at)$ be any point on the parabola

$$y^2=4ax.$$

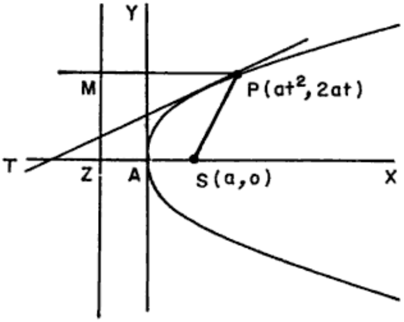


Fig. 7.11

Then equation of the tangent PT at $P(at^2, 2at)$ is

$$ty=x+at^2.$$

Slope of the tangent $PT=\frac{1}{t}$.

Slope of the focal chord $SP=\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1}$.

Now $$\tan SPT=\frac{\frac{2t}{t^2-1}-\frac{1}{t}}{1+\frac{2t}{(t^2-1)}\cdot\frac{1}{t}}=\frac{1}{t}.$$

Thus $$\tan PTS=\tan SPT=\frac{1}{t}$$

$\Rightarrow$ $$\angle PTS=\angle SPT.$$

But $$\angle PTS=\angle TPM.$$

Hence $$\angle TPM=\angle SPT.$$

This prove the proposition.

Proposition II. *The tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.*

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola

$$y^2=4ax.$$

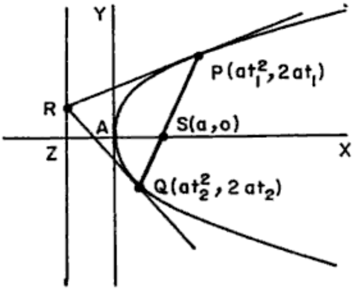


Fig. 7.12

Then equations of the tangents at P and Q are

$$t_1y=x+at_1^2$$

and $$t_2y=x+at_2^2.$$

Let R be the point of intersection of these tangents. Then the coordinates of the point R are given by

$$\{at_1t_2,\ a(t_1+t_2)\}.$$

Further equation of the chord PQ is

$$(t_1+t_2)y=2(x+at_1t_2).$$

This chord will pass through focus $(a, 0)$ if

$$0=2(a+at_1t_2)$$

i.e. if $\qquad t_1t_2=-1.$

Hence the coordinates of the point R become

$$\left\{-a,\ \frac{a(t_1^2-1)}{t_1}\right\}.$$

It is easy to see that the point $R\left\{-a,\ \frac{a(t_1^2-1)}{t_1}\right\}$ lies on the directrix of the parabola.

Further slopes of the tangents at P and Q are $\frac{1}{t_1}$ and $\frac{1}{t_2}$ respectively.

Now slope of the tangent at P × slope of the tangent at Q

$$=\frac{1}{t_1}\cdot\frac{1}{t_2}=-1.$$

This shows that the tangents at the extremities of the focal chord PQ intersect at right angles.

Remarks. (*i*) The extremities of a focal chord of the parabola may be taken as the points 't' and '$-\frac{1}{t}$'.

(*ii*) The tangents to a parabola drawn from any point on the directrix are at right angles.

(*iii*) The locus of the point of intersection of perpendicular tangents to a parabola is the directrix.

Proposition III. *The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.*

Let $P(at^2, 2at)$ be any point on the parabola

$$y^2=4ax. \tag{1}$$

The equation of the tangent at $P(at^2, 2at)$ to the parabola (1) is

$$ty=x+at^2. \tag{2}$$

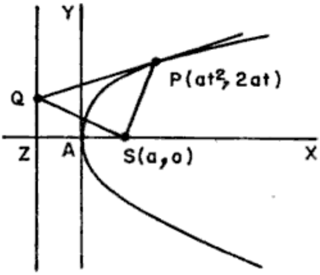


Fig. 7.13

Equation of the directrix of the parabola (1) is

$$x+a=0. \tag{3}$$

By solving (2) and (3) simultaneously, the coordinates of the point of intersection Q of these lines are given by

$$\left\{-a, \frac{a(t^2-1)}{t}\right\}.$$

Now

$$\text{Slope of } QS=\frac{\frac{a(t^2-1)}{t}-0}{-a-a}=-\frac{t^2-1}{2t}.$$

$$\text{Slope of } SP=\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1}.$$

$$\text{The slope of } QS\times\text{slope of } SP=-\left(\frac{t^2-1}{2t}\right)\times\left(\frac{2t}{t^2-1}\right)$$

$$=-1.$$

This shows that QS and SP are at right angles. Hence *the portion PQ of the tangent at P subtends a right angle at the focus.*

Ex. Prove that any tangent to parabola and the perpendicular to it from the focus meet on the tangent at the vertex.

Definition. Let the tangent and the normal at any point P on the parabola meet the axis in the points T and G, respectively. Draw PM perpendicular to the axis. Then TM is called the subtangent and MG the subnormal of the point P.

Proposition IV. *The subtangent of a point on a parabola is bisected at the vertex and the subnormal is of constant length.*

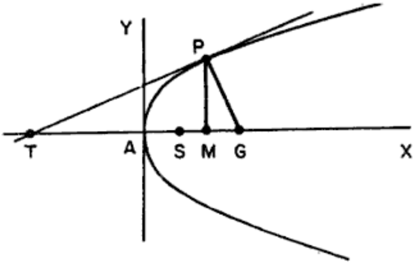


Fig. 7.14

Let $P(at^2, 2at)$ be any point on the parabola

$$y^2=4ax. \tag{1}$$

Then the equation of the tangent at $P(at^2, 2at)$ to the parabola (1) is

$$ty=x+at^2. \tag{2}$$

The line (2) meets the x-axis in the point T (say) whose coordinates are given by

$$(-at^2, 0).$$

$\therefore$ Subtangent $=TM=TA+AM=at^2+at^2=2at^2$,

But $TA=AM=at^2$.

Hence *the subtangent is bisected at the vertex A*.

Further equation of the normal at $P(at^2, 2at)$ is

$$tx+y=2at+at^3.$$

This normal meets the x-axis in the point G (say) whose coordinates are given by

$$(2a+at^2, 0).$$

$$\therefore \quad \text{Subnormal}=MG=AG-AM=2a+at^2-at^2$$
$$=2a=\text{constant}.$$

8.2 Examples

Example 1. The normal at a point t_1 on the parabola $y^2=4ax$ meets it again at the point t_2. Prove that

$$t_2=-t_1-\frac{2}{t_1}.$$

Solution. First Method. The equation of the normal at the point t_1 to the given parabola $y^2=4ax$ is

$$t_1x+y=2at_1+at_1^3.$$

But this normal meets the parabola again at the point t_2.

$$\therefore \quad t_1\cdot at_2^2+2at_2=2at_1+at_1^3$$

$$\Rightarrow \quad at_1(t_2^2-t_1^2)=-2a(t_2-t_1)$$

$$\Rightarrow \quad t_2=-t_1-\frac{2}{t_1}.$$

Second Method. Equation of the chord joining the points t_1 and t_2 on the parabola $y^2=4ax$ is

$$y-2at_1=\frac{2at_2-2at_1}{at_2^2-at_1^2}(x-at_1^2)$$

$$\Rightarrow \quad 2x-(t_1+t_2)y+2at_1t_2=0. \tag{1}$$

The normal at the point t_1

$$t_1x+y-(2at_1+at_1^3)=0 \tag{2}$$

passes through the point t_2. Thus the equations (1) and (2) represent the same line.

$$\therefore \quad 2x-(t_1+t_2)y+2at_1t_2\equiv t_1x+y-(2at_1+at_1^3)$$

$$\Rightarrow \quad \frac{2}{t_1}=\frac{-(t_1+t_2)}{1}=\frac{2at_1t_2}{-(2at_1+at_1^3)}$$

$$\Rightarrow \quad t_2=-t_1-\frac{2}{t_1}.$$

Example 2. If the point $(at^2, 2at)$ is one extremity of a focal chord of the parabola $y^2=4ax$, find the coordinates of the other extremity, and show that the length of the chord is $a\left(t+\frac{1}{t}\right)^2$.

Solution. Let $(at_1^2, 2at_1)$ be the coordinates of the other extremity of the focal chord. Then, we have

$$tt_1=-1.$$

Hence the coordinates of the other extremity are

$$\left(\frac{a}{t^2}, \frac{-2a}{t}\right).$$

Length of the chord $=\sqrt{\left(at^2-\frac{a}{t^2}\right)^2+\left(2at+\frac{2a}{t}\right)^2}$

$$=a\left(t+\frac{1}{t}\right)^2.$$

Example 3. Prove that the locus of the points of intersection of the tangents to the parabola $y^2=4ax$ which intercept a fixed length l on the directrix is

$$(y^2-4ax)(x+a)^2=l^2x^2.$$

Solution. Let the two tangents to the given parabola be

$$t_1y=x+at_1^2 \tag{1}$$

and

$$t_2y=x+at_2^2. \tag{2}$$

Let (h, k) be the point of intersection of (1) and (2). Then

$$h=at_1t_2 \text{ and } k=a(t_1+t_2).$$

The tangents (1) and (2) cut the directrix $x+a=0$ in the points $M\left(-a, \frac{at_1^2-a}{t_1}\right)$ and $N\left(-a, \frac{at_2^2-a}{t_2}\right)$, respectively.

But intercept $MN=l$ (given).

$$\therefore \quad \sqrt{(-a+a)^2+\left(\frac{at_2^2-a}{t_2}-\frac{at_1^2-a}{t_1}\right)^2}=l$$

$$\Rightarrow \quad a\{\sqrt{(t_1+t_2)^2-4t_1t_2}\}\left(1+\frac{1}{t_1t_2}\right)=l$$

$$\Rightarrow \quad a^2\left(\frac{k^2}{a^2}-\frac{4h}{a}\right)\left(1+\frac{a}{h}\right)^2=l^2$$

$$\Rightarrow \quad (k^2-4ah)(h+a)^2=l^2h^2.$$

Hence the locus of (h, k) is

$$(y^2-4ax)(x+a)^2=l^2x^2.$$

Example 4. Prove that the locus of the point of intersection of two tangents to the parabola $y^2=4ax$ which with the tangent at the vertex form a triangle of constant area c^2, is the curve

$$(y^2-4ax)\,x^2=4c^4.$$

Solution. The equations of the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ to the given parabola are

$$t_1y=x+at_1^2, \tag{1}$$

and
$$t_2y=x+at_2^2. \tag{2}$$

Let (h, k) be the point of intersection of these tangents. Then

$$h=at_1t_2 \quad \text{and} \quad k=a(t_1+t_2).$$

The tangents (1) and (2) meet the tangent at the vertex (*i.e.* $x=0$) at $(0, at_1)$ and $(0, at_2)$, respectively. Now

$$\frac{1}{2}\begin{vmatrix} h & k & 1 \\ 0 & at_1 & 1 \\ 0 & at_2 & 1 \end{vmatrix}=c^2$$

$$\Rightarrow \qquad \tfrac{1}{2}h\,(at_1-at_2)=c^2$$

$$\Rightarrow \qquad ah\sqrt{(t_1+t_2)^2-4t_1t_2}=2c^2$$

$$\Rightarrow \qquad a^2h^2\left(\frac{k^2}{a^2}-\frac{4h}{a}\right)=4c^4$$

$$\Rightarrow \qquad h^2\,(k^2-4ah)=4c^4.$$

Hence the locus of (h, k) is

$$(y^2-4ax)\,x^2=4c^4.$$

Example 5. Find the locus of the point of intersection of normals at the ends of a focal chord of the parabola $y^2=4ax$.

Solution. Let $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ be the ends of the focal chord. Then $t_1t_2=-1$.

The equations of the normals at these points are

$$t_1x+y=2at_1+at_1^3, \tag{1}$$

$$t_2x+y=2at_2+at_2^3. \tag{2}$$

Let (h, k) be the point of intersection of (1) and (2). Then

$$h=2a+a(t_1^2+t_2^2+t_1t_2)$$

and
$$k=-at_1t_2(t_1+t_2)=a\,(t_1+t_2).$$

Now

$$h=2a+a\,\{(t_1+t_2)^2-t_1t_2\}$$

$$\Rightarrow \qquad h-2a=a\left(\frac{k^2}{a^2}+1\right)$$

$$\Rightarrow \qquad k^2=a\,(h-3a).$$

Hence the locus of (h, k) is

$$y^2=a(x-3a).$$

Example 6. Prove that the semi-latus rectum is a harmonic mean between the segments of any focal chord of a parabola.

Solution. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the ends of any focal chord of the parabola $y^2=4ax$. Then $t_1t_2=-1$. Focus of the parabola is $S(a, 0)$. Now

$$SP=\sqrt{(at_1^2-a)^2+(2at_1-0)^2}=a(t_1^2+1),$$

and similarly $$SQ=a(t_2^2+1).$$

Therefore, the harmonic mean of SP and SQ

$$=\frac{2SP\cdot SQ}{SP+SQ}$$

$$=\frac{2a^2(t_1^2+1)(t_2^2+1)}{a(t_1^2+1)+a(t_2^2+1)}$$

$$=\frac{2a(t_1^2+t_2^2+t_1^2t_2^2+1)}{t_1^2+t_2^2+2}$$

$$=\frac{2a(t_1^2+t_2^2+2)}{t_1^2+t_2^2+2} \qquad \because \quad t_1t_2=-1$$

$$=2a=\text{semi-latus rectum}.$$

EXERCISES

1. Find the equation of the chord joining two points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2=4ax$. Prove that this chord will pass through the focus of the parabola if $t_1t_2=-1$.
2. Prove that the tangent drawn at one extremity of a focal chord of the parabola $y^2=4ax$ is parallel to the normal drawn at the other extremity.
3. Find the locus of the point of intersection of two normals to the parabola $y^2=4ax$ which are at right angles to one another.
4. Prove that the locus of the point of intersection of two tangents to a parabola which intercept a given distance $4c$ on the tangent at the vertex is an equal parabola.
5. Prove that the locus of the middle points of chords of the parabola $y^2=4ax$ which are of given length l is

$$(y^2-4ax)(y^2+4a^2)+a^2l^2=0.$$

6. Prove that the tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.
7. Find the equation of the normal to $y^2=4ax$ at the point $(at^2, 2at)$ and find the point where this normal meets the parabola again.
8. Find the locus of the mid-point of the chord QR of the parabola $y^2=4ax$ when the tangents at its extremities Q and R are perpendicular to each other. Also, find the locus of the point of intersection of the normals to the parabola at Q and R.
9. Prove that if the difference of the squares of the perpendiculars on a moving line from two fixed points is constant, the line touches a fixed parabola.
10. If the perpendiculars be drawn on any tangent to a parabola from two fixed points on the axis, which are equidistant from the focus, prove that the difference of their squares is constant.
11. The normals at two points P, Q on the parabola $y^2=4ax$ intersect on the curve. Prove that the ordinates of P, Q are the roots of the equation $y^2+ky+8a^2=0$.
12. Two lines AP, AQ are drawn through the vertex of a parabola at right angles to one another, meeting the curve in P, Q. Prove that the line PQ cuts the axis in a fixed point.
13. If the circle $x^2+y^2+2gx+2fy+c=0$ cut the parabola in four points, prove that the algebraic sum of the ordinates of those points will be zero.
14. If PSQ be a focal chord of a parabola, and PA meets the directrix in M; prove that MQ will be parallel to the axis of the parabola.
15. Prove that if two tangents to a parabola intercept a constant length on any fixed tangent, the locus of their point of intersection is another equal parabola.
16. From any point on the latus rectum of a parabola perpendiculars are drawn to the tangents at its extremities. Prove that the line joining the feet of these perpendiculars touches the parabola.
17. Two parabolas have the same axis, tangents are drawn from points on the first to the second. Prove that the middle points of the chords of contact with the second lie on a fixed parabola.

18. If the normals at two points on a parabola intersect on a curve, the line joining the points will pass through a fixed point on the axis.
19. If through a fixed point any chord of a parabola be drawn, and normals be drawn at the ends of the chord. Prove that the locus of the point of intersection of the normals is another parabola.
20. Prove that the locus of the point of intersection of the tangents to the parabola

$$y^2=4ax \text{ at } (at^2, 2at), \left(\frac{a}{t^2}, \frac{2a}{t}\right)$$

is a line parallel to y-axis.
21. If a chord of the parabola $y^2=4ax$ subtends a right angle at the vertex, the tangents at its extremities meet on the line $x+4a=0$.
22. The normal to the parabola $y^2=4ax$ at a point $P(at^2, 2at)$ makes an angle θ with the x-axis and meets the parabola again at Q. Prove that

$$PQ=4a \sec\theta \operatorname{cosec}^2\theta.$$

23. Prove that, if the difference of the ordinates of two points on a parabola is constant, then the locus of the point of intersection of the tangents at these points is an equal parabola.
24. Find the locus of the point of intersection of the normals at two points on a parabola when the chord joining them subtends a right angle at the vertex.
25. Prove that the length of the intercept on the normal at the point $(at^2, 2at)$ made by the circle which is described on the focal distance of the given point as diameter is $a\sqrt{1+t^2}$.
26. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola is

$$y^2(2x+a)=a(3x+a)^2.$$

27. Two equal parabolas have the same focus and their axes are at right angles. A normal to one is perpendicular to a normal to the other. Prove that the locus of the point of intersection of these normals is another parabola.
28. Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

29. Prove that the circle described on any focal chord of a parabola as diameter touches the directrix.

30. A circle on any focal chord of a parabola as diameter cuts the curve again in P and Q. Prove that PQ passes through a fixed point.

31. The normals at the extremities of a chord of the parabola $y^2=4ax$ meet on the parabola. Prove that the middle point of the chord lies on the parabola

$$y^2=2a(x+2a).$$

9. DIAMETER

A line parallel to the axis of the parabola is called a **diameter** *of the parabola.*

As we have seen in § 7.2 that the locus of the middle points of a system of parallel chords to a parabola is a line parallel to the axis of the parabola, it is a diameter of the parabola. Thus

$$y=\frac{2a}{m}$$

is a diameter of parabola $y^2=4ax$, where m is the slope of the parallel chords of the parabola. In fact,

$$y=\frac{2a}{m}$$

represents the family of diameters of the parabola $y^2=4ax$, m being the parameter.

On the other hand it is noted that any line $y=k$ parallel to x-axis (the axis of the parabola) represents the locus of the middle points of a system of parallel chords having slope

$$m=\frac{2a}{k}.$$

Therefore, the diameter of a parabola can alternatively be defined as *the locus of the middle points of a system of parallel chords of the parabola*.

9.1 Propositions on diameters

Proposition I. *To prove that the tangent at the extremity of a diameter of a parabola is parallel to the chords which are bisected by that diameter.*

Let the equation of the parabola be

$$y^2=4ax. \tag{1}$$

Let $y=mx+c$ be a member of a system of parallel chords of the parabola (1). Then equation of the diameter is

$$y=\frac{2a}{m}.$$

The coordinates of the point of intersection of the parabola (1) and the diameter $y=\frac{2a}{m}$ are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The equation of the tangent at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ to the parabola (1) is

$$y\cdot\frac{2a}{m}=2a\left(x+\frac{a}{m^2}\right)$$

$$\Rightarrow \qquad y=mx+\frac{a}{m}\cdot$$

But the line $y=mx+\frac{a}{m}$ is parallel to each member of the system of parallel chords. Hence the proposition follows.

Proposition II. *To prove that the tangents at the ends of any chord of a parabola meet on the diameter which bisects the chord.*

Let the equation of the parabola be

$$y^2=4ax. \tag{1}$$

Let

$$y=mx+c \tag{2}$$

be the equation of any chord of the parabola (1).

Let (x_1, y_1) be the point of intersection of the tangents at the extremities of the chord (2). Then equation of the chord of contact of (x_1, y_1) with respect to the parabola (1) is

$$yy_1=2a(x+x_1). \tag{3}$$

But equations (2) and (3) represent the same line. Thus we have

$$mx-y+c\equiv 2ax-yy_1+2ax_1$$

$$\Rightarrow \qquad \frac{2a}{m}=\frac{-y_1}{-1}=\frac{2ax_1}{c}$$

$$\Rightarrow \qquad x_1=\frac{c}{m},\ y_1=\frac{2a}{m}.$$

Hence locus of (x_1, y_1) is

$$y=\frac{2a}{m},$$

which is the equation of the diameter. Hence the proposition follows.

EXERCISES

1. If the diameter through any point O of a parabola meet any chord in P, and the tangents at the ends of that chord meet the diameter in Q, Q'. Prove that $OP^2=OQ\cdot OQ'$.
2. The normal at a point P of a parabola meets the curve again in Q, and T is the pole of PQ. Prove that T lies on the diameter passing through the other end of the focal chord passing through P, and that PT is bisected by the directrix.
3. Prove that a diameter intersects a parabola at the point of contact of the conjugate tangent line.
4. Prove that the normal at any point of a parabola bisects the angle between the focal chord and the diameter through that point.

MISCELLANEOUS EXERCISES

1. Find the equations of the two parabolas whose latus rectum is 6 and the axis and the tangent at the vertex are the lines whose equations are

   $$3x+4y+1=0, \quad 4x-3y=0.$$

   respectively.
2. A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; the axes being AB and a perpendicular diameter. Prove that the locus of the focus of the parabola is

   $$\frac{x^2}{b^2}+\frac{y^2}{b^2-a^2}=1.$$
3. Prove that the locus of a point, which moves, so that its dis-

14. Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is a line, which is normal to the curve.

15. The normals at P, Q, R on the parabola $y^2=4ax$ meet in a point on the line $y=k$. Prove that the sides of the triangle PQR touch the parabola $x^2=2ky$.

16. Tangents are drawn to the parabola $y^2=4ax$ at points whose abscissae are in the ratio $m^2:1$. Prove that the locus of their points of intersection is the curve

$$y^2=(m^{1/2}+m^{-1/2})^2ax.$$

17. Prove that the length of any focal chord of a parabola is four times the distance of the focus from the point where the diameter bisecting the chord meets the parabola.

18. If P, Q, R are the feet of the normals drawn from a point to the parabola $y^2=4ax$, prove that
    (*i*) the centroid of the triangle PQR lies on the axis of the parabola;
    (*ii*) the circumcircle of the triangle PQR passes through the vertex of the parabola.

19. Find the coordinates of the point of intersection of the tangents $y=mx+\frac{a}{m}$, $y=m'x+\frac{a}{m'}$. Prove that the locus of their intersection is a line whenever mm' is constant; and that, when $mm'=-1$, this line is the directrix.

20. The perpendicular TN from any point T on its polar with respect to a parabola meets the axis in M. Prove that if $TN.TM$ is constant the locus of T is a parabola.

21. If the normals at two points of a parabola be inclined to the axis at angles θ, ϕ such that $\tan\theta\tan\phi=2$. Prove that the locus of their intersection is a parabola.

22. Prove that the locus of the feet of the perpendiculars from the point $(a, 0)$ to the normals of the parabola is $y=a(x-a)$.

23. From a fixed point P on the parabola $y^2=4ax$ chords PQ, PQ' are drawn making equal angles ϕ with the tangent at P. Prove that for all values of ϕ, QQ' will pass through the same point R. Prove further that if P moves along the parabola, the locus of R is

$$(x+2a)y^2+4a^3=0.$$

24. A parabola is drawn with its axis parallel to a fixed line and passes through a fixed point and touches a given line. Prove that the locus of its vertex is a parabola.

25. From any point P on a parabola $y^2=4ax$, lines PQ and PR are drawn normal to the parabola at Q and R, respectively. Prove that QR passes through the point $(-2a, 0)$.

26. A chord PQ of a parabola is normal at P, and subtends a right angle at the vertex. If S is the focus, prove that

$$SQ=3SP.$$

27. If the variable chord PQ of the parabola $y^2=4ax$ subtends a right angle at the vertex, prove that it cuts the axis at a fixed point. Prove also that the tangents at P and Q meet on a fixed line.

28. If two points of the axis of a parabola are equidistant from the focus, prove that the difference of the squares of their distances from any tangent is independent of its position.

29. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

30. If the normals at P and Q meet on the parabola, prove that the point of intersection of the tangents at P and Q lies either on a certain line, which is parallel to the tangent at the vertex, or on the curve whose equation is

$$(x+2a)y^2+4a^3=0.$$

31. Prove that the locus of the centre of the circle, which passes through the vertex of a parabola and through its intersections with a normal chord, is the parabola

$$2y^2=a(x-a).$$

32. The sides of a triangle touch a parabola, and two of its angular points lie on another parabola with its axis in the same direction. Prove that the locus of the third angular point is another parabola.

33. PP' is any one of a system of a parallel chords of a parabola, O is a point on PP' such that $PO.OP'$ is constant. Prove that the locus of O is a parabola.

34. If a tangent to a parabola cut two given parallel lines in P, Q, the locus of the point of intersection of the other tangents from P, Q to the curve will be a parabola.

35. TP, TQ are tangents to a parabola, and p_1, p_2, p_3 are the lengths of the perpendiculars from P, T, Q respectively on any other tangent to the curve. Prove that $p_1p_3=p_2^2$.

36. A circle cuts the parabola $y^2=4ax$ at right angles and passes through the focus. Prove that its centre lies on the curve

$$y^2(2x+a)=a(3x+a)^2.$$

37. The product of the tangents drawn from a point P to the parabola $y^2=4ax$ is equal to the product of the focal distance of P and the latus rectum. Prove that the locus of P is the parabola

$$y^2=4a(x+a).$$

38. Prove that if a chord of a parabola meets the axis in a fixed point, the normals at its extremities intersect on another fixed parabola. When does this parabola coincide with the original curve?

CHAPTER VIII

THE ELLIPSE

1. DEFINITION

An **ellipse** *is the locus of a point which moves such that its distance from a fixed point is in a constant ratio* (<1) *to the perpendicular distance of it from a fixed line.*

The fixed point is called the **focus**, constant ratio the **eccentricity** and the fixed line the **directrix.**

1.1 Standard equation of an ellipse. *To find the equation of an ellipse in standard form.*

Let S be the focus, ZK the directrix and SZ the perpendicular from S to the directrix. Let e be the eccentricity of the ellipse.

Let A divide SZ such that

$$SA:AZ=e:1$$

Then A lies on the curve.

Since $e<1$, there is another point A' on ZS produced such that

$$SA':ZA'=e:1.$$

Thus $$SA=e.AZ$$

and $$SA'=e.ZA'.$$

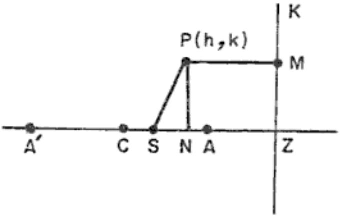


Fig. 8.1

Let C be the middle point of AA' and let $AA'=2a$.

Then $$SA+SA'=e(AZ+ZA')$$

$$\Rightarrow \quad AA'=e.2CZ$$

$$\Rightarrow \quad 2a=e.2CZ$$

$$\Rightarrow \quad CZ=\frac{a}{e}.$$

Further

$$SA'-SA=e(ZA'-AZ)$$

$$\Rightarrow \quad SA'+SA-2SA=e.AA'$$

$$\Rightarrow \quad 2CA-2SA=e.AA'$$

$$\Rightarrow \quad CS=ae.$$

Let us take C to be the origin, CA the x-axis and a line through C perpendicular to CA the y-axis.

Let $P(h, k)$ be any point on the locus (ellipse). Draw PM perpendicular from P to the directrix ZK.

Then, be definition of ellipse, we have

$$SP=e.PM.$$

But $$SP=\sqrt{(h-ae)^2+k^2}$$

and $$PM=NZ=CZ-CN$$

$$=\frac{a}{e}-h.$$

$$\therefore \quad (h-ae)^2+k^2=e^2\left(\frac{a}{e}-h\right)^2$$

$$\Rightarrow \quad h^2(1-e^2)+k^2=a^2(1-e^2)$$

$$\Rightarrow \quad \frac{h^2}{a^2}+\frac{k^2}{a^2(1-e^2)}=1.$$

Since $e<1$, $1-e^2$ is positive, so that we may write

$$a^2(1-e^2)=b^2$$

and choose b to be positive. Hence the locus of (h, k) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is the required equation of an ellipse in its standard form.

1.2 General equation of an ellipse. *To find the general equation of an ellipse whose focus is* (α, β) *and the directrix is the line*

$$ax+by+c=0.$$

Let e be the eccentricity of the ellipse. Let $P(h, k)$ be any point on the locus (ellipse).

Then, by definition

$$SP=e.PM.$$

$$\Rightarrow \qquad (h-\alpha)^2+(k-\beta)^2=e^2\left(\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right)^2$$

Hence locus of (h, k) is

$$(x-\alpha)^2+(y-\beta)^2=\frac{e^2(ax+by+c)^2}{a^2+b^2}.$$

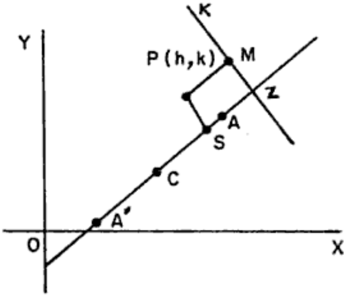


Fig. 8.2

Ex. Find the equation of an ellipse whose focus is $(-1, 1)$, directrix is $x-y+3=0$ and eccentricity is $\frac{1}{2}$.

1.3 Shape of an ellipse. *To trace the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

In order to sketch the graph of the given ellipse we notice the following:

(*i*) The curve is symmetrical about both axes and the origin since if (x, y) lies on the curve, the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$ also lie on the curve.

(ii) The curve meets the x-axis in the points $A(a, 0)$, $A'(-a, 0)$ and the y-axis in the points $B(0, b)$, $B'(0, -b)$.

(iii) Writing the equation of the ellipse in either of the forms

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

or

$$x = \pm\frac{a}{b}\sqrt{b^2-y^2}.$$

We see that y is real if and only if $-a \leqslant x \leqslant a$ whereas x is real if and only if $-b \leqslant y \leqslant b$. Thus the curve is limited and bounded. In fact, it lies entirely inside the rectangle $\{(x, y) : -a \leqslant x \leqslant a, -b \leqslant y \leqslant b\}$.

Therefore in view of the points noticed above, we need only to examine its shape in the first quadrant. We find that as x increases from 0 to a, y decreases from b to 0

Hence the ellipse has the shape as shown in Fig. 8.3.

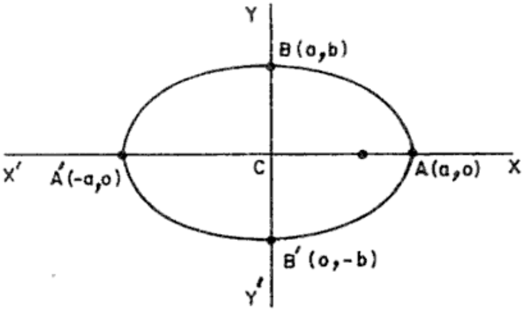


Fig. 8.3

Note. The ellipse occurs commonly in everyday life. It is the shape seen when any circle is viewed obliquely, the section of a circular cylinder by any plane not perpendicular to the axis of the cylinder. It is also the general path of any planet, including our own, in its orbit round the sun.

The points A and A' are called the vertices of the ellipse.

The two lines AA' and BB' with respect to which the curve is symmetrical are called the axes of the ellipse. The axis AA' is of length $2a$ and is called the **major axis** whereas the axis BB' is of length $2b$ and is called the **minor axis** since $a>b$. □

Since the curve is symmetrical about the origin, the point C (the origin) bisects every chord through it. Therefore, the point C is referred to the **centre** of the ellipse.

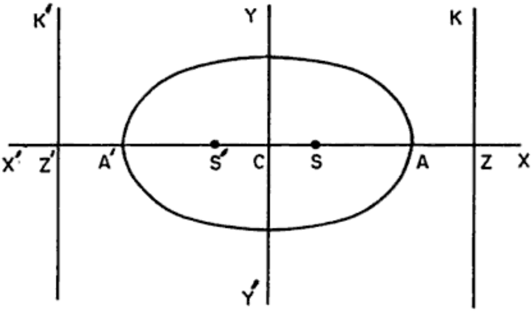


Fig. 8.4

Further, the symmetry of the curve about its minor axis (y-axis) verifies that there must be a second focus S' situated on the major axis at the same distance from C as S, and a second directrix corresponding to S' and parallel to the original directrix which would be cutting the major axis produced in Z' such that

$$CZ'=CZ.$$

Thus the coordinates of the foci S and S' are $(ae, 0)$ and $(-ae, 0)$, respectively.

The equations of the directrices ZK and $Z'K'$ are

$$x=\frac{a}{e} \quad \text{and} \quad x=-\frac{a}{e},$$

respectively. □

If from a point P on the curve PN be drawn perpendicular to the major axis, and is produced to meet the curve again in P', PN is called the **ordinate** and PNP' the **double ordinate** of the point P.

A double ordinate through a focus is called a **latus rectum**[*].

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[*] In general latus rectum of an ellipse is the chord through a focus perpendicular to the major axis.

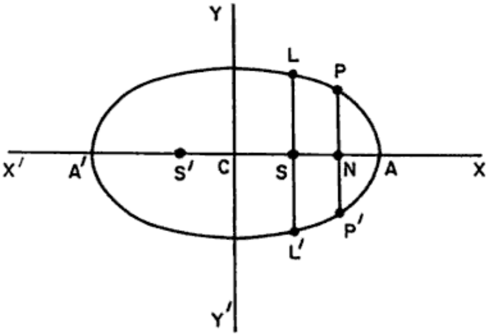

Fig. 8.5

Since there are two foci for an ellipse, there must be two latera recta for the ellipse. The equations of the latera recta are $x=\pm ae$.

To find the length of the semi-latus rectum.

Let LSL' be the latus rectum through the focus $S(ae, 0)$ and let $SL=l$. Then the coordinates of the point L are (ae, l). But L lies on the ellipse. Therefore

$$\frac{a^2e^2}{a^2}+\frac{l^2}{b^2}=1$$

$$\Rightarrow \qquad l^2=b^2(1-e^2)$$

$$=\frac{b^4}{a^2}.$$

Hence the length of the semi-latus rectum is $\frac{b^2}{a}$.

Note. If $b>a$, then for the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we note the following:

(*i*) The major axis is along y-axis and is of length $2b$.

(*ii*) The minor axis is along x-axis and is of length $2a$.

(*iii*) The eccentricity is given by

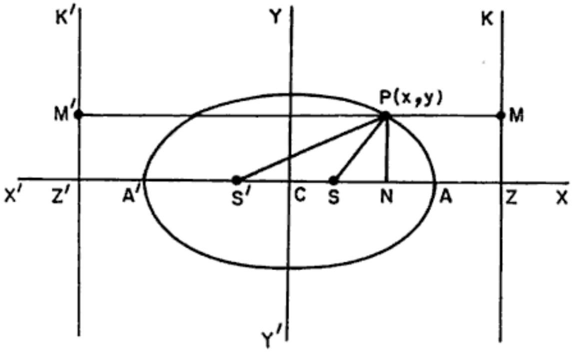

Fig. 8.7

$$=e\cdot\left(\frac{a}{e}-x\right)$$

$$=a-ex.$$

and $$S'P=e\cdot M'P$$

$$=e\cdot Z'N$$

$$=e\cdot(Z'C+CN)$$

$$=e\left(\frac{a}{e}+x\right)$$

$$=a+ex.$$

Thus $$SP+S'P=2a. \quad \square$$

An ellipse may also be defined as *the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

To find the equation of the curve from the above definition.

Let the distance between the two fixed points S and S' be $2ae$.

Let C be the middle point of SS'. Taking CS as x-axis and a line perpendicular to it through C as y-axis. Let $P(h, k)$ be the moving point so that

$$SP+S'P=\text{constant}=2a \text{ (say)}.$$

But $$SP=\sqrt{(h-ae)^2+k^2}$$

and $$S'P=\sqrt{(h+ae)^2+k^2}.$$

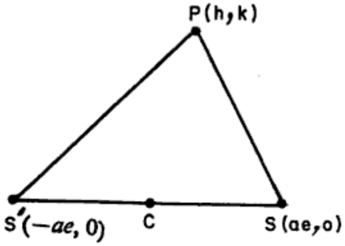

Fig. 8.8

$$\therefore \qquad \sqrt{(h-ae)^2+k^2}+\sqrt{(h+ae)^2+k^2}=2a$$

$$\Rightarrow \qquad h^2(1-e^2)+k^2=a^2(1-e^2).$$

Hence the locus of (h, k) is

$$x^2(1-e^2)+y^2=a^2(1-e^2)$$

$$\Rightarrow \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $\qquad b^2=a^2(1-e^2).$

EXERCISES

1. Find the equation of an ellipse referred to its axes as coordinate axes

   (*i*) whose eccentricity is $\frac{1}{2}$ and foci are $(\pm\sqrt{2}, 0)$.

   (*ii*) whose focus is $(2, 0)$ and vertex is $(5, 0)$.

   (*iii*) whose eccentricity is $\frac{\sqrt{3}}{2}$ and latus rectum is 2.

   (*iv*) which passes through $(3, 5)$ and $\left(7, \frac{5}{3}\right)$.

   (*v*) whose end of the minor axis is $(5, 0)$ and length of the latus rectum is $\frac{50}{13}$.

2. Find the eccentricities, foci, directrices and latera recta of the following ellipses:

   (*i*) $4x^2+9y^2=144$.

(*ii*) $x^2 \tan^2 \alpha + y^2 \sec^2 \alpha = 1.$

(*iii*) $\frac{x^2}{169} + \frac{y^2}{25} = 1.$

(*iv*) $x^2 + 4y^2 = 9.$

(*v*) $25x^2 + 4y^2 = 100.$

3. Find the locus of a point which moves so that the sum of its distances from $(-4, 3)$ and $(4, 3)$ is 12.
4. The perimeter of a triangle is 20, and the points $(-2, -3)$ and $(-2, 3)$ are two of the vertices of it. Find the equation of the locus of the third vertex.

2. CIRCLE AND PARABOLA AS THE LIMITING CASES OF THE ELLIPSE

We shall see that by taking e sufficiently small so much so that by taking $e \to 0$ and keeping a constant the ellipse tends to become a circle while taking e sufficiently near to 1 and moving the centre off to a great distance in such a way that the vertex A' and focus at S' remains unchanged the ellipse becomes the parabola.

2.1 The circle as a limiting case of the ellipse. It is clear that for $b=a$, the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

takes the form $x^2 + y^2 = a^2$ which is a circle of radius a, and in this case the eccentricity $e=0$. •

Further, we note that if e is made sufficiently small keeping a unchanged, the quantity ae becomes very small and so the foci $(\pm ae, 0)$ approach the centre of the ellipse. Also b^2 $[=a^2(1-e^2)]$ approaches to a^2. Thus, as the foci approach the centre, the ellipse tends to become more circular in appearance, and when e is sufficiently small, rather, nearly equal to zero the ellipse becomes almost a circle. Hence, we say that a circle of radius a is the limiting case of an ellipse whose major axis is of length $2a$ and eccentricity tends to zero.

2.2 The parabola as the limiting case of the ellipse. Shifting the origin to the vertex $A'(-a, 0)$, the equation of the ellipse

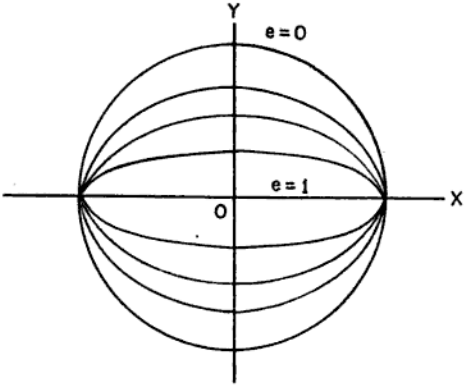

Fig. 8.9

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

takes the form

$$\frac{(X-a)^2}{a^2}+\frac{Y^2}{b^2}=1$$

i.e. $$\frac{X^2}{a}-2X+\frac{Y^2}{a\,(1-e^2)}=0. \qquad (1)$$

Now $A'S'=a\,(1-e)$. Let us denote the distance $A'S'$ by d so that

$$a=\frac{d}{1-e}.$$

Thus, the equation (1) becomes

$$\frac{X^2\,(1-e)}{d}-2X+\frac{Y^2}{d\,(1+e)}=0.$$

Now suppose A' and S' remain fixed so that d remains constant and finite. Make e sufficiently near to 1 so that a becomes large and large *i.e.* the centre moves off very far. In these circumstances the above equation approximates to

$$Y^2=4dX.$$

This represents a parabola. Thus, under the limiting circumstances described above, the ellipse approximates to a parabola.

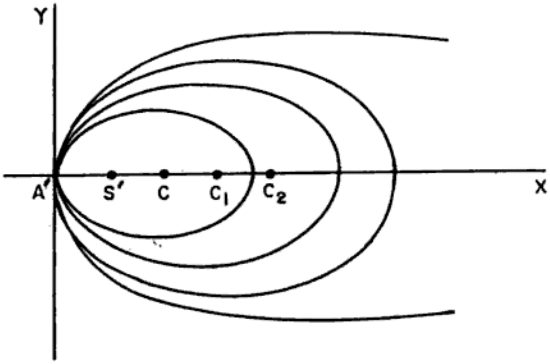

Fig. 8.10

Hence a parabola may be regarded as a limiting case of the ellipse.

3. SOME IMPORTANT RESULTS

Many of the results developed for circle and parabola in the preceding chapters also hold good for the ellipse. We shall, therefore, only enumerate them and the reader is advised to work out the detailed proofs of them.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(*i*) The equation of the tangent at the point (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

(*ii*) The line $y=mx+c$ is tangent to the ellipse if

$$c=\sqrt{a^2m^2+b^2}.$$

(*iii*) The equation of the normal at (x_1, y_1) to the ellipse is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}.$$

(*iv*) The equation of the pair of tangents from (x_1, y_1) to the ellipse is

$$T^2=SS_1,$$

where
$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

$$S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1.$$

(*v*) The equation of the chord of contact of the tangents drawn from (x_1, y_1) to the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

(*vi*) The equation of the polar of the point (x_1, y_1) with respect to the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

(*vii*) The pole of the line $lx+my+n=0$ with respect to the ellipse is

$$\left(-\frac{a^2l}{n},\ -\frac{b^2m}{n}\right).$$

(*viii*) The equation of the chord with given middle point (x_1, y_1) is

$$T=S_1.$$

where S_1 and T are as defined in (*iv*) above.

(*ix*) The locus of the middle points of all chords parallel to $y=mx$ of the ellipse is

$$y=-\frac{b^2}{a^2m}\,x$$

which is a line through the centre of the ellipse.

Ex. 1. Prove that the tangent at the point (x_1, y_1) to the ellipse is parallel to the tangent at the point $(-x_1, -y_1)$.

Ex. 2. Prove that the polar of the focus $(ae, 0)$ is the corresponding directrix.

3.1 Examples

Example 1. If the normal at one end of a latus rectum of an ellipse passes through one extremity of the minor axis, prove that the eccentricity of the curve is given by the equation

$$e^4+e^2-1=0.$$

Solution. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The coordinates of one end of a latus rectum of the ellipse are $\left(ae, \frac{b^2}{a}\right)$. Then the equation of the normal at the point $\left(ae, \frac{b^2}{a}\right)$ to the ellipse is

$$\frac{x-ae}{\frac{ae}{a^2}}=\frac{y-\frac{b^2}{a}}{\frac{\frac{b^2}{a}}{b^2}}$$

$$\Rightarrow \qquad ax-a^2e=eay-b^2e$$

$$\Rightarrow \qquad x-ey=ae^3.$$

This normal will pass through the extremity $(0, -b)$ of the minor axis if

$$0-e(-b)=ae^3$$

$$\Rightarrow \qquad ea\sqrt{1-e^2}=ae^3$$

$$\Rightarrow \qquad e^4+e^2-1=0.$$

Example 2. A variable chord subtends a right angle at the centre of the ellipse. Find the locus of the point of intersection of the tangents at the ends.

Solution. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let (h, k) be the point of intersection of the tangents drawn at the ends of the variable chord. Then it is the chord of contact and hence its equation is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1.$$

The equation of the lines joining the centre (origin) to the points of intersections of the ellipse and the chord of contact is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xh}{a^2}+\frac{yk}{b^2}\right)^2$$

i.e. $$\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)x^2-\frac{2hk}{a^2b^2}xy+\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)=0.$$

These lines are at right angles if

$$\text{coeff. of } x^2+\text{coeff. of } y^2=0$$

i.e. $$\text{if } \frac{1}{a^2}-\frac{h^2}{a^4}+\frac{1}{b^2}-\frac{k^2}{b^4}=0.$$

Hence the locus of (h, k) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Example 3. Find the locus of the poles of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the parabola $y^2=4ax$.

Solution. Any tangent to the given ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}. \tag{1}$$

Let (h, k) be the pole of (1). Then its polar with respect to the parabola $y^2=4ax$ is

$$ky=2a(x+h). \tag{2}$$

But equations (1) and (2) are identical. Therefore we have

$$\frac{m}{2a}=\frac{1}{k}=\frac{\sqrt{a^2m^2+b^2}}{2ah}$$

$$\Rightarrow \qquad m=\frac{2a}{k} \text{ and } \sqrt{a^2m^2+b^2}=\frac{2ah}{k}.$$

$$\therefore \qquad a^2\left(\frac{4a^2}{k^2}\right)+b^2=\frac{4a^2h^2}{k^2}$$

$$\Rightarrow \qquad b^2k^2=4a^2(h^2-a^2)$$

Hence the locus of (h, k) is

$$b^2y^2=4a^2(x^2-a^2).$$

Example 4. Prove that the locus of the middle points of chords

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}.$$

Solution. Let (h, k) be the point on the locus. Then its polar with respect to the given ellipse is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1.$$

Since c is the perpendicular distance of the polar from the centre of the ellipse, we have

$$\frac{1}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}}=c$$

$$\Rightarrow \qquad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{c^2}.$$

Hence the locus of (h, k) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}.$$

Example 6. From a point on the circle $x^2+y^2=a^2$, tangents are drawn to the ellipse $x^2/a^2+y^2/b^2=1$. Prove that the locus of the middle points of the chords of contact is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2}.$$

Solution. Let (h, k) be the middle point of the chord of contact of the ellipse. Then its equation is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$$\Rightarrow \qquad \frac{x}{a^2}\left(\frac{h}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\right)+\frac{y}{b^2}\left(\frac{k}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\right)=1.$$

Clearly this is the chord of contact of the tangents drawn from the point

$$\left(\frac{h}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}, \frac{k}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\right)$$

to the given ellipse. But this point lies on the circle $x^2+y^2=a^2$. Therefore

$$\frac{h^2}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2}+\frac{k^2}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2}=a^2$$

$$\Rightarrow \qquad \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=\frac{h^2+k^2}{a^2}.$$

Hence the locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2}.$$

Example 7. Prove that the locus of the middle points of chords of constant length $2c$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0.$$

Solution. Let (h, k) be the middle point of the chord of length $2c$. Let α be the inclination of this chord with x-axis. Then $(h+c \cos \alpha, k+c \sin \alpha)$ and $(h-c \cos \alpha, k-c \sin \alpha)$ are the coordinates of the extremities of the chord. But these points lie on the given ellipse. Therefore

$$\frac{(h+c\cos\alpha)^2}{a^2}+\frac{(k+c\sin\alpha)^2}{b^2}=1 \tag{1}$$

and

$$\frac{(h-c\cos\alpha)^2}{a^2}+\frac{(k-c\sin\alpha)^2}{b^2}=1. \tag{2}$$

Subtracting (2) from (1), we get

$$\frac{4ch\cos\alpha}{a^2}+\frac{4ch\sin\alpha}{b^2}=0$$

$$\Rightarrow \qquad \tan\alpha=-\frac{\frac{h}{a^2}}{\frac{k}{b^2}}$$

$$\Rightarrow \qquad \sin\alpha=-\frac{\frac{h}{a^2}}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}},\ \cos\alpha=-\frac{\frac{k}{b^2}}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}}.$$

On adding (1) and (2), we get

$$2\left[\frac{h^2}{a^2}+\frac{k^2}{b^2}+c^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}\right)\right]=2$$

$$\Rightarrow \quad \frac{h^2}{a^2}+\frac{k^2}{b^2}+c^2\left[\frac{1}{a^2}\cdot\frac{\frac{k^2}{b^4}}{\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)}+\frac{1}{b^2}\cdot\frac{\frac{h^2}{a^2}}{\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)}\right]=1$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)=0.$$

Hence the locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0.$$

EXERCISES

1. Find the equation of the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which makes an angle of 60° with x-axis.

2. Prove that the line $x \cos\alpha+y\sin\alpha=p$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$a^2\cos^2\alpha+b^2\sin^2\alpha=p^2.$$

3. Prove that the line $lx+my+n=0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$a^2l^2+b^2m^2=n^2.$$

4. Find the locus of the foot of the perpendicular drawn from a focus to the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

5. Prove that the product of the perpendiculars drawn from the points $(c, 0)$, $(-c, 0)$ on the tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is equal to b^2 if $c^2=a^2-b^2$.

recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and prove that they pass through the intersections of the axis and the directrices.

22. Tangents are drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the circle $x^2+y^2=r^2$. Prove that the chords of contact are tangents to the ellipse $a^2x^2+b^2y^2=r^4$.

23. Find the locus of the poles of tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with respect to the concentric ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$.

4. DIRECTOR CIRCLE

To find the locus of the point of intersection of tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *which meet at right angles.*

Let (h, k) be any point on the locus. The equation of a tangent to the ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}.$$

This will pass through the point (h, k) if

$$k=mh+\sqrt{a^2m^2+b^2}$$

$$\Rightarrow \qquad (h^2-a^2)m^2-2hkm+k^2-b^2=0. \qquad (1)$$

Equation (1) is quadratic in m. This gives two roots say m_1 and m_2. Thus there are two tangents passing through (h, k). These tangents are at right angles if

$$m_1m_2=-1$$

i.e. if

$$\frac{k^2-b^2}{h^2-a^2}=-1$$

$$\Rightarrow \qquad h^2+k^2=a^2+b^2.$$

Hence the locus of (h, k) is

$$x^2+y^2=a^2+b^2.$$

This is the required equation of the locus which represents a circle.

The circle $x^2+y^2=a^2+b^2$ *is called the* **director circle** *or* **orthoptic circle** *of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

5. AUXILIARY CIRCLE

The circle described on the major axis of an ellipse as diameter is called the **auxiliary circle.**

If the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{1}$$

the equation of the auxiliary circle will be

$$x^2+y^2=a^2. \tag{2}$$

Thus, if any ordinate NP of the ellipse be produced to meet the auxiliary circle in Q, we have from (1) and (2)

$$\frac{CN^2}{a^2}+\frac{NP^2}{b^2}=1$$

and $$CN^2+NQ^2=a^2.$$

$$\therefore \qquad \frac{a^2-NQ^2}{a^2}+\frac{NP^2}{b^2}=1$$

$$\Rightarrow \qquad \frac{NQ}{NP}=\frac{a}{b}.$$

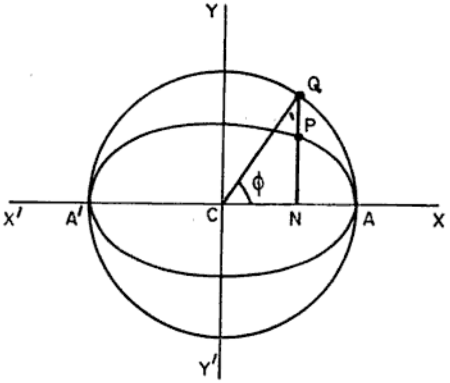

Fig. 8.11

Hence the ordinates of the circle and of the ellipse are in a constant ratio to one another.

Ex. Perpendiculars are drawn from the points on a given circle upon a diameter. Prove that the locus of the points which divide these perpendiculars in a constant ratio is an ellipse of which the given circle is the auxiliary circle.

One may define the ellipse alternatively as follows:

Given a circle and from each point on it draw perpendiculars upon a diameter. The locus of the points dividing these perpendiculars in a given ratio is an ellipse, of which the given circle is the auxiliary circle.

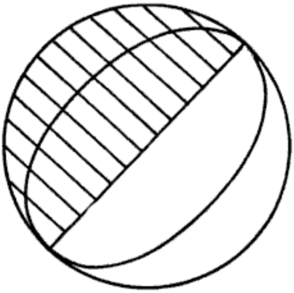

Fig. 8.12

6. ECCENTRIC ANGLES

Let P be a point on an ellipse and Q the point on the auxiliary circle such that Q lies on the ordinate produced through the point P. Then the angle ACQ is called the eccentric angle of P. The eccentric angle is usually denoted by 'ϕ'.

Now, we introduce a very important and useful method of representing any point P on an ellipse in terms of the eccentric angle of that point.

We note that

$$CN = a \cos \phi.$$

To find the y-coordinate of the point P, we have

$$\frac{a^2 \cos^2 \phi}{a^2} + \frac{PN^2}{b^2} = 1$$

$$\Rightarrow \qquad PN = \pm b \sin \phi.$$

It is easy to see that the positive sign must be taken in all cases, for the y-coordinate of P it is positive if ϕ lies between 0 and π, and negative if ϕ lies between π and 2π.

Thus, the coordinates of any point on the ellipse can be expressed as $(a \cos \phi, b \sin \phi)$, ϕ being the eccentric angle of the point.

The point $(a \cos \phi, b \sin \phi)$ is also referred to the point ϕ.

The equations

$$x=a \cos \phi,$$

$$y=b \sin \phi$$

can be regarded as the parametric form of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where ϕ is taken as a parameter.

6.1 *To find the equation of the chord joining two points whose eccentric angles are given.*

Let ϕ_1, ϕ_2 be the eccentric angles of the two points A and B on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then the coordinates of the points are $(a \cos \phi_1, b \sin \phi_1)$ and $(a \cos \phi_2, b \sin \phi_2)$, respectively.

The equation of the chord AB is

$$y-b \sin \phi_1=\frac{b \sin \phi_2-b \sin \phi_1}{a \cos \phi_2-a \cos \phi_1}(x-a \cos \phi_1)$$

$$\Rightarrow \qquad y-b \sin \phi_1=\frac{b \cos \dfrac{\phi_1+\phi_2}{2}}{-a \sin \dfrac{\phi_1+\phi_2}{2}}(x-a \cos \phi_1)$$

$$\Rightarrow \qquad bx \cos \frac{\phi_1+\phi_2}{2}+ay \sin \frac{\phi_1+\phi_2}{2}=ab \cos \frac{\phi_1-\phi_2}{2}.$$

Hence the equation of the chord joining the points ϕ_1 and ϕ_2 on the ellipse is

$$\frac{x}{a} \cos \frac{\phi_1+\phi_2}{2}+\frac{y}{b} \sin \frac{\phi_1+\phi_2}{2}=\cos \frac{\phi_1-\phi_2}{2}.$$

Corollary 1. The equation of the tangent at the point $(a \cos \phi, b \sin \phi)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a} \cos \phi+\frac{y}{b} \sin \phi=1$.

$$\Rightarrow \qquad \phi=2n\pi+2\tan^{-1} t_1.$$

These all give the same point on the ellipse, for by adding any multiple of 2π to the eccentric angle we reach the same point.

Hence each value of $\tan\frac{\phi}{2}$ gives just one point on the ellipse, real or imaginary.

Thus there are four normals that can be drawn from a point (x_1, y_1) to the ellipse.

7. PROPOSITIONS ON ELLIPSE

Proposition I. *The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.*

Let $P(x_1, y_1)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

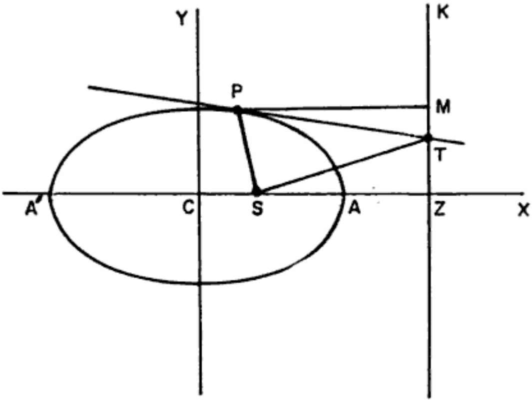

Fig. 8.13

The equation of the tangent at $P(x_1, y_1)$ to the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

Equation of the directrix ZK is

$$x=\frac{a}{e}.$$

The coordinates of the point of intersection of the tangent and the directrix are

$$\left(\frac{a}{e}, \frac{a(1-e^2)(ae-x_1)}{ey_1}\right).$$

Now $$m_1=\text{slope of } SP=\frac{y_1-0}{x_1-ae}=-\frac{y_1}{ae-x_1}.$$

$$m_2=\text{slope of } ST=\frac{\frac{a(1-e^2)(ae-x_1)}{ey_1}}{\frac{a}{e}-ae}=\frac{ae-x_1}{y_1}.$$

$\therefore$ $$m_1m_2=-\left(\frac{y_1}{ae-x_1}\right)\left(\frac{ae-x_1}{y_1}\right)=-1.$$

Hence $\angle TSP$ is a right angle.

Proposition II. *The tangent and the normal at any point of an ellipse bisect the angles between the focal radii to that point.*

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

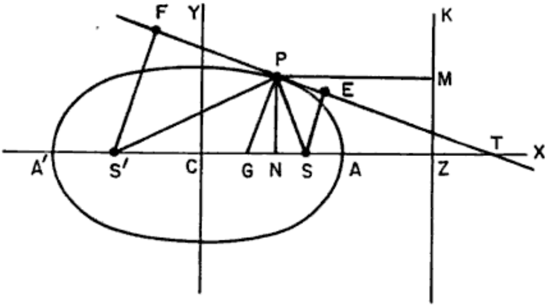

Fig. 8.14

Let $P(x_1, y_1)$ be any point on the ellipse. Let the tangent and normal at $P(x_1, y_1)$ meet the major axis in the points T and G respectively.

The equation to the normal PG is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}.$$

The coordinates of G are $(e^2x_1, 0)$.

$$\therefore \qquad GS=CS-CG=ae-e^2x_1=e(a-ex_1).$$

Also $$SP=e.PM=e.NZ=e(CZ-CN)=e\left(\frac{a}{e}-x_1\right)$$
$$=a-ex_1.$$

$$\therefore \qquad GS=e.SP.$$

Similarly $$S'G=e.S'P.$$

$$\therefore \qquad SP:S'P=GS:S'G.$$

Hence by geometry, PG bisects the angle SPS'.

It follows that the tangent PT bisects the exterior angle between the focal radii.

Proposition III. *If the normal at any point P meet the major and minor axes in G and H, and if CM be the perpendicular upon this normal, then*

$$PM.PG=b^2 \text{ and } PM.PH=a^2.$$

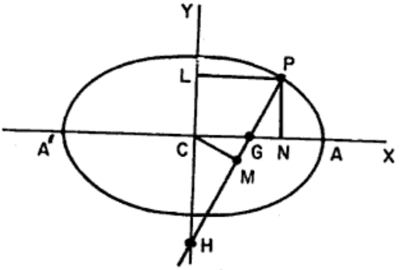

Fig. 8.15

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the tangent at at $P(a\cos\phi, b\sin\phi)$ to the ellipse is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

$\therefore$ PM=perpendicular distance of the tangent from C

7.1 Examples

Example 1. Prove that the locus of the poles of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Solution. The equation of the normal at the point $(a\cos\theta, b\sin\theta)$ on the given ellipse is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2. \tag{1}$$

Let (h, k) be the pole of the line (1). Then its polar with respect to the given ellipse is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1. \tag{2}$$

Since (1) and (2) represent the same line, we have

$$-ax\sec\theta+by\operatorname{cosec}\theta+a^2-b^2\equiv\frac{xh}{a^2}+\frac{yk}{b^2}-1$$

$$\Rightarrow\quad \frac{-a\sec\theta}{\frac{h}{a^2}}=\frac{b\operatorname{cosec}\theta}{\frac{k}{b^2}}=\frac{a^2-b^2}{-1}$$

$$\Rightarrow\quad \cos\theta=\frac{a^3}{h(a^2-b^2)},\ \sin\theta=\frac{-b^3}{k(a^2-b^2)}.$$

$$\therefore\quad \cos^2\theta+\sin^2\theta=\frac{a^6}{h^2(a^2-b^2)^2}+\frac{b^6}{k^2(a^2-b^2)^2}$$

$$\Rightarrow\quad \frac{a^6}{h^2}+\frac{b^6}{k^2}=(a^2-b^2)^2.$$

Hence locus of (h, k) is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Example 2. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on the ellipse such that the normals at them are concurrent, prove that $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

Solution. Let the normals be concurrent at (x_1, y_1). Then $\tan\frac{\alpha}{2}$,

$\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$ and $\tan\frac{\delta}{2}$ are the roots of the equation (1) in § 6.2.

By theory of equations, we have

$$s_1=\Sigma\tan\frac{\alpha}{2}=\frac{-2(ax_1+a^2-b^2)}{by_1},$$

$$s_2=\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=0,$$

$$s_3=\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=\frac{-2(ax_1-a^2+b^2)}{by_1}$$

and $$s_4=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=-1.$$

Now $$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}$$

$$=\frac{s_1-s_3}{0}=\infty$$

$$\Rightarrow \quad \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}=\text{an odd multiple of }\frac{\pi}{2}$$

$$\Rightarrow \quad \alpha+\beta+\gamma+\delta=\text{an odd multiple of }\pi.$$

Example 3. A perpendicular is drawn from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to any tangent. Prove that the locus of the foot of the perpendicular is

$$(x^2+y^2)^2=a^2x^2+b^2y^2.$$

Solution. Any tangent to the given ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}. \tag{1}$$

The equation of the perpendicular drawn from the centre (0, 0) to (1) is

$$y=-\frac{1}{m}x. \tag{2}$$

The locus of the foot of the perpendicular is obtained by eliminating m between (1) and (2). Hence the required locus is

$$y=-\frac{x}{y}x+\sqrt{a^2\frac{x^2}{y^2}+b^2}$$

$$\Rightarrow \quad (x^2+y^2)^2=a^2x^2+b^2y^2.$$

Let (h, k) be the middle point of the line (1). Then the equation of the chord with middle point (h, k) is

$$\frac{xh}{a^2}+\frac{yk}{a^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}. \tag{2}$$

Since (1) and (2) represent the same line, we have

$$-ax \sec\theta+by \operatorname{cosec}\theta+a^2-b^2\equiv\frac{xh}{a^2}+\frac{yk}{b^2}-\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

$$\Rightarrow \qquad \frac{-a \sec\theta}{\dfrac{h}{a^2}}=\frac{b \operatorname{cosec}\theta}{\dfrac{k}{b^2}}=\frac{a^2-b^2}{-\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}$$

$$\Rightarrow \qquad \cos\theta=\frac{a^3\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}{h\,(a^2-b^2)},\ \sin\theta=-\frac{b^3\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}{k\,(a^2-b^2)}$$

$$\Rightarrow \qquad \cos^2\theta+\sin^2\theta=\frac{a^6\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)^2}{h^2\,(a^2-b^2)^2}+\frac{b^6\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)^2}{k^2\,(a^2-b^2)^2}$$

$$\Rightarrow \qquad \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2\left(\frac{a^6}{h^2}+\frac{b^6}{k^2}\right)=(a^2-b^2)^2.$$

Hence the locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2.$$

Example 6. If the chord joining two points whose eccentric angles are α and β cut the major axis of an ellipse at a distance d from the centre, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}.$$

Solution. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the chord joining the points whose eccentric angles are α and β, is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}.$$

drawn at the extremities of a normal chord is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

4. If θ and ϕ are the eccentric angles of two points collinear with a focus of an ellipse, prove that

$$\cos\frac{\theta-\phi}{2}=\pm e\cos\frac{\theta+\phi}{2}.$$

5. Prove that the product of the distances of the foci from any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is b^2.

6. Prove that the line joining the points α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through a focus if

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1} \text{ or } \frac{e+1}{e-1},$$

where e is the eccentricity.

7. P and Q are the points α and β on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If PQ subtends a right angle at $(a, 0)$, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-\frac{b^2}{a^2}$$

and deduce that PQ passes through a fixed point. Further, obtain the coordinates of this fixed point.

8. Tangents are drawn to an ellipse at the points whose eccentric angles are α, β, γ. Prove that the area of the triangle formed by them is

$$ab\tan\frac{\beta-\gamma}{2}\tan\frac{\gamma-\alpha}{2}\tan\frac{\alpha-\beta}{2}.$$

9. If the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\alpha, b\sin\alpha)$ meets the coordinate axes at P and Q, and M is the mid-point of PQ, find the coordinates of M and hence prove that as α varies, the locus of M is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4.$$

10. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and Q is the corresponding point on the auxiliary circle. C is the centre of the ellipe and the normal to the ellipse at P meets CQ in R. Find the coordinates of R in terms of a, b and θ, Hence prove that, as θ varies, the locus of R is the circle $x^2+y^2=(a+b)^2$.

11. Any tangent to an ellipse is cut by the tangents at the ends of the major axis in the points P, Q. Prove that the circle whose diameter is PQ will pass through the foci.

12. Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant is an ellipse.

13. If Q be the point on the auxiliary circle corresponding to the point P on an ellipse, prove that the normals at P and Q meet on a fixed circle.

14. If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1.$$

where, α, β, γ, δ are the eccentric angles of the extremities of the chords.

15. If the normals be drawn at the extremities of any focal chord of an ellipse, a line through their intersection parallel to the major axis will bisect the chord.

16. If SY and $S'Y'$ be the perpendiculars from the foci upon the tangent at any point P of the ellipse, then Y and Y' lie on the auxiliary circle, and $SY.S'Y'=b^2$. Also CY and $S'P$ are parallel.

17. Two tangents to the ellipse intersect at right angles. Prove that the sum of the squares of the chords which the auxiliary circle intercepts on them is constant, and equal to the square on the line joining the foci.

18. Prove that the circle on any focal distance as diameter touches the auxiliary circle.

19. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=a+b$ in the points P and Q. Prove that the tangents at P and Q are at right angles.

20. If the normals at four points (x_r, y_r), $r=1, 2, 3, 4$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, prove that

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4.$$

21. If the normals at the points whose eccentric angles are α, β, γ are concurrent, prove that

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$$

8. DIAMETER

A line through the centre of an ellipse is called a **diameter** *of the ellipse.*

Note. The definitions of a diameter in respect of a circle, parabola and ellipse are consistent. In each case it is a line through the centre of the corresponding curve. Though at first sight it appears that a diameter of a parabola is quite different from that of a circle or an ellipse, yet it is not so actually since a line drawn through a point P on a parabola parallel to the axis is the limiting case of a line joining P to a point on the axis of the parabola at a very great distance, and so that lines parallel to the axis of the parabola may be regarded as lines through its centre at infinity.

8.1 Equation of diameter. *To find the equation of a diameter of an ellipse.*

Let equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be any two points on the ellipse.

Then equation of the chord PQ is

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$$

This chord is parallel to the line $y=mx$ if

$$m=-\frac{b}{a}\cot\frac{\theta+\phi}{2}. \tag{1}$$

Let the coordinates of the middle point of the chord PQ be (h, k).

Then $$h=\frac{a\cos\theta+a\cos\phi}{2}=a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

and $$k=\frac{b\sin\theta+b\sin\phi}{2}=b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}.$$

$$\therefore \qquad \frac{h}{k}=\frac{a}{b}\cot\frac{\theta+\phi}{2} \tag{2}$$

Thus from (1) and (2), we have

$$k=-\frac{b^2}{a^2m}\,h.$$

Hence locus of (h, k) is

$$y=-\frac{b^2}{a^2m}\,x.$$

This is the equation of the diameter. Clearly all the diameters of the ellipse pass through its centre.

8.2 Alternative definition of a diameter. Following the arguements as in case of parabola (see § 9 of Chapter VII) one can give an alternative definition to a diameter of an ellipse as

The locus of the middle points of a system of parallel chords of an ellipse is called a diameter of the ellipse.

Let $y=m'x$ be the diameter of the ellipse which bisects all the chords parallel to the lime $y=mx$. Then

$$m'=-\frac{b^2}{a^2m}$$

$$\Rightarrow \qquad mm'=-\frac{b^2}{a^2}.$$

The symmetry of the result shows that the diameter $y=mx$ bisects all the chords parallel to the diameter $y=m'x$. Hence, if one diameter of an ellipse bisects chords parallel to a second, the second diameter will bisect all chords parallel to the first.

8.3 Conjugate diameters.

Two diameters of an ellipse are said to be **conjugate** *when each bisects chords parallel to the other.*

Let $P(h, k)$ be any point on the ellipse and the tangent at P be parallel to the chord QR whose equation is

$$y=mx+c.$$

But the equation of the tangent at P is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1.$$

$$\therefore \qquad m=-\frac{b^2h}{a^2k}$$

$$\Rightarrow \qquad k=-\frac{b^2}{a^2m}\,h.$$

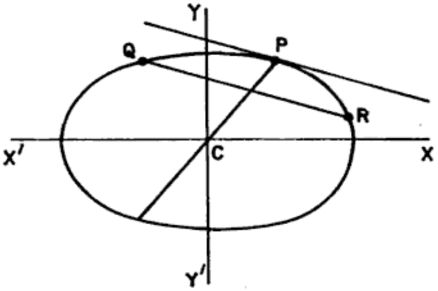

Fig. 8.17

Hence the locus of the point $P(h, k)$ is

$$y=-\frac{b^2}{a^2m}\,x.$$

This is the equation of the diameter which bisects the chord QR and all chords parallel to it.

Proposition II. *The tangents at the ends of any chord meet on the diameter which bisects the chord.*

Let Q and R be the ends of a given chord

$$y=mx+c.$$

Let $P(h, k)$ be the point of intersection of the tangents at Q and R. Then QR is the chord of contact of $P(h, k)$ and so its equation is *i.e.* if $$\frac{b \sin \theta}{a \cos \theta} \cdot \frac{b \sin \phi}{a \cos \phi} = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \tan \theta = -\cot \phi = \tan \left(\frac{\pi}{2} + \phi\right)$$

$$\Rightarrow \quad \theta - \phi = \pm \frac{\pi}{2}.$$

Corollary 1. If $(a \cos \theta,\ b \sin \theta)$ be the coordinates of the extremity of a diameter, then $(-a \sin \theta,\ b \cos \theta)$ will be the coordinates of the extremity of its conjugate.

Corollary 2. The line joining the points on the auxiliary circle corresponding to the extremities P and D of a pair of conjugate diameters subtends a right angle at the centre.

Proposition IV. *The sum of the squares of the lengths of two conjugate semi-diameters is constant.*

Let P and D be the extremities of two conjugate diameters of the ellipse. Let $(a \cos \theta,\ b \sin \theta)$ be the coordinates of P. Then the coordinates of D are $(-a \sin \theta,\ b \cos \theta)$.

Now $$CP^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

and $$CD^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta.$$

$$\therefore \quad CP^2 + CD^2 = a^2 + b^2 = \text{constant}.$$

Proposition V. *The area of the parallelogram which touches an ellipse at the ends of conjugate diameters is constant.*

Let PCP' and DCD' be the conjugate diameters of the ellipse. Let $(a \cos \theta,\ b \sin \theta)$ be the coordinates of the point P.

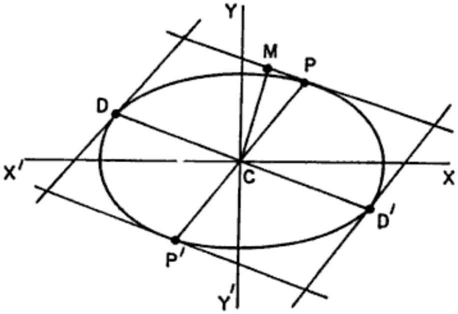

Fig. 8.19

The area of the parallelogram which touches the ellipse at the points P, P', D, D' is

$$4CD \cdot CM,$$

where CM is the perpendicular distance of the tangent at P from C.

But the equation of the tangent at $P(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1.$$

$$CM = \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}}.$$

$$= \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}.$$

Since the coordinates of D are $(-a \sin \theta, b \cos \theta)$, we have

$$CD = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}.$$

Thus $4CD \cdot CM = 4\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \cdot \dfrac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$

$$= 4ab,$$

which is constant.

Proposition VI. *The product of the focal distances of a point P on an ellipse is equal to the square of the length of the semi-diameter parallel to the tangent at P.*

Let $(a \cos \theta, b \sin \theta)$ be the coordinates of P.

Then $\quad SP = a - ae \cos \theta$

and $\quad S'P = a + ae \cos \theta$.

$\therefore \quad SP \cdot S'P = (a - ae \cos \theta)(a + ae \cos \theta)$

$\Rightarrow \quad SP \cdot S'P = a^2 \sin^2 \theta + b^2 \cos^2 \theta. \quad (1)$

The semi-diameter CD parallel to the tangent at P is conjugate to CP. Therefore the coordinates of D are $(-a \sin \theta, b \cos \theta)$. Thus,

$$CD^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta. \quad (2)$$

From equations (1) and (2) the result follows.

8.6 Equi-conjugate diameters. *Two conjugate diameters of an ellipse are said to be* **equi-conjugate** *when they are equal in length.*

$$\Rightarrow \qquad \frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=2.$$

Example 3. *CP* and *CD* are conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the orthocentre of the triangle *CPD* is the curve

$$2\,(b^2\,y^2+a^2\,x^2)^3=(a^2-b^2)^2\,(b^2\,y^2-a^2\,x^2)^2.$$

Solution. Let ϕ be the eccentric angle of *P*. Then the coordinates of *P* and *D* are, respectively

$$(a\cos\phi,\ b\sin\phi) \quad \text{and} \quad (-a\sin\phi,\ b\cos\phi).$$

Since the tangent at *P* is parallel to *CD*, the perpendicular from *P* on *CD* is the normal at *P* and its equation is

$$ax\sec\phi-by\,\text{cosec}\,\phi=a^2-b^2. \tag{1}$$

Similarly the equation of the perpendicular from *D* on *CP* is

$$-ax\,\text{cosec}\,\phi-by\sec\phi=a^2-b^2. \tag{2}$$

The locus of the orthocentre of the triangle *CPD* is obtained by eliminating ϕ between (1) and (2). Now from equations (1) and (2), we have

$$\frac{\sec\phi}{(by-ax)(a^2-b^2)}=\frac{\text{cosec}\,\phi}{(by+ax)(a^2-b^2)}=\frac{1}{-(b^2y^2+a^2x^2)}$$

$$\Rightarrow \qquad \cos\phi=\frac{-(b^2y^2+a^2x^2)}{(by-ax)(a^2-b^2)} \text{ and } \sin\phi=\frac{-(b^2y^2+a^2x^2)}{(by+ax)(a^2-b^2)}$$

Now $\cos^2\phi+\sin^2\phi=\frac{(b^2y^2+a^2x^2)^2}{(by-ax)^2(a^2-b^2)^2}+\frac{(b^2y^2+a^2x^2)^2}{(by+ax)^2(a^2-b^2)^2}$

$$\Rightarrow \qquad 1=\frac{(b^2y^2+a^2x^2)^2}{(a^2-b^2)^2}\left\{\frac{1}{(by-ax)^2}+\frac{1}{(by+ax)^2}\right\}$$

$$\Rightarrow \qquad 2(b^2y^2+a^2x^2)^3=(a^2-b^2)^2(b^2y^2-a^2x^2)^2.$$

Example 4. If *P* and *D* be the ends of conjugate diameters of the elliipse $x^2/a^2+y^2/b^2=1$, prove that the locus of the foot of the perpendicular from the centre of the ellipse on *PD* is

$$2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

Solution. Let ϕ be the eccentric angle of *P*. Then the coordinates of *P* and *D* are, respectively $(a\cos\phi,\ b\sin\phi)$ and

$\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$ and the equation of *PD* is

$$\frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right)+\frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right)=\frac{1}{\sqrt{2}}. \tag{1}$$

The equation of the perpendicular drawn from the centre (0, 0) to (1) is

$$\frac{x}{b}\sin\left(\frac{\pi}{4}+\phi\right)-\frac{y}{a}\cos\left(\frac{\pi}{4}+\phi\right)=0. \tag{2}$$

The locus of the foot of the perpendicular is obtained by eliminating ϕ between (1) and (2). Now from equation (2), we have

$$\tan\left(\frac{\pi}{4}+\phi\right)=\frac{by}{ax}$$

$$\Rightarrow \qquad \sin\left(\frac{\pi}{4}+\phi\right)=\frac{by}{\sqrt{a^2x^2+b^2y^2}}$$

and
$$\cos\left(\frac{\pi}{4}+\phi\right)=\frac{ax}{\sqrt{a^2x^2+b^2y^2}}.$$

Substituting these values in (1), we get

$$\frac{x}{a}\cdot\frac{ax}{\sqrt{a^2x^2+b^2y^2}}+\frac{y}{b}\cdot\frac{by}{\sqrt{a^2x^2+b^2y^2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \qquad 2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

Example 5. *CP* and *CD* are conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the point of intersection of normal at *P* with *CD* is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=\left(\frac{a^2-b^2}{x^2+y^2}\right)^2.$$

Solution. Let ϕ be the eccentric angle of *P*. Then the coordinates of *P* and *D* are, respectively

$$(a\cos\phi, b\sin\phi)$$

and
$$\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right).$$

The equation of the normal at *P* is

$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2. \tag{1}$$

Therefore

$$m=\frac{b\left(\frac{ay}{bx}-1\right)}{a\left(1+\frac{ay}{bx}\right)}=\frac{b(ay-bx)}{a(bx+ay)}.$$

Substituting this value of m in (1), we get

$$y=\frac{b(ay-bx)}{a(ay+bx)}\cdot x+\sqrt{a^2\cdot\frac{b^2(ay-bx)^2}{a^2(ay+bx)^2}+b^2}$$

$$\Rightarrow \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

This is the locus of the point L. Similarly the locus of M can be obtained as the same ellipse. Hence the result follows.

EXERCISES

1. P and D are the extremities of a pair of conjugate radii of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that

   (*i*) PD touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

   (*ii*) The locus of the middle point of PD is
   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$$

   (*iii*) The tangents at P and D intersect on the ellipse
   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

2. PCP', DCD' are conjugate diameters of an ellipse; PD is produced to R so the $DR=2PD$. Find the locus of R.
3. Prove that the locus of the intersections of the normals at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve
   $$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$
4. In an ellipse, a pair of conjugate diameters is produced to meet the directrix. Prove that the orthocentre of the triangle so formed is the focus of the ellipse.
5. If $x\cos\alpha+y\sin\alpha=p$ is a chord joining the ends of conjugate

semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

$$a^2\cos^2\alpha+b^2\sin^2\alpha=2p^2.$$

6. Lines are drawn through the foci of an ellipse perpendicular respectively to a pair of conjugate diameters and intersect in *R*. Prove that the locus of *R* is a concentric ellipse.
7. Lines are drawn through the origin perpendicular to the tangents from a point *P* to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If the lines are conjugate diameters of the ellipse, prove that *P* lies on the curve $a^2x^2+b^2y^2=a^4+b^4$.
8. If *P*, *D* be the extremities of conjugate diameters of an ellipse, and *PP'*, *DD'* be chords parallel to an axis of the ellipse; prove that *PD'* and *P'D* are parallel to the equi-conjugate diameters.
9. *CP* and *CD* are conjugate semi-diameters of an ellipse, and the tangent at *P* meets any other pair of conjugate diameters in *T* and *T'*. Prove that $TP\cdot PT'=CD^2$.
10. If *P*, *D* are extremities of conjugate diameters, and the tangent at *P* cut the major axis in *T*, and the tangent at *D* cut the minor axis in *T'*; prove that *TT'* will be parallel to one of the equi-conjugates.
11. Two fixed conjugate diameters of an ellipse are met in the points *P*, *Q* respectively by two lines *OP*, *OQ* which pass through a fixed point *O* and are parallel to any other pair of conjugate diameters. Prove that the locus of the middle point of *PQ* is a line.
12. *CP* and *CD* are conjugate semi-diameters of an ellipse whose foci are *S* and *S'*. The tangents to the ellipse at *P* and *D* meet in *T*. Prove that *SP*, *S'P*, *SD*, *S'D* touch a circle whose centre is *T*. Find the radius of the circle.
13. A variable line meets an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at *A* and *B*. *P* is a fixed point on the ellipse. If *PA* and *PB* are parallel to conjugate diameters of the fixed ellipse

$$\frac{x^2}{a^2+c^2}+\frac{y^2}{b^2+c^2}=1,$$

prove that *AB* passes through a fixed point on the normal at *P* to the first ellipse.

centre of the ellipse to this normal is equal to the product of the focal distances from P.

6. The tangent to an ellipse cuts the directrix corresponding to the focus S at Q. Prove that the angle PSQ is a right angle.

7. Prove that the lines $lx+my+n=0$, and $l'x+m'y+n'=0$, are conjugate lines of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$a^2ll'+b^2mm'=nn'.$$

If two conjugate lines cut at right angles at (x_1,y_1), and one of them make an angle θ with the x-axis, then

$$\tan 2\theta=\frac{2x_1y_1}{(x_1^2-a^2)-(y_1^2-b^2)}.$$

8. Prove that the area of the triangle with vertices at the points α, β and γ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$2ab\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}.$$

9. Tangents are drawn to an ellipse at the points whose eccentric angles are α, β, γ. Find the condition that the circle circumscribing the triangle formed by them should pass through the centre of the ellipse.

10. The point S is a focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and P is a point on the ellipse such that SP is perpendicular to the x-axis. The tangent and normal to the ellipse at P meet the y-axis in Q and R, respectively. If S' is the other focus of the ellipse, prove that $QR=S'P$.

11. A circle $x^2+y^2+2gx+2fy+c=0$ meets the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in the points $(a\cos\phi, b\sin\phi)$, where $\phi=\alpha, \beta, \gamma, \delta$. Prove that

$$\alpha+\beta+\gamma+\delta=2n\pi,$$

where n is an integer.

12. If tangents at P and Q to an ellipse meet in T, and PQ meets the directrix in K, prove that SK bisects the angle PSQ externally. Deduce that the angle KST is a right angle.

13. A system of ellipses is described with a given focus and corresponding directrix. Prove that the locus of the extremities of their minor axes is a parabola.

14 PNP' is a double ordinate of an ellipse whose centre is C, and the normal at P meets the CP' in R. Prove that the locus of R is an ellipse.

15. If the perpendicular from the centre C of an ellipse to the tangent at any point P meet the focal distance, produced if necessary, in R, prove that the locus of R is a circle.

16. Prove that the locus of the point of intersection of normals at the ends of chords drawn through any fixed point on the major axis of an ellipse is an ellipse, which cuts the major axis in two points the product of whose distances from the centre is a^2e^4.

17. Prove that the locus of the middle points of chords of the conic

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

which touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is}$$

$$\left(\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}\right)^2=\frac{a^2x^2}{(a^2+\lambda)^2}+\frac{b^2y^2}{(b^2+\lambda)^2}.$$

18. Prove that if (x', y') be the middle point of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (x_1, y_1) the point of intersection of the normals and (x_2, y_2) that of the tangents at its extremities, then

$$\frac{a^2x_1}{x_2}+\frac{b^2y_1}{y_2}=(a^2-b^2)\left(\frac{x_2x'}{a^2}-\frac{y_2y'}{b^2}\right).$$

19. From any point of eccentric angle ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

three normals are drawn to the curve. Prove that the equation of the circum circle of the triangle formed by their feet is

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi=a^2+b^2.$$

20. A point P moves so that the chord of contact of the tangents from P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}.$$

Find the locus of P.

21. P and Q are variable points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that the lines joining them to its centre are perpendicular, and the tangents at P and Q meet in T. Find the locus of T.

22. P, Q, R are points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentric angles θ, α, β. PQ goes through $(ae, 0)$ and PR through $(-ae, 0)$. Prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{(1-e)^2}{(1+e)^2}.$$

23. The area of a quadrilateral formed by joining the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are α, β, γ, δ is

$$\tfrac{1}{2}ab\,\{\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\delta-\gamma)+\sin(\alpha-\delta)\}.$$

24. From any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ normals are drawn to the curve. Prove that the locus of the orthocentre of the triangle formed by joining their feet is the ellipse

$$a^2x^2+b^2y^2=\left(\frac{a^4+b^4}{a^2-b^2}\right)^2.$$

25. If PSQ, $PS'R$ be focal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^3}{b^2}=1,$$

where S and S' are the foci, prove that the equation of the chord QR is

$$\frac{x}{a}\cos\theta+\frac{y(1+e^2)}{b(1-e^2)}\sin\theta+1=0,$$

where θ is the eccentric angle of P.

26. If the pole of the normal at P lies on the normal at Q, then

prove that the pole of the normal at Q lies on the normal at P.

27. Find the locus of the point of intersection of the tangent at one end of a focal chord of an ellipse with the normal at the other end.

28. If a pair of tangents to a conic be at right angles to one another, prove that the product of the perpendiculars from the centre and the intersection of the tangents to the chord of contact is constant.

29. The lines PS, PS' joining any point P on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

to the foci S and S' meet the curve again in Q, Q'. The tangents at Q and Q' meet in T. Prove that the locus of T as P moves round the curve is

$$(1+e^2)^2\frac{x^2}{a^2}+(1-e^2)^2\frac{y^2}{b^2}=(1+e^2)^2.$$

30. If a point P on an ellipse be such that all chords drawn through a given point Q subtend a right angle at P, prove that P must be the foot of one of the normals that pass through Q.

31. Focal chords are drawn through any point on an ellipse, prove that the radical axis of the two circles described on them as diameters is normal to a coaxal and concentric ellipse.

32. A point P of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is joined to the points $(\pm c, 0)$, and the joints meet the ellipse again in Q and R. Prove that the locus of the point of intersection of the tangents to the ellipse at Q and R is the ellipse

$$\frac{x^2}{a^2}+\frac{(a^2-c^2)^2}{(a^2+c^2)^2}\cdot\frac{y^2}{b^2}=1.$$

33. The lines joining the point P to the foci S and S' of an ellipse cut it again at A and B. Prove that the tangents at A and B and the normal at P are concurrent.

34. SP and SQ are radii vectors from a focus of an ellipse, parallel to two conjugate diameters. Prove that the tangents at P and Q intersect in an ellipse having the same eccentricity

as the original curve with the centre at a distance $\frac{ae}{1-e^2}$ from its centre.

35. If PL and PM be the perpendiculars from the point P to two fixed conjugate diameters of an ellipse, prove that the perpendicular from P to its polar divides LM in a constant ratio.

36. PQ is any chord of an ellipse parallel to one of the equi-conjugates and the tangents at P and Q meet in T. Prove that the circle PTQ passes through the centre.

37. The line joining two extremities of any two diameters of an ellipse is either parallel or conjugate to the line joining two extremities of their conjugate diameters.

38. If a length PQ be taken in the normal at any paint P of an ellipse whose centre is C, equal in length to the semi-diameter which is conjugate to CP, prove that Q lies on one, or the other of two circles.

39. Prove that the locus of the centre of a circle which cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the fixed point (α, β) and in two other points at the extremities of a diameter is the ellipse

$$2a^2x^2+2b^2y^2=(a^2-b^2)(\alpha x-\beta y).$$

40. Prove that a chord which joins the ends of a pair of conjugate diameters of an ellipse always touches a similar ellipse.

41. A pair of conjugate diameters is produced to meet the directrix. Prove that the orthocentre of the triangle so formed is at the focus.

CHAPTER IX

THE HYPERBOLA

1. DEFINITION

The **hyperbola** *is the locus of a point which moves so that its distance from a fixed point bears a constant ratio* (> 1) *to its distance from a fixed line.*

The fixed point is called the **focus**, constant ratio the **eccentricity** and the fixed line the **directrix**.

1.1 Standard equation of a hyperbola. *To find the equation of a hyperbola.*

Let S be the focus and ZK the directrix. Draw SZ perpendicular to the directrix. Let e be the eccentricity of the hyperbola. Let A divide ZS such that $AS \,.\, ZA=e:1$. Then A lies on the curve.

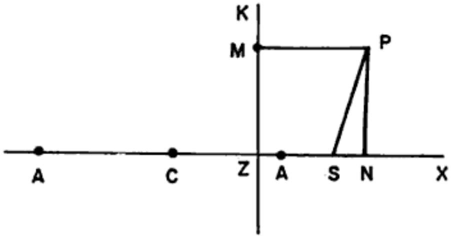

Fig. 9.1

Since $e>1$, there is another point A' on SZ produced such that $A'S : A'Z=e:1$. Thus $AS=e\,.\,ZA$ and $A'S=e\,.A'Z$.

Let C be the middle point of AA' and let $AA'=2a$.

$$\text{Now}\qquad AS+A'S=e(ZA+A'Z)$$

$$\Rightarrow\qquad 2CS=e\,.\,AA'=2ae$$

$$\Rightarrow\qquad CS=ae.$$

Further $A'S-AS=e(A'Z-ZA)$

$\Rightarrow \quad AA'=e\,.\,2CZ$

$\Rightarrow \quad 2a=e\,.\,2CZ$

$\Rightarrow \quad CZ=\frac{a}{e}.$

Let us take C to be the origin, CA the x-axis and a line through C perpendicular to CA the y-axis.

Let $P(h, k)$ be any point on the locus (hyperbola). Draw PM perpendicular from P to the directrix ZK.

Then, by definition of hyperbola, we have

$$SP=e\,.\,MP.$$

But $$SP=\sqrt{(h-ae)^2+k^2}$$

and $$MP=ZN=CN-CZ=h-\frac{a}{e}$$

$$\therefore \quad (h-ae)^2+k^2=e^2\left(h-\frac{a}{e}\right)^2$$

$$\Rightarrow \quad h^2(1-e^2)+k^2=a^2(1-e^2)$$

$$\Rightarrow \quad \frac{h^2}{a^2}+\frac{k^2}{a^2(1-e^2)}=1.$$

Hence locus of (h, k) is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

Since $e>1$, $a^2(1-e^2)$ is negative. If we put $-b^2$ for $a^2(1-e^2)$, the equation of the curve takes the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

This is the required equation of a hyperbola in its standard form.

Ex. Find the equation of a hyperbola whose focus is the point (2, 2), directrix is the line $x+2y=3$ and eccentricity is $\frac{3}{2}$.

1.2 Shape of a hyperbola. *To trace the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

In order to sketch the graph of the given hyperbola we notice the following:

(*i*) The curve is symmetrical about both axes and the origin since if (x, y) lies on the curve, the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$ also lie on the curve.

(*ii*) The curve meets the x-axis in the points $A(a, 0)$ and $A'(-a, 0)$.

(*iii*) The curve meets the y-axis in imaginary points.

(*iv*) Writing the equation of the curve in either of the forms

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2}$$

$$x=\pm\frac{a}{b}\sqrt{b^2+y^2}.$$

We see that there are no points on the curve if $-a<x<a$ and the points $(\pm a, 0)$ lie on it. There is no limitation on the possible values of y.

Futher, from the second equation we note that y may have any real value and from the first that x may have any real value except those for which $x^2<a^2$. Hence the hyperbola extends indefinitely far from the axis in each quadrants. Accordingly, the hyperbola consists of two separate parts or branches as shown in Fig. 9.2.

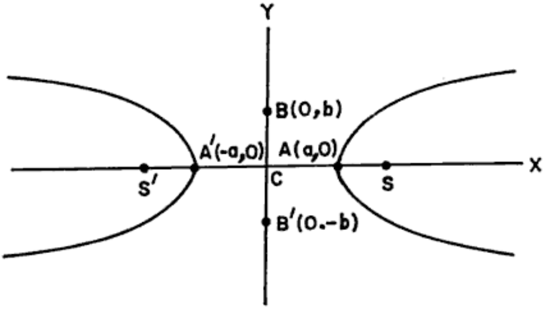

Fig. 9.2

1.3 Pair of lines as a limiting case of a hyperbola. The equation of a hyperbola referred to its axes is

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

i.e.
$$x^2-\frac{y^2}{e^2-1}=a^2.$$

If we keep e constant and make a smaller and smaller, the equation of the hyperbola approximates to

$$x^2-\frac{y^2}{e^2-1}=0.$$

This represents a pair of lines.

Hence a pair of lines may be regarded as a limiting case of a hyperbola whose axes are infinitely small, while their ratio is finite.

Note. An application of the hyperbola is that of locating the place from which a sound, such as gun fire, emanates. From the difference in the times at which the sound reaches two listening posts, the difference between distances of the posts from the gun can be determined. Then the gun is known to be located on a branch of a hyperbola of which the posts are foci.

The position of the gun on this curve can be found by the use of a third listening post. Either of the two posts and the third are foci of a branch of another hyperbola on which the gun is located. Hence the gun is at the intersection of the two branches.

The principle used in finding the location of a gun is also employed by a radar-equipped airplane to determine its location. In this case the plane receives signals from three stations of known locations. ☐

The points $A(a, 0)$, $A'(-a, 0)$ are called the vertices of the hyperbola. The points $B(0, b)$, $B'(0,-b)$ do not lie on the hyperbola.

The two lines $A'A$ and BB' with respect to which the curve is symmetrical are called the axes of the hyperbola. The axis $A'A$ is called the transverse axis whereas the axis BB' is called the conjugate axis.

Since the curve is symmetrical about the origin, the point C (the origin) bisects every chord through it. Therefore, the point C is referred to the centre of the hyperbola.

Further, the symmetry of the curve about its conjugate axis (y-axis) verifies that there must be a second focus S' situated on *moves so that the difference of its distances from two fixed points is constant.*

To find the equation of the curve from the above definition.

Let the distance between the two fixed points S and S' be $2a$.

Let C be the middle point of SS'. Taking CS as x-axis and a line perpendicular to it through C as y-axis. Let $P(h, k)$ be the moving point so that

$$S'P-SP=\text{constant}=2a \text{ (say)}.$$

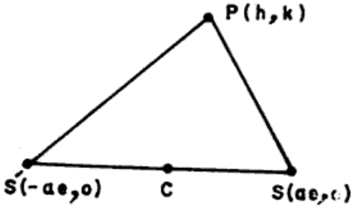

Fig. 9.5

But $$SP=\sqrt{(h-ae)^2+k^2}$$

and $$S'P=\sqrt{(h+ae)^2+k^2}.$$

$\therefore$ $$\sqrt{(h+ae)^2+k^2}-\sqrt{(h-ae)^2+k^2}=2a$$

$\Rightarrow$ $$h^2(e^2-1)-k^2=a^2(e^2-1).$$

Hence the locus of (h, k) is

$$x^2(e^2-1)-y^2=a^2(e^2-1)$$

$\Rightarrow$ $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where $$b^2=a^2(e^2-1).$$

1.5 Polar equation of the hyperbola. *To find the polar equation of the hyperbola referred to the centre as pole.*

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Putting $x=r\cos\theta$, $y=r\sin\theta$, we get

6. Find the locus of a point which moves so that its distance from (4, 0) is twice its distance from the line $x=1$.

2. SOME IMPORTANT RESULTS

Most of the results obtained in the preceding chapter hold good for the hyperbola and in the proofs it is only necessary to change the sign of b^2.

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then we have the following:

(*i*) The equation of the tangent at the point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(*ii*) The line $y=mx+c$ is tangent to the hyperbola if

$$c=\sqrt{a^2m^2-b^2}.$$

(*iii*) The equation of the normal at (x_1, y_1) to the hyperbola is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{-\frac{y_1}{b^2}}.$$

(*iv*) The equation of the pair of tangents from (x_1, y_1) to the hyperbola is

$$T^2=SS_1,$$

where
$$S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$$

$$S_1\equiv\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

$$T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1.$$

(*v*) The equation of the chord of contact of the tangents drawn from (x_1, y_1) to the hyperbola is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(*vi*) The equation of the polar of the point (x_1, y_1) with respect to the hyperbola is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(*vii*) The pole of the line $lx+my+n=0$ with respect to the hyperbola is

$$\left(-\frac{a^2l}{n},\ \frac{b^2m}{n}\right).$$

(*viii*) The equation of the chord with given middle point (x_1, y_1) is

$$T=S_1,$$

where S_1 and T are as defined in (*iv*) above.

(*ix*) The locus of the point of intersection of the tangents at right angles is the circle

$$x^2+y^2=a^2-b^2.$$

This is known as **director circle** of the hyperbola and is real if $a>b$

(*x*) The locus of the middle points of all chords parallel to $y=mx$ of the hyperbola is

$$y=\frac{b^2}{a^2m}x$$

which is a line through the centre of the hyperbola.

(*xi*) The lines $y=mx$ and $y=m'x$ are conjugate if

$$mm'=\frac{b^2}{a^2}.$$

Note. *The theory of conjugate diameters of a hyperbola differs in several respects from that for an ellipse. We note that of two conjugate diameters of a hyperbola one meets the curve in real points and the other in imaginary points.*

Let $y=mx$ and $y=m'x$ be a pair of conjugate diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then the abscissae of the points of intersection of the hyperbola with the diameters $y=mx$ and $y=m'x$ are respectively given by

$$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)=1$$

and $$x^2\left(\frac{1}{a^2}-\frac{m'^2}{b^2}\right)=1.$$

Each of the above equations gives the real values of x if and only if the corresponding slope m (or m') is numerically less than $\frac{b}{a}$. This is not possible since $mm'=\frac{b^2}{a^2}$ and as such if the slope of one diameter is numerically less than $\frac{b}{a}$, the slope of the other is bounded to be numerically greater than $\frac{b}{a}$.

Hence, *of two conjugate diameters of a hyperbola one meets the curve in real points, and the other in imaginary points.*

3. PARAMETRIC FORM OF THE HYPERBOLA

Draw the circle, called the **auxillary circle** in analogy with that of an ellipse on the transverse axis $A'A$ as diameter. Let PN be the perpendicular from $P(x, y)$ to the x-axis and construct the tangent NT to the circle. Denote the angle NCT by θ. Then

$$x=CN=CT\sec\theta=a\sec\theta.$$

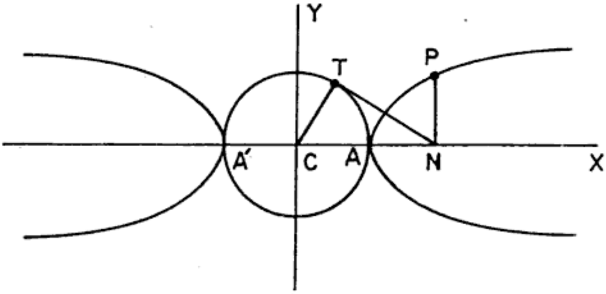

Fig. 9.6

Substituting the value of x in the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ of the hyperbola, we get $y=b\tan\theta$. As θ varies from 0 to $\frac{\pi}{2}$ the point P traces the portion of the hyperbola which lies in the first quadrant starting from A and tending to infinity as θ tends to $\frac{\pi}{2}$. Next, as

θ varies from $\frac{\pi}{2}$ to π, the point P traces the portion of the hyperbola in the third quadrant from infinity to A' since both sec θ and tan θ are negative for $\frac{\pi}{2}<\theta<\pi$. As θ varies from π to $\frac{3\pi}{2}$, P traces out the portion of the hyperbola in the second quadrant from A' to infinity. Finally, as θ varies from $\frac{3\pi}{2}$ to 2π, the point P returns to A from infinity in the fourth quadrant. That is, P traces out every point of the hyperbola as θ ranges from 0 to 2π. Consequently we may say that the hyperbola has the parametric equations.

$$x=a \sec \theta, \ y=b \tan \theta.$$

The point $(a \sec \theta, \ b \tan \theta)$ is also referred to the point θ on the hyperbola.

Note. Instead of using sec θ and tan θ, we may use 'hyperbolic sines and cosines' and write

$$x=a \cosh u, \ y=b \sinh u,$$

where
$$\cosh u=\frac{e^u+e^{-u}}{2},$$

and
$$\sinh u=\frac{e^u-e^{-u}}{2}.$$

so that
$$\cosh^2 u-\sinh^2 u=1.$$

It may be pointed out that from the identity $\cosh^2 u-\sinh^2 u=1$, clearly the point $(a \cosh u, \ b \sinh u)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, but $\cosh u > 1$ for all values of u, the point $(a \cosh u, \ b \sinh u)$ lies only on one branch of the hyperbola, and the other branch is entirely lost. Therefore, the use of $(a \sec \theta, b \tan \theta)$ is usually preferred.

3.1 *To find the equation of the chord joining two points α and β.*

The equation of the chord joining the points α and β is given by

$$y-b \tan \alpha=\frac{b \tan \beta-b \tan \alpha}{a \sec \beta-a \sec \alpha}(x-a \sec \alpha)$$

$$\Rightarrow \qquad y-b\tan\alpha=\frac{b\sin(\alpha-\beta)}{a(\cos\beta-\cos\alpha)}(x-a\sec\alpha)$$

$$\Rightarrow \qquad \frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$$

Corollary 1. The equation of the tangent at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta$.

Corollary 2. The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $by\cot\theta+ax\cos\theta=a^2+b^2$.

Note. Proceeding as in §6.2, Chapter VIII, one may easily verify that four normals can be drawn to a hyperbola from a point.

3.2 Examples

Example 1. The chord of contact of the tangents through P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ subtends a right angle at the origin. Prove that the locus of P is the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}.$$

Solution. Let (h, k) be the coordinates of P. Then the equation of the chord of contact of the given hyperbola is

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1.$$

The equation of the lines joining the origin to the points of intersections of the chord and the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{xh}{a^2}-\frac{yk}{b^2}\right)^2$$

i.e.
$$\left(\frac{h^2}{a^4}-\frac{1}{a^2}\right)x^2-\frac{2hk}{a^2b^2}xy+\left(\frac{k^2}{b^4}+\frac{1}{b^2}\right)y^2=0. \qquad (1)$$

But the chord subtends a right at the origin. Therefore the lines represented by (1) are at right angles. Thus

$$\text{coeff. of } x^2+\text{coeff. of } y^2=0$$

i.e.
$$\left(\frac{h^2}{a^4}-\frac{1}{a^2}\right)+\left(\frac{k^2}{b^4}+\frac{1}{b^2}\right)=0.$$

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1. \tag{2}$$

But equations (1) and (2) are identical. Therefore on comparing (1) and (2), we get

$$\frac{a\cos\theta}{\dfrac{h}{a^2}}=\frac{b\cot\theta}{-\dfrac{k}{b^2}}=\frac{a^2+b^2}{1}$$

$$\Rightarrow \qquad \frac{a^3}{h}=(a^2+b^2)\sec\theta \ \text{ and } \ -\frac{b^3}{k}=(a^2+b^2)\tan\theta.$$

$$\therefore \qquad \frac{a^6}{h^2}-\frac{b^6}{k^2}=(a^2+b^2)^2\,(\sec^2\theta-\tan^2\theta).$$

Hence the locus of (h, k) is

$$\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2.$$

Example 4. Prove that the locus of the middle points of chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passing through the fixed point (α, β) is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Solution. Let (h, k) be the middle point of the chord of the given hyperbola. Then its equation is

$$\frac{xh}{a^2}-\frac{yk}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}.$$

This chord will pass through (α, β) if

$$\frac{\alpha h}{a^2}-\frac{\beta k}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}.$$

Hence the locus of (h, k) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}.$$

This equation can be written as

$$\frac{\left(x-\frac{\alpha}{2}\right)^2}{a^2}-\frac{\left(y-\frac{\beta}{2}\right)^2}{b^2}=\frac{1}{4}\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}\right).$$

Clearly it is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

EXERCISES

1. Prove that the line $x \cos \alpha + y \sin \alpha = p$ touches the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
if $$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha.$$

2. Prove that the tangents to a hyperbola at the extremities of a focal chord intersect on the corresponding directrix.

3. Prove that the product of the perpendiculars from the two foci on any tangent to the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } -b^2.$$

4. Find the equation of the locus of middle points of chords of the hyperbola $4x^2 - 9y^2 = 18$ parallel to the line $x + y = 0$.

5. Find the equation of the locus of the foot of the perpendicular from the origin to a variable tangent to the hyperbola.

6. The normal at a variable point on a hyperbola cuts the principal axes in P, Q. Prove that the locus of the mid-points of PQ is a hyperbola. Can it coincide with the original hyperbola?

7. Prove that the locus of the mid-points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which are tangents to the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is $$a^2 b^2 (b^2 x^2 + a^2 y^2) = (a^2 y^2 - b^2 x^2)^2.$$

8. If the polars of (x_1, y_1), (x_2, y_2) with respect to the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
are at right angles; prove that
$$\frac{x_1 x_2}{y_1 y_2} + \frac{a^4}{b^4} = 0.$$

9. A series of chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle described on the line joining the foci of the

hyperbola as diameter. Prove that the locus of their poles with respect to the hyperbola is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$

10. Prove that the line $lx+my+n=0$ is a normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

if
$$\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}.$$

11. P is any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. A tangent is drawn at P which cuts the directrix at Q. Prove the PQ subtends a right angle at a focus S.

12. A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is the hyperbola

$$a^2x^2-b^2y^2=(a^2+b^2)^2.$$

13. A variable chord of the circle $x^2+y^2=a^2$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Prove that the locus of the middle point of the chord is the curve

$$(x^2+y^2)^2=a^2x^2-b^2y^2.$$

14. Prove that the locus of the middle points of chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which subtend a right angle at the centre, is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2.$$

4. ASYMPTOTES

An **asymptote** *is a line which meets a curve in two coincident points at infinity, but which is not altogether at infinity.*

4.1 Equation of an asymptote.

To find the equation of an asymptote of a hyperbola.

Let equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

First Method

Any line $y=mx+c$ meets the hyperbola at points where

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

i.e. $$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)-\frac{2mc}{b^2}x-\frac{c^2}{b^2}-1=0. \tag{1}$$

The line $y=mx+c$ meets the hyperbola in two coincident points at infinity if the coefficients of x^2 and x in (1) both are zero, *i.e.* if

$$\frac{1}{a^2}-\frac{m^2}{b^2}=0 \text{ and } mc=0$$

$$\Rightarrow \qquad m=\pm\frac{b}{a} \text{ and } c=0.$$

Hence $y=\pm\frac{b}{a}x$ are two real asymptotes of the hyperbola whose combined equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

Second Method

Consider the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola, and suppose that $0<\theta<\frac{\pi}{2}$. Then both x and y coordinates of P are positive and as such P lies in the first quadrant. As $\theta\to\frac{\pi}{2}$, $a\sec\theta\to\infty$ so that $P\to\infty$. Similarly, as $\theta\to\frac{3\pi}{2}$, $P\to\infty$.

The asymptote, if exists, is the limiting form of the tangent as $\theta\to\frac{\pi}{2}$. Therefore, as $\theta\to\frac{\pi}{2}$, the tangent

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

i.e. $$\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta$$

becomes $\frac{x}{a}-\frac{y}{b}=0.$

Similarly, as $\theta \to \frac{3\pi}{2}$, we have as another asymptote

$$\frac{x}{a}+\frac{y}{b}=0.$$

Hence the required asymptotes are

$$\frac{x}{a}\pm\frac{y}{b}=0.$$

Remarks. (*i*) If we draw lines through B, B' parallel to the transverse axis and through A, A' parallel to the conjugate axis, then from $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$, it is clear that the asymptotes are the diagonals of the rectangle so formed. This rectangle is called **associated rectangle** of the hyperbola.

(*ii*) The asymptotes of a hyperbola are helpful in sketching the hyperbola. A rough drawing can be made from the associated rectangle and its extended diagonals. The accuracy may be improved considerably, however, by plotting the end points of each latus rectum.

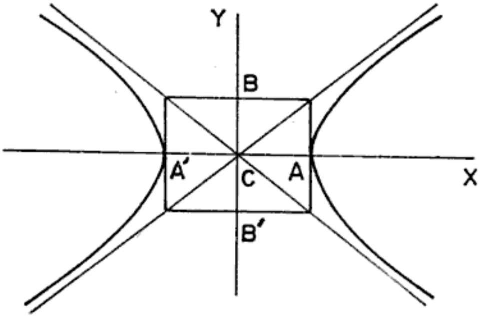

Fig. 9.7

(*iii*) Each asymptote lies along a pair of coincident conjugate diameters.

Corollary. Any line parallel to an asymptote will meet the curve in one point at infinity.

One root of the equation (1) will be infinite, if the coefficient of x^2 is zero. This will be the case if $m=\pm\frac{b}{a}$. So the line

$$y=\pm\frac{b}{a}x+c$$

meets the curve in one point at infinity, whatever the value of c may be.

4.2 Certain important observations about the asymptotes. *The asymptotes of a hyperbola are the pair of tangents from its centre.*

(*I*) The equation of the pair of tangents from the centre of the hyperbola is obtained by putting $x_1=0$ and $y_1=0$ in the equation $T^2=SS_1$, *i.e.*,

$$(-1)^2=(-1)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)$$

$$\Rightarrow \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

(*II*) By actually solving simultaneously the equation of the hyperbola and any equation of its asymptotes, it appears that the asymptotes do not meet the hyperbola at all in the points, real or imaginary. But each asymptote being a tangent drawn from the centre to the hyperbola, there must be two coincident points in which each asymptote meets the hyperbola. We would now like to give an explanation as below so as to remove this apparent inconsistency.

Consider the line

$$y=\left(\frac{b}{a}-\epsilon\right)x,$$

where ϵ is very small. This line is inclined at a very small angle to the asymptote $y=\frac{b}{a}x$. We see that the line meets the hyperbola in two points where

$$x^2\left\{\frac{1}{a^2}-\left(\frac{1}{a}-\frac{\epsilon}{b}\right)^2\right\}=1$$

$$\Rightarrow \qquad x^2=\frac{1}{\frac{2\epsilon}{ab}-\frac{\epsilon^2}{b^2}}.$$

are those diameters each of which meets the hyperbola in two coincident points at infinity. Therefore, they represent the asymptotes to the hyperbola and they divide all the diameters into two classes. One class consists of those diameters, of slope numerically less than $\frac{b}{a}$, which intersect both the branches of the hyperbola. The second class consists of those diameters, of slope numerically greater than $\frac{b}{a}$, which do not intersect the hyperbola.

Hence *the asymptotes can be regarded as the diameters lying on the boundary of the two classes of diameters, intersecting and non-intersecting.*

Note. Since the equations of an ellipse and a hyperbola are alike in the sense that b^2 is replaced by $-b^2$ in case of hyperbola, one is interested to study the asymptotes for an ellipse also. In fact the equations of the pair of tangents from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

In analogy with the hyperbola case, the lines given by this equation are to be regarded as the asymptotes of the ellipse. But these lines being imaginary having no real point upon them except the origin, they do not have the geometrical importace. However, they possess the same algebraic properties as the asymptotes of the hyperbola.

5. CONJUGATE HYPERBOLA

For a given hyperbola the hyperbola, having for, respectively, its transverse and conjugate axes the conjugate and transverse axes of the given hyperbola, is called the **conjugate hyperbola.**

If the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

then the equation of the conjugate hyperbola will be

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1.$$

Both of these hyperbolas have the same asymptotes, *viz.*,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

Note. If the axes of coordinates be changed in any manner, we should, in order to obtain the new equations of a hyperbola, of the asymptotes and of the conjugate hyperbola, have to make the same substitutions in all three cases.

Hence, for all positions of the axes of coordinates, the equations of a hyperbola and of the conjugate hyperbola will only differ from the equation of the asymptotes by constants, and the two constants will be equal and opposite for the two hyperbolas.

5.1 Properties of a pair of conjugate hyperbolas. (*i*) The two hyperbolas have the same asymptotes.

(*ii*) If two diameters be conjugate with respect to one of the hyperbolas, they will be conjugate with respect to the other.

(*iii*) The equations of the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

can be written in polar forms

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}$$

and $$-\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2},$$

respectively. It is evident that if, for any value of θ, r^2 is positive for one curve it is negative for the other.

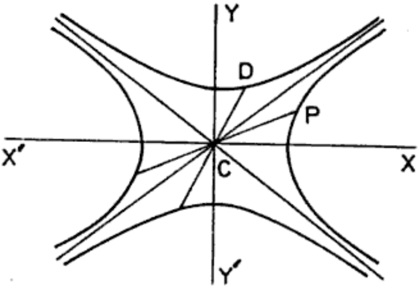

Fig. 9.8

Hence every diameter meets one curve in real points and the other in imaginary points. Moreover the lengths of semi-diameters of the two curves are connected by the relation $r_1^2=-r_2^2$ for all values of θ.

(*iv*) *If two conjugate diameters cut the hyperbola and its conjugate in P and D, respectively, then* $CP^2-CD^2=a^2-b^2$.

Let $P(a\sec\theta, b\tan\theta)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{1}$$

$$\therefore \quad m=\text{slope of the diameter } CP=\frac{b\tan\theta-0}{a\sec\theta-0}=\frac{b}{a}\sin\theta.$$

Let m' be the slope of the diameter CD. Since CP and CD are conjugate diameters, we have

$$mm'=\frac{b^2}{a^2}$$

$$\Rightarrow \qquad m'=\frac{b}{a\sin\theta}.$$

$\therefore$ Equation of CD is

$$y=\frac{b}{a\sin\theta}x. \tag{2}$$

The equation of the hyperbola conjugate to (1) is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1. \tag{3}$$

On solving (2) and (3) we get two points of intersection whose coordinates are $(a\tan\theta, b\sec\theta)$ and $(-a\tan\theta, -b\sec\theta)$.

Hence the coordinates of D are $(a\tan\theta, b\sec\theta)$.

Now $CP^2-CD^2=a^2\sec^2\theta+b^2\tan^2\theta-a^2\tan^2\theta-b^2\sec^2\theta$

$$=a^2-b^2.$$

(*v*) *The parallelogram formed by the tangents at P, P′, D, D′ is of constant area.*

Area of the parallelogram$=4CD.CM$, where CM is perpendicular from C to the tangent at P.

Let $(a\sec, b\tan\theta)$ be the coordinates of P.

The equation of the tangent at P is

$$\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta.$$

The equations of the polars of any point (x_1, y_1) with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{1}$$

and its conjugate are

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

and $$\frac{yy_1}{b^2}-\frac{xx_1}{a^2}=1.$$

Clearly these polars are parallel and equidistant from the centre. ☐

Further, let $P(x_1, y_1)$ be any point on the hyperbola (1).

Then polar of $P(x_1, y_1)$ with respect to the conjugate hyperbola

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

is $$\frac{yy_1}{b^2}-\frac{xx_1}{a^2}=1.$$

This can be written as

$$\frac{x(-x_1)}{a^2}-\frac{y(-y_1)}{b^2}=1.$$

But this equation is the tangent to the hyperbola (1) at $(-x_1, -y_1)$, which is the other extremity of the diameter through P.

Hence, if from any point on a hyperbola the tangents PQ, PQ' be drawn to the conjugate hyperbola, the line QQ' will touch the original hyperbola at the other end of the diameter through P.

5.2 Examples

Example 1. If e, e' be the eccentricities of a hyperbola and of the conjugate hyperbola; prove that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

Solution. Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

hyperbola and with respect to its auxiliary circle, are perpendicular. Prove that P lies on one of the asymptotes of the hyperbola.

Solution. Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then the equation of the auxiliary circle is

$$x^2+y^2=a^2.$$

Let (h, k) be the coordinates of P. Then its polars with respect to the hyperbola and the auxiliary circle are, respectively

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1 \text{ and } xh+yk=a^2.$$

But these polar lines are perpendicular. Therefore

$$\left(\frac{\frac{h}{a^2}}{\frac{k}{b^2}}\right)\left(-\frac{h}{k}\right)=-1$$

$$\Rightarrow \qquad \frac{h^2}{a^2}-\frac{k^2}{b^2}=0.$$

Hence the locus of (h, k) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

Thus the result follows.

EXERCISES

1. Find the asymptotes of $2xy+7x-6y-18=0$. What is the equation of the conjugate hyperbola?
2. Prove that the line $x=0$ is an asymptote of the hyperbola $3x^2+2xy+4x=9$. What is the equation of the other asymptote?
3. Find the equation of the hyperbola which has $3x-4y+7=0$ and $4x+3y+1=0$ for its asymptotes and passes through the origin.
4. Prove that the portion of any tangent to a hyperbola inter-

cepted by the asymptotes is bisected at the point of contact.

5. Prove that the polar of any point on an asymptote of a hyperbola with respect to the hyperbola is parallel to that asymptote.
6. Prove that a line parallel to an asymptote intersects a hyperbola in only one point.
7. P is any point on and C is the centre of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The diameter conjugate to CP intersects the conjugate hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ at Q. Prove that the locus of the point of intersection of the normals at P and Q is a pair of lines represented by $a^2x^2-b^2y^2=0$.

8. Prove that the product of the perpendiculars from any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the two asymptotes is equal to

$$\frac{a^2b^2}{a^2+b^2}.$$

9. The tangent at the point $(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the asymptotes at P and Q. Find the coordinates of P and Q. Prove that the locus of the point which divides PQ in the ratio $\lambda : 1$ is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{4\lambda}{(\lambda+1)^2}.$$

10. The normal at a point P to a hyaerbola meets the transverse axis in G; the line perpendicular to this axis through P meets an asymptote in Q. Prove that GQ is perpendicular to the asymptote.
11. The locus of the centre of a circle which circumscribes the triangle formed by the asymptotes and any tangent to a given hyperbola is another hyperbola whose asomptotes are perpendicular to those of the given hyperbola.
12. Prove that the two lines joining the points in which any two tangents to a hyperbola meet the asymptotes are parallel to the chord of contact of the tangents and are equidistant from it.

13. A variable tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the asymptotes in the points P and Q. Prove that the locus of the centre of the circle CPQ, where C is the origin of coordinates, is given by the equation

$$4(a^2x^2-b^2y^2)=(a^2+b^2)^2.$$

6. RECTANGULAR HYPERBOLA

A hyperbola whose asymptotes are perpendibular is called **rectangular hyperbola.**

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The angle between the asymptotes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

of this hyperbola is equal to $2 \tan^{-1} \frac{b}{a}$. Thus, when the angle is a right angle, we must have $b=a$. On this account the curve is sometimes referred to an **equilateral hyperbola.**

The equation of the rectangular hyperbola, referred to its axes as coordinate axes, is

$$x^2-y^2=a^2.$$

From the relation $b^2=a^2(e^2-1)$ it follows that the eccentricity of the rectangular hyperbola will be $\sqrt{2}$.

Note. If $a=b$, the associated rectangle is a square and the asymptotes are perpendicular to each other. For this reason the hyperbola is said to be equilateral because its axes are equal or is said to be rectangular because its asymptotes intersect at right angles. □

We now proceed to simplify the equation $x^2-y^2=a^2$ of a rectangular hyperbola, by selecting the asymptotes as coordinate axes.

In order to reduce the given equation to the more usual form, we rotate the axes through an angle $-\frac{\pi}{4}$ and we get

$$X=x \cos(-45^\circ)+y \sin(-45^\circ)=\frac{x-y}{\sqrt{2}}$$

and $$Y=-x\sin(-45°)+y\cos(-45°)=\frac{x+y}{\sqrt{2}}.$$

$\therefore$ $$x^2-y^2=2XY.$$

But $$x^2-y^2=a^2.$$

$\therefore$ $$XY=\frac{a^2}{2}.$$

Hence the equation of the rectangular hyperbola can also be written as

$$xy=c^2,$$

where $$c^2=\frac{a^2}{2}.$$

6.1 Parametric equations. If we substitute $x=ct$ in the equation $xy=c^2$ of the rectangular hyperbola, we get $y=\frac{c}{t}$. That is, as t varies, the point $\left(ct, \frac{c}{t}\right)$ always lies on the hyperbola $xy=c^2$. Accordingly, the rectangular hyperbola can be represented by the parametric equations

$$x=ct,\ y=\frac{c}{t}.$$

Thus any point on the curve $xy=c^2$ can be represented by

$$\left(ct, \frac{c}{t}\right).$$

This point is also referred to the **point t** on the curve.

6.2 Equation of a chord. *To find the equation of the chord joining any two points t_1 and t_2 on the rectangular hyperbola $xy=c^2$.*

The equation of the chord joining the points t_1 and t_2 is

$$y-\frac{c}{t_1}=\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1}\left(x-ct_1\right)$$

$$\Rightarrow \qquad x+t_1t_2\,y-c(t_1+t_2)=0.$$

Corollary 1. The equation of the tangent at the point t to the curve $xy=c^2$ is $x+t^2y-2ct=0$.

Corollary 2. The equation of the normal at the point t to $xy=c^2$ is $t^3x-ty+c(1-t^4)=0$.

6.3 Chord of contact. *To find the equation of the chord of contact of tangents drawn from* (x_1, y_1) *to the hyperbola* $xy=c^2$.

Let the tangents drawn from $A(x_1, y_1)$ to the given hyperbola meet it in the points P and Q.

Let $\left(ct_1, \dfrac{c}{t_1}\right)$ and $\left(ct_2, \dfrac{c}{t_2}\right)$ be the coordinates of P and Q, respectively. The equations of the tangents at P and Q are

$$x+t_1^2y=2ct_1,$$
$$x+t_2^2y=2ct_2$$

respectively. Since these tangents pass through $A(x_1, y_1)$, we have

$$x_1+t_1^2y_1=2ct_1,$$
$$x_1+t_2^2y_1=2ct_2.$$

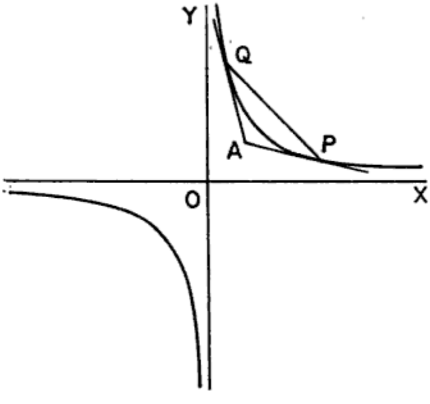

Fig. 9.10

These equations may be written as

$$ct_1y_1+\frac{c}{t_1}\,x_1=2c^2,$$

$$ct_2y_1+\frac{c}{t_2}\,x_1=2c^2.$$

Thus the points $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ both lie on the line

$$xy_1+yx_1=2c^2.$$

This equation represents the chord joining the points of contact of tangents drawn from the point (x_1, y_1) to the hyperbola.

6.4 *To find the locus of the middle points of a system of parallel chords of the hyperbola* $xy=c^2$.

Let AB be one of the parallel chords of the given hyperbola.

Let the coordinates of A and B be $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$, respectively. Let $P(h, k)$ be the middle point of AB. Then

$$h=\frac{c(t_1+t_2)}{2},\quad k=\frac{\frac{c}{t_1}+\frac{c}{t_2}}{2}=\frac{c(t_1+t_2)}{2t_1t_2}.$$

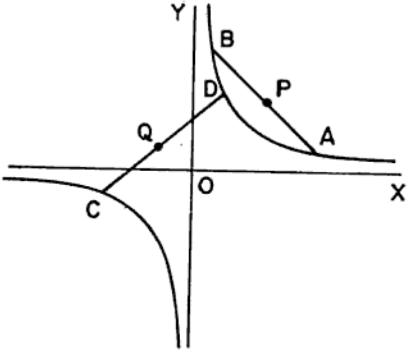

Fig. 9.11

Now $\qquad m=\text{slope of } AB=\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1}=-\frac{1}{t_1t_2}.$

$\therefore \qquad \frac{k}{h}=\frac{1}{t_1t_2}=-m.$

Hence locus of (h, k) is

$$y+mx=0.$$

But $y+mx=0$ is the equation of a diameter of the curve. Thus *the locus of the middle points of a system of parallel chords is a diameter of the hyperbola.*

Let us denote the slope of the above diameter by m'. Then $m'=-m$. Since $m=-m'$, it follows by symmetry that the locus of the middle points Q of all chords parallel to CD with slope m' is the diameter OP with slope m. That is, each of the diameters $y=mx$ and $y=m'x$ bisects the chords parallel to the other. These are called **conjugate diameters** and the relation between their slopes is

$$m+m'=0.$$

6.5 *To prove that four normals can be drawn to the hyperbola $xy=c^2$ from any given point.*

Let $P(x_1, y_1)$ be any given point. Let $\left(ct, \frac{c}{t}\right)$ be the coordinates of any point on the given hyperbola

$$xy=c^2. \tag{1}$$

The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola (1) is

$$t^3x-ty+c(1-t^4)=0.$$

This will pass through $P(x_1, y_1)$ if

$$t^3x_1-ty_1+c(1-t^4)=0$$

i.e. $$ct^4-x_1t^3+y_1t-c=0. \tag{2}$$

Equation (2) is biquadratic in t, it has four values of t. Corresponding to each value of t there is a point on the hyperbola (1) the normal at which passes through (x_1, y_1).

Hence *four normals can be drawn to a hyperbola from a given point.*

6.6 Intersection of a circle and a hyperbola. Let the equation of the hyperbola be

$$xy=c^2 \tag{1}$$

and the equation of the circle be

$$x^2+y^2+2gx+2fy+k=0. \tag{2}$$

The equation (1) can also be written as

$$x=ct,\ y=\frac{c}{t}. \tag{3}$$

On solving (2) and (3), we get

$$c^2t^2+\frac{c^2}{t^2}+2gct+2f\frac{c}{t}+k=0$$

$$\Rightarrow \qquad c^2t^4+2gct^3+kt^2+2fct+c^2=0. \tag{4}$$

Equation (4) is biquadratic in t, it has four roots say t_1, t_2, t_3, t_4. Corresponding to each value of t there is a common point of the circle and the hyperbola. Thus there are four points of intersections of a circle and a hyperbola. ☐

Further, by theory of equations, we have

$$t_1+t_2+t_3+t_4=-\frac{2g}{c},$$

$$t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\frac{k}{c^2},$$

$$t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4=-\frac{2f}{c},$$

and

$$t_1t_2t_3t_4=1.$$

The fourth equation yields the necessary condition that four points are **concyclic.**

In other words, if the parameters t_1, t_2, t_3, t_4 of four points on the hyperbola satisfy $t_1t_2t_3t_4=1$ these points shall be concyclic, since one condition is enough to ensure that four points should be on a circle.

Conversely, if t_1, t_2, t_3, t_4 be the parameters of any four points on a hyperbola, then for the points to be concyclic the parameters must satisfy equation (4) above which requires that $t_1t_2t_3t_4=1$.

6.7 Certain results on the rectangular hyperbola. In this section we state certain results of the rectangular hyperbola $xy=c^2$ without using parametric forms. The proofs may be obtained on the same lines as in previous chapters.

(*i*) The equation of the tangent at the point (x_1, y_1) to the hyperbola is

$$xy_1+yx_1=2c^2.$$

Solution. Let $\left(ct, \frac{c}{t}\right)$ be any point on the hyperbola

$$xy=c^2.$$

Then equation of the normal at $\left(ct, \frac{c}{t}\right)$ to this hyperbola is

$$ct^4-xt^3+yt-c=0.$$

This normal will pass through (α, β) if

$$ct^4-\alpha t^3+\beta t-c=0.$$

The above equation being biquadratic in t has four values say t_1. t_2, t_3, t_4; hence by theory of equations, we have

$$t_1+t_2+t_3+t_4=\frac{\alpha}{c}, \quad (1)$$

$$t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=0, \quad (2)$$

$$t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4=-\frac{\beta}{c}, \quad (3)$$

$$t_1t_2t_3t_4=-1. \quad (4)$$

From (1), we get

$$\alpha=ct_1+ct_2+ct_3+ct_4$$

i.e. $$\alpha=x_1+x_2+x_3+x_4.$$

On dividing equation (3) by equation (4), we get

$$\frac{\beta}{c}=\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}$$

$$\Rightarrow \quad \beta=\frac{c}{t_1}+\frac{c}{t_2}+\frac{c}{t_3}+\frac{c}{t_4}$$

$$\Rightarrow \quad \beta=y_1+y_2+y_3+y_4.$$

Further

$$\begin{aligned} x_1x_2x_3x_4&=ct_1\cdot ct_2\cdot ct_3\cdot ct_4\\ &=c^4t_1t_2t_3t_4\\ &=-c^4 \text{ from (4)}, \end{aligned}$$

and $$y_1y_2y_3y_4=\frac{c}{t_1}\cdot\frac{c}{t_2}\cdot\frac{c}{t_3}\cdot\frac{c}{t_4}$$

Hence the locus of (h, k) is

$$y^2(x-a)=x^3.$$

Example 5. A line through the origin meets the circle $x^2+y^2=a^2$ at P and the hyperbola $x^2-y^2=a^2$ at Q. Prove that the locus of the point of intersection of the tangent at P to the circle with the tangent at Q to the hyperbola is the curve

$$(a^4+4y^4)\,x^2=a^6.$$

Solution. Any line through the origin is $y=mx$. This line meets the circle $x^2+y^2=a^2$ at $P\left(\dfrac{a}{\sqrt{1+m^2}}, \dfrac{am}{\sqrt{1+m^2}}\right)$ and the hyperbola $x^2-y^2=a^2$ at $Q\left(\dfrac{a}{\sqrt{1-m^2}}, \dfrac{am}{\sqrt{1-m^2}}\right)$. Tangents at P and Q are, respectively,

$$x+my=a\sqrt{1+m^2}$$

and $$x-my=a\sqrt{1-m^2}.$$

On solving these equations, we get $m=\dfrac{2xy}{a^2}$. Substituting the value of m in either of the equations, we get

$$x+\frac{2xy}{a^2}\cdot y=a\sqrt{1+\left(\frac{2xy}{a^2}\right)^2}$$

$$\Rightarrow \quad (a^4+4y^4)\,x^2=a^6.$$

Example 6. Prove that the locus of the poles of normal chords of the rectangular hyperbola $xy=c^2$ is the curve

$$(x^2-y^2)^2+4c^2xy=0.$$

Solution. Any normal chord of the given rectangular hyperbola is

$$t^3x-ty+c(1-t^4)=0. \qquad (1)$$

Let (h, k) be the pole of (1). Then its polar with respect to $xy=c^2$ is

$$xk+yh=2c^2. \qquad (2)$$

But equations (1) and (2) represent the same locus. Therefore, on comparing (1) and (2), we get

$$\frac{t^3}{k}=\frac{-t}{h}=\frac{c(1-t^4)}{-2c^2}$$

$$\Rightarrow \qquad t^2=-\frac{k}{h} \text{ and } t=\frac{h(1-t^4)}{2c}.$$

$$\therefore \qquad -\frac{k}{h}=\frac{h^2\left(1-\frac{k^2}{h^2}\right)^2}{4c^2}$$

$$\Rightarrow \qquad (h^2-k^2)^2+4c^2hk=0.$$

Hence the locus of (h, k) is

$$(x^2-y^2)^2+4c^2xy=0.$$

EXERCISES

1. P and Q are the points (x_1, y_1) and $(x_1, -y_1)$ on the hyperbola
$$x^2-y^2=a^2.$$
Prove that the tangent at P is perpendicular to the line joining Q to the centre C of the hyperbola. If the tangent at P meets CQ at R, prove that $CQ\cdot CR=a^2$.

2. If the polar of (h, k) with respect to $y^2=4ax$ touches
$$x^2+y^2=4a^2;$$
prove that the locus of (h, k) is the rectangular hyperbola
$$x^2-y^2=4a^2.$$

3. If the polar of (h, k) with respect to $y^2=4ax$ touches $x^2=4by$; prove that the locus of (h, k) is the rectangular hyperbola
$$xy+2ab=0.$$

4. In a rectangular hyperbola, prove that
$$SP\cdot S'P=CP^2,$$
where S, S' are the foci, C is the centre and P is any point on the hyperbola.

5. From points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$. Prove that the locus of the middle points of the chords of contact is the curve
$$(x^2-y^2)^2=a^2(x^2+y^2).$$

MISCELLANEOUS EXERCISES

1. Find the equation to the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose mid-point is (α, β), and interpret the result when the point (α, β) lies on the hyperbola.
2. Prove that the locus of the poles of a given line with respect to the circles of a coaxal system is a hyperbola.
3. Find the locus of the point of intersection of tangents to the hyperbola $x^2-y^2=a^2$ at points with parameters θ, $\theta+\alpha$, as θ varies. What is the locus when $\alpha=\pi$?
4. A circle cuts two fixed perpendicular lines so that each intercept is of given length. Prove that the locus of the centre of the circle is a rectangular hyperbola.
5. If tangents are drawn to a system of coaxal circles parallel to a given line, prove that the locus of their points of contact is a rectangular hyperbola.
6. AOB, COD are two lines which bisect one another at right angles. Prove that the locus of a point which moves so that $PA \cdot PB=PC \cdot PD$ is a rectangular hyperbola.
7. A line has its extremities on two fixed lines and passes through a fixed point. Find the locus of the middle point of the line.
8. The two lines $x=\alpha$, $y=\beta$ are conjugate with respect to the hyperbola $xy=c^2$. Prove that (α, β) lies on the hyperbola $xy=2c^2$.
9. Prove that the distance of any point from the centre of a rectangular hyperbola varies inversely as the perpendicular distance of its polar from the centre.
10. If four points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if α, β, γ, δ be the inclinations to either asymptote of the lines joining these points respectively to the centre; prove that $\tan\alpha \tan\beta \tan\gamma \tan\delta=1$.
11. The normals at three points P, Q, R on a rectangular hyperbola intersect at a point S on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.
12. A rectangular hyperbola whose centre is C is cut by any circle of radius r in the four points P, Q, R, S; prove that

$$CP^2+CQ^2+CR^2+CS^2=4r^2.$$

CHAPTER X

GENERAL EQUATION OF THE SECOND DEGREE TRACING OF CONICS

1. CONIC SECTION

In general the **conic section** is defined as the locus of a point which moves so that its distance from a fixed point is in a constant ratio to the perpendicular distance of it from a fixed line.

The fixed point is called the focus, the fixed line the directrix and the constant ratio the eccentricity of the conic section. Let us denote the focus by S, the directrix by ZK and eccentricity by e.

We note the following important particular cases of conic sections which we have already dealt with in the preceding chapters independently.

(*I*) When the focus does not lie on the directrix, we find that the locus (conic section) is an ellipse, a parabola or a hyperbola according as the eccentricity $e <=$ or >1.

The circle is further a particular case of an ellipse. In case of a circle $e=0$, focus is at the centre and directrix is at an infinite distance.

Note. In case of an ellipse it may happen that both the axes be zero, so that the conic section reduces to a point.

(*II*) *When the focus lies on the directrix, the conic section is a pair of lines, real or imaginary.*

Since $SP=e \, . \, MP$, we have

$$\sin PSM=\frac{MP}{SP}=\frac{1}{e}.$$

There are four cases:

(*i*) When $e>1$, the point P lies on one or the other of the two lines passing through S inclined to KK' at an angle $\sin^{-1} \frac{1}{e}$.

(*ii*) When $e=1$, the angle PSM is a right angle and the conic section represents a pair of coincident lines coinciding with SX.

(*iii*) When $e<1$, the angle PSM is imaginary and the conic section represents a pair of imaginary lines.

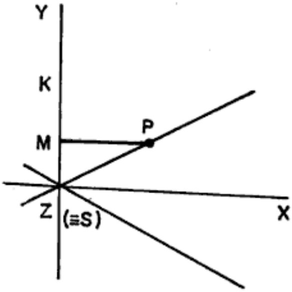

Fig. 10.1

(*iv*) When both KK' and S lie at infinity and S on KK', the lines passing through S will meet at infinity and the conic section will represent a pair of parallel lines.

Thus each of the curves, a pair of lines, a circle, a parabola, an ellipse and a hyperbola can be regarded as a particular case of a conic section. We have seen in the preceding chapters that the equation of each of these particular conic sections is always a second degree in x and y. Now we shall demonstrate the converse that every equation of second degree in x and y represents one of the above conic sections. We shall also show that how to determine from any such equation the nature and the position of the conic section which it represents.

1.1 *To prove that every curve whose equation is of the second degree is a conic.*

Let
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$
be the equation of the curve.

Since this is the most general form of the equation of second degree it will include all possible cases.

In order to eliminate the term containing xy, we turn the axes through a certain angle for which we have to substitute

$$X\cos\theta-Y\sin\theta \text{ and } X\sin\theta+Y\cos\theta$$

for x and y, respectively.

On taking the origin at $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, this equation takes the form

$$AX^2+BY^2=K. \tag{6}$$

If the right side of (6) be zero, this equation will represent a pair of lines. If however the right side of (6) be not zero, then it may be written as

$$\frac{X^2}{\frac{K}{A}}+\frac{Y^2}{\frac{K}{B}}=1.$$

This is the equation of an ellipse if both the denominators are positive, and a hyperbola if one denominator is positive and the other is negative. If both the denominators are negative, it is clear that no real values of X and Y will satisfy the equation. In this case the curve is an imaginary ellipse.

Hence in all cases the curve represented by the general equation of the second degree is a conic section.

Note. From the above it is clear that the general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents

(*i*) a parabola if $h^2=ab$,
(*ii*) an ellipse if $h^2<ab$ and
(*iii*) a hyperbola if $h^2>ab$.

2. CENTRE OF A CONIC SECTION

The **centre** *of a conic section is a point such that all chords of the conic which pass through it are bisected.*

2.1 *To find the coordinates of the centre of a conic section.*

Let $$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$

be the equation of the conic and let (x', y') be its centre. —

Transferring the origin to the point (x', y') by taking the axes through the point (x', y') parallel to the original axes and substituting $X+x'$ for x and $Y+y'$ for y, equation (1) will become

$$a(X+x')^2+2h(X+x')(Y+y')+b(Y+y')^2$$
$$+2g(X+x')+2f(Y+y')+c=0$$
$$\Rightarrow \quad aX^2+2hXY+bY^2+2(ax'+hy'+g)X+2(hx'+by'+f)Y$$

$$+ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0 \tag{2}$$

In this equation the coefficients of X and Y will both be zero, if x' and y' be so chosen that

$$ax'+hy'+g=0 \tag{3}$$

and
$$hx'+by'+f=0. \tag{4}$$

Thus equation (2) becomes

$$aX^2+2hXY+bY^2+c'=0, \tag{5}$$

where
$$c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c \tag{6}$$

By solving equations (3) and (4) simultaneously, the centre of the conic can be obtained as

$$\left(\frac{fh-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right).$$

Notes. (*i*) If $ab-h^2\neq0$, the coordinates of the centre are finite. Thus, the conic sections—circle, ellipse and hyperbola have a finite centre and are referred to **central conics.**

(*ii*) If $ab-h^2=0$, the conic section is a parabola and the centre is at infinity. Thus, a parabola has its centre at infinity.

(*iii*) If $ab-h^2=0$ and $fh-bg=0$ *i.e.*, if

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f},$$

the equations (3) and (4) represent the same line, and any point of that line is a centre. In this case the locus is a pair of parallel lines. The conic section—pair of lines is referred to **degenerate conic.**

2.2 *To find the equation of a conic referred to the axes through the centre and parallel to the original axes.*

From § 2.1 the equation of a conic referred to the axes through the centre and parallel to the original axes is

$$aX^2+2hXY+bY^2+c'=0,$$

where
$$c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$$

$$=x'(ax'+hy'+g)+y'(hx'+by'+f)+gx'+fy'+c$$

$$=gx'+fy'+c \qquad \text{by using (3) and (4)}$$

$$=g\left(\frac{fh-bg}{ab-h^2}\right)+f\left(\frac{gh-af}{ab-h^2}\right)+c$$

$$=\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}$$

$$=\frac{\Delta}{ab-h^2},$$

where $\Delta\equiv abc+2fgh-af^2-bg^2-ch^2$.

Now equation (5) in § 2.1 can be written as

$$aX^2+2hXY+bY^2+\frac{\Delta}{ab-h^2}=0.$$

This is the required equation of the conic referred to the new axes through the centre.

Ex. Find the centre of the conic

$$3x^2-5xy+6y^2+11x-17y+13=0$$

and its equation when transformed to the centre.

Corollary. The equation of the conic section with its centre at the origin is $Ax^2+2Hxy+By^2=1$.

From § 2.2, we know that the equation of the conic referred to the axes through the centre parallel to the original axes is

$$aX^2+2hXY+bY^2+c'=0,$$

where $c'=\dfrac{\Delta}{ab-h^2}$.

The centre of the conic is $\left(\dfrac{fh-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$. This centre will lie at the origin if

$$fh-bg=0 \text{ and } gh-af=0.$$

This gives $c'=c$. The above equation reduces to

$$ax^2+2hxy+by^2+c=0$$

which can be written as

$$Ax^2+2Hxy+By^2=1.$$

3. PRINCIPAL AXES AND ECCENTRICITY OF A CONIC

The study of principal axes and eccentricity is in fact relevant for the conic section which is either an ellipse or a hyperbola, and each one of these is a central conic. Further, there is no loss of generality if the equation of a central conic, with origin at the centre which being a finite one, is taken in the form

$$ax^2+2hxy+by^2=1.$$

3.1 Length and position of the principal axes. *To find the length and position of the principal axes of the conic section whose equation is*

$$ax^2+2hxy+by^2=1. \tag{1}$$

Let this conic be cut by a concentric circle

$$x^2+y^2=r^2. \tag{2}$$

The equation of the lines joining the origin to the points of intersection of (1) and (2) is

$$ax^2+2hxy+by^2=\frac{x^2+y^2}{r^2}$$

$$\Rightarrow \quad \left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0. \tag{3}$$

Clearly these lines are the diameters passing through the points of intersection of (1) and (2). Also these diameters will be equally inclined to the axes of the conic and will be coincident if the radius of the circle be equal to either of the semi-axes of the conic. Hence the lines will be coincident if

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2$$

$$\Rightarrow \quad \frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0, \tag{4}$$

and then they will coincide with one or the other of the axes of the conic. Thus the lengths of the semi-axes of the conic are the roots of the equation (4).

Now multiply equation (3) by $a-\frac{1}{r^2}$, we get

$$\left(a-\frac{1}{r^2}\right)^2 x^2+2h\left(a-\frac{1}{r^2}\right)xy+\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)y^2=0$$

$$\Rightarrow \quad \left(a-\frac{1}{r^2}\right)^2 x^2+2h\left(a-\frac{1}{r^2}\right)xy+h^2y^2=0$$

$$\Rightarrow \quad \left\{\left(a-\frac{1}{r^2}\right)x+hy\right\}^2=0$$

$$\Rightarrow \quad \left(a-\frac{1}{r^2}\right)x+hy=0. \tag{5}$$

Hence the axes of the conic will be obtained by substituting the either root of the equation (4) in equation (5).

Ex. Find the principal axes of the conic

$$22x^2-12xy+17y^2-112x+92y+178=0.$$

3.2 Eccentricity. *To find the eccentricity of a conic section whose equation is*

$$ax^2+2hxy+by^2=1.$$

Let e be an eccentricity of the given conic. Let the given equation become on transforming to principal axes

$$Ax^2+By^2=1.$$

$$\therefore \quad \frac{1}{B}=\frac{1}{A}(1-e^2)$$

$$\Rightarrow \quad A=B(1-e^2).$$

Also we know that

$$A+B=a+b$$

and

$$AB=ab-h^2.$$

On eliminating A and B, we get

$$e^4+\left\{\frac{(a-b)^2+4h^2}{ab-h^2}\right\}(e^2-1)=0. \tag{1}$$

Now there arise two cases:

Case I. When the conic is an ellipse, *i.e.* $ab-h^2>0$, the two values of e^2 in equation (1) have opposite sign. The positive value of e^2 gives the real eccentricity.

Case II. When the conic is a hyperbola, *i.e.* $ab-h^2<0$, the two values of e^2 in equation (1) are positive, which will give the real eccentricity.

3.3 Asymptotes of a conic. *To find the equation of the asymptotes of a conic.*

Let equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \tag{1}$$

We know that the equations of a conic and of the asymptotes only differ by a constant [see Chapter IX]. Therefore the equations of the asymptotes are given by

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0, \tag{2}$$

where λ is so chosen that the equation (2) represents a pair of lines.

The condition that equation (2) may represent a pair of lines is [see Chapter IV § 3]

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix}=0$$

$$\Rightarrow \qquad \lambda(ab-h^2)+\Delta=0.$$

Hence the equations of the asymptotes of the conic (1) are given by

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0.$$

The equations of the hyperbola and its conjugate differ from the equation of their asymptotes by constants which are equal and opposite to one another [see Chapter IX]. Thus, the equation of the hyperbola conjugate to (1) is

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{2\Delta}{ab-h^2}=0.$$

Corollary. The lines represented by the equation

$$ax^2+2hxy+by^2=0$$

are parallel to the asymptotes of the conic.

Ex. Find the asymptotes of the conic

$$x^2-3xy+y^2+10x-10y+21=0.$$

3.4 Condition that the conic represented by the general equation of the second degree may be a rectangular hyperbola.

Let equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \tag{1}$$

The equation of the lines parallel to the asymptotes of the conic (1) is

$$ax^2+2hxy+by^2=0. \tag{2}$$

The conic (1) will represent a rectangular hyperbola if the lines given by (2) be at right angles *i.e.* if

$$a+b=0.$$

4. AXIS, LATUS RECTUM, VERTEX AND FOCUS OF A PARABOLA

To find the axis, latus rectum, vertex and focus of a parabola.

If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1}$$

represents a parabola, then $h^2=ab$ *i.e.* second degree terms form a perfect square.

Hence equation (1) can be written as

$$(\alpha x+\beta y)^2+2gx+2fy+c=0, \tag{2}$$

where $$\alpha^2=a \text{ and } \beta^2=b.$$

By introducing a new constant λ, equation (2) may be rewritten as

$$(\alpha x+\beta y+\lambda)^2=2(\lambda\alpha-g)x+2(\lambda\beta-f)y+\lambda^2-c. \tag{3}$$

Choose λ such that the lines

$$\alpha x+\beta y+\lambda=0$$

and $$2(\lambda\alpha-g)x+2(\lambda\beta-f)y+\lambda^2-c=0$$

are at right angles for which

$$\alpha(\lambda\alpha-g)+\beta(\lambda\beta-f)=0$$

$$\Rightarrow \qquad \lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

Putting this value of λ in equation (3), we get

$$(\alpha x+\beta y+\lambda)^2=\frac{2(\alpha f-\beta g)}{\alpha^2+\beta^2}(\beta x-\alpha y)+\lambda^2-c$$

$$\Rightarrow \qquad \left(\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}\right)^2=\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}\left(\frac{\beta x-\alpha y+c'}{\sqrt{\alpha^2+\beta^2}}\right).$$

where
$$c'=\frac{(\lambda^2-c)(\alpha^2+\beta^2)}{2(\alpha f-\beta g)}.$$

The above equation with respect to new axes X and Y can be written as

$$Y^2=4AX,$$

where
$$Y=\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}},\ X=\frac{\beta x-\alpha y+c'}{\sqrt{\alpha^2+\beta^2}}$$

and latus rectum $4A=\dfrac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}$.

Hence the axis of the parabola is $Y=0$ *i.e.*

$$\alpha x+\beta y+\lambda=0$$

and tangent at the vertex is

$$\beta x-\alpha y+c'=0.$$

Vertex. On solving the equations

$$\alpha x+\beta y+\lambda=0 \text{ and } \beta x-\alpha y+c'=0$$

simultaneously, we get the coordinates of the vertex.

Equation of latus rectum. The equation of the latus rectum is $X=A$ *i.e.*

$$\frac{\beta x-\alpha y+c'}{\sqrt{\alpha^2+\beta^2}}=\frac{1}{2}\frac{(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}$$

$$\Rightarrow \qquad \beta x-\alpha y+c'=\frac{1}{2}\frac{(\alpha f-\beta g)}{(\alpha^2+\beta^2)}.$$

Focus. On solving the equations of the axis of the parabola and the latus rectum, simultaneously, we get the coordinates of the focus.

Ex. Find the axis, latus rectum, vertex and focus of the parabola

$$x^2+2xy+y^2-2x-1=0.$$

5. TRACING OF CONICS

Example 1. Trace the conic

$$9x^2+24xy+16y^2-2x+14y+1=0.$$

Solution. Here $a=9$, $h=12$, $b=16$. Since $h^2=ab=144$, the given conic is a parabola. The equation of the parabola can be written as

$$(3x+4y)^2=2x-14y-1.$$

Introducing a new constant λ in this equation, we get

$$(3x+4y+\lambda)^2=2(1+3\lambda)x+2(4\lambda-7)y+\lambda^2-1. \qquad (1)$$

Choose λ such that the lines

$$3x+4y+\lambda=0$$

and $$2(1+3\lambda)x+2(4\lambda-7)y+\lambda^2-1=0$$

are at right angles for which

$$3(1+3\lambda)+4(4\lambda-7)=0$$

$$\Rightarrow \qquad \lambda=1.$$

Hence equation (1) becomes

$$(3x+4y+1)^2=8x-6y.$$

This can be written as

$$25\cdot\left(\frac{3x+4y+1}{\sqrt{3^2+4^2}}\right)^2=10\cdot\frac{(8x-6y)}{\sqrt{8^2+6^2}}.$$

$$\Rightarrow \qquad \left(\frac{3x+4y+1}{5}\right)^2=\frac{2}{5}\,\frac{(4x-3y)}{5}.$$

This is of the form $Y^2=4AX$, where

$$Y=\frac{3x+4y+1}{5},\ X=\frac{4x-3y}{5},\ 4A=\frac{2}{5}.$$

Hence the axis of the parabola is $Y=0$ *i.e.*

$$3x+4y+1=0 \qquad (3)$$

and tangent at the vertex is $X=0$ *i.e.*

$$4x-3y=0. \qquad (4)$$

On solving the equations (3) and (4), simultaneously the coordinates of the vertex are given by

$$\left(-\frac{3}{25}, \frac{-4}{25}\right).$$

The parabola meets x-axis, where $y=0$ *i.e.* in the points given by

$$9x^2-2x+1=0$$

which are imaginary.

$$2\left(x-\frac{3}{2}\right)-(y-2)=0 \quad i.e. \quad 2x-y-1=0$$

and $$\left(x-\frac{3}{2}\right)+2(y-2)=0 \quad i.e. \quad 2x+4y-11=0.$$

The conic meets the x-axis in the points where $y=0$ *i.e.* in the points given by

$$8x^2-16x+17=0$$

which are imaginary.

Further, the conic meets the y-axis, where $x=0$, *i.e.* in the points given by the equation

$$5y^2-14y+17=0.$$

This also gives imaginary values.

Hence the ellipse meets neither with x-axis nor with y-axis in real points.

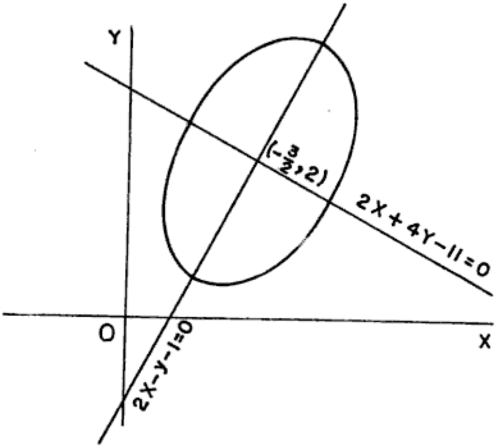

Fig. 10.3

Example 3. Trace the conic

$$x^2-4xy-2y^2+10x+4y=0.$$

Solution. Here $a=1$, $h=-2$. $b=-2$. Since $h^2>ab$, the given conic is a hyperbola.

The lengths of the transverse and conjugate axes are, respectively $\frac{2}{\sqrt{2}}$ and $\frac{2}{\sqrt{3}}$.

Equation of the transverse axis referred to centre as origin is

$$(1-2)x-2y=0 \quad i.e. \quad x+2y=0.$$

Equation of the conjugate axis referred to centre as origin is

$$(1+3)x-2y=0 \quad i.e. \quad 2x-y=0.$$

The equations of the transverse and conjugate axes referred to original axes are, respectively

$$(x+1)+2(y-2)=0$$

and $$2(x+1)-(y-2)=0$$

i.e. $$x+2y-3=0$$

and $$2x-y+4=0.$$

The hyperbola meets x-axis, where $y=0$ *i.e.*

$$x^2+10x=0$$

$$\Rightarrow \quad x=0, -10.$$

The hyperbola meets y-axis, where $x=0$ *i.e.*

$$-2y^2+4y=0$$

$$\Rightarrow \quad y=0, 2.$$

EXERCISES

1. Trace the following conics:

(*i*) $x^2+4xy+y^2-2x+2y+4=0.$

(*ii*) $6x^2-5xy-6y^2+14x+5y+4=0.$

(*iii*) $9x^2-24xy+16y^2-18x-101y+19=0.$

(*iv*) $x^2+4xy+y^2-2x+2y-6=0.$

(*v*) $x^2+xy+y^2+x-4y+1=0.$

(*vi*) $36x^2+24xy+29y^2-72x+126y+81=0.$

(*vii*) $9x^2-6xy+17y^2+30x-74y+17=0.$

(*viii*) $16x^2-24xy+9y^2+77x-64y+95=0.$

(*ix*) $17x^2-12xy+8y^2+46x-28y+17=0.$

(*x*) $4x^2-4xy+y^2-8x-6y+5=0.$

2. Trace the parabola

$$x^2-4xy+4y^2+10x-8y+13=0.$$

Also find the vertex, directrix and the focus.

3. Trace the conic

$$2x^2+3xy-2y^2-7x+y-2=0.$$

Calculate the eccentricity of the conic.

4. Trace the conic

$$25x^2+120xy+144y^2-146x+89y-25=0,$$

and find the coordinates of its foci.

5. Find the equation to the hyperbola which has $3x-4y+7=0$ and $4x+3y+1=0$ for asymptotes and passes through the origin. Find its centre and trace the curve.

6. Obtain the foci and directrices of the conic whose equation is

$$x^2+12xy-4y^2-6x+4y+9=0.$$

7. Trace the conic

$$9x^2+6xy+y^2+2x+3y+4=0,$$

and find its latus rectum.

8. Trace the conic

$$32x^2-52xy-7y^2-64x-52y-148=0$$

and find the coordinates of its foci.

9. Prove that the conic

$$(a^2+b^2)(x^2+y^2)=(bx+ay-ab)^2$$

is a parabola of latus rectum $\dfrac{2ab}{\sqrt{a^2+b^2}}$.

10. Prove that the lengths of the semi-axes of the conic

$$ax^2+2hxy+by^2=d$$

are

$$\sqrt{\frac{d}{a+h}} \text{ and } \sqrt{\frac{d}{a-h}}$$

respectively, and that their equations are $x^2-y^2=0$.

11. Find the equation to the directrix and coordinates of focus of the parabola

$$x^2-2xy+y^2-2x-2y+3=0.$$

12. Trace the conic

$$9x^2-24xy+16y^2-2x-39y-11=0.$$

13. If r be the length of the semi-axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

prove that

$$(ab-h^2)^3r^4+\Delta(a+b)(ab-h^2)r^2+\Delta^2=0,$$

where

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

CHAPTER XI

POLAR EQUATION OF A CONIC

1. INTRODUCTION

The polar equation of a conic can always be obtained from the corresponding cartesian equation referred to rectangular coordinates by writing $x=r\cos\theta$, $y=r\sin\theta$. Thus, for example the polar equations of a line, a circle etc. have been obtained in the preceding chapters.

The polar equations of conics are of much use in the case when the pole is at the focus of the conic. In this case the polar equation of the conic takes very simple form which is usually used in the problems. Now we proceed to obtain the same independently.

2. POLAR EQUATION OF A CONIC

2.1 *To find the polar equation of a conic, the focus being at the pole.*

Let S be the focus, ZK the directrix and e the eccentricity of the conic.

Draw SZ perpendicular to the directrix and consider SZ as initial line.

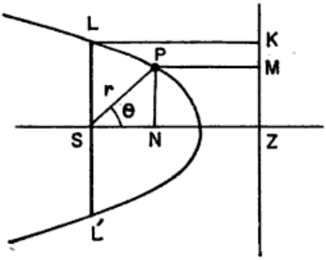

Fig. 11.1

Let $SL(=l)$ be the semi-latus rectum.

Then $$l=SL=e.LK=e.SZ.$$

Let $P(r, \theta)$ be any point on the conic. Draw PM and PN perpendiculars to the directrix and to SZ, respectively. Then, by the definition of the conic, we have

$$SP=e.PM.$$

But $$SP=r,$$

and $$PM=NZ=SZ-SN$$

$$=\frac{l}{e}-r\cos\theta$$

$\therefore$ $$r=e\left(\frac{l}{e}-r\cos\theta\right)$$

$\Rightarrow$ $$\frac{l}{r}=1+e\cos\theta.$$

This is the required equation of the conic in polar form.

Corollary. The equation of the conic whose axis makes an angle α with the initial line is

$$\frac{l}{r}=1+e\cos(\theta-\alpha).$$

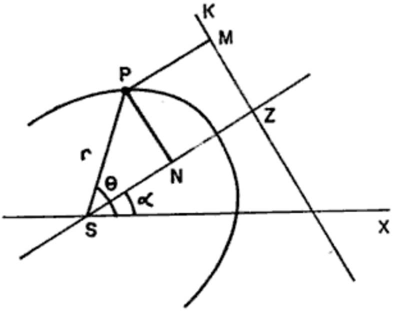

Fig. 11.2

3.2 *To find the equation of a directrix to the conic*

$$\frac{l}{r}=1+e\cos\theta.$$

Let $P(r, \theta)$ be any point on the directrix ZK. Then

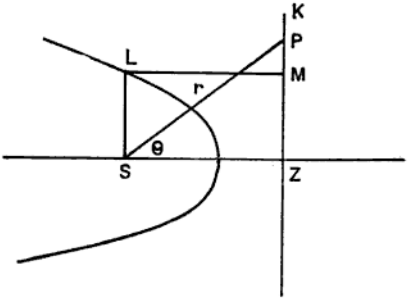

Fig. 11.3

$$SZ=SP\cos\theta=r\cos\theta.$$

But $$SZ=LM=\frac{SL}{e} \quad \because \quad SL=e.LM$$

$$=\frac{l}{e}.$$

$\therefore$ $$\frac{l}{e}=r\cos\theta.$$

$\Rightarrow$ $$\frac{l}{r}=e\cos\theta.$$

This is the equation of the directrix.

3. TRACING OF THE CONIC $\frac{1}{r}=1+e\cos\theta$

Case I. If $e=0$, the equation of the conic reduces to $r=l$, which represents a circle of radius l and centre at the pole.

Case II. If $e=1$, the equation of the conic reduces to

$$\frac{l}{r}=1+\cos\theta,$$

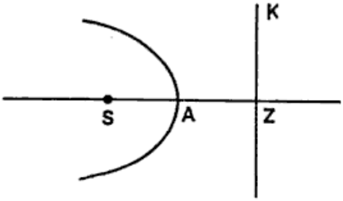

Fig. 11.4

which represents a parabola. Writing the equation of the parabola in the form

$$r=\frac{l}{1+\cos\theta}.$$

When $\theta=0$, $r=\frac{l}{2}$ *i.e.* the curve meets the axis at some point A such that

$$r=\frac{l}{2}.$$

This value of r is minimum.

As θ increases, $1+\cos\theta$ decreases, and therefore r increases until $\theta=\pi$. When $\theta=\pi$, r becomes infinite. Further, as θ increases beyond π, $1+\cos\theta$ increases continuously until $\theta=2\pi$. When $\theta=2\pi$, r takes the value $\frac{l}{2}$. Thus the curve extends to an infinite distance in the direction of AS, see Fig. 11.4.

Case III. If $e<1$, the equation of the conic represents an ellipse which can also be written in the form

$$r=\frac{l}{1+e\cos\theta}.$$

When $\theta=0$, $r=\frac{l}{1+e}$ *i.e.* the curve meets the axis at the point A such that $SA=\frac{l}{1+e}$

As θ increases, $1+e\cos\theta$ decreases, and therefore r increases until $\theta=\pi$. When $\theta=\pi$, $r=\frac{l}{1-e}$ which is positive since $e<1$.

becomes negative, and when $\theta=\pi$, $r=\frac{-l}{e-1}$ *i.e.* the curve meets the axis at some point A' other than A such that

$$SA'=\frac{-l}{e-1}.$$

The value of $1+e\cos\theta$ will remain negative until $\theta=2\pi-\alpha$ (the angle ASK' in Fig. 11.6). When $\theta=2\pi-\alpha$, r becomes infinite again. If $\theta<2\pi-\alpha$, r becomes very large and negative, and if $\theta>2\pi-\alpha$, r becomes very large and positive. The values of r will remain positive while θ varies from $2\pi-\alpha$ to 2π.

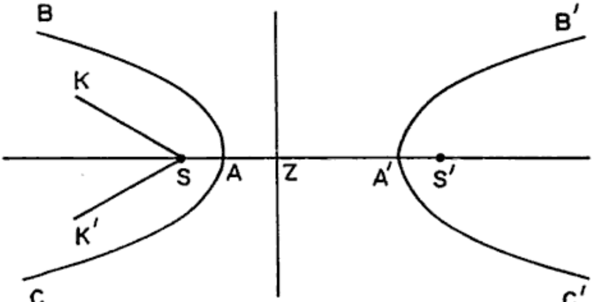

Fig. 11.6

Thus the portion AB is described first, then the portions $C'A'$, $A'B'$ and CA.

3.1 Examples

Example 1. Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.

Solution. Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

Let PSQ be the focal chord of the conic. Let α be the vectorial angle of P. Then vectorial angle of Q will be $\pi+\alpha$. Then

$$\frac{l}{SP}=1+e\cos\alpha$$

and $$\frac{l}{SQ}=1+e\cos(\pi+\alpha)=1-e\cos\alpha.$$

5. If PSQ is a focal chord of a conic; prove that the locus of its middle point is another conic of the same nature as the original conic.

6. Prove that the directrix of the conic $\frac{l}{r}=1+e\cos\theta$ corresponding to the focus other than the pole is

$$\frac{l}{r}=-\frac{e(1-e^2)}{(1+e^2)}\cos\theta.$$

7. A point moves so that the sum of its distances from two fixed points S, S' is constant and equal to $2a$. Show that P lies on the conic

$$\frac{a(1-e^2)}{r}=1-e\cos\theta,$$

referred to S as pole and SS' as the initial line, SS' being equal to $2ae$.

4. CHORD JOINING TWO POINTS

To find the equation of the chord joining two points on a conic.

Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

Let $\alpha-\beta$ and $\alpha+\beta$ be the vectorial angles of any two points P and Q on the conic, respectively.

The general equation of any line is

$$\frac{l}{r}=A\cos\theta+B\cos(\theta-\alpha).$$

This will pass through the points P and Q if

$$1+e\cos(\alpha-\beta)=A\cos(\alpha-\beta)+B\cos\beta$$

and

$$1+e\cos(\alpha+\beta)=A\cos(\alpha+\beta)+B\cos\beta.$$

These equations imply that

$$A=e \quad\text{and}\quad B=\sec\beta.$$

Substituting the values of A and B in the equation of the line, we get

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta.$$

Corollary. The equation of the chord joining the points whose vectorial angles are $\alpha-\beta$ and $\alpha+\beta$ on the conic

$$\frac{l}{r}=1+e\cos(\theta-\gamma) \text{ is}$$

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

5. TANGENT AND NORMAL

5.1 Equation of tangent. *To find the equation of the tangent at a point whose vectorial angle is α to a conic.*

Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

The equation of the chord joining the two points whose vectorial angles are $\alpha-\beta$, $\alpha+\beta$ is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)+e\cos\theta.$$

This chord will become the tangent at the point whose vectorial angle is α if we take $\beta=0$ in its equation, and we get

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta.$$

Corollary. The equation of the tangent at the point whose vectorial angle is α to the conic

$$\frac{l}{r}=1+e\cos(\theta-\gamma)$$

is
$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos(\theta-\gamma).$$

5.2 Equation of normal. *To find the equation of the normal at a point whose vectorial angle is α to a conic.*

Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

The equation of the tangent at the point whose vectorial angle is α, is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta.$$

The coordinates of the point of contact of the tangent are

$$\left(\frac{l}{1+e\cos\alpha}, \alpha\right).$$

The equation of the line perpendicular to the tangent is

$$\frac{A}{r}=\cos\left(\theta+\frac{\pi}{2}-\alpha\right)+e\cos\left(\theta+\frac{\pi}{2}\right)$$

$$\Rightarrow \qquad \frac{A}{r}=-\sin(\theta-\alpha)-e\sin\theta.$$

This line will pass through the point

$$\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$$

if

$$\frac{A(1+e\cos\alpha)}{l}=-e\sin\alpha$$

$$\Rightarrow \qquad A=\frac{-l\,e\sin\alpha}{1+e\cos\alpha}.$$

Hence the equation of the normal at the point whose vectorial angle is α, is

$$\frac{l\,e\sin\alpha}{r(1+e\cos\alpha)}=\sin(\theta-\alpha)+e\sin\theta.$$

5.3 Examples

Example 1. Find the condition that the line

$$\frac{l}{r}=A\cos\theta+B\sin\theta$$

may be a tangent to the conic

$$\frac{l}{r}=1+e\cos\theta.$$

Solution. The equation of the tangent at the point whose vectorial angle is α to the given conic is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta$$

$$\Rightarrow \qquad \frac{l}{r}=(e+\cos\alpha)\cos\theta+\sin\alpha\sin\theta.$$

Thus when $e=1$, the locus is tangent at the vertex to the parabola

$$\frac{l}{r}=1+\cos\theta.$$

Example 3. The normal to the conic

$$\frac{l}{r}=1+e\cos\theta$$

at the point whose vectorial angle is α meets the curve again at the point whose vectorial angle is β. Prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-\frac{1+2e\cos^2\frac{\alpha}{2}+e^2}{1-2e\sin^2\frac{\alpha}{2}+e^2}.$$

Solution. The equation of the normal to the given conic at the point whose vectorial angle is α, is

$$\frac{le\sin\alpha}{r(1+e\cos\alpha)}=\sin(\theta-\alpha)+e\sin\theta.$$

This normal will pass through the point

$$\left(\frac{l}{1+e\cos\beta}, \beta\right)$$

if $$\frac{e\sin\alpha(1+e\cos\beta)}{1+e\cos\alpha}=\sin(\beta-\alpha)+e\sin\beta$$

$$\Rightarrow \quad (1+e\cos\alpha+e^2)\sin(\alpha-\beta)+e(\sin\alpha-\sin\beta)=0$$

$$\Rightarrow \quad (1+e\cos\alpha+e^2).2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$+2e\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=0$$

$$\Rightarrow \quad (1+e\cos\alpha+e^2)\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$
$$+e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=0$$

$$\Rightarrow \quad (1+e\cos\alpha+e^2)\left(1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$
$$+e\left(1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)=0.$$

$$\Rightarrow \quad (1+e\cos\alpha+e^2-e)\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+(1+e\cos\alpha+e^2+e)=0$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-\frac{1+2e\cos^2\frac{\alpha}{2}+e^2}{1-2e\sin^2\frac{\alpha}{2}+e^2}.$$

EXERCISES

1. Prove that the line $\frac{l}{r}=A\cos\theta+B\sin\theta$ may be a tangent to the conic $\frac{l}{r}=1+e\cos(\theta-\gamma)$ if

$$A^2+B^2-2e(A\cos\gamma+B\sin\gamma)+e^2-1=0.$$

2. Prove that the two conics $\frac{l_1}{r}=1+e_1\cos\theta$ and

$$\frac{l_2}{r}=1+e_2\cos(\theta-\alpha)$$

will touch one another if

$$l_1^2(1-e_2^2)+l_2^2(1-e_1^2)=2l_1l_2(1-e_1e_2\cos\alpha).$$

3. If a chord of the conic $\frac{l}{r}=1+e\cos\theta$ subtends a constant angle 2α at the focus; prove that the locus of the point where it meets the internal bisector of the angle 2α is the conic

$$\frac{l\cos\alpha}{r}=1+e\cos\alpha\cos\theta.$$

4. PQ is a variable chord of a conic having a focus at S and the angle PSQ is constant. Prove that the locus of the point of intersection of the tangents at P and Q is a conic having S for a focus, and the corresponding directrix in common with the given conic.

5. In any conic prove that the portion of the tangent intercepted between a directrix and the point of contact subtends a right angle at the corresponding focus.

The equations of the tangents at P and Q are

$$\frac{l}{r}=\cos(\theta-\overline{\alpha-\beta})+e\cos\theta \tag{3}$$

and
$$\frac{l}{r}=\cos(\theta-\overline{\alpha+\beta})+e\cos\theta. \tag{4}$$

These tangents will pass through (r_1, θ_1) if

$$\frac{l}{r_1}=\cos(\theta_1-\overline{\alpha-\beta})+e\cos\theta_1 \tag{5}$$

and
$$\frac{l}{r_1}=\cos(\theta_1-\overline{\alpha+\beta})+e\cos\theta_1. \tag{6}$$

From equations (5) and (6), we get

$$\cos(\theta_1-\overline{\alpha-\beta})=\cos(\theta_1-\overline{\alpha+\beta})$$

$$\Rightarrow \qquad \theta_1-(\alpha-\beta)=2n\pi\pm\{\theta_1-(\alpha+\beta)\}.$$

We consider here lower sign, since upper sign would give a special value of β.

$$\therefore \qquad \theta_1-(\alpha-\beta)=2n\pi-\{\theta_1-(\alpha+\beta)\}$$

$$\Rightarrow \qquad \theta_1=n\pi+\alpha.$$

Substituting this value of α in (5), we get

$$\frac{l}{r_1}=\cos(n\pi+\beta)+e\cos\theta_1$$

$$\Rightarrow \qquad \frac{l}{r_1}-e\cos\theta_1=(-1)^n\cos\beta.$$

Thus equation (2) becomes

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=(-1)^n\cos(\theta-\overline{\theta_1-n\pi})$$

$$\Rightarrow \qquad \left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\theta_1).$$

7. DIRECTOR CIRCLE

To find the polar equation of the director circle of a conic.

Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

The equations of the tangents to the conic at the points whose vectorial angles are α and β are

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta$$

and $$\frac{l}{r}=\cos(\theta-\beta)+e\cos\theta.$$

The vectorial angle of the point of intersection of these tangents is given by the equation

$$\cos(\theta-\alpha)=\cos(\theta-\beta)$$

$$\Rightarrow \quad \theta-\alpha=\pm(\theta-\beta).$$

We consider here lower sign, since upper sign would give $\alpha=\beta$ which is a contradiction.

$$\therefore \quad \theta=\frac{\alpha+\beta}{2}. \tag{1}$$

Substituting this value of θ in either equation of the tangent and we obtain

$$\frac{l}{r}=\cos\frac{\alpha-\beta}{2}+e\cos\frac{\alpha+\beta}{2}. \tag{2}$$

Transforming the equations of the tangents into cartesian co-ordinates and taking that the tangents are at right angles, then

$$(e+\cos\alpha)(e+\cos\beta)+\sin\alpha\sin\beta=0$$

$$\Rightarrow \quad e^2+e(\cos\alpha+\cos\beta)+\cos(\alpha-\beta)=0$$

$$\Rightarrow \quad e^2+2e\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos^2\frac{\alpha-\beta}{2}-1=0. \tag{3}$$

Eliminating α and β from (3) with the help of (1) and (2), we get

$$e^2+2e\cos\theta\left(\frac{l}{r}-e\cos\theta\right)+2\left(\frac{l}{r}-e\cos\theta\right)^2-1=0$$

$$\Rightarrow \quad r^2(1-e^2)+2elr\cos\theta-2l^2=0.$$

This is the required equation of the director circle.

8. ASYMPTOTES

To find the polar equations of the asymptotes of a conic.

Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta.$$

Let $P(R, \alpha)$ be any point on the conic. Then equation of the tangent to the conic at $P(R, \alpha)$ is

$$\frac{l}{r}=\cos(\theta-\alpha)+e\cos\theta$$

$$\Rightarrow \qquad \frac{l}{r}=(e+\cos\alpha)\cos\theta+\sin\theta\sin\alpha. \tag{1}$$

Since the point $P(R, \alpha)$ lies on the conic, we have

$$\frac{l}{R}=1+e\cos\alpha.$$

The tangent at $P(R, \alpha)$ will become the asymptote if P lies at infinity for which $R\to\infty$ *i.e.* if

$$0=1+e\cos\alpha$$

$$\Rightarrow \qquad \cos\alpha=-\frac{1}{e}. \tag{2}$$

Eliminating α from (1) with the help of (2), we get

$$\frac{l}{r}=\left(e-\frac{1}{e}\right)\cos\theta\pm\sin\theta\sqrt{1-\frac{1}{e^2}}$$

$$\Rightarrow \qquad \left\{\frac{le}{r}-(e^2-1)\cos\theta\right\}^2=(e^2-1)\sin^2\theta.$$

These are the equations of the asymptotes.

MISCELLANEOUS EXERCISES

1. Prove that the equations $\frac{l}{r}=1-e\cos\theta$ and $\frac{l}{r}=-1-e\cos\theta$ represent the same conic.
2. If a chord of an ellipse makes an angle α with the axis; prove that the angle between the tangents at its extremities is

$$\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right).$$

3. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.
4. Prove that the locus of the point of intersection of two tangents to a parabola which cut one another at a constant angle is a

hyperbola having the same focus and the directrix as the original parabola.

5. A focal chord PSP' of an ellipse is inclined at an angle α to the major axis. Prove that the perpendicular from the focus on the tangent at P makes an angle $\tan^{-1}\left(\frac{\sin\alpha}{e+\cos\alpha}\right)$ with the axis.

6. Prove that the locus of the pole of a chord of the conic $\frac{l}{r}=1+e\cos\theta$ which subtends a constant angle 2α at the focus is

$$\frac{l\sec\alpha}{r}=1+e\sec\alpha\cos\theta.$$

Also distinguish between the cases for which $\cos\alpha >=< e$.

7. The tangents at the points P and Q to a parabola meet in T. Prove that $ST^2=SP.SQ$, where S is the focus of the parabola.

8. An ellipse and a parabola have a common focus S and intersect in two real points P and Q, of which P is the vertex of the parabola. If e be the eccentricity of the ellipse and α the angle made by SP with the major axis; prove that

$$\frac{SQ}{SP}=1+\frac{4e^2\sin^2\alpha}{(1-e\cos\alpha)^2}.$$

9. If A, B, C be any three points on a parabola, and the tangents at these points form a triangle $A'B'C'$; prove that

$$SA.SB.SC=SA'.SB'.SC',$$

where S is the focus of the parabola.

10. A conic is described having the same focus and eccentricity as the conic $\frac{l}{r}=1+e\cos\theta$, and the two conics touch at the point $\theta=\alpha$; prove that the length of its latus rectum is

$$\frac{2l(1-e^2)}{e^2+2e\cos\alpha+1}.$$

11. If the tangent at any point of an ellipse makes an angle α with its major axis and an angle β with the focal radius to the point of contact; prove that $e\cos\alpha=\cos\beta$.

20. Given the focus and directrix of a conic, prove that the polar of a given point with respect to it passes through a fixed point.

21. Prove that the equation of the pair of tangents which can be drawn to the conic $\frac{l}{r}=1+e\cos\theta$ from the point (r_1, θ_1) is

$$\left\{\left(\frac{l}{r}-e\cos\theta\right)^2-1\right\}\left\{\left(\frac{l}{r_1}-e\cos\theta_1\right)^2-1\right\}$$
$$=\left\{\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos\left(\theta-\theta_1\right)\right\}^2.$$

22. If the normals at three points whose vectorial angles are α, β, γ to the parabola $r=a\operatorname{cosec}^2\frac{\theta}{2}$ meet in the point (R, δ); prove that $2\delta=\alpha+\beta+\gamma-\pi$.

23. Find the equation to the circle circumscribing the triangle formed by the tangents to a parabola and prove that it passes through the focus.

24. If the normals to the parabola $\frac{l}{r}=1+\cos\theta$ at the points P, Q, R whose vectorial angles are α, β, γ meet in the point $T(\rho, \delta)$; prove that the diameter of the circumcircle of the triangle formed by the tangents at P, Q, R is equal to ST, where S is the focus of the parabola.

25. From the focus S of the conic $\frac{l}{r}=1+e\cos\theta$, radii SP and SQ are drawn at right angles to one another. Prove that the locus of the pole of PQ is a conic having the same focus which is hyperbola, parabola or ellipse according as

$$e>=<\frac{1}{\sqrt{2}}.$$

APPENDIX

OBLIQUE AXES

1. INTRODUCTION

In Chapters I—XI we have studied different loci viz., straight line, circle, parabola etc. in different forms by taking rectangular axes. In the present chapter we study the above loci by taking oblique axes *i.e.* the axes inclined at an angle other than right angle.

2. DISTANCE BEWTEEN TWO POINTS

To express the distance between two points in terms of their coordinates.

Let the axes be inclined at an angle ω and let $P(x_1, y_1)$, $Q(x_2, y_2)$ be any two given points. Draw PM, QN parallel to y-axis and PL parallel to x-axis. Then

$$OM=x_1,\ MP=y_1,\ ON=x_2 \text{ and } NQ=y_2.$$

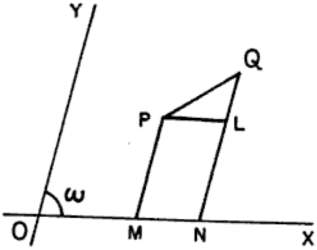

Fig. 1

In $\triangle\ PLQ$, we have

$$PQ^2=PL^2+LQ^2-2PL.LQ\cos QLP.$$

But $QP=LP-LQ=LP-ON=y-c,$

and $NQ=OL=x.$

$$\therefore \quad \frac{y-c}{x}=m$$

$$\Rightarrow \quad y=mx+c.$$

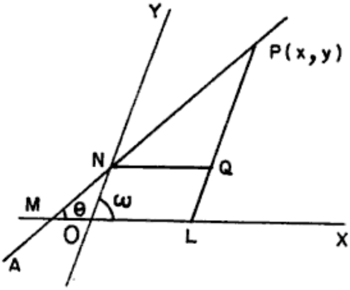

Fig. 2

Now
$$m=\frac{\sin\theta}{\sin(\omega-\theta)}=\frac{\sin\theta}{\sin\omega\cos\theta-\cos\omega\sin\theta}$$
$$=\frac{\tan\theta}{\sin\omega-\cos\omega\tan\theta}$$

which gives
$$\tan\theta=\frac{m\sin\omega}{1+m\cos\omega}.$$

Thus $y=mx+c$ is the equation of the line which cuts off an intercept c on y-axis and makes an angle $\tan^{-1}\left(\frac{m\sin\omega}{1+m\cos\omega}\right)$ with x-axis.

4. ANGLE BETWEEN TWO LINES

To find the angle between two lines referred to the axes inclined at an angle ω.

Let $y=m_1x+c_1$ and $y=m_2x+c_2$

be the equations of the two lines which make angles θ_1, θ_2, respectively with x-axis. Then from § 3, we have are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2-2a_1b_1\cos\omega}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2-2a_2b_2\cos\omega}}.$$

6. ANGLE BETWEEN THE PAIR OF LINES

To find the angle between the lines $ax^2+2hxy+by^2=0$, referred to the axes inclined at an angle ω.

Let $y=m_1x$ and $y=m_2x$ be the lines represented by the given equation and let θ be angle between them. Then

$$m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}.$$

By § 4, we have

$$\begin{aligned}\tan\theta&=\frac{(m_1-m_2)\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2}\\&=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}.\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2}\\&=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}.\sin\omega}{1+\left(-\frac{2h}{b}\right)\cos\omega+\frac{a}{b}}\\&=\frac{2\sqrt{h^2-ab}.\sin\omega}{a+b-2h\cos\omega}.\end{aligned}$$

Corollary 1. The lines represented by $ax^2+2hxy+by^2=0$ will be parallel if $h^2=ab$ and perpendicular if $a+b-2h\cos\omega=0$.

Corollary 2. The equations of the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$ are given by

$$h(x^2-y^2)-(a-b)xy+(ax^2-by^2)\cos\omega=0$$

i.e. $(-a\cos\omega-h)x^2+(b\cos\omega+h)y^2-(b-a)xy=0.$

7. OBLIQUE AXES FROM RECTANGULAR AXES

To transform rectangular axes into oblique axes.

Let (x', y') be the coordinates of a point P referred to rectangular axes OX, OY' and (x, y) the coordinates of the same point P referred to oblique axes OX. OY.

Draw PN parallel to OY' and PL parallel to OY.
Then $ON=x'$, $NP=y'$, $OL=x$ and $LP=y$.
Further draw PM' perpendicular to OX', LL' perpendicular to OX' and LM perpendicular to PM'. Then

$$y' \sin \omega' = M'P = L'L + MP = x \sin \alpha + y \sin \theta.$$

Also OX and OY make angles $\omega' - \alpha$ and $\omega' - \theta$ with OY'. Therefore

$$x' \sin \omega' = x \sin (\omega' - \alpha) + y \sin (\omega' - \theta).$$

Thus $$x' = \frac{x \sin (\omega' - \alpha)}{\sin \omega'} + \frac{y \sin (\omega' - \theta)}{\sin \omega'},$$

$$y' = \frac{x \sin \alpha}{\sin \omega'} + \frac{y \sin \theta}{\sin \omega'},$$

where $$\theta - \alpha = \omega.$$

9. INVARIANTS

If by any change of axes the expression $ax^2 + 2hxy + by^2$ *be changed into* $a'x^2 + 2h'xy + b'y^2$, *then*

$$\frac{a + b - 2h \cos \omega}{\sin^2 \omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2 \omega'}$$

and $$\frac{ab - h^2}{\sin^2 \omega} = \frac{a'b' - h'^2}{\sin^2 \omega'},$$

where ω *and* ω' *are the angles of inclination of the two sets of axes.*

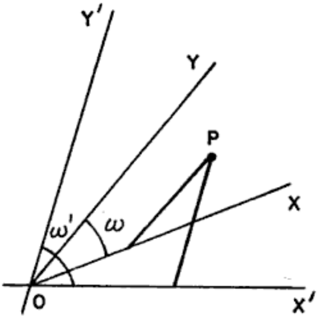

Fig. 6

Let (x', y') be the coordinates of a point P referred to the axes OX' and OY' and (x, y) the coordinates of the same point referred to the axes OX and OY.

Then $\quad OP^2=x^2+y^2+2xy\cos\omega.$

Also $\quad OP^2=x'^2+y'^2+2x'y'\cos\omega'.$

This shows that the expression $x^2+y^2+2xy\cos\omega$ is changed into $x'^2+y'^2+2x'y'\cos\omega'$. But it is given that $ax^2+2hxy+by^2$ is changed into $a'x'^2+2h'x'y'+b'y'^2$. Therefore, if λ be any constant, then

$$ax^2+2hxy+by^2+\lambda(x^2+y^2+2xy\cos\omega)$$

will be changed into

$$a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+y'^2+2x'y'\cos\omega').$$

Thus, if λ be so chosen that one of these expressions is a perfect square, the other will also be a perfect square for the same value of λ.

The expressions will be perfect square if

$$(a+\lambda)(b+\lambda)-(h+\lambda\cos\omega)^2=0$$

and $$(a'+\lambda)(b'+\lambda)-(h'+\lambda\cos\omega')^2=0,$$

respectively.

i.e. $$\lambda^2\sin^2\omega+(a+b-2h\cos\omega)\lambda+ab-h^2=0$$

and $$\lambda^2\sin^2\omega'+(a'+b'-2h'\cos\omega')\lambda+a'b'-h'^2=0.$$

For finding the value of λ these equations must have the same roots. Thus, we get

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

and $$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}.$$

10. EQUATION TO A CIRCLE

To find the equation to a circle when the axes are inclined at an angle ω.

Let $C(h, k)$ be the centre of the circle and $P(x, y)$ be any point on it. Then from §2, we have

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=a^2$$

$$\Rightarrow \quad x^2+y^2+2xy\cos\omega-2(h+k\cos\omega)x-2(k+h\cos\omega)y$$
$$+h^2+k^2+2hk\cos\omega-a^2=0.$$

This is the required equation of the circle.

Corollary. The equation of the circle with centre at origin is

$$x^2+y^2+2xy\cos\omega=a^2.$$

Remark. The general equation of the circle may be considered is

$$x^2+y^2+2xy\cos\omega+2gx+2fy+c=0.$$

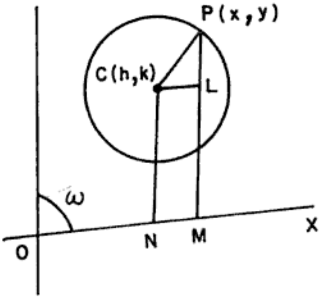

Fig. 7

11. EQUATION TO A PARABOLA

To find the equation to a parabola when the axes are any diameter and the tangent at the extremity of that diameter.

Let $P(at^2, 2at)$ be any point on the parabola $y^2=4ax$. Transferring the origin to the point P by taking axes parallel to the original axes. Let (x', y') be the coordinates of a point Q with respect to these axes. Then equation of the parabola becomes

$$(y'+2at)^2=4a(x'+at^2).$$

Now consider the new axes as PX' and the tangent at P (*i.e.* PY') so that $\angle X'PY'=\omega$. Let (x'', y'') be the coordinates of Q with respect to the axes PX' and PY'. Then

$$x'=x''+y''\cos\omega,$$

$$y'=y''\sin\omega.$$

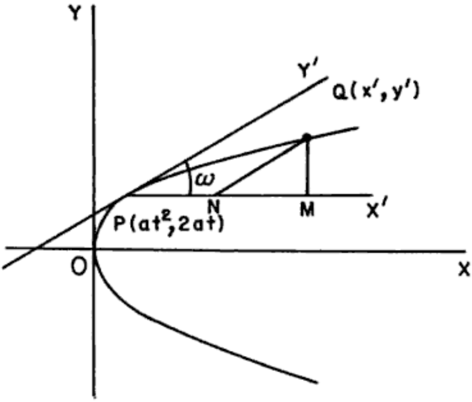

Fig. 8

Therefore the equation of the parabola referred to PX' and PY' as axes becomes

$$(y'' \sin\omega + 2at)^2 = 4a(x'' + y'' \cos\omega + at^2)$$

$$\Rightarrow \qquad y''^2 \sin^2\omega + 4ay''(t \sin\omega - \cos\omega) = 4ax''. \qquad (1)$$

But $\tan\omega$ = slope of the tangent at $P = \frac{1}{t}$.

Hence the equation (1) becomes

$$y''^2 = \frac{4a\,x''}{\sin^2\omega}.$$

This equation may be written as

$$y^2 = 4bx,$$

where
$$b = \frac{4a}{\sin^2\omega}.$$

This shows that the equation of the parabola $y^2 = 4ax$ is a particular case of the equation of the parabola referred to the diameter and the tangent at the extremity of that diameter.

12. EQUATION OF AN ELLIPSE

To find the equation of an ellipse referred to any pair of conjugate diameters as axes.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

Also the line $x'=a'$ is the tangent at P. Substituting $x'=a'$ in (2), we get

$$Aa'^2+2Ha'y'+By'^2=1$$

$$\Rightarrow \qquad 2Ha'y'+By'^2=0.$$

The two roots of this equation have to be zero, therefore $H=0$. Hence the equation of the ellipse is

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1.$$

13. EQUATION OF A HYPERBOLA

To find the equation of a hyperbola referred to any pair of conjugate diameters as axes.

Proof is similar to above.

EXERCISES

1. Prove that the equation of the line which passes through the point (h, k) and is perpendicular to x-axis is $x+y\cos\omega=h+k\cos\omega$.
2. Find the length of the perpendicular drawn from the point $(4, -3)$ upon the line $6x+3y-10=0$, the angle between the axes being 60°.
3. The coordinates of a point P referred to the axes meeting at an angle ω are (h, k), prove that the length of the line joining the feet of the perpendiculars from P upon the axes is
$$\sin\omega\sqrt{h^2+k^2+2hk\cos\omega}.$$
4. From a given point (h, k) perpendiculars are drawn to the axes and their feet are joined, prove that the length of the perpendicular from (h, k) upon this line is
$$\frac{hk\sin^2\omega}{\sqrt{h^2+k^2+2hk\cos\omega}},$$
and that its equation is $hx-ky=h^2-k^2$, ω being the angle between the axes.
5. Prove that the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ make equal

15. Parabolas are drawn to touch the axes, which are inclined at an angle ω, and their directrices all pass through a fixed point (h, k). Prove that all the parabolas touch the line

$$\frac{x}{h+k\sec\omega}+\frac{y}{k+h\sec\omega}=1.$$

16. Prove that the equation to the director circle of the conic $xy=c^2$ is

$$x^2+2xy\cos\omega+y^2=4c^2\cos\omega.$$

ANSWERS

Pages 6—8

1. (*i*) $\sqrt{2}$. (*ii*) 11. (*iii*) $2\sqrt{2}$. (*iv*) $3\sqrt{5}$. (*v*) $\sqrt{10}$.

2. 10, 5, $5\sqrt{5}$. 6. $2\sqrt{2}$.

7. (*i*) $\left(-3, -\frac{1}{9}\right)$. (*ii*) $\left(5, \frac{25}{7}\right)$. (*iii*) $(-19, 8)$.

(*iv*) $(11, 3)$. (*v*) $(2, -1)$. 8. $\left(\frac{9}{5}, \frac{26}{5}\right)$. 9. $(7, 7)$.

10. $(0, 3)$. 11. $2:3$. 12. (*i*) 30. (*ii*) 6. (*iii*) $\frac{1}{2}(a^2-b^2)$.

(*iv*) $a \sin(\alpha-\beta)$. 13. $\frac{11}{2}$. 17. 7.

Pages 12—13

1. (*i*) $\sqrt{34}$. (*ii*) $\sqrt{76}$. (*iii*) $2\sqrt{3}$. (*iv*) $2a\sqrt{5}$.

3. (*i*) $\frac{9}{2}-2\sqrt{3}$. (*ii*) $\frac{13}{2}\sqrt{3}$. (*iii*) $a^2(8-3\sqrt{3})$.

4. (*i*) $r(3\cos\theta+\sin\theta)=0$. (*ii*) $r=4$.

(*iii*) $r=4a \operatorname{cosec}\theta \cot\theta$. (*iv*) $r^2=a^2\sin 2\theta$.

5. (*i*) $(x^2+y^2)^2=a^2(x^2-y^2)$. (*ii*) $x^2+y^2=8x$.

(*iii*) $x^2+y^2=(4-x)^2$.

Pages 17—18

1. $x-3y-1=0$. 3. $x^2+y^2=4$.

5. $x^2+y^2-6x+2y+1=0$. 6. $x^2-3y^2=0$.

7. $x^2-3y^2+2x-2y+2=0$. 9. $x^2-6y+9=0$.

10. $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Pages 28—29

1. (*i*) $x+2y-6=0$. (*ii*) $3x-y+2=0$. (*iii*) $x+5y+13=0$.

(*iv*) $x+y-2=0$.

2. $x+y-2=0$. 3. (*i*) $x+y+2=0$. (*ii*) $2x+3y-13=0$.

(*iii*) $x+y-(a+b)=0$. (*iv*) $2x-(t_1+t_2)y+2at_1t_2=0$.

(*v*) $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$.

4. (*i*) $4x+3y=12$. (*ii*) $3x-5y=15$.

(*iii*) $4x-3y+12=0$. (*iv*) $x-4y=8$. (*v*) $15x-16y=10$.

6. $3x-y=3$.

8. $\frac{4}{3}$; $3x+4y-5=0$; $4x-3y-15=0$.

9. $5x+y-9=0$; $4x+5y-17=0$; $x-4y+8=0$.

10. $3x+4y-2=0$.

11. $3x-y-24=0$.

Pages 32—33

2. (*i*) $(0, 6)$. (*ii*) $(1, 3)$. (*iii*) $(1, -2)$.

3. $2x+y-8=0$.

Page 34

1. (*i*) $\tan^{-1}\frac{1}{7}$. (*ii*) $\tan^{-1}\frac{17}{19}$. (*iii*) $\tan^{-1}31$. (*iv*) $\frac{\pi}{4}$.

2. $y=x+5$.

3. $\tan^{-1}\frac{2}{9}$, $\tan^{-1}\frac{-1}{7}$, $\tan^{-1}5$.

Pages 37—38

1. (*i*) $\frac{9}{5}$. (*ii*) $\frac{66}{13}$. (*iii*) $\frac{12}{5}$

2. $\frac{14}{5}$. 3. $\frac{17}{\sqrt{11}}$.

Page 42

3. (*i*) $\frac{-2x-3y+1}{\sqrt{13}}=\pm\frac{2x-y+3}{\sqrt{5}}$.

(*ii*) $x+y+2=0$, $7x-7y-8=0$.

4. $2x+16y+13=0$.

Page 46

1. $8x-3y-19=0$.

Pages 48—49

1. -8. 2. $\frac{-12}{5}$. 3. $5x-12y+26=0$.

4. $3x-5y+21=0$, $5x+3y+1=0$.

5. $x+y-9=0$.

12. -3. 13. $(4, 3)$. 14. $x=y$.

Pages 56—57

1. (*i*) $x^2+y^2=4$. (*ii*) $x^2+2y^2+16=0$.

3. $x^2+14xy+49y^2+2\sqrt{5}(7x-y)=0$. 4. $150x^2+101y^2=875$.

Pages 65—66

1. (*i*) $3x^2+8xy-3y^2=0$. (*ii*) $xy-4y^2=0$.

3. -3, 5. $(ab'-a'b)^2+4(ah'-a'h)(bh'-b'h)=0$.

7. (*i*) $4x^2+xy-4y^2=0$. (*ii*) $x^2+42xy-y^2=0$.
(*iii*) $7x^2+26xy-7y^2=0$.

9. -1.

Page 71

2. (*i*) 2. (*ii*) $\frac{1}{3}$. (*iii*) 28. (*iv*) 1, $\frac{5}{3}$.

3. (*i*) $\frac{\pi}{4}$. (*ii*) $\tan^{-1}\frac{3}{5}$. (*iii*) $\frac{\pi}{4}$. 4. $\lambda=-10$, $\tan^{-1}\frac{23}{2}$.

Pages 85—86

1. (*i*) $x^2+y^2-2x-4y-11=0$. (*ii*) $x^2+y^2+6x+8y+16=0$.
(*iii*) $9x^2+9y^2-12x-6y-22=0$.

2. (*i*) $(-2, -4)$, $\sqrt{61}$. (*ii*) $(2, 0)$, $2\sqrt{5}$.

(*iii*) $\left(-1, \frac{5}{6}\right)$. $\frac{\sqrt{61}}{6}$.

3. (*i*) $x^2+y^2-12x-6y+40=0$. (*ii*) $x^2+y^2+3x+y-6=0$.
   (*iii*) $x^2+y^2-2x+2y-11=0$.
4. (*i*) $x^2+y^2-19x-3y=0$. (*ii*) $x^2+y^2-ax-by=0$.
   (*iii*) $7x^2+7y^2-27x-25y+18=0$.
5. $x^2+y^2+14x+22y-35=0$.
6. $x^2+y^2-4x-5y=0$.
7. (*i*) $x^2+y^2-8x-8y+16=0$. (*ii*) $x^2+y^2+6x-2y-58=0$.
8. $x^2+y^2-17x-19y+50=0$.
9. ± 5.

Pages 91—92

1. $2x+3y=13$. 5. $y=x\sqrt{3}\pm 8$.

7. $\frac{lh+mk+n}{\sqrt{l^2+m^2}}=r$. 10. $(-3, 5), (-3, -\frac{1}{3}), (-\frac{1}{2}, 3)$.

11. $\lambda=-3$. 12. $2x+y=0$. 13. $n=0$.

14. $y=(x-a)\tan\alpha$.

Pages 97—98

2. $(g^2-c)x^2+(f^2-c)y^2+2fgxy=0$.
4. $(2a^2+2ab+b^2)^{1/2}$.

Page 100

1. $5x+18y+36=0$. 2. $ax-by+a^2=0$.

4. $x_1^2+y_1^2=2a^2$. 5. $x^2+y^2=2a^2$.

Pages 105—106

1. $3x-y=4$. 2. $3x+3y-7=0$.

3. $\left(\frac{5}{3}, \frac{5}{2}\right)$. 4. $\left(\frac{5}{4}, \frac{1}{2}\right)$.

5. $y^2+2ax=a^2$. 8. $b^2(x^2+y^2)=(a^2-cx-dy)^2$.

Page 108

1. $8x-5y+21=0$. 2. $\left(\frac{-ln}{l^2+m^2}, \frac{-mn}{l^2+m^2}\right)$.

3. $2(x^2+y^2)-2cx+c^2-a^2=0$. 5. Circle.

Page 110

1. $r^2-r\{r_1\cos(\theta-\theta_1)+r_2\cos(\theta-\theta_2)\}+r_1r_2\cos(\theta_1-\theta_2)=0$.

2. $\left(\frac{1}{2}\sqrt{A^2+B^2}, \tan^{-1}\frac{B}{A}\right)$.

3. $r^2-2ar\operatorname{cosec}\alpha\cos(\theta-\alpha)+a^2\cot^2\alpha=0$.

4. $r\cos(\theta-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2$,
$r\cos(\theta-2\theta_1)=2a\cos^2\theta_1$.

5. $2aA+a^2B^2=1$.

Pages 110—113

1. $x^2+y^2-2x-2y+1=0$.

2. $\left(-\frac{3}{2}, \frac{7}{4}\right)$; $\frac{1}{4}\sqrt{85}$.

5. $\sqrt{l^2+m^2}(x^2+y^2-a^2)=c(lx+my+n)$.

6. Circle.

9. $(x-2)^2+\left(y-\frac{3}{2}\right)^2=1$. 14. $r=a(1+\cos\theta)$.

22. Same circle in both cases.

25. $(ax-by)(ay-bx)+ab(a\pm b)(x\pm y)=0$.

Pages 118—119

2. $3(x^2+y^2)-8x+29y=0$.

3. $3(x^2+y^2)-14x+23y-15=0$.

4. $x^2+y^2-6x-6y+9=0$.

5. $x^2+y^2-16x-18y-4=0$.

6. $x^2+y^2-cx-by+a^2=0$.

7. $x^2+y^2-5x-4y+4=0$. 9. Radical axis.

12. $x^2+y^2-2x-4y+3=0$; $x^2+y^2-8x-6y+17=0$.

13. $25(x^2+ {}^2)+49x+18y-20=0$.

Pages 127—128

1. (*i*) $2x+1=0$. (*ii*) $x+10y-2=0$.
3. (*i*) $(1, 1)$. (*ii*) $(3, 0)$. (*iii*) $(-2, -1)$.

Pages 136—138

1. (*i*) $(1, -2), (-2, 1)$. (*ii*) $(0, -3), (-2, -1)$.
   (*iii*) $(1, 2), (3, 1)$. (*iv*) $(1, -2), (-1, 2)$.
2. $x+y+4=0, (-5, -6)$.
3. $(3, 4)$, $x^2+y^2-2x+4y-115=0$.
5. $(-1, 1), \left(\frac{1}{5}, \frac{8}{5}\right)$; $x^2+y^2+102x+48y-23=0$.
7. $3(x^2+y^2)-11x+3y=0$; $4x-3y+3=0$.
8. $4(x^2+y^2)-3x+41y=0$.
9. $x^2+y^2+4y=0$.
11. $x^2+y^2+2x-2y+2=0$, $x^2+y^2-2x+6y=0$.
13. $(1, 1), (3, 3)$.

Pages 138—141

2. $x^2+y^2+x+2y=0$.
3. $3x+y=0$; $4(x^2+y^2)-9x-3y+4=0$.
4. $2x+1=0$; $(1, -2)$.

Pages 149—150

1. (*i*) $y^2-4y-8x+28=0$.
   (*ii*) $16x^2-24xy+9y^2-100x-50y+125=0$.
   (*iii*) $x^2-2xy+y^2-4y+6=0$.
   (*iv*) $16x^2+24xy+9y^2-164x+102y+589=0$.
2. (*i*) $x^2=-8y$. (*ii*) $y^2-6y+24x-63=0$.
   (*iii*) $y^2+4y+28x-136=0$.

3. (*i*) $x^2+2x-2y-1=0$. (*ii*) $x^2-3x-y=0$.

   (*ii*) $y^2-3y+x-4=0$.

4. (*i*) Vertex (0, 0), focus $\left(0, -\frac{6}{5}\right)$, directrix $y=\frac{6}{5}$,

   latus rectum$=\frac{24}{5}$.

   (*ii*) Vertex $\left(\frac{1}{2}, 2\right)$, focus (0, 2), directrix $x=1$,

   latus rectum$=2$.

   (*iii*) Vertex (2, −1), focus $\left(2, \frac{1}{4}\right)$, directrix $4y+9=0$,

   latus rectum$=5$.

   (*iv*) Vertex $\left(\frac{9}{4}, -1\right)$, focus $\left(\frac{5}{4}, -1\right)$,

   directrix $4x-13=0$, latus rectum$=4$.

Pages 161–162

1. $x-y+a=0$; $x+y+a=0$; $x+y=3a$; $x-y=3a$.

11. $x(x-h)^2+y(x-h)(y-k)+a(y-k)^2=0$.

15. $y^2=a(x-3a)$.

16. $27ay^2=(x-5a)^2(x-a)$.

Page 165

1. $y^2-4ax=(a+x)^2$.

3. (*i*) $\frac{y}{x}=$constant. (*ii*) $y-\lambda x^2=2ax$, λ is constant.

Page 174

1. (*i*) $y^2=2ax$. (*ii*) $y^2=2a(x-a)$.

   (*iii*) $y^2=2a(x-4a)$.

   (*iv*) $(y^2-2ax+8a^2)^2 \tan^2\alpha+16a^2(y^2-4ax)=0$.

Pages 183—186

3. $y^2=a(x-3a)$.

8. $y^2=2a(x-a)$; $y^2=a(x-3a)$.

24. $y^2=16a\,(x-6a)$.

Pages 188—192

6. $al(2m^2+l^2)+m^2n=0$.

Pages 201—202

1. (*i*) $\dfrac{x^2}{32}+\dfrac{y^2}{30}=1$. (*ii*) $\dfrac{x^2}{25}+\dfrac{y^2}{21}=1$.

(*iii*) $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$. (*iv*) $\dfrac{x^2}{54}+\dfrac{y^2}{30}=1$.

(*v*) $13x^2+5y^2=125$.

2. (*i*) $\dfrac{\sqrt{5}}{3}$, $(\pm 2\sqrt{5}, 0)$, $x=\pm\dfrac{18}{\sqrt{5}}$, $\dfrac{16}{3}$.

(*ii*) $\cos\alpha$, $\left(\pm\dfrac{\cos^2\alpha}{\sin\alpha}, 0\right)$, $x=\pm\operatorname{cosec}\alpha$, $\sin 2\alpha$.

(*iii*) $\dfrac{12}{13}$, $(\pm 12, 0)$, $x=\pm\dfrac{169}{12}$, $\dfrac{50}{13}$.

(*iv*) $\dfrac{\sqrt{3}}{2}$, $\left(\pm\dfrac{3\sqrt{3}}{2}, 0\right)$, $x=\pm 2\sqrt{3}$, $\dfrac{3}{2}$.

(*v*) $\dfrac{\sqrt{21}}{5}$, $(0, \pm\sqrt{21})$, $y=\pm\dfrac{25}{\sqrt{21}}$, $\dfrac{8}{5}$.

3. $\dfrac{5x^2}{324}+\dfrac{(y-3)^2}{36}=1$.

4. $\dfrac{(x+2)^2}{40}+\dfrac{y^2}{54}=1$.

Pages 211—214

1. $y=x\sqrt{3}+\sqrt{3a^2+b^2}$.

23. $\dfrac{a^2x^2}{\alpha^4}+\dfrac{b^2y^2}{\beta^4}=1$.

Pages 245—250

3. $x\pm y\pm\sqrt{a^2+b^2}=0$.

20. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4.$

Pages 258—259

1. $\frac{x^2}{4}-\frac{y^2}{5}=\frac{4}{9}.$

2. (*i*) $\frac{x^2}{16}-\frac{y^2}{20}=1.$ (*ii*) $\frac{x^2}{49}-\frac{y^2}{9}=1.$

(*iii*) $\frac{25x^2}{81}-\frac{y^2}{9}=1.$

3. (*i*) Foci $(\pm\sqrt{6}, 0)$; Latus Rectum $=2\sqrt{6}$; Eccentricity $=\sqrt{2}$.

(*ii*) Foci $(\pm 5, 0)$; Latus Rectum $=21$; Eccentricity $=\frac{5}{2}$.

(*iii*) Foci $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$; Latus Rectum $=2\sqrt{7}$;

Eccentricity $=\sqrt{3}$.

4. Foci $(\pm\sqrt{13}, 0)$; Latus Rectum $=\frac{8}{3}$; Eccentricity $=\frac{\sqrt{13}}{3}$.

5. $7x^2-9y^2+54y-144=0.$

6. $3x^2-y^2=12.$

Pages 266—267

4. $4x+9y=0.$

5. $(x^2+y^2)^2=a^2x^2-b^2y^2.$

Pages 278—280

1. $2xy+7x-6y-21=0$; $2xy+7x-6y-24=0.$

2. $3x+2y+4=0.$

3. $12x^2-7xy-12y^2+31x+17y=0.$

Pages 290—291

10. $(xy-c^2)(x^2+y^2)=c^2xy.$

Pages 310—312

1. (*i*) Hyperbola; centre $(-1, 1)$, transverse axis $x+y=0$; length $2\sqrt{6}$; conjugate axis $x-y+2=0$, length $2\sqrt{2}$.

(*iii*) Parabola; axis $3x-4y+7=0$; tangent at the vertex $4x+3y+2=0$; latus rectum 3.

(*iv*) Hyperbola; centre $(-1, 1)$, transverse axis $x-y+2=0$, length $\frac{4}{\sqrt{3}}$; conjugate axis $x+y=0$, length 4.

(*v*) Ellipse; centre $(-2, 3)$; major axis $x+y-1=0$, length $4\sqrt{3}$; minor axis $x-y+5=0$, length 4.

(*vi*) Ellipse; centre $(2, -3)$; major axis $x-y+3=0$, length 6; minor axis $3x-4y-18=0$, length 4.

(*viii*) Parabola; axis $4x-3y+10=0$; tangent at the vertex $3x+4y+5=0$; latus rectum $\frac{1}{5}$.

(*ix*) Ellipse; centre $(-1, 1)$; major axis $2x-y+3=0$, length 4; minor axis $x+2y-1=0$, length 2; ecc. $\frac{\sqrt{3}}{2}$.

(*x*) Parabola; axis $2x-y-1=0$; tangent at the vertex $x+2y-1=0$; latus rectum $\frac{4}{\sqrt{5}}$.

2. Parabola; axis $5x-10y+13=0$; tangent at vertex $10x+5y+13=0$; latus rectum $\frac{12}{\sqrt{125}}$; vertex $\left(\frac{-39}{25}, \frac{13}{25}\right)$; focus $\left(-\frac{9}{5}, \frac{2}{5}\right)$.

3. Hyperbola; centre $(1, 1)$; transverse axis $x-3y+2=0$, length $2\sqrt{2}$, conjugate axis $3x+y-4=0$, length $2\sqrt{2}$, ecc. $\sqrt{2}$.

5. $(3x-4y+7)(4x+3y+1)=7$; centre $(-1, 1)$; transverse axis $x+7y-6=0$, length $\frac{2}{5}\sqrt{14}$; conjugate $7x-y+8=0$, length $\frac{2}{5}\sqrt{14}$; asymptotes $3x-4y+7=0$ and $4x+3y+1=0$.

6. Hyperbola; centre $\left(0, \frac{1}{2}\right)$; transverse axis $3x+2y-1=0$, length $\sqrt{5}$; conjugate axis $2x-3y+\frac{3}{2}=0$, length $2\sqrt{2}$; foci $(1, -1)$, $(-1, 2)$.

8. Hyperbola; centre $(1, 0)$: transverse axis $x-2y-1=0$, length 4, conjugate axis $2x+y-2=0$, length 6; foci

$$\left(1\pm2\sqrt{\frac{13}{5}}, \pm\sqrt{\frac{13}{5}}\right).$$

INDEX

Textbook of
Analytical Geometry of Two Dimensions, 2/e

The book is intended to serve as a textbook for B.A./B.Sc. Hons. and Pass Course students of Indian Universities and abroad. It is also meant for the Engineering students and other professional competitive examinations such as IAS, IES, PCS etc.

The text starts with the introduction of coordinates of a point in a plane, distance formula, area of triangle, polar coordinates, locus and followed by the study of the straight line, transformation of axes, pair of straight lines, circle, systems of circles, parabola, ellipse, hyperbola, polar equation of a conic and tracing of conics. An appendix has been given on oblique axes. The salient features of the book are:

- Presentation of the subject in a natural way
- Description of the concepts with justification
- Grading of exercises
- Exercises (solved and unsolved) after each section and a miscellaneous set of exercises at the end of each chapter
- Notes and remarks at proper places

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