MATHEMATICS

Volume 1

for

IIT-JEE

R.D. SHARMA



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MATHEMATICS

FOR

IIT-JEE

Volume 1

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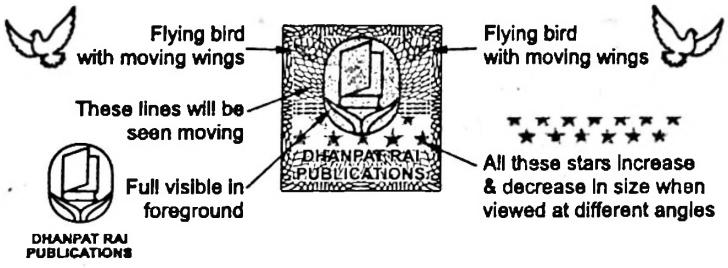
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Dedicated
to the memory of
my Dear friend
Late ***Dr. K.L. Vithal***
whose guidance and insight
always encouraged me
to excel.

SYLLABUS
FOR
IIT - JEE - 2004
COMMON FOR SCREENING TEST AND MAIN EXAMINATION

ALGEBRA

Algebra of Complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.

Quadratic inequalities in one real variable.

Arithmetic and geometric progressions, arithmetic and geometric means, sums of finite arithmetic and geometric progressions, infinite geometric series, sums of squares and cubes of the first n natural numbers.

Logarithms and their properties.

Permutations and combinations, Binomial theorem for a positive integral index, properties of binomial coefficients.

Determinants of order up to three

Matrices as a rectangular array of real numbers, equality of matrices, addition, scalar multiplication and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.

Addition and multiplication rules of probability, conditional probability, independence of events, computation of probability of events using permutations and combinations.

TRIGONOMETRY

Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles, general solution of trigonometric equations.

Relations between sides and angles of a triangle, sine rule, cosine rule, projection rule, Napier's rule, half-angle formula and the area of a triangle, inverse trigonometric functions (principal value only).

ANALYTICAL GEOMETRY

Cartesian coordinates, distance between two points, section formulae, shift of origin.

Equation of a straight line in various forms, angle between two lines, distance of a point from a line.

Lines through the point of intersection of two given lines, equations of the bisector of the angle between two lines, concurrency of lines, centroid, orthocentre, incentre and circumcentre of a triangle.

Equation of a circle in various forms, equations of tangent, normal and chords.

Parametric equations of a circle, intersection of a circle with a straight line of a circle, equation of a circle through the points of intersection of two circles and that of a circle and a straight line.

Equation of a parabola, ellipse and hyperbola in standard form, their foci, directrices and eccentricity, parametric equations, equations of tangent and normal.

Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

Locus Problems.

DIFFERENTIAL CALCULUS

Real valued function of a real variable, into, onto and one-to-one functions, sum, difference product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions.

Limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions.

Even and odd functions, inverse of a function, continuity of composite functions, intermediate value property of continuous functions.

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational trigonometric, inverse trigonometric, exponential and logarithmic functions.

Derivatives of implicit functions, derivatives up to order three, geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, applications of Rolle's Theorem and Lagrange's Men Value Theorem.

INTEGRAL CALCULUS

Integration as the inverse process of differentiation, indefinite integrals of standard functions, definite integrals and their properties, application of the fundamental Theorem of Integral Calculus.

Integration by parts, integration by the methods of substitution and partial fractions, application of definite integrals to the determination of areas involving simple curves.

Formation of ordinary differential equations, solution of homogeneous differential equations, variable separable method, Formation of linear first order differential equations.

VECTORS

Addition of vectors, scalar multiplication, scalar products, dot and cross products, scalar triple products and their geometrical interpretations.

Preface

The Joint Entrance Examination (JEE) conducted jointly by seven Indian Institutes of Technology (IITs) is one of the toughest and most sought after examinations in India. This book "MATHEMATICS FOR IIT-JEE" has been written as per the latest syllabii for the students preparing for IIT-JEE. There is a plethora of books in the market on the subject cited above. These books provide an inarticulated collection of problems and their solutions only as the same have been written with the supposition that mastery can be achieved in a topic merely by working out a certain number of problems on it. However, *I am of the strong opinion that the best way to achieve mastery in a topic is that one should first understand the concepts and theory properly and then the same should be applied in solving problems.*

In the last 22 years of my teaching, I have realized that the sole thing that makes certain topic imperceptible is the inability of the concerned books to put forth clarity of concepts and failure to lay due stress on theoretical and conceptual aspects. In this book *I have made a humble attempt to enlighten the readers in terms of deep comprehension of concepts by incorporating a systematic approach involving essential theory followed by meticulously selected illustrations to explain the applications of concepts.*

This book consists of 41 chapters and hence paying due concern to the convenient handling, it has been divided into two volumes. The broad divisions are as follows :

	Chapters	Topics
Volume 1	1 - 22	Functions, Calculus, Vectors and their applications to Geometry, Complex Numbers, Theory of Equations, Progressions and Permutations and Combinations.
Volume 2	23 - 41	Binomial Theorem, Matrices and Determinants, Inequalities, Co-ordinate Geometry, Trigonometry, Logarithms Probability and Three Dimensional Geometry

Each chapter is divided into sections and sub-sections. In each section theory and concepts pertaining to a topic are discussed in a lucid and unambiguous manner without any prerequisite which are followed by suitably chosen illustrations to illustrate the applications of the concepts learnt in the topic. In order to have good command on the topic a set consisting of large number of illustrative examples has been given at the end of each topic. These examples are based upon the concepts learnt in the concerned section and sections prior to it. At the end of each section or in some cases two or more sections a set of large number of unsolved problems have been given. The difficulty level of these problems is slightly on the lower side in comparison to the illustrative examples. This has been particularly done to boost the confidence of the students in solving problems independently. I hope that students will not find any difficulty in solving unsolved problems in case all illustrative examples have already been worked out independently.

During the preparation of manuscript I had very fruitful discussions with my daughter Shikha (presently studying at IIT, Roorkee). She gave very incisive suggestions and provided very short and logical solutions to many typical problems. Her valuable contribution to improve the quality of the book is worth appreciating. My special thanks and love goes to her. I am also thankful to my son Ravish (studying at Delhi College of Engineering) and wife Sharda for providing moral support during the course of writing this book.

Lastly, but of course not the least, I wish to convey my thanks to the publisher of this book Sh. Ish Kapur for bringing out this voluminous book in an excellent form.

All efforts have been made to keep the book free from errors. Inspite of my best possible efforts some printing errors might have crept in inadvertently because of such a large volume of the book. I shall be grateful to the readers if the same are brought to my notice.

R.D. SHARMA

March, 2003

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CHAPTER

1

FUNCTIONS

1.1 FUNCTION

Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function". If f is a function from a set A to a set B, then we write $f : A \to B$ or $A \xrightarrow{f} B$, which is read as *f is a function from A to B* or *f maps A to B.*

If an element $a \in A$ is associated to an element $b \in B$, then b is called *the 'f-image of a'* or *'image of a under f'* or *'the value of the function f at a'*. Also, a is called the pre-image of b under the function f. We write it as $b = f(a)$.

ILLUSTRATION *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements (A to elements of) B as shown in the following figures :*

We observe that f_1 is not a function from set A to set B, since there is an element $3 \in A$ which is not associated to any element of B. Also, f_2 is not a function from A to B because an element $4 \in A$ is associated to two elements c and e in B. But, f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated to a unique element in B.

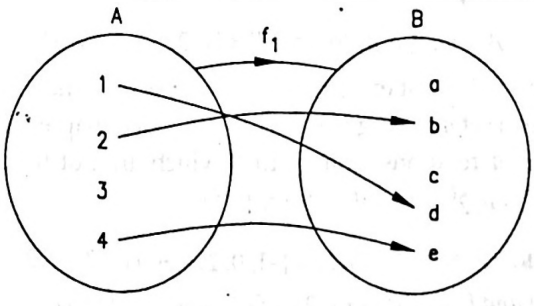


Fig. 1.1

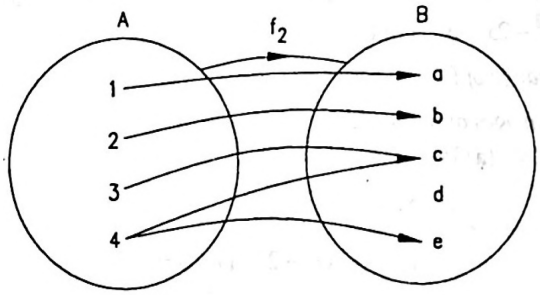


Fig. 1.2

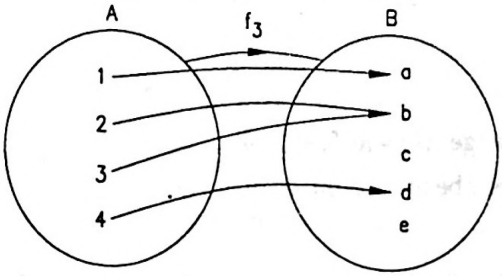


Fig. 1.3

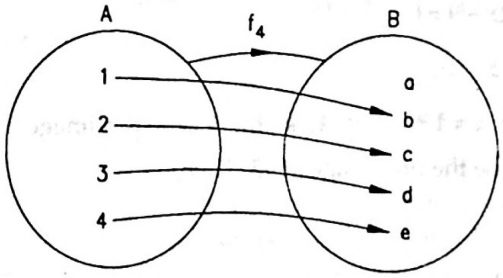


Fig. 1.4

1.2 DOMAIN, CO-DOMAIN AND RANGE

Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f.

Clearly, $f(A) \subseteq B$.

ILLUSTRATION 1 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $B = \{0, 1, 2, 3, 4, 5, 6\}$. *Consider a rule* $f(x) = x^2$. *Then,*

$$f(-2) = (-2)^2 = 4,\ f(-1) = (-1)^2 = 1,$$
$$f(0) = 0^2 = 0,\quad f(1) = 1^2 = 1$$
and $$f(2) = 2^2 = 4$$

Clearly, each element of A is associated to a unique element of B. So, $f: A \to B$ *given by* $f(x) = x^2$ *is a function.*

Clearly, domain $(f) = A = \{-2, -1, 0, 1, 2\}$ *and range* $(f) = \{0, 1, 4\}$.

ILLUSTRATION 2 *Consider a rule* $f(x) = 2x - 3$ *associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because* $f(1) = 2 \times 1 - 3 = -1 \notin N$ *i.e.* $1 \in N$ *is not associated to any element of N.*

ILLUSTRATION 3 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $f: A \to Z$ *given by* $f(x) = x^2 - 2x - 3$. *Find*

(a) the range of f

(b) pre-images of 6, –3 and 5.

SOLUTION (a) We have,

$$f(-2) = (-2)^2 - 2(-2) - 3 = 5,$$
$$f(-1) = (-1)^2 - 2(-1) - 3 = 0,$$
$$f(0) = -3,$$
$$f(1) = 1^2 - 2 \times 1 - 3 = -4$$

and,

$$f(2) = 2^2 - 2 \times 2 - 3 = -3$$

So, range $(f) = \{0, 5, -3, -4\}$

(b) Let x be the pre-image of 6. Then,

$f(x) = 6$

$\Rightarrow x^2 - 2x - 3 = 6$

$\Rightarrow x^2 - 2x - 9 = 0$

$\Rightarrow x = 1 \pm \sqrt{10}$.

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.

Let x be the pre-image of –3. Then,

$f(x) = -3$

$\Rightarrow x^2 - 2x - 3 = -3$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x = 0, 2$.

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of –3.

Let x be the pre-image of 5. Then,

$f(x) = 5$

$\Rightarrow x^2 - 2x - 3 = 5$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x - 4)(x + 2) = 0$

$\Rightarrow x = 4, -2$.

Since, $-2 \in A$. So, –2 is the pre-image of 5.

1.3 DESCRIPTION OF A FUNCTION

Let $f: A \to B$ be a function such that its domain A consists of finite number of elements. Then, $f(x)$ can be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1, 1, 2, 3\}$ and B is the set of real numbers, the a function $f: A \to B$ can be described as $f(-1) = 3$, $f(1) = 0$, $f(2) = 3/2$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases functions are generally described by a formula. For example, $f: Z \to Z$ given by $f(x) = x^2 + 1$ or, $f: R \to R$ given by $f(x) = e^x$ etc.

1.4 EQUAL FUNCTION

Two functions f and g are said to be equal iff

(i) the domain of f = domain of g,

(ii) the co-domain of f = the co-domain of g, and

(iii) $f(x) = g(x)$ *for every x belonging to their common domain.*

If two functions f and g are equal, then we write $f = g$.

ILLUSTRATION *Let* $A = \{1, 2\}$, $B = \{3, 6\}$, *and* $f: A \to B$ *given by* $f(x) = x^2 + 2$ *and* $g: A \to B$ *given by* $g(x) = 3x$.

We observe that f and g have the same domain and co-domain. Also,

$$f(1) = 3 = g(1) \quad \text{and} \quad f(2) = 6 = g(2)$$

Hence, $f = g$.

1.5 NUMBER OF FUNCTIONS

Let A and B be two finite sets having m and n elements respectively. Then, each element of set A can be associated to any one of n elements of set B. So, total number of functions from set A to set B is equal to the number of ways of doing m jobs where each job can be done in n ways. The total number of such ways is $n \times n \times n \ldots\ldots \times n = n^m$.
(m-times)

Hence, the total number of functions from A to B is n^m i.e., $[O(B)]^{O(A)}$.

For example, the total number of functions from a set $A = \{a, b, c, d\}$ to a set $B = \{1, 2, 3\}$ is $3^4 = 81$.

The total number of relations from a set A having m elements to a set B having n elements is 2^{mn} (see chapter 2). So, the number of relations from A to B which are not functions is $2^{mn} - n^m$ i.e. $2^{O(A) \times O(B)} - [O(B)]^{O(A)}$.

ILLUSTRATION 1 *Given* $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, -1, 0, 18, 28, 108\}$ *and* $f(x) = x^2 - x - 2$. *Is* $f(A) = B$? *Find* $f(A)$.

SOLUTION We have,

$$f(-1) = (-1)^2 - (-1) - 2 = 0,$$
$$f(0) = 0^2 - 0 - 2 = -2$$
$$f(2) = 2^2 - 2 - 2 = 0,$$
$$f(5) = 5^2 - 5 - 2 = 18,$$
$$f(6) = 6^2 - 6 - 2 = 28$$

and, $f(11) = 11^2 - 11 - 2 = 108.$

$\therefore$ $f(A) = \{f(x) : x \in A\}$
$= \{f(-1), f(0), f(2), f(5), f(6), f(11)\}$
$= \{0, -2, 18, 28, 108\}$

We observe that $-1 \in B$ but $-1 \notin f(A)$. So, $f(A) \neq B$.

ILLUSTRATION 2 *Let $f : R \to R$ be given by $f(x) = x^2 + 3$. Find (a) $\{x : f(x) = 28\}$*

(b) the pre-images of 39 and 2 under f.

SOLUTION (a) We have,

$f(x) = 28$
$\Rightarrow x^2 + 3 = 28$
$\Rightarrow x^2 = 25$
$\Rightarrow x = \pm 5.$

$\therefore$ $\{x : f(x) = 28\} = \{-5, 5\}$.

(b) Let x be the pre-image of 39. Then,

$$f(x) = 39 \Rightarrow x^2 + 3 = 39 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$$

So, pre-images of 39 are -6 and 6.

Let x be the pre-image of 2. Then,

$$f(x) = 2 \Rightarrow x^2 + 3 = 2 \Rightarrow x^2 = -1$$

Since, no real value of x satisfies the equation $x^2 = -1$. So, 2 does not have any pre-image under f.

ILLUSTRATION 3 *If $f : R \to R$ be defined as follows :*

$$f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q. \end{cases}$$

Find (a) $f(1/2), f(\pi), f(\sqrt{2})$

(b) Range of f

(c) pre-images of 1 and -1.

SOLUTION (a) It is evident from the definition of f that at every rational point the function attains value 1 and at every irrational point it attains value -1. So,

$$\frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right) = 1,$$
$$\pi \notin Q \Rightarrow f(\pi) = -1$$

and, $\sqrt{2} \notin Q \Rightarrow f(\sqrt{2}) = -1.$

(b) We have,

Range of $f = \{f(x) : x \in R\}$.

Also, by definition $f(x)$ attains values 1 or -1 according as x is rational or irrational and a real number is either rational or irrational.

$\therefore$ Range of $f = \{1, -1\}$.

(c) Since $f(x) = 1$ for all $x \in Q$. Therefore, pre-images of 1 are rational numbers i.e.

$$f^{-1}(1) = Q.$$

Also, -1 is the image of every real number which is not rational. So,

$f^{-1}(1) = R - Q =$ Set of irrational numbers.

ILLUSTRATION 4 *Let $f : R \to R$ be such that $f(x) = 2^x$. Determine (a) Range of f*

(b) $\{x : f(x) = 1\}$

(c) *whether $f(x + y) = f(x) \cdot f(y)$ holds.*

SOLUTION (a) Since 2^x is positive for every $x \in R$. So, $f(x) = 2^x$ is a positive real number for every $x \in R$. Moreover, for every positive real number x, there exist $\log_2 x \in R$ such that

$$f(\log_2 x) = 2^{\log_2 x} = x$$

Hence, we conclude that the range of f is the set of all positive real numbers.

(b) $\because$ $f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0.$

$\therefore$ $\{x : f(x) = 1\} = \{0\}$.

(c) $\because$ $f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x) f(y)$

$\therefore$ $f(x + y) = f(x) f(y)$ holds for all $x, y \in R$.

1.6 FUNCTION AS A RELATION

Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B if

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$,*

(ii) $(a, b) \in f$ *and* $(a, c) \in f \Rightarrow b = c$.

Thus, a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b is called the image of a under f.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given below :*

$f_1 = \{(1, 2), (2, 3), (3, 4)\}$,

$f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$,

$f_3 = \{(1, 3), (2, 4)\}$.

Then f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B, because

$1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f i.e.,

Domain of $f = \{a : (a, b) \in f\}$

Range of $f = \{b : (a, b) \in f\}$

ILLUSTRATION 2 *If $a, b \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set ?*

(a) $f_1 = \{(x, y) : y = x + 1\}$ *(b)* $f_2 = \{(x, y) : x + y > 4\}$

(c) $f_3 = \{(x, y) : y < x\}$ *(d)* $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

SOLUTION If we express f_1, f_2, f_3 and f_4 as sets of ordered pairs. Then, we have

$f_1 = \{(1, 2), (2, 3), (3, 4)\}$,

$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$

$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$

and,

$f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

(a) We have,

$f_1 = \{(1, 2), (2, 3), (3, 4)\}$.

We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f_1. So, f_1 is not a function from the given set to itself.

(b) We have,

$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$.

We observe that 2, 3, 4 have appeared more than once as first components of the ordered pairs in f_2. So, f_2 is not a function.

(c) We have,

$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$.

We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f_3. So, f_3 is not a function.

(d) We have,

$f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

We observe that each element of the given set has appeared as first components in one and only one ordered pair of f_4. So, f_4 is a function in the given set. In this case

Range of $f = \{1, 2, 3, 4\}$

ILLUSTRATION 3 *Express the following functions as sets of ordered pairs and determine their ranges :*

(a) $f : A \to R, f(x) = x^2 + 1$, *where* $A = \{-1, 0, 2, 4\}$.

(b) $g : A \to N, g(x) = 2x$, *where* $A = \{x : x \in N, x \le 10\}$.

SOLUTION (a) We have,

$f(-1) = (-1)^2 + 1 = 2$,

$f(0) = 0^2 + 1 = 1$

$f(2) = 2^2 + 1 = 5$

and, $f(4) = 4^2 + 1 = 17$

So, $f = \{(x, f(x)) : x \in A\}$

$= \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$

(b) We have, $A = \{1, 2, 3, ..., 10\}$. So,

$g(1) = 2 \times 1 = 2$, $g(2) = 2 \times 2 = 4$ $g(3) = 2 \times 3 = 6$,

$g(4) = 2 \times 4 = 8$, $g(5) = 2 \times 5 = 10$, $g(6) = 2 \times 6 = 12$,

$g(7) = 2 \times 7 = 14$, $g(8) = 2 \times 8 = 16$, $g(9) = 2 \times 9 = 18$

and, $g(10) = 2 \times 10 = 20$.

$\therefore$ $g = \{(x, g(x) : x \in A\}$

$= \{(1, 2), (2, 4), (3, 6), ..., (10, 20)\}$

Range of $g = g(A) = \{g(x) : x \in A\}$

$= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

ILLUSTRATION 4 *Find the domain for which the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.*

SOLUTION We have,

$f(x) = g(x)$

$\Rightarrow 2x^2 - 1 = 1 - 3x$

$\Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow (x + 2)(2x - 1) = 0$

$\Rightarrow x = -2, 1/2$.

Thus, $f(x)$ and $g(x)$ are equal on the set $\{-2, 1/2\}$.

ILLUSTRATION 5 *Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function ? If this is described by the formula, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β ?*

SOLUTION Since no two ordered pairs in g have the same first component. So, g is a function such that

$g(1) = 1, g(2) = 3, g(3) = 5$ and $g(4) = 7$.

It is given that $g(x) = \alpha x + \beta$. Therefore,

$g(1) = 1$ and $g(2) = 3$

$\Rightarrow \alpha + \beta = 1$ and $2\alpha + \beta = 3$

$\Rightarrow \alpha = 2, \beta = -1$

ILLUSTRATION 6 *Let $f : R \to R$ be a function given by $f(x) = x^2 + 1$. Find :*

(i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$.

SOLUTION Recall that if $f : A \to B$ such that $y \in B$. Then, $f^{-1}\{y\} = \{x \in A : f(x) = y\}$. In other words, $f^{-1}\{y\}$ is the set of pre-images of y.

(i) Let $f^{-1}(-5) = x$. Then,

$$f(x) = -5 \Rightarrow x^2 + 1 = -5$$

Clearly, this equation is not solvable in R. So,

$$f^{-1}\{-5\} = \phi.$$

(ii) Let $f^{-1}\{26\} = x$. Then,

$$f(x) = 26 \Rightarrow x^2 + 1 = 26 \Rightarrow x = \pm 5.$$

$$\therefore f^{-1}\{26\} = \{-5, 5\}.$$

(iii) Let $f^{-1}\{10, 37\} = x$. Then $f(x) = 10$ or, $f(x) = 37$.

$$\Rightarrow x^2 + 1 = 10 \text{ or, } x^2 + 1 = 37$$

$$\Rightarrow x = \pm 3, \quad \text{or, } x = \pm 6$$

So, $f^{-1}\{10, 37\} = \{3, -3, 6, -6\}$.

1.7 KINDS OF FUNCTIONS

If $f: A \to B$ is a function, then f associates all elements of set A to elements in set B such that an element of set A is associated to a unique element of set B. Following these two conditions we may associate different elements of set A to different elements of set B or more than one element of set A may be associated to the same element of set B. Similarly, there may be some elements in B which do not have their pre-images in A or all elements in B may have their pre-images in A. Corresponding to each of these possibilities we define a type of a function as given below.

ONE-ONE FUNCTION (INJECTION) : *A function $f: A \to B$ is said to be a one-one function or an injection if different elements of A have different images in B.*

Thus, $f: A \to B$ is one-one

$$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b) \Rightarrow a = b \text{ for all } a, b \in A$$

ILLUSTRATION 1 *A function which associates to each country in the world, its capital, is one-one because different countries have their different capitals.*

ILLUSTRATION 2 *Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams :*

Clearly, $f: A \to B$ is a one-one function. But, $g: X \to Y$ is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

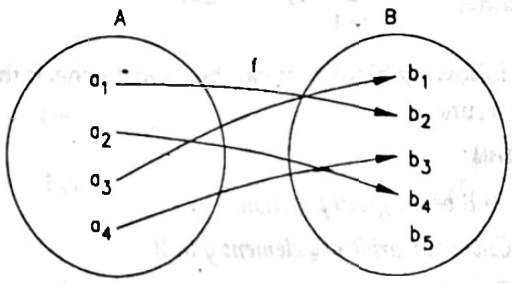


Fig. 1.5

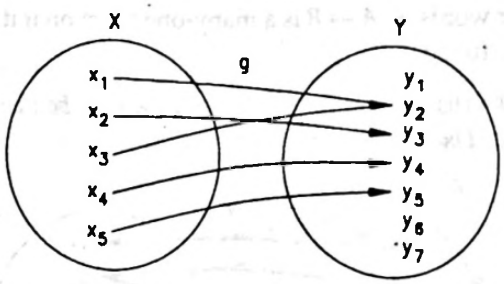


Fig. 1.6

ILLUSTRATION 3 *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $f: A \to B$ be a function defined by $f(x) = x + 2$ for all $x \in A$.*

We have, $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$

Clearly, different elements in A have different images under function f. So, $f: A \to B$ is an injection.

Let $f: A \to B$ be a function such that A is an infinite set and we wish to check the injectivity of f. In such a case it is not possible to list the images of all elements of set A to see whether different elements of A have different images or not. The following algorithm provides a systematic procedure to check the injectivity of a function.

ALGORITHM

STEP I *Take two arbitrary elements x, y (say) in the domain of f.*

STEP II *Put $f(x) = f(y)$*

STEP III *Solve $f(x) = f(y)$.*

If $f(x) = f(y)$ gives $x = y$ only, then $f: A \to B$ is a one-one function (or an injection). Otherwise it is not.

REMARK 1 *Injectivity of a function can also be checked from its graph. If any straight line parallel to x-axis intersects the curve $y = f(x)$ exactly at one point, then the function $f(x)$ is one-one or an injection. Otherwise it is not.*

REMARK 2 *If $f: A \to B$, then $x = y \Rightarrow f(x) = f(y)$ is true for all $x, y \in A$. However, $f(x) = f(y) \Rightarrow x = y$ is true only when f is an injective map.*

REMARK 3 *If $f: R \to R$ is an injective map, then the graph of $y = f(x)$ is either a strictly increasing curve or a strictly decreasing curve. Consequently,*

$$\frac{dy}{dx} > 0 \text{ or } \frac{dy}{dx} < 0 \text{ for all } x.$$

NUMBER OF ONE-ONE FUNCTIONS *If A and B are finite sets having m and n elements respectively, then*

Number of one-one functions from A to B is the number of arrangements of n items by taking m at a time i.e. ${}^nP_m$.

Thus,

$$\text{Number of one-one functions from } A \text{ to } B = \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

MANY-ONE FUNCTION *A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.*

Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function.

ILLUSTRATION 4 *Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams :*

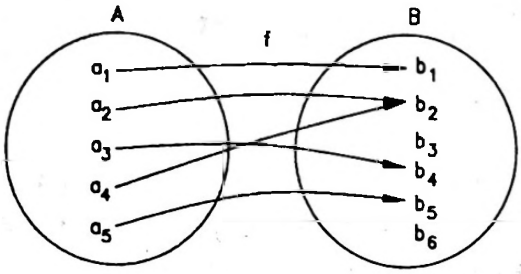


Fig. 1.7

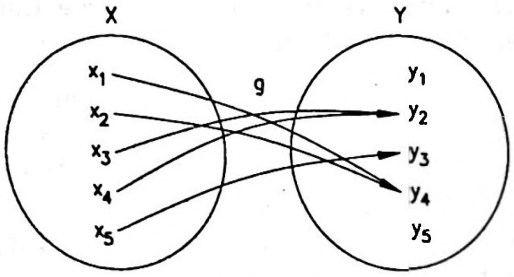


Fig. 1.8

Clearly, $a_2 \neq a_4$ but $f(a_2) = f(a_4)$ and $x_1 \neq x_2$ but $g(x_1) = g(x_2)$. So, f and g are many-one functions.

ILLUSTRATION 5 *Find whether the following functions are one-one or not :*

(i) $f: R \to R$ *given by* $f(x) = x^3 + 2$ *for all* $x \in R$.

(ii) $f: Z \to Z$ *given by* $f(x) = x^2 + 1$ *for all* $x \in Z$

SOLUTION (i) Let x, y be two arbitrary elements of R (domain of f) such that $f(x) = f(y)$. Then,

$$f(x) = f(y) \Rightarrow x^3 + 2 = y^3 + 2 \Rightarrow x^3 = y^3 \Rightarrow x = y$$

Hence, f is a one-one function from R to itself.

(ii) Let x, y be two arbitrary elements of Z such that $f(x) = f(y)$. Then,

$$f(x) = f(y) \Rightarrow x^2 + 1 = y^2 + 1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

Here, $f(x) = f(y)$ does not provide the unique solution $x = y$ but it provides $x = \pm y$. So, f is not a one-one function. Infact, $f(2) = 2^2 + 1 = 5$ and $f(-2) = (-2)^2 + 1 = 5$. So, 2 and –2 are two distinct elements having the same image.

ILLUSTRATION 6 *Let* $A = \{-1, 1, -2, 2\}$ *and* $B = \{1, 4, 9, 16\}$. *Consider* $f: A \to B$ *given by* $f(x) = x^2$. *Then,*

$$f(-1) = 1, f(1) = 1, f(-2) = 4 \text{ and } f(2) = 4.$$

Clearly, 1 and –1 have the same image. Similarly, 2 and –2 also have the same image. So, f is a many-one function.

ILLUSTRATION 7 *Consider a function* $f: Z \to Z$ *given by* $f(x) = |x|$.

Then, f is a many-one function because for every $a \in Z, a \neq 0, a \neq -a$ but $f(a) = f(-a)$.

ONTO FUNCTION (SURJECTION) *A function $f: A \to B$ is said to be an onto function or a surjection if every element of B is the f-image of some element of A i.e., if $f(A) = B$, or range of f is the co-domain of f.*

Thus, $f: A \to B$ is a surjection iff for each $b \in B$, $\exists\, a \in A$ such that $f(a) = b$.

INTO FUNCTION *A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A.*

In other words, $f: A \to B$ is an into function if it is not an onto function.

ILLUSTRATION 8 *Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams :*

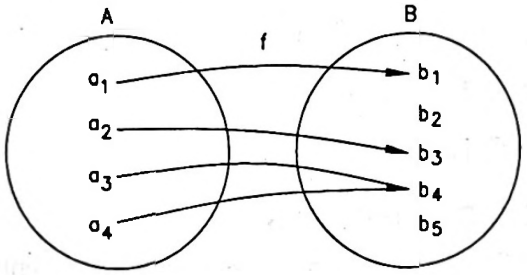


Fig. 1.9

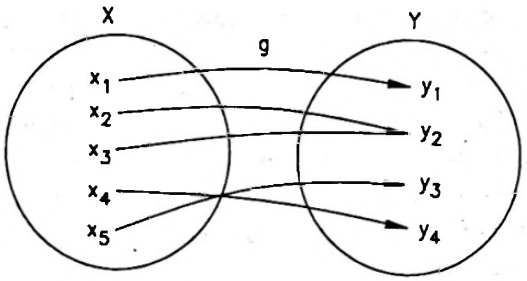


Fig. 1.10

Clearly, b_2 and b_5 are two elements in B which do not have their pre-images in A. So, $f: A \to B$ is an into function. Under function g every element in Y has its pre-image X. So, $g: X \to Y$ is an onto function.

ILLUSTRATION 9 *Let* $A = \{-1, 1, 2, -2\}, B = \{1, 4\}$ *and* $f: A \to B$ *be a function defined by* $f(x) = x^2$. *Then f is onto, because*

$$f(A) = \{f(-1), f(1), f(2), f(-2)\} = \{1, 4\} = B$$

ILLUSTRATION 10 *A function $f: N \to N$ defined by $f(x) = 2x$ is not an onto function, because $f(N) = \{2, 4, 6, \ldots\} \neq N$ (Co-domain).*

NUMBER OF ONTO FUNCTIONS If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is given by

$$\sum_{r=1}^{n} (-1)^{n-r} \, {}^nC_r \, r^m.$$

The following algorithm can be used to check the surjectivity of a function.

ALGORITHM

Let $f: A \to B$ be the given function.

STEP I *Choose an arbitrary element y in B.*

STEP II *Put* $f(x) = y$

STEP III *Solve the equation $f(x) = y$ for x and obtain x in terms of y. Let $x = g(y)$*

STEP IV *If for all values of $y \in B$, the values x obtained from $x = g(y)$ are in A, then f is onto.*

If there are some $y \in B$ for which x, given by $x = g(y)$, is not in A. Then, f is not onto.

BIJECTION (ONE-ONE ONTO FUNCTION): *A function $f: A \to B$ is a bijection if it is one-one as well as onto.*

In other words, a function $f: A \to B$ is a bijection if

(i) it is one-one i.e. $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) it is onto i.e. for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

ILLUSTRATION 11 *Let $f: A \to B$ be a function represented by the following diagram:*

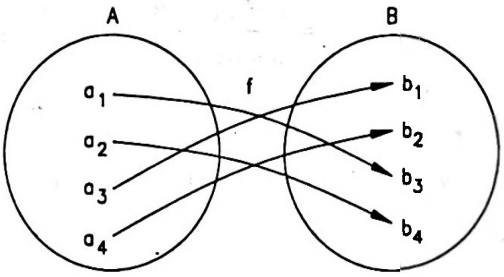


Fig. 1.11

Clearly, f is a bijection since it is both injective as well as surjective.

NUMBER OF BIJECTIONS If A and B are finite sets and $f: A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time i.e. $n!$

IDENTITY FUNCTION: *Let A be a non-empty set. A function $f: A \to A$ is said to be an identity function on set A if f associates every element of set A to the element itself.*

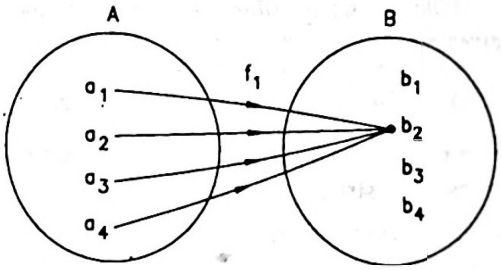
Thus, $f: A \to A$ is an identity function iff $f(x) = x$, for all $x \in A$.

The identity function on a set A is generally denoted by I_A.

It is evident from the definition that I_A is a bijection on A.

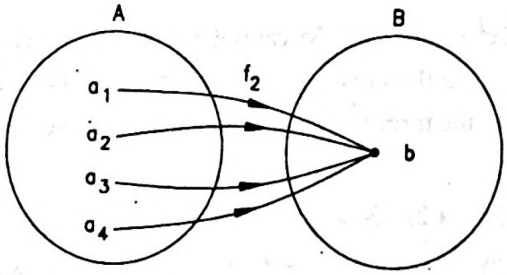
CONSTANT FUNCTION *A function $f: A \to B$ is a said to be a constant function if every element of A has the same image under function f i.e. $f(x) = c$ for all $x \in A$, where $c \in B$.*

It should be noted that the range of a constant function contains only one element. Moreover, a constant function may be one-one or many-one, onto or into as shown in the following diagrams.



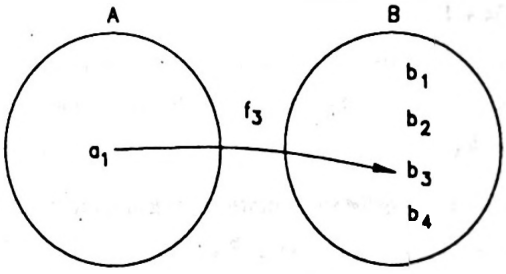
f_1 is many-one and into function

Fig. 1.12



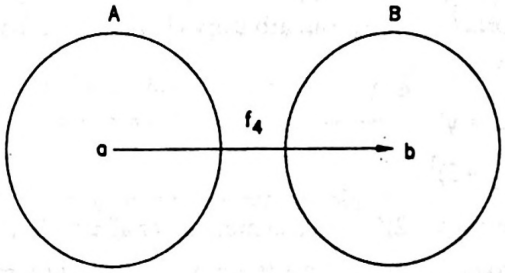
f_2 is many-one and onto function

Fig. 1.13



f_3 is many-one and into function

Fig. 1.14



f_4 is many-one and onto function

Fig. 1.15

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the function $f: Z \to Z$ defined by $f(x) = x^2 + x$ for all $x \in Z$, is a many-one function.*

SOLUTION Let $x, y \in Z$ such that $f(x) = f(y)$. Then,

$$f(x) = f(y)$$

$$\Rightarrow x^2 + x = y^2 + y$$

$$\Rightarrow (x^2 - y^2) + (x - y) = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0$$

$$\Rightarrow x = y \text{ or, } y = -x - 1$$

Since $f(x) = f(y)$ does not provide the unique solution $x = y$ but it also provides $y = -x - 1$. This means that $x \neq y$ but $f(x) = f(y)$ when $y = -x - 1$. For example, if we put $x = 1$ in

$y = -x - 1$ we obtain $y = -2$. This shows that 1 and –2 have the same image under f. Hence, f is a many-one function.

EXAMPLE 2 *Find the set of values of a for which the function* $f: R \to R$ *given by*

$$f(x) = x^3 + (a+2)x^2 + 3ax + 5 \text{ is one-one.}$$

SOLUTION Since $f: R \to R$ is one-one. Therefore, $f(x)$ is either strictly increasing or strictly decreasing.

$\Rightarrow f'(x) > 0$ or, $f'(x)$ for all x.

$\Rightarrow 3x^2 + 2x(a+2) + 3a > 0$ for all $x \in R$

or, $3x^2 + 2x(a+2) + 3a < 0$ for all $x \in R$.

But, $3x^2 + 2x(a+2) + 3a$ cannot be less than zero for all $x \in R$, because the curve $y = 3x^2 + 2x(a+2) + 3a$ represents a parabola of the form $x^2 = 4ay$ for which $y > 0$ for some x.

So,

$3x^2 + 2x(a+2) + 3a > 0$ for all x.

$\Rightarrow 4(a+2)^2 - 36a < 0 \quad [\because ax^2 + bx + c > 0$ for all $x \Rightarrow$ Disc $< 0]$

$\Rightarrow 4(a^2 + 4a + 4 - 9a) < 0$

$\Rightarrow (a^2 - 5a + 4) < 0$

$\Rightarrow (a-1)(a-4) < 0$

$\Rightarrow 1 < a < 4.$

Hence, $f(x)$ is one-one if $a \in (1, 4)$.

EXAMPLE 3 *Discuss the surjectivity of the following functions:*

(i) $f: R \to R$ *given by* $f(x) = x^3 + 2$ *for all* $x \in R$

(ii) $f: R \to R$ *given by* $f(x) = x^2 + 2$ *for all* $x \in R$.

(iii) $f: Z \to Z$ *given by* $f(x) = 3x + 2$ *for all* $x \in Z$.

SOLUTION (i) Let y be an arbitrary element of R. Then,

$f(x) = y$

$\Rightarrow x^3 + 2 = y$

$\Rightarrow x = (y-2)^{1/3}$

Clearly, $(y-2)^{1/3}$ is a real number for all $y \in R$.. Thus, for all $y \in R$ (co-domain) there exists $x = (y-2)^{1/3}$ in R(domain) such that $f(x) = x^3 + 2 = y$.

Hence, $f: R \to R$ is an onto function.

(ii) Clearly, $f(x) = x^2 + 2 \geq 2$ for all $x \in R$. So, negative real numbers in R (domain) do not have their pre-images in R(domain).

Hence, f is not an onto function.

(iii) Let y be an arbitrary element of Z (co-domain). Then,

$$f(x) = y \Rightarrow 3x + 2 = y \Rightarrow x = \frac{y-2}{3}.$$

Clearly, if $y = 0$, then $x = -2/3 \notin Z$.

Thus, $y = 0 \in Z$ does not have its pre-image in Z(domain).

Hence, f is not an onto function.

EXAMPLE 4 *Find the number of surjections from* $A = \{1, 2, ..., n\}$, *to* $B = \{a, b\}$ *for* $n \geq 2$.

SOLUTION Clearly, number of surjections from A to B

$$= \sum_{r=1}^{n} (-1)^{2-r} \, {}^2C_r \, r^n.$$

$$= (-1)^1 \, {}^2C_1 \, (1)^n + (-1)^0 \, {}^2C_2 \cdot 2^n$$

$$= 2^n - 2$$

ALITER The total number of functions from A to B is 2^n of which two functions $f(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in A$ are not surjections. Thus, the number of surjections from A to B is $2^n - 2$.

EXAMPLE 5 *A function* $f: R \to R$ *is defined by*

$$f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2} \qquad \text{[IIT 1996]}$$

find the interval of values of α *for which f is onto. Is the funtion one-to-one for* $\alpha = 3$? *Justify your answer.*

SOLUTION Since $f: R \to R$ is an onto function. Therefore,

Range of $f = R$

$\Rightarrow \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ assumes all real values for real values of x.

Let $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Then, y assumes all real values for real values of x.

Now,

$$y = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$$

$\Rightarrow x^2(\alpha + 8y) + 6x(1-y) - (8 + y\alpha) = 0$ for all $y \in R$

$\Rightarrow 36(1-y)^2 + 4(\alpha + 8y)(8 + y\alpha) \geq 0$ for all $y \in R$

$\Rightarrow 9(y^2 - 2y + 1) + 8\alpha + 8y^2\alpha + y(\alpha^2 + 64) > 0$ for all $y \in R$

$\Rightarrow y^2(8\alpha + 9) + y(\alpha^2 + 46) + (8\alpha + 9) \geq 0$ for all $y \in R$.

$\Rightarrow (\alpha^2 + 46)^2 - 4(8\alpha + 9)^2 \leq 0 \quad [\because$ Disc $\leq 0]$

$\Rightarrow [(\alpha^2 + 46) - 2(8\alpha + 9)][(\alpha^2 + 46) + 2(8\alpha + 9)] \leq 0$

$\Rightarrow (\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0$

$\Rightarrow (\alpha - 14)(\alpha - 2)(\alpha + 8)^2 \leq 0$

$\Rightarrow (\alpha - 14)(\alpha - 2) \leq 0 \quad [\because (\alpha + 8)^2 \geq 0]$

$\Rightarrow 2 \leq \alpha \leq 14$

$\Rightarrow \alpha \in [2, 14]$.

Thus, $f: R \to R$ is onto for $\alpha \in [2, 14]$

When $\alpha = 3$, we have

$$f(x) = \frac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$$

Clearly, $f(1) = f(-1)$ but $1 \neq -1$.

So, $f(x)$ is not one-to-one for $\alpha = 3$.

EXAMPLE 6 *Prove that the function $f: Q \to Q$ given by $f(x) = 2x - 3$ for all $x \in Q$ is a bijection.*

SOLUTION Let us check the injectivity and surjectivity of $f(x)$.

Injectivity : Let x, y be two arbitrary elements in Q. Then,

$$f(x) = f(y)$$
$$\Rightarrow 2x - 3 = 2y - 3$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

Thus,

$f(x) = f(y) \Rightarrow x = y$ for all $x, y \in Q$.

So, f is an injective map.

Surjectivity : Let y be an arbitrary element of Q. Then,

$$f(x) = y$$
$$\Rightarrow 2x - 3 = y$$
$$\Rightarrow x = \frac{y+3}{2}$$

Clearly, for all $y \in Q$, $x = \frac{y+3}{2} \in Q$. Thus, for all $y \in Q$ (co-domain) there exists $x \in Q$ (domain) such that

$$f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y.$$

Therefore, every element in the co-domain has its pre-image in x. So, f is a surjection.

Hence, $f: Q \to Q$ is a bijection.

EXAMPLE 7 *Show that the function $f: R \to R$ defined by $f(x) = 3x^3 + 5$ for all $x \in R$ is a bijection.*

SOLUTION

Injectivity : Let x, y be any two elements of R(domain).

Then,

$$f(x) = f(y)$$
$$\Rightarrow 3x^3 + 5 = 3y^3 + 5$$
$$\Rightarrow x^3 = y^3$$
$$\Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$.

So, f is an injective map.

Surjectivity : Let y be an arbitrary element of R(co-domain). Then,

$$f(x) = y$$
$$\Rightarrow 3x^3 + 5 = y$$
$$\Rightarrow x^3 = \frac{y-5}{3}$$
$$\Rightarrow x = \left(\frac{y-5}{3}\right)^{1/3}$$

Thus, we find that for all $y \in R$ (co-domain) there exists $x = \left(\frac{y-5}{3}\right)^{1/3} \in R$ (domain) such that

$$f(x) = f\left\{\left(\frac{y-5}{3}\right)^{1/3}\right\} = 3\left[\left(\frac{y-5}{3}\right)^{1/3}\right]^3 + 5 = y - 5 + 5 = y$$

This shows that every element in the co-domain has its pre-image in the domain. So, f is a surjection.

Hence, f is a bijection.

EXAMPLE 8 *Let $A = \{x \in R \mid -1 \le x \le 1\} = B$. Show that $f: A \to B$ given by $f(x) = x \mid x \mid$ is a bijection.*

SOLUTION

Injectivity : Let x, y be any two elements in A. Then,

$$x \neq y$$
$$\Rightarrow x \mid x \mid \neq y \mid y \mid$$
$$\Rightarrow f(x) \neq f(y)$$

So, $f: A \to B$ is an injective map.

Surjectivity : Clearly, range of $f = f(A) = B$.

So, $f: A \to B$ is a surjective map.

Hence, f is a bijection.

EXAMPLE 9 *Let C and R denote the sets of all complex numbers and all real numbers respectively. Then show that $f: C \to R$ given by $f(z) = |z|$ for all $z \in C$ is neither one-one nor onto.*

SOLUTION

Injectivity : We find that $z_1 = 1 - i$ and $z_2 = 1 + i$ are two distinct complex numbers such that $|z_1| = |z_2|$

i.e. $z_1 \neq z_2$ but, $f(z_1) = f(z_2)$

This shows that different elements may have the same image. So, f is not an injection.

Surjectivity : f is not a surjection, because negative real numbers in R do not have their pre-images in C. In other words, for every negative real number a there is no complex number $z \in C$ such that $f(z) = |z| = a$. So, f is not a surjection.

EXAMPLE 10 *Show that the function $f: R \to R$ given by $f(x) = ax + b$, where $a, b \in R$, $a \neq 0$ is a bijection.*

SOLUTION

Injectivity : Let x, y be any two real numbers.

Then, $f(x) = f(y)$

$$\Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$ (domain).

So, f is an injection.

Surjectivity : Let y be an arbitrary element of R(co-domain).

Then,

$$f(x) = y$$
$$\Rightarrow ax + b = y$$
$$\Rightarrow x = \left(\frac{y-b}{a}\right)$$

Clearly, $x = \frac{y-b}{a} \in R$(domain) for all $y \in R$ (co-domain)

Thus, for all $y \in R$ (co-domain) there exists $x = \frac{y-b}{a} \in R$ (domain) such that

$$f(x) = f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y.$$

This shows that every element in co-domain has its pre-image in domain. So, f is a surjection. Hence, f is a bijection.

EXAMPLE 11 *Show that the function $f: R \to R$ given by $f(x) = \cos x$ for all $x \in R$, is neither one-one nor onto.*

SOLUTION

Injectivity: We know that : $f(0) = \cos 0 = 1$

and, $f(2\pi) = \cos 2\pi = 1.$

So, different elements in R may have the same image. Hence, f is not an injection.

Surjectivity: Since the values of $\cos x$ lie between –1 and 1, it follows that the range of $f(x)$ is not equal to its co-domain. So, f is not a surjection.

EXAMPLE 12 *Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f: A \to B$ is a mapping defined by $f(x) = \frac{x-1}{x-2}$. Show that f is bijective.*

SOLUTION

Injectivity: Let x, y be any two elements of A. Then,

$$f(x) = f(y)$$

$$\Rightarrow \quad \frac{x-1}{x-2} = \frac{y-1}{y-2}$$

$$\Rightarrow \quad (x-1)(y-2) = (x-2)(y-1)$$

$$\Rightarrow \quad xy - y - 2x + 2 = xy - x - 2y + 2$$

$$\Rightarrow \quad x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of B. Then,

$$f(x) = y$$

$$\Rightarrow \quad \frac{x-1}{x-2} = y$$

$$\Rightarrow \quad (x-1) = y(x-2)$$

$$\Rightarrow \quad x = \frac{1-2y}{1-y}$$

Clearly, $x = \frac{1-2y}{1-y}$ is a real number for all $y \neq 1$.

Also, $\frac{1-2y}{1-y} \neq 2$ for any y, for, if we take $\frac{1-2y}{1-y} = 2$, then we get $1 = 0$, which is wrong.

Thus, every element y in B has its pre-image x in A given by $x = \frac{1-2y}{1-y}$.

So, f is a surjective map.

Hence, f is a bijective map.

EXAMPLE 13 *Let f be an injective map with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$ such that exactly one of the following statements is correct and the remaining are false :*

$$f(x) = 1, \; f(y) \neq 1, \; f(z) \neq 2$$

Find $f^{-1}(1)$.

SOLUTION Three cases arise.

CASE I *When $f(x) = 1$ is true :*

In this case, the remaining two statements are false.

$\therefore \; f(y) = 1$ and $f(z) = 2$

This means that x and y have the same image. So, $f(x)$ is not an injection, which is a contradiction.

Hence, $f(x) = 1$ is not true.

CASE II *When $f(y) \neq 1$ is true :*

If $f(y) \neq 1$ is true, then the remaining statements are false.

$\therefore \; f(x) \neq 1$ and $f(z) = 2$.

This shows that x and y both are not mapped to 1. So, either both are associated to 3 or both are associated to 2 or one of them is mapped to 3 and other to 2. In all the three cases, f cannot be an injective map.

So, $f(y) \neq 1$ is not true.

CASE III *When $f(z) \neq 2$ is true :*

If $f(z) \neq 2$ is true, then the remaining statements are false.

$\therefore \; f(x) \neq 1$ and $f(y) = 1$.

But, f is an injective map.

Thus, we have,

$f(y) = 1$, $f(z) = 3$ and $f(x) = 2$.

Hence, $f^{-1}(1) = y$.

EXAMPLE 14 *Let $f: X \to Y$ be a function defined by $f(x) = a \sin(x + \pi/4) + b \cos x + c$. If f is a bijection, find the sets X and Y.*

SOLUTION We have,

$$f(x) = a \sin(x + \pi/4) + b \cos x + c$$

$$\Rightarrow f(x) = \frac{a \sin x}{\sqrt{2}} + \left(\frac{a}{\sqrt{2}} + b\right)\cos x + c$$

$$\Rightarrow f(x) = r \sin(x + \theta) + c, \text{ where}$$

$$r = \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}} + b\right)^2} \text{ and } \theta = \tan^{-1}\left(\frac{a + \sqrt{2}b}{a}\right)$$

For f to be one-one, we must have

$$-\frac{\pi}{2} \le x + \theta \le \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} - \theta \le x \le \frac{\pi}{2} - \theta$$

$$\Rightarrow x \in \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right]$$

Hence, $X = \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right]$, where $\theta = \tan^{-1}\left(\frac{a + \sqrt{2}b}{a}\right)$

For f to be onto, we must have

Range of $f = y$

But, range of $f = [c-r, c+r]$

Hence, $Y = [c-r, c+r]$, where $r = \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}} + b\right)^2}$

—❑❑❑—

1.8 COMPOSITION OF FUNCTIONS

Let A, B and C be three non-void sets and let $f: A \to B, g: B \to C$ be two functions. Since f is a function from A to B, therefore for each $x \in A$ there exists a unique element $f(x) \in B$. Again, since g is a function from B to C, therefore corresponding to $f(x) \in B$ there exists a unique element $g(f(x)) \in C$. Thus, for each $x \in A$ there exists a unique element $g(f(x)) \in C$.

It follows from the above discussion that f and g when considered together define a new function from A to C. This function is called the composition of f and g and is denoted by gof. We define it formally as follows :

DEFINITION *Let $f: A \to B$ and $g: B \to C$ be two functions. Then a function $gof: A \to C$ defined by*

$$(gof)(x) = g\left(f(x)\right), \text{ for all } x \in A$$

is called the composition of f and g.

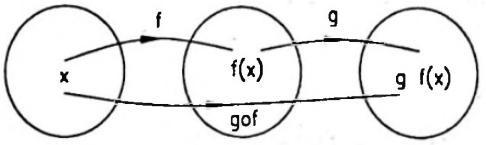


Fig. 1.16

NOTE 1 *It is evident from the definition that gof is defined only if for each $x \in A$, $f(x)$ is an element of g so that we can take its g-image. Hence, for the composition gof to exist, the range of f must be a subset of the domain of g.*

NOTE 2 *It should be noted that gof exists iff the range of f is a subset of domain of g. Similarly, fog exists if range of g is a subset of domain of f.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Let R be the set of real numbers. If $f: R \to R$; $f(x) = x^2$ and $g: R \to R$; $g(x) = 2x + 1$. Then, find fog and gof. Also, show that $fog \neq gof$.*

SOLUTION Clearly, range of f is a subset of domain of g and range of g is a subset of domain of f. So, fog and gof both exist.

Now, $(gof)(x) = g(f(x)) = g(x^2) = 2(x^2) + 1 = 2x^2 + 1$.

And, $(fog)(x) = f(g(x)) = f(2x+1) = (2x+1)^2$.

Since, $2x^2 + 1 \neq (2x+1)^2$.

$\therefore$ $gof \neq fog$.

EXAMPLE 2 *Let $f: R \to R$; $f(x) = \sin x$ and $g: R \to R$; $g(x) = x^2$ find fog and gof.*

SOLUTION Clearly, fog and gof both exist.

Now, $(gof)(x) = g(f(x)) = g(\sin x) = (\sin x)^2 = \sin^2 x$

And, $(fog)(x) = f(g(x)) = f(x^2) = \sin x^2$.

EXAMPLE 3 *If the functions f and g are given by*

$$f = \{(1, 2), (3, 5), (4, 1)\}$$

and

$$g = \{(2, 3), (5, 1), (1, 3)\},$$

find range of f and g. Also, write down fog and gof as sets of ordered pairs.

SOLUTION We have,

Range of f = Set of second components of ordered pairs in

$= \{2, 5, 1\}$

Similarly,

Range of $g = \{3, 1\}$

We have, domain $f = \{1, 3, 4\}$, domain $g = \{2, 5, 1\}$

Clearly, range $f \subset$ domain g and range $g \subset$ domain f.

So, fog and gof both exist.

Now,

$fog(2) = f(g(2)) = f(3) = 5$;

$fog(5) = f(g(5)) = f(1) = 2$;

and $fog(1) = f(g(1)) = f(3) = 5$.

$\therefore$ $fog = \{(2, 5), (5, 2), (1, 5)\}$

Also, $gof(1) = g(f(1)) = g(2) = 3$;

$gof(3) = g(f(3)) = g(5) = 1$

and, $gof(4) = g(f(4)) = g(1) = 3$

$\therefore$ $gof = \{(1, 3), (3, 1), (4, 3)\}$

EXAMPLE 4 *If the function $f: R \to R$ be given by $f(x) = x^2 + 2$ and $g: R \to R$ be given by $g(x) = \frac{x}{x-1}$.*

Find fog and gof.

SOLUTION Clearly, range f = domain g

and, range g = domain f.

So fog and gof both exist.

Now,

$$(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2}{(x-1)^2} + 2$$

and, $(gof)(x) = g(f(x)) = g(x^2+2) = \dfrac{x^2+2}{(x^2+2)-1} = \dfrac{x^2+2}{x^2+1}$

Hence, $gof : R \to R$ and $fog : R \to R$ are given by

$(gof)(x) = \dfrac{x^2+2}{x^2+1}$ and, $(fog)(x) = \dfrac{x^2}{(x-1)^2} + 2$ respectively

EXAMPLE 5 *If $f : R \to R$ is defined by*

$$f(x) = x^2 - 3x + 2, \text{ find } f(f(x)).$$

SOLUTION We have,

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= f(y), \text{ where } y = x^2 - 3x + 2. \\ &= y^2 - 3y + 2 \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

EXAMPLE 6 *If $f, g : R \to R$ are defined respectively by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find*

(i) fog (ii) gof (iii) fof (iv) gog.

SOLUTION Since range f = domain g and range g = domain f.

Therefore, *fog, gof, fof* and *gog* exist.

(i) For any $x \in R$, we have

$$\begin{aligned} (fog)(x) &= f(g(x)) = f(2x-3) \\ &= (2x-3)^2 + 3(2x-3) + 1 \\ &= 4x^2 - 6x + 1 \end{aligned}$$

$\therefore$ $fog : R \to R$ is defined by

$$(fog)(x) = 4x^2 - 6x + 1, \quad \text{for all } x \in R.$$

(ii) For any $x \in R$, we have

$$\begin{aligned} (gof)(x) &= g(f(x)) \\ &= g(x^2 + 3x + 1) \\ &= 2(x^2 + 3x + 1) - 3 \\ &= 2x^2 + 6x - 1 \end{aligned}$$

$\therefore$ $gof : R \to R$ is defined by

$$(gof)(x) = 2x^2 + 6x - 1.$$

(iii) For any $x \in R$, we have

$$\begin{aligned} (fof)(x) &= f(f(x)) \\ &= f(x^2 + 3x + 1) \\ &= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1 \\ &= x^4 + 6x^3 + 14x^2 + 15x + 5 \end{aligned}$$

$\therefore$ $fof : R \to R$ is defined by

$$(fof)(x) = x^4 + 6x^3 + 14x^2 + 15x + 5.$$

(iv) For any $x \in R$, we have

$$\begin{aligned} (gog)(x) &= g(g(x)) \\ &= g(2x-3) \\ &= 2(2x-3) - 3 \\ &= 4x - 9 \end{aligned}$$

$\therefore$ $gog : R \to R$ is defined by

$$(gog)(x) = 4x - 9.$$

EXAMPLE 7 *If $f(x)$ is defined on $[0, 1]$ by the rule*

$$f(x) = \begin{cases} x, & \text{if } x \in Q \\ 1-x, & \text{if } x \notin Q \end{cases}$$

Prove that $fof(x) = x$ for all $x \in [0, 1]$

SOLUTION We have two cases.

CASE I *When $x \in Q$:*

In this case, we have

$$f(x) = x$$

$$fof(x) = f(f(x)) = f(x) = x \qquad [\because f(x) = x, \text{when } x \in Q]$$

CASE II *When $x \notin Q$.*

In this case, we have

$$f(x) = 1 - x$$

$$\begin{aligned} fof(x) &= f(f(x)) \\ &= f(1-x) = 1 - (1-x) = x \end{aligned}$$

Thus, $fof(x) = x$ for all $x \in [0, 1]$

EXAMPLE 8 *Let* $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$

find fof (x).

SOLUTION We have,

$$fof(x) = f(f(x))$$

$$= \begin{cases} f(1+x), & 0 \le x \le 2 \\ f(3-x), & 2 < x \le 3 \end{cases}$$

$$= \begin{cases} f(1+x), & 0 \le x < 1 \\ f(1+x), & 1 < x \le 2 \\ f(3-x), & 2 < x \le 3 \end{cases}$$

$$= \begin{cases} 1+(1+x), & 0 < x \le 1 \\ 3-(1+x), & 1 < x \le 2 \\ 1+(3-x), & 2 < x \le 3 \end{cases}$$

$$= \begin{cases} 2+x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 4-x, & 2 < x \le 3 \end{cases}$$

EXAMPLE 9 *If* $f(x) = \begin{cases} x^3+1, & x < 0 \\ x^2+1, & x \ge 0 \end{cases}$, $g(x) = \begin{cases} (x-1)^{1/3}, & x < 1 \\ (x-1)^{1/2}, & x \ge 1 \end{cases}$

compute gof(x).

SOLUTION We have,

$$gof(x) = g(f(x)) = \begin{cases} g(x^3+1), & x<0 \\ g(x^2+1), & x\geq 0 \end{cases}$$

$$= \begin{cases} (x^3+1-1)^{1/3}, & x<0 \\ (x^2+1-1)^{1/2}, & x\geq 0 \end{cases}$$

$$= \begin{cases} x, & x<0 \\ x, & x\geq 0 \end{cases}$$

$$= x \text{ for all } x$$

Hence, $gof(x) = x$ for all x.

EXAMPLE 10 *If*

$$f(x) = \sin^2 x + \sin^2(x+\pi/3) + \cos x \cos(x+\pi/3)$$

and $g(5/4) = 1$, *then find* $gof(x)$ **[IIT 1996]**

SOLUTION We have,

$$f(x) = \sin^2 x + \sin^2(x+\pi/3) + \cos x \cos(x+\pi/3)$$

$$= \frac{1-\cos 2x}{2} + \frac{1-\cos(2x+2\pi/3)}{2} + \frac{1}{2}[2\cos x \cos(x+\pi/3)]$$

$$= \frac{1}{2}[1-\cos 2x + 1 - \cos(2x+2\pi/3) + \cos(2x+\pi/3) + \cos\pi/3]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \left\{\cos 2x + \cos\left(2x+\frac{2\pi}{3}\right)\right\} + \cos\left(2x+\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x+\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x+\frac{\pi}{3}\right)\right]$$

$= 5/4$ for all x.

$\therefore\ gof(x) = g(f(x)) = g(5/4) = 1$ $[\because\ g(5/4) = 1 \text{ (given)}]$

Hence, $gof(x) = 1$, for all x

—❑❑❑—

1.9 PROPERTIES OF COMPOSITION OF FUNCTIONS

PROPERTY I *The composition of functions is not commutative i.e.* $fog \neq gof$.

Let $f: A \to B$ and $g: B \to C$ be two functions. Then, the function gof exists because the range of f is a subset of the domain of g. But, fog cannot exist unless the range of g is a subset of domain of f i.e. unless $C \subset A$. As such we find that fog does not exist if $C \not\subset A$ but fog will be a function from B to itself if $A = C$. Thus, if $A = C$.

$f: A \to B$ and $g: B \to A \Rightarrow gof: A \to A$ and $fog: B \to B$

Now we find that both fog and gof exist but they cannot be equal if A and B are two distinct sets, which are their domains. However, if $A = B = C$, then both gof and fog exist and both are from A to itself, even then they may not be equal as shown in Example 1 on page 1.9.

Hence, in general the composition of functions is not necessarily commutative.

PROPERTY II *The composition of functions is associative i.e. if f, g, h are three functions such that (fog)oh and fo (goh) exist, then*

$$(fog)\,oh = fo\,(goh)$$

PROOF Let A, B, C, D be four non-void sets. Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be three functions. Then,

$f: A \to B, g: B \to C, h: C \to D$

$\Rightarrow fog: A \to C$ and $h: C \to D$

$\Rightarrow (fog)\,oh: A \to D$

Again,

$f: A \to B, g: B \to C, h: C \to D$

$\Rightarrow f: A \to B$ and $goh: B \to D$

$\Rightarrow fo\,(goh): A \to D$

Thus, $(fog)oh$ and $fo\,(goh)$ are functions from A to D.

Now, we shall show that

$$\{(fog)oh\}(x) = \{fo\,(goh)\}(x) \text{ for all } x \in A.$$

Let x be an arbitrary element of A. Let $y \in B, z \in C$ such that $h(x) = y$ and $g(y) = z$.

Then,

$$\{(fog)\,oh\}(x) = (fog)\{h(x)\}$$

$$= (fog)(y) \qquad [\because\ h(x) = y]$$

$$= f(g(y))$$

$$= f(z) \qquad \text{...(i)} \qquad [\because\ g(y) = z]$$

And, $\{fo\,(goh)\}(x) = f\{(goh)(x)\}$

$$= f\{g(h(x))\}$$

$$= f\{g(y)\} \qquad [\because\ h(x) = y]$$

$$= f(z) \qquad \text{...(ii)} \qquad [\because\ g(y) = z]$$

From (i) and (ii), we have

$$\{(fog)\,oh\}(x) = \{fo\,(goh)\}(x) \quad \text{for all } x \in A.$$

Hence, $(fog)\,oh = fo\,(goh)$

PROPERTY III *The composition of two bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijection.*

PROOF Let $f: A \to B$ and $g: B \to C$ be two bijections. Then, gof exists such that $gof: A \to C$.

We have to prove that gof is an injective as well as a surjective map.

Injectivity: Let x, y be two arbitrary elements of A. Then,

$$(gof)(x) = (gof)(y)$$

$$\Rightarrow \quad g(f(x)) = g(f(y))$$

$$\Rightarrow \quad f(x) = f(y) \qquad [\because\ g \text{ is an injective map}]$$

$\Rightarrow x = y$ [$\because$ f is an injective map]

Thus, $(gof)(x) = (gof)(y)$ for all $x, y \in A$

So, gof is an injective map.

Surjectivity : In order to prove the surjectivity of gof, we have to show that every element in C has its pre-image in A i.e. for all $z \in C$, there exists $x \in A$ such that $(gof)(x) = z$.

Let z be an arbitrary element of C. Then,

$z \in C$

$\Rightarrow$ There exists $y \in B$ s.t. $g(y) = z$ [$\because$ g is a surjective map]

and, $y \in B$

$\Rightarrow$ There exists $x \in A$ s.t. $f(x) = y$ [$\because$ f is a surjective map]

Thus, we find that for every $z \in C$, there exists $x \in A$ such that $(gof)(x) = g(f(x)) = g(y) = z$. i.e. every element of C is the gof-image of some element of A.

So, gof is a surjective map.

Hence, gof being both injective as well as surjective, is a bijective map.

PROPERTY IV *Let $f: A \to B$. Then $fo\, I_A = I_B\, of = f$ i.e. the composition of any function with the identity function is the function itself.*

PROOF Since $I_A : A \to A$ and $f: A \to B$, therefore $fo\, I_A : A \to B$.

Now let x be an arbitrary element of A. Then,

$(f o I_A)(x) = f(I_A(x)) = f(x)$ [$\because$ $I_A(x) = x$ for all $x \in A$]

$\therefore$ $fo\, I_A = f$

Again, $f: A \to B$ and $I_B : B \to B$

$\Rightarrow$ $I_B\ of : A \to B$.

Now, let x be an arbitrary element of B. Let $f(x) = y$. Then, $y \in B$

$\therefore$ $(I_B\, of)(x) = I_B(f(x))$

$= I_B(y)$ [$\because$ $f(x) = y$]

$= y$ [$\because$ $I_B(y) = y$ for all $y \in B$]

$= f(x)$

$\therefore$ $I_B\, of = f$

Hence, $fo\, I_A = I_B\, of = f$

1.10 INVERSE OF AN ELEMENT

Let A and B be two sets and let $f: A \to B$ be a mapping. As we have discussed earlier that if $a \in A$ is associated to $b \in B$ under the function f, then 'b' is called the f image of 'a' and we write it as $b = f(a)$. We also say that 'a' is the pre-image or inverse element of 'b' under f and we write $a = f^{-1}(b)$.

It should be noted that the inverse of an element under a function may consist of a single element, two or more elements or no element depending on whether function is injective or many-one; onto or into. If $f: A \to B$ is a many-one and into function, then the inverse of some elements of B may or may not exist or the inverse of some element of B may consists of more than one element. If f is a bijection, then for each $b \in B$, $f^{-1}(b)$ exists and it consists of a single element only.

If f is represented by Fig. 1.17, then we find that

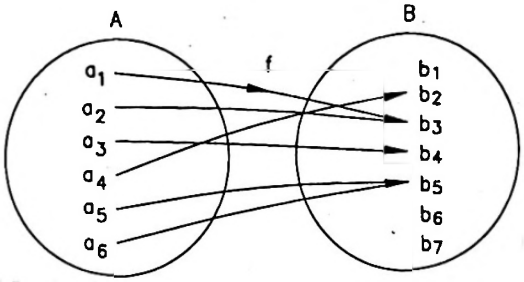


Fig. 1.17

$f^{-1}(b_1) = \phi, f^{-1}(b_2) = a_4,$

$f^{-1}(b_3) = \{a_1, a_2\}, f^{-1}(b_4) = a_3$

$f^{-1}(b_5) = \{a_5, a_6\}, f^{-1}(b_6) = \phi$

and $f^{-1}(b_7) = \phi$

ILLUSTRATION 1 *If $f: Q \to Q$ is given by $f(x) = x^2$, then find*

(i) $f^{-1}(9)$ (ii) $f^{-1}(-5)$ (iii) $f^{-1}(0)$

SOLUTION (i) Let $f^{-1}(9) = x$. Then,

$f(x) = 9$

$\Rightarrow$ $x^2 = 9 \Rightarrow x = \pm 3$.

$\therefore$ $f^{-1}(9) = \{-3, 3\}$.

(ii) Let $f^{-1}(-5) = x$. Then,

$f(x) = -5 \Rightarrow x^2 = -5$, which is not possible for $x \in Q$.

$\therefore$ $f^{-1}(-5) = \phi$

(iii) Let $f^{-1}(0) = x$. Then,

$f(x) = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$.

$\therefore$ $f^{-1}(0) = \{0\}$.

ILLUSTRATION 2 *If the function $f: R \to R$ be defined by $f(x) = x^2 + 5x + 9$, find $f^{-1}(8)$ and $f^{-1}(9)$.*

SOLUTION Let $f^{-1}(8) = x$. Then,

$f(x) = 8 \Rightarrow x^2 + 5x + 9 = 8 \Rightarrow x = \dfrac{-5 \pm \sqrt{21}}{2}$, which are in R.

$\therefore$ $f^{-1}(8) = \left\{\dfrac{-5+\sqrt{21}}{2}, \dfrac{-5-\sqrt{21}}{2}\right\}$

Again, let

$f^{-1}(9) = x$

$\Rightarrow f(x) = 9$

$\Rightarrow x^2 + 5x + 9 = 9$

$\Rightarrow x^2 + 5x = 0$

$\Rightarrow x(x+5) = 0$

$\Rightarrow x = 0, -5$, which are in R

$\therefore f^{-1}(9) = \{0, 5\}$

ILLUSTRATION 3 *If the function $f: C \to C$ be defined by $f(x) = x^2 - 1$, find $f^{-1}(-5), f^{-1}(8)$.*

SOLUTION Let $f^{-1}(-5) = x$. Then,

$f(x) = -5$

$\Rightarrow x^2 - 1 = -5$

$\Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4} \Rightarrow x = \pm 2i$, which are in C.

$\therefore f^{-1}(-5) = \{2i, -2i\}$

Again, let $f^{-1}(8) = x$. Then,

$f(x) = 8$

$\Rightarrow x^2 - 1 = 8 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ which are in C.

$\therefore f^{-1}(8) = \{-3, 3\}$

1.11 INVERSE OF A FUNCTION

Let A and B be two sets and let $f: A \to B$ be a function. If we follow a rule in which elements of B are associated to their pre-images, then we find that under such a rule there may be some elements in B which are not associated to elements in A. This happens when f is not an onto map. Therefore all elements in B will be associated to some elements in A if f is an onto map. Also, if it is a many-one function then under the said rule an element in B may be associated to more than one element in A. Therefore an element in B will be associated to a unique element in A if f is an injective map.

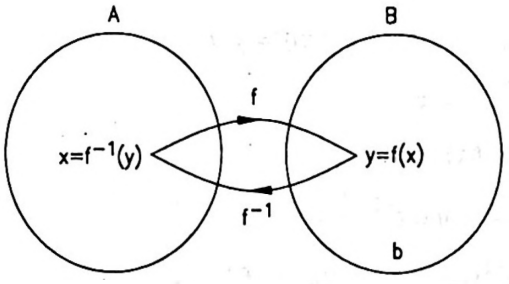


Fig. 1.18

It follows from the above discussion that if $f: A \to B$ is a bijection, we can define a new function from B to A which associates each element $y \in B$ to its pre-image $f^{-1}(y) \in A$. Such a function is known as the inverse of function f and is denoted by f^{-1}.

ILLUSTRATION 1 *If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $f: A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.*

SOLUTION We have, $f(1) = 2, f(2) = 4, f(3) = 6$ and $f(4) = 8$.

$\therefore f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$, which is clearly a bijection.

$\therefore f^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$.

In order to find the inverse of a bijection $f: A \to B$, we follow the following algorithm.

ALGORITHM

STEP I *Put $f(x) = y$, where $y \in B$ and $x \in A$.*

STEP II *Solve $f(x) = y$ to obtain x in terms of y.*

STEP III *In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the inverse of f.*

ILLUSTRATION 2 *If $f: R \to R$ is defined by $f(x) = 2x + 7$. Prove that f is a bijection. Also, find the inverse of f.*

SOLUTION *Injectivity*: Let x, y be any two elements of R. Then,

$f(x) = f(y)$

$\Rightarrow 2x + 7 = 2y + 7 \Rightarrow x = y$

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of R (co-domain).

Then, $y = f(x)$

$\Rightarrow y = 2x + 7 \Rightarrow x = \dfrac{y-7}{2}$.

Clearly, $x = \dfrac{y-7}{2} \in R$ for all $y \in R$.

Thus, for all $y \in R$ (co-domain) there exists $x = \dfrac{y-7}{2} \in R$ such that $f(x) = y$. In other words every element in R (co-domain) has its pre-image in R (domain).

So, f is a surjective map.

Hence, f is a bijection. Consequently, f^{-1} exists.

Inverse of f : Let $x \in R$ (domain) and $y \in R$ (co-domain) such that $f(x) = y$.

$\Rightarrow 2x + 7 = y$

$\Rightarrow x = \dfrac{y-7}{2}$

$\Rightarrow f^{-1}(y) = \dfrac{y-7}{2}$.

Thus, $f^{-1}: R \to R$ is given by

$f^{-1}(x) = \dfrac{x-7}{2}$ for all $x \in R$.

ILLUSTRATION 3 *If $f: R \to R$ is a bijection given by $f(x) = x^3 + 3$, find $f^{-1}(x)$.*

SOLUTION Let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow x^3 + 3 = y$

$\Rightarrow x = (y-3)^{1/3}$

$\Rightarrow f^{-1}(y) = (y-3)^{1/3}$

Thus, $f^{-1}: R \to R$ is given by

$f^{-1}(x) = (x-3)^{1/3}$ for all $x \in R$.

1.12 PROPERTIES OF INVERSE OF A FUNCTION

PROPERTY I *The inverse of a bijection is unique.*

PROOF Let $f: A \to B$ be a bijection. If possible, let $g: B \to A$ and $h: B \to A$ be two inverses of f. We have to prove that $g = h$. In order to prove this it is sufficient to show that $g(y) = h(y)$ for all $y \in B$. Let y be an arbitrary element of B.

Let $g(y) = x_1$ and $h(y) = x_2$.

Then, $g(y) = x_1 \Rightarrow f(x_1) = y$ [$\because$ g is inverse of f]

and, $h(y) = x_2 \Rightarrow f(x_2) = y$ [$\because$ h is inverse of f]

$\therefore$ $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 = x_2$ [$\because$ f is one-one]

$\Rightarrow$ $g(y) = h(y)$

Thus, $g(y) = h(y)$ for all $y \in B$.

Hence, $g = h$

PROPERTY II *The inverse of a bijection is also a bijection.*

PROOF Let $f: A \to B$ be a bijection and let $g: B \to A$ be its inverse. We have to show that g is a bijection.

Injectivity : Let $y_1, y_2 \in B$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$.

Since g is the inverse of f. Therefore,

$$g(y_1) = x_1 \Rightarrow f(x_1) = y_1$$

and, $$g(y_2) = x_2 \Rightarrow f(x_2) = y_2$$

Now,

$g(y_1) = g(y_2)$

$\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ $f(x_1) = f(x_2)$

$\Rightarrow$ $y_1 = y_2$

So, g is an injective map.

Surjectivity : In order to prove that $g: B \to A$ is a surjection, we have to show that every element in A has its pre-image in B under g.

So, let x be an arbitrary element of A. Then,

$x \in A$

$\Rightarrow \exists\, y \in B$ such that $f(x) = y$ [$\because$ f is a function from A to B]

$\Rightarrow \exists\, y \in B$ such that $g(y) = x$ [$\because$ g is inverse of f]

Thus, for each $x \in A$, there exists $y \in B$ such that $g(y) = x$. So, g is a surjective map.

Hence, g is a bijection.

PROPERTY III *If $f: A \to B$ is a bijection and $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.*

PROOF In order to prove that $gof = I_A$ and $fog = I_B$, we have to prove that

$(gof)(x) = x$ for all $x \in A$ and $(fog)(y) = y$ for all $y \in B$

Let x be an element of A such that $f(x) = y$. Then,

$g(y) = x$ [$\because$ g is inverse of f]

Now, $(gof)(x) = g(f(x)) = g(y) = x$

$\Rightarrow$ $(gof)(x) = x$ for all $x \in A$

$\Rightarrow$ $gof = I_A$.

We have, $(fog)(y) = f(g(y)) = f(x) = y$

$\Rightarrow$ $fog(y) = y$ for all $y \in B$

$\Rightarrow$ $fog = I_B$.

Hence, $gof = I_A$ and $fog = I_B$.

REMARK *If in the above property, we have $B = A$. Then we find that for every bijection $f: A \to A$ there exists a bijection $g: A \to A$ such that $fog = gof = I_A$.*

PROPERTY IV *If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is a bijection and $(gof)^{-1} = f^{-1}og^{-1}$.*

PROOF We have,

$f: A \to B$ is a bijection, $g: B \to C$ is a bijection $\Rightarrow$ $gof: A \to C$ is a bijection

$\Rightarrow$ $(gof)^{-1}: C \to A$ exists.

Again,

$f: A \to B$ is a bijection $\Rightarrow f^{-1}: B \to A$ is a bijection

$g: B \to C$ is a bijection $\Rightarrow g^{-1}: C \to B$ is a bijection

$\therefore$ $f^{-1}og^{-1}: C \to A$ exists

Let $x \in A$, $y \in B$ and $z \in C$ such that $f(x) = y$ and $g(y) = z$

Then, $(gof)(x) = g(f(x)) = g(y) = z$

$\Rightarrow (gof)^{-1}(z) = x$ ...(i)

Now, $f(x) = y$ and $g(y) = z$

$\Rightarrow f^{-1}(y) = x$ and $g^{-1}(z) = y$

$\therefore$ $(f^{-1}og^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$ ...(ii)

From (i) and (ii), we have

$(gof)^{-1}(z) = (f^{-1}og^{-1})(z)$ for all $z \in C$.

$\Rightarrow$ $(gof)^{-1} = f^{-1}og^{-1}$.

PROPERTY V *Let $f: A \to B$ and $g: B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.*

PROOF *f is one-one* : Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$f(x) = f(y)$

$\Rightarrow$ $g(f(x)) = g(f(y))$

$\Rightarrow$ $(gof)(x) = (gof)(y)$

$\Rightarrow I_A(x) = I_A(y)$ $[\because gof = I_A]$

$\Rightarrow x = y$

$\therefore f$ is a one-one map.

f is onto : Let $y \in B$ and let $g(y) = x$. Then,

$g(y) = x$

$\Rightarrow f(g(y)) = f(x)$

$\Rightarrow (fog)(y) = f(x)$

$\Rightarrow I_B(y) = f(x)$ $[\because fog = I_B]$

$\Rightarrow y = f(x)$ $[\because I_B(y) = y]$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$. So, f is onto.

Hence, f is a bijection.

Similarly, it can be proved that g is a bijection.

Now we shall show that $g = f^{-1}$.

Since $f: A \to B$ is a bijection. Therefore, f^{-1} exists.

Now,

$fog = I_B$

$\Rightarrow f^{-1} o (fog) = f^{-1} o I_B$

$\Rightarrow (f^{-1}of) og = f^{-1} o I_B$ [By associativity]

$\Rightarrow I_A o g = f^{-1} o I_B$ $[\because f^{-1} of = I_A]$

$\Rightarrow g = f^{-1}$ $[\because I_A og = g$ and $f^{-1} o I_B = f^{-1}]$

Hence, $g = f^{-1}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the inverse of the function* $f: R \to R$ *given by* $f(x) = 3x + 5$.

SOLUTION Clearly, $f: R \to R$ given by $f(x) = 3x + 5$ is a bijection. So, it is invertible.

Let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow 3x + 5 = y$

$\Rightarrow x = \dfrac{y-5}{3}$

$\Rightarrow f^{-1}(y) = \dfrac{y-5}{3}$.

Hence, $f^{-1}(x) = \dfrac{x-5}{3}$.

EXAMPLE 2 *Show that the function* $f: R - \{0\} \to R - \{0\}$ *given by* $f(x) = \dfrac{k}{x}$, *where k is a non-zero real number, is inverse of itself.*

SOLUTION Clearly, $f: R - \{0\} \to R - \{0\}$ given by $f(x) = \dfrac{k}{x}$ is a bijection and hence it is invertible.

Let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow \dfrac{k}{x} = y$

$\Rightarrow x = \dfrac{k}{y}$

$\Rightarrow f^{-1}(y) = \dfrac{k}{y}$

$\therefore f^{-1}(x) = \dfrac{k}{x}$

Hence, $f(x) = f^{-1}(x)$ i.e. $f(x)$ is inverse of itself.

EXAMPLE 3 *Show that the function* $f: R \to R$ *given by*

$$f(x) = \log_a (x + \sqrt{x^2 + 1}), a > 0, a \neq 1$$

is invertible and find its inverse.

SOLUTION In order to prove that $f(x)$ is invertible, it is sufficient to show that f is a bijection.

f is an injection : Let x, y be any two distinct real numbers. Then,

$x \neq y$

$\Rightarrow x + \sqrt{x^2+1} \neq y + \sqrt{y^2+1}$

$\Rightarrow \log_a (x + \sqrt{x^2+1}) \neq \log_a (y + \sqrt{y^2+1})$

$\Rightarrow f(x) \neq f(y)$

$\Rightarrow f$ is an injection.

f is a surjection : Let $f(x) = y$. Then,

$\log_a (x + \sqrt{x^2+1}) = y$

$\Rightarrow x + \sqrt{x^2+1} = a^y$

$\Rightarrow -x + \sqrt{x^2+1} = a^{-y}$

$\therefore 2x = a^y - a^{-y}$

$\Rightarrow x = \left(\dfrac{a^y - a^{-y}}{2}\right)$

Thus, for every $y \in R$, there exists $x = \dfrac{a^y - a^{-y}}{2} \in R$ such that $f(x) = y$. Therefore, f is a surjection.

Hence, f is a bijection. Consequently, it is invertible.

In order to find f^{-1}, let

$$f(x) = y$$

$$\Rightarrow \log_a (x+\sqrt{x^2+1}) = y$$

$$\Rightarrow x+\sqrt{x^2+1} = a^y$$

$$\Rightarrow -x+\sqrt{x^2+1} = a^{-y}$$

$$\therefore 2x = (a^y - a^{-y})$$

$$\Rightarrow x = \frac{1}{2}(a^y - a^{-y})$$

$$\Rightarrow f^{-1}(y) = \frac{1}{2}(a^y - a^{-y})$$

Hence, $f^{-1}(x) = \frac{1}{2}(a^x - a^{-x})$.

EXAMPLE 4 *Find the inverse of the function $f(x)$ given by*

$$f(x) = \begin{cases} x & , x < 1 \\ x^2 & , 1 \le x \le 4 \\ 8\sqrt{x} & , x > 4 \end{cases}$$

SOLUTION Let $f(x) = y$. Then,

$$y = f(x)$$

$$\Rightarrow y = \begin{cases} x & , x < 1 \\ x^2 & , 1 \le x \le 4 \\ 8\sqrt{x} & , x > 4 \end{cases}$$

$$\Rightarrow x = \begin{cases} y & , y < 1 \\ \sqrt{y} & , 1 \le y \le 16 \\ \dfrac{y^2}{64} & , y > 16 \end{cases}$$

$$\Rightarrow f^{-1}(y) = \begin{cases} y & , y < 1 \\ \sqrt{y} & , 1 \le y \le 16 \\ \dfrac{y^2}{64} & , y > 16 \end{cases}$$

$$\Rightarrow f^{-1}(x) = \begin{cases} x & , x < 1 \\ \sqrt{x} & , 1 \le x \le 16 \\ \dfrac{x^2}{64} & , x > 16 \end{cases}$$

—❑❑❑—

1.13 INTERVALS

CLOSED INTERVAL *Let a and b be two given real numbers such that $a < b$. Then the set of all real numbers x such that $a \le x \le b$ is called a closed interval and is denoted by $[a, b]$.*

i.e. $[a, b] = \{x \in R \mid a \le x \le b\}$

For example, $[1, 2] = \{x \in R \mid 1 \le x \le 2\}$ i.e., the set of all real numbers lying between 1 and 2, including the end points.

OPEN INTERVAL *Let a and b be two given real numbers such that $a < b$. Then the set of all real numbers x such that $a \le x \le b$ is called a closed interval and is denoted by (a, b).*

i.e. $(a, b) = \{x \in R \mid a \le x \le b\}$

For example, $(1, 2) = \{x \in R \mid 1 \le x \le 2\}$ i.e., the set of all real numbers lying between 1 and 2, excluding the end points.

SEMI-CLOSED OR SEMI OPEN INTERVAL *If a, b are two given real numbers such that $a < b$, then the sets $(a, b) = \{x \in R \mid a < x \le b\}$ and $(a, b) = \{x \in R \mid a \le x < b\}$ are known as semi-closed or semi-open intervals and are also denoted by $]a, b[$ and $[a, b]$ respectively.*

1.14 REAL FUNCTIONS

REAL FUNCTION *If the domain and co-domain of a function are subsets of R (set of all real numbers). It is called a real valued function or in short a real function.*

Let A and B be two non-empty subsets of R, and let $f: A \to B$ be a real function. Let x be an element of A. The element in B that is associated to x by f is denoted by $f(x)$ and is known as the image of x under f or the value of f at x. Sometimes we also say that f takes value $f(x)$ at x.

DESCRIPTION OF AREAL FUNCTION *If f is a real valued function with finite domain, then f can be described by listing the values which it attains at different points of its domain. However, if the domain of a real function is an infinite set, then, f cannot be described by listing the values at points in its domain. In such cases real functions are generally described by some general formula or rule like $f(x) = x^2 + 1$ or $f(x) = 2\sin x + 3$ etc. In calculus almost all real functions are described by some general formula or rule.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.*

SOLUTION We have,

$$f(x) = x + \frac{1}{x}$$

$$\Rightarrow f(x^3) = x^3 + \frac{1}{x^3}$$

Now,

$$[f(x)]^3 = \left(x + \frac{1}{x}\right)^3$$

$$= \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$= f(x^3) + 3f(x)$$

$$= f(x^3) + 3f\left(\frac{1}{x}\right) \qquad \left[\because f(x) = f\left(\frac{1}{x}\right)\right]$$

EXAMPLE 2 *If* $f(x) = \cos(\log x)$, *then prove that*

$$f(x)f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\} = 0$$

SOLUTION We have,

$$f(x) = \cos(\log x)$$

$$\therefore \quad f\left(\frac{x}{y}\right) = \cos\left\{\log\left(\frac{x}{y}\right)\right\} = \cos(\log x - \log y)$$

and, $f(xy) = \cos\{\log(xy)\} = \cos(\log x + \log y)$

$$\Rightarrow f\left(\frac{x}{y}\right) + f(xy) = \cos(\log x - \log y) + \cos(\log x + \log y)$$

$$\Rightarrow f\left(\frac{x}{y}\right) + f(xy) = 2\cos(\log x)\cos(\log y)$$

$$\Rightarrow \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\} = \cos(\log x)\cos(\log y) = f(x)f(y)$$

$$\Rightarrow f(x)(f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\} = 0$$

EXAMPLE 3 *If* $f(x) = (a - x^n)^{1/n}$, $a > 0$ *and* $n \in N$, *then prove that* $f(f(x)) = x$ *for all* x.

SOLUTION We have,

$$f(f(x)) = [a - (f(x))^n]^{1/n}$$

$$= [a - \{(a - x^n)^{1/n}\}^n]^{1/n}$$

$$= [a - (a - x^n)]^{1/n} = x$$

Hence, $f(x) = x$ for all x.

EXAMPLE 4 *If for non-zero* x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, *where* $a \neq b$, *then find* $f(x)$. [IIT - 1995]

SOLUTION We have,

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad ...(i)$$

On replacing x by $\frac{1}{x}$, we get

$$af\left(\frac{1}{x}\right) + bf(x) = x - 5$$

$$\Rightarrow bf(x) + af\left(\frac{1}{x}\right) = x - 5 \qquad ...(ii)$$

Multiplying (i) by a, (ii) by b and then subtracting, we get

$$(a^2 - b^2)f(x) = \left(\frac{a}{x} - bx\right) - 5(a - b)$$

$$\Rightarrow \quad f(x) = \frac{1}{a^2 - b^2}\left(\frac{a}{x} - bx\right) - \frac{5}{a + b}$$

EXAMPLE 5 *If* $f(x)$ *is defined on* $[0, 1]$ *by the rule*

$$f(x) = \begin{cases} x & , \text{if } x \in Q \\ 1 - x & , \text{if } x \notin Q \end{cases}$$

Then, for all $x \in [0, 1]$ *prove that* $f(f(x)) = x$.

SOLUTION We have,

$$f(x) = \begin{cases} x & , \text{if } x \in Q \\ 1 - x & , \text{if } x \notin Q \end{cases}$$

$$\therefore f(f(x)) = \begin{cases} f(x) & , x \in Q \\ f(1 - x), & x \notin Q \end{cases}$$

$$= \begin{cases} f(x) & , x \notin Q \\ 1 - (1 - x), & x \notin Q \end{cases} \qquad \left[\begin{array}{l} \because x \in Q \Rightarrow 1 - x \notin Q \\ \therefore f(1 - x) = 1 - (1 - x) \end{array}\right]$$

$$= \begin{cases} x, & x \notin Q \\ x, & x \notin Q \end{cases}$$

$$= x \text{ for all } x \in [0, 1].$$

EXAMPLE 6 *If* $f(x)$ *is a polynomial function satisfying* $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ *and* $f(3) = 28$, *then find* $f(4)$.

SOLUTION Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$$

Then,

$$f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow (a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n) \times \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + ... + \frac{a_{n-1}}{x} + a_n\right)$$

$$= (a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n) + \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + ... + \frac{a_{n-1}}{x} + a_n\right)$$

On comparing the coefficients of like powers of x on both sides, we get

$$a_0 = \pm 1, a_n = 1 \text{ and } a_1 = a_2 = ... = a_{n-1} = 0$$

$$\therefore \quad f(x) = x^n + 1, \text{ or } f(x) = -x^n + 1$$

If $f(x) = -x^n + 1$, then

$$f(3) = -3^n + 1 \neq 28 \qquad [\because f(3) = 28]$$

So, $f(x) = x^n + 1$

$$\Rightarrow \quad f(3) = 3^n + 1$$

$$\Rightarrow \quad 28 = 3^n + 1$$

$$\Rightarrow \quad n = 3$$

Thus, $f(x) = x^3 + 1$

$$\Rightarrow \quad f(4) = 4^3 + 1 = 65.$$

EXAMPLE 7 *Let $g(x)$ is a polynomial function satisfying $g(x)g(y) = g(x) + g(y) + g(xy) - 2$ for all $x, y \in R$. If $g(2) = 5$, then find $g(3)$.*

SOLUTION Putting $x = 2$ and $y = 1$ in the given relation, we obtain

$$g(2)g(1) = g(2) + g(1) + g(2) - 2$$
$$\Rightarrow \quad 5g(1) = 5 + g(1) + 5 - 2$$
$$\Rightarrow \quad g(1) = 2$$

Putting $y = \frac{1}{x}$ in the given relation, we get

$$g(x)g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right) + g(1) - 2$$
$$\Rightarrow g(x)g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right) \qquad [\because g(1) = 2]$$
$$\Rightarrow \quad g(x) = x^n + 1 \qquad \text{[See Example 6]}$$
$$\Rightarrow \quad g(2) = 2^n + 1$$
$$\Rightarrow \quad 5 = 2^n + 1$$
$$\Rightarrow \quad n = 2$$

Thus, $g(x) = x^2 + 1$

$$\Rightarrow \quad g(5) = 25 + 1 = 26$$

EXAMPLE 8 *Let $f_1(x) = \frac{x}{3} + 10$ for all $x \in R$, and $f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$. Then, find $f_n(x)$.*

SOLUTION We have,

$$f_n(x) = f_1(f_{n-1}(x)), \; n \geq 2$$
$$\therefore \; f_2(x) = f_1(f_1(x))$$
$$= \frac{1}{3}f_1(x) + 10$$
$$= \frac{1}{3}\left(\frac{x}{3} + 10\right) + 10$$
$$= \frac{x}{3^2} + \frac{10}{3} + 10$$
$$f_3(x) = f_1\left(f_2(x)\right)$$
$$= \frac{1}{3}f_2(x) + 10$$
$$= \frac{1}{3}\left(\frac{x}{3^2} + \frac{10}{3} + 10\right) + 10$$
$$= \frac{x}{3^3} + \frac{10}{3^2} + \frac{10}{3} + 10$$
$$f_4(x) = f_1\left(f_3(x)\right)$$
$$= \frac{1}{3}f_3(x) + 10$$
$$= \frac{1}{3}\left(\frac{x}{3^3} + \frac{10}{3^2} + \frac{10}{3} + 10\right) + 10$$
$$= \frac{x}{3^4} + \frac{10}{3^3} + \frac{10}{3^2} + \frac{10}{3} + 10$$

Continuing in this manner, we obtain

$$f_n(x) = \frac{x}{3^n} + \frac{10}{3^{n-1}} + \frac{10}{3^{n-2}} + \ldots\ldots + \frac{10}{3} + 10$$
$$= \frac{x}{3^n} + 10\left(\frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}}\right)$$
$$= \frac{x}{3^n} + 15\left(1 - \frac{1}{3^n}\right)$$
$$= \frac{x - 15}{3^n} + 15$$

EXAMPLE 9 *If $3f(x) - f\left(\frac{1}{x}\right) = \log_e x^4$ for $x > 0$; find $f(e^x)$.*

SOLUTION We have,

$$3f(x) - f\left(\frac{1}{x}\right) = \log_e x^4$$
$$\Rightarrow 3f(x) - f\left(\frac{1}{x}\right) = 4\log_e x \qquad \ldots(i)$$
$$\Rightarrow 3f\left(\frac{1}{x}\right) - f(x) = 4\log_e\left(\frac{1}{x}\right) \qquad \text{[Replacing } x \text{ by } 1/x\text{]}$$
$$\Rightarrow 3f\left(\frac{1}{x}\right) - f(x) = -4\log_e x \qquad \ldots(ii)$$

Solving (i) and (ii), we get

$$8f(x) = 8\log_e x$$
$$\Rightarrow f(x) = \log_e x.$$
$$\Rightarrow f(e^x) = x.$$

EXAMPLE 10 *If $f: R \to R$ be a function defined by $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. Show that*

$$f(x) = xf(1) \quad \text{for all } x \in R.$$

SOLUTION We shall discuss the following cases :

CASE I *When $x \in N$:*

In this case, we have

$$f(x) = f(\underbrace{1 + 1 + 1 + \ldots. + 1}_{x\text{-times}})$$
$$= \underbrace{f(1) + f(1) + \ldots. + f(1)}_{x\text{-times}}$$
$$= xf(1).$$

CASE II *When x is a negative integer :*

Let $x = -y, y \in N$.

Now,

$$f(x+y) = f(x)+f(y) \text{ for all } x, y \in R$$

$$\Rightarrow f(0+0) = f(0)+f(0) \qquad [\text{Putting } x = y = 0]$$

$$\Rightarrow f(0) = f(0)+f(0)$$

$$\Rightarrow f(0) = 0$$

Again,

$$f(x+y) = f(x)+f(y) \text{ for all } x, y$$

$$\Rightarrow f\left((-1)+1\right) = f(-1)+f(1) \qquad [\text{Putting } x = -1 \text{ and } y = 1]$$

$$\Rightarrow f(0) = f(-1)+f(1)$$

$$\Rightarrow 0 = f(-1)+f(1) \qquad [\because f(0) = 0]$$

$$\Rightarrow f(-1) = -f(1)$$

Now,

$$f(x) = f(-y)$$

$$= f[(-1)+f(-1)+\ldots+(-1)] \quad (y\text{-times})$$

$$= f(-1)+f(-1)+\ldots.+f(-1) \quad (y\text{-times})$$

$$= yf(-1)$$

$$= -yf(1) \qquad [\because f(-1) = -f(1)]$$

$$= xf(1)$$

CASE III *When x is a rational number other than an integer.*

Let $x = \frac{m}{n}$, where $m, n \in Z, n \neq 0$.

We have,

$$f(1) = f\left(n \times \frac{1}{n}\right)$$

$$= f\left(\underbrace{\frac{1}{n}+\frac{1}{n}+\ldots..+\frac{1}{n}}_{n\text{-times}}\right)$$

$$= f\left(\frac{1}{n}\right)+f\left(\frac{1}{n}\right)+\ldots.+f\left(\frac{1}{n}\right)$$

$$= nf\left(\frac{1}{n}\right)$$

$$\therefore \quad f\left(\frac{1}{n}\right) = \frac{1}{n}f(1)$$

Now,

$$f\left(\frac{m}{n}\right) = f\left(m \times \frac{1}{n}\right)$$

$$= f\left(\underbrace{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}}_{m\text{-times}}\right)$$

$$= \underbrace{f\left(\frac{1}{n}\right)+f\left(\frac{1}{n}\right)+\ldots+f\left(\frac{1}{n}\right)}_{m\text{-times}}$$

$$= mf\left(\frac{1}{n}\right)$$

$$= m\left(\frac{1}{n}f(1)\right) \qquad \left[\because f\left(\frac{1}{n}\right) = \frac{1}{n}f(1)\right]$$

$$= \frac{m}{n}f(1)$$

$$= xf(1)$$

CASE IV *When $x \notin Q$.*

The prooft of this part is beyond the scope of this book.

Hence, $f(x) = xf(1)$ for all $x \in R$.

EXAMPLE 11 *Let $f: R \to R$ be a function given by*

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in R.$$

Prove that $f(x) = [f(1)]^x$ for all $x \in R$.

SOLUTION Proceed as in the solution of Ex. 10 by replacing addition (+) by multiplication (·).

EXAMPLE 12 *If $f: R \to R$ is given by*

$$f(x) = \frac{4^x}{4^x+2} \text{ for all } x \in R$$

Prove that $f(x)+f(1-x) = 1$. Hence, prove that

$$f\left(\frac{1}{1997}\right)+f\left(\frac{2}{1997}\right)+\ldots+f\left(\frac{1996}{1997}\right) = 998$$

SOLUTION We have,

$$f(x) = \frac{4^x}{4^x+2}$$

$$\Rightarrow f(1-x) = \frac{4^{1-x}}{4^{1-x}+2} = \frac{4}{4+2\times 4^x} = \frac{2}{2+4^x}$$

$$\therefore f(x)+f(1-x) = \frac{4^x}{4^x+2}+\frac{2}{2+4^x} = \frac{4^x+2}{4^x+2} = 1$$

Now,

$$f\left(\frac{1}{1997}\right)+f\left(\frac{2}{1997}\right)+\ldots.+f\left(\frac{1996}{1997}\right)$$

$$= \left[f\left(\frac{1}{1997}\right)+f\left(\frac{1996}{1997}\right)\right]+\left[f\left(\frac{2}{1997}\right)+f\left(\frac{1995}{1997}\right)\right]+\ldots.$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots+\left[f\left(\frac{998}{1997}\right)+f\left(\frac{999}{1997}\right)\right]$$

$$= \left[f\left(\frac{1}{1997}\right)+f\left(1-\frac{1}{1997}\right)\right]+\left[f\left(\frac{2}{1997}\right)+f\left(1-\frac{2}{1997}\right)\right]$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots+\left[f\left(\frac{998}{1997}\right)+f\left(1-\frac{998}{1997}\right)\right]$$

$$= \sum_{n=1}^{998} \left[f\left(\frac{n}{1997}\right) + f\left(1 - \frac{n}{1997}\right) \right]$$

$$= \sum_{n=1}^{998} 1 \qquad [\because f(x) + f(1-x) = 1 \text{ for all } x]$$

$$= 998.$$

EXAMPLE 13 *Determine all real valued functions satisfying functional relation*

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} \text{ for all } x \in R - \{0, 1\}$$

SOLUTION We have,

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$$

$$\Rightarrow f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x} \qquad \ldots(i)$$

Putting $\frac{1}{1-x} = y$ in (i), we get

$$f(x) + f(y) = \frac{2}{x} - 2y \qquad \ldots(ii)$$

Again from (i), we get

$$f(y) + f\left(\frac{1}{1-y}\right) = \frac{2}{y} - \frac{2}{1-y} \qquad \ldots(iii) \quad [\text{Replacing } x \text{ by } y]$$

Putting $\frac{1}{1-y} = z$ in (iii), we get

$$f(y) + f(z) = \frac{2}{y} - 2z \qquad \ldots(iv)$$

Now, $\frac{1}{1-x} = y$ and $\frac{1}{1-y} = z$

$$\Rightarrow \qquad x = \frac{1}{1-z}$$

From (ii), we obtain

$$f(x) + f(z) = \frac{2}{z} - 2x \qquad \ldots(v) \quad [\text{Replacing } x \text{ by } z \text{ and } y \text{ by } x]$$

From (iii) we get

$$f(z) + f\left(\frac{1}{1-z}\right) = \frac{2}{z} - \frac{2}{1-z} \qquad \ldots(vi) \quad [\text{Replacing } y \text{ by } z]$$

Adding corresponding sides of (ii), (iv) and (v), we get

$$2[f(x) + f(y) + f(z) = \frac{2}{x} - 2y + \frac{2}{y} - 2z + \frac{2}{z} - 2x$$

$$\Rightarrow \quad f(x) + f(y) + f(z) = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - (x + y + z)$$

$$\Rightarrow f(x) + \frac{2}{y} - 2z = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - (x + y + z) \qquad [\text{Using (iv)}]$$

$$\Rightarrow f(x) = \left(\frac{1}{x} - x\right) - \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right)$$

$$\Rightarrow f(x) = \left(\frac{1}{x} - x\right) - \left(\frac{1}{1-x} + 1 - x\right) + \left(1 - \frac{1}{x} + \frac{x}{x-1}\right)$$

$$\left[\because y = \frac{1}{1-x} \text{ and } z = 1 - \frac{1}{x}\right]$$

$$\Rightarrow f(x) = \frac{x+1}{x-1}$$

EXAMPLE 14 *Let $g : R \to R$ be given by $g(x) = 4x + 3$.*
If $g^n(x) = \underbrace{gogog \ldots og}_{n\text{-times}}(x)$, show that

$$g^n(x) = 4^n x + (4^n - 1).$$

If $g^{-n}(x)$ denotes the inverse of $g^n(x)$, prove that
$g^{-n}(x) = 4^{-n} x + (4^{-n} - 1)$ for all $x \in R$.

SOLUTION We shall prove the result by mathematical induction on n.

We have,

$$g(x) = 4x + 3$$

$$\therefore \quad gog(x) = g(g(x))$$

$$= g(4x + 3)$$

$$= 4(4x + 3) + 3$$

$$= 4^2 x + 15$$

$$= 4^2 x + (4^2 - 1)$$

So the result is true for $n = 2$.

Let the result be true for $n = m$. Then,

$$g^m(x) = 4^m x + (4^m - 1) \text{ for all } x \in R.$$

Now,

$$g^{m+1}(x) = g^m \, 0 \, g(x)$$

$$= g^m(g(x))$$

$$= g^m(4x + 3)$$

$$= 4^m(4x + 3) + 4^m - 1$$

$$= 4^{m+1} x + 4^m \times 3 + 4^m - 1$$

$$= 4^{m+1} x + 4^{m+1} - 1$$

So, the result is true for $n = m + 1$.

Hence, by the principle of mathematical induction on n, we have

$$g^n(x) = 4^n x + (4^n - 1) \text{ for all } n \in N.$$

In order to find the inverse of $g^n(x)$, we proceed as follows:

Let $g^n(x) = y$. Then,

$$g^n(x) = y$$

$$\Rightarrow 4^n x + (4^n - 1) = y$$

$\Rightarrow \quad 4^n x = y - 4^n + 1$

$\Rightarrow \quad x = 4^{-n} y - 1 + 4^{-n}$

$\Rightarrow \quad g^{-n}(y) = 4^{-n} y + 4^{-n} - 1$

Hence, $g^{-n}(x) = 4^{-n} x + 4^{-n} - 1$.

EXAMPLE 15 *Let $f(x)$ be a real valued function satisfying $2f(xy) = [f(x)]^y + [f(y)]^x$ for all $x, y \in R$ and $f(1) = a$, where $a \neq 1$. Prove that*

$$(a-1)\sum_{r=1}^{n} f(r) = a^{n+1} - a.$$

SOLUTION We have,

$2f(xy) = [f(x)]^y + [f(y)]^x$ for all $x, y \in R$.

$\Rightarrow \quad 2f(x) = f(x) + [f(1)]^x$ [Replacing y by 1]

$\Rightarrow f(x) = a^x$

$\therefore \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} a^r$

$\Rightarrow \sum_{r=1}^{n} f(r) = a\left(\frac{a^n - 1}{a - 1}\right)$

$\Rightarrow (a-1)\sum_{r=1}^{n} f(r) = a^{n+1} - a.$

EXAMPLE 17 *Let $f(x) = 1 + x^2$. Find a function $g(x)$ so that $f(g(x)) = 1 + x^2 - 2x^3 + x^4$.*

SOLUTION We have,

$$f(g(x)) = 1 + x^2 - 2x^3 + x^4$$
$$= 1 + x^2(1 - 2x + x^2)$$
$$= 1 + x^2(x-1)^2$$
$$= 1 + [x(x-1)]^2$$

$\Rightarrow \quad g(x) = \pm x(x-1)$

SOME IMPORTANT RESULTS TO LEARN

I *If $f(x)$ is a polynomial function of degree n satisfying*

$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ *for all $x \neq 0$, then*

$f(x) = 1 \pm x^n$.

II *If $f: R \to R$ is a function satisfying*

$f(x + y) = f(x) + f(y)$ *for all $x, y \in R$,* then

$f(x) = xf(1)$ for all $x \in R$.

III *If $f: R \to R$ is a real function given by*

$f(x + y) = f(x) \cdot f(y)$ *for all $x, y \in R$, then*

$f(x) = [f(1)]^x$ for all $x \in R$.

EXERCISE 1.1

1. Let $f: R \to R$ be a function given by $f(x) = ax + b$ for all $x \in R$. Find the constants a and b such that $fof = I$ (I being the identity function on R).

2. If $f(x) = \frac{9^x}{9^x + 3}$ for all $x \in R$, find the sum

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right)$$

3. For any positive integer n, let

$$f(n) = 1! + 2! + \ldots + n!$$

Find polynomials $g(x)$ and $h(x)$ satisfying

$f(n+2) = f(n+1)\,g(n) + f(n)\,h(n)$ for all $n \geq 1$.

4. Let $f(x) = \frac{1}{1-x}$ and let $f^n(x) = \underbrace{fofof \ldots of}_{n\text{-times}}(x)$.

Then, find $f_{3n}(x)$.

5. Let $f: N \to R$ be a function given by

(i) $f(1) = 1$

and

(ii) $f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n+1)f(n)$ for $n \geq 2$.

Find $f(1994)$.

6. If $f(x) = (2\cos x - 1)(2\cos 2x - 1) \ldots (2\cos 2^{n-1}x - 1)$, $n \geq 1$, prove that $f\left(\frac{2r\pi}{2^r \pm 1}\right) = 1$ for all $r \in Z$.

7. Let $f(x)$ be a polynomial satisfying

$f(x)f(1/x) = f(x) + f(1/x)$ for all $x \in R - \{0\}$

and $f(5) = 126$, find $f(4)$.

8. If a function $f: R \to R$ is given by $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = a$. Find

$$\sum_{r=1}^{n} f(r).$$

9. Show that the functions

$$f(x) = x^2 - x + 1, x \geq \frac{1}{2} \text{ and } g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

are mutually inverse, and solve the equation

$$x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

10. Find a function $f: R \to R$ satisfying

$f(x)f(y) - f(xy) = x + y$ for all $x, y \in R$.

11. Find a function f satisfying

$$f(x)+f\left(\frac{x-1}{x}\right)=1+x \text{ for all } x\neq 0,1$$

12. If for all real values of u and v, we have

$$2f(u)\cos v=f(u+v)+f(u-v).$$

Prove that, for all real values of x

(i) $f(x)+f(-x)=2a\cos x$

(ii) $f(\pi-x)+f(-x)=0$,

(iii) $f(\pi-x)+f(x)=-2b\sin x$

Deduce that $f(x)=a\cos x-b\sin x$, where a, b are arbitrary constants.

13. If $f: R\to R$ be defined by $f(x)=x^3+3x^2+10x+2\sin x$, then prove that $f(x)$ is an onto function.

14. Let $f: X\to[1, 27]$ be given by $f(x)=5\sin x+12\cos x+14$. Find X so that f is a bijection.

ANSWERS

1. $a=1, b=0$; $a=-1, b\in R$
2. 997.5
3. $g(x)=x+3, h(x)=-(x+2)$
4. $f_{3n}(x)=x$
5. $\frac{1}{3988}$
7. 65
8. $\frac{an(n+1)}{2}$
10. $f(x)=1+x$
11. $f(x)=\frac{x^3-x^2-1}{2x(x-1)}$
14. $[-\alpha-\pi/2, \pi/2-\alpha]$, $\alpha=\tan^{-1}(12/5)$

HINTS TO SELECTED PROBLEMS

1. We have, $fof=1$

$\Rightarrow fof(x)=I(x)$ for all $x\in R$

$\Rightarrow f(f(x))=x$ for all $x\in R$

$\Rightarrow f(ax+b)=x$ for all $x\in R$

$\Rightarrow a(ax+b)+b=x$ for all $x\in R$

$\Rightarrow x(a^2-1)+ab+b=0$ for all $x\in R$

$\Rightarrow a^2-1=0$ and $ab+b=0$

$\Rightarrow a=\pm 1$ and $b(a+1)=0$

When $a=1, b(a+1)=0\Rightarrow b=0$

When $a=-1, b(a+1)=0\Rightarrow b$ can be any real number.

2. Proceed as in Example 13.

3. We have,

$$f(n)=1!+2!+3!+\ldots+n!$$

$\therefore f(n+1)=1!+2!+\ldots+n!+(n+1)!$

$\Rightarrow f(n+1)-f(n)=(n+1)!$ ...(i)

Now,

$$\begin{aligned}f(n+2)&=1!+2!+..+n!+(n+1)!+(n+2)!\\&=f(n+1)+(n+2)!\\&=f(n+1)+(n+2)(n+1)!\\&=f(n+1)+(n+2)[f(n+1)-f(n)] \quad \text{[Using (i)]}\\&=(n+3)f(n+1)-(n+2)f(n)\end{aligned}$$

But, $f(n+2)=f(n+1)g(n)+f(n)h(n)$. Therefore, on compairing, we get

$$g(n)=n+3 \text{ and } h(n)=-(n+2)$$

Hence, $g(x)=x+3$ and $h(x)=-(x+2)$.

5. We have

$$f(1)+2f(2)+3f(3)+\ldots.+nf(n)=n(n+1)f(n) \quad \ldots(i)$$

Replacing n by $(n+1)$, we get

$$f(1)+2f(2)+3f(3)+\ldots+nf(n)+(n+1)f(n+1)$$
$$=(n+1)(n+2)f(n+1) \quad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$(n+1)f(n+1)=(n+1)(n+2)f(n+1)-n(n+1)f(n)$$

$\Rightarrow nf(n)=(n+1)f(n+1)$

$\Rightarrow 2f(2)=3f(3)=\ldots=nf(n)$

Using this result in (i), we get

$$f(1)+(n-1)nf(n)=n(n+1)f(n)$$

$\Rightarrow f(1)=2nf(n)$

$\Rightarrow f(n)=\frac{1}{2n}$ $[\because f(1)=1]$

$\Rightarrow f(1994)=\frac{1}{3998}$.

9. Since the graphs of a given function and its inverse can intersect only on the straight line $y=x$. Therefore, to solve the given equation, it is sufficient to solve the equation $x^2-x+1=x$ which gives $x=1$.

10. Putting $x=y=0$, we get

$$(f(0))^2-f(0)=0\Rightarrow f(0)=0 \text{ or } f(0)=1.$$

Putting $y=0$, we get

$$f(x)f(0)-f(0)=x \text{ for all } x\in R \quad \ldots(i)$$

If $f(0)=0$, then the above result does not hold good. So, $f(0)=1$. Putting $f(0)=1$ in (i), we get

$\therefore f(x)-1=x\Rightarrow f(x)=1+x$ for all $x\in R$.

11. Replace x by $\frac{1}{1-x}$ and solve the two equations for $f(x)$].

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1.15 DOMAIN AND RANGE OF A REAL FUNCTION

DOMAIN Generally real functions in calculus are described by some formula and their domains are not explicitly stated. In such cases to find the domain of a function f (say) we use the fact that the domain is the set of all real numbers x for which $f(x)$ is a real number.

In other words, determining the domain of a function f means finding all real numbers x for which $f(x)$ is real. For example, if $f(x) = \sqrt{2-x}$, then $f(x)$ is real for all $x \le 2$. For $x > 2$, $f(x)$ is not real. So, domain of $f(x)$ is the set of all real numbers less than or equal to 2 i.e. $(-\infty, 2]$.

RANGE As discussed in section 1.2, the range of a function $f(x)$ is the set of values of $f(x)$ which it attains at points in its domain. For a real function the co-domain is always a subset of R. So, range of a real function f is the set of all points y such that $y = f(x)$, where $x \in Dom\, f(x)$.

The following algorithm is very helpful in determining the range of real functions.

ALGORITHM

STEP I *Put $f(x) = y$*

STEP II *Solve the equation in step I for x to obtain $x = \phi(y)$.*

STEP III *Find the values of y for which the values of x, obtained from $x = \phi(y)$ are in the domain of f*

STEP IV *The set of values of y obtained in step III is the range of f.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the domain of the function $f(x) = \sqrt{(x-1)(x-2)}$.*

SOLUTION Clearly, $f(x)$ is real if

$(x-1)(x-2) \ge 0$

$\Rightarrow x \le 1$ or $x \ge 2$

$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$

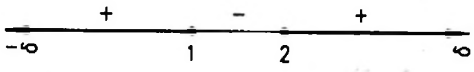


Fig. 1.19

Hence, domain of $f = (-\infty, 1] \cup [2, \infty)$.

EXAMPLE 2 *Find the domain of the function $f(x) = \frac{1}{\sqrt{9-x^2}}$*

SOLUTION Clearly $f(x)$ is a real number for all x satisfying $9 - x^2 > 0$.

Now,

$9 - x^2 > 0$

$\Rightarrow x^2 - 9 < 0$

$\Rightarrow (x-3)(x+3) < 0$

$\Rightarrow -3 < x < 3$

$\Rightarrow x \in (-3, 3)$

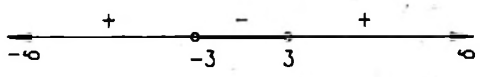


Fig. 1.20

Hence, domain of f is $(-3, 3)$.

EXAMPLE 3 *Find the domain of the function $f(x) = \frac{x}{x^2 - 3x + 2}$.*

SOLUTION Clearly, $f(x)$ is not defined, when $x^2 - 3x + 2 = 0$.

Now,

$x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$

So, $f(x)$ is not defined for $x = 1, 2$.

Hence, domain of $f(x)$ is $R - \{1, 2\}$.

EXAMPLE 4 *Find the domain of the function $f(x) = \sqrt{x^2 - 4}$.*

SOLUTION Clearly, $f(x)$ is a real number for all $x^2 - 4 \ge 0$. So, domain of $f(x)$ is the set of all real numbers for which $x^2 - 4 \ge 0$.

Now,

$x^2 - 4 \ge 0$

$\Rightarrow (x-2)(x+2) \ge 0$

$\Rightarrow x \le -2$ or $x \ge 2$

$\Rightarrow x \in (-\infty, -2] \cup [2, \infty)$

Hence, domain of f is $(-\infty, -2] \cup [2, \infty)$.

EXAMPLE 5 *Find the domain of the function $f(x) = \frac{x}{1+x^2}$.*

SOLUTION Clearly $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

EXAMPLE 6 *Find the domain and range of the following :*

(i) $f(x) = \frac{1}{x-5}$ (ii) $f(x) = x^2$ (iii) $f(x) = \frac{x}{1-x}$

SOLUTION (i) We have,

$$f(x) = \frac{1}{x-5}.$$

Clearly, $f(x)$ is not defined, when $x - 5 = 0$ i.e., $x = 5$.

So, Dom $(f) = R - \{5\}$.

Range : Let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow \frac{1}{x-5} = y$

$\Rightarrow x - 5 = \frac{1}{y}$

$\Rightarrow x = 5 + \frac{1}{y}$

$\Rightarrow x = \frac{5y+1}{y}$.

Clearly, x is not defined, when $y = 0$.

So, range $(f) = R - \{0\}$.

(ii) We have, $f(x) = x^2$.

Domain : Clearly, $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

Range : Let $f(x) = y$. Then,

$$x^2 = y \Rightarrow x = \pm\sqrt{y}.$$

Clearly, x is not a real number for negative real values of y. So, range (f) = set of all non-negative real numbers i.e. $[0, \infty)$.

(iii) We have, $f(x) = \dfrac{x}{1-x}$

Domain: Clearly, $f(x)$ is not defined, when $1 - x = 0$, i.e. when $x = 1$.

So, domain $(f) = R - \{1\}$.

Range : Let $f(x) = y$. Then,

$$\frac{x}{1-x} = y$$

$$\Rightarrow \quad x = \frac{y}{1+y}$$

This shows that x is not a real number, when $1 + y = 0$, i.e. when $y = -1$.

So, range $(f) = R - \{-1\}$.

EXAMPLE 7 *Find the domain and range of the function*

$$f(x) = \frac{x}{1+x^2}.$$

SOLUTION We have,

$$f(x) = \frac{x}{1+x^2}.$$

Domain : Clearly, $f(x)$ is defined for all real values of x. So, domain $(f) = R$.

Range : Let $f(x) = y$. Then,

$$\frac{x}{1+x^2} = y$$

$$\Rightarrow x^2 y - x + y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

Now,

$\dfrac{1 \pm \sqrt{1-4y^2}}{2y}$ is a real number *iff*

$1 - 4y^2 \geq 0$ and $y \neq 0$

$\Leftrightarrow 4y^2 - 1 \leq 0$ and $y \neq 0$

$\Leftrightarrow (y^2 - 1/4) \leq 0$ and $y \neq 0$

$\Leftrightarrow (y - 1/2)(y + 1/2) \leq 0$ and $y \neq 0$

$\Leftrightarrow -1/2 \leq y \leq 1/2$ and $y \neq 0$

$\Leftrightarrow y \in [-1/2, 0) \cup (0, 1/2]$

For $x = 0$, we have $y = 0$.

Hence, range of f is $[-1/2, 1/2]$.

EXAMPLE 8 *Find the domain and range of the function*

$$f(x) = \frac{1}{2 - \cos 3x}.$$

SOLUTION We have, $f(x) = \dfrac{1}{2 - \cos 3x}$.

Domain : We have,

$-1 \leq \cos 3x \leq 1$ for all $x \in R$,

$\Rightarrow -1 \leq -\cos 3x \leq 1$

$\Rightarrow 1 \leq 2 - \cos 3x \leq 3$ for all $x \in R$

$\Rightarrow f(x)$ is defined for all $x \in R$.

So, domain $(f) = R$.

Range : Let $f(x) = y$. Then,

$$\Rightarrow \frac{1}{2 - \cos 3x} = y$$

$$\Rightarrow 2 - \cos 3x = \frac{1}{y}$$

$$\Rightarrow \cos 3x = 2 - \frac{1}{y}$$

$$\Rightarrow -1 \leq 2 - \frac{1}{y} \leq 1 \qquad [\because -1 \leq \cos 3x \leq 1]$$

$$\Rightarrow -3 \leq -\frac{1}{y} \leq -1$$

$$\Rightarrow 3 \geq \frac{1}{y} \geq 1$$

$$\Rightarrow 1/3 \leq y \leq 1$$

$$\Rightarrow y \in [1/3, 1]$$

So, range of $(f) = [1/3, 1]$.

EXAMPLE 9 *Find the domain and range of* $f(x) = \sqrt{9 - x^2}$.

SOLUTION We have,

$$f(x) = \sqrt{9 - x^2}$$

Domain Clearly, $f(x)$ is not defined for $9 - x^2 < 0$. So, $f(x)$ is defined only when $9 - x^2 \geq 0$.

Now,

$$9 - x^2 \geq 0$$

$$\Rightarrow x^2 - 9 \leq 0$$

$$\Rightarrow (x-3)(x+3) \leq 0$$

$$\Rightarrow -3 \leq x \leq 3$$

$\Rightarrow x \in [-3, 3]$

So, domain $f(x) = [-3, 3]$.

Range Let $f(x) = y$. Then,

$\Rightarrow \sqrt{9 - x^2} = y$

$\Rightarrow \quad x = \pm\sqrt{9 - y^2}$

Clearly, x is defined, when $9 - y^2 \geq 0$.

Now,

$9 - y^2 \geq 0$

$\Rightarrow y^2 - 9 \leq 0$

$\Rightarrow (y - 3)(y + 3) \leq 0$

$\Rightarrow -3 \leq y \leq 3$

But, $f(x)$ attains only non-negative values. Therefore,

$y \in [0, 3]$. Hence, range $(f) = [0, 3]$.

EXAMPLE 10 *Find the domain and range of the function* $f(x) = \dfrac{2 + x}{2 - x}$.

SOLUTION We have,

$$f(x) = \frac{2 + x}{2 - x}.$$

Domain : Clearly, $f(x)$ is not defined, when $2 - x = 0$, i.e., when $x = 2$.

So, domain $(f) = R - \{2\}$.

Range Let $f(x) = y$. Then,

$\Rightarrow \dfrac{2 + x}{2 - x} = y$

$\Rightarrow x = 2\left(\dfrac{y - 1}{y + 1}\right)$

Clearly, x is not defined for $y + 1 = 0$, i.e., when $y = -1$.

So, range $(f) = R - \{-1\}$.

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1.16 SOME STANDARD REAL FUNCTIONS

In this section, we shall learn about some standard real functions. These functions and their graphs will be used to draw the graphs of other functions obtained from the standard functions.

CONSTANT FUNCTION *Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

Clearly, the domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in Fig. 1.21 and it is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.

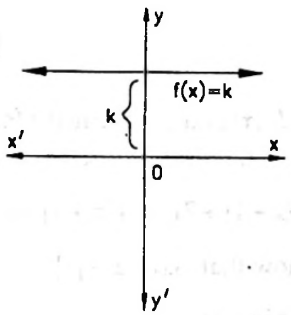


Fig. 1.21

IDENTITY FUNCTION *The function defined by $I(x) = x$ for all $x \in R$, is called the identity function on R.*

Clearly, the domain and range of the identity function are both equal to R. The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with x-axis.

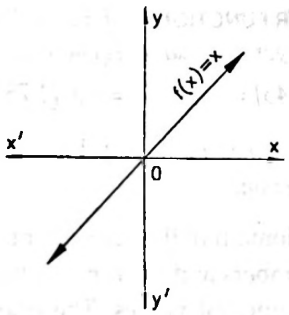


Fig. 1.22

(III) MODULUS FUNCTION *The function defined by*

$$f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$$

is called the modulus function.

The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers. The graph of the modulus function is as shown in Fig. 1.23.

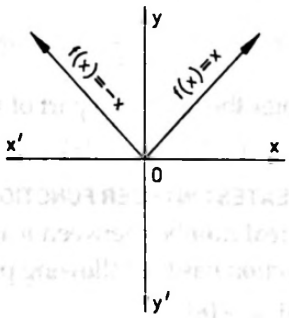


Fig. 1.23

PROPERTIES OF MODULUS FUNCTION *The modulus function has the following properties :*

(i) For any real number x, we have

$$\sqrt{x^2} = |x|.$$

(ii) If a, b are positive real numbers then

$$x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$$

$$x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$$

$$x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a \text{ or } x \ge a$$

$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a \text{ or } x > a$$

$$a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$$

$$a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$$

(iii) $|x+y| = |x| + |y|$

$\Leftrightarrow (x \ge 0 \text{ and } y \ge 0)$ or $(x < 0 \text{ and } y < 0)$

(iv) $|x-y| = |x| - |y|$

$\Leftrightarrow (x \ge 0 \text{ and } |x| \ge |y| \text{ or } (x \le 0, y \le 0 \text{ and } |x| \ge |y|)$

(v) $|x \pm y| \le |x| + |y|$

(vi) $|x \pm y| \ge \big||x| - |y|\big|$

GREATEST INTEGER FUNCTION *For any real number x, we denote $[x]$, the greatest integer less than or equal to x.*

For example, $[2.45] = 2, [-2.1] = -3, [1.75] = 1, [0.32] = 0$ etc.

The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.

Clearly, the domain of the greatest integer function is the set R of all real numbers and the range is the set of all integers as it attains only integral values. The graph of the greatest integer function is shown in Fig. 1.24.

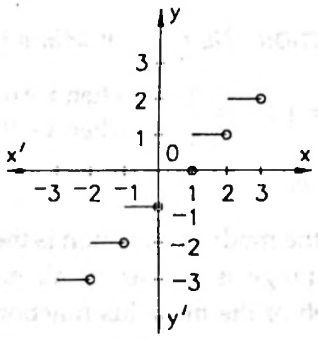


Fig. 1.24

In general, if n is an integer and x is any number satisfying $n \le x < n+1$, then

$$[x] = n$$

Also, if $\{x\}$ denotes the fractional part of x, then

$$[x] = x - \{x\} \text{ or } x = [x] + \{x\}.$$

PROPERTIES OF GREATEST INTEGER FUNCTION If n is an integer and x is any real number between n and $n+1$, then the greatest integer function has the following properties :

(i) $[-n] = -[n]$

(ii) $[x+n] = [x] + n$

(iii) $[-x] = -[x] - 1$

(iv) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x] - [-x] = \begin{cases} 2[x] & ; \text{ if } x \in Z \\ 2[x]+1, & \text{if } x \notin Z \end{cases}$

(vi) $[x] \ge n \Rightarrow x \ge n$, where $n \in Z$

(vii) $[x] \le n \Rightarrow x < n+1, n \in Z$

(viii) $[x] > n \Rightarrow x \ge n+1, n \in Z$

(ix) $[x] < n \Rightarrow x < n, n \in Z$

(x) $[x+y] = [x] + [y + x - [x]]$ for all $x, y \in R$.

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$

ILLUSTRATION 1 *If $\{x\}$ and $[x]$ denote respectively the fractional and integral parts of a real number x. Solve the eqution*

$$4\{x\} = x + [x] \qquad \textbf{[IIT 1994]}$$

SOLUTION We know that : $x = [x] + \{x\}$

$\therefore\ 4\{x\} = x + [x]$

$\Rightarrow\ 4\{x\} = [x] + \{x\} + [x]$

$\Rightarrow\ 3\{x\} = 2[x]$ ...(i)

But,

$0 \le \{x\} < 1$

$\therefore\ 0 \le 3\{x\} < 3$

$\Rightarrow\ 0 \le 2[x] < 3$ [Using (i)]

$\Rightarrow\ 0 \le [x] < \frac{3}{2}$

$\Rightarrow\ [x] = 0, 1$

From (i), we have

$3\{x\} = 2[x]$

$\Rightarrow\ 3(x - [x]) = 2[x]$ $[\because \{x\} = x - [x]]$

$\Rightarrow\ x = \frac{5}{3}[x]$

$\Rightarrow\ x = 0, \frac{5}{3}$ [Putting $[x] = 0, 1$]

ILLUSTRATION 2 *If $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x, respectively, solve*

$$\{x+1\} + 2x = 4[x+1] - 6$$

SOLUTION We know that : $\{x\} = x - [x]$

$\therefore\ \{x+1\} + 2x = 4[x+1] - 6$

$\Rightarrow\ x + 1 - [x+1] + 2x = 4[x+1] - 6$

$\Rightarrow\ 3x + 1 = 5[x+1] - 6$

$\Rightarrow\ 3x + 1 = 5([x] + 1) - 6$

$\Rightarrow\ 3x = 5[x] - 2$ ...(i)

$\Rightarrow\ 3([x] + \{x\}) = 5[x] - 2$

$\Rightarrow 3\{x\} = 2[x] - 2$ ...(ii)

Now,

$0 \le \{x\} < 1$

$\Rightarrow 0 \le 3\{x\} < 3$

$\Rightarrow 0 \le 2[x] - 2 < 3$ [Using (ii)]

$\Rightarrow 2 \le 2[x] < 5$

$\Rightarrow 1 \le [x] < \frac{5}{2}$

$\Rightarrow [x] = 1, 2$

From (i), we have

$[x] = 1 \Rightarrow x = 1$ and, $[x] = 2 \Rightarrow x = \frac{8}{3}$

Hence, $x = 1, 8/3$ are the solutions of the given equation.

ILLUSTRATION 3 *If $\{x\}$ and $[x]$ denote the fractional and integral parts of x and (x) is defined as follows :*

$$(x) = \begin{cases} 2[x] - \{x\}, & x < 0 \\ [x] + 3\{x\}, & x \ge 0 \end{cases}$$

then solve the equation :

$$(x) = x + \{x\}$$

SOLUTION Case I *When $x < 0$:*

We have,

$(x) = 2[x] - \{x\}$

$\Rightarrow x + \{x\} = 2[x] - \{x\}$ $[\because (x) = x + \{x\}]$

$\Rightarrow x + 2\{x\} = 2[x]$

$\Rightarrow [x] + \{x\} + 2\{x\} = 2[x]$ $[\because x = [x] + \{x\}]$

$\Rightarrow 3\{x\} = [x]$ ...(i)

Now,

$0 \le \{x\} < 1$

$\Rightarrow 0 \le 3\{x\} < 3$

$\Rightarrow 0 \le [x] < 3$ [Using (i)]

$\Rightarrow [x] = 0, 1, 2$ ...(ii)

Again,

$3\{x\} = [x]$ [From (i)]

$\Rightarrow 3(x - [x]) = [x]$

$\Rightarrow 3x = 4[x]$

$\Rightarrow x = \frac{4}{3}[x]$

$\Rightarrow x = 0, \frac{4}{3}, \frac{8}{3}$ [Using (ii)]

But, $x < 0$. Therefore, the given equation has no solution for $x < 0$.

CASE II *When $x \ge 0$*

We have,

$(x) = [x] + 3\{x\}$

$\Rightarrow x + \{x\} = [x] + 3\{x\}$ $[\because (x) = x + \{x\}]$

$\Rightarrow x - [x] = 2\{x\}$

$\Rightarrow \{x\} = 2\{x\}$

$\Rightarrow \{x\} = 0$

$\Rightarrow x \in Z.$

But, $x \ge 0$.

$\therefore x = 0, 1, 2, 3, \ldots$

Hence, $x = 0, 1, 2, 3, ..$ are the solutions of the given equation.

ILLUSTRATION 4 *Solve the equation $(x)^2 = [x]^2 + 2x$, where $[x]$ and (x) are integes just less than or equal to x and just greater than or equal to x respectively.*

SOLUTION Let $\{x\}$ denote the fractional part of x.

CASE I When $0 \le \{x\} < \frac{1}{2}$

In this case, we have

$(x) = [x]$

So, the given equation reduces to

$[x]^2 = [x]^2 + 2x$

$\Rightarrow x = 0$

CASE II When $\frac{1}{2} \le \{x\} < 1$

In this case, we have

$(x) = [x] + 1$

$\therefore (x)^2 = [x]^2 + 2x$

$\Rightarrow \{[x] + 1\}^2 = [x]^2 + 2x$

$\Rightarrow 2[x] + 1 = 2x$

$\Rightarrow x = [x] + \frac{1}{2}$

$\Rightarrow x = n + \frac{1}{2}, n \in Z$

Hence, the solution set of the given equation is given by $x = 0, \ n + \frac{1}{2}, n \in Z$

SIGNUM FUNCTION *The function defined by*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

or,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function.

The domain of the signum function is R and the range is the set $\{-1, 0, 1\}$. The graph of this function is shown in Fig. 1.25.

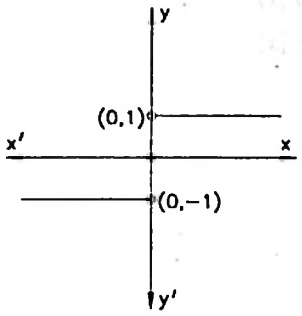


Fig. 1.25

RECIPROCAL FUNCTION *The function that associates each non-zero real number x to its reciprocal $1/x$ is called the reciprocal function.*

The domain and range of the reciprocal function are both equal to $R - \{0\}$ i.e. the set of all non-zero real numbers. The graph is as shown in Fig. 1.26.

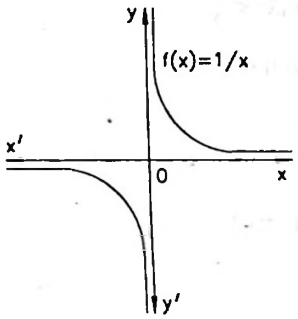


Fig. 1.26

LOGARITHMIC FUNCTION *If 'a' is a positive real number, then the function that associates every positive real number to $\log_a x$ i.e. $f(x) = \log_a x$ is called the logarithmic function.*

The domain of the logarithmic function is the set of all positive real numbers and the range is the set R of all real numbers. Graphs of this function for different values of a are shown in Figs. 1.27 and 1.28.

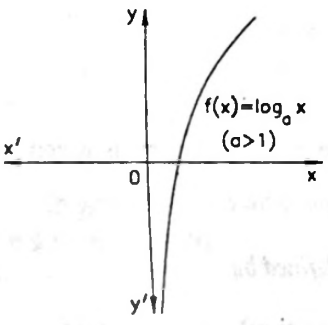


Fig. 1.27

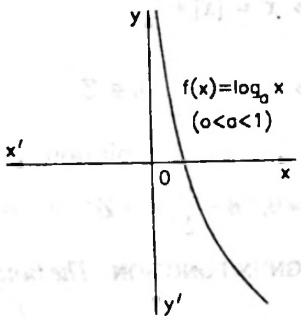


Fig. 1.28

EXPONENTIAL FUNCTION *If a is positive real number, then the function which associates every real number x to a^x i.e. $f(x) = a^x$ is called the exponential function.*

Clearly, the domain of the exponential function is R and the range is the set of all positive real numbers. The graphs of the exponential function for different values of a are as shown in Figs. 1.29 and 1.30.

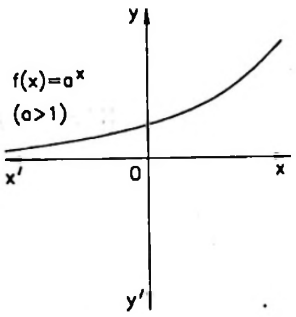


Fig. 1.29

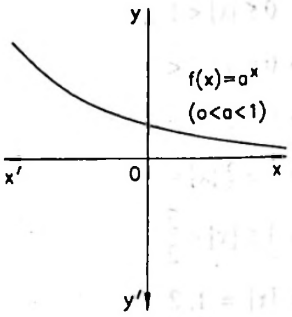


Fig. 1.30

SQUARE ROOT FUNCTION *The function that associates every positive real number x to $+\sqrt{x}$ is called the square root function, i.e., $f(x) = +\sqrt{x}$.*

Since negative real numbers do not have real square roots. So, $f(x)$ is not defined when x is a negative real number. Therefore, domain of f is the set of all non-negative real numbers $= [0, \infty)$.

Clearly, range $(f) = \{+\sqrt{x} \mid x \in [0, \infty)\} = [0, \infty)$.

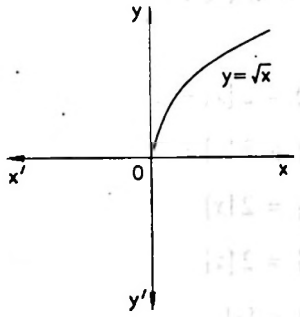


Fig. 1.31

The graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 1.31.

POLYNOMIAL FUNCTION *A function of the form $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, where $a_0, a_1, a_2, \ldots a_n$ are real numbers, $a_0 \neq 0$ and $n \in N$, is called a polynomial function of degree n.*

The domain of a polynomial function is always R.

RATIONAL FUNCTION *A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.*

The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers except points where $q(x) = 0$.

SINE FUNCTION *The function that associates to each real number x to $\sin x$ is called the sine function. Here, x is the radian measure of the angle.*

The domain of the sine function is R and the range is $[-1, 1]$. Graph of the sine function is shown in Fig. 1.32

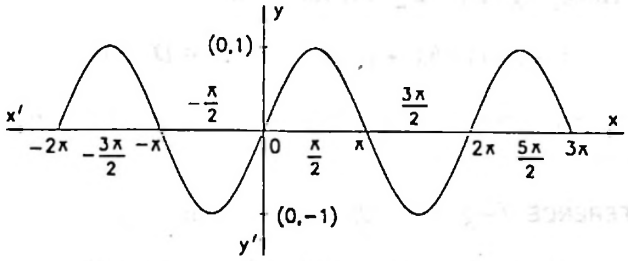


Fig. 1.32

COSINE FUNCTION *The function that associates to each real number x to cos x is called the cosine function. Here, x is the radian measure of the angle.*

The domain of the cosine function is R and the range is $[-1, 1]$. Graph of the cosine function is shown in Fig. 1.33

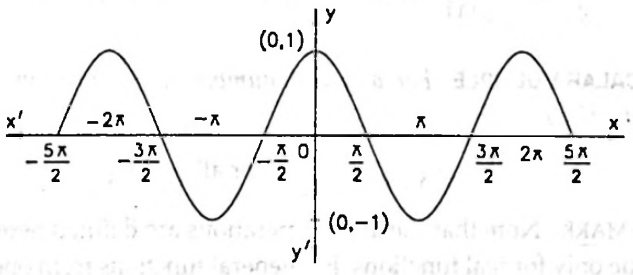


Fig. 1.33

TANGENT FUNCTION *The function that associates a real number x to tan x is called the tangent function.*

Clearly, the tangent function is not defined at odd multiples of $\pi/2$ i.e. $\pm\pi/2, \pm 3\pi/2$ etc. So, the domain of the tangent function is $R - \{(2n+1)\pi/2 \mid n \in Z\}$. Since it takes every value between $-\infty$ and ∞. So, the range is R. Graph of $f(x) = \tan x$ is shown in Fig. 1.34

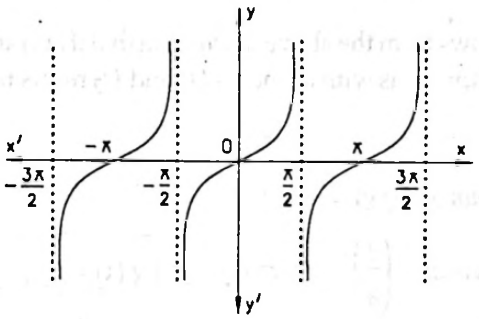


Fig. 1.34

COSECANT FUNCTION *The function that associates a real number x to cosec x is called the cosecant function.*

Clearly, cosec x is not defined at $x = n\pi, n \in Z$ i.e. $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So, its domain is $R - \{n\pi \mid n \in Z\}$.

Since cosec $x \geq 1$ or cosec $x \leq -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \text{cosec } x$ is shown in Fig. 1.35.

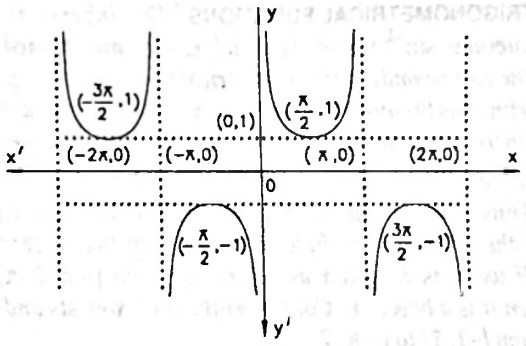


Fig. 1.35

SECANT FUNCTION *The function that associates a real number x to sec x is called the secant function.*

Clearly, sec x is not defined at odd multiples of $\pi/2$ i.e. $(2n+1)\pi/2$, where $n \in Z$. So, its domain is

$$R - \{(2n+1)\pi/2 \mid n \in Z\}.$$

Also, $|\sec x| \geq 1$, therefore its range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \sec x$ is shown in Fig. 1.36.

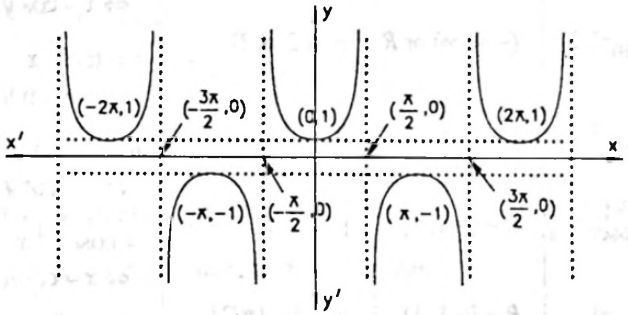


Fig. 1.36

COTANGENT FUNCTION *The function that associates a real number x to cot x is called the cotangent function.*

Clearly, cot x is not defined at $x = n\pi, n \in Z$ i.e. at $n = 0$, $\pm\pi, \pm 2\pi$ etc. So, domain of cot x is $R - \{n\pi \mid n \in Z\}$. The range of $f(x) = \cot x$ is R as is evident from its graph shown in Fig. 1.37.

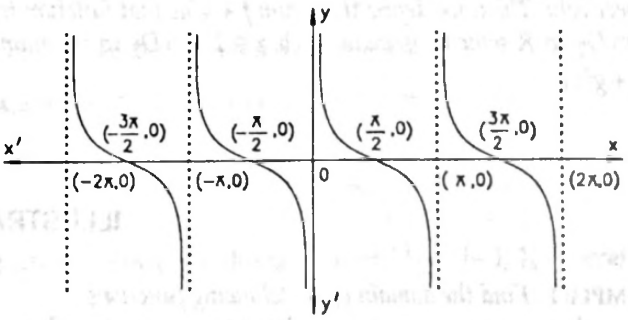


Fig. 1.37

INVERSE TRIGONOMETRICAL FUNCTIONS *The inverse trigonometrical functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc. are defined as the inverse of the corresponding trigonometrical functions. For example, $\sin^{-1} x$ is defined as the angle whose sine is x. But, since for each value of $\sin^{-1} x$, there corresponding an unlimited number of values of the angles whose sine is x and for values of x with $|x| > 1$, $\sin^{-1} x$ does not exist. Thus, we cannot define the inverse function of sin x unless we modify the definition of sin x in such a way that it becomes a bijection. If we consider sin x as a function from $[-\pi/2, \pi/2]$ to $[-1, 1]$, then it is a bijection. Consequently $\sin^{-1} x$ exists and it is a function from $[-1, 1]$ to $[-\pi/2, \pi/2]$.*

Similarly, the other inverse trigonometrical functions can be defined.

The domain and range of the inverse trigonometrical functions are as stated below :

Function	*Domain*	*Range*	*Definition of the function*
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\text{cosec}^{-1} x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec } y$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

1.17 OPERATIONS ON REAL FUNCTIONS

In the previous section, we have listed some important real functions. In this section we will introduce some operations on real functions to define new functions.

SUM *Let f and g be two real functions with domain D_1 and D_2 respectively. Then, we define their sum f + g as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

Thus, $f + g : D_1 \cap D_2 \to R$ such that

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2.$$

Similarly, we define the difference, product and quotient as follows :

DIFFERENCE $f - g : D_1 \cap D_2 \to R$ such that

$$(f-g)(x) = f(x) - g(x) \text{ for all } x \in D_1 \cap D_2$$

PRODUCT $fg : D_1 \cap D_2 \to R$ such that

$$(fg)(x) = f(x)\, g(x) \text{ for all } x \in D_1 \cap D_2$$

QUOTIENT $\frac{f}{g} : D_1 \cap D_2 - \{x \mid g(x) = 0\} \to R$ such that

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \in D_1 \cap D_2 - \{x \mid g(x) = 0\}.$$

SCALAR MULTIPLE *For any real number c, the function cf is defined by*

$$(cf)(x) = c\,.f(x) \text{ for all } x \in D_1.$$

REMARK Note that the above operations are defined here are true only for real functions. For general functions from one set to another, these do not make sense.

COMPOSITION OF FUNCTIONS *Let f and g be two functions with domain D_1 and D_2 respectively. If range $(f) \subset$ domain(g), we define gof by the rule*

$$(gof)(x) = g(f(x)) \text{ for all } x \in D_1.$$

Also, if range $(g) \subset$ domain (f), we define *fog* by the rule

$$(fog)(x) = f(g(x)) \text{ for all } x \in D_2.$$

It follows from the above discussion that if $f(x)$ and $g(x)$ are two real functions with domains D_1 and D_2 respectively. Then

(i) *Domain of* $(f \pm g) = D_1 \cap D_2$

(ii) *Domain of* $(fg) = D_1 \cap D_2$

(iii) *Domain of* $\left(\frac{f}{g}\right) = D_1 \cap D_2 - \{x \mid g(x) = 0\}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the domain of the following functions :*

(i) $\cos^{-1}(3x-1)$ (ii) $\sin^{-1} 2x$ (iii) $\tan^{-1}(2x+1)$

SOLUTION (i) Clearly, $\cos^{-1}(3x-1)$ is defined, if

$-1 \le 3x - 1 \le 1$ [$\because$ Domain of $\cos^{-1} x = [-1, 1]$]

$\Rightarrow -1 + 1 \le 3x \le 1 + 1$

$\Rightarrow 0 \le 3x \le 2$

$\Rightarrow 0 \le x \le 2/3$

$\Rightarrow x \in [0, 2/3]$

Hence, domain of $\cos^{-1}(3x-1)$ is $[0, 2/3]$.

(ii) Clearly, $\sin^{-1} 2x$ exists, if

$-1 \le 2x \le 1$ [$\because$ Domain of $\sin^{-1} x = [-1, 1]$]

$\Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

$\Rightarrow x \in [-1/2, 1/2]$

Hence, domain of $\sin^{-1}(2x)$ is $[-1/2, 1/2]$.

(iii) Since $\tan^{-1} x$ exists, if $x \in (-\infty, \infty)$.

Therefore, $\tan^{-1}(2x+1)$ is defined, if

$-\infty < 2x+1 < \infty \Rightarrow -\infty < x < \infty \Rightarrow x \in (-\infty, \infty) = R$

So, domain of $\tan^{-1}(2x+1)$ is R.

EXAMPLE 2 *Find the domain of the following functions :*

(i) $\frac{\sin^{-1} x}{x}$ (ii) $\frac{\cos^{-1} x}{[x]}$ (iii) $\frac{1}{\sqrt{|x|-x}}$

SOLUTION (i) Let $f(x) = \frac{\sin^{-1} x}{x}, g(x) = \sin^{-1} x$ and $h(x) = x$.

Then, $f(x) = \frac{g(x)}{h(x)}$

$\therefore$ Domain (f) = Domain $(g) \cap$ Domain $h - \{x \mid h(x) = 0\}$.

Clearly, domain $(g) = [-1, 1]$, domain $h = R$,
and, $\{x \mid h(x) = 0\} = \{0\}$

$\therefore$ Domain $(f) = [-1, 1] \cap R - \{0\} = [-1, 0) \cup (0, 1]$.

(ii) Let $f(x) = \frac{\cos^{-1} x}{[x]}, g(x) = \cos^{-1} x$ and $h(x) = [x]$.

Then, $f(x) = \frac{g(x)}{h(x)}$

$\therefore$ Domain (f) = Domain $(g) \cap$ Domain $h - \{x \mid h(x) = 0\}$

Clearly, domain $(g) = [-1, 1]$, domain $h = R$,
and, $\{x : h(x) = 0\} = \{x : [x] = 0\} = [0, 1)$.

$\therefore$ Domain $(f) = [-1, 1] \cap R - [0, 1) = [-1, 0) \cup \{1\}$.

(iii) Let $f(x) = \frac{1}{\sqrt{|x|-x}}$. Clearly, $f(x)$ is defined for

$|x| - x > 0 \Rightarrow |x| > x \Rightarrow x < 0$

Hence, domain $(f) = (-\infty, 0)$.

EXAMPLE 3 *Find the domain of the function*

$$f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2} \qquad \text{[IIT 1983]}$$

SOLUTION Let $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$.

Then, $f(x) = g(x) + h(x)$. Therefore,

Domain (f) = Domain $(g) \cap$ Domain (h)

Now, $g(x) = \frac{1}{\log_{10}(1-x)}$ exists, if $1 - x > 0$ and $1 - x \ne 1$
i.e., $x < 1$ and $x \ne 0$.

So, domain $(g) = (-\infty, 1) - \{0\}$.

Clearly, $h(x) = \sqrt{x+2}$ exists, if $x + 2 \ge 0$, i.e. $x \ge -2$

So, domain $(h) = [-2, \infty)$

Hence, domain $(f) = (-\infty, 1) - \{0\} \cap [-2, \infty)$
$= [-2, 0) \cup (0, 1)$

EXAMPLE 4 *Find the domain of the function*

$$f(x) = \sqrt{1-2x} + 3\sin^{-1}\left(\frac{3x-1}{2}\right).$$

SOLUTION Let $f(x) = g(x) + h(x)$, where

$$g(x) = \sqrt{1-2x} \quad \text{and} \quad h(x) = 3\sin^{-1}\left(\frac{3x-1}{2}\right).$$

Clearly, $g(x)$ is not defined, when $1 - 2x < 0$ i.e. $g(x)$ is defined, when $1 - 2x \ge 0 \Rightarrow x \le 1/2$. Therefore,

Domain $(g) = (-\infty, 1/2]$

Also, $h(x)$ is defined, when

$$-1 \le \frac{3x-1}{2} \le 1$$

$\Rightarrow -2 \le 3x - 1 \le 2$

$\Rightarrow -1 \le 3x \le 3 \Rightarrow \frac{-1}{3} \le x \le 1$

So, domain $(h) = [-1/3, 1]$

Hence, domain $(f) = (-\infty, 1/2] \cap [-1/3, 1] = [-1/3, 1/2]$.

EXAMPLE 5 *Find the domain of the function*

$$\psi(x) = \frac{1}{x} + 2^{\sin^{-1} x} + \frac{1}{\sqrt{x-2}}.$$

SOLUTION Let $\psi(x) = f(x) + g(x) + h(x)$,

where $f(x) = \frac{1}{x}, g(x) = 2^{\sin^{-1} x}$ and $h(x) = \frac{1}{\sqrt{x-2}}$.

Clearly, domain $(f) = R - \{0\}$, domain $(g) = [-1, 1]$, and domain $h = (2, \infty)$.

Domain $(\psi) = R - \{0\} \cap [-1, 1] \cap (2, \infty) = \phi$ (null set).

Hence, $f(x)$ is not defined for any $x \in R$.

EXAMPLE 6 *Find the domain of the function*

$$f(x) = \sin^{-1}\left\{\log_2\left(\frac{x^2}{2}\right)\right\} \qquad \text{[IIT 1984]}$$

SOLUTION Since the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore,

$f(x) = \sin^{-1}\left\{\log_2\left(\frac{x^2}{2}\right)\right\}$ exists if

$-1 \le \log_2\left(x^2/2\right) \le 1$

$\Rightarrow 2^{-1} \le \frac{x^2}{2} \le 2^1$

$\Rightarrow 1 \le x^2 \le 4$

$\Rightarrow 1 \le |x| \le 2$

$\Rightarrow x \in [-2, -1] \cup [1, 2]$

Hence, the domain of $f(x)$ is $[-2, -1] \cup [1, 2]$.

EXAMPLE 7 *Find the domain of the function* $f(x) = \sqrt{\log_2 \sin x}$.

SOLUTION $f(x)$ exists iff

$\log_2 (\sin x) \ge 0$

$\Rightarrow \sin x \ge 2^0$

$\Rightarrow \sin x = 1$

$\Rightarrow x = (4n+1)\frac{\pi}{2}, n \in N$

EXAMPLE 8 *Find the domain of the definition of the function*

$$f(x) = \log_4\left[\log_5\left\{\log_3 (18x - x^2 - 77)\right\}\right].$$

SOLUTION Since $\log x$ is defined for $x > 0$. Therefore,

$f(x) = \log_4\left[\log_5\left\{\log_3 (18x - x^2 - 77)\right\}\right]$ exists, if

$\log_5 \{\log_3 (18x - x^2 - 77)\} > 0$

$\Rightarrow \log_3 (18x - x^2 - 77) > 5^0$

$\Rightarrow (18x - x^2 - 77) > 3^1$

$\Rightarrow x^2 - 18 + 80 < 0$

$\Rightarrow (x-8)(x-10) < 0$

$\Rightarrow 8 < x < 10$

$\Rightarrow x \in (8, 10)$

Hence, the domain of definition of the given function is $(8, 10)$.

EXAMPLE 9 *Find the domain and range of the function*

$$f(x) = \sin\left\{\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}.$$

SOLUTION Since $\log x$ is defined for $x > 0$. Therefore, $f(x)$ is defined for

$\frac{\sqrt{4-x^2}}{1-x} > 0$

$\Rightarrow 4 - x^2 > 0$ and $1 - x > 0$

$\Rightarrow x^2 - 4 < 0$ and $x < 1$

$\Rightarrow -2 < x < 2$ and $x < 1$

$\Rightarrow x \in (-2, 1)$

Hence, the domain of $f(x)$ is $(-2, 1)$

Since $f(x) = \sin\left\{\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$ and $\sin x$ assumes all values between -1 and 1. Therefore, range of $f(x)$ is $[-1, 1]$.

EXAMPLE 10 *Find the domains of definition of each of the following functions :*

(i) $f(x) = \sqrt{\frac{\log_{0.3} |x-2|}{|x|}}$

(ii) $f(x) = \log \{1 - \log_{10} (x^2 - 5x + 16)\}$

(iii) $f(x) = \frac{1}{[|x-1|] + [|7-x|] - 6}$

(iv) $f(x) = \frac{1}{\sqrt{\log_{1/2} (x^2 - 7x + 13)}}$

SOLUTION (i) We have,

$$f(x) = \sqrt{\frac{\log_{0.3} |x-2|}{|x|}}$$

Clearly, $f(x)$ is defined if

$x \ne 0, x - 2 \ne 0$ and $\frac{\log_{0.3} |x-2|}{|x|} \ge 0$

$\Rightarrow x \ne 0, 2$ and $\log_{0.3} |x-2| \ge 0$ $[\because |x| > 0]$

$\Rightarrow x \ne 0, 2$ and $|x-2| \le (0.3)^0$

$\Rightarrow x \ne 0, 2$ and $|x-2| \le 1$

$\Rightarrow x \ne 0, 2$ and $-1 \le x - 2 \le 1$

$\Rightarrow x \ne 0, 2$ and $1 \le x \le 3$

$\Rightarrow x \in [1, 2) \cup (2, 3]$

Hence, the domain of definition of $f(x)$ is $[1, 2) \cup (2, 3]$.

(ii) We have,

$$f(x) = \log_{10} \{1 - \log_{10} (x^2 - 5x + 16)\}$$

Clearly, $x^2 - 5x + 16 > 0$ for all real values of x, because its discriminant is negative and coefficient of x^2 is positive. Therefore, $f(x)$ is defined if

$1 - \log_{10} (x^2 - 5x + 16) > 0$

$\Rightarrow \log_{10} (x^2 - 5x + 16) < 1$

$\Rightarrow x^2 - 5x + 16 < 10^1$

$\Rightarrow x^2 - 5x + 6 < 0$

$\Rightarrow (x-2)(x-3) < 0$

$\Rightarrow 2 < x < 3$

$\Rightarrow x \in (2, 3)$

Hence, the domain of definition of $f(x)$ is $(2, 3)$.

(iii) We have,

$$f(x) = \frac{1}{[|x-1|] + [|7-x|] - 6}$$

Clearly, $f(x)$ is defined if

$[|x-1|] + [|7-x|] - 6 \ne 0$

$\Rightarrow [|x-1|]+[|7-x|] \neq 6$

$\Rightarrow [|x-1|]+[|x-7|] \neq 6$

$\Rightarrow [1-x]+[7-x] \neq 6$, if $x<1$

$[x-1]+[7-x] \neq 6$, if $1 \leq x<7$

and,

$[x-1]+[x-7] \neq 6$, if $x \geq 7$.

$\Rightarrow 1+[-x]+7+[-x] \neq 6$, if $x<1$

$[x]-1+7+[-x] \neq 6$, if $1 \leq x<7$ $[\because [n+x]=n+[x]]$

and,

$[x]-1+[x]-7 \neq 6$, if $x \geq 7$.

$\Rightarrow [-x] \neq -1$, if $x<1$

$[x]+[-x] \neq 0$, if $1 \leq x<7$

and,

$[x] \neq 7$, if $x \geq 7$

Now,

$[-x] \neq -1$, if $x<1$

$\Rightarrow x \notin (0, 1]$ ...(i)

$[x]+[-x] \neq 0$, if $1 \leq x<7$

$\Rightarrow x \notin \{1, 2, 3, 4, 5, 6\}$ ...(ii)

and, $[x] \neq 7$, if $x \geq 7$

$\Rightarrow x \notin [7, 8)$ ...(iii)

From (i), (ii) and (iii), we obtain

$x \in R-(0, 1] \cup \{1, 2, 3, 4, 5, 6\} \cup [7, 8)$.

Hence, domain of

$f(x)=R-(0, 1] \cup \{1, 2, 3, 4, 5, 6\} \cup [7, 8)$.

(iv) Let $f(x)=\dfrac{1}{\sqrt{\log_{1/2}(x^2-7x+13)}}$

Clearly, $f(x)$ is defined if

$\log_{1/2}(x^2-7x+13)>0$ and $x^2-7x+13 \neq 0$

$\Rightarrow x^2-7x+13<\left(\dfrac{1}{2}\right)^0$ $[\because x^2-7x+13>0$ for all $x]$

$\Rightarrow x^2-7x+13<1$

$\Rightarrow x^2-7x+12<0$

$\Rightarrow (x-3)(x-4)<0$

$\Rightarrow 3<x<4$

$\Rightarrow x \in (3, 4)$.

Hence, domain of $f(x)=(3, 4)$.

EXAMPLE 11 *Find the domain of definition of each of the following functions :*

(i) $f(x)=\dfrac{1}{\sqrt{|\,[|x|-1]\,|-5}}$ (ii) $f(x)=\sqrt{\log_{1/2}\{\log_2[x^2+4x+5]\}}$

(iii) $f(x)=\dfrac{3}{[x/2]}-5\cos^{-1}x^2+\dfrac{(2x+1)!}{\sqrt{x+1}}$

Here, $[\cdot]$ denotes the greatest integer function.

SOLUTION (i) We have,

$f(x)=\dfrac{1}{\sqrt{|\,[|x|-1]\,|-5}}$

Clearly, $f(x)$ is defined for

$\Big|[|x|-1]\Big|-5>0$

$\Rightarrow \Big|[|x|-1]\Big|>5$

$\Rightarrow [|x|-1]>5$ or, $[|x|-1]<-5$

$\Rightarrow |x|-1 \geq 6$ or, $|x|-1<-5$

$\Rightarrow |x| \geq 7$ or, $|x|<-4$

$\Rightarrow |x| \geq 7$ $[\because |x| \not< -4]$

$\Rightarrow x \in (-\infty, -7] \cup [7, \infty)$.

(ii) We have,

$f(x)=\sqrt{\log_{1/2}\{\log_2[x^2+4x+5]\}}$

Clearly, $f(x)$ is defined for

$\log_{1/2}\{\log_2[x^2+4x+5]\} \geq 0$

$\Rightarrow \log_2[x^2+4x+5] \leq \left(\dfrac{1}{2}\right)^0$

$\Rightarrow \log_2[x^2+4x+5] \leq 1$

$\Rightarrow [x^2+4x+5] \leq 2^1$

$\Rightarrow [x^2+4x+5] \leq 2$

$\Rightarrow 2 \leq x^2+4x+5<3$ $[\because x^2+4x+5=(x+2)^2+1 \geq 1]$

$\Rightarrow x^2+4x+5 \geq 2$ and $x^2+4x+5<3$

$\Rightarrow x^2+4x+3 \geq 0$ and $x^2+4x+2<0$

$\Rightarrow (x+1)(x+3) \geq 0$ and $(x+2-\sqrt{2})(x+2+\sqrt{2})<0$

$\Rightarrow x \notin (-3, -1)$ and $-2-\sqrt{2}<x<-2+\sqrt{2}$

$\Rightarrow x \in (-2-\sqrt{2}, -3] \cup [-1, -2+\sqrt{2})$

(iii) We have,

$f(x)=\dfrac{3}{\left[\dfrac{x}{2}\right]}-5^{\cos^{-1}x^2}+\dfrac{(2x+1)!}{\sqrt{x+1}}$

$=\phi(x)-g(x)+h(x)$, where

$\phi(x)=\dfrac{3}{\left[\dfrac{x}{2}\right]}$, $g(x)=5^{\cos^{-1}x^2}$ and $h(x)=\dfrac{(2x+1)!}{\sqrt{x+1}}$

Clearly, $\phi(x)$ is defined for $\left[\dfrac{x}{2}\right] \neq 0$

i.e. for $\frac{x}{2} \notin [0, 1)$

i.e. for $x \notin [0, 2)$

$\therefore$ Domain of $\phi(x) = R - [0, 2)$ ...(i)

$g(x) = 5^{\cos^{-1} x^2}$ is defined for

$0 \le x^2 \le 1$

$\Rightarrow -1 \le x \le 1$

$\Rightarrow x \in [-1, 1]$

$\therefore$ Domain of $g(x) = [-1, 1]$ ...(ii)

$h(x) = \frac{(2x+1)!}{\sqrt{x+1}}$ is defined if

$2x + 1$ is a non-negative integer and $x + 1 > 0$

$\Rightarrow x = \frac{n-1}{2}$, where n is a non-negative integer and $x > -1$.

$\Rightarrow x = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ ...(iii)

From (i), (ii) and (iii), we get

Domain of $f(x) = \{-1/2\}$.

EXAMPLE 12 *If a function $f(x)$ is defined as*

$$f(x) = \sqrt{\log_{\phi(x)} g(x)},$$

where

$$g(x) = |\sin x| + \sin x, \phi(x) = \sin x + \cos x, 0 \le x \le \pi.$$

Then, find the domain of $f(x)$.

SOLUTION We have,

$$f(x) = \sqrt{\log_{\phi(x)} g(x)}.$$

Clearly, $f(x)$ is defined if

$g(x) > 0, \phi(x) > 0, \phi(x) \ne 1$ and $\log_{\phi(x)} g(x) \ge 0$

Clearly, $g(x) > 0$ for all $x \in [0, \pi]$.

and, $\phi(x) > 0$

$\Rightarrow \sin x + \cos x > 0$

$\Rightarrow \sin(x + \pi/4) > 0$

$\Rightarrow x \in [0, 3\pi/4)$

$\phi(x) = 1 \Rightarrow \sin x + \cos x = 1 \Rightarrow x = 0, \pi/2.$

Now,

$\log_{\phi(x)} g(x) \ge 0$

$\Rightarrow g(x) \ge 1$ and $\phi(x) > 1$

or,

$0 < g(x) \le 1$ and $0 < \phi(x) < 1$.

$\Rightarrow g(x) \ge 1$ and $\phi(x) > 1$

or,

$g(x) \le 1$ and $\phi(x) < 1$ $[\because g(x) > 0\ x \in [0, \pi]]$

$\Rightarrow 2 \sin x \ge 1$ and $\sin x + \cos x > 1 [\because g(x) = 2 \sin x$ for $0 \le x \le \pi]$

or,

$2 \sin x \le 1$ and $0 < \sin x + \cos x < 1$

$\Rightarrow \sin x \ge \frac{1}{2}$ and $\sin x + \cos x > 1$

or,

$\sin x \le \frac{1}{2}$ and $0 < \sin x + \cos x < 1$

$\Rightarrow \sin x \ge \frac{1}{2}$ and $\sin\left(x + \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}}$

or,

$\sin x \le \frac{1}{2}$ and $\sin\left(x + \frac{\pi}{4}\right) < \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{\pi}{6} \le x \le \frac{5\pi}{6}$ and $\frac{\pi}{4} < x + \frac{\pi}{4} < \frac{3\pi}{4}$

or,

$\left(0 \le x \le \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \le x \le \pi\right)$

and

$\left(0 < x + \frac{\pi}{4} < \frac{\pi}{4} \text{ or } \frac{3\pi}{4} < x + \frac{\pi}{4} < \pi\right)$

$\Rightarrow \frac{\pi}{6} \le x < \frac{\pi}{2}$

$\Rightarrow x \in [\pi/6, \pi/2).$

Hence, domain of $f(x) = [\pi/6, \pi/2)$.

EXAMPLE 13 *Find the domains of definition and the ranges of the following functions :*

(i) $f(x) = \log_{10}(3x^2 - 4x + 5)$

(ii) $f(x) = \sqrt{x-1} + 2\sqrt{3-x}$

(iii) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

(iv) $f(x) = \frac{e^x - e^{|x|}}{e^x + e^{|x|}}$

SOLUTION (i) We have,

$$f(x) = \log_{10}(3x^2 - 4x + 5)$$

Clearly, $f(x)$ is defined for $3x^2 - 4x + 5 > 0$

Since discriminant of $3x^2 - 4x + 5$ is negative and coefficient of x^2 is also positive. Therefore,

$3x^2 - 4x + 5 > 0$ for all $x \in R$

Hence, domain of $f(x) = R$.

Let $f(x) = y$. Then,

$\log_{10}(3x^2 - 4x + 5) = y$

$\Rightarrow 3x^2 - 4x + 5 = 10^y$

$\Rightarrow 3x^2 - 4x + (5 - 10^y) = 0$

$\Rightarrow x = \dfrac{4 \pm \sqrt{16 - 12(5 - 10^y)}}{6}$

$\Rightarrow x = \dfrac{2 \pm \sqrt{3 \times 10^y - 11}}{3}$

Since x is real. Therefore,

$3 \times 10^y - 11 \geq 0$

$\Rightarrow 10^y \geq \dfrac{11}{3}$

$\Rightarrow y \geq \log_{10}(11/3)$

$\Rightarrow y \in \left[\log_{10}(11/3), \infty\right)$

Hence, range of $f(x) = \left[\log_{10}(11/3), \infty\right)$.

(ii) We have,

$f(x) = \sqrt{x-1} + 2\sqrt{3-x}$

Clearly, $f(x)$ is defined if

$x - 1 \geq 0$ and $3 - x \geq 0$

$\Rightarrow x \geq 1$ and $x \leq 3$

$\Rightarrow 1 \leq x \leq 3$

$\Rightarrow x \in [1, 3]$.

$\therefore$ Domain of $f(x) = [1, 3]$

Let $f(x) = y$. Then,

$\sqrt{x-1} + 2\sqrt{3-x} = y$

$\Rightarrow \{\sqrt{x-1} + 2\sqrt{3-x}\}^2 = y^2$

$\Rightarrow x - 1 + 4(3 - x) + 4\sqrt{(x-1)(3-x)} = y^2$

$\Rightarrow 11 - 3x + 4\sqrt{-3 + 4x - x^2} = y^2$

$\Rightarrow 4\sqrt{-3 + 4x - x^2} = y^2 + 3x - 11$

$\Rightarrow 16(-3 + 4x - x^2) = (y^2 + 3x - 11)^2$

$\Rightarrow 16(4x - x^2 - 3) = (y^2 - 11)^2 + 9x^2 + 6x(y^2 - 11)$

$\Rightarrow 25x^2 + 2x(3y^2 - 33 - 32) + (y^2 - 11)^2 + 48 = 0$

$\Rightarrow 25x^2 + 2x(3y^2 - 65) + y^4 - 22y^2 + 169 = 0$

Since x is real, therefore

Disc $\geq 0 \Rightarrow y^2 - 10 \leq 0 \Rightarrow -\sqrt{10} \leq y \leq \sqrt{10}$

But, y is always positive and the minimum value of y is $\sqrt{2}$ at $x = 3$. Therefore,

$y \in [\sqrt{2}, \sqrt{10}]$

Hence, range of $f(x)$ is $[\sqrt{2}, \sqrt{10}]$

(iii) We have,

$f(x) = \log_2 \left\{\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right\}$

$\Rightarrow f(x) = \log_2 \left\{\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\}$

We know that :

$-1 \leq \sin(x - \pi/4) \leq 1$ for all $x \in R$

$\Rightarrow 3 - 1 \leq \sin(x - \pi/4) + 3 \leq 1 + 3$ for all $x \in R$

$\Rightarrow 2 \leq \sin(x - \pi/4) + 3 \leq 4$ for all $x \in R$

$\Rightarrow \sin(x - \pi/4) + 3 > 0$ for all $x \in R$.

$\therefore$ Domain of $f(x) = R$.

To find the range of $f(x)$, let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow \log_2 \{\sin(x - \pi/4) + 3\} = y$

$\Rightarrow \sin(x - \pi/4) + 3 = 2^y$

$\Rightarrow \sin(x - \pi/4) = 2^y - 3$

$\Rightarrow -1 \leq 2^y - 3 \leq 1$ $[\because -1 \leq \sin(x - \pi/4) \leq 1]$

$\Rightarrow 2 \leq 2^y \leq 4$

$\Rightarrow 1 \leq y \leq 2$

$\Rightarrow y \in [1, 2]$

Hence, range of $f(x)$ is $[1, 2]$.

(iv) We have,

$f(x) = \dfrac{e^x - e^{|x|}}{e^x + e^{|x|}}$

Clearly, $f(x)$ is defined for all $x \in R$.

Also, $f(x) = \begin{cases} \dfrac{e^x - e^x}{e^x + e^x} = 0, & \text{for all } x \geq 0 \\ \dfrac{e^x - e^{-x}}{e^x + e^{-x}}, & \text{for all } x < 0 \end{cases}$

To find the range of $f(x)$, let $f(x) = y$. Then,

$f(x) = y$

$\Rightarrow \dfrac{e^x - e^{|x|}}{e^x + e^{|x|}} = y$

$\Rightarrow y = \begin{cases} 0 & \text{for all } x \geq 0 \\ \dfrac{e^x - e^{-x}}{e^x + e^{-x}} & \text{for all } x < 0. \end{cases}$ ...(i)

Now,

$y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\Rightarrow y = \frac{e^{2x}-1}{e^{2x}+1}$$

$$\Rightarrow e^{2x} = \frac{y+1}{1-y}$$

$$\Rightarrow 0 < \frac{y+1}{1-y} < 1 \qquad [\because\ 0 < e^{2x} < 1 \text{ for } x < 0]$$

$$\Rightarrow -1 < \frac{y+1}{y-1} < 0$$

$$\Rightarrow -1 < \frac{y+1}{y-1} \text{ and } \frac{y+1}{y-1} < 0$$

$$\Rightarrow \frac{2y}{y-1} > 0 \quad \text{and} \quad \frac{y+1}{y-1} < 0$$

$$\Rightarrow y \in (-1, 0) \qquad \ldots(ii)$$

From (i) and (ii), we have

$$y \in (-1, 0].$$

Hence, range of $f(x)$ is $(-1, 0]$.

EXAMPLE 14 *Find the ranges of the following functions :*

(i) $f(x) = \dfrac{e^{-x}}{1+[x]}$ (ii) $f(x) = \cot^{-1}(2x - x^2)$

(iii) $f(x) = 4^x - 2^x + 1$ (iv) $f(x) = \dfrac{\sin^2 x + \sin x - 1}{\sin^2 x - \sin x + 2}$

SOLUTION (i) We have, $f(x) = \dfrac{e^{-x}}{1+[x]}$

Clearly, $f(x)$ is defined for $1 + [x] \neq 0$ i.e. for $x \notin [-1, 0)$.

Let $f(x) = y$. Then,

$$\frac{e^{-x}}{1+[x]} = y$$

$$\Rightarrow (1+[x])\, y = e^{-x}$$

$$\Rightarrow (1+[x])\, y > 0 \qquad [\because\ e^{-x} > 0 \text{ for all } x]$$

$\Rightarrow y > 0$, if $1 + [x] > 0$ i.e. $x \in [0, \infty)$

and,

$y < 0$, if $1 + [x] < 0$ i.e. $x \in (-\infty, -1)$.

$\Rightarrow y \in R - \{0\}$.

$\therefore$ Range of $f(x) = R - \{0\}$.

(ii) We know that $\cot^{-1} x$ is defined for all $x \in R$. Therefore,

$f(x) = \cot^{-1}(2x - x^2)$ is defined for all $x \in R$.

Let $f(x) = y$. Then,

$$\cot^{-1}(2x - x^2) = y \text{ and } 0 < y < \pi$$

$$\Rightarrow \quad 2x - x^2 = \cot y$$

$$\Rightarrow x^2 - 2x + \cot y = 0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4\cot y}}{2}$$

$$\Rightarrow x = 1 \pm \sqrt{1 - \cot y}$$

But, x is real. Therefore,

$$1 - \cot y \geq 0$$

$$\Rightarrow \cot y \leq 1.$$

$$\Rightarrow \frac{\pi}{4} \leq y < \pi \qquad [\because\ 0 < y < \pi]$$

Hence, range of $f(x)$ is $[\pi/4, \pi)$

(iii) We have,

$$f(x) = 4^x - 2^x + 1.$$

Clearly, $f(x)$ is defined for all $x \in R$.

Let $f(x) = y$. Then,

$$4^x - 2^x + 1 = y$$

$$\Rightarrow (2^x)^2 - 2^x + (1 - y) = 0$$

$$\Rightarrow 2^x = \frac{1 \pm \sqrt{1 - 4(1-y)}}{2}$$

$$\Rightarrow 2^x = \frac{1 \pm \sqrt{4y-3}}{2}$$

$$\Rightarrow x = \log_2 \left\{\frac{1 \pm \sqrt{4y-3}}{2}\right\}$$

But, x is real. Therefore, we must have

$$4y - 3 \geq 0 \text{ and } 1 \pm \sqrt{4y-3} > 0$$

$$\Rightarrow y \geq \frac{3}{4} \quad \text{and } 1 - \sqrt{4y-3} > 0 \qquad [\because\ 1 + \sqrt{4y-3} > 0]$$

$$\Rightarrow y \geq \frac{3}{4} \quad \text{and } \sqrt{4y-3} < 1$$

$$\Rightarrow y \geq \frac{3}{4} \quad \text{and } 4y - 3 < 1$$

$$\Rightarrow y \geq \frac{3}{4} \quad \text{and } y < 1.$$

$$\Rightarrow y \in [3/4, 1).$$

Hence, range of $f(x)$ is $[3/4, 1)$.

(iv) We have,

$$f(x) = \frac{\sin^2 x + \sin x - 1}{\sin^2 x - \sin x + 2}$$

$$\because \quad \sin^2 x - \sin x + 2 = \left(\sin x - \frac{1}{2}\right)^2 + \frac{7}{4} > 0 \text{ for all } x \in R.$$

So, domain of $f(x) = R$.

Let $f(x) = y$. Then,

$$\frac{\sin^2 x + \sin x - 1}{\sin^2 x - \sin x + 2} = y$$

$\Rightarrow (\sin^2 x)(y-1) - (\sin x)(y+1) + 2y + 1 = 0$

$\Rightarrow (y-1)\alpha^2 - (y+1)\alpha + (2y+1) = 0,$ ...(i)

where $\alpha = \sin x$

Since $\alpha = \sin x$ is real. Therefore,

$(y+1)^2 - 4(y-1)(2y+1) \geq 0$

$\Rightarrow -7y^2 + 6y + 5 \geq 0$

$\Rightarrow 7y^2 - 6y - 5 \leq 0$

$\Rightarrow \frac{3-2\sqrt{11}}{7} \leq y \leq \frac{3+2\sqrt{11}}{7}$ ...(ii)

Since $-1 \leq \sin x \leq 1$. This means that the roots of (i) lie between -1 and 1. The conditions for the same are :

$f(1)f(-1) > 0$ and $-1 < \frac{y+1}{2(y-1)} < 1$, where

$f(\alpha) = (y-1)\alpha^2 - (y+1)\alpha + (2y+1).$

Now,

$f(1)f(-1) > 0$

$\Rightarrow (4y+1)(2y-1) > 0$

$\Rightarrow y < -\frac{1}{4}$ or $y > \frac{1}{2}$ ...(iii)

and,

$-1 < \frac{y+1}{2(y-1)} < 1$

$\Rightarrow -2 < \frac{y+1}{y-1} < 2$

$\Rightarrow -2 < \frac{y+1}{y-1}$ and $\frac{y+1}{y-1} < 2$

$\Rightarrow 0 < \frac{3y-1}{y-1}$ and $\frac{3-y}{y-1} < 0$

$\Rightarrow \frac{3y-1}{y-1} > 0$ and $\frac{y-3}{y-1} > 0$

$\Rightarrow y > 3$ or, $y < \frac{1}{3}$ ...(iv)

From (iii) and (iv), we obtain

$y < -\frac{1}{4}$ or, $y > 3$ ...(v)

From (ii) and (v), we get

$\frac{3-2\sqrt{11}}{7} \leq y < -\frac{1}{4}$

$\Rightarrow y \in \left[\frac{3-2\sqrt{11}}{7}, -\frac{1}{4}\right)$

Hence, range of $f(x)$ is $\left[\frac{3-2\sqrt{11}}{7}, -\frac{1}{4}\right)$.

EXAMPLE 15 *Find the range of each of the following functions :*

(i) $f(x) = \log \frac{1}{\sqrt{[\cos x] - [\sin x]}}$

(ii) $f(x) = \log_2 \{\sqrt{x-4} + \sqrt{6-x}\}$

SOLUTION (i) We have,

$f(x) = \log \left\{\frac{1}{\sqrt{[\cos x] - [\sin x]}}\right\}.$

$f(x)$ is defined for $[\cos x] - [\sin x] > 0$

It is evident from the graphs of $[\cos x]$ and $[\sin x]$ that $[\cos x] - [\sin x] > 0$ for all $x \in [2n\pi - \pi/2, 2n\pi]$, where $n \in Z$.

Thus, domain of $f(x) = \left[2n\pi - \frac{\pi}{2}, 2n\pi\right]$

Also, for $x \in [2n\pi - \pi/2, 2n\pi)$, we have

$[\cos x] = 0$ and $[\sin x] = -1$

$\Rightarrow [\cos x] - [\sin x] = 1$

At $x = 2n\pi$, we have

$[\cos x] = 1$ and $[\sin x] = 0$

$\Rightarrow [\cos x] - [\sin x] = 1$

Thus, $[\cos x] - [\sin x] = 1$ for all $x \in [2n\pi - \pi/2, 2n\pi]$

$\therefore f(x) = \log 1 = 0$ for all $x \in [2n\pi - \pi/2, 2n\pi]$

Hence, range of $f(x)$ is $\{0\}$.

(ii) We have,

$f(x) = \log_2 \{\sqrt{x-4} + \sqrt{6-x}\}$

Clearly, $f(x)$ is defined for

$\sqrt{x-4} + \sqrt{6-x} > 0$

$\Rightarrow x - 4 \geq 0$ and $6 - x \geq 0$

$\Rightarrow 4 \leq x \leq 6$

$\therefore$ Domain of $f(x)$ is $[4, 6]$.

Let $y = f(x)$. Then,

$y = \log_2 (\sqrt{x-4} + \sqrt{6-x})$

$\Rightarrow 2^y = \sqrt{x-4} + \sqrt{6-x}$

$\Rightarrow 2^{2y} = 2 + 2\sqrt{(x-4)(6-x)}$

$\Rightarrow 2^{2y-1} - 1 = \sqrt{(x-4)(6-x)}$

$\because \sqrt{(x-4)(6-x)} \geq 0$ for all $x \in [4, 6]$

$\therefore 2^{2y-1} - 1 \geq 0$

$\Rightarrow 2^{2y-1} \geq 1$

$\Rightarrow 2y - 1 \geq 0$

$\Rightarrow 2y \geq 1$

$\Rightarrow y \geq \frac{1}{2}$

Again,

$2^{2y-1} - 1 = \sqrt{(x-4)(6-x)}$

$\Rightarrow (2^{2y-1} - 1)^2 = -24 + 10x - x^2$

$\Rightarrow x^2 - 10x + 24 + (2^{2y-1} - 1)^2 = 0$

As x is real. Therefore,

$100 - 4\{24 + (2^{2y-1} - 1)^2\} \geq 0$

$\Rightarrow 4 - 4(2^{2y-1} - 1)^2 \geq 0$

$\Rightarrow (2^{2y-1} - 1)^2 \leq 1$

$\Rightarrow -1 \leq 2^{2y-1} - 1 \leq 1$

$\Rightarrow 0 \leq 2^{2y-1} \leq 2$

$\Rightarrow 2y - 1 \leq 1$

$\Rightarrow y \leq 1$ ...(ii)

From (i) and (ii), we get

$\frac{1}{2} \leq y \leq 1 \Rightarrow y \in [1/2, 1]$

Hence, range of $f(x)$ is $[1/2, 1]$

EXAMPLE 16 *Find the range of*

$$f(x) = \left[\sin x + \sqrt{\sin^2 x + \sin^2 \alpha}\right] \cos x.$$

SOLUTION Clearly, $f(x)$ is defined for all $x \in R$.

Let $f(x) = y$. Then,

$y = [\sin x + \sqrt{\sin^2 x + \sin^2 \alpha}] \cos x$

$\Rightarrow y = \cos x \sin x + \sqrt{\sin^2 x + \sin^2 \alpha}\ \cos x$

$\Rightarrow (y - \cos x \sin x)^2 = \cos^2 x (\sin^2 x + \sin^2 \alpha)$

$\Rightarrow y^2 - 2y \cos x \sin x - \cos^2 x \sin^2 \alpha = 0$

$\Rightarrow y^2 (1 + \tan^2 x) - 2y \tan x - \sin^2 \alpha = 0$ [Dividing by $\cos^2 x$]

$\Rightarrow y^2 \tan^2 x - 2y \tan x + (y^2 - \sin^2 \alpha) = 0$

Since $\tan x$ assumes real values. Therefore,

Disc ≥ 0

$\Rightarrow 4y^2 - 4y^2 (y^2 - \sin^2 \alpha) \geq 0$

$\Rightarrow y^2 - (1 + \sin^2 \alpha) \leq 0$

$\Rightarrow -\sqrt{1 + \sin^2 \alpha} \leq y \leq \sqrt{1 + \sin^2 \alpha}$

Hence, range of $f(x)$ is $[-\sqrt{1 + \sin^2 \alpha}, \sqrt{1 + \sin^2 \alpha}]$.

EXAMPLE 17 *Find the range of the function*

$$f(x) = \sqrt{\sin\left|\log_e\left(\frac{x^2+e}{x^2+1}\right)\right|} + \sqrt{\cos\left|\log_e\left(\frac{x^2+e}{x^2+1}\right)\right|}$$

SOLUTION We have,

$$\frac{x^2+e}{x^2+1} = 1 + \frac{e-1}{x^2+1}$$

$\Rightarrow \frac{x^2+e}{x^2+1} > 1$ $\left[\because \frac{e-1}{x^2+1} > 0 \text{ for all } x\right]$

Again,

$$\frac{x^2+e}{x^2+1} = 1 + \frac{e-1}{x^2+1}$$

$\Rightarrow \frac{x^2+e}{x^2+1} \leq 1 + e - 1 = e$ $\left[\because \frac{e-1}{x^2+1} \text{ assumes its maximum value at } x = 0\right]$

Thus,

$1 < \frac{x^2+e}{x^2+1} \leq e$ for all $x \in R$.

$\Rightarrow 0 < \log_e\left(\frac{x^2+e}{x^2+1}\right) \leq 1$ for all $x \in R$.

$\Rightarrow 0 < \theta \leq 1$, where $\theta = \log_e\left(\frac{x^2+e}{x^2+1}\right)$.

$\therefore f(x) = \sqrt{\sin\theta} + \sqrt{\cos\theta}$.

Let $g(\theta) = \sqrt{\sin\theta} + \sqrt{\cos\theta}$, where $0 \leq \theta \leq 1$.

Clearly, the range of $f(x)$ will be same as that of $g(\theta)$.

Now,

$g(\theta) = \sqrt{\sin\theta} + \sqrt{\cos\theta}$

$\Rightarrow g'(\theta) = \frac{\cos\theta}{2\sqrt{\sin\theta}} - \frac{\sin\theta}{2\sqrt{\cos\theta}}$

$\Rightarrow g'(\theta) = \frac{(\cos\theta)^{3/2} - (\sin\theta)^{3/2}}{2\sqrt{\sin\theta \cos\theta}}$

$\Rightarrow g'(\theta) \geq 0$ for $0 < \theta \leq \pi/4$ and $g'(\theta) < 0$ for $\pi/4 < \theta \leq 1$

$\Rightarrow g(\theta)$ is increasing on $(0, \pi/4]$ and decreasing on $(\pi/4, 1]$

$\Rightarrow g(\theta) \in [g(0), g(\pi/4)]$ for $\theta \in (0, \pi/4]$

and,

$g(\theta) \in [g(1), g(\pi/4))$ for $\theta \in (\pi/4, 1]$

$\Rightarrow g(\theta) \in (1, 2^{3/4}]$ for $\theta \in (0, \pi/4]$

and,

$g(\theta) \in [\sqrt{\sin 1} + \sqrt{\cos 1}, 2^{3/4})$ for $\theta \in (\pi/4, 1]$

$\Rightarrow g(\theta) \in (1, 2^{3/4}]$, for $\theta \in (0, 1]$ $[\because \sqrt{\sin 1} + \sqrt{\cos 1} > 1]$

Hence, range of $f(x)$ is $(1, 2^{3/4}]$

EXAMPLE 18 *Find the range of the function $f: R \to R$ defined by*

$$f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{1+x^2}}$$

SOLUTION We have,

$$f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{1+x^2}}$$

Clearly, $f(x)$ is an even function on R. Therefore, we shall find the range of $f(x)$ for all $x \geq 0$.

We have,

$$f'(x) = \frac{2}{1+x^2}\left\{\tan^{-1}x - \frac{x}{\sqrt{1+x^2}}\right\}$$

$$= \frac{2}{1+x^2}g(x), \text{ where } g(x) = \tan^{-1}x - \frac{x}{\sqrt{1+x^2}}$$

Now,

$$g'(x) = \frac{1}{1+x^2} - \frac{1}{(1+x^2)^{3/2}}$$

$$\Rightarrow g'(x) = \frac{\sqrt{1+x^2}-1}{(1+x^2)^{3/2}} \geq 0 \qquad \text{for all } x \in R$$

$\Rightarrow g(x)$ is increasing on R

$\Rightarrow g(x)$ is increasing for all $x > 0$

$\Rightarrow g(x) > g(0)$ for all $x > 0$

$\Rightarrow g(x) > 0$ for all $x > 0$

$$\therefore f'(x) = \frac{2}{1+x^2}g(x) > 0 \text{ for all } x > 0 \left[\because \frac{2}{1+x^2} > 0 \text{ for all } x > 0\right]$$

$\Rightarrow f(x)$ is increasing for all $x > 0$

$$\Rightarrow f(x) \in \left(f(0), \lim_{x\to\infty} f(x)\right)$$

$$\Rightarrow f(x) \in \left(2, (\pi/2)^2\right)$$

$\Rightarrow$ Range of $f(x)$ is $\left(2, \pi^2/4\right)$.

EXAMPLE 19 *Find the range of*

$$f(x) = \log_e(2\sin x + \tan x - 3x + 1), x \in [\pi/6, \pi/3].$$

SOLUTION Let $g(x) = 2\sin x + \tan x - 3x + 1$. Then,

$$g'(x) = 2\cos x + \sec^2 x - 3$$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$$= \frac{(\cos x - 1)^2(2\cos x + 1)}{\cos^2 x} > 0 \text{ for all } x \in [\pi/6, \pi/3]$$

$\Rightarrow g(x)$ is increasing function on $[\pi/6, \pi/3]$

$\Rightarrow g(\pi/6) \leq g(x) \leq g(\pi/3)$ for all $x \in [\pi/6, \pi/3]$

$$\Rightarrow \left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right) \leq g(x) \leq (2\sqrt{3} + 1 - \pi) \text{ for all } x \in [\pi/6, \pi/3]$$

$$\Rightarrow \log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right) \leq \log_e g(x) \leq \log(2\sqrt{3} + 1 - \pi)$$

for all $x \in [\pi/6, \pi/3]$

$$\Rightarrow \log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right) \leq f(x) \leq \log(2\sqrt{3} + 1 - \pi)$$

for all $x \in [\pi/6, \pi/3]$

Hence, range of $f(x)$ is

$$\left[\log_e\left(2 + \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right), \log_e(2\sqrt{3} + 1 - \pi)\right]$$

EXAMPLE 20 *Find the range of*

$$f(x) = \log_e\left\{(\cos x)^{\cos x} + 1\right\}, x \in (0, \pi/2).$$

SOLUTION For $0 < x < \pi/2$, we have $0 < \cos x < 1$.

So, the range of $\log_e\{(\cos x)^{\cos x} + 1\}$ in $(0, \pi/2)$ is same as the range of $\log_e(x^x + 1)$ in $(0, 1)$.

Let $y = \log_e(x^x + 1)$ and $g(x) = x^x + 1$. Then,

$$g'(x) = x^x(1 + \log x)$$

$$\Rightarrow g'(x) > 0 \text{ for } x > \frac{1}{e} \text{ and } g'(x) < 0 \text{ for } x < \frac{1}{e}$$

$\Rightarrow g(x)$ is increasing for $x > \frac{1}{e}$ and $g(x)$ is decreasing for $x < \frac{1}{e}$

$\Rightarrow g(x)$ attains its minimum value at $x = 1/e$ and the minimum value is given by $g\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} + 1$

Since $g(x)$ is increasing function for $x > \frac{1}{e}$, therefore maximum value of $g(x)$ in the interval $(0, 1)$ is $g(1) = 1 + 1 = 2$.

Thus,

$$(1/e)^{1/e} + 1 < g(x) < 2 \quad \text{for all } x \in (0, 1)$$

$$\Rightarrow \log_e\left\{(1/e)^{1/e} + 1\right\} < \log_e(g(x)) < \log_e 2$$

$$\Rightarrow \log_e\left\{(1/e^{1/e} + 1\right\} < y < \log_e 2 \text{ for all } x \in (0, 1)$$

Hence, range of $f(x)$ is

$$\left(\log_e\left\{(1/e)^{1/e} + 1\right\}, \log_e 2\right)$$

EXAMPLE 21 *For what real values of 'a' does the range of the function* $f(x) = \dfrac{x-1}{a - x^2 + 1}$ *not contain any values belonging to the interval* $[-1, -1/3]$ *?*

SOLUTION Suppose the function $f(x)$ assumes value $-k \in [-1, -1/3]$, where $k \in [1/3, 1]$ for some value of x. Then,

$$-k = \frac{x-1}{a - x^2 + 1}$$

$$\Rightarrow \frac{x}{k} - \frac{1}{k} = -a + x^2 - 1$$

$$\Rightarrow x^2-\frac{1}{k}x-a-1+\frac{1}{k}=0$$

$$\Rightarrow \left(x-\frac{1}{2k}\right)^2-\frac{1}{4k^2}-a-1+\frac{1}{k}=0$$

$$\Rightarrow \left(x-\frac{1}{2k}\right)^2=\frac{1}{4k^2}-\frac{1}{k}+1+a$$

$$\Rightarrow \left(x-\frac{1}{2k}\right)^2=\left(\frac{1}{2k}-1\right)^2+a$$

$$\Rightarrow x=\frac{1}{2k}\pm\sqrt{\left(1-\frac{1}{2k}\right)^2+a} \qquad ...(i)$$

If the function does not attain any value in the interval $[-1,-1/3]$, then x given by (i) should not exist.

$$\therefore \left(1-\frac{1}{2k}\right)^2+a<0 \text{ for all } k\in[1/3,1]$$

$$\Rightarrow \quad a<-\left(1-\frac{1}{2k}\right)^2 \text{ for all } k\in[1/3,1]$$

$\Rightarrow a <$ minimum value of $-\left(1-\frac{1}{2k}\right)^2$ in the interval $[1/3,1]$

Let $g(x)=-\left(1-\frac{1}{2x}\right)^2, x\in[1/3,1]$. Then,

$$g(x)=-\frac{(2x-1)^2}{4x^2}.$$

Clearly, $g(x)$ attains its maximum value 0 at $x=\frac{1}{2}$ and it attains its minimum value at $x=\frac{1}{3}, 1$. The minimum value is given by $g(1/3)=g(1)=-\frac{1}{4}$.

Therefore, from (i), we have

$$a<-\frac{1}{4} \Rightarrow a\in\left(-\infty,-\frac{1}{4}\right)$$

EXAMPLE 22 *For what real values of 'a' does the range of the function* $f(x)=\frac{x-1}{1-x^2-a}$ *not attain any value from the interval* $[-1,1]$ *?*

SOLUTION Let there be a value $k\in[-1,1]$ such that the functions attains this value at some point x. Then,

$$f(x)=k$$

$$\Rightarrow \frac{x-1}{1-x^2-a}=k$$

$$\Rightarrow x-1=k(1-x^2-a)$$

$$\Rightarrow kx^2+x-(1+k-ka)=0$$

$$\Rightarrow x=\frac{-1\pm\sqrt{1+4k(1+k-ka)}}{2k}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{4k^2(1-a)+4k+1}}{2k} \qquad ...(i)$$

If the function does not attain any value in the interval $[-1,1]$, then x given by (i) should not exist.

$\therefore\ 4k^2(1-a)+4k+1<0$ for all $k\in[-1,1]$

$\Rightarrow 4k^2a>4k^2+4k+1$ for all $k\in[-1,1]$

$$\Rightarrow a>\left(\frac{2k+1}{2k}\right)^2 \qquad \text{for all } k\in[-1,1]$$

$$\Rightarrow a>\left(1+\frac{1}{2k}\right)^2 \qquad \text{for all } k\in[-1,1]$$

$\Rightarrow a >$ maximum value of $\left(1+\frac{1}{2k}\right)^2$ in the interval $[-1,1]$

But $\left(1+\frac{1}{2k}\right)^2\to\infty$ as $k\to 0$. Therefore, there is no alue of a for which the function does not attain any value in $[-1,1]$.

ALITER We have,

$$f(x)=\frac{x-1}{1-x^2-a}.$$

$\Rightarrow f(1)=0\in[-1,1]$ for all values of a except for those values for which denominator is zero.

Therefore, $f(x)$ assumes a value in $[-1,1]$ for all values of a except for those values of a for which the denominator of the expression for $f(x)$ becomes zero.

For $a=0$, we have

$$f(x)=\frac{x-1}{1-x^2}=-\frac{1}{1+x}$$

Clearly, $f(x)$ assumes some values in $[-1,1]$

Thus, for all a the given function has values belonging to the interval $[-1,1]$.

Hence, there is no value of a for which the function does not attain any value belonging to the interval $[-1,1]$.

EXAMPLE 23 *For what real values of a does the range of the function* $f(x)=\frac{x+1}{a+x^2}$ *contain the interval* $[-1,1]$ *?*

SOLUTION CASE I: *When* $a=-1$.

For $a=-1$, we have

$$f(x)=\frac{x+1}{x^2-1}$$

Clearly, $f(x)$ is defined for all except for $x=\pm1$. If $x=\pm1$, then

$$f(x)=\frac{1}{x-1}.$$

The range of this function is $R-\{0\}$.

Thus, for $a=-1$, the function does not contain the value 0. Consequently, the range of $f(x)$ cannot contain the interval $[-1,1]$.

CASE II *When* $a<0$, $a\neq -1$

Let $a=-b^2$, where $b\in R$ and $b\neq 1$. Then,

$$f(x)=\frac{x+1}{x^2-b^2}$$

Clearly, $f(x)\to\pm\infty$ for $x=\pm b$.

Thus, the range of $f(x)$ contains the interval $[-1,1]$ for $a<0$, and $a\neq -1$.

CASE III *When* $a\geq 0$. In this case, $f(x)$ is defined for all $x\in R$.

Let $y=f(x)$. Then,

$$y=\frac{x+1}{x^2+a}$$

$$\Rightarrow x^2y-x+ay-1=0$$

$$\Rightarrow x=\frac{1\pm\sqrt{1-4ay^2+4y}}{2y}$$

Now, x will be real, if $1-4ay^2+4y\geq 0$

Also, the range must contain the interval $[-1,1]$.

$$\therefore\ 1-4ay^2+4y\geq 0 \quad \text{for all } y\in[-1,1]$$

$$\Rightarrow 4ay^2-4y-1\leq 0 \quad \text{for all } y\in[-1,1]$$

$\Rightarrow -1$ and 1 lie between the roots of $\phi(y)=4ay^2-4y-1$.

$$\Rightarrow \phi(-1)<0 \text{ and } \phi(1)<0$$

$$\Rightarrow (4a+3)<0 \text{ and } (4a-5)<0$$

$$\Rightarrow a<-\frac{3}{4} \quad \text{and } a<5/4$$

$$\Rightarrow 0\leq a<5/4 \qquad [\because a\geq 0]$$

$$\Rightarrow a\in(0,5/4).$$

From the above three cases we conclude that

$$a\in(-\infty,-1)\cup(-1,5/4).$$

EXAMPLE 24 *If* $f\left(\frac{2\tan x}{1+\tan^2 x}\right)=\frac{(1+\cos 2x)(\sec^2 x+2\tan x)}{2}$, *find the domain and range of* $f(x)$.

SOLUTION We have,

$$f\left(\frac{2\tan x}{1+\tan^2 x}\right)=\frac{(1+\cos 2x)(\sec^2 x+2\tan x)}{2}$$

$$\Rightarrow f(\sin 2x)=\frac{2\cos^2 x(\sec^2 x+2\tan x)}{2}$$

$$\Rightarrow f(\sin 2x)=1+\sin 2x$$

$$\Rightarrow f(\alpha)=1+\alpha, \quad \text{where } \alpha=\sin 2x$$

Since $-1\leq\sin 2x\leq 1$ for all x. Therefore, $-1\leq\alpha\leq 1$.

Thus, we have

$$f(\alpha)=1+\alpha \text{ where } \alpha\in[-1,1]$$

Hence, domain of f is $[-1,1]$

Now,

$$-1\leq\alpha\leq 1$$

$$\Rightarrow 0\leq 1+\alpha\leq 2$$

$$\Rightarrow 0\leq f(\alpha)\leq 2$$

$$\Rightarrow f(\alpha)\in[0,2].$$

So, range of f is $[0,2]$.

EXAMPLE 25 *Find the domain and range of the real valued function* $f(x)$ *satisfying*

$$2f(\sin x)+f(\cos x)=x \qquad \text{for all } x\in R.$$

SOLUTION We have,

$$2f(\sin x)+f(\cos x)=x \qquad \text{for all } x\in R \qquad \ldots(i)$$

Replacing x by $(\pi/2-x)$, we get

$$2f\left\{\sin\left(\frac{\pi}{2}-x\right)\right\}+f\left\{\cos\left(\frac{\pi}{2}-x\right)\right\}=\frac{\pi}{2}-x$$

$$\Rightarrow 2f(\cos x)+f(\sin x)=\frac{\pi}{2}-x \qquad \ldots(ii)$$

Solving (i) and (ii), we get

$$f(\sin x)=x-\frac{\pi}{6}$$

$$\Rightarrow f(y)=\sin^{-1}y-\frac{\pi}{6} \quad [\text{Putting } \sin x=y, \text{ and } x=\sin^{-1}y]$$

Thus,

$$f(x)=\sin^{-1}x-\frac{\pi}{6}$$

Clearly, domain of $f=[-1,1]$

Now,

$$-\frac{\pi}{2}\leq\sin^{-1}x\leq\frac{\pi}{2} \text{ for all } x\in[-1,1]$$

$$\Rightarrow -\frac{\pi}{2}-\frac{\pi}{6}\leq\sin^{-1}x-\frac{\pi}{6}\leq\frac{\pi}{2}-\frac{\pi}{6} \quad \text{for all } x\in[-1,1]$$

$$\Rightarrow \frac{-2\pi}{3}\leq f(x)\leq\frac{\pi}{3} \quad \text{for all } x\in[-1,1]$$

$$\Rightarrow f(x)\in[-2\pi/3,\pi/3]$$

Hence, range of $f(x)$ is $[-2\pi/3,\pi/3]$.

EXERCISE 1.2

Find the domains of definition of the following functions :

1. $f(x) = \sqrt{5 - x - \frac{6}{x}}$

2. $f(x) = \left(\frac{1}{2}\right)^{\sqrt{4-x^2}} + \frac{1}{x-1}$

3. $f(x) = \sqrt{\frac{\sqrt{17 - 15x - 2x^2}}{x+3}}$

4. $f(x) = \sqrt{\frac{7-x}{\sqrt{4x^2 - 19x + 12}}}$

5. $f(x) = \sqrt{\frac{x^4 - 3x^2 + x + 7}{x^4 - 2x^2 + 1} - 1}$

6. $f(x) = \frac{\sqrt{x+5}}{\log_{10}(9-x)}$

7. $f(x) = \frac{\sqrt{3\log_{64} x - 1}}{\sqrt[3]{2x - 11}}$

8. $f(x) = \sqrt{4x - x^2} - \log_3 (x-2)$

9. $f(x) = \frac{\log_{10}(3 - 2x - x^2)}{\sqrt{x}}$

10. $f(x) = \sqrt{\log_{10}\left(\frac{1-2x}{x+3}\right)}$

11. $f(x) = \sqrt[4]{x - |x|} + \log_{10}(x+2)$

12. $f(x) = \sqrt{1 - \log(x-1)} + \sqrt{\frac{4-x}{x+2}}$

13. $f(x) = \sqrt{\log_{0.5}(-x^2 + x + 6)} + \frac{1}{x^2 + 2x}$

14. $f(x) = \sqrt{\frac{-\log_{0.3}(x-1)}{\sqrt{-x^2 + 2x + 8}}}$

15. $f(x) = \sqrt[4]{\frac{1}{2}\log_4 16 - \log_8 (x^2 - 4x + 3)}$

16. $f(x) = \log_4 \left\{2 - \sqrt[4]{x} - \frac{2\sqrt{x}+1}{\sqrt{x}+2}\right\}$

17. $f(x) = \sqrt{\frac{3^x - 4^x}{2x^2 - x - 8}}$

18. $f(x) = \frac{1}{[|x-1|] + [|12-x|] - 11}$, where $[\cdot]$ denotes the greatest integer function.

19. $f(x) = \sin^{-1}[2 - 3x^2]$, where $[\cdot]$ denotes the grestest integer function.

20. $f(x) = \log_2 \left\{-\log_{1/2}\left(1 + \frac{6}{\sqrt[4]{x}}\right) - 2\right\}$

21. $f(x) = \log_{(2x-5)} (x^2 - 3x - 10)$

22. $f(x) = \sqrt[6]{4^x + 8^{(2/3)(x-2)} - 52 - 2^{2(x-1)}}$.

23. $f(x) = \log_{1.7} \frac{\sqrt{2 - f'(x)}}{x+1}$, where $f(x) = \frac{x^3}{3} - \frac{3}{2}x^2 - 2x + \frac{3}{2}$.

24. $f(x) = \log_{0.5}\left\{-\log_2\left(\frac{3x-1}{3x+2}\right)\right\}$

25. $f(x) = \log\left\{\sqrt{8^{-2+\log x} - \sqrt[3]{4^{2-\log x}}}\right\}$.

26. $f(x) = \log_{100 x}\left\{\frac{2\log x + 1}{-x}\right\}$

27. $f(x) = \left(x + \frac{1}{2}\right)\log_{\left(x+\frac{1}{2}\right)}\left|\frac{x^2 + 2x - 3}{4x^2 - 4x - 3}\right|$

28. $f(x) = \log_{100 x}\left\{\frac{2\log x + 2}{-x}\right\}$

29. $f(x) = \sqrt{\log_{0.5}(3x-8) - \log_{0.5}(x^2+4)}$

30. $f(x) = \frac{1}{\sin^4 x + \cos^4 x}$

31. $f(x) = \sin^{-1}(3^x)$

32. $f(x) = \frac{\sqrt{\cos x - 1/2}}{\sqrt{6 + 35x - 6x^2}}$

33. $f(x) = \frac{\log_3 (x^2+1)}{\sin^2 x - \sin x + 0.25}$

34. $f(x) = \sqrt{(\sin x + \cos x)^2 - 1}$

35. $f(x) = \log_{10} \sin(x-3) - \sqrt{16 - x^2}$

36. $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

37. $f(x) = \sqrt{3-x} + \sin^{-1}\left\{\frac{3-2x}{5}\right\}$

38. $f(x) = \cos^{-1}\left\{\frac{2x+1}{2\sqrt{2x}}\right\}$

39. $f(x) = \cos^{-1}\left\{\frac{2}{2+\sin x}\right\}$

40. $f(x) = \sqrt{-2\cos^2 x + 3\cos x - 1}$

41. $f(x) = \sqrt{\sin^2 x - \sin x}$

42. $f(x) = \sqrt{\cos\left(\frac{\pi}{x^2}\right)}$

43. $f(x) = [x]\sin\left\{\frac{\pi}{[x+1]}\right\}$

44. $f(x) = e^x + \sin^{-1}\left(\frac{x}{2} - 1\right) + \log_e \sqrt{x - [x]}$

45. $f(x) = \frac{1}{[x]} + e^{\sin^{-1} x} + \frac{1}{\sqrt{x+1}}$

46. $f(x) = \sqrt{-\log_{\frac{x+4}{2}} \left\{\log_2 \frac{2x-1}{3+x}\right\}}$

47. Show that image of $[\pi/6, \pi/3]$ under the mapping $f(x) = \cos 2x + 2(\sin x - \cos x) - x$ is precisely

$$[f(\pi/3), f(\pi/6)]$$

48. Find the domain of the function

$$f(x) = \log_4\left[\log_3\left\{\log_2 (x^2 - 2x + 3) - \log_{\sqrt{2}} (2x-1)\right\}\right]$$

49. Find the set of values of x for which

$$f(x) = \sin^{-1}\left|\frac{4x^2 - 5x + 1}{x^2 + 5x - 2}\right| + \log_e (46x^2 - 9 - 5x^4) \text{ is real.}$$

50. Find the domain of the function $f(x) = \frac{x^2 + x - 6}{x^2 + 2x - 8}$. Find the inverse of this function, if it exists, and its domain also.

51. Find the domain of definition of the function

$$f(x) = \cos\{\sqrt{42 + x - x^2}\} + \sqrt{\sin x}$$

52. Find the domain of definition of each of the following functions :

(i) $\sqrt{\cos 2x} + \sqrt{16 - x^2}$ (ii) $\log_7 \log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x)$

53. Find the domain and the range of $f(x) = \sin x + \cos^2 x$

54. Find the image of the following sets under the mapping

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 10$$

(*i*) $[1, 2]$ (*ii*) $[1, \infty]$

55. Find the domains of definition of the single valued branches of the function $y = \phi(x)$ specified by the equation :

$$y^2 - 1 + \log_2 (x - 1) = 0$$

56. Find the domain of the following functions :

(i) $f(x) = \sqrt{1 - 2x} + 3\sin^{-1}\left\{\frac{3x-1}{2}\right\}$

(ii) $f(x) = \sqrt{4^x + 8^{(2/3)(x-2)} - 13 - 2^{2(x-1)}}$

57. Find the range of the following functions :

(i) $f(x) = \frac{x - [x]}{1 + x - [x]}$ (ii) $f(x) = \frac{e^x}{1 + [x]}, x \geq 0$

, where $[x]$ denotes the grestest integer less than or equal to x.

58. Find the domain of

$$f(x) = \left\{\frac{1}{2}\log_4 16 - \log_8 (x^2 - 4x + 3)\right\}^{1/4}.$$

59. Find the range of $f(x) = 9^x - 3^x + 1$.

60. Find the domain of $f(x) = \sqrt{\log_{1/2} \frac{x}{x^2 - 1}}$.

61. Find the range of $f(x) = \left[\frac{1}{[x - \pi]}\right]$, where $[\cdot]$ denotes the greatest integer function.

62. Find the range of $f(x) = \frac{\{x\}}{1 + \{x\}}$

63. Let $f(x)$ be a function defined in $[-2, 2]$ such that

$$f(x) = \begin{cases} [x] & , -2 \leq x \leq -\frac{1}{2} \\ 2x^2 - 1 & , -\frac{1}{2} \leq x \leq 2 \end{cases}$$

($[x]$ denotes the greatest integer less than or equal to x), then determine $g(x) = f(|x|) + |f(x)|$.

64. Find all linear functions from $[-2, 4]$ to $[10, 15]$ which are onto.

65. Let $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1$. Then determine $f^{-1}(x)$ and its domain and range.

ANSWERS

1. $(-\infty, 0) \cup [2, 3]$ **2.** $[-2, 1) \cup (1, 2]$

3. $(-3, 1]$ **4.** $(-\infty, 3/4) \cup (4, 7]$

5. $[-2, -1) \cup (-1, 1) \cup (1, 3]$ **6.** $[-5, 8) \cup (8, 9)$

7. $[4, 11/2) \cup (11/2, \infty)$ **8.** $(2, 4]$

9. $(0, 1)$ **10.** $(-3, -2/3]$

11. $[0, \infty)$ **12.** $(1, 4)$

13. $\left(-2, \frac{1-\sqrt{21}}{2}\right] \cup \left[\frac{1+\sqrt{21}}{2}, 3\right)$ **14.** $[2, 4)$

15. $[-1, 1) \cup (3, 5]$ **16.** $[0, 1)$

17. $\left(-\infty, \frac{1-\sqrt{65}}{4}\right) \cup \left[0, \frac{1+\sqrt{65}}{4}\right)$

18. $R - \{(0, 1] \cup \{1, 2, 3, ..., 12\} \cup (12, 13)\}$

19. $[-1, 1] - \{0\}$ **20.** $(0, 16)$

21. $(5, \infty)$ **22.** $[3, \infty)$

23. $(-\infty, -1) \cup (-1, 4)$ **24.** $(1/3, \infty)$

25. $(100, \infty)$ **26.** $(0, 10^{-2}) \cup (10^{-2}, 10^{-1/2})$

27. $(-1/2, 1/2) \cup (1/2, 1) \cup (3/2, \infty)$

28. $(0, 10^{-2}) \cup (10^{-2}, 10^{-1})$ **29.** $(8/3, \infty)$

30. R **31.** $(-\infty, 0]$ **32.** $(-1/6, \pi/3] \cup [5\pi/3, 6)$

33. $R - \left\{n\pi + (-1)^n \frac{\pi}{6} : n \in Z\right\}$

34. $\left[n\pi, (2n+1)\frac{\pi}{2}\right], n \in Z$

35. $(3-2\pi, 3-\pi) \cup (3, 4]$ 36. $[1, 4)$ 37. $[-1, 3]$

38. $\{1/2\}$ 39. $[2n\pi, (2n+1)\pi], n \in Z$

40. $\left((6n-1)\frac{\pi}{3}, (6n+1)\frac{\pi}{3}\right), n \in Z$

41. $[(2n+1)\pi, (2n+2)\pi], \cup \{n\pi + (-1)^n \pi/2, n \in Z\}$

42. $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

$$\cup\left[-\sqrt{\frac{2}{4n-1}}, -\sqrt{\frac{2}{4n+1}}\right]\cup\left[\sqrt{\frac{2}{4n+1}}, \sqrt{\frac{2}{4n-1}}\right] n = 1, 2, \ldots$$

43. $(-1, 0]$ 44. $(0, 4) - \{1, 2, 3\}$

45. $(-1, 0) \cup \{1\}$ 46. $(-4, -3) \cup (4, \infty)$

48. $\{1/2, 1)$ 49. $(1/\sqrt{5}, 3)$

50. $R - \{-4, 2\}, f^{-1}(x) = \frac{3-4x}{x-1}$,

Domain of $f^{-1}(x) = R - \left\{\frac{5}{6}, 1\right\}$

51. $[-6, -\pi] \cup [0, \pi] \cup [2\pi, 7]$

52. (i) $\left[-\frac{5\pi}{4}, -\frac{3\pi}{4}\right]\cup\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

(ii) $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

53. $[-1, 5/4]$ 54. (i) $[1, 2]$ (ii) $[1, \infty)$

55. $(1, 3]$

56. (i) $[-1/3, 1/2]$ (ii) $[2, \infty)$ 57. (i) $[0, 1/2]$ (ii) $[1, 1)$

58. $[-1, 1) \cup (3, 5]$ 59. $[3/4, 1)$

60. $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

61. $\{-1, 0, 1\}$ 62. $[0, 1/2)$

63. $g(x) = \begin{cases} 2x^2+1 & , -2 \le x \le -1 \\ 2x^2 & , -1 \le x \le -\frac{1}{2} \\ 0 & , -\frac{1}{2} < x < \frac{1}{\sqrt{2}} \\ 2(2x^2-1) & , \frac{1}{\sqrt{2}} \le x \le 2 \end{cases}$

64. $f(x) = \frac{5}{6}x + \frac{35}{3}, f(x) = -\frac{5}{6}x + \frac{50}{3}$

65. $f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\frac{x-2}{2} - \frac{\pi}{6}\right)$

or, $f^{-1}(x) = \frac{1}{2}\left(\cos^{-1}\frac{x-2}{2} + \frac{\pi}{3}\right)$.

HINTS TO SELECTED PROBLEMS

62. Let $f(x) = y$. Then,

$$y = \frac{\{x\}}{1+\{x\}}$$

$$\Rightarrow \{x\} = \frac{y}{1-y}$$

$$\Rightarrow 0 \le \frac{y}{1-y} < 1$$

$$\Rightarrow 0 \le y < 1-y$$

$$\Rightarrow y \ge 0 \text{ and } y < 1/2$$

$$\Rightarrow y \in [0, 1/2)$$

—❑ ❑ ❑—

1.18 PERIODIC FUNCTIONS

PERIODIC FUNCTIONS *A function $f(x)$ is said to be a periodic function if there exists a positive real number T such that*

$$f(x+T) = f(x) \text{ for all } x \in R.$$

We know that

$$\sin(x+2\pi) = \sin(x+4\pi) = \ldots = \sin x$$

and,

$$\cos(x+2\pi) = \cos(x+4\pi) = \ldots = \cos x \quad \text{for all } x \in R.$$

Therefore, $\sin x$ and $\cos x$ are periodic functions.

PERIOD *If $f(x)$ is a periodic function, then the smallest positive real number T is called the period or fundamental period of function $f(x)$ if*

$$f(x+T) = f(x) \text{ for all } x \in R.$$

In order to check the periodicity of a function $f(x)$, we follow the following algorithm:

ALGORITHM

STEP I *Write $f(x+T) = f(x)$*

STEP II *Simplify the equation obtained in step I and solve it for T.*

STEP III *If the values of T obtained in step II are positive and independent of x, then $f(x)$ is periodic, otherwise not.*

STEP IV *If $f(x)$ is periodic, then choose the smallest value of T obtained in step III. The value of T so obtained is the period of $f(x)$.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that* $\sin x$ *is a periodic function. Also, find its period.*

SOLUTION Let $f(x) = \sin x$ and T be a positive real number. If possible, let $f(x)$ be a periodic function with period T , then

$$f(x+T) = f(x) \text{ for all } x \in R$$

$$\Rightarrow \sin(x+T) = \sin x \text{ for all } x \in R$$

$$\Rightarrow \sin T = \sin 0 \qquad [\text{Putting } x = 0]$$

$$\Rightarrow T = n\pi, n \in N \qquad [\because T > 0]$$

Now,

$$\sin(n\pi + x) = (-1)^n \sin x$$

$$\Rightarrow \sin(n\pi + x) = \sin x, \text{ if n is even.}$$

Thus, $f(x+T) = f(x)$ for all $x \in R$

$$\Rightarrow T = n\pi, \text{ where n is even natural number}$$

$$\Rightarrow T = 2\pi, 4\pi, 6\pi, \ldots.$$

But, period of a function $f(x)$ is the smallest positive real number T satisfying $f(x+T) = f(x)$ for all $x \in R$. Therefore,

$$T = 2\pi.$$

Hence, sin x is periodic with period 2π.

EXAMPLE 2 *Prove that the function* $f(x) = x - [x]$ *is a periodic function. Also, find its period.* **[IIT 1983]**

SOLUTION Let T be a positive real number. If possible, let $f(x)$ be periodic with period T. Then,

$$f(x+T) = f(x) \qquad \text{for all } x \in R$$

$$\Rightarrow x + T - [x+T] = x - [x] \text{ for all } x \in R$$

$$\Rightarrow [x+T] - [x] = T \qquad \text{for all } x \in R$$

$$\Rightarrow T = 1, 2, 3, 4, \ldots\ldots \qquad [\because [x+n] - [x] = n \text{ for all } n \in N]$$

Thus, there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in R$.

So, $f(x)$ is a periodic function.

The smallest value of T satisfying $f(x+T) = f(x)$ for all $x \in R$ is 1.

Hence, $f(x) = x - [x]$ has period 1.

EXAMPLE 3 *Let* $f(x)$ *be a function and k be a positive real number such that*

$$f(x+k) + f(x) = 0 \text{ for all } x \in R.$$

Prove that $f(x)$ is a periodic function with period 2k.

SOLUTION We have,

$$f(x+k) + f(x) = 0 \qquad \text{for all } x \in R.$$

$$\Rightarrow f(x+k) = -f(x) \qquad \text{for all } x \in R$$

$$\Rightarrow f(x+2k) = -f(x+k) \qquad \text{for all } x \in R$$

[On replacing x by $x+k$]

$$\Rightarrow f(x+2k) = f(x) \text{ for all } x \in R \qquad [\because f(x+k) = -f(x)]$$

Hence, $f(x)$ is a periodic function with period 2k.

EXAMPLE 4 *If a function* $f(x)$ *satisfies the equation*

$$f(x+1) + f(x-1) = \sqrt{3}\, f(x) \text{ for all } x \in R.$$

Prove that $f(x)$ *is a periodic function. Also, find its period.*

SOLUTION We have,

$$f(x+1) + f(x-1) = \sqrt{3} f(x) \quad \text{for all } x \in R$$

Replacing x by $x+1$ and $x-1$ respectively, we get

$$f(x+2) + f(x) = \sqrt{3} f(x+1) \qquad \ldots(i)$$

and, $$f(x) + f(x-2) = \sqrt{3} f(x-1) \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$f(x+2) + f(x-2) + 2f(x) = \sqrt{3}\,[f(x+1) + f(x-1)]$$

$$\Rightarrow f(x+2) + f(x-2) + 2f(x) = \sqrt{3}\,[\sqrt{3}\, f(x)]$$

$$[\because f(x+1) + f(x-1) = \sqrt{3} f(x)]$$

$$\Rightarrow f(x+2) + f(x-2) = f(x) \qquad \ldots(iii)$$

Replacing x by $x+2$, we get

$$f(x+4) + f(x) = f(x+2) \qquad \ldots(iv)$$

Adding (iii) and (iv), we get

$$f(x+2) + f(x-2) + f(x+4) + f(x) = f(x) + f(x+2)$$

$$\Rightarrow f(x+4) + f(x-2) = 0 \text{ for all } x \in R$$

Replacing x be $x+2$, we get

$$f(x+6) + f(x) = 0 \text{ for all } x \in R$$

$$\Rightarrow f(x+6) = -f(x) \quad \text{for all } x \in R \qquad \ldots(v)$$

Replacing x by $x+6$, we get

$$f(x+12) = -f(x+6) \text{ for all } x \in R$$

$$\Rightarrow f(x+12) = f(x) \qquad \text{for all } x \in R \qquad [\text{Using (v)}]$$

Hence, $f(x)$ is a periodic function with period 12.

EXAMPLE 5 *Let* $f(x)$ *be a real valued function with domain R such that*

$$f(x+p) = 1 + [2 - 3f(x) + 3(f(x))^2 - (f(x))^3]^{1/3}$$

holds good for all $x \in R$ *and some positive constant p, then prove that* $f(x)$ *is a periodic function.*

SOLUTION We have,

$$f(x+p) = 1 + \left[2 - 3f(x) + 3(f(x))^2 - (f(x))^3\right]^{1/3}$$

$$\Rightarrow f(x+p) = 1 + \left[1 + (1 - f(x))^3\right]^{1/3}$$

$$\Rightarrow f(x+p) - 1 = \left[1 - \{f(x) - 1\}^3\right]^{1/3}$$

$\Rightarrow g(x+p) = \left[1-\{g(x)\}^3\right]^{1/3}$ ...(i)

where $g(x) = f(x) - 1$.

$\Rightarrow g(x+2p) = \left[1-\{g(x+p)\}^3\right]^{1/3}$ [On replacing x by $x+p$]

$\Rightarrow g(x+2p) = \left[1-\{1-g(x)\}^3\right]^{1/3}$ [Using (i)]

$\Rightarrow g(x+2p) = g(x)$ for all $x \in R$

$\Rightarrow f(x+2p) - 1 = f(x) - 1$

$\Rightarrow f(x+2p) = f(x)$

Hence, $f(x)$ is a period function with period $2p$.

Students are advised to learn the periods of following standard functions.

Function	*Period*
$\sin x$	2π
$\cos x$	2π
$\tan x$	π
$\operatorname{cosec} x$	2π
$\sec x$	2π
$\cot x$	π
$x - [x]$	1
$\sin^2 x,\ \cos^2 x$	π
$\lvert \sin x \rvert,\ \lvert \cos x \rvert$	π
$\sin^4 x + \cos^4 x$	$\pi/2$

EXAMPLE 6 *If the function $f(x) = \sin x + \cos ax$ is periodic, then prove that a is rational.*

SOLUTION Let T be the period of $f(x)$. Then,

$$f(x+T) = f(x) \text{ for all } x.$$

$\Rightarrow \sin(T+x) + \cos a(T+x) = \sin x + \cos ax$ for all $x \in R$

Putting $x = 0$ and $x = -T$ respectively, we get

$$\sin T + \cos aT = 1$$

and, $$-\sin T + \cos aT = 1$$

Solving these two equations, we get

$$\sin T = 0 \text{ and } \cos aT = 1$$

$\Rightarrow T = n\pi$ and $aT = 2m\pi$, where $m, n \in Z$.

$\Rightarrow \dfrac{aT}{T} = \dfrac{2m\pi}{n\pi}$

$\Rightarrow a = \dfrac{2m}{n}$, which is a rational number.

EXAMPLE 7 *Prove that $\cos x^2$ is not a periodic function.*

SOLUTION If possible, let $f(x) = \cos x^2$ be a periodic function with period T. Then,

$f(x+T) = f(x)$ for all $\in R$

$\Rightarrow \cos(x+T)^2 = \cos x^2$ for all $x \in R$

$\Rightarrow (x+T)^2 = 2n\pi \pm x^2$, for all $x \in R$

$\Rightarrow x^2 + 2Tx + T^2 \pm x^2 = 2n\pi$, for all $x \in R$

This relation is not possible, because the right hand side is an integral multiple of 2π whereas the left hand side is a quadratic function of x.

For example, for $x = 0$ and $x = T$, we obtain

$T^2 = 2n\pi$ and $4T^2 \mp T^2 = 2n\pi$

$\Rightarrow 4T^2 \mp T^2 = T^2$

$\Rightarrow 4 \mp 1 = 1$ [$\because (T > 0)$, which is not true.]

Hence, $f(x) = \cos^2 x$ is not a periodic function.

EXAMPLE 8 *Let f be a real valued function defined for all real numbers x such that for some fixed $a > 0$,*

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2} \text{ for all } x.$$

Show that the function $f(x)$ is periodic with period $2a$.

SOLUTION We have,

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2}$$

$$\therefore f(x+a+a) = \frac{1}{2} + \sqrt{f(x+a) - \{f(x+a)\}^2}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{f(x+a)\{1 - f(x+a)\}}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\left\{\frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2}\right\}\left\{\frac{1}{2} - \sqrt{f(x) - \{f(x)\}^2}\right\}}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\frac{1}{4} - \{f(x) - \{f(x)\}^2\}}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\frac{1}{4} - f(x) + \{f(x)\}^2}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\left\{f(x) - \frac{1}{2}\right\}^2}$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + \left|f(x) - \frac{1}{2}\right| \quad [\because \sqrt{x^2} = |x|]$$

$$\Rightarrow f(x+2a) = \frac{1}{2} + f(x) - \frac{1}{2} \quad \left[\because f(x) \geq \frac{1}{2}\right]$$

$\Rightarrow f(x+2a) = f(x)$ for all x.

Hence, $f(x)$ is a periodic function with period 2a.

EXAMPLE 9 *Let $f(x, y)$ be a periodic function satisfying*

$$f(x, y) = f(2x + 2y, 2y - 2x) \text{ for all } x, y \in R.$$

Define $g(x) = f(2^x, 0)$. Show that $g(x)$ is a periodic function with period 12.

SOLUTION We have,

$f(x, y) = f(2x + 2y, 2y - 2x)$ for all $x, y \in R$ ...(i)

$\Rightarrow f(x, y) = f\Big(2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x + 2y)\Big)$ [Using (i)]

$\Rightarrow f(x, y) = f(8y, -8x)$ ...(ii)

$= f(8 \times -8x, -8 \times 8y)$ [Using (ii)]

$\Rightarrow f(x, y) = f(-64x, -64y)$ ...(iii)

$= f(-64 \times -64x, -64 \times -64y)$ [Using (iii)]

$= f(2^{12}x, 2^{12}y)$

$\Rightarrow f(x, 0) = f(2^{12}x, 0)$ [Replacing y by 0]

$\Rightarrow f(2^y, 0) = f(2^{12} \times 2^y, 0)$ [Replacing x by 2^y]

$\Rightarrow f(2^y, 0) = f(2^{12+y}, 0)$

$\Rightarrow f(2^x, 0) = f(2^{12+x}, 0)$

$\Rightarrow g(x) = g(12 + x)$ for all x [$\because g(x) = f(2^x, 0)$]

Hence, $g(x)$ is a peridoic function with period 12.

SOME USEFUL RESULTS ON PERIODIC FUNCTIONS

RESULT 1 *If $f(x)$ is a periodic function with periodic T and $a, b \in R$ such that $a \neq 0$, then $af(x) + b$ is periodic with period T.*

RESULT 2 *If $f(x)$ is a periodic function with period T and $a, b \in R$ such that $a \neq 0$, then $f(ax + b)$ is periodic with period $T/|a|$.*

RESULT 3 *Let $f(x)$ and $g(x)$ be two periodic functions such that :*

Period of $f(x) = \dfrac{m}{n}$, *where $m, n \in N$ and m, n are co-prime.*

and,

Period of $g(x) = \dfrac{r}{s}$, *where $r \in N$ and $s \in N$ are coprime*

Then, $(f + g)(x)$ is periodic with period T given by

$$T = \frac{\text{LCM of } (m, r)}{\text{HCF of } (n, s)},$$

provided that there does not exist a positive number $k < T$ for which $f(k + x) = g(x)$ and $g(k + x) = f(x)$, else k will be the period of $(f + g)(x)$.

The above result is also true for functions $\dfrac{f}{g}$, $f - g$ and fg.

EXAMPLE 10 *Since sin x is a periodic function with period 2π. Therefore,* $\sin(3x + 2)$, $\sin\left(\dfrac{x}{2} - 3\right)$, $\sin\left(\dfrac{\pi x + 2}{3}\right)$ *are periodic functions wit periods $2\pi/3$, 4π and 6 respectively.*

EXAMPLE 11 *Find the period of* $f(x) = \sin^2 x$.

SOLUTION We have,

$$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2}.$$

Since $\cos 2x$ is a periodic function with period $\dfrac{2\pi}{2} = \pi$.

Therefore, $f(x)$ is a periodic function with period π.

EXAMPLE 12 *Prove that $f(x) = \sin^4 x + \cos^4 x$ is a periodic function with period $\pi/2$.*

SOLUTION We have,

$$f(x) = \sin^4 x + \cos^4 x$$

$$= \{(\sin^2 x)^2 + (\cos^2 x)\}^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x$$

$$= (\sin^2 x + \cos^2 x)^2 - \frac{1}{2}(2 \sin x \cos x)^2$$

$$= 1 - \frac{1}{2}(\sin^2 2x)$$

$$= 1 - \frac{1}{2}\left\{\frac{1 - \cos 4x}{2}\right\}$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x.$$

Since $\cos 4x$ is a periodic function with period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

So, $f(x)$ is periodic with period $\pi/2$.

EXAMPLE 13 *Find the period of* $f(x) = \sin\dfrac{2\pi x}{3} + \cos\dfrac{\pi x}{2}$.

SOLUTION Since $\sin\dfrac{2\pi}{3}x$ and $\cos\dfrac{\pi}{2}x$ are periodic functions with periods $\dfrac{2\pi}{2\pi/3} = 3$ and $\dfrac{2\pi}{\pi/2} = 4$ respectively. Therefore, $f(x)$ is periodic with period equal to LCM of 3 and 4 i.e. 12.

EXAMPLE 14 *Find the period of function*

$$f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \ldots\ldots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}.$$

SOLUTION We have,

$$f(x) = \sum_{r=1}^{n}\left(\sin\frac{x}{2^{r-1}} + \tan\frac{x}{2^r}\right)$$

$$= \sum_{r=1}^{n} f_r(x), \text{ where } f_r(x) = \sin\frac{x}{2^{r-1}} + \tan\frac{x}{2^r}$$

Since each of $\sin\dfrac{x}{2^{r-1}}$ and $\dfrac{x}{2^r}$ are periodic functions with period $2^r\pi$. Therefore, $f_r(x)$ is periodic with period $2^r\pi$.

But, $f(x) = \sum_{r=1}^{n} f_r(x)$. Therefore, $f(x)$ is periodoc with period

$$T = \text{LCM of } (2\pi, 2^2\pi, \ldots., 2^n\pi)$$

$$= 2^n\pi$$

EXAMPLE 15 *Find the period of* $f(x) = 5 \sin 3x - 7 \sin 8x$.

SOLUTION We observe that :

Period of $5 \sin 3x$ is $\frac{2\pi}{3}$

Period of $7 \sin 8x$ is $\frac{2\pi}{8} = \frac{\pi}{4}$

$\therefore$ Period of $f(x) = \frac{\text{LCM of } 2\pi \text{ and } \pi}{\text{HCF of 3 and 4}} = \frac{2\pi}{1} = 2\pi$

EXAMPLE 16 *Find the period of each of the following functions, if exists :*

(i) $f(x) = \tan 3x + \cos \frac{5x}{2}$

(ii) $f(x) = \cos x + \{x\}$, where $\{x\}$ denotes the fractional part of x.

(iii) $f(x) = \cos(\cos x) + \cos(\sin x)$

(iv) $f(x) = |\sin x| + |\cos x|$

(v) $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{\sin x}{|\cos x|}\right\}$

SOLUTION (i) We have,

$$f(x) = \tan 3x + \cos \frac{5x}{2}$$

Clearly, period of $\tan 3x$ is $\frac{\pi}{3}$ and that of $\cos 5x$ is $\frac{2\pi}{5/2} = \frac{4\pi}{5}$

$\therefore$ Period of $f(x) = \frac{\text{LCM of } \pi \text{ and } 4\pi}{\text{HCF of 3 and 5}} = 4\pi$

(ii) We have,

$$f(x) = \cos x + \{x\}$$
$$= \cos x + x - [x] \quad [\because x = [x] + \{x\}]$$

Here, $\cos x$ is periodic with period 2π and $x - [x]$ is periodic with period 1. But, LCM of 2π and 1 does not exist.

Hence, $f(x)$ is not periodic.

(iii) We have,

$$f(x) = \cos(\cos x) + \cos(\sin x)$$

$\because \quad \cos\{\cos(\pi + x)\} = \cos(-\cos x) = \cos(\cos x)$

and,

$$\cos\{\sin(\pi + x)\} = \cos(-\sin x) = \cos(\sin x)$$

Therefore, $\cos(\cos x)$ and $\cos(\sin x)$ are periodic functions with period π. So, $f(x)$ should have period π. But, we find that

$$\cos\{\cos(\pi/2) + x\} = \cos(-\sin x) = \cos(\sin x)$$

and,

$$\cos\{\sin(\pi/2) + x\} = \cos(\cos x)$$

Therefore, by result 3, $f(x)$ is periodic with period $\pi/2$.

(iv) We have,

$$f(x) = |\sin x| + |\cos x|.$$

We know that $|\sin x|$ and $|\cos x|$ are periodic with period π. Therefore, $f(x)$ should be periodic with period π. But $|\sin(\pi/2 + x)| = \cos x$ and $|\cos(\pi/2 + x)| = \sin x$.

Therefore, by result 3, $f(x)$ is periodic with period $\frac{\pi}{2}$.

(v) We have,

$$f(x) = \frac{1}{2}\left\{\frac{\sin x}{|\cos x|} + \frac{|\sin x|}{\cos x}\right\}$$

Since $\sin x$ and $|\cos x|$ are periodic with periods 2π and π respectively. Therefore, $\frac{\sin x}{|\cos x|}$ is periodic with period 2π.

Similarly, $\frac{|\sin x|}{\cos x}$ is periodic with period 2π.

Hence, $f(x)$ is periodic with period 2π.

EXAMPLE 17 *Prove that* $f(x) = \sin^{-1}(\sin x)$ *is a periodic function*

SOLUTION We have,

$$f(x) = \sin^{-1}(\sin x) = \begin{cases} x & , -\pi/2 \le x \le \pi/2 \\ \pi - x & , \pi/2 \le x \le 3\pi/2 \\ 2\pi - x & , 3\pi/2 \le x \le 2\pi \end{cases}$$

and so on

It is evident from the graph that $f(x)$ is periodic with period 2π.

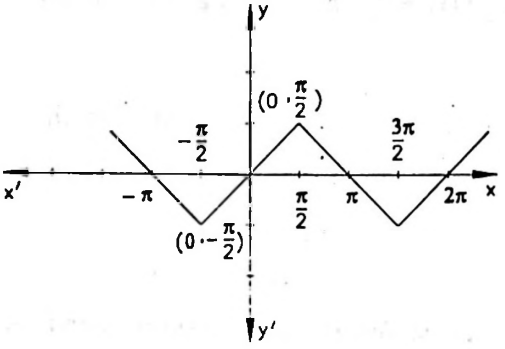


Fig. 1.38

SOME IMPORTANT POINTS TO REMEMBER

(i) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$ are periodic functions with period 2π and π according as n is odd or even.

(ii) $\tan^n x$, $\cot^n x$ are periodic functions with period π whether n is even or odd.

(iii) $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\text{cosec } x|$ are periodic with period π

(iv) $|\sin x| + |\cos x|$, $|\tan x| + |\cot x|$, $|\sec x| + |\text{cosec } x|$ are periodic with period $\pi/2$.

(v) $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$, $\text{cosec}^{-1}(\text{cosec } x)$, $\sec^{-1}(\sec x)$ are periodic with period 2π whereas $\tan^{-1}(\tan x)$ and $\cot^{-1}(\cot x)$ are periodic with period π.

EXAMPLE 18 *Find the period of the function*

$$f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+\ldots+|\cos n\pi x|}$$

SOLUTION We observe that

Period of $x-[x]$ is 1

Period of $|\cos \pi x|$ is $\frac{\pi}{\pi} = 1$

Period of $|\cos 2\pi x|$ is $\frac{\pi}{2\pi} = \frac{1}{2}$

Period of $|\cos 3\pi x|$ is $\frac{\pi}{3\pi} = \frac{1}{3}$

and so on.

Finally, Period of $|\cos n\pi x| = \frac{\pi}{n\pi} = \frac{1}{n}$

$\therefore$ Period of $f(x)$ = LCM of $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right) = 1$.

EXERCISE 1.3

1. Find the periods of the following functions :
   (i) $f(x) = 2\sin 3x + 3\sin 2x$ (ii) $f(x) = \sin x + \cos 2x$
   (iii) $f(x) = \sin\frac{\pi x}{3} + \sin\frac{\pi x}{4}$
   (iv) $f(x) = \sin(2\pi x + \pi/3) + 2\sin(3\pi x + \pi/4) + 3\sin 5\pi x$
   (v) $f(x) = [x] + \left[x+\frac{1}{3}\right] + \left[x+\frac{2}{3}\right] - 3x + 10$
   (vi) $f(x) = x + a - [x+b]$, where $a, b \in R$.
   (vii) $f(x) = \sin\left(\frac{\pi[x]}{4}\right)$
2. Find the periods of the followings functions :
   (i) $f(x) = \frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$
   (ii) $f(x) = \frac{1}{|\sin x| + |\cos x|}$
   (iii) $f(x) = e^{x-[x]+|\sin \pi x|+|\sin 2\pi x|+\ldots+|\sin n\pi x|}$
   (iv) $f(x) = \sin^{-1}(\sin 2\pi x)$
   (v) $f(x) = \cos^{-1}(\cos 3\pi x)$
3. For what integral value of n is 3π period of the function $\cos n\, x \sin 5x/n$?
4. Prove that $\cos^{-1}(\cos x)$ is a periodic function. Also, find its period.
5. Prove that the following functions are not periodic :
   (i) $f(x) = \cos\sqrt{x}$ (ii) $f(x) = \sin\sqrt{x}$
   (iii) $f(x) = \sin x^2$ (iv) $f(x) = x\cos x$
   (v) $f(x) = x\sin x$ (vi) $x + \sin x$
6. Prove that $f(x) = \cos x + \{x\}$ is not a periodic function even if $\cos x$ and $\{x\}$ are periodic functions.
7. If the function $f(x) = \sin x + \cos ax$ is periodic, then prove that a is a rational number.
8. Let $f: R \to R$ and $g: R - \{(2n+1)\pi/2 \mid n \in Z\} \to R$ be given by
   $f(x) = \sin x$ and $g(x) = \tan x$. Find :
   (i) the periods of *fog* and *gof*.
   (ii) the ranges of *fog* and *gof*.
9. Find the periods of the following functions :
   (i) $f(x) = e^{\log_e(\sin x) + \tan^3 x - \operatorname{cosec}(3x-5)}$
   (ii) $f(x) = x - [x-b]$, $b \in R$.

ANSWERS

1. (i) 2π (ii) 2π (iii) 24 (iv) 2
   (v) 1 (vi) 1 (vii) 8
2. (i) π (ii) $\pi/2$ (iii) 1 (iv) 1 (v) 2/3
3. $n = \pm 1, \pm 3, \pm 5, \pm 15$
8. (i) $\pi, 2\pi$ (ii) $[-1, 1], [-\tan 1, \tan 1]$
9. (i) 2π (ii) 1

HINTS TO SELECTED PROBLEMS

1. (v) We have,

$$f(x) = [x] + \left[x+\frac{1}{3}\right] + \left[x+\frac{2}{3}\right] - 3x + 10$$

$$= \left\{[x]-x\right) + \left(\left[x+\frac{1}{3}\right] - \left(x+\frac{1}{3}\right)\right) + \left(\left[x+\frac{2}{3}\right] - \left(x+\frac{2}{3}\right)\right) + 11$$

Since each of $[x]-x$, $\left[x+\frac{1}{3}\right] - \left(x+\frac{1}{3}\right)$ and $\left[x+\frac{2}{3}\right] - \left(x+\frac{2}{3}\right)$ is a periodic function with period 1. Therefore, period of $f(x)$ is 1.

(vi) We have,

$$f(x) = x + a - [x+b]$$

$$= (x+b) - [x+b] + a - b$$

Since $x-[x]$ is periodic with period 1. So, $f(x)$ is periodic with period 1.

(vii) We have,

$$f(x) = \begin{cases} 0 & , 0 \le x < 1 \\ \sin(\pi/4) = \frac{1}{\sqrt{2}} & , 1 \le x < 2 \\ \sin(2\pi/4) = 1 & , 2 \le x < 3 \\ \sin(3\pi/4) = \frac{1}{\sqrt{2}} & , 3 \le x < 4 \\ \sin(4\pi/4) = 0 & , 4 \le x < 5 \\ \sin(5\pi/4) = -\frac{1}{\sqrt{2}} & , 5 \le x < 6 \\ \sin(6\pi/4) = -1 & , 6 \le x < 7 \\ \sin(7\pi/4) = -\frac{1}{\sqrt{2}} & , 7 \le x < 8 \\ \sin(8\pi/4) = 0 & , 8 \le x < 9 \\ \sin(\pi/4) = \frac{1}{\sqrt{2}} & , 9 \le x < 10 \text{ and so on} \end{cases}$$

Clearly, $f(x)$ is periodic with period 8 units.

3. For $f(x) = \cos n\,x \sin(5x/n)$ to be periodic with period 3π, we must have

$$f(3\pi + x) = f(x) \quad \text{for all } x$$

$$\Rightarrow f(3\pi + 0) = f(0)$$

$$\Rightarrow \cos 3n\,\pi \sin\frac{15\pi}{n} = 0$$

$$\Rightarrow \sin\frac{15\pi}{n} = 0 \qquad [\because \cos 3n\,\pi = (-1)^{3n}]$$

$$\Rightarrow \frac{15\pi}{n} \text{ is a multiple of } \pi$$

$$\Rightarrow \frac{15}{n} \text{ is an integer}$$

$$\Rightarrow n = \pm 1, \pm 3 \pm 5, \pm 15$$

5. (i) If possible, let $f(x)$ be periodic with period T. Then,

$$f(x + T) = f(x) \qquad \text{for all } x.$$

$$\Rightarrow \cos\sqrt{x + T} = \cos\sqrt{x} \quad \text{for all } x$$

$$\Rightarrow \sqrt{x + T} = 2n\pi \pm \sqrt{x} \quad \text{for all } x$$

$$\Rightarrow \sqrt{x + T} \mp \sqrt{x} = 2n\pi \quad \text{for all } x \text{ and } n \in Z.$$

For fixed n, we observe that the RHS of the above equation is a constant whereas LHS is a variable which is not possible. Therefore, $f(x)$ is not a periodic function.

7. Let T be the period of $f(x)$. Then,

$$f(x + T) = f(x) \text{ for all } x.$$

$$\Rightarrow \sin(x + T) + \cos a\,(x + T) = \sin x + \cos ax \text{ for all } x$$

$$\Rightarrow \sin T + \cos a\,T = 1 \qquad [\text{Putting } x = 0]$$

and,

$$-\sin T + \cos aT = 1 \qquad [\text{Putting } x = -T]$$

$$\Rightarrow \sin T = 0 \text{ and } \cos aT = 1$$

$$\Rightarrow T = n\pi, n \in N \text{ and } aT = 2m\pi, m \in N$$

$$\Rightarrow a\,(n\,\pi) = 2m\,\pi, \text{ where } m, n \in N$$

$$\Rightarrow a = \frac{2m}{n}, \text{ which is a rational number.}$$

—❑ ❑ ❑—

1.19 EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS *A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x.*

Here, it is assumed that $-x$ is in the domain of $f(x)$ whenever x is in the domain of $f(x)$.

The graph of an even function is always symmetrical about y-axis, i.e., the curve on the left hand side of y-axis is the mirror image of the curve on its right hand side.

ODD FUNCTION *A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x.*

The graph of an odd function is always symmetric in opposite quadrants, i.e., the curve in the first quadrant is identical to the curve in third quadrant and the curve in second quadrant is identical to the curve in fourth quadrant.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Determine the nature of the following functions for even and odd :*

(i) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ (ii) $f(x) = \log(x + \sqrt{x^2 + 1})$

(iii) $f(x) = \sin x + \cos x$ (iv) $f(x) = x^2 - |x|$

SOLUTION (i) We have,

$$f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$$

$$\therefore f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right)$$

$$= -x\left(\frac{1 - a^x}{1 + a^x}\right)$$

$$= x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

(ii) We have,

$$f(x) = \log\{x+\sqrt{x^2+1}\}$$

$$\Rightarrow f(-x) = \log\{-x+\sqrt{x^2+1}\}$$

$$= \log\left\{\frac{-x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}.(x+\sqrt{x^2+1})\right\}$$

$$= \log\left\{\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right\}$$

$$= \log\left\{\frac{1}{x+\sqrt{x^2+1}}\right\}$$

$$= \log 1 - \log\{x+\sqrt{x^2+1}\}$$

$$= -\log\{x+\sqrt{x^2+1}\} = -f(x)$$

So, $f(x)$ is an odd function.

(iii) We have,

$$f(x) = \sin x + \cos x$$

$$\Rightarrow f(-x) = \sin(-x)+\cos(-x)$$

$$= -\sin x + \cos x$$

Clearly, $-\sin x + \cos x$ is neither equal to $f(x)$ nor equal to $-f(x)$.

So, $f(x)$ is neither even nor odd.

(iv) We have,

$$f(x) = x^2 - |x|$$

$$\Rightarrow f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x)$$

So, $f(x)$ is an even function.

EXAMPLE 2 *If f is an even function defined on the interval $[-5, 5]$, then find the real values of x satisfying the equation* $f(x) = f\left(\frac{x+1}{x+2}\right)$. **[IIT 1996]**

SOLUTION Since $f(x)$ is an even function defined on $[-5, 5]$.

Therefore, $f(-x) = f(x)$ for all $x \in [-5, 5]$

Thus, $f(x) = f\left(\frac{x+1}{x+2}\right)$

$$\Rightarrow x = \frac{x+1}{x+2} \quad \text{or,} \quad x = -\left(\frac{x+1}{x+2}\right)$$

$$\Rightarrow x^2+x-1 = 0 \text{ or, } x^2+3x+1 = 0$$

$$\Rightarrow x = \frac{-1\pm\sqrt{5}}{2} \quad \text{or,} \quad x = \frac{-3\pm\sqrt{5}}{2}$$

Hence, the values of x are

$$\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2} \text{ and } \frac{-3+\sqrt{5}}{2}.$$

EXAMPLE 3 *Let f, g be two functions. Then prove that*

(i) *f is even, g is even* $\Rightarrow$ *fog is an even function*

(ii) *f is odd, g is odd* $\Rightarrow$ fog is an odd function

(iii) *f is even, g is odd* $\Rightarrow$ *fog is an even function*

(iv) *f is odd, g is even* $\Rightarrow$ *fog is an even function*

SOLUTION (i) We have,

$$(fog)(-x) = f(g(-x))$$

$$= f(g(x)) \qquad [\because g \text{ is even } \therefore g(-x) = g(x)]$$

$$= fog(x) \quad \text{for all } x$$

$\therefore$ fog is an even function.

(ii) We have,

$$(fog)(-x) = f(g(-x))$$

$$= f(-g(x)). \qquad [\because g \text{ is odd} \Rightarrow g(-x) = -g(x)]$$

$$= -f(g(x)) \qquad [\because f \text{ is odd}]$$

$$= -fog(x) \quad \text{for all } x$$

$\therefore$ fog is an odd function.

(iii) We have,

$$fog(-x) = f(g(-x))$$

$$= f(-g(x)) \qquad [\because g \text{ is odd } \therefore g(-x) = -g(x)]$$

$$= f(g(x)) \qquad [\because f \text{ is even } \therefore f(-g(x)) = f(g(x))]$$

$$= fog(x) \quad \text{for all } x.$$

$\therefore$ fog is an even function.

(iv) We have,

$$fog(-x) = f(g(-x))$$

$$= f(g(x)) \qquad [\because g \text{ is even } \therefore g(-x) = g(x)]$$

$$= fog(x)$$

$\therefore$ *fog* is an even function.

EXAMPLE 4 *Determine the nature of the function.*

$$f(x) = \sin(\log(x+\sqrt{x^2+1})).$$

SOLUTION Let $g(x) = \sin x$. and $f(x) = \log(x+\sqrt{x^2+1})$. Then,

$$gof(x) = \sin\left\{(\log(x+\sqrt{x^2+1}))\right\}.$$

Since $f(x)$ and $g(x)$ both are odd functions, therefore $gof(x)$ is an odd function.

EXAMPLE 5 *Let $f: R \to R$ be a function given by*

$$f(x+y)+f(x-y) = 2f(x)f(y) \text{ for all } x, y \in R$$

and $f(0) \neq 0$. Prove that $f(x)$ is an even function. What can be said about $f(x)$ if $f(0) = 0$?

SOLUTION We have,

$f(x+y)+f(x-y) = 2f(x)f(y)$ for all $x, y \in R$ ...(i)

$\therefore\ f(0+0)+f(0-0)=2f(0)f(0)$ [Replacing both x and y by 0]

$\Rightarrow f(0)+f(0) = 2f(0)f(0)$

$\Rightarrow 2f(0)\{f(0)-1\} = 0$

$\Rightarrow f(0) = 1.$ [$\because f(0) \neq 0$]

Putting $x = 0$ in (i), we get

$f(y)+f(-y) = 2f(0)f(y)$ for all $y \in R$ ...(ii)

$\Rightarrow f(y)+f(-y) = 2f(y)$ for all $y \in R$ [$\because f(0)=1$]

$\Rightarrow f(-y) = f(y)$ for all $y \in R$

$\Rightarrow f(x)$ is an even function.

If $f(0)=0$, then, from (ii), we get

$f(y)+f(-y)=0$ for all $y \in R$

$\Rightarrow f(-y)=-f(y)$ for all $y \in R$

$\Rightarrow f$ is an odd function.

EXAMPLE 6 *Prove that* $f(x)=\dfrac{2x(\sin x+\tan x)}{2\left[\dfrac{x+2\pi}{\pi}\right]-3}$ *is an odd function, where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION We have,

$$f(x) = \frac{2x(\sin x+\tan x)}{2\left[\frac{x}{\pi}+2\right]-3}$$

$$\Rightarrow f(x) = \frac{2x(\sin x+\tan x)}{2\left[\frac{x}{\pi}\right]+4-3}$$

$$\Rightarrow f(x) = \frac{2x(\sin x+\tan x)}{2\left[\frac{x}{\pi}\right]+1}$$

$$\therefore\ f(-x) = \frac{-2x[\sin(-x)+\tan(-x)]}{2\left[\frac{-x}{\pi}\right]+1}$$

$$\Rightarrow f(-x) = \frac{x(\sin x+\tan x)}{\left[\frac{-x}{\pi}\right]+\frac{1}{2}}$$

Now, we shall discuss the following two cases :

CASE I *When* $\dfrac{x}{\pi}$ *is an integer* n *(say) i.e.* $x = n\pi$.

In this case, we have

$$\left[\frac{-x}{\pi}\right] = -\left[\frac{x}{\pi}\right]$$

and, $x(\sin x+\tan x) = n\pi(\sin n\pi+\tan n\pi) = 0$

$\therefore\ f(-x) = 0$ for all x.

Thus, we have $f(x) = 0$ for all x.

CASE II *When* $\dfrac{x}{\pi}$ *is not an integer :*

In this case, we have

$$\left[\frac{-x}{\pi}\right] = -\left[\frac{x}{\pi}\right]-1$$

$$\therefore\ f(-x) = \frac{x(\sin x+\tan x)}{-\left[\frac{x}{\pi}\right]-1+\frac{1}{2}}$$

$$\Rightarrow f(-x) = \frac{x(\sin x+\tan x)}{-\left[\frac{x}{\pi}\right]-\frac{1}{2}} = -f(x)$$

Thus, $f(-x) = -f(x)$ in both the cases.

Hence, $f(x)$ is an odd function.

EXAMPLE 7 *Find out whether the following function is even or odd:*

$$f(x) = \begin{cases} x|x| & , x \le -1 \\ [1+x]+[1-x] & , -1 < x < 1 \\ -x|x| & , x \ge 1 \end{cases}$$

SOLUTION We have,

$$f(x) = \begin{cases} x|x| & , x \le -1 \\ [1+x]+[1-x] & , -1 < x < 1 \\ -x|x| & , x \ge 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -x^2 & , x \le -1 \\ 1+[x]+1+[-x] & , -1 < x < 1 \\ -x^2 & , x \ge 1 \end{cases} \left[\because \begin{matrix}[1+x]=[x]+1 \\ [1-x]=[-x]+1\end{matrix}\right]$$

$$\Rightarrow f(x) = \begin{cases} -x^2 & , x \le -1 \\ 2+[x]+[-x] & , -1 < x < 1 \\ -x^2 & , x \ge -1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -x^2 & , x \le -1 \\ 1 & , -1 < x < 0 \\ 2 & , x = 0 \\ 1 & , 0 < x < 1 \\ -x^2 & , x \ge 1 \end{cases} \quad \left[\because \begin{matrix}[x]+[-x]=-1, \text{if } x \notin Z \\ [x]+[-x]=0, \text{if } x \in Z\end{matrix}\right]$$

It is evident from the definition of the function or its graph that $f(x)$ is an even function.

EXERCISE 1.4

1. Find out whether the following functions are even or odd:

(i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

(ii) $f(x) = \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}}$

(iii) $f(x) = \begin{cases} x\,|x| & , x \le -1 \\ [2+x] + [2-x] & , -1 < x < 1 \\ -x\,|x| & , x \ge 1 \end{cases}$

(iv) $f(x) = \displaystyle\int_0^x \log_e\left(\frac{1-x}{1+x}\right) dx$

(v) $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$

2. Let $f: [-2, 2] \to R$ be a function defined on $[0, 2]$ as follows:

$$f(x) = \begin{cases} x \tan x & , 0 < x \le \pi/2 \\ \dfrac{\pi}{2}[x] & , \dfrac{\pi}{2} < x \le 2 \end{cases}$$

Define $f(x)$ on $[-2, 0]$ so that it becomes an

(i) even function on $[-2, 2]$

(ii) odd function on $[-2, 2]$.

3. Define a function $f(x)$ suitably in the interval $[0, \infty)$ so that $f(x)$ may be an

(i) even function (ii) odd function

whose definition is as follows :

$$f(x) = \begin{cases} 1 & , x < -1 \\ -x & , -1 \le x \le 0 \end{cases}$$

4. Extend $f(x) = x^2 + x$ defined in $[0, 3]$ onto the interval $[-3, 3]$ so that $f(x)$ is (i) even (ii) odd.

5. Which of the following functions is (are) even and which is (are) odd ?

(i) $f(x) = 4 - 2x^4 + \sin^2 x$

(ii) $f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$

(iii) $f(x) = \dfrac{1 + a^{nx}}{1 - a^{nx}}$

(iv) $f(x) = \sin x + \cos x$

(v) $f(x) =$ constant.

(vi) $f(x) = \dfrac{(1 + 2^x)^2}{2^x}$

6. Represent each of the following functions as the sum of an even and an odd function :

(i) $f(x) = a^x$ (ii) $f(x) = (1 + x)^{100}$

(iii) $f(x) = \sin 2x + \tan x + \cos\dfrac{x}{2}$

(iv) $f(x) = 1 - x^3 - x^4 - 2x^5$

7. If a function f satisfies the condition

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \text{ for all } x, y \in R \text{ and } xy \ne 1.$$

Prove that $f(x)$ is an odd function.

8. Let $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ for all x and y belonging to the domain of $f(x)$. Find whether $f(x)$ is even, odd, neither even nor odd.

9. Let $f: [-2, 2] \to R$ be a function given by

$$f(x) = \begin{cases} x \tan x & , 0 < x \le \pi/2 \\ \dfrac{\pi}{2}[x] & , \dfrac{\pi}{2} < x \le 2 \end{cases}$$

Define $f(x)$ for $x \in [-2, 0]$ such that

(a) $f(x)$ is an odd function on $[-2, 2]$

(b) $f(x)$ is an even function on $[-2, 2]$

ANSWERS

1. (i) even (ii) odd (iii) even
(iv) even (v) even

2. (i) $f(x) = \begin{cases} x \tan x & , 0 < x \le \pi/2 \\ \dfrac{\pi}{2}[x] & , \pi/2 < x \le 2 \\ x \tan x & , -\pi/2 \le x < 0 \\ \dfrac{\pi}{2}[-x] & , -2 \le x < -\pi/2 \end{cases}$

(ii) $f(x) = \begin{cases} x \tan x & , 0 < x \le \pi/2 \\ \dfrac{\pi}{2}[x] & , \pi/2 < x \le 2 \\ -x \tan x & , -\pi/2 \le x < 0 \\ \dfrac{\pi}{2}[-x] & , -2 \le x < -\pi/2 \end{cases}$

3. (i) $f(x) = \begin{cases} x & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$ (ii) $f(x) = \begin{cases} -x & , 0 \le x \le 1 \\ -1 & , x > 1 \end{cases}$

4. (i) $f(x) = x^2 - x, -3 \le x < 0$
(ii) $f(x) = -x^2 + x, -3 \le x < 0$

5. (i) even (ii) odd (iii) odd
(iv) Neither even, nor odd (v) even
(vi) even

8. odd

9. (a) $f(x) = \begin{cases} -x \tan x & , -\pi/2 \le x < 0 \\ -\dfrac{\pi}{2}[-x] & , -2 \le x \le -\pi/2 \end{cases}$

(b) $f(x) = \begin{cases} x \tan x & , -\pi/2 \le x < 0 \\ \dfrac{\pi}{2}[-x] & , -2 \le x \le -\pi/2 \end{cases}$

CHAPTER

2

GRAPHS OF FUNCTIONS

2.1 INTRODUCTION

The knowledge of graphs is very useful in mathematics. Particularly, in calculus graphs play a very important role in understanding continuity, differentiability, maximum and minimum values etc. In this chapter, we shall first discuss the construction the graphs of some basic functions and then we shall learn about some simple transformations of graphs. These transformations will enable us to draw the graphs of functions obtained from basic functions.

2.2 THE GRAPH OF A FUNCTION

The graph of a function $f(x)$ is the graph of the equation $y = f(x)$. It consists of all points in the Cartesian plane whose coordinates are $(x, f(x))$ or (x, y), where x is any point in the domain of the function $f(x)$.

Recall that in the point $(x, f(x))$, x is the directed distance of the point from y-axis and $f(x)$ is the directed distance of the point from x-axis as shown in Fig. 2.1

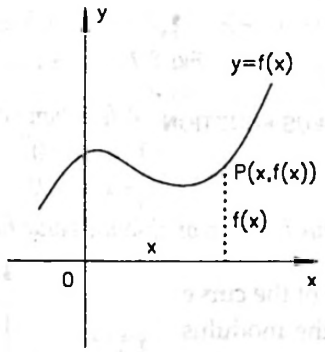


Fig. 2.1

Note that every curve we draw in the Cartesian plane is not necessarily the graph of a function. There may be graphs not representing a function. Since a function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once. We can use this as a convenient visual test to check whether a given graph represents a function or not. It is called the vertical test for functions. For example, in Fig. 2.2 (i), we see that the graph does not represent a function. In other words, it does not define y as a function of x, since a vertical line intersects the graph twice. Graphs in Fig. 2.2 (ii) and (iii) represent functions.

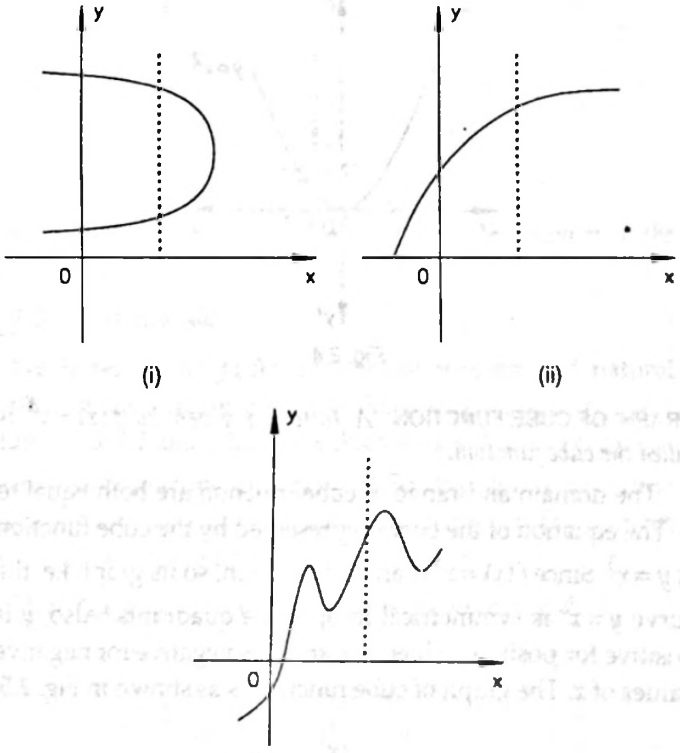


Fig. 2.2

2.3 GRAPHS OF SOME BASIC FUNCTIONS

In this section, we shall discuss the graphs of some basic functions.

GRAPH OF IDENTITY FUNCTION *A function f defined by $f(x) = x$ for all $x \in R$ is called the identity function.*

The equation of the curve represented by the identity function is $y = x$. Clearly, it represents a straight line passing through the origin and inclined at an angle of 45° with x-axis.

Its graph is as shown in Fig. 2.3. The domain and range of the identity function are both equal to R.

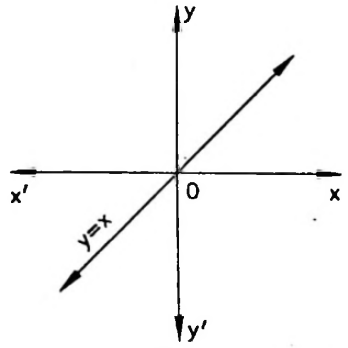


Fig. 2.3

GRAPH OF SQUARE FUNCTION *A function given by* $f(x) = x^2$ *is called the square function.*

The domain of the square function is R and its range is $[0, \infty)$. The equation of the curve represented by the square function is $y = x^2$.

Clearly, $y = x^2$ represents a parabola. Since $f(x) = x^2$ is an even function, so its graph is symmetrical about y-axis and is as shown in Fig. 2.4.

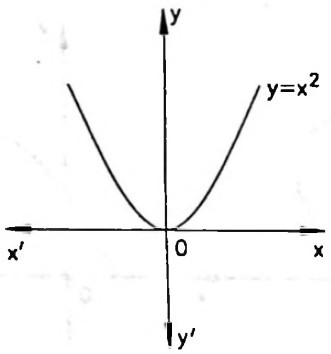


Fig. 2.4

GRAPH OF CUBE FUNCTION *A function given by* $f(x) = x^3$ *is called the cube function.*

The domain and range of cube function are both equal to R. The equation of the curve represented by the cube function is $y = x^3$. Since $f(x) = x^3$ is an odd function, so its graph i.e. the curve $y = x^3$ is symmetrical in opposite quadrants. Also, y is positive for positive values of x and it is negative for negative values of x. The graph of cube function is as shown in Fig. 2.5.

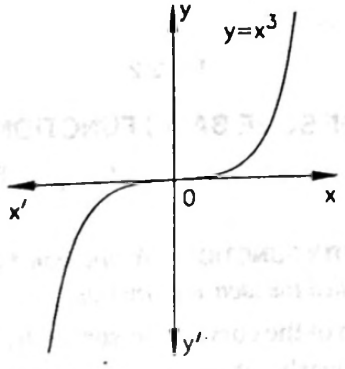


Fig. 2.5

GRAPH OF $y = x^{2n}, n \in N$ If $n \in N$, then function f given by $f(x) = x^{2n}$ is an even function. So, its graph is always symmetrical about y-axis. Also,

$$x^2 > x^4 > x^6 > x^8 \ldots \text{ for all } x \in (-1, 1)$$

and, $$x^2 < x^4 < x^6 < x^8 \ldots \text{ for all } x > 1 \text{ or } x < -1$$

The graphs of $f(x) = x^2, f(x) = x^4, f(x) = x^6$ etc. are shown in Fig. 2.6.

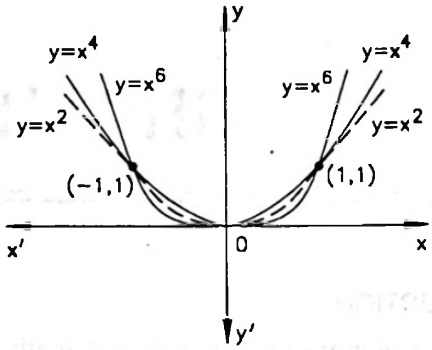


Fig. 2.6

GRAPH OF $y = x^{2n-1}, n \in N$ If $n \in N$, then the function f given by $f(x) = x^{2n-1}$ is an odd function. So, its graph is symmetrical in opposite quadrants. Graphs of $f(x) = x, f(x) = x^3, f(x) = x^5$ etc. are shown in Fig. 2.7.

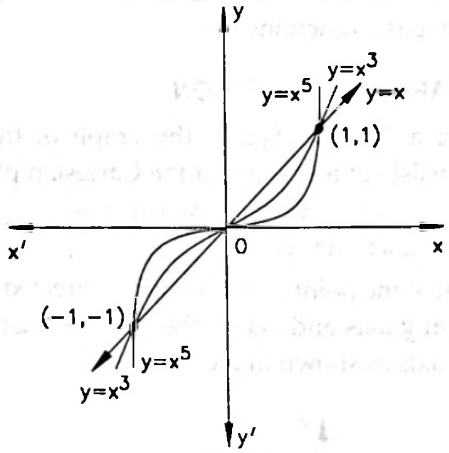


Fig. 2.7

GRAPH OF MODULUS FUNCTION *A function f defined by*

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is called the modulus function or absolute value function.

The equation of the curve represented by the modulus function is $y = |x|$ or, $y = x$ when $x \geq 0$ and $y = -x$ when $x < 0$. The domain of the modulus function is R and its range is $[0, \infty)$. The graph of the modulus function is shown in Fig. 2.8.

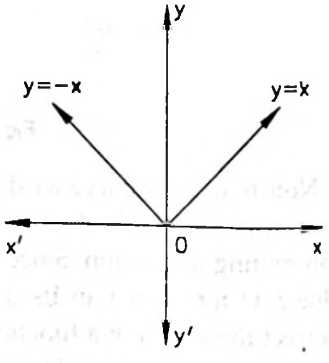


Fig. 2.8

GRAPH OF RECIPROCAL FUNCTION *A function defined by* $f(x)=\frac{1}{x}$ *is called the reciprocal function.*

The domain of reciprocal function is $R-\{0\}$. Its range is also $R-\{0\}$. Clearly, $f(x)=\frac{1}{x}$ is an odd function so its graph is symmetrical in opposite quadrants. Also, we observe that

$x\to 0^+ \Rightarrow f(x)\to +\infty,\ x\to 0^- \Rightarrow f(x)\to -\infty,$

and $x\to \pm\infty \Rightarrow f(x)\to 0.$

Thus, the graph of $f(x)=\frac{1}{x}$ is as shown in Fig. 2.9.

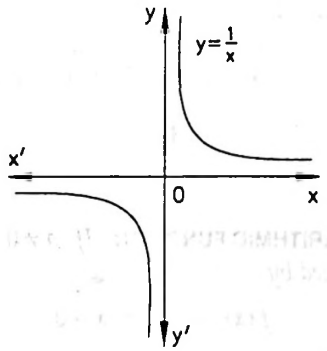


Fig. 2.9

GRAPH OF $f(x)=\frac{1}{x^{2n-1}}, n\in N$ Clearly, function $f(x)=\frac{1}{x^{2n-1}}$ is an odd function , so its graph is symmetrical in opposite quadrants. We also observe that y is positive for positive values of x and it is negative for negative values of x. Also, $y\to\infty$ when $x\to 0^+$ and $y\to -\infty$ when $x\to 0^-$. Thus, the graphs of $f(x)=1/x^3$, $f(x)=1/x^5$ etc. will be similar to the graph of $f(x)=\frac{1}{x}$.

GRAPH OF $f(x)=\frac{1}{x^{2n}}, n\in N$ We observe that the function $f(x)=\frac{1}{x^{2n}}$ is an even function, so its graph is symmetrical about y-axis. Domain of definition of $f(x)$ is $R-\{0\}$ and $f(x)$ assumes positive values only for all $x\in R$. Also, $y\to\infty$ as $x\to 0^+$ or 0^- and $y\to 0$ as $x\to -\infty$ or $+\infty$. The values of y decrease as the values of x increase. Thus, the graph of $f(x)=\frac{1}{x^2}$ is as shown in Fig. 2.10.

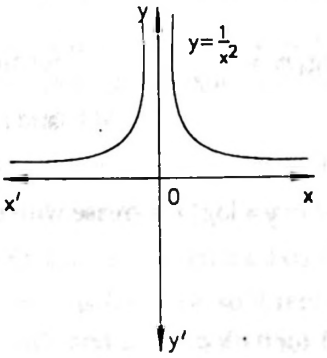


Fig. 2.10

REMARK *Since the characteristics of functions* $f(x)=\frac{1}{x^4}$, $f(x)=\frac{1}{x^6}$ *etc. are exactly same as that of* $f(x)=\frac{1}{x^2}$. *So, their graphs are identical to the graph of* $f(x)=\frac{1}{x^2}$.

GRAPHS OF $f(x)=x^{1/n}, n\in N$

CASE I *When n is even :*

Clearly, $f(x)=x^{1/n}$ is defined for all $x\in[0,\infty)$ and the values taken by $f(x)$ are positive. So, domain and range of $f(x)$ are both $[0,\infty)$. We also observe that $f(x)$ is the inverse of the function $g(x)=x^n$ when its domain is taken as $[0,\infty)$. Therefore, the graph of $f(x)=x^{1/n}$ is the mirror image of the graph of $g(x)=x^n$ in the line mirror $y=x$. That is the graphs of $f(x)$ and $g(x)$ are symmetrical about $y=x$. Thus, the graph of $f(x)=x^{1/2}$ is as shown in Fig. 2.11.

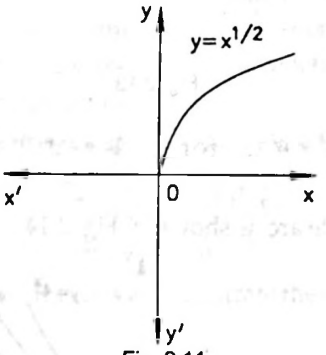


Fig. 2.11

REMARK *Graphs of* $f(x)=x^{1/4}, x^{1/6}, x^{1/8}$ *etc. are similar to the graph of* $f(x)=x^{1/2}$.

CASE II *When n is odd :*

We observe that $f(x)=x^{1/n}$, when n is an odd natural number, is defined for all $x\in R$. Also, $f(x)$ assumes positive values for $x>0$ and negative values for $x<0$. Clearly, $f(x)$ is the inverse of the function $g(x)=x^n$. So, its graph is the mirror image of the graph of $g(x)$ in the line mirror $y=x$. The domain and range of $f(x)$ are both equal to R.

Thus, the graph of $f(x)=x^{1/3}$ is as shown in Fig. 2.12.

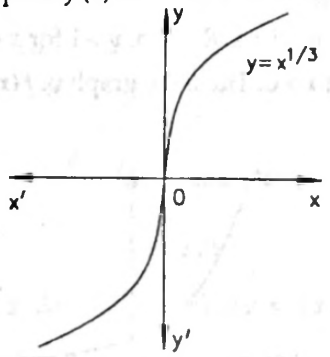


Fig. 2.12

REMARK *Graphs of* $f(x)=x^{1/5}, x^{1/7}$ *etc. are similar to the graphs of* $f(x)=x^{1/3}$.

GRAPH OF THE EXPONENTIAL FUNCTION If $a > 0$ and $a \neq 1$, then for any $x \in R$, $f(x) = a^x$ is called the exponential function. Clearly, $f(x)$ is defined for all $x \in R$ and it takes only positive values. Thus, domain of $f(x)$ is R and its range is $(0, \infty)$.

CASE I *When* $a > 1$.

We observe that the values of $y = f(x)$ increase as the values of x increase i.e. $f(x)$ is an increasing function on R. Also, $y = 1$ at $x = 0$, $y < 1$ for $x < 0$ and $y > 1$ for $x > 0$. Thus, the graph of $f(x) = a^x$, $a > 1$ is as shown in Fig. 2.13.

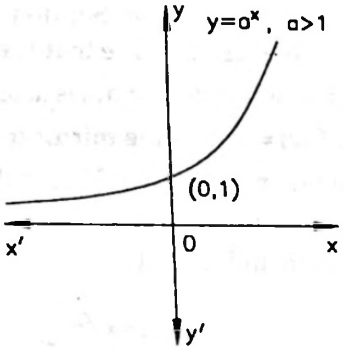


Fig. 2.13

Since $2^x < 3^x < 4^x$ .... for $x > 1$ and $2^x > 3^x > 4^x > \ldots$ for $0 < x < 1$. Therefore, graphs of $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = 4^x$ etc. on the same scale are as shown in Fig. 2.14.

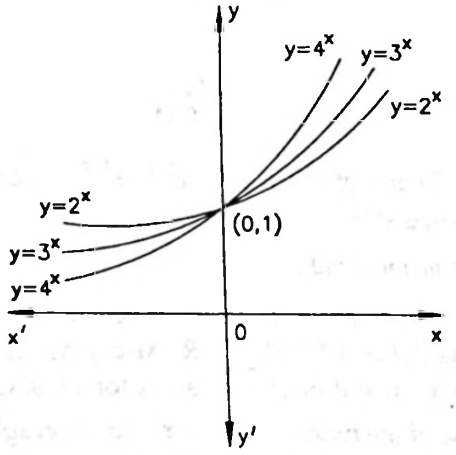


Fig. 2.14

CASE II When $0 < a < 1$

In this case, the values of $y = f(x)$ decrease with the increase of x and $y > 0$ for all $x \in R$. Also, $y > 1$ for $x < 0$, $y = 1$ for $x = 0$ and $y < 1$ for all $x > 0$. Thus, the graph of $f(x) = a^x$ is as shown in Fig. 2.15.

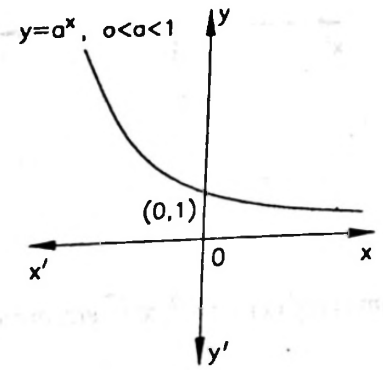


Fig. 2.15

The graphs of $f(x) = a^x$, $0 < a < 1$ for various values of a are shown in Fig. 2.16.

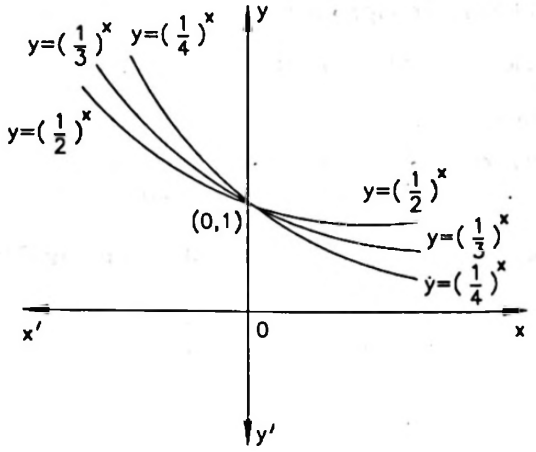


Fig. 2.16

GRAPH OF LOGARITHMIC FUNCTION *If* $a \neq 0, 1$ and $a > 0$, *then the function defined by*

$$f(x) = \log_a x, x > 0$$

is called the logarithmic function.

In the previous chapter, we have seen that the logarithmic and the exponential functions are inverse of each other. Thus for $a \neq 0, 1$ and $a > 0$, we have

$$\log_a x = y \Leftrightarrow a^y = x.$$

Some important properties of the logarithmic function are listed below :

(i) $\log_a 1 = 0$

(ii) $\log_a a = 1$

(iii) $\log_a (mn) = \log_a m + \log_a n$ for all $m, n > 0$

(iv) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$ for all $m, n > 0$

(v) $\log_a m^n = n \log_a m$ for all $m > 0$

(vi) $\log_a (a^m) = m$ for all $m > 0$

(vii) $\log_{a^l} m^n = \frac{n}{l} \log_a m$ for all $m > 0$

(viii) $\log_a (1/m) = -\log_a m$ for all $m > 0$

(ix) $\log_a m = \frac{\log_b m}{\log_b a}$ for all $m > 0, a > 0, b > 0$ and $a \neq 1, b \neq 1$

(x) If $a > 1$, then $m < n \Rightarrow \log_a m < \log_a n$

i.e. the value of $y = \log_a x$ increase with the increase in the values of x.

Also,

$\log_a x < 0$ for $0 < x < 1$

and, $\log_a x > 0$ for $x > 1$

(xi) if $0 < a < 1$, then

$m < n \Rightarrow \log_a m > \log_a n$ for all $m, n > 0$ i.e. the values of $y = \log_a x$ decrease with the increase in the values of x.

Also,

$\log_a x > 0$ for $0 < x < 1$

and, $\log_a x < 0$ for $x > 1$.

(xii) $\log_b a = \dfrac{1}{\log_a b}$ for all $a, b > 0$ and $a, b \neq 1$.

We find that the domain of $f(x) = \log_a x, a \neq 0, 1$ is $(0, \infty)$ and its range is R. Also, logarithmic and exponential functions are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$. Thus, the graph of $f(x) = \log_a x$ for $a > 1$ is as shown in Fig. 2.17 and for $0 < a < 1$ in Fig. 2.18.

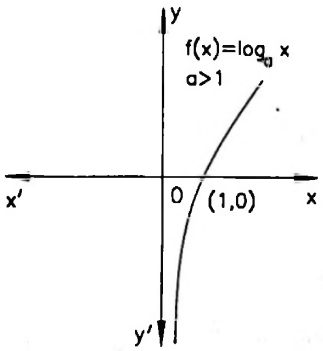


Fig. 2.17

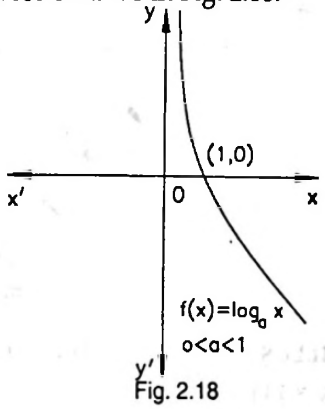


Fig. 2.18

GRAPH OF $y = ax^2 + bx + c$, where $a, b, c \in R$ and $b^2 - 4ac > 0$

In order to draw the graph of $y = ax^2 + bx + c$, we proceed as follows :

STEP I *Put $ax^2 + bx + c = 0$ and obtain the values of x.*

STEP II *Mark the points obtained in step I on x-axis.*

STEP III *Check the sign of a. If $a > 0$, draw a parabola opening upward and passing through the points marked in step II.*

If $a < 0$, draw a parabola opening downward and passing through the points marked in step II

Following illustration will illustrate the above procedure :

ILLUSTRATION 1 *Graph the following functions :*

(i) $y = x^2 - 3x + 2$ (ii) $y = -x^2 + 4x$

SOLUTION (i) We have,

$$x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2.$$

Thus, the curve $y = x^2 - 3x + 2$ cuts x-axis at (1, 0) and (2, 0).

Since the coefficient of x^2 is positive, so $y = x^2 - 3x + 2$ represents a parabola opening upward and passing through (1, 0) and (2, 0) as shown in Fig. 2.19.

(ii) We have,

$$-x^2 + 4x = 0 \Rightarrow x = 0, 4.$$

Thus, the curve $y = -x^2 + 4x$ cuts x-axis at (0, 0) and (4, 0). Since coefficient of x^2 is negative, so it represents a parabola opening downwards and passing through (0, 0) and (4, 0) as shown in Fig. 2.20.

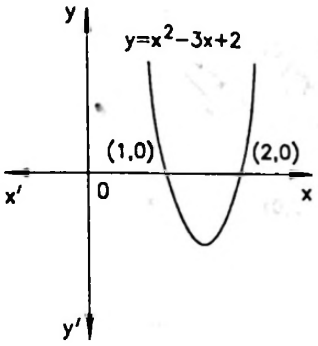


Fig. 2.19

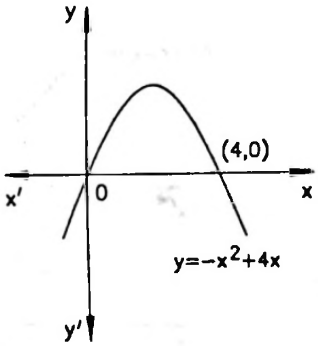


Fig. 2.20

In the previous chapter, we have learnt about trigonometrical functions and their inverses. We have also discussed about their graphs. So, reader is advised to refer chapter 1 for the same.

2.4 SIMPLE TRANSFORMATIONS OF GRAPHS

In the previous section, we have learnt about some simple basic elementary functions and their graphs. In this section, we intend to discuss some simple transformations of graphs which will enable us to draw the graphs of functions derived from basic elementary functions.

I TRANSLATION ALONG x-AXIS. The graph of the function $y = f(x + a)$ is obtained from the graph of the function $y = f(x)$ by translating the graph of $y = f(x)$ along the x-axis by $|a|$ units in the direction opposite to the sign of a.

Following illustrations will illustrate the above transformation.

ILLUSTRATION 1 *Draw the graphs of $f(y) = x + 1$ and $y = x - 1$ by translating the graph of $y = x$ along x-axis.*

SOLUTION The graph of $y = x$ is a straight line passing through the origin and making an angle of 45° with x–axis as shown in Fig. 2.21.

In order to draw the graphs of $y = x + 1$ and $y = x - 1$, the graph of $y = x$ is translated parallel to itself along OX' and OX respectively by 1 unit as shown in Fig. 2.21.

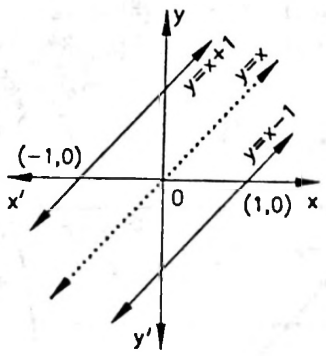


Fig. 2.21

ILLUSTRATION 2 *Sketch the graph of $y = |x - 2|$.*

SOLUTION We observe that the function $y = |x - 2|$ is derived from $y = |x|$ by replacing x by $x - 2$. Thus, the graph of $y = |x - 2|$ is obtained from the graph of $y = |x|$ by translating it along OX-axis by 2 units as shown in Fig. 2.22.

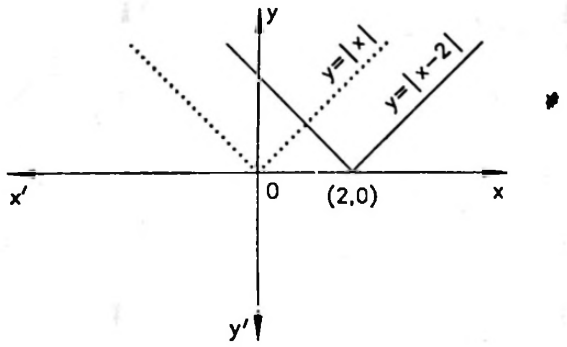


Fig. 2.22

ILLUSTRATION 3 *Graph the following functions :*

(i) $y = (x+2)^3$ (ii) $y = \log_a (x-2)$, $a > 1$

SOLUTION We observe that the function $y = (x+2)^3$ can be obtained from $y = x^3$ by replacing x by $x+2$. Thus, the graph of $y = (x+2)^3$ can be obtained from the graph of $y = x^3$ by translating it along OX' by 2 units as shown in Fig. 2.23.

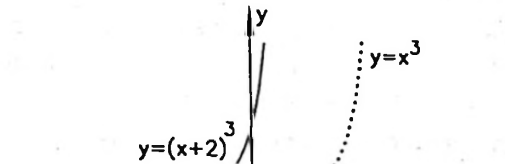


Fig. 2.23

(ii) We observe that $y = \log_a (x-2)$, $a > 1$ is obtained from $y = \log_a x$, $a > 1$ by replacing x by $x-2$. So, the graph of $y = \log_a (x-2)$ is obtained from the graph of $y = \log_a x$ by translating it along OX by 2 units as shown in Fig. 2.24.

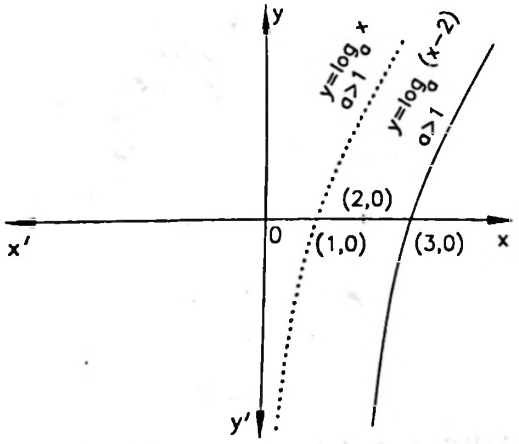


Fig. 2.24

II TRANSLATION ALONG Y-AXIS *The graph of $y = f(x) + b$ or, $y - b = f(x)$ is obtained from the graph of $y = f(x)$ by translating the graph of $y = f(x)$ along y-axis by $|b|$ units in the vertically upward or downward direction according as $b > 0$ or $b < 0$.*

Following illustrations will illustrate the above transformation.

ILLUSTRATION 1 *Draw the graphs of $y = x + 1$ and $y = x - 1$ by translating the graph of $y = x$ along y-axis.*

SOLUTION Clearly $y = x + 1$ and $y = x - 1$ can be obtained from $y = x$ by replacing x by $x + 1$ and $x - 1$ respectively. Therefore the graphs of $y = x + 1$ and $y = x - 1$ can be obtained by translating the graph of $y = x$ by 1 unit along OY and OY' respectively as shown in Fig. 2.25.

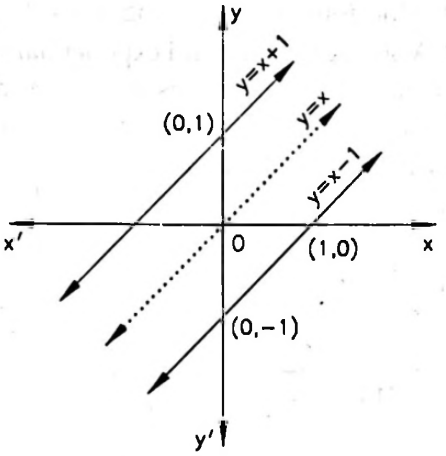


Fig. 2.25

ILLUSTRATION 2 *Draw the graphs of $y = |x| + 1$ and $y = |x| - 2$.*

SOLUTION We observe that $y = |x| + 1$ and $y = |x| - 2$ can be obtained from $y = |x|$ by replacing $|x|$ by $|x| + 1$ and $|x| - 2$ respectively. Thus, the graph of $y = |x| + 1$ can be obtained from the graph of $y = |x|$ by translating it along OY by 1 unit. Similarly, the graph of $y = |x| - 2$ can be obtained by translating the graph of $y = |x|$ along OY' by 2 units.

Graphs of $y = |x| + 1$ and $y = |x| - 2$ are shown in Fig. 2.26.

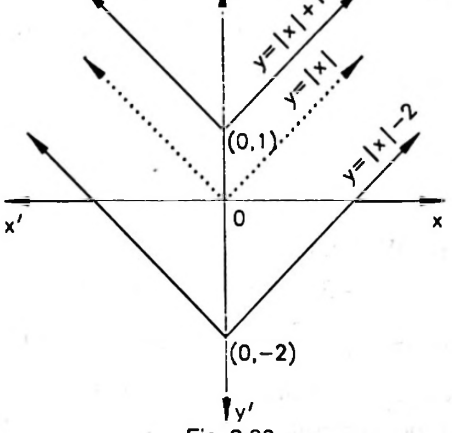


Fig. 2.26

ILLUSTRATION 3 *Graph the following functions :*

(i) $y = -x^2 + 1$ (ii) $y = x^2 - 1$

SOLUTION (i) We observe that the function $y = -x^2 + 1$ is derived from $y = -x^2$. Clearly, the graph of $y = -x^2 + 1$ can be obtained from the graph of $y = -x^2$ by translating it along OY by 1 unit as shown in Fig. 2.27.

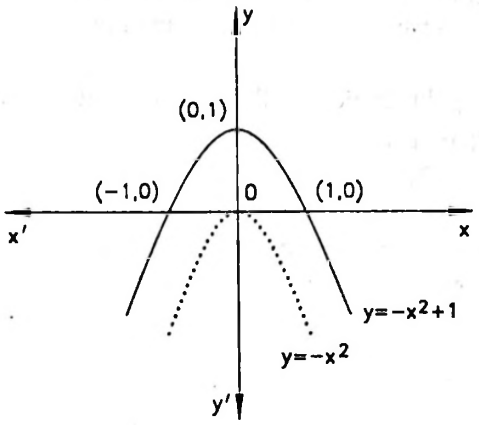


Fig. 2.27

(ii) Clearly, $y = x^2 - 1$ is derived from $y = x^2$. So, the graph of $y = x^2 - 1$ is obtained from the graph of $y = x^2$ by translating it by 1 unit along OY' as shown in Fig. 2.28.

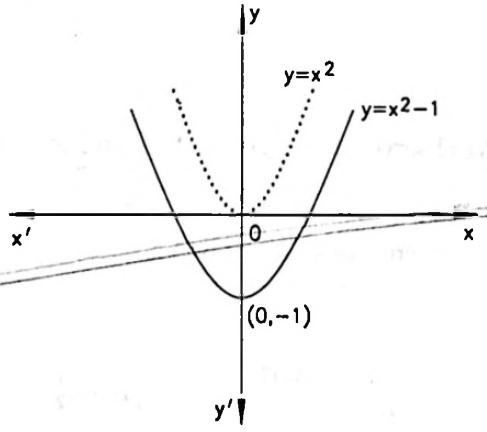


Fig. 2.28

ILLUSTRATION 4 *Graph the following functions :*

(i) $y = |x-1| + 1$ (ii) $y + 2 = |x+1|$

SOLUTION (i) Clearly, $y = |x-1| + 1$ is obtained from $y = |x|$ by performing the following steps :

STEP I Translate along OX by 1 unit to obtain the graph of $y = |x-1|$.

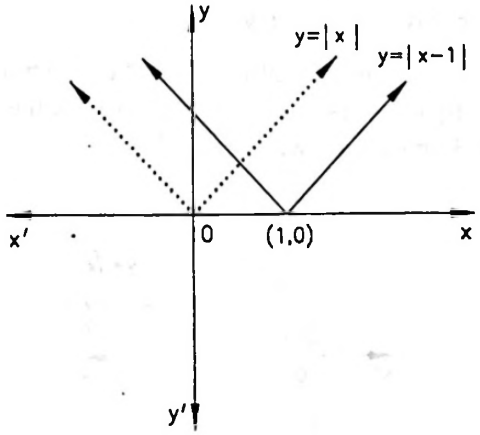


Fig. 2.29

STEP II Translate the graph of $y = |x-1|$ along OY by 1 unit.

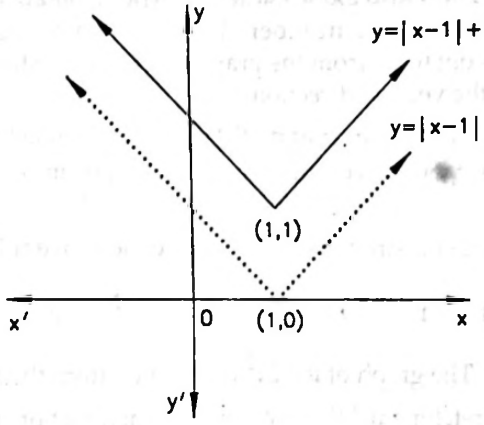


Fig. 2.30

The graph of $y = |x-1| + 1$ is shown in Fig. 2.30.

(ii) We have,

$$y + 2 = |x+1|$$

$$\Rightarrow \quad y = |x+1| - 2$$

The graph of $y = |x+1| - 2$ is obtained from the graph of $y = |x|$ by using the following steps :

STEP I The graph of $y = |x+1|$ is obtained from the graph of $y = |x|$ by translating it along OX' by 1 unit as shown in Fig. 2.31.

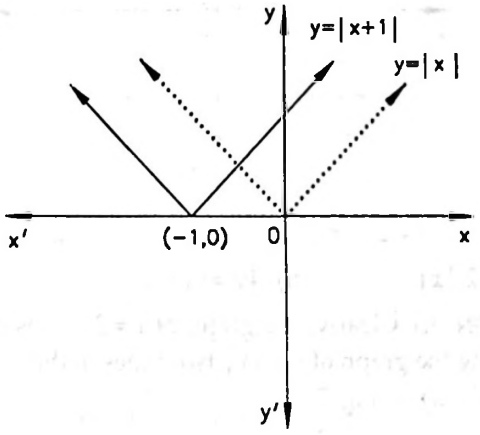


Fig. 2.31

STEP II Translate the graph of $y = |x+1|$ along OY' by 2 units to obtain the graph of $y = |x+1| - 2$ as shown in Fig. 2.32.

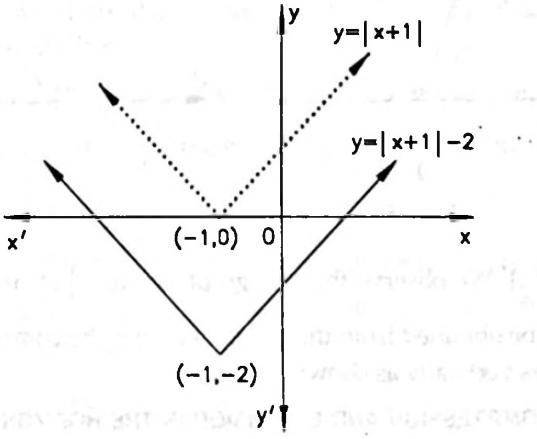


Fig. 2.32

III COMPRESSION AND EXPANSION IN VERTICAL DIRECTION Let k be a positive real number. If $k>1$, then the graph of $y=kf(x)$ is obtained from the graph of $y=f(x)$ by stretching it k-times in the vertical direction.

If $0<k<1$, then the graph of $y=kf(x)$ is obtained from the graph of $y=f(x)$ by compressing it $1/k$ times in the vertical direction.

Following illustrations will illustrate the above rule.

ILLUSTRATION 1 *Sketch the graphs of* $y=2x^2$ *and* $y=\frac{x^2}{3}$.

SOLUTION The graph of $y=2x^2$ is obtained from the graph of $y=x^2$ by stretching it 2 times in the vertical direction, whereas the graph of $y=\frac{1}{3}x^2$ is obtained by compressing the graph of $y=x^2$, 3 times in the vertical direction as shown in Fig. 2.33.

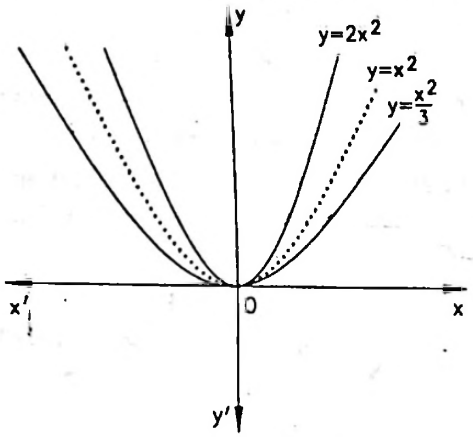


Fig. 2.33

ILLUSTRATION 2 *Graph the following functions :*

(i) $y=2|x|$ (ii) $4y=|x|$.

SOLUTION (i) Clearly, the graph of $y=2|x|$ is obtained by stretching the graph of $y=|x|$ two times in the vertical direction as shown in Fig. 2.34.

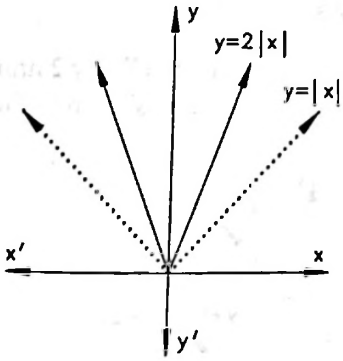


Fig. 2.34

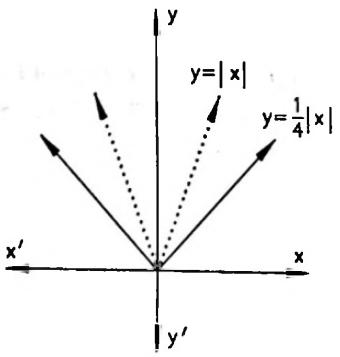


Fig. 2.35

(ii) We observe that the graph of $4y=|x|$ or, $y=\frac{1}{4}|x|$ can be obtained from the graph of $y=|x|$ by compressing it 4 times vertically as shown in Fig. 2.35.

IV COMPRESSION AND EXPANSION IN THE HORIZONTAL DIRECTION Let k be a positive real number. If $k>1$, then the graph of $y=f(kx)$ is obtained from the graph of $y=f(x)$ by compressing it k times against y-axis in the horizontal direction. If $0<k<1$, then the graph of $y=f(kx)$ is obtained from the graph of $y=f(x)$ by stretching it $1/k$ times in the horizontal direction from y-axis.

Following illustrations will illustrate the above rule.

ILLUSTRATION 1 *Graph the following functions :*

(i) $y=\sin 2x$ (ii) $y=\sin x/2$

SOLUTION (i) Clearly, the graph of $y=\sin 2x$ is obtained from the graph of $y=\sin x$ by compressing it 2 times against y-axis in the horizontal direction as shown in Fig. 2.36.

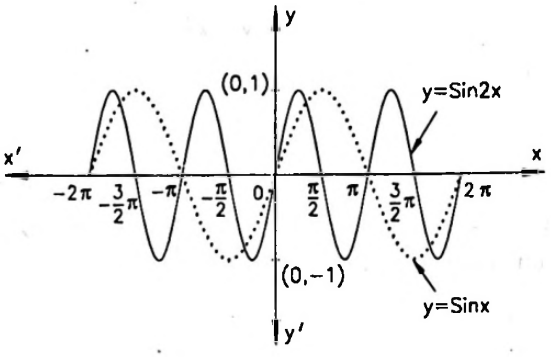


Fig. 2.36

(ii) We observe that the graph of $y=\sin\frac{x}{2}$ is obtained from the graph of $y=\sin x$ by stretching it 2 times from y-axis in the horizontal direction as shown in Fig. 2.37.

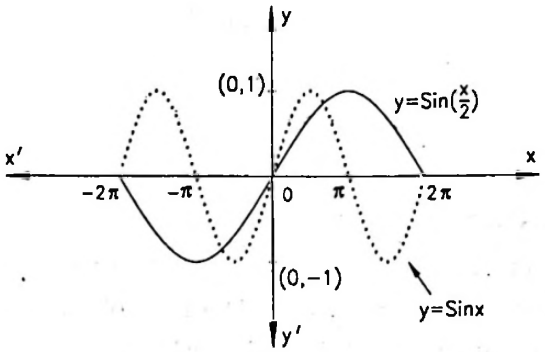


Fig. 2.37

ILLUSTRATION 2 *Graph the following functions :*

(i) $y=\sqrt{3x}$ (ii) $y=\sqrt{\frac{x}{2}}$

SOLUTION (i) The graph of $y=\sqrt{3x}$ is obtained from the graph of $y=\sqrt{x}$ by compressing it against y-axis in the horizontal direction 3 times as shown in Fig. 2.38.

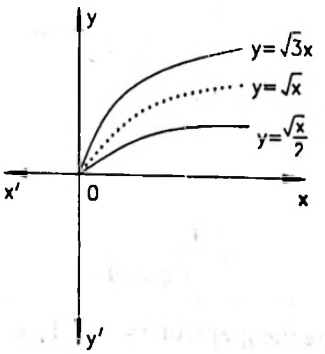


Fig. 2.38

(ii) The graph of $y = \sqrt{\frac{x}{2}}$ is obtained from the graph of $y = \sqrt{x}$ by stretching it 2 times in the horizontal direction from y-axis as shown in Fig. 2.38.

V REFLECTION ABOUT X-AXIS *The graph of the function $y = -f(x)$ can be obtained from the graph of $y = f(x)$ by reflecting it about the x-axis.*

In other words, to draw the graph of $y = -f(x)$, we first draw the graph of $y = f(x)$ and then its mirror image is taken in x-axis.

The above rule is illustrated by the following illustrations.

ILLUSTRATION 1 *Graph the following functions :*

(i) $y = -|x|$ (ii) $y = -e^x$

SOLUTION (i) In order to draw the graph of $y = -|x|$, we first draw the graph of $y = |x|$ and then we take its reflection about x-axis as shown in Fig. 2.39.

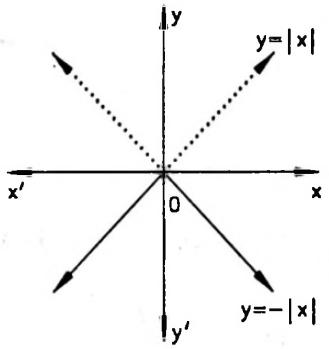


Fig. 2.39

(ii) The graph of $y = -e^x$ is obtained from the graph of $y = e^x$ by using the following steps :

STEP I Draw the graph of $y = e^x$

STEP II Take its reflection about x-axis.

The graph of $y = -e^x$ is shown in Fig. 2.40.

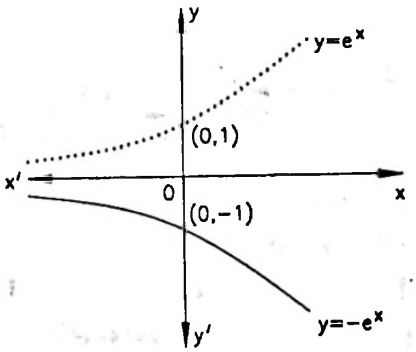


Fig. 2.40

ILLUSTRATION 2 *Graph the following functions :*

(i) $y = -\frac{1}{x}$ (ii) $y = -\frac{1}{x^2}$

SOLUTION (i) If we take $f(x) = \frac{1}{x}$, then $-f(x) = -\frac{1}{x}$. So, the graph of $y = -\frac{1}{x}$ is obtained from the graph of $y = \frac{1}{x}$ by taking its reflection about x-axis as shown in Fig. 2.41.

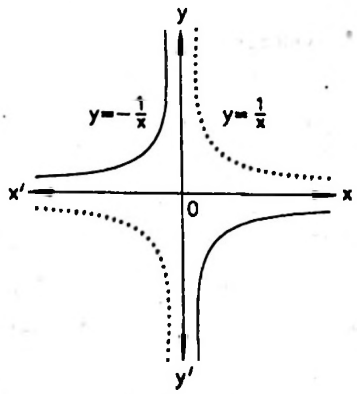


Fig. 2.41

(ii) The graph of $y = -\frac{1}{x^2}$ is obtained from the graph of $y = \frac{1}{x^2}$ by reflecting it about x-axis. Thus, to draw the graph of $y = -\frac{1}{x^2}$, we first draw the graph of $y = \frac{1}{x^2}$ and then we take its reflection about x-axis as shown in Fig. 2.42.

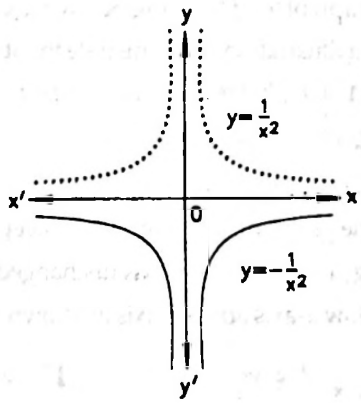


Fig. 2.42

VI REFLECTION ABOUT Y-AXIS The graph of the function $y = f(-x)$ can be obtained from that of $y = f(x)$ by reflecting the graph of $y = f(x)$ about y-axis.

In other words, to draw the graph of $y = f(-x)$, we first draw the graph of $y = f(x)$ and then its reflection is taken about y-axis.

NOTE *If $f(x)$ is an even function, then $y = f(x)$ and $y = f(-x)$ will represent the same curve.*

The following illustrations will illustrate the above rule.

ILLUSTRATION 1 *Draw the graphs of the following functions :*

(i) $y = \log_e(-x)$ (ii) $y = [-x]$

SOLUTION (i) In order to draw the graph of $y = \log_e(-x)$, we first draw the graph of $y = \log_e x$ as shown in Fig. 2.43 and then its reflection is taken about y-axis.

(ii) To obtain the graph of $y = [-x]$, we take the relfection of the graph of $y = [x]$ about y-axis as shown in Fig. 2.44. Note that $y = [x]$ is represented by dotted lines and $y = [-x]$ by continuous lines.

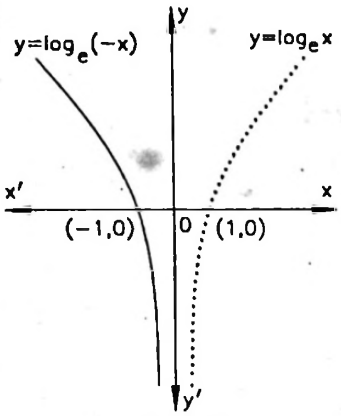


Fig. 2.43

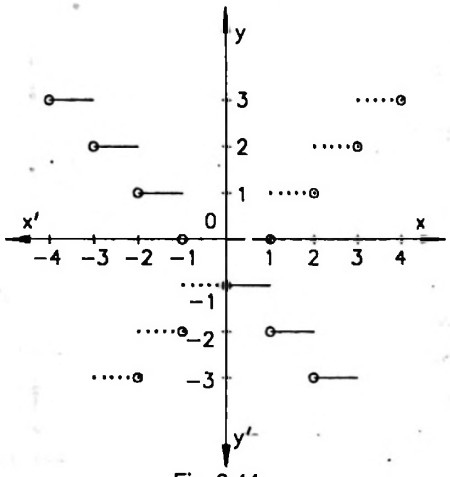


Fig. 2.44

VII GRAPH OF $y = |f(x)|$ The graph of $y = |f(x)|$ is constructed from the graph of $y = f(x)$ by performing the following steps :

STEP I *Draw the graph of $y = f(x)$.*

STEP II *To draw the graph of $y = |f(x)|$, keep the portion of the graph of $y = f(x)$ lying above the x-axis unchanged and reflect the other portion located below x-axis about x-axis.*

In other words, to obtain the graph of $y = |f(x)|$, reflect the portion of the graph of $y = f(x)$ lying below x-axis about x-axis.

The following illustrations will illustrate the above procedure.

ILLUSTRATION 1 *Graph the following functions :*

(i) $y = \left|\log_e x\right|$ (ii) $y = \left||x| - 1\right|$

SOLUTION (i) The graph of $y = \log_e x$ is shown in Fig. 2.45. In order to draw the graph of $y = |\log_e x|$ we keep the portion of the curve $y = \log_e x$ lying above x-axis unchanged and reflect the portion lying below x-axis about x-axis as shown in Fig. 2.46.

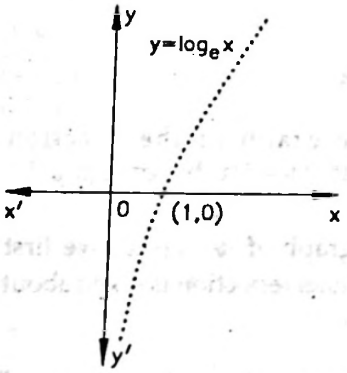


Fig. 2.45

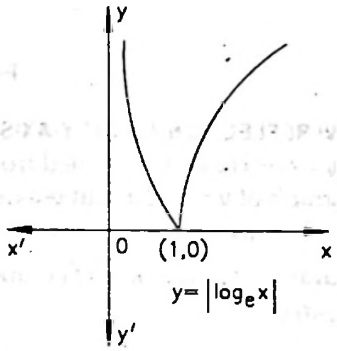


Fig. 2.46

(ii) In order to draw the graph of $y = \left||x| - 1\right|$, we follow the following steps :

STEP I Draw the graph of $y = |x|$ as shown in Fig. 2.47.

STEP II Translate the graph obtained in step I along OY' by 1 unit as shown in Fig. 2.48.

STEP III Now to draw the graph of $y = \left||x| - 1\right|$, we keep the portion lying above x-axis unchanged and take the reflection of the portion lying below x-axis about x-axis as shown in Fig. 2.49.

ILLUSTRATION 2 *Graph the following functions :*

(i) $y = |4 - x^2|$ (ii) $y = \dfrac{1}{|x|}$

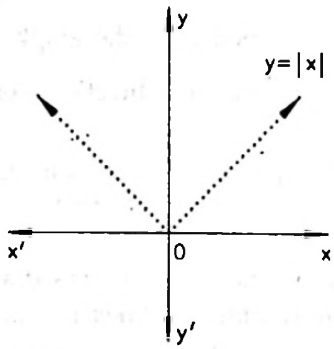


Fig. 2.47

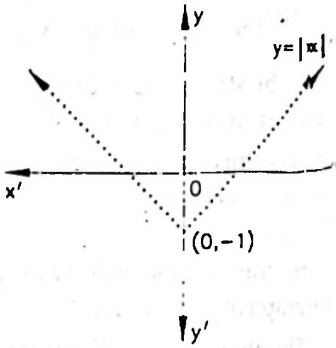


Fig. 2.48

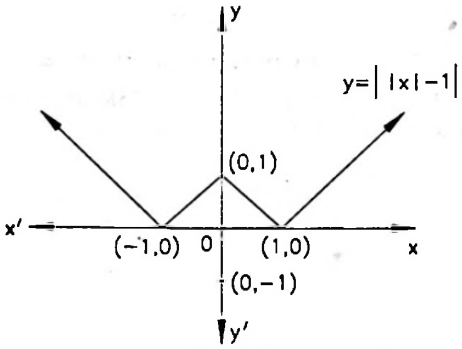


Fig. 2.49

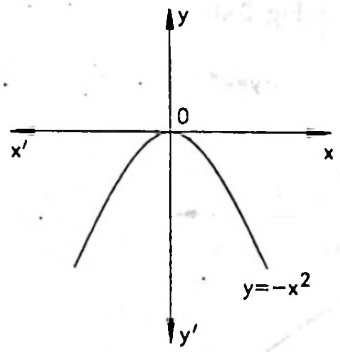
SOLUTION *(i) To draw the graph of $y = |4 - x^2|$, we follow th following steps :*

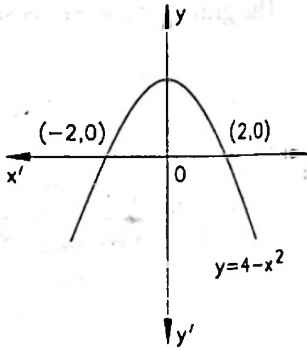
STEP I Draw the graph of $y = -x^2$.

STEP II Translate the graph of $y = -x^2$ by 4 units along OY' t obtain the graph of $y = 4 - x^2$.

STEP III Keep the portion of the graph of $y = 4 - x^2$ lying abov x-axis unchanged and reflect the portion below x-axis abou x-axis.

The graph of $y = |4 - x^2|$ is shown in Fig. 250 (iii).





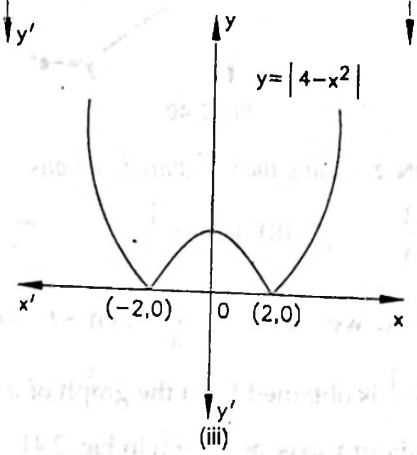


Fig. 2.50

(ii) To draw the graph of $y = \frac{1}{|x|}$, we follow the following steps :

STEP I Draw the graph of $y = \frac{1}{x}$ as shown in Fig. 2.51 (i)

STEP II Keeping the portion of the graph lying above x-axis unchanged, reflect the portion lying below x-axis about x-axis.

The graph of $y = \frac{1}{|x|}$ is shown in Fig. 2.51 (ii).

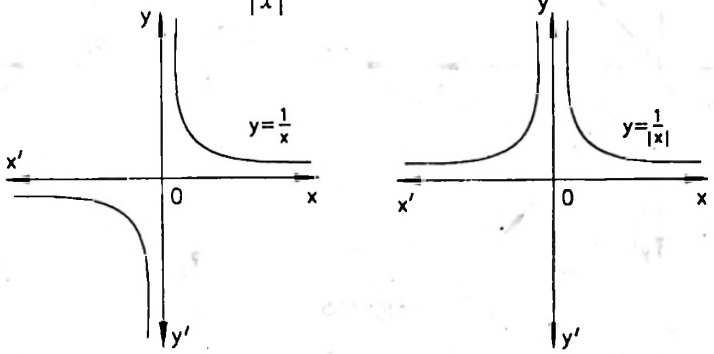


Fig. 2.51

GRAPH OF $y = f(|x|)$ The graph of $y = f(|x|)$ is obtained from the graph of the function $y = f(x)$ by using the following steps:

STEP I *Draw the graph of* $y = f(x)$ *for* $x \geq 0$.

STEP II *Reflect the graph obtained in step I about y-axis to obtain the graph of* $y = f(|x|)$ *for* $x < 0$.

Following illustrations will illustrate the above steps.

ILLUSTRATION 1 *Graph the following functions :*

(i) $y = x^2 - 2|x|$ (ii) $y = e^{-|x|}$

SOLUTION (i) We have,

$y = x^2 - 2|x|$

$\Rightarrow y = |x|^2 - 2|x|$

$\Rightarrow y = f(|x|)$, where $f(x) = x^2 - 2x$.

Thus, to draw the graph of $y = x^2 - 2|x|$, we follow the following steps :

STEP I Draw the graph of $y = x^2 - 2x$ for $x \geq 0$ as shown in Fig. 2.52 (i)

STEP II Reflect the graph obtained in step I about y-axis to obtain the graph of $y = x^2 - 2x$ for $x < 0$ as shown in Fig. 2.52 (ii).

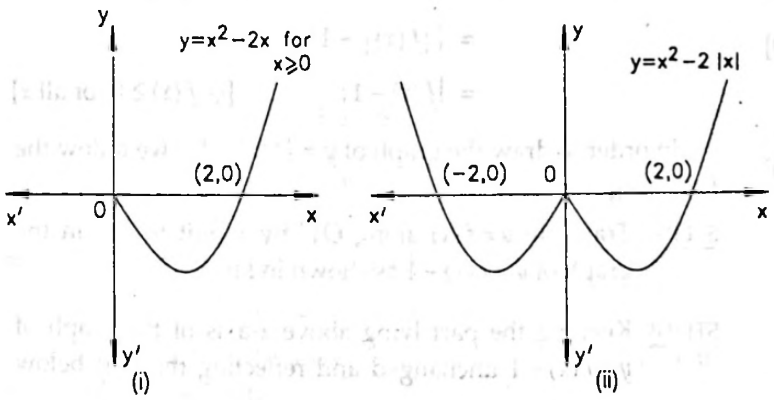


Fig. 2.52

(ii) To draw the graph of $y = e^{-|x|}$ the following steps are used :

STEP I Draw the graph of $y = e^{-|x|}$ for $x \geq 0$ as shown in Fig. 2.53 (i). This gives the graph of $y = e^{-|x|}$ for $x \geq 0$.

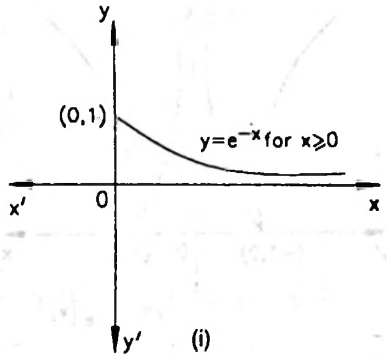


Fig. 2.53

STEP II Reflect the graph obtained in step I about y-axis to obtain the graph of $y = e^{-|x|}$ for $x < 0$. The graph of $y = e^{-|x|}$ is shown in Fig. 2.53 (ii).

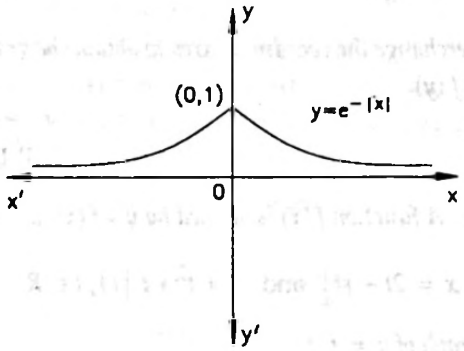


Fig. 2.53 (ii)

ILLUSTRATION 2 *Graph the function* $y = \left|\log_e |x|\right|$

SOLUTION In order to draw the graph of $y = \left|\log_e |x|\right|$, we follow the following steps :

STEP I Draw the graph of $y = \log_e x$ for $x > 0$ to obtain the graph of $f(x) = \log_e |x|$ for $x > 0$.

STEP II Reflect the graph obtained in step I about y-axis to obtain the graph of $y = \log_e |x|$ for $x < 0$.

The graph of $y = \log_e |x|$ is shown in Fig. 2.54 (i).

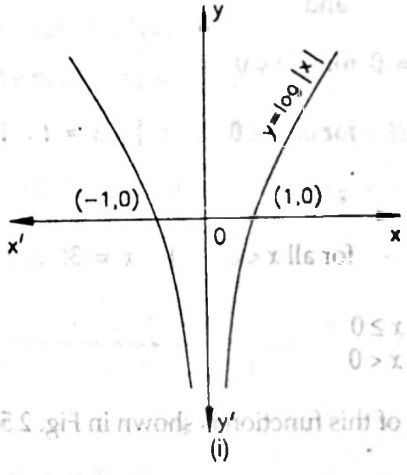


Fig. 2.54

STEP III Keeping the portion lying above x-axis of the graph shown in Fig. 2.54 (i) unchanged reflect the portion lying below x-axis about x-axis. This will give the graph of $y = \left|\log_e |x|\right|$ as shown in Fig. 2.54 (ii).

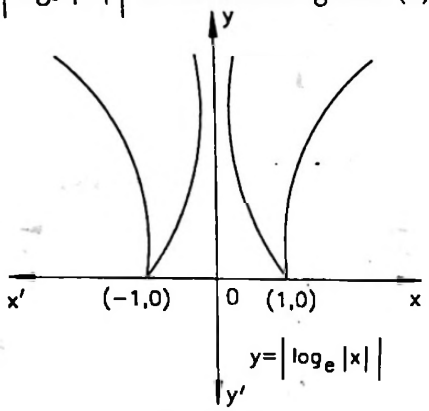


Fig. 2.54 (ii)

VIII GRAPH OF $x = f(y)$ In order to draw the graph of $x = f(y)$, we follow the following steps :

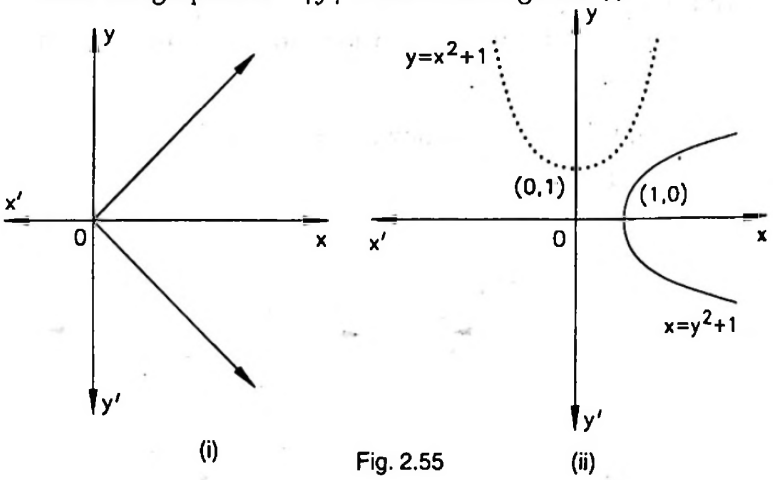
STEP I *Draw the graph of* $y = f(x)$.

STEP II *Interchange the coordinate axes to obtain the graph of* $x = f(y)$.

ILLUSTRATION 1 *Graph the following functions :*

(i) $x = |y|$ (ii) $x = y^2 + 1$

SOLUTION (i) In order to draw the graph of $x = |y|$, we first draw the graph of $y = |x|$ and then interchange the coordiante axes. The graph of $x = |y|$ is shown in Fig. 2.55 (i).



(i) Fig. 2.55 (ii)

(ii) To draw the graph of $x = y^2 + 1$, we first draw the graph of $y = x^2 + 1$ and then we interchange the coordinate axes to obtain the required graph as shown in Fig. 2.55 (ii).

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *A function $f(x)$ is defined by $y = f(x)$, where*

$$x = 2t - |t| \text{ and } y = t^2 + t|t|, t \in R$$

Draw the graph of $y = f(x)$.

SOLUTION We have,

$$x = 2t - |t| \text{ and } y = t^2 + t|t|, t \in R$$

$$\Rightarrow x = 2t - t, y = t^2 + t^2 \quad \text{for all } t \geq 0$$

and

$$x = 2t + t, y = t^2 - t^2 \quad \text{for all } t < 0$$

$$\Rightarrow x = t, y = 2t^2 \quad \text{for all } t \geq 0$$

and

$$x = 3t, y = 0 \quad \text{for all } t < 0$$

$$\Rightarrow y = 2x^2 \quad \text{for all } x \geq 0 \qquad [\because x = t \therefore t \geq 0 \Rightarrow x \geq 0]$$

and

$$y = 0 \quad \text{for all } x < 0 \qquad [\because x = 3t \therefore t < 0 \Rightarrow x < 0]$$

$$\Rightarrow y = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The graph of this function is shown in Fig. 2.56.

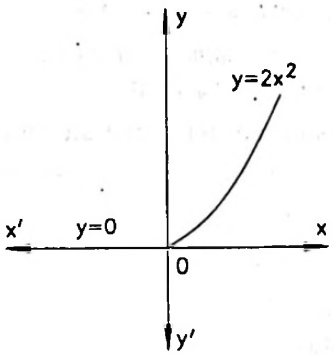


Fig. 2.56

EXAMPLE 2 *Construct the graph of $f(x) = \left||x| - 1\right|$. Also, draw the graph of $fof(x)$.*

SOLUTION The graph of $f(x) = ||x| - 1|$ is shown in Fig. 2.57. It is evident from the graph that $f(x) \geq 0$ for all $x \in R$.

Now,

$$y = fof(x) = f(f(x))$$
$$= ||f(x)| - 1|$$
$$= |f(x) - 1| \qquad [\because f(x) \geq 0 \text{ for all } x]$$

In order to draw the graph of $y = |f(x) - 1|$, we follow the following steps :

STEP I Translate $y = f(x)$ along OY' by 1 unit to obtain the graph of $y = f(x) - 1$ as shown in Fig. 2.57.

STEP II Keeping the part lying above x-axis of the graph of $y = f(x) - 1$ unchanged and reflecting the part below

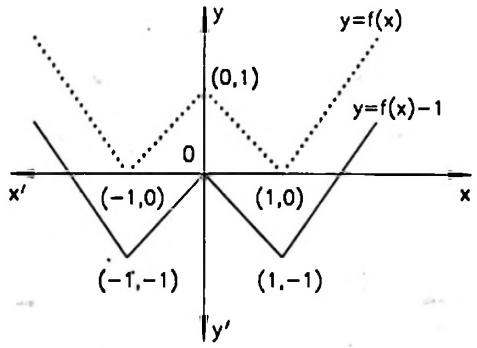


Fig. 2.57

x-axis about x-axis, we obtain the graph of $y = |f(x) - 1|$ i.e. $y = fof(x)$ as shown in Fig. 2.58.

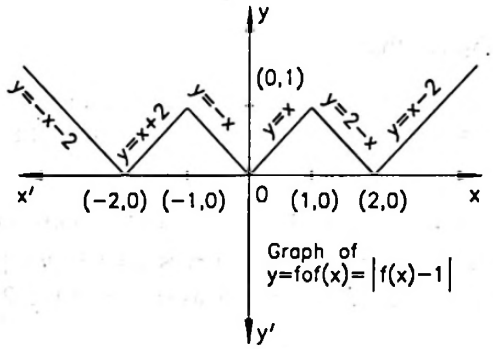


Fig. 2.58

EXAMPLE 3 *Draw the graph of* $f(x) = x - [x]$ *for all* $x \in R$.

SOLUTION In the previous chapter we have learnt that $f(x) = x - [x]$ is a periodic function with period 1 unit. So, it is sufficient to draw the graph of $f(x) = x - [x]$ in $[0, 1]$ only.

We have,

$$f(x) = \begin{cases} x - [x] = x & , \text{ for all } x \in [0, 1) \\ 1 - 1 = 0 & , \text{ for } x = 1 \end{cases}$$

Using periodicity of the function, we obtain the graph of $y = f(x)$ as shown in Fig. 2.59.

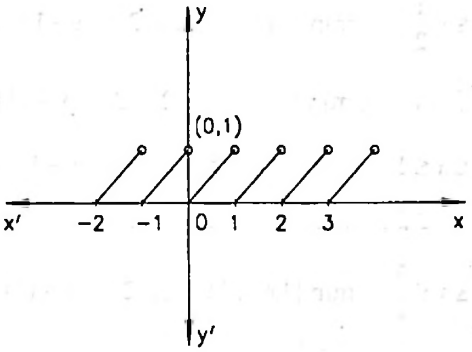


Fig. 2.59

EXAMPLE 4 *If* $f(x) = x^3 - x$ *for all* $x \in R$, *then draw the graph of* $y = f(x)$.

SOLUTION We have, $y = f(x) = x^3 - x$

$$\Rightarrow \quad y = x(x-1)(x+1)$$

We observe that the signs of the values of y for different values of x are as under :

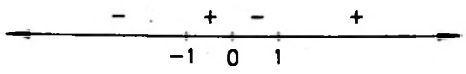


Fig. 2.60

Also,

$$y = x^3 - x$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 1$$

Now,

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

Clearly, $\left(\frac{d^2y}{dx^2}\right)_{x = \frac{1}{\sqrt{3}}} > 0$ and $\left(\frac{d^2y}{dx^2}\right)_{x = -\frac{1}{\sqrt{3}}} < 0$

So, $x = \frac{1}{\sqrt{3}}$ and $x = -\frac{1}{\sqrt{3}}$ are points of local minimum and local maximum respectively.

Clearly, the values of y increase for all $x > 1$. At $x = 1$, $y = 0$ and for all $x \in (0, 1)$, y is negative. At $x = 0$, y becomes zero and remains positive for all $x \in (-1, 0)$. The values of y are negative for all $x < -1$ and decrease with the decrease in the values of x. Thus, the graph of $y = x^3 - x$ is as given in Fig. 2.61.

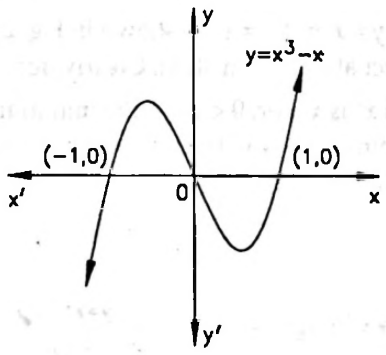


Fig. 2.61

EXAMPLE 5 *If* $f(x) = x^3 - 6x^2 + 11x - 6$ *for all* $x \in R$ *and*

$$g(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases}$$

Then draw the graph of $gof(x)$.

SOLUTION We have, $f(x) = x^3 - 6x^2 + 11x - 6$

$$= (x-1)(x-2)(x-3)$$

The signs of $f(x)$ for different values of x are shown in Fig. 2.62.

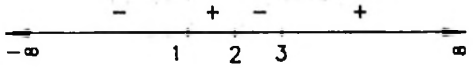


Fig. 2.62

Now, $gof(x) = g(f(x))$

$$= \begin{cases} 1 & , \text{if } f(x) > 0 \\ 0 & , \text{if } f(x) = 0 \\ -1 & , \text{if } f(x) < 0 \end{cases}$$

$$= \begin{cases} 1 & , \text{if } x \in (1,2) \cup (3,\infty) \\ 0 & , \text{if } x = 1, 2, 3 \\ -1 & , \text{if } x \in (-\infty, 1) \cup (2,3) \end{cases}$$

The graph of $gof(x)$ is as shown in Fig. 2.63.

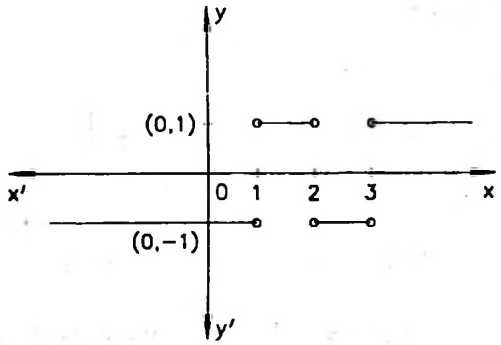


Fig. 2.63

EXAMPLE 6 *Graph the following functions :*

(i) $f(x) = min\{x, x^2\}$

(ii) $f(x) = max\{1-x, 1+x, 2\}$

(iii) $f(x) = min\{|x|, |x-2|, 2-|x-1|\}$

SOLUTION (i) In order to draw the graph of $f(x)$, we first draw the graphs of $y = x$ and $y = x^2$ as shown in Fig. 2.64. These two curves intersect at (0, 0) and (1, 1). Clearly, for $x < 0$, the minimum of x and x^2 is x. For, $0 < x < 1$, the minimum is x^2 and for all $x > 1$, the minimum is x. Thus, the graph of $f(x)$ is the thick lined curve shown in Fig. 2.64.

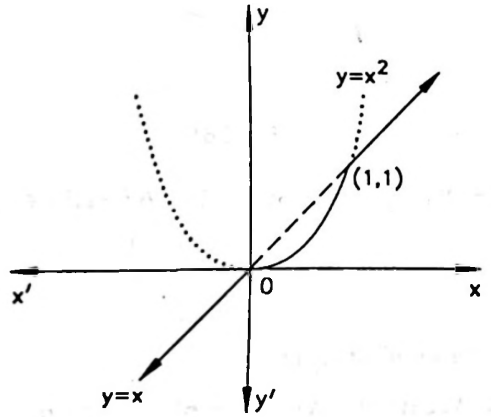


Fig. 2.64

It follows from the graph that

$$f(x) = \begin{cases} x & , x \le 0 \\ x^2 & , 0 \le x \le 1 \\ x & , x \ge 1 \end{cases}$$

(ii) First we draw the graphs of $y = 1-x$, $y = 1+x$ and $y = 2$ on the same scale as shown in Fig. 2.65.

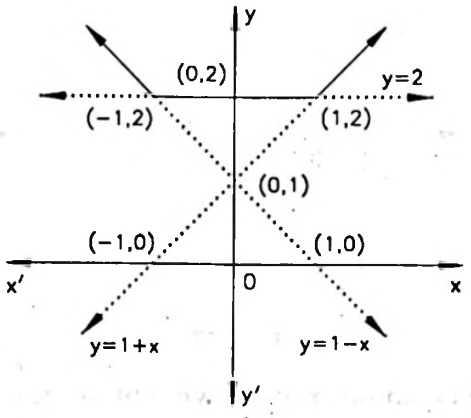


Fig. 2.65

We observe that

$$f(x) = \max\{1-x, 1+x, 2\} = \begin{cases} 1-x & , \text{for } x \le -1 \\ 2 & , \text{for } -1 \le x \le 1 \\ 1+x & , \text{for } x \ge 1 \end{cases}$$

The graph of $f(x)$ is the thick lined curve shown in Fig. 2.65

(iii) Let us first draw the curves $y = |x|$, $y = |x-2|$ and $y = 2 - |x-1|$ on the same scale as shown in Fig. 2.66.

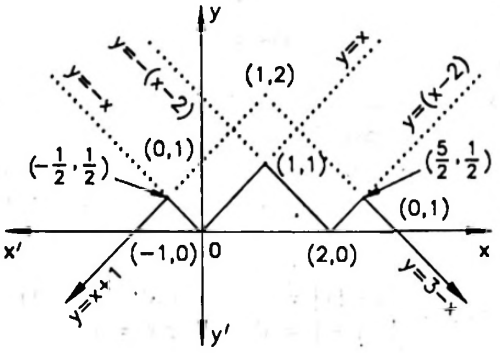


Fig. 2.66

We observe that :

For $x \le -\frac{1}{2}$, $\min\{|x|, |x-2|, 2-|x-1|\} = x+1$

For $-\frac{1}{2} \le x \le 0$, $\min\{|x|, |x-2|, 2-|x-1|\} = -x$

For $0 \le x \le 1$, $\min\{|x|, |x-2|, 2-|x-1|\} = x$

For $1 \le x \le 2$, $\min\{|x|, |x-2|, 2-|x-1|\} = -(x-2)$

For $2 \le x \le \frac{5}{2}$, $\min\{|x|, |x-2|, 2-|x-1|\} = x-2$

For $\frac{5}{2} \le x \le 3$, $\min\{|x|, |x-2|, 2-|x-1|\} = 3-x$

The graph of $f(x)$ is the thick-lined curve shown in Fig. 2.66

EXAMPLE 7 *Construct the graph of the function*

$$y = f(x-1) + f(x+1), \text{ where}$$

$$f(x) = \begin{cases} 1-|x| & , \text{for } |x| \le 1 \\ 0 & , \text{for } |x| > 1 \end{cases}$$

SOLUTION In order to construct the graph of $y = f(x-1) + f(x+1)$, we follow the following steps:

STEP I First we draw the graph of $f(x)$ as shown in Fig. 2.67. For this we first draw the graph of $y = -|x|$ and then it is translated vertically upward by 1 unit. For $|x| > 1$, the values of y are zero.

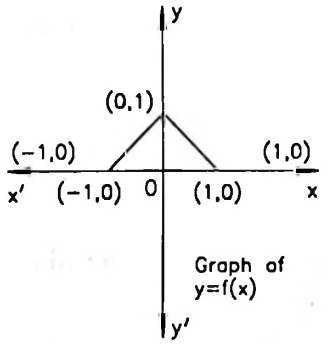


Fig. 2.67

STEP II In this step, we draw the graph of $y = f(x-1)$. For this the graph of $y = f(x)$ is translated along OX by 1 unit as shown in Fig. 2.68.

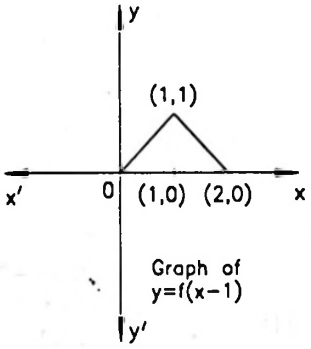


Fig. 2.68

STEP III In this step, we draw the graph of $y = f(x+1)$ by translating the graph of $y = f(x)$ along OX' by 1 unit as shown in Fig. 2.69.

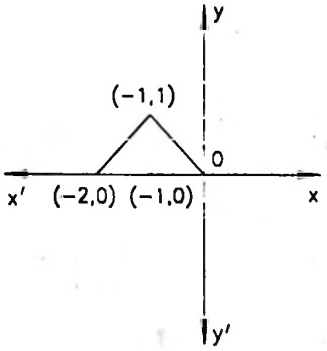


Fig. 2.69

STEP IV In this step, we draw the graph of $y = f(x-1) + f(x+1)$ by adding the graphs in Figs. 2.68 and 2.69.

We observe that

$f(x+1) = 0$ and $f(x-1) = 0$ for all x satisfying $|x| \geq 2$

$\therefore\ y = f(x-1) + f(x+1) = 0$ for all x satisfying $|x| \geq 2$.

For $-2 < x < 0$, we have

$f(x-1) = 0$ and $f(x+1) \neq 0$

$\therefore\ y = f(x-1) + f(x+1) = f(x+1)$

For $0 < x < 2$, we have

$f(x-1) \neq 0$ and $f(x+1) = 0$

$\therefore\ y = f(x-1) + f(x+1) = f(x-1)$

Thus, the graph of $y = f(x-1) + f(x+1)$ is shown in Fig. 2.70.

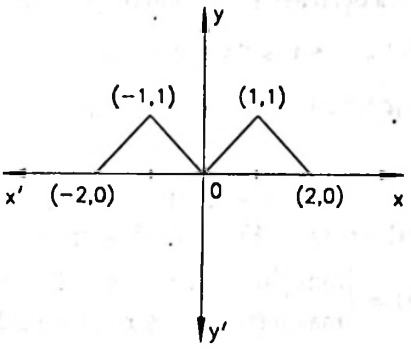


Fig. 2.70

EXAMPLE 8 *Let* $f(x) = \begin{cases} x - [x], & x \notin I \\ 1, & x \in I \end{cases}$ *where I is the set of integers and $[x]$ represents the greatest integer less than or equal to x. If*

$$g(x) = \lim_{n \to \infty} \frac{\{f(x)\}^{2n} - 1}{\{f(x)\}^{2n} + 1}$$

then draw the graphs of $g(x)$ and $gog(x)$.

SOLUTION Since $0 < x - [x] < 1$ for all $x \notin I$. Therefore,

$$\lim_{n \to \infty} \{f(x)\}^{2n} = 0 \text{ for all } x \notin I.$$

Also, $f(x) = 1$ for all $x \in I$.

Thus,

$$g(x) = \begin{cases} \dfrac{0-1}{0+1} = -1, & \text{for all } x \notin I \\ \dfrac{1-1}{1+1} = 0, & \text{for all } x \in I. \end{cases}$$

The graph of $g(x)$ is shown in Fig. 2.71.

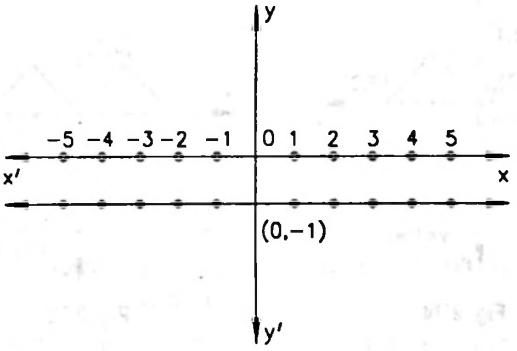


Fig. 2.71

Now, .

$$gog(x) = g\{g(x)\} = \begin{cases} g(-1) = 0, & \text{for all } x \notin I \\ g(0) = 0, & \text{for all } x \in I \end{cases}$$

$= 0$ for all $x \in R$.

Clearly, graph of $y = gog(x)$ is x-axis itself.

AN IMPORTANT RESULT Let $f(x)$ be an increasing function defined on $[a, b]$. Then,

$$\max\{f(t) \mid a \le t \le x, a \le x \le b\} = f(x)$$

$$\min\{f(t) \mid a \le t \le x, a \le x \le b\} = f(a)$$

Also, if $f(x)$ is a decreasing function on $[a, b]$, then

$$\max\{f(t) \mid a \le t \le x, a \le x \le b\} = f(a)$$

$$\min\{f(t) \mid a \le t \le x, a \le x \le b\} = f(x)$$

EXAMPLE 9 *If* $f(x) = \begin{cases} |x+1|-1, -2 \le x \le 0 \\ 1-|x-1|, 0 < x \le 2 \end{cases}$, *then draw the graph of* $|f(x)| - f(|x|)$. *Also, draw the graph of* $g(x)$ *given by*

$$g(x) = \begin{cases} \min\{f(t) \mid -2 \le t \le x, -2 \le x \le 0\} \\ \max\{f(t) \mid 0 \le t \le x, 0 < x \le 2\} \end{cases}$$

SOLUTION In order to draw the graph of $f(x)$, we first draw the graphs of $y = |x+1| - 1$ and $y = 1 - |x-1|$ as shown in Figs. 2.72 and 2.73.

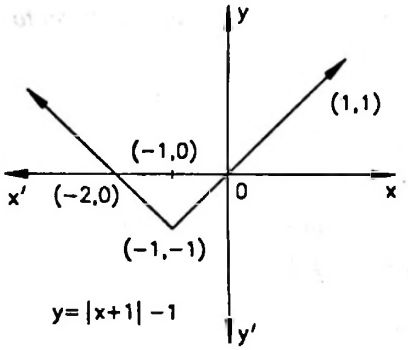


Fig. 2.72

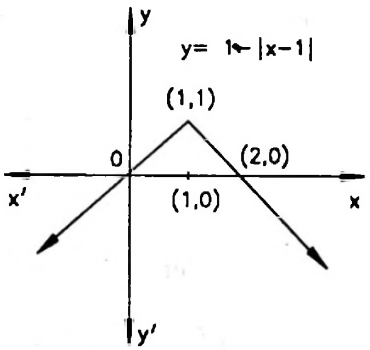


Fig. 2.73

In order to obtain the graph of $y = |f(x)|$ from the graph of $y = f(x)$ the portion of the graph of $y = f(x)$ lying below x-axis is reflected above x-axis as shown in Fig. 2.75.

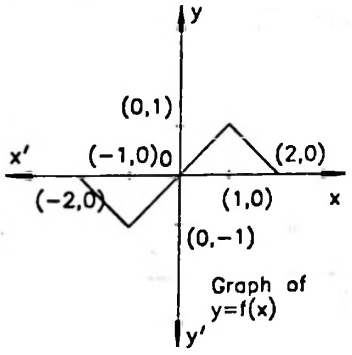


Fig. 2.74

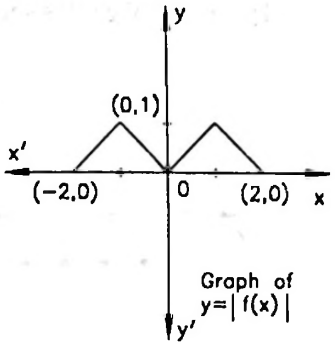


Fig. 2.75

To obtain the graph of $y = f(|x|)$, we follow the following steps :

I Take the graph of $y = f(x)$ for $x \ge 0$.

II Take its mirror-image in y-axis.

The graph of $y = f(|x|)$ is shown in Fig. 2.76.

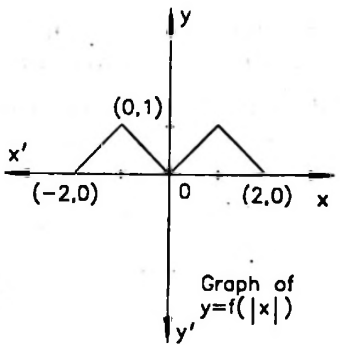


Fig. 2.76

Clearly, graphs of $y = |f(x)|$ and $y = f(|x|)$ are exactly identical.

So, $y = |f(x)| - f(|x|) = 0$ for all x

Hence, the graph of $y = |f(x)| - f(|x|)$ is x-axis itself.

It is evident from the graph of $f(x)$ that :

$f(x)$ is decreasing on $[-2, -1]$

$f(x)$ is increasing on $[-1, 1]$,

and, $f(x)$ is decreasing on $[1, 2]$.

$$\min\left\{f(t) \mid -2 \le t \le x, -2 \le x \le 0\right\} = \begin{cases} f(x), -2 \le x \le -1 \\ -1, -1 \le x \le 0 \end{cases}$$

and,

$$\max\left\{f(t) \mid 0 \le t \le x, 0 < x \le 2\right\} = \begin{cases} f(x), 0 < x \le 1 \\ 1, 1 \le x \le 2 \end{cases}$$

Thus,

$$g(x) = \begin{cases} f(x), -2 \le x \le -1 \\ -1, -1 \le x \le 0 \\ f(x), 0 < x \le 1 \\ 1, 1 \le x \le 2 \end{cases}$$

So, the graph of $y = g(x)$ is as shown in Fig. 2.77.

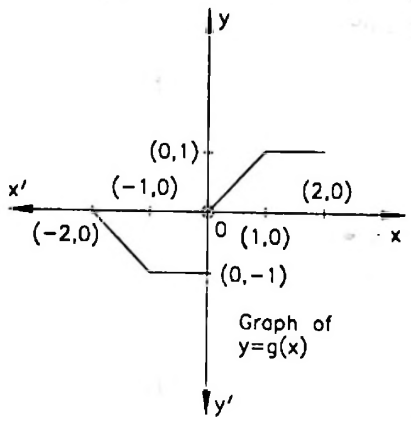


Fig. 2.77

EXAMPLE 10 *Draw rough sketches of the following curves :*

(i) $y = \sqrt{x - [x]}$

(ii) $y = [x] + \sqrt{x - [x]}$

(iii) $y = \left|[x] + \sqrt{x - [x]}\right|$

SOLUTION (i) We know that $0 \le x - [x] < 1$ for all $x \in R$. Also, for any $x \in [0, 1]$, we have

$$x^2 \le x \le \sqrt{x}$$

$$\therefore \quad x - [x] \le \sqrt{x - [x]}$$

Now,

$$y = \sqrt{x - [x]} = \begin{cases} \sqrt{x+1}, & \text{for } -1 \le x < 0 \\ \sqrt{x}, & \text{for } 0 \le x < 1 \\ \sqrt{x-1}, & \text{for } 1 \le x < 2 \text{ and so on.} \end{cases}$$

In general, $y = \sqrt{x - [x]} = \sqrt{x - k}$ for $k \le x < k + 1$, where $k \in Z$.

Thus,

Between 0 and 1, the graph of $y = \sqrt{x - [x]}$ is same as that of $y = \sqrt{x}$.

Between 1 and 2, the graph of $y = \sqrt{x - [x]}$ is same as that of $y = \sqrt{x - 1}$ and is obtained by shifting the graph of $y = \sqrt{x}$ by 1 unit on the right side.

Continuing in this manner, we obtain the graph as shown in Fig. 2.78.

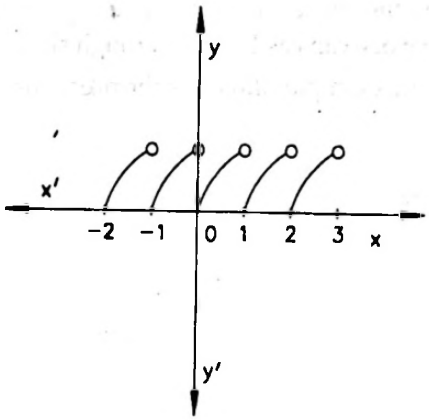


Fig. 2.78

(ii) We have, $y = [x] + \sqrt{x - [x]}$

$$\Rightarrow \qquad y = k + \sqrt{x - k} \quad \text{for } k \le x < k + 1, k \in Z.$$

Thus, the graph of $y = k + \sqrt{x - k}$ is obtained from the graph of $y = \sqrt{x - k}$ by translating it by k units in upward or downward directions according as $k > 0$ or $k < 0$.

Hence, the graph of $y = [x] + \sqrt{x - [x]}$ is as shown in Fig. 2.79.

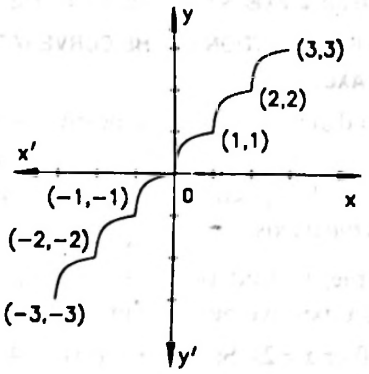


Fig. 2.79

(iii) The graph of $y = \left|[x] + \sqrt{x - [x]}\right|$ is obtained by reflecting the portion lying below x-axis of the graph of $y = [x] + \sqrt{x - [x]}$. Thus the graph of $y = \left|[x] + \sqrt{x - [x]}\right|$ is shown in Fig. 2.80.

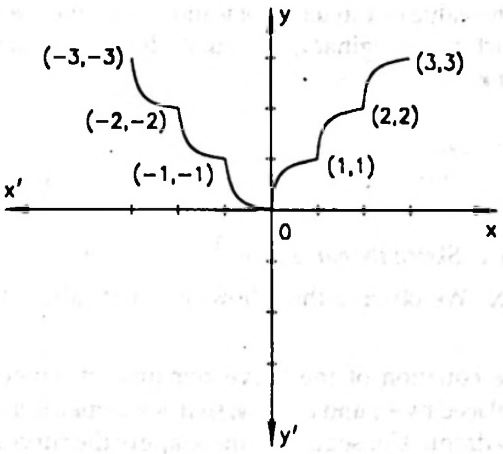


Fig. 2.80

—❑❑❑—

2.5 SKETCHING OF CURVES USING CALCULUS

Following points are very helpful to draw a rough sketch of a curve.

I SYMMETRY

(i) *Symmetry about x-axis* : If all powers of y in the equation of the given curve are even, then it is symmetric about x-axis i.e., the shape of the curve above x-axis is exactly identical to its shape below x-axis.

For example, $y^2 = 4ax$ is symmetric about x-axis.

(ii) *Symmetry about y-axis* : If all powers of x in the equation of the given curve are even, then it is symmetric about y-axis.

For example, $x^2 = 4ay$ is symmetric about y-axis.

(iii) *Symmetry in opposite quadrants* : If by putting $-x$ for x and $-y$ for y, the equation of a curve remains same, then it is symmetric in opposite quadrants.

For example, $x^2 + y^2 = a^2$ and $xy = a^2$ are symmetric in opposite quadrants.

(iv) *Symmetry about the line* $y = x$: If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45° with the positive direction of x-axis.

II ORIGIN AND TANGENTS AT THE ORIGIN

See whether the curve passes through the origin or not. If the

point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such a case to find the equation(s) of the tangent(s) at the origin, equate the lowest degree term to zero.

For example, $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$. So, $x = 0$ i.e. y-axis is tangent at the origin to $y^2 = 4ax$.

III POINTS OF INTERSECTION OF THE CURVE WITH THE COORDINATE AXES

By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the x-axis. Similarly, by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the y-axis.

For example, to find points where the curve $xy^2 = 4a^2(2a - x)$ meets x-axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or $x = 2a$. So the curve $xy^2 = 4a^2(2a - x)$, meets x-axis at $(2a, 0)$. This curve does not intersect y-axis, because by putting $x = 0$ in the equation of the given curve get an absurd result.

IV REGIONS WHERE THE CURVE DOES NOT EXIST

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the values of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

For example, the values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of y-axis. Similarly, the curve $a^2y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

V SPECIAL POINTS

Find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to x-axis.

Find the points at which $dx/dy = 0$. At these points the tangent to the curve is parallel to y-axis.

VI SIGN OF $\frac{dy}{dx}$ AND POINTS OF MAXIMA AND MINIMA

Find the interval in which $dy/dx > 0$. In this interval, the function is monotonically increasing.

Find the interval in which $dy/dx < 0$. In this interval, the function is monotonically decreasing.

Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the points so obtained to find the points of maxima or minima.

Keeping the above facts in mind and plotting some points on the curve one can easily draw a rough sketch of the curve.

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Sketch the curve* $y = x^3$.

SOLUTION We observe the following points about the given curve.

(i) The equation of the curve remains unchanged if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants. Consequently the shape of the curve is similar in the first and the third quadrants.

(ii) The curve passes through the origin. Equating the lowest degree term y to zero, we get $y = 0$ i.e. x-axis is the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x = 0$. Similarly, when $x = 0$, we get $y = 0$. So, the curve meets the coordinate axes at $(0, 0)$ only.

(iv) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$ and $\frac{d^3y}{dx^3} = 6$.

Clearly, $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$ at the origin but $\frac{d^3y}{dx^3} = 6 \neq 0$. So, origin is a point of inflexion.

(v) As x increases from 0 to ∞, y also increases from 0 to ∞. Keeping all the above points in mind, we obtain a sketch of the curve as shown in Fig. 2.81.

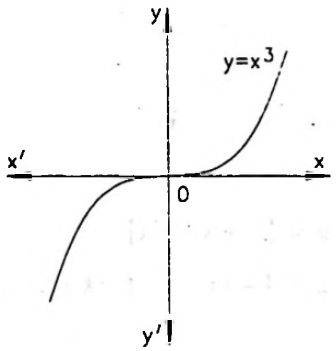


Fig. 2.81

EXAMPLE 2 *Sketch the curve* $y = x^3 - 4x$.

SOLUTION We make the following observations about the given curve.

(i) The equation of the curve remains same if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants. Consequently, the curve in first quadrant is identical to the curve in third quadrant and the curve in second quadrant is similar to the curve in fourth quadrant.

(ii) The curve passes through the origin. Equating the lowest degree term $y + 4x$ to zero, we get $y + 4x = 0$ or $y = -4x$. So, $y = -4x$ is tangent to the curve at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $x^3 - 4x = 0 \Rightarrow x = 0, \pm 2$. So, the curve meets x-axis at (0, 0) (2, 0) and (–2, 0).
Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve meets y-axis at (0, 0) only.

(iv) $y = x^3 - 4x \Rightarrow \frac{dy}{dx} = 3x^2 - 4$

Now,

$$\frac{dy}{dx} > 0 \Rightarrow 3x^2 - 4 > 0$$

$$\Rightarrow x^2 - \frac{4}{3} > 0$$

$$\Rightarrow \left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) > 0$$

$$\Rightarrow x < -\frac{2}{\sqrt{3}} \text{ or } x > \frac{2}{\sqrt{3}}$$

And,

$$\frac{dy}{dx} < 0 \Rightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$$

So, the curve is decreasing in the interval $(-2/\sqrt{3}, 2/\sqrt{3})$ and increasing for $x > 2/\sqrt{3}$ or $x < -2/\sqrt{3}$. $x = -2/\sqrt{3}$ is a point of local maximum and $x = 2/\sqrt{3}$, is a point of local minimum. When $x = 2/\sqrt{3}$, $y = 8/3\sqrt{3} - 8/\sqrt{3} = -16/3\sqrt{3}$. When $x = -2/\sqrt{3}$, $y = 16/3\sqrt{3}$. Keeping the above points in mind, we sketch the curve as shown in Fig. 2.82.

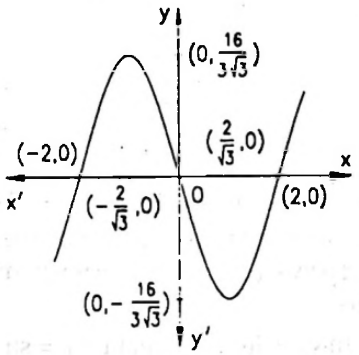


Fig. 2.82

EXAMPLE 3 *Sketch the curve* $y = (x - 1)(x - 2)(x - 3)$.

SOLUTION We note the following points about the given curve.

(i) The curve does not have any type of symmetry about the coordinate axes and also in the opposite quadrants.

(ii) The curve does not pass through the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $(x - 1)(x - 2)(x - 3) = 0 \Rightarrow x = 1, 2, 3$. So, the curve meets x-axis at (1, 0), (2, 0) and (3, 0).

Putting $x = 0$ in the equation of the curve, we get $y = -6$. So, the curve crosses y-axis at (0, – 6).

We observe that

$$x < 1 \Rightarrow y < 0$$
$$1 < x < 2 \Rightarrow y > 0$$
$$2 < x < 3 \Rightarrow y < 0$$

and, $$x > 3 \Rightarrow y > 0.$$

Clearly, y decreases as x decreases for all $x < 1$ and y increases as x increases for $x > 3$.

Keeping all the above points in mind, we sketch the curve as shown in Fig. 2.83

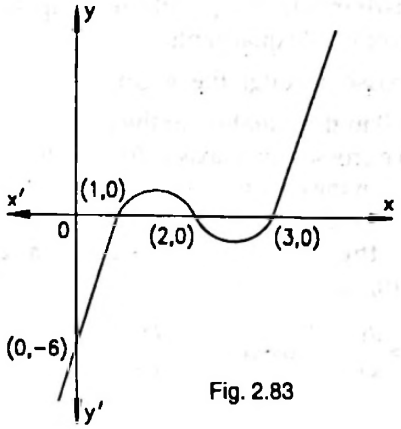


Fig. 2.83

EXAMPLE 4 *Sketch the curve* $y = x^2 - x$.

SOLUTION We note the following points about the curve.

(i) The curve does not have any kind of symmetry.

(ii) The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero. The lowest degree term is $x + y$. Equating it to zero, we get $x + y = 0$ as the equation of the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, 1$. So, the curve crosses x-axis at (0, 0) and (1, 0).
Putting $x = 0$, in the equation of the curve, we obtain $y = 0$. So, the curve meets y-axis at (0, 0) only.

(iv) $y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$ and $\frac{d^2y}{dx^2} = 2$.

Now, $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$. At $x = \frac{1}{2}$, we have $\frac{d^2y}{dx^2} = 2 > 0$.

So, $x = 1/2$ is a point of local minima.

(v) $\frac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > 1/2$

So, the curve is increasing for all $x > 1/2$ and decreasing for all $x < 1/2$.

Keeping the above points in mind, we obtain the sketch of the curve as shown in Fig. 2.84.

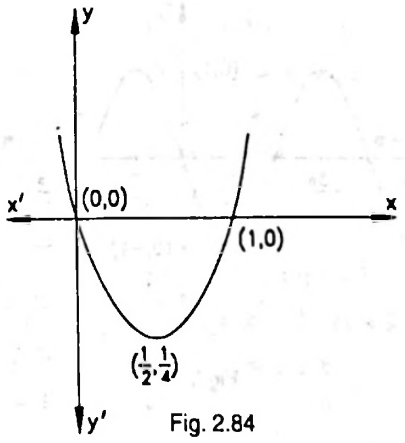


Fig. 2.84

EXAMPLE 5 *Sketch the curve* $y = \sin 2x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains unchanged if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in opposite quadrants.

(ii) The curve passes through the origin.

(iii) Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve crosses the y-axis at $(0, 0)$ only.
Putting $y = 0$ in the equation of the curve, we get $\sin 2x = 0 \Rightarrow 2x = n\pi, n \in Z \Rightarrow x = n\pi/2, n \in Z$. So, the curve cuts x-axis at the points ..., $(-\pi, 0)$ $(-\pi/2, 0)$ $(0, 0)$, $(\pi/2, 0)$,$(\pi, 0)$, ...

(iv) $y = \sin 2x \Rightarrow \dfrac{dy}{dx} = 2\cos 2x$ and $\dfrac{d^2y}{dx^2} = -4\sin 2x$

Now,

$$\frac{dy}{dx} = 0 \Rightarrow 2\cos 2x = 0$$

$$\Rightarrow \cos 2x = 0$$

$$\Rightarrow 2x = \pm\pi/2, \pm 3\pi/2, \ldots$$

$$\Rightarrow x = \pm\pi/4, \pm 3\pi/4, \ldots$$

Clearly, $\dfrac{d^2y}{dx} < 0$ at $x = \pi/4, 5\pi/4 \ldots$ and at $x = -3\pi/4, -7\pi/4, \ldots$

And,

$\dfrac{d^2y}{dx^2} > 0$ at $x = 3\pi/4, 7\pi/4, \ldots$ and at $x = -\pi/4, -5\pi/4 \ldots$

So, the points $x = \pi/4, 5\pi/4, 7\pi/4$, and $x = -3\pi/4, -7\pi/4, \ldots$ are the points of local maximum and local maximum value at these points is 1. Similarly the points $x = -\pi/4, -5\pi/4 \ldots$ and $x = 3\pi/4, 7\pi/4, \ldots$ are points of local minimum and local minimum value at these points is -1.

(v) $\sin 2(x + \pi) = \sin 2x$ for all x. So, the periodicity of the function is π. This means that the pattern of the curve repeats at intervals of length π.

Thus, keeping the above ideas in mind, we sketch the curve $y = \sin 2x$ as shown in Fig. 2.85.

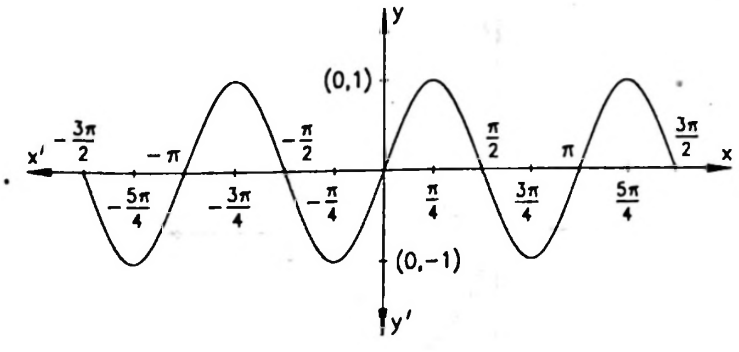


Fig. 2.85

EXAMPLE 6 *Sketch the curve* $y = 2\sin 2x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains same if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants.

(ii) The curve passes through the origin.

(iii) The curve meets the coordinate axes at the same points where $y = \sin 2x$ meets them.

(iv) $\dfrac{dy}{dx} = 0$ gives the same points as given by $\dfrac{d}{dx}(\sin 2x) = 0$ and the signs of $\dfrac{d^2}{dx^2}(2\sin 2x)$ and $\dfrac{d^2}{dx^2}(\sin 2x)$ are same for any value of x. This means that the points of local maxima or minima of $y = 2\sin 2x$ are the same as the points of local maxima or minima of $y = \sin 2x$, but the maximum value will be 2 and minimum value will be -2.

Keeping the above ideas in mind, we obtain the rough sketch of the curve as shown in Fig. 2.86

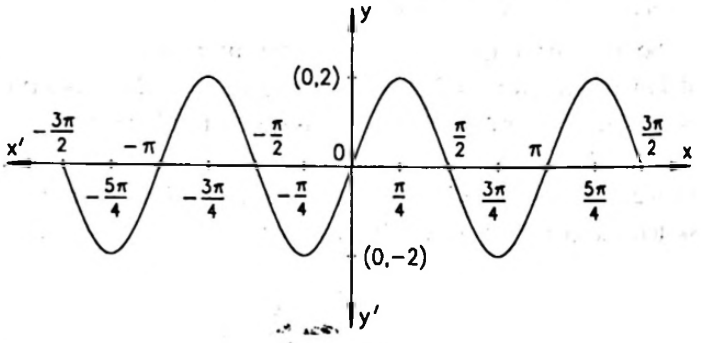


Fig. 2.86

EXAMPLE 7 *Sketch the curve* $y = \sin x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve does not alter if x is replaced by $-x$ and y by $-y$. So, the curve is symmetric in opposite quadrants.

(ii) Point $(0, 0)$ satisfies the equation $y = \sin x$. So, the curve passes through the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $\sin x = 0 \Rightarrow x = n\pi, n \in Z$. So, the curve meets the x-axis at ... $(3\pi, 0)$, $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, ... Putting $x = 0$ in the equation of the curve, we obtain $y = \sin 0 = 0$. So, the curve meets y-axis at $(0, 0)$ only.

(iv) $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x$ and $\dfrac{d^2y}{dx^2} = -\sin x$.

Now, $\dfrac{dy}{dx} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

Clearly, $\dfrac{d^2y}{dx^2} < 0$ at $x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}, \ldots$ and at $x = -\dfrac{3\pi}{2}, -\dfrac{7\pi}{2}, \ldots$

And,

$\dfrac{d^2y}{dx^2} > 0$ at $x = -\dfrac{\pi}{2}, -\dfrac{5\pi}{2}, -\dfrac{9\pi}{2}, \ldots$ and $x = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \ldots$

So, the points $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$, and $x = -\frac{3\pi}{2}, -\frac{7\pi}{2}, \ldots$ are the points of local maximum and the local maximum value at these points is 1. Similarly, the points $x = -\frac{\pi}{2}, -\frac{5\pi}{2}, \ldots$, and $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ are the points of local minimum and the local minimum value at these points is -1.

(v) $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$.

Clearly, $\frac{dy}{dx} > 0$ when $-\pi/2 < x < \pi/2$ and $\frac{dy}{dx} < 0$ when $\pi/2 < x < 3\pi/2$. So, the given curve is increasing for $-\pi/2 < x < \pi/2$ and decreasing for $\pi/2 < x < 3\pi/2$.

(vi) $\sin(2\pi + x) = \sin x$ for all x. So, the periodicity of the function is 2π. This means that the pattern of the curve repeats at the intervals of length 2π.

Keeping the above facts in mind, we may sketch the curve $y = \sin x$ as shown in Fig. 2.87.

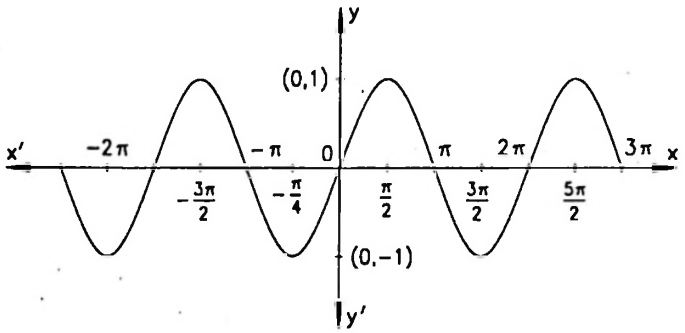


Fig. 2.87

EXAMPLE 8 *Sketch the curve* $y = \sin^2 x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains same if x is replaced by $-x$. So, the curve is symmetric about y-axis i.e. the curve on the left side of y-axis is identical to the curve on its right side.

(ii) The curve meets the coordinate axes at the same points where $y = \sin x$ meets them.

(iii) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$ and $\frac{d^2y}{dx^2} = 2\cos 2x$

Now,

$$\frac{dy}{dx} = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi,\ n \in Z \Rightarrow x = n\pi/2, n \in Z$$

$$\Rightarrow x = \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi, \ldots$$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

And, $\frac{d^2y}{dx^2} > 0$ at $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

So, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$ are the points of local maximum and the local maximum value at these points is 1. Points $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are points of local minimum and the local minimum value at these points is zero.

(iv) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$

Clearly, $\frac{dy}{dx} > 0$ when $0 < x < \pi/2$ and $\frac{dy}{dx} < 0$ when $\pi/2 < x < \pi$.

So, the given curve is increasing in the interval $[0, \pi/2]$ and decreasing in the interval $[\pi/2, \pi]$.

(v) Since $\sin^2(\pi + x) = \sin^2 x$ for all x. So, the periodicity of the function is π. This means that the shape of the curve repeats at the intervals of length 2π.

With these ideals, we may sketch the curve as shown in Fig. 2.88.

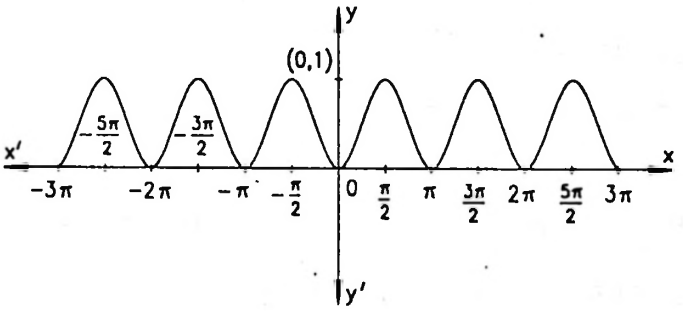


Fig. 2.88

EXERCISE 2.1

Sketch the graph of each of the following functions :

1. $y = x|x|$
2. $y = \sqrt{x-2}$
3. $y = \frac{1}{x-2}$
4. $y + 1 = \frac{1}{x-1}$
5. $y = \frac{1}{|x-3|}$
6. $y = |\sin x|$
7. $y = |\cos x|$
8. $y = |\tan x|$
9. $y = e^{|x|}$
10. $y = [x-3]$
11. $y = [3-x]$
12. $y = x^2 - 3|x|$
13. $y = |x^2 + x|$
14. $x = |y^2 + y|$
15. $y = 1 + |\sin x|$
16. $y = 1 + \sin x$
17. $y - 1 = x^2 + |x|$
18. $y = \log|x-1|$
19. $y = [3x]$
20. $y = \left[\frac{x}{2}\right]$
21. $y = \sin^2 x$
22. $y = \cos^2 x$
23. $y = |2\sin x - 1|$
24. $y = |\tan^{-1} x|$
25. $y = x^2 - 2|x| + 3$
26. $y = |16 - 9x^2|$
27. $|y| = \log x$
28. $|x| - |y| = 1$
29. $|y| - |x| = 1$

30. $|x|+2|y|=1$ 31. $y=\max\{x,x^2\}$

32. $y=\min\{x-[x],-x-[-x]\}$

33. $|x-1|=y^2+1$

34. $|y|=\dfrac{1}{x+1}$ 35. $x=|e^{-y}-1|$

36. $y=3-\dfrac{1}{|x-1|}$ 37. $y=\dfrac{x+2}{x-2}$

38. $y=x^{\frac{1}{\log_{10}x}}$ 39. $y=\sin^{-1}(\sin x)$

40. $y=\cos^{-1}(\cos x)$ 41. $y=\tan^{-1}(\tan x)$

42. $y=\cot^{-1}(\cot x)$ 43. $y=\cos x+|\cos x|$

44. $y=|x+2|x$ 45. $y=2|x-2|-|x+1|+x$

46. $y=2^x-2^{-x}$ 47. $y=x\sin x$

48. $y=|x-1|x$ 49. $y=x^3-6x^2+11x-6$

50. $y=x(1-x^2)$ 51. $y=\dfrac{1}{x^2+1}$

52. $y=m\cos\{|x|,|x-2|,2-|x-1|\}$

53. $y=\sqrt{\{x\}}$, where $\{x\}$ denotes the fractional part of x.

54. $y=\{x\}^2$, where $\{x\}$ denotes the fractional part of x.

55. $y=[x]+(x-[x])^2$

56. If $f(x)=\begin{cases}|x+1|-1, & -2<x\le 0\\ 1-|x-1|, & 0<x\le 2\end{cases}$,

then find and draw the graph of $g(x)$ given by

$$g(x)=\begin{cases}\text{Min}\{f(t)\mid -2\le t\le x\}, & -2<x\le 0\\ \text{Max}\{f(t)\mid 0\le t\le x\}, & 0<x\le 2\end{cases}$$

CHAPTER

3

LIMITS

3.1 INFORMAL APPROACH TO LIMIT

Consider the function $f(x)=\frac{x^2-4}{x-2}, x\neq 2$. This function is defined for all x except $x=2$. At $x=2$ we cannot determine the value of the function simply by substituting $x=2$ in $f(x)$ as it assumes the form $\frac{0}{0}$ which is known as an indeterminate form. However, if $x\neq 2$, then

$$f(x)=\frac{(x-2)(x+2)}{x-2}=x+2.$$

The following table exhibits the values of $f(x)$ at points which are close to 2 on its two sides viz. left and right on the real line.

x	1.4	1.5	1.6	1.7	1.8	1.9	1.99	2	2.01	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	3.4	3.5	3.6	3.7	3.8	3.9	3.99	$\frac{0}{0}$	4.01	4.1	4.2	4.3	4.4	4.5	4.6

It is evident from the above table that as x increases and comes closer to 2 from left hand side of 2, the values of $f(x)$ increase and come closer to 4. This is interpreted as:

When x approaches to 2 from its left hand side, the function $f(x)$ tends to the limit 4.

If we use the notation '$x\to 2^-$' to denote 'x tends to 2 from left hand side', the above statement can be restated as

$$x\to 2^-,\ f(x)\to 4$$

or $\lim\limits_{x\to 2^-} f(x)=4$ or Left hand limit of $f(x)$ at $x=2$ is 4.

Thus, $\lim\limits_{x\to 2^-} f(x)=4$ means that as x tends to 2 from left hand side, $f(x)$ is tending to 4.

From the above table we observe that as x decreases and comes closer to 2 from right hand side 2, the values of $f(x)$ decrease and come closer to 4. This is interpreted as:

When x approaches to 2 from its right hand side, the function f(x) tends to the limit 4.

Using the notation '$x\to 2^+$' to denote 'x tends to 2 from right hand side', the above statement can be re-stated as

as $x\to 2^+$, $f(x)\to 4$ or $\lim\limits_{x\to 2^+} f(x)=4$ or Right hand limit of $f(x)$ at $x=2$ is 4.

Thus, $\lim\limits_{x\to 2^+} f(x)=4$ means that as x tends to 2 from right hand side, $f(x)$ is tending to 4.

Now, consider the function $f(x)=\frac{|x-3|}{x-3}, x\neq 3$. This function is defined for all x except $x=3$. At $x=3$ it assumes the form $\frac{0}{0}$ (an indeterminate form). The following table shows the values of $f(x)$ at points which are close to 3 and are on its two sides.

x	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.99	3	3.0	3.1	3.2	3.3	3.4	3.5	3.6
$f(x)$	−1	−1	−1	−1	−1	−1	−1	−1	$\frac{0}{0}$	1	1	1	1	1	1	1

It is evident from the table that: as $x=3$ from its left hand side the values of $f(x)$ are everywhere −1 i.e.,

$\lim\limits_{x\to 3^-} f(x)=-1$ or, left hand limit (LHL) of $f(x)$ at $x=3$ is −1.

We also observe that at every point on the right hand side of 3, the function assumes value 1, therefore

$$\lim_{x\to 3^+} f(x)=1$$

Let us now consider the function $f(x)=\frac{1}{x-4}, x\neq 4$. Here also the function is undefined at $x=4$ as f(4) assumes the form 1/0. In this case as x approaches to 4 from the left hand side, $f(x)$ decreases to $-\infty$, i.e.

$$\lim_{x \to 4^-} f(x) = -\infty$$

and $f(x)$ increases to $+\infty$ as x approaches to 4 from the right, i.e.

$$\lim_{x \to 4^+} f(x) = +\infty$$

So, we say that $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ both do not exist.

It follows from the above discussion that as we can approach to a given number 'a' (say) on the real line either from its left hand side by increasing numbers which are less than 'a' or from right hand side by decreasing numbers which are greater than 'a'. So, there are two types of limits viz. (i) left hand limit and (ii) right hand limit. We also observe that for some functions at a given point 'a' (say) left hand and right hand limits are equal whereas for some functions these two limits are not equal and even sometimes either left hand limit or right hand limit or both do not exist. If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ i.e. (LHL at $x = a$) = (RHL at $x = a$), then we say that $\lim_{x \to a} f(x)$ exists. Otherwise, $\lim_{x \to a} f(x)$ does not exist.

3.2 FORMAL APPROACH TO LIMIT

Before we proceed to formulate a definition of the limit of a function at a point on the basis of the discussion made in the previous article, we intend to discuss some basic concepts which will be used in further discussions.

NEIGHBOURHOOD (nbd) OF A POINT *Let a be a real number and let δ be a positive real number. Then the set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a of radius 'δ' and is denoted by $N_\delta(a)$.*

Thus, $N_\delta(a) = (a - \delta, a + \delta) = \{x \,.\, R \mid a - \delta < x < a + \delta\}$

The set $N_\delta(a) - \{a\}$ is called deleted nbd of a of radius δ. The set $(a - \delta, a)$ is called the left nbd of a and the set $(a, a + \delta)$ is known as the right nbd of a.

If δ is very small and x lies in the interval $(a - \delta, a)$, then x is said to approach to a from the left and we write $x \to a^-$. If $x \,.\, (a, a + \delta)$, then x is said to approach to a from the right which is denoted by $x \to a^+$.

Consider the statement $|x - a| < \delta$. We have

$|x - a| < \delta \Leftrightarrow -\delta < x - a < \delta \Leftrightarrow a - \delta < x < a + \delta \Leftrightarrow x \,.\, N_\delta(a)$.

Thus, $|x - a| < \delta$ means that x lies in the *nbd* of 'a' of radius δ as shown in Fig. 3.1

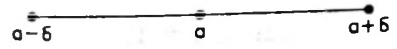


Fig. 3.1

Let $f(x)$ be a function with domain D and let 'a' be a point such that every nbd of a contains infinitely many points of D. A real number l is limit of $f(x)$ as x tends to a, if for every nbd of l, there exists a nbd of 'a', such that images of all points in the deleted nbd of a are in the nbd of l.

In other words, a real number l is limit of $f(x)$ as x tends to a, if for every . > 0 there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

$$\Leftrightarrow \quad x \,.\, (a - \delta, a + \delta),\ x \neq a \Rightarrow f(x) \,.\, (l - \varepsilon,\ l + \varepsilon).$$

If l is the limit of $f(x)$ as x tends to a, then we write $\lim_{x \to a} f(x) = l$

From Fig. 3.2 it is evident that l is the limit of $f(x)$ as x approaches to a if for every pair of horizontal lines equidistant from l there exists a pair of vertical lines equidistant from a such that every point on x-axis between vertical lines has its image between horizontal lines.

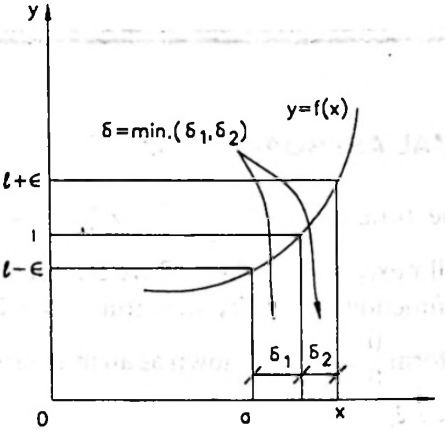


Fig. 3.2

ILLUSTRATION *Show that* $\lim_{x \to 2} \dfrac{4 - 2x}{x^2 - 4} = -\dfrac{1}{2}$.

SOLUTION We have to show that for every nbd of $-\frac{1}{2}$ there exists a nbd of 2 such that points in the deleted nbd of 2 have their images in the nbd of $-\frac{1}{2}$.

Let $\left(-\frac{1}{2} - \varepsilon, -\frac{1}{2} + \varepsilon\right)$ be an arbitrary nbd of $-\frac{1}{2}$ of radius $\varepsilon > 0$. Then

$$-\frac{1}{2} - \varepsilon < f(x) < -\frac{1}{2} + \varepsilon \Leftrightarrow -\frac{1}{2} - \varepsilon < \frac{4 - 2x}{x^2 - 4} < -\frac{1}{2} + \varepsilon$$

$$\Leftrightarrow \frac{1}{2} + \varepsilon > \frac{2x - 4}{x^2 - 4} > \frac{1}{2} - \varepsilon \quad \Leftrightarrow \frac{1}{2} - \varepsilon < \frac{2x - 4}{x^2 - 4} < \frac{1}{2} + \varepsilon$$

$$\Leftrightarrow \frac{2}{1 + 2\varepsilon} < \frac{x^2 - 4}{2x - 4} < \frac{2}{1 - 2\varepsilon} \Leftrightarrow \frac{2}{1 + 2\varepsilon} < \frac{1}{2}x + 1 < \frac{2}{1 - 2\varepsilon}$$

$$\Leftrightarrow 2\left(\frac{1 - 2\varepsilon}{1 + 2\varepsilon}\right) < x < 2\left(\frac{1 + 2\varepsilon}{1 - 2\varepsilon}\right) \qquad \ldots(i)$$

Let $x_1 = 2\left(\frac{1 - 2\varepsilon}{1 + 2\varepsilon}\right)$, $x_2 = 2\left(\frac{1 + 2\varepsilon}{1 - 2\varepsilon}\right)$. Then $x_1 < 2$ and $x_2 > 2$.

Now $x_2 - 2 = \frac{8\varepsilon}{1 - 2\varepsilon}$; $2 - x_1 = \frac{8\varepsilon}{1 + 2\varepsilon}$. Clearly $2 - x_1 < x_2 - 2$.

Let $\delta = \min\{x_2 - 2,\ 2 - x_1\} = \frac{8\varepsilon}{1 + 2\varepsilon}$. Then from (i)

$$-\frac{1}{2}-\varepsilon < f(x) < -\frac{1}{2}+\varepsilon, \text{ whenever } 2-\delta < x < 2+\delta,\ x \neq 2$$

$$\Rightarrow \left| f(x)-\left(-\frac{1}{2}\right)\right| < \varepsilon \text{ whenever } 0 < |x-2| < \delta.$$

Hence, $\lim_{x\to 2} f(x) = -\frac{1}{2}$.

3.3 EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

The statement $x \to a^-$ means that x is tending to a from the left hand side, i.e., x is a number less than a but very very close to a. Therefore $x \to a^-$ is equivalent to $x = a - h$ where $h > 0$ such that $h \to 0$. Similarly, $x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$. Thus, we have the following algorithms for finding left hand and right hand limits at $x = a$.

ALGORITHM FOR FINDING LEFT HAND LIMIT

To evaluate LHL of $f(x)$ at $x = a$ i.e. $\lim_{x\to a^-} f(x)$ we proceed as follows:

STEP I *Write* $\lim_{x\to a^-} f(x)$

STEP II *Put* $x = a - h$ *and replace* $x \to a^-$ *by* $h \to 0$ *to obtain* $\lim_{h\to 0} f(a-h)$.

STEP III *Simplify* $\lim_{h\to 0} f(a-h)$ *by using the formula for the given function.*

STEP IV *The value obtained in step III is the LHL of* $f(x)$ *at* $x = a$.

ALGORITHM FOR FINDING RIGHT HAND LIMIT

To evaluate RHL of $f(x)$ at $x = a$ i.e. $\lim_{x\to a^+} f(x)$ we proceed as follows:

STEP I *Write* $\lim_{x\to a^+} f(x)$

STEP II *Put* $x = a + h$ *and replace* $x \to a^+$ *by* $h \to 0$ *to obtain* $\lim_{h\to 0} f(a+h)$.

STEP III *Simplify* $\lim_{h\to 0} f(a+h)$ *by using the formula for the given function.*

STEP IV *The value obtained in step III is the RHL of* $f(x)$ *at* $x = a$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the left hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$$

at $x = 4$.

SOLUTION We have,

(LHL of $f(x)$ at $x = 4$)

$= \lim_{x\to 4^-} f(x)$

$= \lim_{h\to 0} f(4-h)$

$= \lim_{h\to 0} \dfrac{|4-h-4|}{4-h-4}$

$= \lim_{h\to 0} \dfrac{|-h|}{-h}$

$= \lim_{h\to 0} \dfrac{h}{-h} = \lim_{h\to 0} -1 = -1.$

EXAMPLE 2 *Evaluate the right hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$$

at $x = 4$.

SOLUTION We have,

(RHL of $f(x)$ at $x = 4$)

$= \lim_{x\to 4^+} f(x)$

$= \lim_{h\to 0} f(4+h)$

$= \lim_{h\to 0} \dfrac{|4+h-4|}{4+h-4}$

$= \lim_{h\to 0} \dfrac{|h|}{h}$

$= \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1.$

EXAMPLE 3 *Evaluate the left hand and right hand limits of the function defined by*

$$f(x) = \begin{cases} 1+x^2, & \text{if } 0 \le x \le 1 \\ 2-x, & \text{if } x > 1 \end{cases}$$

at x = 1. Also show that $\lim_{x\to 1} f(x)$ does not exist.

SOLUTION We have,

(LHL of $f(x)$ at $x = 1$)

$= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h \to 0} 1 + (1-h)^2$

$= \lim_{h \to 0} 2 - 2h + h^2 = 2.$

And,

(RHL of $f(x)$ at $x = 1$)

$= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} 2 - (1+h) = \lim_{h \to 0} 1 - h = 1.$

Clearly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. So, $\lim_{x \to 1} f(x)$ does not exist.

EXAMPLE 4 *If* $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ *show that* $\lim_{x \to 0} f(x)$ *does not exist.*

SOLUTION We have,

(LHL of $f(x)$ at $x = 0$)

$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \dfrac{-h - |-h|}{(-h)} = \lim_{h \to 0} \dfrac{-h-h}{-h}$

$= \lim_{h \to 0} \dfrac{-2h}{-h} = \lim_{h \to 0} 2 = 2.$

And,

(RHL of $f(x)$ at $x = 0$)

$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \dfrac{h - |h|}{h} = \lim_{h \to 0} \dfrac{h-h}{h}$

$= \lim_{h \to 0} \dfrac{0}{h} = \lim_{h \to 0} 0 = 0.$

Since, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. So, $\lim_{x \to 0} f(x)$ does not exist.

EXAMPLE 5 *If* $f(x) = \begin{cases} 5x - 4, & 0 < x \leq 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, *show that* $\lim_{x \to 1} f(x)$ *exists.*

SOLUTION We have,

(LHL of $f(x)$ at $x = 1$)

$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1.$

(RHL of $f(x)$ at $x = 1$)

$= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} 4(1+h)^3 - 3(1+h)$

$= 4(1)^3 - 3(1) = 1$

Clearly, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$. So, $\lim_{x \to 1} f(x)$ exists and is equa to 1.

EXAMPLE 6 *Find the left hand and right hand limits of the greates integer function* $f(x) = [x]$ = *greatest integer less than or equal to* x *at* $x = k$, *where* k *is an integer. Also, show that* $\lim_{x \to k} f(x)$ *does not exist*

SOLUTION We have,

(LHL at $x = k$)

$= \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h]$

$= \lim_{h \to 0} k - 1 = k - 1$ $\quad [\because k-1 < k-h < k \\ \therefore [k-h] = k-1$

(RHL at $x = k$)

$= \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$

$= \lim_{h \to 0} k = k$ $\quad [\because k < k+h < k+1 \\ \therefore [k+h] = k$

Clearly, $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$.

So, $\lim_{x \to k} f(x)$ does not exist.

EXAMPLE 7 *Show that* $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ *does not exist.*

SOLUTION Let $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$. Then,

(LHL of $f(x)$ at $x = 0$)

$= \lim_{x \to 0^-} f(x))$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \dfrac{e^{-1/h} - 1}{e^{-1/h} + 1}$

$= \lim_{h \to 0} \left(\dfrac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} \right) = -1$ $\quad [\because h \to 0 \Rightarrow \frac{1}{h} \to \infty \\ \Rightarrow e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0$

[RHL of $f(x)$ at $x = 0$]

$= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$

$= \lim_{h \to 0} \left(\frac{1 - \frac{1}{e^{1/h}}}{1 + e^{1/h}} \right)$ [Dividing Nr and Dr by $e^{1/h}$]

$= \frac{1-0}{1+0} = 1$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

EXAMPLE 8 *Let* $f(x) = \begin{cases} \cos x, & \text{if } x \geq 0 \\ x + k, & \text{if } x < 0 \end{cases}$. *Find the value of constant* k, *given that* $\lim_{x \to 0} f(x)$ *exists.*

SOLUTION It is given that

$\lim_{x \to 0} f(x)$ exists

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x$

$\Rightarrow 0 + k = \cos 0 \Rightarrow k = 1.$

EXAMPLE 9 *If* f *is an odd function and if* $\lim_{x \to 0} f(x)$ *exists. Prove that this limit must be zero.*

SOLUTION Since $\lim_{x \to 0} f(x)$ exists

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h)$

$\Rightarrow \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h)$

$\Rightarrow -\lim_{h \to 0} f(h) = \lim_{h \to 0} f(h)$ [$\because f(x)$ is odd $\therefore f(-h) = -f(h)$]

$\Rightarrow 2 \lim_{h \to 0} f(h) = 0$

$\Rightarrow \lim_{h \to 0} f(h) = 0$

$\Rightarrow \lim_{x \to 0} f(x) = 0$

EXAMPLE 10 *If* f *is an even function, then prove that* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

SOLUTION We have,

$\lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h)$ [$\because f$ is even $\therefore f(-h) = f(h)$]

$= \lim_{h \to 0} f(0+h)$

$= \lim_{x \to 0^+} f(x).$

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3.4 DIFFERENCE BETWEEN THE VALUE OF A FUNCTION AT A POINT AND THE LIMIT AT A POINT

Let $f(x)$ be a function and let a be a point. Then we have the following possibilities:

(I) *$\lim_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) does not exist*

Consider the function $f(x)$ defined by $f(x) = \frac{x^2 - 9}{x - 3}$

Clearly this function is not defined at $x = 3$ i.e., $f(3)$ does not exist, because it attains the form $\frac{0}{0}$. But it can be easily seen that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists.

Thus, $\lim_{x \to 3} f(x)$ exists but the value $f(3)$ does not exist.

(II) *The value $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.*

In example 4 of section 3.3 we have seen that $\lim_{x \to k} f(x)$ does not exist but $f(x) = k$ exists.

(III) *$\lim_{x \to a} f(x)$ and $f(a)$ both exist but are unequal*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

It can be easily seen that $\lim_{x \to 2^-} f(x) = 4 = \lim_{x \to 2^+} f(x)$.

So, $\lim_{x \to 2} f(x)$ exists and is equal to 4. Also the value $f(2)$ exists and is equal to 3.

Thus, $\lim_{x \to 2} f(x)$ and $f(2)$ both exist but are unequal.

(IV) *$\lim_{x \to a} f(x)$ and $f(a)$ both exist and are equal*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

For this function, it can be easily seen that $\lim_{x \to 2} f(x)$ and $f(2)$ both exist and are equal to 4.

3.5 THE ALGEBRA OF LIMITS

Let f and g be two real functions with domain D. We define four new functions $f \pm g$, fg, f/g on domain D by setting

$$(f \pm g)(x) = f(x) \pm g(x), \quad (fg)(x) = f(x)\,g(x)$$

$$(f/g)(x) = f(x)/g(x), \text{ if } g(x) \neq 0 \text{ for any } x \in D.$$

Following are some results concerning the limits of these functions.

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. If l and m exist, then

(i) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m$

(ii) $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = lm$

(iii) $\lim_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(iv) $\lim_{x \to a} Kf(x) = K \cdot \lim_{x \to a} f(x)$, where K is constant

(v) $\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)| = |l|$

(vi) $\lim_{x \to a} |f(x)|^{g(x)} = l^m$

(vii) If $f(x) \leq g(x)$ for every x in the deleted nbd of a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

(viii) If $f(x) \leq g(x) \leq h(x)$ for every x in the deleted nbd of a and, $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = l$.

(ix) $\lim_{x \to a} fog(x) = f\left(\lim_{x \to a} g(x)\right) = f(m)$

In particular

(i) $\lim_{x \to a} \log f(x) = \log\left(\lim_{x \to a} f(x)\right) = \log l$

(ii) $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)} = e^l$.

(iii) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \dfrac{1}{f(x)} = 0$.

3.6 EVALUATION OF LIMITS

In the previous sections, we have discussed the notion of left hand limit (LHL), right hand limit (RHL) and the existence of the limit of a function $f(x)$ at a given point. In the evaluation of limits, it is assumed that $\lim_{x \to a} f(x)$ always exists i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

In this section, we will discuss various methods of evaluating limits. To facilitate the job of evaluation of limits we categorize problems on limits in the following categories:

(I) Algebraic limits
(II) Trigonometric limits
(III) Exponential and logarithmic limits

VARIOUS METHODS OF EVALUATION OF ALGEBRAIC LIMITS

To evalue algebraic limits, we have the following methods :

(i) Direct substitution method
(ii) Factorization method
(iii) Rationalisation method
(iv) Using some standard results
(v) Method of evaluating limits when variable tends to ∞ or $-\infty$.

We shall now discuss these methods with suitable illustrative examples.

3.6.1 DIRECT SUBSTITUTION METHOD

If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

ILLUSTRATION 1 *Evaluate the following limits :*

(i) $\lim_{x \to 1} (3x^2 + 4x + 5)$ (ii) $\lim_{x \to 0} \dfrac{\cos x}{1 + \sin x}$

SOLUTION We have,

(i) $\lim_{x \to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$

(ii) $\lim_{x \to 0} \dfrac{\cos x}{1 + \sin x} = \dfrac{\cos 0}{1 + \sin 0} = 1$

3.6.2 FACTORISATION METHOD

Consider $\lim_{x \to a} \dfrac{f(x)}{g(x)}$. If by putting $x = a$ the rational function $\dfrac{f(x)}{g(x)}$ takes the form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ etc., then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. In such a case we factorise the numerator and denominator and then cancel out the common factor $(x - a)$. After cancelling out the common factor $x - a$ we again put $x = a$ in the given expression and see whether we get a meaningful number or not. This process is repeated till we get a meaningful number.

ILLUSTRATION 1 *Evaluate :* $\lim_{x\to 2} \dfrac{x^3-6x^2+11x-6}{x^2-6x+8}$.

SOLUTION We have,

$$\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x^2-6x+8}$$

$$= \lim_{x\to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)}$$

$$= \lim_{x\to 2} \frac{(x-1)(x-3)}{(x-4)} = \frac{1}{2}.$$

ILLUSTRATION 2 *Evaluate :* $\lim_{x\to 3} \dfrac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$.

SOLUTION We have,

$$\lim_{x\to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$$

$$= \lim_{x\to 3} \frac{(x-3)(x^2-4x+3)}{(x-3)(x^3-2x^2-6x+9)}$$

$$= \lim_{x\to 3} \frac{x^2-4x+3}{x^3-2x^2-6x+9}$$

$$= \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x^2+x-3)}$$

$$= \lim_{x\to 3} \frac{x-1}{x^2+x-3} = \frac{2}{9}.$$

EXERCISE 3.1

Evaluate the following limits :

1. $\lim_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$

2. $\lim_{x\to 2}\left(\dfrac{2}{x(x-2)}-\dfrac{1}{x^2-3x+2}\right)$

3. $\lim_{x\to 1}\left(\dfrac{x+2}{x^2-5x+4}+\dfrac{x-4}{3(x^2-3x+2)}\right)$

4. $\lim_{x\to 0} \dfrac{(1+x)(1+2x)(1+3x)-1}{x}$

5. $\lim_{x\to 1}\left[\left(\dfrac{4}{x^2-x-1}-\dfrac{1-3x+x^2}{1-x^3}\right)^{-1}+3\left(\dfrac{x^4-1}{x^3-x^{-1}}\right)\right]$

6. $\lim_{x\to 2}\left[\left(\dfrac{x^3-4x}{x^3-8}\right)^{-1}-\left(\dfrac{x+\sqrt{2x}}{x-2}-\dfrac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]$

7. $\lim_{x\to 1} \dfrac{x^4-3x+2}{x^5-4x+3}$

8. $\lim_{x\to 0} \dfrac{(1+x)^5-(1+5x)}{x^2+x^5}$

9. $\lim_{x\to\infty} \{(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})\}$, if $|x|<1$

ANSWERS

1. -1 2. $\frac{1}{2}$ 3. 0 6. $\frac{1}{2}$ 7. 1 8. 10

4. 6 5. 3 9. $\frac{1}{1-x}$

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3.6.3 RATIONALISATION METHOD

This method is generally used when one of numerator and denominator or both of them consist of expressions involving square roots.

Following illustrations will illustrate the above procedure :

ILLUSTRATION 1 *Evaluate :* $\lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

SOLUTION We have,

$$\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$$

$$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x\to a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{3(a-x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x\to a} \frac{(\sqrt{3a+x}+2\sqrt{x})}{3(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{4\sqrt{a}}{2(3\sqrt{3a})} = \frac{2}{3\sqrt{3}}$$

ILLUSTRATION 2 *Evaluate :* $\lim_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

SOLUTION We have,

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+1}-1)(\sqrt{x^2+1}+1)}{(\sqrt{x^2+16}-4)(\sqrt{x^2+16}+4)} \times \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}$$

$$= \lim_{x\to 0} \frac{(x^2+1-1)(\sqrt{x^2+16}+4)}{(x^2+16-16)(\sqrt{x^2+1}+1)}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}$$

$$= \frac{4+4}{1+1} = \frac{8}{2} = 4.$$

EXERCISE 3.2

Evaluate the following limits :

1. $\lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$
2. $\lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{3}-\sqrt{2}}{x^2-6}$
3. $\lim_{x\to\sqrt{2}} \frac{\sqrt{3+2x}-\sqrt{2}-1}{x^2-2}$
4. $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$
5. $\lim_{x\to 9} \left| \frac{3-\sqrt{x}}{9-x} + \frac{1}{3-\sqrt{x}} - \frac{6(x^2+162)}{729-x^3} \right|$

ANSWERS

1. $\frac{\sqrt{5}-\sqrt{2}}{6\sqrt{10}}$
2. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{6}}$
3. $\frac{\sqrt{2}-1}{2\sqrt{2}}$
4. $\frac{4}{3}$
5. $\frac{-2}{9}$

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3.6.4 EVALUATION OF LIMITS USING THE RESULT

$\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$, where $n \in Q$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n}$.

SOLUTION We have,

$$\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n}$$

$$= \lim_{x\to a} \left| \frac{x^m-a^m}{x-a} \cdot \frac{x-a}{x^n-a^n} \right|$$

$$= \lim_{x\to a} \left\{ \frac{x^m-a^m}{x-a} \quad \frac{x^n-a^n}{x-a} \right\}$$

$$= \lim_{x\to a} \frac{x^m-a^m}{x-a} + \lim_{x\to a} \frac{x^n-a^n}{x-a}$$

$$= ma^{m-1} + na^{n-1} = \frac{m}{n} a^{m-n}$$

EXAMPLE 2 *Find the value of k , if* $\lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to k} \frac{x^3-k^3}{x^2-k^2}$.

SOLUTION We have,

$$\lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to 1} \frac{x^4-1^4}{x-1} = 4(1)^{4-1} = 4$$

and,

$$\lim_{x\to k} \frac{x^3-k^3}{x^2-k^2}$$

$$= \lim_{x\to k} \frac{x^3-k^3}{x-k} \cdot \frac{x-k}{x^2-k^2}$$

$$= \lim_{x\to k} \frac{x^3-k^3}{x-k} \div \frac{x^2-k^2}{x-k}$$

$$= \lim_{x\to k} \frac{x^3-k^3}{x-k} \div \lim_{x\to k} \frac{x^2-k^2}{x-k}$$

$$= (3k^{3-1}) \div (2k^{2-1}) = \frac{3}{2}k$$

$$\therefore \lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to k} \frac{x^3-k^3}{x^2-k^2}$$

$$\Rightarrow \quad 4 = \frac{3k}{2} \Rightarrow k = \frac{8}{3}.$$

EXAMPLE 3 *Evaluate :* $\lim_{x\to 1} \frac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}$.

SOLUTION $\lim_{x\to 1} \frac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}$ $\left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x\to 1} \frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^n-1)}{x-1}$$

$$= \lim_{x\to 1} \frac{x-1}{x-1} + \lim_{x\to 1} \frac{x^2-1^2}{x-1} + \lim_{x\to 1} \frac{x^3-1}{x-1} + \ldots + \lim_{x\to 1} \frac{x^n-1^n}{x-1}$$

$= 1 + 2(1)^{2-1} + 3(1)^{3-1} + \ldots + n(1)^{n-1}$

$= 1 + 2 + 3 + \ldots + n$

$= \frac{n(n+1)}{2}$.

EXAMPLE 4 *If* $\lim_{x \to -a} \frac{x^9 + a^9}{x + a} = 9$, *find the value of* a.

SOLUTION We have,

$$\lim_{x \to -a} \frac{x^9 + a^9}{x + a} = 9$$

$$\Rightarrow \lim_{x \to -a} \frac{x^9 - (-a)^9}{x - (-a)} = 9$$

$$\Rightarrow \quad 9(-a)^{9-1} = 9$$

$$\Rightarrow \quad 9a^8 = 9$$

$$\Rightarrow \quad a^8 = 1 \Rightarrow a = \pm 1$$

EXERCISE 3.3

Evaluate the following limits :

1. $\lim_{x \to -a} \frac{x^n + a^n}{x + a}$, n is an odd natural number

2. $\lim_{x \to 0} \frac{(1+x)^5 - 1}{3x + 5x^2}$

3. $\lim_{h \to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h}$

4. $\lim_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$

5. $\lim_{x \to 0} \frac{x + (x+1)^2 + (x+1)^3 + \ldots + (x+1)^n - n + 1}{x}$

6. $\lim_{x \to 1} \frac{(1-x)(1-x^2)\ldots(1-x^{2n})}{[(1-x)(1-x^2)\ldots(1-x^n)]^2}$

ANSWERS

1. na^{n-1}. 2. $\frac{5}{3}$ 3. $\frac{1}{n} x^{\frac{1-n}{n}}$ 4. 3 5. $\frac{n(n+1)}{2}$ 6. ${}^{2n}C_n$

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3.6.5 METHOD OF EVALUTING ALGEBRAIC LIMITS WHEN VARIABLE TENDS TO INFINITY

To evaluate this type of limits we follow the following procedure:

STEP I *Write down the given expression in form of a rational function, i.e.,* $\frac{f(x)}{g(x)}$, *if it is not so.*

STEP II *If* k *is the highest power of* x *in numerator and denominator both, then divide each term in numerator and denominator by* x^k.

STEP III *Use the result* $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$.

SOLUTION We have,

$$\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$$

$$= \lim_{x \to \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a + 0 + 0}{d + 0 + 0} = \frac{a}{d}.$$

EXAMPLE 2 *Evaluate :* $\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} - \sqrt{2x^2 - 1}}{4x + 3}$

SOLUTION Dividing each term in the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} - \sqrt{2x^2 - 1}}{4x + 3}$$

$$= \lim_{x \to \infty} \frac{\sqrt{3 - 1/x^2} - \sqrt{2 - 1/x^2}}{4 + 3/x}$$

$$= \frac{\sqrt{3} - \sqrt{2}}{4}$$

EXAMPLE 3 *Evaluate :* $\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}$.

SOLUTION We have,

$$\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1}$$

$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+x+1}-\sqrt{x^2+1})}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}\ (\sqrt{x^2+x+1}+\sqrt{x^2+1})$$

$$=\lim_{x\to\infty}\frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}}$$ [Dividing *Nr* and *Dr* by x]

$$=\frac{1}{1+1}=\frac{1}{2}.$$

AN IMPORTANT RESULT If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers, then

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$$

$$=\begin{cases}\frac{a_0}{b_0}, \text{ if } m=n\\ 0\ , \text{ if } m<n\\ \infty\ , \text{ if } m>n \text{ and as } b_0>0\\ -\infty, \text{ if } m>n \text{ and as } b_0<0\end{cases}$$

EXAMPLE 4 *If* $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-ax-b\right\}=0$, *find the values of a and b*

SOLUTION We have,

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=0$$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial, so, numerator must be a constant i.e. a zero degree polynomial.

$\therefore\ 1-a=0$ and $a+b=0 \Rightarrow a=1$ and $b=-1$.

Hence, $a=1$ and $b=-1$.

EXAMPLE 5 *If* $\lim_{x\to\infty}\left\{\frac{x^2-1}{x+1}-ax-b\right\}=2$, *find the values of a and b.*

SOLUTION We have,

$$\lim_{x\to\infty}\left\{\frac{x^2-1}{x+1}-ax-b\right\}=2$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=2 \quad \ldots(i)$$

Since the limit of the given expression is a finite non-zero number, therefore numerator and denominator are of the same degree.

$\therefore\ 1-a=0 \Rightarrow a=1.$

Putting $a=1$ in (i), we obtain

$$\lim_{x\to\infty}\frac{-x(1+b)+(1-b)}{x+1}=2$$

$\Rightarrow -(1+b)=2 \Rightarrow b=-3.$

Hence, $a=1$ and $b=-3$

EXAMPLE 6 *If* $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-ax-b\right\}=\infty$, *find a and b.*

SOLUTION We have,

$$\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-ax-b\right\}=\infty$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=\infty$$

The limit of the given expression will be infinity, if degree of numerator is more than that of denominator.

$$\therefore\ 1-a\neq 0 \Rightarrow a\neq 1.$$

Hence, $a\neq 1$ and b can assume any real value.

EXAMPLE 7 *Evaluate :*

$$\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}$$

Here, Π *stands for the product.*

SOLUTION We have,

$$\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}$$

$$=\lim_{n\to\infty}\left(\frac{3^3-8}{3^3+8}\right)\left(\frac{4^3-8}{4^3+8}\right)\left(\frac{5^3-8}{5^3+8}\right)\ldots\left(\frac{n^3-8}{n^3+8}\right)$$

$$=\lim_{n\to\infty}\left[\left(\frac{3-2}{3+2}\right)\left(\frac{3^2+4+3\times2}{3^2+4-3\times2}\right)\right]\left[\left(\frac{4-2}{4+2}\right)\left(\frac{4^2+4+4\times2}{4^2+4-4\times2}\right)\right]$$

$$\left[\left(\frac{5-2}{5+2}\right)\left(\frac{5^2+4+5\times2}{5^2+4-5\times2}\right)\right]\ldots\left[\left(\frac{n-2}{n+2}\right)\left(\frac{n^2+4+2n}{n^2+4-2n}\right)\right]$$

$$=\left[\lim_{n\to\infty}\left\{\frac{1\,.\,2\,.\,3\,\ldots\,(n-2)}{5\,.\,6\,.\,7\,\ldots\,(n-2)\,(n-1)\,n\,(n+1)\,(n+2)}\right\}\right.$$

$$\left.\times\left\{\frac{19\,.\,28\,\ldots\,(n^2+2n+4)}{7\,.\,12\,.\,19\,\ldots\,(n^2-2n+4)}\right\}\right]$$

$$=\lim_{n\to\infty}\left\{\frac{1\,.\,2\,.\,3\,.\,4}{(n-1)\,(n)\,(n+1)\,(n+2)}\times\frac{(n^2+3)\,(n^2+2n+4)}{7\,.\,12}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\frac{(n^2+3)(n^2+2n+4)}{(n-1)\,n\,(n+1)(n+2)}$$

$$= \frac{2}{7}\lim_{n\to\infty}\frac{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}{\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{7}$$

EXAMPLE 8 *Evaluate :*

$$\lim_{n\to\infty}\left\{\frac{1\left(\sum_{r=1}^{n} r\right)+2\left(\sum_{r=1}^{n-1} r\right)+3\left(\sum_{r=1}^{n-2} r\right)+\ldots+n\,.\,1}{n^4}\right\}$$

SOLUTION We have,

$$\lim_{n\to\infty}\left\{\frac{1\left(\sum_{r=1}^{n} r\right)+2\left(\sum_{r=1}^{n-1} r\right)+3\left(\sum_{r=1}^{n-2} r\right)+\ldots+n\,.\,1}{n^4}\right\}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n} k\left(\sum_{r=1}^{n-k+1} r\right)}{n^4}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n} k\,\frac{(n-k+1)(n-k+2)}{2}}{n^4}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n} k\,\{(n+1)-k\}\,\{(n+2)-k\}}{2n^4}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\{(n+1)(n+2)\,k-k^2(2n+3)+k^3\}}{2n^4}$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+2)\sum_{k=1}^{n} k-(2n+3)\sum_{k=1}^{n} k^2+\sum_{k=1}^{n} k^3}{2n^4}$$

$$= \lim_{n\to\infty}\frac{n(n+1)^2(n+2)}{4n^4}-\frac{(2n+3)\,n\,(n+1)(2n+1)}{12n^4}+\frac{n(n+1)}{8n^4}$$

$$= \lim_{n\to\infty}\frac{6n(n+1)^2(n+2)-2n(n+1)(2n+1)(2n+3)+3n^2(n+1)^2}{24n^4}$$

$$= \lim_{n\to\infty}\frac{(n+1)\{6(n+1)(n+2)-2(2n+1)(2n+3)+3n(n+1)\}}{24n^3}$$

$$= \lim_{n\to\infty}\frac{(n+1)(6n^2+18n+12-8n^2-12n-6+3n^2+3n)}{24\,n^3}$$

$$= \lim_{n\to\infty}\frac{(n+1)(n^2+9n+6)}{24n^3}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{9}{n}+\frac{6}{n^2}\right)}{24} = \frac{1}{24}$$

EXAMPLE 9 *Evaluate :*

$$\lim_{n\to\infty}\prod_{r=2}^{n}\frac{r^3-1}{r^3+1}$$

Here, '$\prod$' stands for the product.

SOLUTION We have,

$$\lim_{n\to\infty}\prod_{r=2}^{n}\frac{r^3-1}{r^3+1}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\left(\frac{4^3-1}{4^3+1}\right)\cdots\left(\frac{n^3-1}{n^3+1}\right)\right\}$$

$$= \lim_{n\to\infty}\left[\left\{\left(\frac{2-1}{2+1}\right)\left(\frac{2^2+2+1}{2^2-2+1}\right)\right\}\left\{\left(\frac{3-1}{3+1}\right)\left(\frac{3^2+3+1}{3^2-3+1}\right)\right\}\right.$$
$$\left.\times\left\{\left(\frac{4-1}{4+1}\right)\left(\frac{4^2+4+1}{4^2-4+1}\right)\right\}\cdots\left\{\left(\frac{n-1}{n+1}\right)\left(\frac{n^2+n+1}{n^2-n+1}\right)\right\}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1\,.\,2\ldots(n-1)}{3\,.\,4\,.\,5\ldots(n+1)}\times\frac{7\,.\,13\,.\,21\ldots(n^2+n+1)}{3\,.\,7\,.\,13\ldots(n^2-n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1\,.\,2}{n(n+1)}\times\frac{n^2+n+1}{3}\right]$$

$$= \frac{2}{3}\lim_{n\to\infty}\frac{n^2+n+1}{n^2+n}$$

$$= \frac{2}{3}\lim_{n\to\infty}\frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n}} = \frac{2}{3}.$$

EXAMPLE 10 *Evaluate :*

$$\lim_{k\to\infty}\left[\frac{e^{1/k}}{k^2}+2.\frac{(e^{1/k})^2}{k^2}+3\,.\,\frac{(e^{1/k})^3}{k^2}+\ldots.+k\,.\,\frac{(e^{1/k})^k}{k^2}\right]$$

SOLUTION We have,

$$\lim_{k\to\infty}\left[\frac{e^{1/k}}{k^2}+2.\frac{(e^{1/k})}{k^2}+3\,.\,\frac{(e^{1/k})^3}{k^2}+\ldots.+k\,.\,\frac{(e^{1/k})^k}{k^2}\right]$$

$$= \lim_{k\to\infty}\left[\frac{e^{1/k}+2\,.\,(e^{1/k})^2+3\,.\,(e^{1/k})^3+\ldots.+k\,(e^{1/k})^k}{k^2}\right]$$

$$= \lim_{k\to\infty}\frac{S_k}{k^2}\text{, where}$$

$$S_k = e^{1/k}+2\,.\,(e^{1/k})^2+3\,.\,(e^{1/k})^3+\ldots+k\,.\,(e^{1/k})^k$$

Now,

$$S_k = e^{1/k} + 2\,.\,(e^{1/k})^2 + 3\,.\,(e^{1/k})^3 + \ldots + k\,(e^{1/k})^k$$

$$\Rightarrow e^{1/k} S_k = (e^{1/k})^2 + 2\,(e^{1/k})^3 + \ldots + (k-1)\,(e^{1/k})^k + k\,(e^{1/k})^{k+1}$$

$$\therefore S_k (1 - e^{1/k}) = e^{1/k} + (e^{1/k})^2 + \ldots + (e^{1/k})^k - k\,(e^{1/k})^{k+1}$$

$$\Rightarrow S_k (1 - e^{1/k}) = e^{1/k} \left\{\frac{1 - (e^{1/k})^k}{1 - e^{1/k}}\right\} - k\ e\,.\,e^{1/k}$$

$$\Rightarrow S_k = \frac{e^{1/k}(1-e)}{(1-e^{1/k})^2} - \frac{k\,e\,e^{1/k}}{(1-e^{1/k})}$$

$$\therefore \lim_{k\to\infty} \frac{S_k}{k^2} = \lim_{k\to\infty} \frac{e^{1/k}(1-e)}{\left(\frac{1-e^{1/k}}{1/k}\right)^2} - \frac{e\ e^{1/k}}{\left(\frac{1-e^{1/k}}{1/k}\right)}$$

$$= \lim_{k\to\infty} \frac{e^{1/k}(1-e)}{\left(\frac{e^{1/k}-1}{1/k}\right)^2} + \frac{e\,e^{1/k}}{\left(\frac{e^{1/k}-1}{1/k}\right)}$$

$$= 1 - e + e = 1$$

EXAMPLE 11 *Evaluate :*

$$\lim_{n\to\infty} \frac{[x] + [2x] + \ldots + [nx]}{n^2},\ x \in R$$

SOLUTION For any integer k, we have

$$kx - 1 < [kx] \le kx \quad \text{for all } x \in R$$

$$\therefore \sum_{k=1}^{n} (kx - 1) < \sum_{k=1}^{n} [kx] \le \sum_{k=1}^{n} kx$$

$$\Rightarrow \sum_{k=1}^{n} kx - n < \sum_{k=1}^{n} [kx] \le \sum_{k=1}^{n} kx$$

$$\Rightarrow \frac{n(n+1)x}{2} - n < \sum_{k=1}^{n} [kx] \le \frac{n(n+1)}{2} x$$

$$\Rightarrow \frac{n(n+1)x}{2n^2} - \frac{1}{n} < \frac{\sum_{k=1}^{n} [kx]}{n^2} \le \frac{n(n+1)x}{2n^2}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n(n+1)x}{2n^2} - \frac{1}{n}\right) \le \lim_{n\to\infty} \frac{\sum_{k=1}^{n} [kx]}{n^2} \le \lim_{n\to\infty} \frac{n(n+1)x}{2n^2}$$

$$\Rightarrow \frac{x}{2} \le \lim_{n\to\infty} \frac{[x] + [2x] + .. + [nx]}{n^2} \le \frac{x}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{[x] + [2x] + ... + [nx]}{n^2} = \frac{x}{2}$$

EXAMPLE 12 *Evaluate :*

$$\lim_{n\to\infty} \frac{[x] + [2^2 x] + [3^2 x] + \ldots + [n^2 x]}{n^3}$$

SOLUTION For any integer k, we have

$$k^2 x - 1 \le [k^2 x] < k^2 x + 1$$

$$\Rightarrow \sum_{k=1}^{n} (k^2 x - 1) \le \sum_{k=1}^{n} [k^2 x] < \sum_{k=1}^{n} (k^2 x + 1)$$

$$\Rightarrow \frac{1}{n^3} \sum_{k=1}^{n} (k^2 x - 1) \le \frac{1}{n^3} \sum_{k=1}^{n} [k^2 x] < \frac{1}{n^3} \sum_{k=1}^{n} (k^2 n + 1)$$

$$\Rightarrow \frac{1}{n^3} \left\{\frac{n(n+1)(2n+1)}{6}\right\} x - \frac{n}{n^3} < \frac{1}{n^3} \sum_{k=1}^{n} [k^2 x] < \left\{\frac{n(n+1)(2n+1)}{6}\right\} x + \frac{n}{n^3}$$

$$\Rightarrow \left|\frac{2n^2 + 3n + 1}{6n^2}\right| x - \frac{1}{n^2} < \frac{1}{n^3} \sum_{k=1}^{n} [k^2 x] \le \left|\frac{2n^2 + 3n + 1}{6n^2}\right| x + \frac{1}{n^2}$$

Taking limit as $n \to \infty$, we obtain

$$\frac{x}{3} - 0 < \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} [k^2 x] \le \frac{x}{3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} [k^2 x] = \frac{x}{3}.$$

EXAMPLE 13 *If*

$$\lim_{x\to\infty} \left(\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} - \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}\right) = 4,$$

find the vlaues of a, b, c and d.

SOLUTION We have,

$$\lim_{x\to\infty} \left(\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} - \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}\right) = 4$$

$$\Rightarrow \lim_{x\to\infty} \frac{(a-2)x^3 + (3+c)x^2 + (b-3)x + 2 + d}{\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} + \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}} = 4$$

Since the limit is finite non-zero number, so numerator and denominator must be of the same degree. Clearly, the degree of denominator is 2. So, numerator must also be of degree 2.

$\therefore\ a - 2 = 0 \Rightarrow a = 2$

When $a = 2$, we have

$$\lim_{x\to\infty} \frac{(3+c)x^2 + (b-3)x + 2 + d}{\sqrt{x^4 + 2x^3 + 3x^2 + bx + 2} + \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}} = 4$$

$$\Rightarrow \lim_{x\to\infty} \frac{(3+c) + (b-3)/x + 2 + d/x^2}{\sqrt{1 + \frac{2}{x} + \frac{3}{x^2} + \frac{b}{x^3} + \frac{2}{x^4}} + \sqrt{1 + \frac{2}{x} - \frac{c}{x^2} + \frac{3}{x^3} - \frac{d}{x^4}}} = 4$$

$$\Rightarrow \frac{3+c}{1+1} = 4$$

$$\Rightarrow 3+c = 8 \Rightarrow c = 5$$

Hence, $a = 2, c = 5$ and b, d may take any real value.

EXAMPLE 14 *Evaluate :* $\lim_{x\to\infty} \frac{\log x^n - [x]}{[x]}$ *where,* $n \in N$ *and* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION We have,

$$\lim_{x\to\infty} \frac{\log x^n - [x]}{[x]}$$

$$= \lim_{x\to\infty} \frac{n\log x - [x]}{[x]}$$

$$= \lim_{x\to\infty} \frac{n\log x}{[x]} - 1$$

Now,

$$x - 1 < [x] \le x \quad \text{for all } x \in R$$

$$\Rightarrow \frac{1}{x} \le \frac{1}{[x]} < \frac{1}{x-1} \text{ for all } x \in R - \{0, 1\}$$

$$\Rightarrow \frac{\log x}{x} \le \frac{\log x}{[x]} < \frac{\log x}{x-1} \qquad [\because x \to \infty \therefore \log x > 0]$$

$$\Rightarrow \lim_{x\to\infty} \frac{\log x}{x} \le \lim_{x\to\infty} \frac{\log x}{[x]} < \lim_{x\to\infty} \frac{\log x}{x-1}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\log x}{[x]} = 0 \qquad \left[\because \lim_{x\to\infty} \frac{\log x}{x} = 0 \text{ and } \lim_{x\to\infty} \frac{\log x}{x-1} = 0\right]$$

Hence, $\lim_{x\to\infty} \frac{\log x^n - [x]}{[x]} = n \times 0 - 1 = -1$

EXAMPLE 15. *Evaluate :*

$$\lim_{n\to\infty} \frac{1}{n^3}\left[\frac{1}{1.3} + \frac{16}{3.5} + \frac{81}{5.7} + \frac{256}{7.9} + \ldots \text{ to } n \text{ terms}\right]$$

SOLUTION We have,

$$\frac{1}{1.3} + \frac{16}{3.5} + \frac{81}{5.7} + \frac{256}{7.9} + \ldots\ldots \text{ to } n \text{ terms}$$

$$= \sum_{r=1}^{n} \frac{r^4}{(2r-1)(2r+1)}$$

$$= \frac{1}{16}\sum_{r=1}^{n} \frac{16r^4}{(2r-1)(2r+1)}$$

$$= \frac{1}{16}\sum_{r=1}^{n} \frac{16r^4 - 1 + 1}{4r^2 - 1}$$

$$= \frac{1}{16}\sum_{r=1}^{n} 4r^2 + 1 + \frac{1}{16}\sum_{r=1}^{n} \frac{1}{4r^2 - 1}$$

$$= \frac{1}{16}\sum_{r=1}^{n} (4r^2 + 1) + \frac{1}{32}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{4}{16}\sum_{r=1}^{n} r^2 + \frac{1}{16}\sum_{r=1}^{n} 1 + \frac{1}{32}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} + \frac{n}{16} + \frac{1}{32}\left(1 - \frac{1}{2n+1}\right)$$

$$= \frac{1}{24} n(n+1)(2n+1) + \frac{n}{16} + \frac{1}{32} - \frac{1}{32(2n+1)}$$

$$\therefore \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{1}{1.3} + \frac{16}{3.5} + \frac{81}{5.7} + \frac{256}{7.9} + \ldots + \text{to } n \text{ terms}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{24}\frac{n(n+1)(2n+1)}{n^3} + \frac{1}{16n^2} + \frac{1}{32n^3} - \frac{1}{32n^3(2n+1)}\right]$$

$$= \frac{1}{24}\lim_{n\to\infty}\left[\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] + 0 + 0 - 0$$

$$= \frac{2}{24} = \frac{1}{12}$$

EXERCISE 3.4

Evaluate the following limits :

1. $\lim_{n\to\infty} \frac{(n+1)^4 - (n-1)^4}{(n+1)^4 + (n-1)^4}$

2. $\lim_{n\to\infty} \frac{(2n+1)^4 - (n-1)^4}{(2n+1)^4 + (n-1)^4}$

3. $\lim_{n\to\infty} \frac{\sqrt[3]{n^3 + 2n - 1}}{n+2}$

4. $\lim_{n\to\infty} \frac{n!}{(n+1)! - n!}$

5. $\lim_{n\to\infty} \frac{\sqrt{n^3 - 2n^2 + 1} + \sqrt{n^4 + 1}}{\sqrt[4]{n^6 + 2n^5 + 2} - \sqrt[5]{n^7 + 3n^3 + 1}}$

6. $\lim_{n\to\infty} \frac{\sqrt[4]{n^5 + 2} - \sqrt[3]{n^2 + 1}}{\sqrt[5]{n^4 + 2} - \sqrt{n^3 + 1}}$

7. $\lim_{n\to\infty} \frac{(n+2)! + (n+1)!}{(n+3)!}$

8. $\lim_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$

9. $\lim_{n\to\infty}\left[\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{(n-1)n}\right]$

10. $\lim_{n\to\infty}\left[\frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]$

11. $\lim_{x\to\infty}\left(\frac{x^3}{2x^2-1}-\frac{x^2}{2x+1}\right)$

12. $\lim_{x\to\infty}\left[\frac{3x^2}{2x+1}-\frac{(2x-1)(3x^2+x+2)}{4x^2}\right]$

13. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\ldots.+(x+100)^{10}}{x^{10}+10^{10}}$

14. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^3+x}-x}$

15. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

16. $\lim_{x\to\infty}\frac{\sqrt[5]{x^7+3}+\sqrt[4]{2x^3-1}}{\sqrt[6]{x^8+x^7+1}-x}$

17. $\lim_{x\to\infty}\frac{\sqrt[3]{x^4+3}-\sqrt[5]{x^3+4}}{\sqrt[3]{x^7+1}}$

18. $\lim_{x\to\infty}\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}$

19. $\lim_{x\to\infty}x^{3/2}(\sqrt{x^3+1}-\sqrt{x^3-1})$

20. $\lim_{x\to\infty}\sqrt[4]{(x+a)(x+b)(x+c)(x+d)}-x$

21. If $\lim_{x\to\infty}(\sqrt{x^2-x+1}-(ax+b))=0$, then find the constants a and b.

22. If $\lim_{x\to-\infty}(\sqrt{x^2-x+1}+ax-b)=0$, find the constants a and b.

23. If $\lim_{\alpha\to\infty}\left(\frac{\alpha^2+1}{\alpha+1}-a\alpha\right)$ be finite, find $a\in R$ and the limit.

24. Evaluate :

$\lim_{n\to\infty}\left\{\log_{n-1}n\,.\,\log_n(n+1)\,.\,\log_{n+1}(n+2)\ldots\log_{n^k-1}n^k\right.$

where $k\in N$.

25. Evaluate : $\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

26. $\lim_{x\to\infty}x(\sqrt{x^2+\sqrt{x^4+1}}-x\sqrt{2})$

27. $\lim_{n\to\infty}\frac{1\,.\,n+2\,.\,(n-1)+3.\,(n-2)+..+(n-1)\cdot 2+n\,.\,1}{1^2+2^2+\ldots+n^2}$

28. $\lim_{n\to\infty}\frac{[1^3x]+\frac{1}{2}[2^3x]+\frac{1}{3}[3^3x]+\ldots.+\frac{1}{n}[n^3x]}{1^2+2^2+\ldots+n^2}$

where $[\cdot]$ denotes the greatest integer function.

ANSWERS

1. 0 2. $\frac{15}{17}$ 3. 1 4. 0
5. 1 6. 0 7. 0 8. 1
9. 1 10. $\frac{1}{2}$ 11. $\frac{1}{4}$ 12. $\frac{-1}{2}$
13. 100 14. -1 15. 1 16. ∞
17. 0 18. 0 19. 1 20. $\frac{a+b+c+d}{4}$
21. $a=1, b=-\frac{1}{2}$ 22. $a=1, b=\frac{1}{2}$
23. $a=1$, Limit $=-1$ 24. k 25.
26. 0, if $x\to\infty,-\infty$, if $x\to-\infty$ 27. $\frac{1}{2}$ 28.

—❑ ❑ ❑—

3.7 EVALUATION OF TRIGONOMETRIC LIMITS

In order to evaluate trigonometric limits the following results are very useful :

(i) $\lim_{x\to0}\frac{\sin x}{x}=1$ (ii) $\lim_{x\to0}\frac{\tan x}{x}=1$

(iii) $\lim_{x\to0}\frac{\sin^{-1}x}{x}=1$ (iv) $\lim_{x\to0}\frac{\tan^{-1}x}{x}=1$

(v) $\lim_{x\to0}\frac{\sin x^\circ}{x}=\frac{\pi}{180}$ (vi) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$

(vii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$ (viii) $\lim_{x\to0}\frac{|\sin x|}{x}$ does not exist

(ix) $\lim_{x\to0}\frac{|\tan x|}{x}$ does not exist

(x) $\lim_{x\to a}\frac{|\sin(x-a)|}{x-a}$ does not exist.

(xi) $\lim_{x\to a}\dfrac{|\tan(x-a)|}{x-a}$ does not exist.

Sometimes following expansions are very useful to evaluate limits.

(a) $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+\ldots$

(b) $e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots$

(c) $a^x = 1+x(\log_e a)+\dfrac{x^2}{2!}(\log_e a)^2+\ldots$

(d) $\log(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots$

(e) $\log(1-x) = -x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots$

(f) $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots$

(g) $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots$

(h) $\tan x = x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+\ldots$

(i) $\sin^{-1}x = x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5}+\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\dfrac{x^7}{7}+\ldots$

(j) $\tan^{-1}x = x-\dfrac{1}{3}x^3+\dfrac{1}{5}x^5+\ldots$

(k) $\sec^{-1}x = 1+\dfrac{x^2}{2!}+5\dfrac{x^4}{4!}+\ldots.$

Following examples will illustrate the use of these formulae.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\lim_{x\to\pi/4}\dfrac{\cos x-\sin x}{(\pi/4-x)(\cos x+\sin x)}$

SOLUTION We have,

$$\lim_{x\to\pi/4}\frac{\cos x-\sin x}{(\pi/4-x)(\cos x+\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)}{(\pi/4-x)(\cos x+\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{\sqrt{2}(\sin\pi/4\cos x-\cos\pi/4\sin x)}{(\pi/4-x)(\cos x+\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{\sqrt{2}\sin(\pi/4-x)}{(\pi/4-x)}\cdot\frac{1}{(\cos x+\sin x)}$$

$$=\sqrt{2}\,\frac{1}{\cos\pi/4+\sin\pi/4}$$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=1$$

EXAMPLE 2 *Evaluate :* $\lim_{x\to a}\dfrac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$

SOLUTION We have,

$$\lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$$

$$=\lim_{x\to a}\frac{-2\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)\sin\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)}{2\times\frac{\sqrt{x}-\sqrt{a}}{2}(\sqrt{x}+\sqrt{a})}$$

$$=-\lim_{x\to a}\frac{\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}\cdot\frac{\sin\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)}{(\sqrt{x}+\sqrt{a})}$$

$$=-1\times\frac{\sin\sqrt{a}}{2\sqrt{a}}=-\frac{\sin\sqrt{a}}{2\sqrt{a}}$$

EXAMPLE 3 *Evaluate :* $\lim_{x\to 2}\dfrac{\sqrt{1-\cos 2(x-2)}}{x-2}$ **[IIT 1998]**

SOLUTION We have,

$$\lim_{x\to 2}\frac{\sqrt{1-\cos 2(x-2)}}{x-2}=\lim_{x\to 2}\frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$$

Now,

$$\lim_{x\to 2^-}\frac{\sqrt{1-\cos 2(x-2)}}{x-2}$$

$$=\lim_{x\to 2^-}\frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$$

$$=-\sqrt{2}\lim_{x\to 2}\frac{\sin(x-2)}{x-2}\left[\because x<2\Rightarrow x-2<0\Rightarrow \sin(x-2)<0\right.$$
$$\left.\therefore\ |\sin(x-2)|=-\sin(x-2)\right]$$

$$=-\sqrt{2}$$

and,

$$\lim_{x\to 2^+}\frac{\sqrt{1-\cos 2(x-2)}}{x-2}$$

$$= \lim_{x \to 2^+} \frac{|\sqrt{2}\sin(x-2)|}{x-2}$$

$$= \sqrt{2} \lim_{x \to 2} \frac{\sin(x-2)}{x-2} \qquad \left[\because x > 2 \Rightarrow x-2 > 0 \Rightarrow \sin(x-2) > 0 \right.$$
$$\left. \therefore \ |\sin(x-2)| = \sin(x-2)\right]$$

$$= \sqrt{2}$$

$$\therefore \lim_{x \to 2^-} \frac{\sqrt{1-\cos 2(x-2)}}{x-2} \neq \lim_{x \to 2^+} \frac{\sqrt{1-\cos 2(x-2)}}{x-2}.$$

Hence, $\lim_{x \to 2} \frac{\sqrt{1-\cos 2(x-2)}}{x-2}$ does not exist.

EXAMPLE 4 *Evaluate* : $\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$.

SOLUTION We have,

$$\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$$

$$= \lim_{x \to \pi/2} \frac{\cot x (1 - \sin x)}{(\pi - 2x)^3}$$

$$= \lim_{x \to \pi/2} \frac{\tan(\pi/2 - x)\ [1 - \cos(\pi/2 - x)]}{(\pi - 2x)^3}$$

$$= \lim_{x \to \pi/2} \frac{\tan(\pi/2 - x) \cdot 2\sin^2(\pi/4 - x/2)}{2(\pi/2 - x) \cdot 16(\pi/4 - x/2)^2}$$

$$= \frac{2}{32} \lim_{x \to \pi/2} \frac{\tan(\pi/2 - x)}{(\pi/2 - x)} \times \lim_{x \to \pi/2} \left\{\frac{\sin(\pi/4 - x/2)}{(\pi/4 - x/2)}\right\}^2$$

$$= \frac{2}{32} = \frac{1}{16}.$$

EXAMPLE 5 *Evaluate* : $\lim_{x \to \tan^{-1} 3} \frac{\tan^2 x - 2\tan x - 3}{\tan^2 x - 4\tan x + 3}$

SOLUTION We have,

$$\lim_{x \to \tan^{-1} 3} \frac{\tan^2 x - 2\tan x - 3}{\tan^2 x - 4\tan x + 3}$$

$$= \lim_{x \to \tan^{-1} 3} \frac{(\tan x - 3)(\tan x + 1)}{(\tan x - 3)(\tan x - 1)}$$

$$= \lim_{x \to \tan^{-1} 3} \frac{\tan x + 1}{\tan x - 1} = \frac{3+1}{3-1} = 2$$

EXAMPLE 6 *Evaluate* : $\lim_{x \to \pi/3} \frac{\tan^3 x - 3\tan x}{\cos(\pi/6 + x)}$

SOLUTION We have,

$$\lim_{x \to \pi/3} \frac{\tan^3 x - 3\tan x}{\cos(\pi/6 + x)}$$

$$= \lim_{x \to \pi/3} \frac{\tan x(\tan x - \sqrt{3})(\tan x + \sqrt{3})}{\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x}$$

$$= \lim_{x \to \pi/3} \frac{\tan x(\tan x - \sqrt{3})(\tan x + \sqrt{3})}{-\frac{1}{2}\cos x(\tan x - \sqrt{3})}$$

$$= \lim_{x \to \pi/3} \frac{-2\sin x(\tan x + \sqrt{3})}{\cos^2 x}$$

$$= \frac{-2 \times (\sqrt{3}/2)(\sqrt{3} + \sqrt{3})}{1/4}$$

$$= -24$$

EXAMPLE 7 *Evaluate the following limits :*

(i) $\lim_{x \to 0} \frac{\tan x + 4\tan 2x - 3\tan 3x}{x^2 \tan x}$

(ii) $\lim_{n \to \infty} \left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\tan x + 4\tan 2x - 3\tan 3x}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{(\tan x + \tan 2x) - 3(\tan 3x - \tan 2x)}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin 3x}{\cos x \cos 2x} - \frac{3\sin x}{\cos 3x \cos 2x}}{x^2 \tan x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin 6x - 3\sin 2x}{\cos x \cos 2x \cos 3x\, x^2 \tan x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{-4\sin^3 2x}{\cos x \cos 2x \cos 3x\, x^2 \tan x}$$

$$= -16 \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{2x}\right)^3}{\cos x \cos 2x \cos 3x \left(\frac{\tan x}{x}\right)}.$$

$$= -16$$

(ii) $\lim_{n \to \infty} \left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$

$$= \lim_{n \to \infty} \frac{\sin\left(2^n \cdot \frac{x}{2^n}\right)}{2^n \sin\left(\frac{x}{2^n}\right)} \qquad \left[\because \cos A \cos 2A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}\right]$$

$$= \lim_{n \to \infty} \frac{\sin x}{x \left(\frac{\sin(x/2^n)}{\frac{x}{2^n}}\right)}$$

$$= \frac{\sin x}{x}$$

EXAMPLE 8 *Without using L'Hospital's rule or series expansion, evaluate the following limits :*

(i) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

(ii) $\lim\limits_{x\to \pi/4} \dfrac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$

(iii) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$

SOLUTION (i) Let $l = \lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$. Then,

$$l = \lim_{x\to 0} \frac{3x-3\sin x}{3x^3}$$

$$\Rightarrow l = \lim_{x\to 0} \frac{3x-(\sin 3x+4\sin^3 x)}{3x^3}$$

$$\Rightarrow l = \lim_{x\to 0} \left[\frac{3x-\sin 3x}{3x^3} - \frac{4}{3}\left(\frac{\sin x}{x}\right)^3\right]$$

$$\Rightarrow l = 9\lim_{x\to 0} \frac{3x-\sin 3x}{(3x)^3} - \frac{4}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3$$

$$\Rightarrow l = 9l - \frac{4}{3}$$

$$\Rightarrow 8l = \frac{4}{3}$$

$$\Rightarrow l = \frac{1}{6}$$

Hence, $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} = \dfrac{1}{6}$

(ii) We have,

$$\lim_{x\to \pi/4} \frac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$$

$$= \lim_{x\to \pi/4} \frac{2^{5/2}-[(\cos x+\sin x)^2]^{5/2}}{2-(1+\sin 2x)}$$

$$= \lim_{x\to \pi/4} \frac{2^{5/2}-(1+\sin 2x)^{5/2}}{2-(1+\sin 2x)}$$

$$= \lim_{x\to \pi/4} \frac{(1+\sin 2x)^{5/2}-2^{5/2}}{(1+\sin 2x)-2}$$

$$= \lim_{u\to 2} \frac{u^{5/2}-2^{5/2}}{u-2}, \text{ where } u = 1+\sin 2x$$

$$= \frac{5}{2}\times(2)^{5/2-1} = \frac{5}{2}\times 2^{3/2} = 5\sqrt{2}.$$

(iii) Let $l = \lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$. Then,

$$l = \lim_{x\to 0} \frac{3\tan x - 3x}{3x^3}$$

Using : $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$, we have

$$l = \lim_{x\to 0} \frac{\tan 3x - 3\tan^2 x\tan 3x + \tan^3 x - 3x}{3x^3}$$

$$\Rightarrow l = \lim_{x\to 0} \frac{\tan 3x-3x}{3x^3} - 3\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2\left(\frac{\tan 3x}{3x}\right) + \frac{1}{3}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^3$$

$$\Rightarrow l = 9\lim_{x\to 0} \frac{\tan 3x-3x}{(3x)^3} - 3 + \frac{1}{3}$$

$$\Rightarrow l = 9l - 3 + \frac{1}{3}$$

$$\Rightarrow 8l = \frac{8}{3}$$

$$\Rightarrow l = \frac{1}{3}$$

EXAMPLE 9 *Evaluate the following limits :*

(i) $\lim\limits_{x\to 0} \dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

(ii) $\lim\limits_{x\to 0} \dfrac{1-\cos(1-\cos x)}{x^4}$

(iii) $\lim\limits_{x\to 0} \dfrac{\cos(\sin x)-\cos x}{x^4}$

SOLUTION (i) We have,

$$\lim_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}}{x^2}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x\cos 2x}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x(2\cos^2 x-1)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= -\lim_{x\to 0} \frac{2\cos^4 x-\cos^2 x-1}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= -\lim_{x\to 0} \frac{(2\cos^2 x+1)(\cos^2 x-1)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0} \frac{(2\cos^2 x+1)}{(1+\cos x\sqrt{\cos 2x})}\times\frac{\sin^2 x}{x^2}$$

$= \frac{3}{2}$

(ii) We have,

$$\lim_{x\to 0} \frac{1-\cos(1-\cos x)}{x^4}$$

$$= \lim_{x\to 0} \frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{x^4}$$

$$= 2\lim_{x\to 0} \left\{\frac{\sin\left(\sin^2\frac{x}{2}\right)}{\sin^2\frac{x}{2}}\right\}^2 \times \frac{\sin^4\frac{x}{2}}{16\times\left(\frac{x}{2}\right)^4}$$

$$= \frac{2}{16} = \frac{1}{8}$$

(iii) We have,

$$\lim_{x\to 0} \frac{\cos(\sin x)-\cos x}{x^4}$$

$$\lim_{x\to 0} \frac{2\sin\left(\frac{x+\sin x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$$

$$= 2\lim_{x\to 0} \frac{\sin\left(\frac{x+\sin x}{2}\right)}{\frac{x+\sin x}{2}} \times \frac{\sin\left(\frac{x-\sin x}{2}\right)}{\frac{x-\sin x}{2}} \times \frac{x+\sin x}{2x} \times \frac{x-\sin x}{2x^3}$$

$$= \frac{2}{4}\lim_{x\to 0}\left[\left\{\frac{\sin\left(\frac{x+\sin x}{2}\right)}{\frac{x+\sin x}{2}}\right\} \times \left\{\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\frac{x-\sin x}{2}}\right\} \times \left(\frac{x+\sin x}{x}\right) \times \left\{\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^3}\right\}\right]$$

$$= \frac{2}{4}\times 1\times 1\times(1+1)\times\frac{1}{3!}$$

$$= \frac{1}{6}$$

EXAMPLE 10 *Evaluate the following limits :*

(i) $\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-\sqrt{\cos x}}{\tan^2 x/2}$

(ii) $\lim_{x\to 0} \frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$

SOLUTION (i) We have,

$$\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-\sqrt{\cos x}}{\tan^2 x/2}$$

$$= \lim_{x\to 0} \frac{1+x\sin x-\cos x}{\tan^2 x/2} \times \frac{1}{(\sqrt{1+x\sin x}+\sqrt{\cos x})}$$

$$= \lim_{x\to 0}\left[\frac{2\sin^2\frac{x}{2}+2x\sin\frac{x}{2}\cos\frac{x}{2}}{\tan^2\frac{x}{2}} \times \frac{1}{\sqrt{1+x\sin x}+\sqrt{\cos x}}\right]$$

$$= \lim_{x\to 0}\left[\left\{2\cos^2\frac{x}{2}+2\cos^3\frac{x}{2}\times\frac{x}{\sin\frac{x}{2}}\right\} \times \frac{1}{\sqrt{1+x\sin x}+\sqrt{\cos x}}\right]$$

$$= \lim_{x\to 0}\left[\left\{2\cos^2\frac{x}{2}+4\cos^3\frac{x}{2}\times\frac{x/2}{\sin\frac{x}{2}}\right\} \times \frac{1}{\sqrt{1+x\sin x}+\sqrt{\cos x}}\right]$$

$$= (2+4)\times\frac{1}{2} = 3.$$

(ii) Using : $a-b = \frac{a^3-b^3}{a^2+ab+b^2}$, we have

$$\lim_{x\to 0} \frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$$

$$= \lim_{x\to 0} \frac{\{(1+\tan^{-1}3x)-(1-\sin^{-1}3x)\}}{\{(1-\sin^{-1}2x)-(1+\tan^{-1}2x)\}} \times \frac{(\sqrt{1-\sin^{-1}2x}+\sqrt{1+\tan^{-1}2x})}{[(1+\tan^{-1}3x)^{2/3}+(1-\sin^{-1}3x)^{2/3}+(1+\tan^{-1}3x)^{1/3}\times(1-\sin^{-1}3x)^{1/3}]}$$

$$= -\lim_{x\to 0}\left[\frac{\tan^{-1}3x+\sin^{-1}3x}{\tan^{-1}2x+\sin^{-1}2x} \times \frac{\sqrt{1-\sin^{-1}2x}+\sqrt{1+\tan^{-1}2x}}{[(1+\tan^{-1}3x)^{2/3}+(1-\sin^{-1}3x)^{2/3}+(1+\tan^{-1}3x)^{1/3}(1-\sin^{-1}3x)^{1/3}]}\right]$$

$$= -\lim_{x\to 0}\left[\frac{3\left(\frac{\tan^{-1}3x}{3x}+\frac{\sin^{-1}3x}{3x}\right)}{2\left(\frac{\tan^{-1}2x}{2x}+\frac{\sin^{-1}2x}{2x}\right)} \times \frac{\sqrt{1-\sin^{-1}2x}+\sqrt{1+\tan^{-1}2x}}{[(1+\tan^{-1}3x)^{2/3}+(1-\sin^{-1}3x)^{2/3}+(1+\tan^{-1}3x)^{1/3}(1-\sin^{-1}3x)^{1/3}]}\right]$$

$$= -\left(\frac{3\times 2}{2\times 2}\right)\times\left(\frac{2}{1+1+1}\right)$$

$$= -\frac{6}{4} \times \frac{2}{3} = -1.$$

EXAMPLE 11 *If* $\Delta = \begin{vmatrix} \tan x & \tan(x+h) & \tan(x+2h) \\ \tan(x+2h) & \tan x & \tan(x+h) \\ \tan(x+h) & \tan(x+2h) & \tan x \end{vmatrix}$,

find $\lim_{h \to 0} \frac{\Delta}{h^2}$.

SOLUTION We have,

$$\Delta = \begin{vmatrix} \tan x & \tan(x+h) & \tan(x+2h) \\ \tan(x+2h) & \tan x & \tan(x+h) \\ \tan(x+h) & \tan(x+2h) & \tan x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we obtain

$$\Delta = \begin{vmatrix} \tan x & \tan(x+h) - \tan x & \tan(x+2h) - \tan x \\ \tan(x+2h) & \tan x - \tan(x+2h) & \tan(x+h) - \tan(x+2h) \\ \tan(x+h) & \tan(x+2h) - \tan(x+h) & \tan x - \tan(x+h) \end{vmatrix}$$

$$\Rightarrow \frac{\Delta}{h^2} = \begin{vmatrix} \tan x & \frac{\tan(x+h) - \tan x}{h} & \frac{\tan(x+2h) - \tan x}{h} \\ \tan(x+2h) & \frac{\tan x - \tan(x+2h)}{h} & \frac{\tan(x+h) - \tan(x+2h)}{h} \\ \tan(x+h) & \frac{\tan(x+2h) - \tan(x+h)}{h} & \frac{\tan x - \tan(x+h)}{h} \end{vmatrix}$$

$\because \lim_{h \to 0} \frac{\tan(x+h) - \tan x)}{h} = \sec^2 x,$

$$\lim_{h \to 0} \frac{\tan(x+2h) - \tan(x+h)}{h} = \sec^2 x,$$

and, $\lim_{h \to 0} \frac{\tan(x+2h) - \tan x}{h} = 2\sec^2 x$

$$\therefore \lim_{h \to 0} \frac{\Delta}{h^2} = \begin{vmatrix} \tan x & \sec^2 x & 2\sec^2 x \\ \tan x & -2\sec^2 x & -\sec^2 x \\ \tan x & \sec^2 x & -\sec^2 x \end{vmatrix}$$

$$= \tan x \sec^4 x \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \tan x \sec^4 x \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & -3 \end{vmatrix} \quad [\text{Using} : R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1]$$

$$= 9 \tan x \sec^4 x.$$

EXAMPLE 12 *Evaluate the following limits :*

(i) $\lim_{x \to -1} \frac{\cos 2 - \cos 2x}{x^2 - |x|}$

(ii) $\lim_{x \to 0} \frac{\tan([-2\pi^2]x^2) - x^2 \tan[-2\pi^2]}{\sin^2 x}$,

where $[\cdot]$ *is the greatest integer function.*

SOLUTION (i) We have,

$$\lim_{x \to -1} \frac{\cos 2 - \cos 2x}{x^2 - |x|}$$

$$= \lim_{x \to -1} \frac{\cos 2 - \cos 2x}{x^2 + x} \qquad [\because x < 0 \;\therefore\; |x| = -x]$$

$$= \lim_{x \to -1} \frac{2\sin(1+x)\sin(x-1)}{x(1+x)}$$

$$= \lim_{x \to -1} \frac{2\sin(x-1)}{x} \times \frac{\sin(1+x)}{1+x}$$

$$= \frac{2 \times \sin(-2)}{(-1)} \times 1$$

$$= 2\sin 2$$

(ii) $\lim_{x \to 0} \frac{\tan([-2\pi^2]x^2) - x^2 \tan[-2\pi^2]}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{\tan(-20)x^2 - x^2 \tan(-20)}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{-\tan 20\, x^2 + x^2 \tan 20}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{-20\left(\frac{\tan 20x^2}{20x^2}\right) + \tan 20}{\left(\frac{\sin x}{x}\right)^2} = -20 + \tan 20$$

EXAMPLE 13 *If* α, β *are the roots of* $ax^2 + bx + c$, *then evaluate*

(i) $\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$

(ii) $\lim_{x \to 1/\alpha} \frac{1 - \cos(cx^2 + bx + a)}{(1 - \alpha x)^2}$

SOLUTION (i) Since α, β are the roots of the given equation. Therefore,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

Now,

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$$

$$= \lim_{x \to \beta} \frac{1 - \cos a(x - \alpha)(x - \beta)}{(x - \beta)^2}$$

$$= \lim_{x \to \beta} \frac{2\sin^2\left\{\frac{a(x - \alpha)(x - \beta)}{2}\right\}}{(x - \beta)^2}$$

$$= 2\lim_{x\to\beta}\left[\frac{\sin\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}\right]^2 \times \frac{a^2(x-\alpha)^2}{4}$$

$$= 2\times a^2\frac{(\beta-\alpha)^2}{4} = \frac{a^2(\beta-\alpha)^2}{2}$$

(ii) Since α, β are the roots of $ax^2+bx+c=0$, therefore $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of $cx^2+bx+a=0$.

So, $cx^2+bx+a = c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)$

Now,

$$\lim_{x\to 1/\alpha}\frac{1-\cos(cx^2+bx+a)}{(1-\alpha x)^2}$$

$$= \lim_{x\to 1/\alpha}\frac{1-\cos c\left(x-\alpha^{-1}\right)\left(x-\beta^{-1}\right)}{\alpha^2\left(\alpha^{-1}-x\right)^2}$$

$$= \lim_{x\to 1/\alpha}\frac{2\sin^2\left\{\frac{c\left(x-\alpha^{-1}\right)\left(x-\beta^{-1}\right)}{2}\right\}\times\frac{\left(x-\beta^{-1}\right)^2}{4}}{\alpha^2\left\{\frac{\left(x-\alpha^{-1}\right)^2\left(x-\beta^{-1}\right)^2}{4}\right\}}$$

$$= \frac{2}{\alpha^2}\lim_{x\to\alpha^{-1}}\left[\frac{\sin\left\{\frac{c(x-\alpha^{-1})(x-\beta^{-1})}{2}\right\}}{c\left(x-\alpha^{-1}\right)\left(x-\beta^{-1}\right)}\right]^2\times\frac{c^2\left(x-\beta^{-1}\right)^2}{4}$$

$$= \frac{2}{\alpha^2}\times\frac{c^2\left(\alpha^{-1}-\beta^{-1}\right)^2}{4} = \frac{c^2}{2\alpha^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$$

EXAMPLE 14 *Prove that :*

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}+\lim_{x\to\pi/4}\frac{\sqrt{2}-2\cos x}{\sin(x-\pi/4)} > \log_{10}0.005$$

SOLUTION We have,

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}+\lim_{x\to\pi/4}\frac{\sqrt{2}-2\cos x}{\sin(x-\pi/4)}$$

$$= \lim_{x\to 0}\frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}+2\lim_{x\to\pi/4}\frac{\cos\pi/4-\cos x}{\sin(x-\pi/4)}$$

$$= \lim_{x\to 0}\frac{1+x}{\sqrt{1+x+x^2}+1}+2\lim_{x\to\pi/4}\frac{2\sin\left(\frac{\pi}{8}+\frac{x}{2}\right)\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)}{2\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)\cos\left(\frac{x}{2}-\frac{\pi}{8}\right)}$$

$$= \frac{1}{2}+2\sin\frac{\pi}{4}$$

$$= \frac{1}{2}+\sqrt{2} > 0$$

and,

$\log_{10}0.005$

$= \log_{10}(5\times10^{-3})$

$= \log_{10}5+\log_{10}10^{-3} = \log_{10}5-3<0$

$$\therefore \lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}+\lim_{x\to\pi/4}\frac{\sqrt{2}-2\cos x}{\sin(x-\pi/4)} > \log_{10}0.005$$

EXAMPLE 15 *(i) If* $\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$, *find the values of* a, b, c.

(ii) *If* $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$, *find the values of* a and b.

SOLUTION We have,

$$\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$$

$$\Rightarrow \lim_{x\to 0}\frac{a\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right\}-b\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right\}+c\left\{1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\ldots\right\}}{x\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right\}} = 2$$

$$\Rightarrow \lim_{x\to 0}\frac{(a-b+c)+x(a-c)+x^2\left\{\frac{a}{2!}+\frac{b}{2!}+\frac{c}{2!}\right\}+\ldots}{x^2\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\right\}} = 2$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{a-b+c}{x^2}+\frac{a-c}{x}+\left\{\frac{a}{2!}+\frac{b}{2!}+\frac{c}{2!}\right\}+x(\ldots)+\ldots.}{1-\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots.} = 2$$

Clearly, limit on LHS will exist, if

$$a-b+c = 0 \text{ and } a-c = 0$$

and in that case the limit will be $\frac{a}{2!}+\frac{b}{2!}+\frac{c}{2!}$

$$\therefore \frac{a}{2!}+\frac{b}{2!}+\frac{c}{2!} = 2$$

Thus, we have $a-b+c = 0, a-c = 0$ and $a+b+c = 4$.

Solving these equations, we get $a = 1, b = 2$ and $c = 1$.

(ii) We have,

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\right\}-b\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots.\right\}}{x^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{(1+a-b)+x^2\left(-\frac{a}{2!}+\frac{b}{3!}\right)+x^4\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots}{x^2}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{1+a-b}{x^2}+\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots=1$$

Clearly, the limit on LHS will exist if $1+a-b=0$ and in that case, we have

$$\lim_{x\to 0}\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots..=1$$

$$\Rightarrow -\frac{a}{2!}+\frac{b}{3!}=1$$

$$\Rightarrow -3a+b=6$$

Solving $1+a-b=0$ and $-3a+b=6$, we get

$$a=-5/2 \text{ and } b=-3/2.$$

EXAMPLE 16 *Evaluate :*

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\ldots\infty}}}}\right\}$$

SOLUTION Let $\frac{1}{n}=x$. Then,

$$\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\ldots\infty}}}}\right\}$$

$$=\lim_{x\to 0}\frac{(1-\cos x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty}}{x^2}$$

$$=\lim_{x\to 0}\frac{(1-\cos x)}{x^2}\qquad\left[\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots..=\frac{1/2}{1-\frac{1}{2}}=1\right]$$

$$=\lim_{x\to 0}\frac{2\sin^2 x/2}{x^2}=\frac{1}{2}$$

EXAMPLE 17 *Let $f(x)$ be a function such that $\lim_{x\to 0}\frac{f(x)}{x}=1$. Find the values of a and b without using L' Hospital's rule so that*

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{[f(x)]^3}=1$$

SOLUTION We have, $\lim_{x\to 0}\frac{f(x)}{x}=1$.

$$\therefore \lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{[f(x)]^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots.\right)\right\}-b\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right\}}{[f(x)]^3}$$

$$\Rightarrow \lim_{x\to 0}\frac{x(1+a-b)+x^3\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^5\left(\frac{a}{4!}-\frac{b}{5!}\right)\ldots}{[f(x)]^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{1+a-b}{x^2}+\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots.}{\left\{\frac{f(x)}{x}\right\}^3}=1$$

Clearly, the limit on LHS will exist, if $1+a-b=0$ and in that case, we have

$$\lim_{x\to 0}\frac{\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots.}{\left(\frac{f(x)}{x}\right)^3}=1$$

$$\Rightarrow \frac{-a}{2!}+\frac{b}{3!}=1 \qquad \left[\because \lim_{x\to 0}\frac{f(x)}{x}=1\right]$$

$$\Rightarrow -3a+b=6$$

Solving $1+a-b=0$ and $-3a+b=6$, we get

$$a=-5/2 \text{ and } b=-3/2.$$

EXAMPLE 18 *Evaluate :*

$$\lim_{x\to 0}\frac{-1+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\ldots..\infty}}}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots.....\sqrt{x}}}}}$$

SOLUTION Let

$$y=\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\sqrt{(\tan x-\sin x)+\ldots..\infty}}}$$

Then,

$$y=\sqrt{(\tan x-\sin x)+y}$$

$$\Rightarrow y^2=(\tan x-\sin x)+y$$

$$\Rightarrow y^2-y-(\tan x-\sin x)=0$$

$$\Rightarrow y=\frac{1\pm\sqrt{1+4(\tan x-\sin x)}}{2}$$

$$\Rightarrow y=\frac{1+\sqrt{1+4(\tan x-\sin x)}}{2} \qquad [\because y>0]$$

Now, let

$$z=\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots.....\infty}}}$$

$$\Rightarrow z=\sqrt{x^3+z}$$

$\Rightarrow z^2 = x^3 + z$

$\Rightarrow z^2 - z - x^3 = 0$

$\Rightarrow z = \dfrac{1 \pm \sqrt{1+4x^3}}{2}$

$\Rightarrow z = \dfrac{1+\sqrt{1+4x^3}}{2}$ $[\because z > 0]$

$$\therefore \lim_{x\to 0} \frac{-1+\sqrt{(\tan x - \sin x)+\sqrt{(\tan x - \sin x)+\sqrt{(\tan x - \sin x)+\ldots..\sqrt{\infty}}}}}{-1+\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+\ldots..\sqrt{x}}}}}$$

$$\lim_{x\to 0} \frac{-1+\dfrac{1+\sqrt{1+4(\tan x - \sin x)}}{2}}{-1+\dfrac{1+\sqrt{1+4x^3}}{2}}$$

$$= \lim_{x\to 0} \frac{-1+\sqrt{1+4(\tan x - \sin x)}}{-1+\sqrt{1+4x^3}}$$

$$= \lim_{x\to 0} \frac{4(\tan x - \sin x)}{4x^3} \times \frac{1+\sqrt{1+4x^3}}{1+\sqrt{1+4(\tan x - \sin x)}}$$

$$= \lim_{x\to 0} \frac{\tan x - \sin x}{x^3} \times \frac{2}{2}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \times \frac{1-\cos x}{x^2}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2}$$

EXAMPLE 19 *Evaluate :* $\lim_{x\to 0} \dfrac{\sin^{-1} x - \tan^{-1} x}{x^3}$.

SOLUTION We have,

$$\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}.$$

$$\therefore \lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan^{-1} \dfrac{x}{\sqrt{1-x^2}} - \tan^{-1} x}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{x^3} \left\{ \tan^{-1} \left(\frac{\dfrac{x}{\sqrt{1-x^2}} - x}{1+\dfrac{x^2}{\sqrt{1-x^2}}} \right) \right\} \quad \left[\text{Using : } \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right]$$

$$= \lim_{x\to 0} \frac{1}{x^3} \tan^{-1} \left| \frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|$$

$$= \lim_{x\to 0} \frac{1}{x^3} \left\{ \frac{\tan^{-1} \left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|}{\left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|} \right\} \cdot \frac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \left\{ \frac{\tan^{-1} \left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|}{\left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|} \right\} \times \frac{1-\sqrt{1-x^2}}{x^2} \times \frac{1}{x^2+\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \left\{ \frac{\tan^{-1} \left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|}{\left| \dfrac{x(1-\sqrt{1-x^2})}{x^2+\sqrt{1-x^2}} \right|} \right\} \times \frac{x^2}{x^2(1+\sqrt{1-x^2})} \times \frac{1}{x^2+\sqrt{1-x^2}}$$

$$= 1 \times \frac{1}{1+1} \times \frac{1}{1} = \frac{1}{2}.$$

EXAMPLE 20 *Evaluate :* $\lim_{x\to\infty} x \left\{ \tan^{-1} \dfrac{x+1}{x+4} - \dfrac{\pi}{4} \right\}$

SOLUTION We have,

$$\lim_{x\to\infty} x \left[\tan^{-1} \frac{x+1}{x+4} - \frac{\pi}{4} \right]$$

$$= \lim_{x\to\infty} x \left[\tan^{-1} \frac{x+1}{x+4} - \tan^{-1} 1 \right]$$

$$= \lim_{x\to\infty} x \tan^{-1} \left\{ \frac{\dfrac{x+1}{x+4} - 1}{1+\dfrac{x+1}{x+4}} \right\} \quad \left[\text{Using : } \tan^{-1} x - \tan^{-1} = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right.$$

$$= \lim_{x\to\infty} x \tan^{-1} \left(\frac{-3}{2x+5} \right)$$

$$= \lim_{x\to\infty} x \left\{ \frac{\tan^{-1} \left(\dfrac{-3}{2x+5} \right)}{\dfrac{-3}{2x+5}} \right\} \cdot \left(\frac{-3}{2x+5} \right)$$

$$= \lim_{x\to\infty} \left\{ \frac{\tan^{-1} \left(\dfrac{-3}{2x+5} \right)}{\dfrac{-3}{2x+5}} \right\} \cdot \lim_{x\to\infty} \left(\frac{-3x}{2x+5} \right)$$

$$= \lim_{x\to\infty} \left\{ \frac{\tan^{-1} \left(\dfrac{-3}{2x+5} \right)}{\dfrac{-3}{2x+5}} \right\} \cdot \lim_{x\to\infty} \left(\frac{-3}{2+\dfrac{5}{x}} \right)$$

$= 1 \times -\frac{3}{2} = -\frac{3}{2}$

EXAMPLE 21 *Prove that*

$$\lim_{x\to 2} \frac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}} - \lim_{x\to \pi/3} \frac{\tan^3 x - 3\tan x}{\cos(x + \pi/6)} > e^3$$

SOLUTION Let $l_1 = \lim_{x\to 2} \frac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}}$. Then,

$$l_1 = \lim_{x\to 2} \frac{(2^x)^2 + 2^3 - 6.2^x}{\sqrt{2^x} - 2}$$

$$\Rightarrow l_1 = \lim_{x\to 2} \frac{\{(2^x)^2 - 6.2^x + 8\}(\sqrt{2^x} + 2)}{(2^x - 4)}$$

$$\Rightarrow l_1 = \lim_{x\to 2} \frac{(2^x - 4)(2^x - 2)(\sqrt{2^x} + 2)}{(2^x - 4)}$$

$$\Rightarrow l_1 = \lim_{x\to 2} (2^x - 2)(\sqrt{2^x} + 2)$$

$$\Rightarrow l_1 = 2 \times 4 = 8$$

Let $l_2 = \lim_{x\to \pi/3} \frac{\tan^3 x - 3\tan x}{\cos(x + \pi/6)}$. Then,

$$l_2 = \lim_{x\to \pi/3} \frac{\tan x(\tan x - \sqrt{3})(\tan x + \sqrt{3})}{\frac{1}{2}(\sqrt{3} - \tan x)\cos x}$$

$$\Rightarrow l_2 = -2 \lim_{x\to \pi/3} \frac{\tan x(\tan x + \sqrt{3})}{\cos x} = -24$$

$\therefore\ l_1 - l_2 = 8 - (-24) = 32.$

Clearly, $l_1 - l_2 > e^3$.

EXAMPLE 22 *Evaluate :* $\lim_{x\to \pi/2} \left[\frac{x - \frac{\pi}{2}}{\cos x}\right]$, *where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION Let $x - \frac{\pi}{2} = \theta$. Then,

$$\lim_{x\to \pi/2} \left[\frac{x - \pi/2}{\cos x}\right]$$

$$= \lim_{\theta\to 0} \left[\frac{-\theta}{\sin\theta}\right]$$

$= -2$ $\left[\because \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1 \text{ and for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \text{ we have } \theta > \sin\theta\right]$

EXERCISE 3.5

Evaluate the following limits : (1 — 12)

1. $\lim_{x\to \pi/6} \frac{\sin(x - \pi/6)}{\sqrt{3}/2 - \cos x}$

2. $\lim_{x\to \pi} \frac{1 - \sin\frac{x}{2}}{\cos x/2\,(\cos x/4 - \sin x/4)}$

3. $\lim_{\alpha\to\beta} \frac{\sin^2\alpha - \sin^2\beta}{\alpha^2 - \beta^2}$

4. $\lim_{h\to 0} \frac{\sin(a + 2h) - 2\sin(a + h) + \sin a}{h^2}$

5. $\lim_{h\to 0} \frac{\tan(a + 2h) - 2\tan(a + h) + \tan a}{h^2}$

6. $\lim_{h\to 0} \frac{\sin(a + 3h) - 3\sin(a + 2h) + 3\sin(a + h) - \sin a}{h^3}$

7. $\lim_{x\to \pi/2} \tan^2 x \left\{\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right\}$

8. $\lim_{x\to\infty} (\cos\sqrt{x+1} - \cos\sqrt{x})$

9. $\lim_{x\to\infty} x\left\{\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}\frac{x}{x+2}\right\}$

10. $\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+2} - \frac{\pi}{4}\right)$

11. $\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$

12. $\lim_{x\to \pi/4} \frac{2\sqrt{2} - (\cos x + \sin x)^3}{1 - \sin 2x}$

13. If α, β are the roots of $x^2 + ax + b = 0$, then evaluate

$$\lim_{x\to\beta} \frac{1 - \cos(x^2 + ax + b)}{(x - \beta)^2}$$

14. If $\Delta = \begin{vmatrix} \sin a & \sin(a+h) & \sin(a+2h) \\ \sin(a+2h) & \sin a & \sin(a+h) \\ \sin(a+h) & \sin(a+2h) & \sin a \end{vmatrix}$, then find $\lim_{h\to 0} \frac{\Delta}{h^2}$

15. $\lim_{x\to -\infty} \frac{x^4 \sin\frac{1}{x} + x^2}{1 + |x|^3}$

16. If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = b$, where $a, b \in R$, then find a and b.

17. $\lim_{x\to 0} \dfrac{\cos x - \sqrt[3]{\cos x}}{\sin^2 x}$

18. $\lim_{x\to 0} \dfrac{\int_0^x t\,dt}{x\tan(\pi+x)}$

19. Find the values of constants a, b and c so that

(i) $\lim_{x\to 0} \dfrac{ax\,e^x - b\log(1+x) + cx\,e^{-x}}{x^2\sin x} = 2$

(ii) $\lim_{x\to 0} \dfrac{x(a+b\cos x) - c\sin x}{x^5} = 1$

(iii) $\lim_{x\to 0} \dfrac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$ is finite and determine it.

20. Find the value of a and b, if

$\lim_{x\to 0} \dfrac{a\cos x + bx\sin x - 5}{x^4}$ is finite. Also, find the limit.

21. If $\lim_{x\to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$, find the value of a.

22. Evaluate :

$\lim_{n\to\infty} \dfrac{1}{2^n}\left\{(1+\cos x)\left(1+\cos\dfrac{x}{2}\right)\left(1+\cos\dfrac{x}{4}\right)..\left(1+\cos\dfrac{x}{2^{n-1}}\right)\right\}$

23. Evaluate : $\lim_{x\to 0} \dfrac{8}{x^8}\left(1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$.

24. Evaluate : $\lim_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ [IIT 1999]

25. Evaluate

$\lim_{x\to 0} \dfrac{\sin\left(\sin\tan(x^2/2)\right)}{\log\cos 3x}$

26. If $S_n = \sin\theta + \sin 2\theta + ... + \sin n\theta$, prove that

$$\lim_{n\to\infty} \frac{S_1 + S_2 + ... + S_n}{n} = \frac{1}{2}\cot\frac{\theta}{2}$$

[**Hint :** Use

$$S_n = \frac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\theta/2} = \frac{\cos\frac{\theta}{2} - \cos\frac{(n+1)\theta}{2}}{2\sin\theta/2}$$

]

27. Evaluate : $\lim_{x\to\infty} \dfrac{x^2\sin\dfrac{1}{x} - x}{1-|x|}$.

ANSWERS

1. 2
2. $\frac{1}{\sqrt{2}}$
3. $\frac{\sin 2\beta}{2\beta}$
4. $-\sin a$
5. $2\tan a\sec^2 a$
6. $-\cos a$
7. $\frac{1}{12}$
8. $-\frac{1}{2}$
9. $\frac{1}{2}$
10. $-\frac{1}{2}$
11. $\frac{1}{120}$
12. $\frac{3}{\sqrt{2}}$
13. $\frac{(\alpha-\beta)^2}{2}$
14. $9\sin a\cos^2 a$
15. -1
16. $a=-2, b=-1$
17. $\frac{-1}{3}$
18. $\frac{1}{2}$
19. (i) $a=3, b=12, c=9$ (ii) $a=120, b=60, c=180$ (iii) $a=6, b=6, c=0$
20. $a=5, b=\frac{5}{2}$, limit $=\frac{-5}{24}$
21. 1
22. $\frac{\sin^2 x}{x^2}$
23. $\frac{1}{32}$
24. $\frac{1}{2}$
25. $-\frac{1}{9}$
27. 0

—❑ ❑ ❑—

3.8 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

The following results are helpful to evaluate exponential and logarithmic limits :

(i) $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a,\ a>0$

(ii) $\lim_{x\to 0} \dfrac{\log_e(1+x)}{x} = 1$

(iii) $\lim_{x\to 0} \dfrac{e^x - 1}{x} = \log_e e = 1$

(iv) $\lim_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$

Following examples will illustrate the applications of these results :

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following limits :*

(i) $\lim_{x\to 0} \dfrac{5^x + 5^{-x} - 2}{x^2}$ (ii) $\lim_{x\to 1} \dfrac{2^{x-1} - 1}{\sin\pi x}$

SOLUTION (i) We have,

$$\lim_{x\to 0} \frac{5^x + 5^{-x} - 2}{x^2}$$

$$= \lim_{x\to 0} \frac{(5^x)^2 - 2.5^x + 1}{x^2.5^x}$$

$$= \lim_{x\to 0}\left(\frac{5^x-1}{x}\right)^2 \times \frac{1}{5^x} = \left(\log_e 5\right)^2$$

(ii) We have,

$$\lim_{x\to 1}\frac{2^{x-1}-1}{\sin \pi x}$$

$$= \lim_{x\to 1}\frac{2^{x-1}-1}{\sin(\pi-\pi x)}$$

$$= \lim_{x\to 1}\frac{2^{x-1}-1}{-\sin \pi(x-1)}$$

$$= -\frac{1}{\pi}\lim_{x\to 1}\frac{2^{x-1}-1}{(x-1)} \times \frac{\pi(x-1)}{\sin \pi(x-1)}$$

$$= -\frac{1}{\pi}\left(\log_e 2 \times 1\right) = -\frac{1}{\pi}\log_e 2$$

EXAMPLE 2 *Evaluate the following limits :*

(i) $\lim_{x\to 2}\dfrac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x-2}$, $x \in (0, \pi/2)$

(ii) $\lim_{x\to 4}\dfrac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}$, $x \in (0, \pi/2)$

SOLUTION : (i) We have,

$$\lim_{x\to 2}\frac{(\cos\alpha)^x + (\sin\alpha)^x - (\cos^2\alpha + \sin^2\alpha)}{x-2}$$

$$= \lim_{x\to 2}\frac{\{(\cos\alpha)^x - (\cos\alpha)^2\} + \{(\sin\alpha)^x - (\sin\alpha)^2\}}{x-2}$$

$$= \lim_{x\to 2}\left[(\cos\alpha)^2\left\{\frac{(\cos\alpha)^{x-2}-1}{x-2}\right\} + (\sin\alpha)^2\left\{\frac{(\sin\alpha)^{x-2}-1}{x-2}\right\}\right]$$

$$= (\cos\alpha)^2(\log_e \cos\alpha) + (\sin\alpha)^2(\log_e \sin\alpha)$$

(ii) We have,

$$\lim_{x\to 4}\frac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}$$

$$= \lim_{x\to 4}\frac{(\cos\alpha)^x - (\sin\alpha)^x - (\cos^4\alpha - \sin^4\alpha)}{x-4}$$

$$= \lim_{x\to 4}\frac{\left((\cos\alpha)^x - (\cos\alpha)^4\right) - \left((\sin\alpha)^x - (\sin\alpha)^4\right)}{x-4}$$

$$= \lim_{x\to 4}(\cos\alpha)^4\left\{\frac{(\cos\alpha)^{x-4}-1}{x-4}\right\} + \lim_{x\to 4}(\sin\alpha)^4\left\{\frac{(\sin\alpha)^{x-4}-1}{x-4}\right\}$$

$$= (\cos\alpha)^4(\log_e \cos\alpha) + (\sin\alpha)^4(\log_e \sin\alpha)$$

EXAMPLE 3 *Evaluate each of the following limits :*

(i) $\lim_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$

(ii) $\lim_{x\to \pi/2}\dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$, $a > 0$

(iii) $\lim_{x\to 0}\dfrac{x - \log_e(1+x)}{x^2}$

(iv) $\lim_{x\to 1}\dfrac{\{\log(1+x) - \log 2\}(3.4^{x-1} - 3x)}{\{(7+x)^{1/3} - (1+3x)^{1/2}\}\sin \pi x}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$$

$$= \lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)^2$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

(ii) We have,

$$\lim_{x\to \pi/2}\frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

$$= \lim_{x\to \pi/2} a^{\cos x}\left\{\frac{a^{\cot x - \cos x} - 1}{(\cot x - \cos x)}\right\}$$

$$= a^0 . \log_e a = \log_e a.$$

(iii) We have,

$$\lim_{x\to \pi/2}\frac{x - \log_e(1+x)}{x^2}$$

$$= \lim_{x\to \pi/2}\frac{x - \left\{x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\right\}}{x^2}$$

$$= \lim_{x\to 0}\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} \ldots = \frac{1}{2}$$

(iv) We have,

$$\lim_{x\to 1}\frac{\{\log(1+x) - \log 2\}\{3.4^{x-1} - 3x\}}{\{(7+x)^{1/3} - (1+3x)^{1/2}\}\sin \pi x}$$

17. $\lim_{x\to 0} \dfrac{\cos x - \sqrt[3]{\cos x}}{\sin^2 x}$

18. $\lim_{x\to 0} \dfrac{\int_0^x t\,dt}{x\tan(\pi + x)}$

19. Find the values of constants a, b and c so that

(i) $\lim_{x\to 0} \dfrac{ax e^x - b\log(1+x) + c x e^{-x}}{x^2 \sin x} = 2$

(ii) $\lim_{x\to 0} \dfrac{x(a + b\cos x) - c\sin x}{x^5} = 1$

(iii) $\lim_{x\to 0} \dfrac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$ is finite and determine it.

20. Find the value of a and b, if

$\lim_{x\to 0} \dfrac{a\cos x + bx\sin x - 5}{x^4}$ is finite. Also, find the limit.

21. If $\lim_{x\to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$, find the value of a.

22. Evaluate :

$\lim_{n\to\infty} \dfrac{1}{2^n}\left\{(1+\cos x)\left(1+\cos\dfrac{x}{2}\right)\left(1+\cos\dfrac{x}{4}\right)..\left(1+\cos\dfrac{x}{2^{n-1}}\right)\right\}$

23. Evaluate : $\lim_{x\to 0} \dfrac{8}{x^8}\left(1 - \cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$.

24. Evaluate : $\lim_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ [IIT 1999]

25. Evaluate

$\lim_{x\to 0} \dfrac{\sin\left(\sin\tan(x^2/2)\right)}{\log\cos 3x}$

26. If $S_n = \sin\theta + \sin 2\theta + ... + \sin n\theta$, prove that

$\lim_{n\to\infty} \dfrac{S_1 + S_2 + ... + S_n}{n} = \dfrac{1}{2}\cot\dfrac{\theta}{2}$

[Hint : Use

$S_n = \dfrac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\theta/2} = \dfrac{\cos\frac{\theta}{2} - \cos\frac{(n+1)\theta}{2}}{2\sin\theta/2}$]

27. Evaluate : $\lim_{x\to\infty} \dfrac{x^2\sin\frac{1}{x} - x}{1 - |x|}$

ANSWERS

1. 2 2. $\frac{1}{\sqrt{2}}$ 3. $\frac{\sin 2\beta}{2\beta}$ 4. $-\sin a$ 16. $a=-2, b=-1$ 17. $\frac{-1}{3}$ 18. $\frac{1}{2}$

5. $2\tan a\sec^2 a$ 6. $-\cos a$ 7. $\frac{1}{12}$ 8. $-\frac{1}{2}$ 19. (i) $a=3, b=12, c=9$ (ii) $a=120, b=60, c=180$ (iii) $a=6, b=6, c=0$

9. $\frac{1}{2}$ 10. $-\frac{1}{2}$ 11. $\frac{1}{120}$ 12. $\frac{3}{\sqrt{2}}$ 20. $a=5, b=\frac{5}{2}$, limit $=\frac{-5}{24}$ 21. 1 22. $\frac{\sin^2 x}{x^2}$

13. $\frac{(\alpha-\beta)^2}{2}$ 14. $9\sin a\cos^2 a$ 15. -1 23. $\frac{1}{32}$ 24. $\frac{1}{2}$ 25. $-\frac{1}{9}$ 27. 0

—❑ ❑ ❑—

3.8 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

The following results are helpful to evaluate exponential and logarithmic limits :

(i) $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a,\ a > 0$ (ii) $\lim_{x\to 0} \dfrac{\log_e(1+x)}{x} = 1$

(iii) $\lim_{x\to 0} \dfrac{e^x - 1}{x} = \log_e e = 1$

(iv) $\lim_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$

Following examples will illustrate the applications of these results :

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following limits :*

(i) $\lim_{x\to 0} \dfrac{5^x + 5^{-x} - 2}{x^2}$ (ii) $\lim_{x\to 1} \dfrac{2^{x-1} - 1}{\sin\pi x}$

SOLUTION (i) We have,

$\lim_{x\to 0} \dfrac{5^x + 5^{-x} - 2}{x^2}$

$= \lim_{x\to 0} \dfrac{(5^x)^2 - 2.5^x + 1}{x^2 . 5^x}$

$$= \lim_{x\to 0}\left(\frac{5^x-1}{x}\right)^2 \times \frac{1}{5^x} = \left(\log_e 5\right)^2$$

(ii) We have,

$$\lim_{x\to 1}\frac{2^{x-1}-1}{\sin \pi x}$$

$$= \lim_{x\to 1}\frac{2^{x-1}-1}{\sin(\pi-\pi x)}$$

$$= \lim_{x\to 1}\frac{2^{x-1}-1}{-\sin \pi(x-1)}$$

$$= -\frac{1}{\pi}\lim_{x\to 1}\frac{2^{x-1}-1}{(x-1)} \times \frac{\pi(x-1)}{\sin \pi(x-1)}$$

$$= -\frac{1}{\pi}\left(\log_e 2 \times 1\right) = -\frac{1}{\pi}\log_e 2$$

EXAMPLE 2 *Evaluate the following limits :*

(i) $\lim_{x\to 2}\dfrac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x-2}, x \in (0, \pi/2)$

(ii) $\lim_{x\to 4}\dfrac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}, x \in (0, \pi/2)$

SOLUTION : (i) We have,

$$\lim_{x\to 2}\frac{(\cos\alpha)^x + (\sin\alpha)^x - (\cos^2\alpha + \sin^2\alpha)}{x-2}$$

$$= \lim_{x\to 2}\frac{\{(\cos\alpha)^x - (\cos\alpha)^2\} + \{(\sin\alpha)^x - (\sin\alpha)^2\}}{x-2}$$

$$= \lim_{x\to 2}\left[(\cos\alpha)^2\left\{\frac{(\cos\alpha)^{x-2}-1}{x-2}\right\} + (\sin\alpha)^2\left\{\frac{(\sin\alpha)^{x-2}-1}{x-2}\right\}\right]$$

$$= (\cos\alpha)^2(\log_e\cos\alpha) + (\sin\alpha)^2(\log_e\sin\alpha)$$

(ii) We have,

$$\lim_{x\to 4}\frac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}$$

$$= \lim_{x\to 4}\frac{(\cos\alpha)^x - (\sin\alpha)^x - (\cos^4\alpha - \sin^4\alpha)}{x-4}$$

$$= \lim_{x\to 4}\frac{\left((\cos\alpha)^x - (\cos\alpha)^4\right) - \left((\sin\alpha)^x - (\sin\alpha)^4\right)}{x-4}$$

$$= \lim_{x\to 4}(\cos\alpha)^4\left\{\frac{(\cos\alpha)^{x-4}-1}{x-4}\right\} + \lim_{x\to 4}(\sin\alpha)^4\left\{\frac{(\sin\alpha)^{x-4}-1}{x-4}\right\}$$

$$= (\cos\alpha)^4(\log_e\cos\alpha) + (\sin\alpha)^4(\log_e\sin\alpha)$$

EXAMPLE 3 *Evaluate each of the following limits :*

(i) $\lim_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$

(ii) $\lim_{x\to \pi/2}\dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}, a > 0$

(iii) $\lim_{x\to 0}\dfrac{x - \log_e(1+x)}{x^2}$

(iv) $\lim_{x\to 1}\dfrac{\{\log(1+x) - \log 2\}(3.4^{x-1} - 3x)}{\{(7+x)^{1/3} - (1+3x)^{1/2}\}\sin \pi x}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$$

$$= \lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)^2$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

(ii) We have,

$$\lim_{x\to \pi/2}\frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

$$= \lim_{x\to \pi/2} a^{\cos x}\left\{\frac{a^{\cot x - \cos x} - 1}{(\cot x - \cos x)}\right\}$$

$$= a^0 . \log_e a = \log_e a.$$

(iii) We have,

$$\lim_{x\to \pi/2}\frac{x - \log_e(1+x)}{x^2}$$

$$= \lim_{x\to \pi/2}\frac{x - \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\}}{x^2}$$

$$= \lim_{x\to 0}\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} \ldots = \frac{1}{2}$$

(iv) We have,

$$\lim_{x\to 1}\frac{\{\log(1+x) - \log 2\}\{3.4^{x-1} - 3x\}}{\{(7+x)^{1/3} - (1+3x)^{1/2}\}\sin \pi x}$$

$$= \lim_{h\to 0} \frac{\{\log(2+h) - \log 2\}\{3.4^h - 3(1+h)\}}{\{(8+h)^{1/3} - (4+3h)^{1/2}\}\sin(\pi+\pi h)}$$

$$= \lim_{h\to 0} \frac{\log\left(1+\frac{h}{2}\right).\{3(4^h-1)-3h\}}{-2\left\{\left(1+\frac{h}{8}\right)^{1/3} - \left(1+\frac{3h}{4}\right)^{1/2}\right\}\sin \pi h}$$

$$= \lim_{h\to 0} \frac{\log\left(1+\frac{h}{2}\right).\{3(4^h-1)-3h\}}{-2\left\{\left(1+\frac{h}{24}+\ldots\right) - \left(1+\frac{3h}{8}+\ldots\right)\right\}\sin \pi h}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{h}{2}\right\}.\left(3(4^h-1)-3h\right|}{-2\left\{-\frac{h}{3} + \text{terms containing } h^2 \text{ and higher power of } h)\right\}\sin \pi h}$$

$$= \lim_{h\to 0} \frac{\frac{\log\left(1+\frac{h}{2}\right)}{h/2}\left\{3.\left(\frac{4^h-1}{h}\right)-3\right\}}{-\left\{-\frac{1}{3} + \text{terms containing } h \text{ and higherpowers of } h)\right\}\frac{\sin \pi h}{h}}$$

$$= \frac{\left(3\log_e 4 - 3\right)}{-\left(-\frac{1}{3}\right)\pi}$$

$$\frac{9}{\pi}\log_e(4/e)$$

EXAMPLE 4 *Evaluate the following limits :*

(i) $\lim_{x\to -1} \frac{\sqrt{x+2}-1}{\log_5(x+2)}$ (ii) $\lim_{x\to 0} \frac{(2^x-1)^3}{\sin\frac{x}{4}\log\left(1+\frac{x^2}{2}\right)}$

SOLUTION (i) We have,

$$\lim_{x\to -1} \frac{\sqrt{x+2}-1}{\log_5(x+2)}$$

$$= \lim_{x\to -1} \frac{(x+2-1)}{\left[\log_5\{1+(x+1)\}\right](\sqrt{x+2}+1)}$$

$$= \lim_{x\to -1} \frac{x+1}{\log_5\{1+(x+1)\}} \times \frac{1}{\sqrt{x+2}+1}$$

$$= \frac{1}{\log_5 e} \times \frac{1}{2} = \frac{1}{2}\log_e 5$$

(ii) We have,

$$\lim_{x\to 0} \frac{(2^x-1)^3}{\sin\frac{x}{4}.\log\left(1+x^2/2\right)}$$

$$= \lim_{x\to 0} \frac{\left(\frac{2^x-1}{x}\right)^3}{\frac{1}{4}\times\left(\frac{\sin x/4}{x/4}\right)\times\left(\frac{\log(1+x^2/2)}{x^2/2}\right)\times\frac{1}{2}}$$

[Dividing N^r and D^r by x^3]

$$= 8\left(\log_e 2\right)^3$$

EXAMPLE 5 *Evaluate :* $\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{(x+\sin x)}$

SOLUTION We have,

$$\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{(x+\sin x)}$$

$$= \lim_{x\to 0} \frac{(e^x-1)-(e^{x\cos x}-1)}{x+\sin x}$$

$$= \lim_{x\to 0} \left(\frac{e^x-1}{x+\sin x} - \frac{e^{x\cos x}-1}{x+\sin x}\right)$$

$$= \lim_{x\to 0} \left[\frac{\frac{e^x-1}{x}}{\left(1+\frac{\sin x}{x}\right)} - \frac{\frac{e^{x\cos x}-1}{x\cos x}}{\left(\frac{1}{\cos x}+\frac{\tan x}{x}\right)}\right]$$

$$= \lim_{x\to 0} \left(\frac{\frac{e^x-1}{x}}{1+\frac{\sin x}{x}}\right) - \lim_{x\to 0}\left(\frac{\frac{e^{x\cos x}-1}{x\cos x}}{\sec x + \frac{\tan x}{x}}\right)$$

$$= \frac{1}{1+1} - \frac{1}{1+1} = \frac{1}{2} - \frac{1}{2} = 0$$

EXAMPLE 6 *Evaluate :* $\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{\tan x}$

SOLUTION We have,

$$\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{\tan x}$$

$$= \lim_{x\to 0} \frac{e-e^{(1/x).\log(1+x)}}{\tan x}$$

$$= \lim_{x\to 0} \frac{e-e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots\right)}}{\tan x}$$

$$= -e \lim_{x\to 0}\left(\frac{e^{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots}-1}{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots}\right)\left(\frac{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots}{\tan x}\right)$$

$$= -e \lim_{x\to 0}\left(\frac{e^{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots}-1}{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots}\right)\left(\frac{-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}\ldots}{\frac{\tan x}{x}}\right)$$

$$= -e \times \frac{-1}{2} = \frac{e}{2}$$

EXAMPLE 7 *Evaluate the following limits :*

(i) $\lim_{x\to 0} \frac{(1+x)^{1/x}-e}{x}$ (ii) $\lim_{x\to 0} \frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$$

$$=\lim_{x\to 0}\frac{e^{(1/x)\log(1+x)}-e}{x}$$

$$=\lim_{x\to 0}\frac{e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)}-e}{x}$$

$$=e\lim_{x\to 0}\frac{e^{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots-1}}{x}$$

$$=e\lim_{x\to 0}\left\{\frac{e^{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots}-1}{\frac{-x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots.}\right\}\times\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots.\right)}{x}$$

$$=e\left(-\frac{1}{2}\right)$$

$$=-\frac{e}{2}$$

(ii) We have,

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{e^{(1/x)\log(1+x)}-e+\frac{ex}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)}-e+\frac{ex}{2}}{x^2}$$

$$=e\lim_{x\to 0}\frac{e^{\frac{-x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots}-1+\frac{x}{2}}{x^2}$$

$$=e\lim_{x\to 0}\frac{1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ldots\right)^2+\ldots-1+\frac{x}{2}}{x^2}$$

$$=e\lim_{x\to 0}\frac{\left(\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots.\right)^2+\ldots}{x^2}$$

$$=e\lim_{x\to 0}\left(\frac{1}{3}-\frac{x}{4}+\ldots.\right)+\frac{1}{2!}\left(-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}\ldots\right)^2+\ldots$$

$$=e\left(\frac{1}{3}+\frac{1}{8}\right)=\frac{11e}{24}.$$

EXAMPLE 8 *If $f(x+y)=f(x)+f(y)$ for all $x, y\in R$ and $f(1)=1$, then evaluate*

$$\lim_{x\to 0}\frac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2 f(\sin x)}$$

SOLUTION We have,

$f(x+y)=f(x)+f(y)$ for all $x, y\in R$.

$\Rightarrow f(x)=xf(1)$ for all $x\in R$

$\Rightarrow f(x)=x$ for all $x\in R$ $[\because f(1)=1$ (given)]

Now,

$$\lim_{x\to 0}\frac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2 f(\sin x)}$$

$$=\lim_{x\to 0}\frac{2^{\tan x}-2^{\sin x}}{x^2\sin x}$$

$$=\lim_{x\to 0}\left(\frac{2^{\tan x-\sin x}-1}{x^2\sin x}\right)\times 2^{\sin x}$$

$$=\lim_{x\to 0}\left(\frac{2^{\tan x-\sin x}-1}{\tan x-\sin x}\right)\times\left(\frac{\tan x-\sin x}{x^2\sin x}\right)\times 2^{\sin x}$$

$$=(\log_e 2)\times\frac{1}{2}\times 2^0 \qquad \left[\because \lim_{x\to 0}\frac{\tan x-\sin x}{x^2\sin x}=\frac{1}{2}\right]$$

$$=\frac{1}{2}\log_e 2$$

EXERCISE 3.6

Evaluate the following limits :

1. $\lim_{x\to e}\dfrac{\log_e x-1}{x-e}$

2. $\lim_{x\to 0}\dfrac{\log_e(5+x)-\log_e(5-x)}{x}$

3. $\lim_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{x^3}$

4. $\lim_{x\to 0}\dfrac{\log_e(2+x)+\log_e 0.5}{x}$

5. $\lim_{x\to \pi/2}\dfrac{a^{2x}-a^{\pi}}{\cos x}$

6. $\lim_{x\to 2}\dfrac{x-2}{\log_a(x-1)}$

7. $\lim_{x\to 0}\dfrac{\log(1+2x-3x^2+4x^3)}{\log(1-x+2x^2-7x^3)}$

8. $\lim_{x\to \pi/2}\dfrac{2^{-\cos x}-1}{x(x-\pi/2)}$

9. $\lim_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

10. $\lim_{x\to\infty}\dfrac{e^{1/x^2}-1}{2\tan^{-1}x^2-\pi}$

11. $\lim_{x\to -1} \dfrac{\log_e (x^2+2x+5)+\cos\dfrac{5(x+1)}{6}-\cos\dfrac{x+1}{6}-\log_e 4}{(x+1)^2}$

12. $\lim_{x\to 0} \dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

13. If $\lim_{n\to\infty} n\left(x^{1/n}-1\right)=f(x)$, prove that for $x>0, y>0$

(i) $f(1)=0$ (ii) $f(x^{-1})=-f(x)$ (iii) $f(xy)=f(x)+f(y)$

ANSWERS

1. $\dfrac{1}{e}$ 2. $\dfrac{2}{5}$ 3. $\dfrac{1}{2}\log_e a$ 7. -2 8. $\dfrac{2}{\pi}\log 2$ 9. $8\sqrt{2}(\log 3)^2$

4. $\dfrac{1}{2}$ 5. $-2a^{\pi}\log a$ 6. $\log_e a$ 10. $-\dfrac{1}{2}$ 11. $-\dfrac{1}{12}$ 12. $8\sqrt{2}(\log 3)^2$

3.9 EVALUATION OF LIMITS OF THE FORM 1^{∞}

To evaluate the exponential limits of the form 1^{∞}, we use the following result.

RESULT If $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ such that $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ exists, then

$$\lim_{x\to a}\{1+f(x)\}^{1/g(x)} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

PROOF Let $A=\lim_{x\to a}\{1+f(x)\}^{\frac{1}{g(x)}}$. Then,

$$\log_e A = \lim_{x\to a}\frac{\log\{1+f(x)\}}{g(x)}$$

$$\Rightarrow \log_e A = \lim_{x\to a}\frac{\log\{1+f(x)\}}{f(x)}\times\frac{f(x)}{g(x)}$$

$$\Rightarrow \log_e A = \lim_{x\to a}\frac{f(x)}{g(x)} \qquad \left[\because \lim_{x\to a}\frac{\log\{1+f(x)\}}{f(x)}=1\right]$$

$$\Rightarrow A = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

REMARK *The above result can also be resated in the following form :*

If $\lim_{x\to a} f(x)=1$ *and* $\lim_{x\to a} g(x)=\infty$ *such that*

$\lim_{x\to a}\{f(x)-1\}g(x)$ exists,

then $\lim_{x\to a}\{f(x)\}^{g(x)} = e^{\lim_{x\to a}\{f(x)-1\}g(x)}$

PARTICULAR CASES

(i) $\lim_{x\to 0}(1+x)^{1/x}=e$ (ii) $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

(iii) $\lim_{x\to 0}(1+\lambda x)^{1/x}=e^{\lambda}$ (iv) $\lim_{x\to\infty}\left(1+\dfrac{\lambda}{x}\right)^x=e^{\lambda}$.

Following examples will illustrate the applications of the above results.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following limits*

(i) $\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$ (ii) $\lim_{x\to 0}(1+\sin x)^{2\cot x}$

(iii) $\lim_{x\to 1}\left(\log_3 3x\right)^{\log_x 3}$ (iv) $\lim_{x\to 0}(\cos x)^{\cot x}$

SOLUTION (i) We have,

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^{\lim_{x\to\infty}\frac{2}{x}\cdot x} = e^2$$

(ii) We have,

$$\lim_{x\to 0}(1+\sin x)^{2\cot x} = e^{\lim_{x\to 0}\sin x\,.\,2\cot x} = e^{\lim_{x\to 0}2\cos x} = e^2$$

(iii) We have,

$$\lim_{x\to 1}\left(\log_3 3x\right)^{\log_x 3}$$

$$= \lim_{x\to 1}\left(\log_3 3 + \log_3 x\right)^{\log_x 3}$$

$$= \lim_{x\to 1}\left(1+\log_3 x\right)^{\frac{1}{\log_3 x}}$$

$$= e^{\lim_{x\to 1}\log_3 x.\frac{1}{\log_3 x}} = e^1 = e$$

(iv) We have,

$$\lim_{x\to 0}(\cos x)^{\cot x}$$

$$= \lim_{x\to 0}(1+\cos x-1)^{\cot x}$$

$$= \lim_{x\to 0}\left\{1-2\sin^2\left(\frac{x}{2}\right)\right\}^{\cot x}$$

$$= e^{\lim_{x\to 0} -2\sin^2(x/2).\cot x}$$

$$= e^{\lim_{x\to 0} -2\frac{\sin^2(x/2)\cos x}{2\sin(x/2)\cos x/2}}$$

$$= e^{\lim_{x\to 0} -\tan(x/2).\cos x}$$

$$= e^0 = 1$$

EXAMPLE 2 *Evaluate the following limits*

(i) $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$ (ii) $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan \pi x/2a}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$$

$$= \lim_{x\to 0}\left\{1+\frac{a^x+b^x+c^x-3}{3}\right\}^{1/x}$$

$$= \lim_{x\to 0}\left\{1+\frac{(a^x-1)+(b^x-1)+(c^x-1)}{3}\right\}^{1/x}$$

$$= e^{\lim_{x\to 0}\frac{a^x-1}{3x}+\frac{b^x-1}{3x}+\frac{c^x-1}{3x}}$$

$$= e^{\frac{1}{3}\left|\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right|}$$

$$= e^{\frac{1}{3}|\log a+\log b+\log c|}$$

$$= e^{\log(abc)^{1/3}} = (abc)^{1/3}$$

(ii) $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan \pi x/2a}$

$$= \lim_{x\to a}\left\{1+\left(1-\frac{a}{x}\right)\right\}^{\tan \pi x/2a}$$

$$= e^{\lim_{x\to a}\left(1-\frac{a}{x}\right)\tan\frac{\pi x}{2a}}$$

$$= e^{\lim_{x\to a}\left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a}} = e^l, \text{ where } l = \lim_{x\to a}\left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a}$$

Now,

$$l = \lim_{x\to a}\left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a}$$

$$= \lim_{x\to a}\left(\frac{x-a}{x}\right)\cot\left(\frac{\pi}{2}-\frac{\pi x}{2a}\right)$$

$$= \lim_{x\to a}\left(\frac{x-a}{x}\right)\cot\frac{\pi}{2a}(a-x)$$

$$= \lim_{x\to a}\left(\frac{x-a}{x}\right)\times\frac{1}{\tan\frac{\pi}{2a}(a-x)}$$

$$= -\lim_{x\to a}\frac{(a-x)}{\tan\frac{\pi}{2a}(a-x)}\times\frac{1}{x}$$

$$= -\lim_{x\to a}\frac{\frac{\pi}{2a}(a-x)}{\tan\frac{\pi}{2a}(a-x)}\times\frac{2a}{\pi x}$$

$$= -\frac{2}{\pi}$$

Hence, required limt = $e^{-2/\pi}$.

EXAMPLE 3 *Evaluate the following limits :*

(i) $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$ **[IIT 1993]**

(ii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

SOLUTION (i) We have,

$$\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$$

$$= \lim_{x\to 0}\left\{\frac{1+\tan x}{1-\tan x}\right\}^{1/x}$$

$$= \lim_{x\to 0}\left\{1+\frac{2\tan x}{1-\tan x}\right\}^{1/x}$$

$$= e^{\lim_{x\to 0}\frac{2\tan x}{1-\tan x}.\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\frac{2}{1-\tan x}\cdot\frac{\tan x}{x}} = e^2.$$

(ii) We have,

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$= \lim_{x\to 0}\left\{\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+....}{x}\right\}^{1/x^2}$$

$$= \lim_{x\to 0}\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+....\right\}^{1/x^2}$$

$$= e^{\lim_{x\to 0}\frac{1}{x^2}\left(\frac{-x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+....\right)}$$

$$= e^{-1/3!} = e^{-1/6}$$

EXAMPLE 4 *Evaluate the following limits*

(i) $\lim_{x\to\infty}\left(\frac{x+5}{x-1}\right)^x$ (ii) $\lim_{x\to\infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x$

SOLUTION We have,

(i) $\lim_{x\to\infty}\left(\frac{x+5}{x-1}\right)^x$

$$= \lim_{x\to\infty}\left(1+\frac{6}{x-1}\right)^x = e^{\lim_{x\to\infty}\frac{6x}{x-1}} = e^6$$

(ii) We have

$$\lim_{x\to\infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x$$

$$= \lim_{x\to\infty}\left\{1+\frac{6x-8}{x^2-2x+5}\right\}^x$$

$$= e^{\lim_{x\to\infty}\frac{6x^2-8x}{x^2-2x+5}} = e^6$$

EXAMPLE 5 *Find the polynomial function $f(x)$ of degree 6 satisfying :*

$$\lim_{x\to 0}\left\{1+\frac{f(x)}{x^3}\right\}^{1/x} = e^2$$

SOLUTION Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6.$$

Then,

$$\lim_{x\to 0}\left\{1+\frac{f(x)}{x^3}\right\}^{1/x} = e^2$$

$$\Rightarrow e^{\lim_{x\to 0}\frac{f(x)}{x^4}} = e^2$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^4} = 2$$

$$\Rightarrow \lim_{x\to 0}\frac{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6}{x^4} = 2$$

$$\Rightarrow a_0 = a_1 = a_2 = a_3 = 0, a_4 = 2$$

$\therefore f(x) = 2x^4 + a_5x^5 + a_6x^6$, where a_5, a_6 are real numbers.

EXAMPLE 6 *Evaluate the following limits :*

(i) $\lim_{x\to\infty}\left\{\sin\frac{1}{x}+\cos\frac{1}{x}\right\}^x$

(ii) $\lim_{x\to\infty}\left\{\frac{a_1^{1/x}+a_2^{1/x}+....+a_n^{1/x}}{n}\right\}^{nx}$, $a_1, a_2, a_3, ..., a_n > 0$

(iii) $\lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$

SOLUTION (i) We have,

$$\lim_{x\to\infty}\left\{\sin\frac{1}{x}+\cos\frac{1}{x}\right\}^x$$

$$= \lim_{x\to\infty}\left\{1+\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)\right\}^x$$

$$= e^{\lim_{x\to\infty} x\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)}$$

$$= e^{\lim_{x\to\infty}\left\{\frac{\sin 1/x}{1/x}-\frac{\sin(1/2x)\sin(1/2x)}{1/2x}\right\}}$$

$$= e^{1-1\times 0} = e$$

(ii) We have,

$$\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+....+a_n^{1/x}}{n}\right)^{nx}$$

$$= \lim_{x\to\infty}\left\{1+\frac{(a_1^{1/x}-1)+(a_2^{1/x}-1)+...+(a_n^{1/x}-1)}{n}\right\}^{nx}$$

$$= e^{\lim_{x\to\infty}\left\{\frac{(a_1^{1/x}-1)+(a_2^{1/x}-1)+...+(a_n^{1/x}-1)}{n}\right\}\times nx}$$

$$= e^{\lim_{x\to\infty}\left\{\frac{a_1^{1/x}-1}{1/x}+\frac{a_2^{1/x}-1}{1/x}+....+\frac{a_n^{1/x}-1}{1/x}\right\}}$$

$$= e^{\log_e a_1+\log_e a_2+....+\log_e a_n}$$

$$= e^{\log_e(a_1a_2...a_n)} = a_1a_2...a_n.$$

(iii) We have,

$$\lim_{x\to 7/2}\left\{1+(2x^2-9x+7)\right\}^{\cot(2x-7)}$$

$$= e^{\lim_{x\to 7/2}(2x^2-9x+37)\cot(2x-7)}$$

$$= e^{\lim_{x\to 7/2}\frac{(2x-7)(x-1)}{\tan(2x-7)}} = e^{5/2}$$

EXAMPLE 7 *Evaluate the following limits :*

(i) $\lim_{x\to 0}\left\{\frac{1^x+2^x+....+n^x}{n}\right\}^{a/x}$

(ii) $\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)....\left(n+\frac{1}{2^{n-1}}\right)\right\}^n$

SOLUTION (i) We have,

$$\lim_{x\to 0}\left\{\frac{1^x+2^x+\ldots+n^x}{n}\right\}^{a/x}$$

$$= \lim_{x\to 0}\left\{1+\frac{1^x+2^x+\ldots+n^x}{n}-1\right\}^{a/x}$$

$$= \lim_{x\to 0}\left\{1+\frac{(1^x-1)+(2^x-1)+....+(n^x-1)}{n}\right\}^{a/x}$$

$$= e^{\lim_{x\to 0}\left\{\frac{(1^x-1)+3(2^x-1)+....+(n^x-1)}{n}\right\}\left(\frac{a}{x}\right)}$$

$$= e^{\lim_{x\to 0}\left\{\frac{1^x-1}{x}+\frac{2^x-1}{x}+\ldots+\frac{n^x-1}{x}\right)\left(\frac{a}{n}\right\}}$$

$$= e^{(\log 1+\log 2+\log 3+\ldots+\log n)\,a/n}$$

$$= e^{\log(n!)^{a/n}} = (n!)^{a/n}$$

(ii) We have,

$$\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)....\left(n+\frac{1}{2^{n-1}}\right)\right\}^n$$

$$= \lim_{n\to\infty}\left\{\frac{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)..\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right\}^n$$

$$= \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{2^2n}\right)\ldots\left(1+\frac{1}{2^{n-1}.n}\right)\right\}^n$$

$$= \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^n\left(1+\frac{1}{2^2n}\right)^n..\left(1+\frac{1}{2^{n-1}.n}\right)^n\right\}$$

$$= e\,.\,e^{1/2}\,e^{1/2^2}\,.....\,e^{1/2^{n-1}}\,....$$

$$= e^{1+\frac{1}{2}+\frac{1}{2^2}+\ldots} = e^2.$$

EXERCISE 3.7

Evaluate the following limits :

1. $\lim_{x\to 0}(\cos x+\sin x)^{1/x}$
2. $\lim_{x\to 0}(\cos x)^{1/\sin x}$
3. $\lim_{x\to 0}(\cos x+a\sin bx)^{1/x}$
4. $\lim_{x\to 0^+}\left\{1+\tan^2\sqrt{x}\right\}^{1/2x}$
5. $\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$
6. $\lim_{x\to\infty}\left|\frac{x^2+2x+3}{2x^2+x+5}\right|^{\frac{3x-2}{3x+2}}$
7. $\lim_{x\to 1}\left|\frac{x^3+2x^2+x+1}{x^2+2x+3}\right|^{\frac{1-\cos(x-1)}{(x-1)^2}}$
8. $\lim_{x\to 0}\left|\frac{e^x+e^{-x}-2}{x^2}\right|^{1/x^2}$
9. $\lim_{x\to a}\left\{\frac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}$
10. If $f(x)$ is a polynomial satisfying
$\lim_{x\to\infty}\left\{1+x+\frac{f(x)}{x}\right\}^{1/x}=1$, then find $f(x)$.
11. $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$
12. $\lim_{x\to\infty}\left|\frac{3x^2+1}{4x^2-1}\right|^{\frac{x^3}{1+x}}$

ANSWERS

1. e
2. 1
3. e^{ab}
4. $\sqrt{e}$
5. $e^{-2/3}$
6. $\frac{1}{2}$
7. $\sqrt{\frac{5}{6}}$
8. $e^{1/12}$
9. $e^{\cot a}$
10. $f(x)=a-x^2, a\in R$
11. 1

—❑ ❑ ❑—

3.10 EVALUATION OF LIMITS BY USING DE' L' HOSPITAL'S RULE

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) $f(x)$ and $g(x)$ *both are continuous at* $x = a$,

(iii) $f(x)$ and $g(x)$ *both are differentiable at* $x = a$,

(iv) $f'(x)$ and $g'(x)$ *are continuous at the point* $x = a$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \text{ provided that } g(a) \neq 0.$$

REMARK *The above rule is also applicable if* $\lim_{x\to a} f(x) = \infty$ *and* $\lim_{x\to a} g(x) = \infty$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* : $\lim_{x\to a} \frac{x^a - a^x}{x^x - a^a}$

SOLUTION We have,

$$\lim_{x\to a} \frac{x^a - a^x}{x^x - a^a} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to a} \frac{a x^{a-1} - a^x \log a}{x^x (1 + \log x) - 0} \qquad \text{[Using : L' Hospital'S Rule]}$$

$$= \frac{a^a - a^a \log a}{a^a (1 + \log a)}$$

$$= \frac{1 - \log a}{1 + \log a}$$

EXAMPLE 2 *Find the values of* a, b, c *so that*

$$\lim_{x\to 0} \frac{a e^x - b\cos x + c e^{-x}}{x \sin x} = 2.$$

SOLUTION We observe that as $x \to 0$, numerator tends to $a - b + c$ whereas the denominator tends to 0. Therefore, for the limit to exist we must have

$$a - b + c = 0 \qquad \ldots(i)$$

Now, if $a - b + c = 0$, then

$$\lim_{x\to 0} \frac{a e^x - b\cos x + c e^{-x}}{x \sin x} \text{ is in the } \frac{0}{0} \text{ form.}$$

$$\therefore \lim_{x\to 0} \frac{a e^x + b \sin x - c e^{-x}}{\sin x + x \cos x} = 2 \qquad \text{[Using L' Hospital 's Rule]}$$

Here, the numerator is tending to $a - c$ as $x \to 0$. Therefore, for the limit to exist, we must have

$$a - c = 0 \qquad \ldots(ii)$$

Now, if $a - c = 0$, then

$$\lim_{x\to 0} \frac{a e^x + b \sin x - c e^{-x}}{\sin x + x \cos x} \text{ is again in the form } \frac{0}{0}.$$

$$\therefore \lim_{x\to 0} \frac{a e^x + b\cos x + c e^{-x}}{2\cos x + x \sin x} = 2 \qquad \text{[Using : L' Hospital's Rule]}$$

$$\Rightarrow \frac{a + b + c}{2} = 2$$

$$\Rightarrow a + b + c = 4 \qquad \ldots(iii)$$

Solving (i), (ii) and (iii), we get

$$a = 1, b = 2, c = 1$$

EXAMPLE 3 *Evaluate :*

$$\lim_{x\to 0^+} \frac{\log_{\sin x} \cos x}{\log_{\sin x/2} \cos(x/2)}$$

SOLUTION We have,

$$\lim_{x\to 0^+} \frac{\log_{\sin x} \cos x}{\log_{\sin x/2} \cos(x/2)}$$

$$= \lim_{x\to 0^+} \frac{\log_e \cos x \,.\, \log_e \sin x/2}{\log_e \sin x \,.\, \log_e \cos x/2}$$

$$= \lim_{x\to 0^+} \frac{\log_e \cos x}{\log_e \cos x/2} \times \lim_{x\to 0^+} \frac{\log_e \sin x/2}{\log_e \sin x}$$

$$= \lim_{x\to 0^+} \frac{-\tan x}{-\frac{1}{2}\tan\frac{x}{2}} \times \lim_{x\to 0^+} \frac{\frac{1}{2}\cot\frac{x}{2}}{\cot x} \qquad \text{[Using L' Hospital's rule]}$$

$$= \left(\lim_{x\to 0^+} \frac{\tan x}{\tan\frac{x}{2}}\right)\left(\lim_{x\to 0^+} \frac{\cot\frac{x}{2}}{\cot x}\right)$$

$$= \left(\lim_{x\to 0^+} \frac{\tan x}{\tan\frac{x}{2}}\right)\left(\lim_{x\to 0^+} \frac{\tan x}{\tan\frac{x}{2}}\right) = 2 \times 2 = 4$$

EXAMPLE 4 *Evaluate :*

$$\lim_{x\to 0}\left|\lim_{n\to\infty}\left(\frac{[1^2 (\sin x)^x] + [2^2 (\sin x)^x] + \ldots + [n^2 (\sin x)^x]}{n^3}\right)\right|$$

SOLUTION We know that :

$$x - 1 < [x] \leq x$$

$\therefore\ 1^2(\sin x)^x - 1 < [1^2(\sin x)^x] \le 1^2(\sin x)^x$

$2^2(\sin x)^x - 1 < [2^2(\sin x)^x] \le 2^2(\sin x)^x$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n^2(\sin x)^x - 1 < [n^2(\sin x)^x] \le n^2(\sin x)^x$

Adding all these inequalities, we get

$$(\sin x)^x \sum n^2 - n < \sum_{k=1}^{n} [k^2(\sin x)^x] \le (\sin x)^x \sum n^2$$

$$\Rightarrow (\sin x)^x \frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n^2} < \frac{\sum_{k=1}^{n} [k^2(\sin x)^x]}{n^3} \le (\sin x)^x \frac{n(n+1)(2n+1)}{6n^3}$$

Taking limit as $n \to \infty$, we get

$$\frac{(\sin x)^x}{3} \le \lim_{n\to\infty} \frac{\sum_{k=1}^{n} [k^2(\sin x)^x]}{n^3} \le \frac{(\sin x)^x}{3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=1}^{n} [k^2(\sin x)^x]}{n^3} = \frac{(\sin x)^x}{3}$$

$$\Rightarrow \lim_{x\to 0} \left[\lim_{n\to\infty} \frac{\sum_{k=1}^{n} [k^2(\sin x)^x]}{n^3} \right] = \frac{1}{3} \lim_{x\to 0} (\sin x)^x$$

Now,

let $A = \lim_{x\to 0} (\sin x)^x$ [0^0 form]

$\Rightarrow \log A = \lim_{x\to 0} x \log \sin x$ [$0 \times \infty$ form]

$\Rightarrow \log A = \lim_{x\to 0} \frac{\log \sin x}{1/x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$\Rightarrow \log A = \lim_{x\to 0} \frac{\cot x}{-1/x^2}$ [Using L' Hospital's rule]

$\Rightarrow \log A = \lim_{x\to 0} \frac{-x^2}{\tan x} = 0$

$\Rightarrow A = e^0 = 1.$

Hence, $\lim_{x\to 0} \left\{ \lim_{n\to\infty} \frac{\sum_{k=1}^{n} [k^2(\sin x)^x]}{n^3} \right\} = \frac{1}{3}$

EXAMPLE 5 *Evaluate :*

$$\lim_{x\to\infty} x \log_e \left\{ \frac{\sin(a+1/x)}{\sin a} \right\}, 0 < a < \frac{\pi}{2}$$

SOLUTION We have,

$$\lim_{x\to\infty} x \log_e \left\{ \frac{\sin(a+1/x)}{\sin a} \right\}$$ [$\infty \times 0$ form]

$$= \lim_{x\to\infty} \frac{\log_e \left\{ \frac{\sin(a+1/x)}{\sin a} \right\}}{\frac{1}{x}}$$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to\infty} \frac{\frac{\sin a}{\sin(a+1/x)} \times \frac{\cos(a+1/x)}{\sin a} \times -\frac{1}{x^2}}{-\frac{1}{x^2}}$$ [Using : L Hospital's rule]

$$= \lim_{x\to\infty} \cot\left(a + \frac{1}{x}\right) = \cot a$$

EXERCISE 3.8

1. Evaluate : $\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$
2. Evaluate : $\lim_{x\to \pi/2} \left\{ x \tan x - \frac{\pi}{2} \sec x \right\}$
3. Evaluate : $\lim_{x\to \pi/2} \tan x \cdot \log_e \sin x$
4. If $\lim_{x\to 0} \frac{\sin 2x + a \sin x}{x^3}$ exists, find a.
5. If $\lim_{x\to 0} \frac{a\cos x + bx \sin x - 5}{x^4}$ exists, find a, b.
6. $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$
7. $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)}$

ANSWERS

1. $\frac{1}{120}$ 2. -1 3. 0 4. -2 5. $a = 5, b = \frac{5}{2}$ 6. $\frac{1}{3}$ 7. 1

CHAPTER

4

CONTINUITY AND DIFFERENTIABILITY

4.1 INTRODUCTION

In this chapter, the meaning of the term *continuous* is same as we use in our daily life. When we say that a function $f(x)$ is continuous at a point $x = a$ it means that at point $(a, f(a))$ the graph of the function has no holes or gaps. That is, its graph is unbroken at point $(a, f(a))$. Consider the graph of a function $f(x)$ as shown in Fig. 4.1. Here, we observe that there are three points $x = a_1$, $x = a_2$ and $x = a_3$ where the function is not continuous

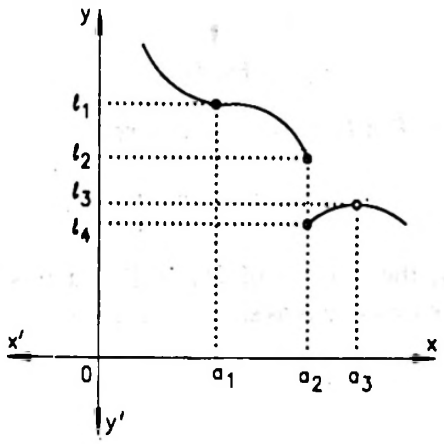


Fig. 4.1

At point $x = a_1$: We observe that the curve $y = f(x)$ is not continuous at $x = a_1$. In fact at $x = a_1$, $f(x)$ is not defined. But, $\lim_{x \to a_1^-} f(x) = \lim_{x \to a_1^+} f(x) = l_1$

i.e. $\lim_{x \to a_1} f(x)$ exists.

Thus, the continuity of a function at a point $x = a_1$ *can be destroyed if the limit of* $f(x)$ *at* $x = a_1$ *exists but* $f(x)$ *is not defined at* $x = a_1$.

At point $x = a_2$: *It is evident from the graph of the function* $f(x)$ *that*

$\lim_{x \to a_2^-} f(x) = l_2$ and $\lim_{x \to a_2^+} f(x) = l_4$, where $l_2 \neq l_4$.

i.e. $\lim_{x \to a_2^-} f(x) \neq \lim_{x \to a_2^+} f(x)$.

Thus, the continuity of a function $f(x)$ *at* $x = a_2$ *can be destroyed if* $\lim_{x \to a_2} f(x)$ *does not exist.*

At point $x = a_3$: At point $x = a_3$, the curve $y = f(x)$ is not continuous, because $\lim_{x \to a_3^-} f(x) = \lim_{x \to a_3^+} f(x) = l_3$

i.e. $\lim_{x \to a_3} f(x)$ exists but it is not equal to $f(a_3)$.

So, the continuity of a function $f(x)$ *can also be disturbed if* $\lim_{x \to a_3} f(x)$ *exists but it is not equal to* $f(a_3)$.

It follows from the above discussion that a function $f(x)$ can be continuous at a point $x = a$ *iff*

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists,

and,

(iii) $\lim_{x \to a} f(x) = f(a)$.

4.2 CONTINUITY AT A POINT

A function $f(x)$ *is said to be continuous at a point* $x = a$ *of its domain, iff* $\lim_{x \to a} f(x) = f(a)$.

Thus, $(f(x)$ is continuous at $x = a) \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

$\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

If $f(x)$ is not continuous at a point $x = a$, then it is said to be discontinuous at $x = a$.

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$, then the discontinuity is known as the *removable discontinuity*, because $f(x)$ can be made continuous by re-defining it at point $x = a$ in such a way that $f(a) = \lim_{x \to a} f(x)$.

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $f(x)$ is said to have a *discontinuity of first kind.*

A function $f(x)$ is said to have a discontinuity of *the second kind* at $x = a$ iff $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both do not exist.

CONTINUITY ON AN OPEN INTERVAL *A function $f(x)$ is said to be continuous on an open interval (a, b) iff it is continuous at every point on the interval (a, b).*

CONTINUITY ON A CLOSED INTERVAL *A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ iff*

(i) f is continuous on the open interval (a, b)

(ii) $\lim_{x \to a^+} f(x) = f(a)$

and,

(iii) $\lim_{x \to b^-} f(x) = f(b)$.

In other words, $f(x)$ is continuous on $[a, b]$ iff it is continuous on (a, b) and it is continuous at a from the right and at b from the left.

4.2 CONTINOUS FUNCTIONS

CONTINUOUS FUNCTION *A function $f(x)$ is said to be continuous, if it is continuous at each point of its domain.*

EVERYWHERE CONTINUOUS FUNCTION *A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line $(-\infty, \infty)$ i.e. on R.*

SOME FUNDAMENTAL RESULTS ON CONTINUOUS FUNCTIONS

Here, we list some fundamental results on continuous functions without giving their proofs.

RESULT I *Let $f(x)$ and $g(x)$ be two continuous functions on their common domain D and let c be a real number. Then*

(i) cf is continuous (ii) $f + g$ is continuous
(iii) $f - g$ is continuous, (iv) fg is continuous
(v) f/g is continuous (vi) $1/g$ is continuous
(vii) f^n, for all $n \in N$ is continuous.

RESULT II *Listed below are some common type of functions that are continuous in their domains.*

(i) *Constant function : Every constant function is every-where continuous.*

Let $f(x) = c$, where c is a constant be a constant function. Its graph is shown in Fig. 4.2. It is evident from the graph that this function is everywhere continuous.

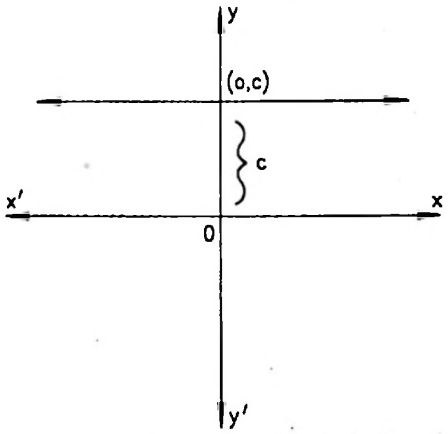


Fig. 4.2

(ii) *Identity Function : The identity function $I(x)$ is defined b $I(x) = x$ for all $x \in R$. This function is everywhere continuous as evident from its graph shown in Fig. 4.3.*

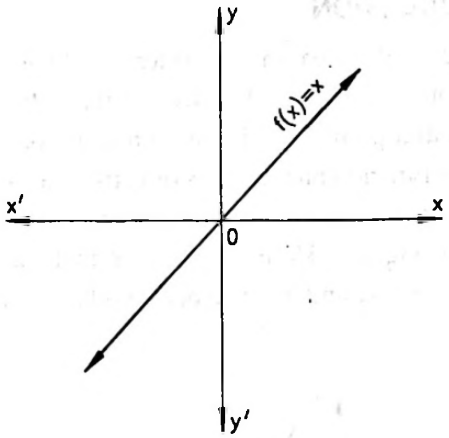


Fig. 4.3

(iii) *Modulus Function: The modulus function $f(x)$ is defined as*

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Clearly, the domain of $f(x)$ is R and this function everywhere continuous as shown in Fig. 4.4.

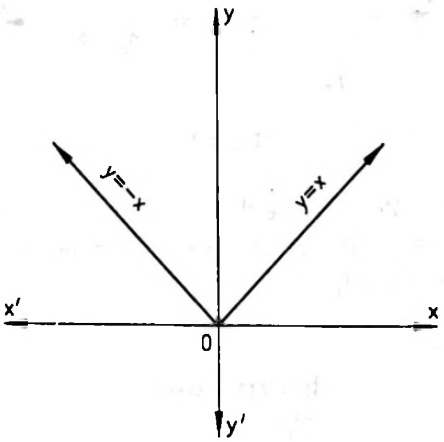


Fig. 4.4

(iv) *Exponential function : If a is a positive real number, other th unity, then the function $f(x)$ defined by $f(x) = a^x$ for all $x \in R$ called the exponential function.*

The domain of this function is R. It is evident from its graph that it is everywhere continuous.

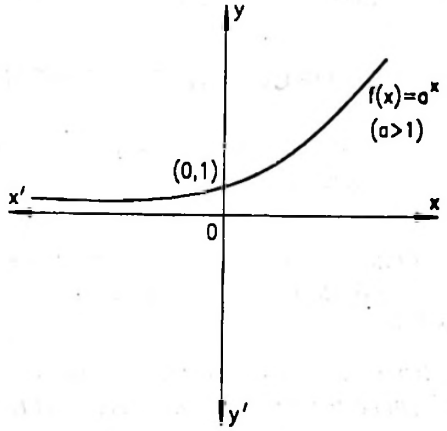


Fig. 4.5

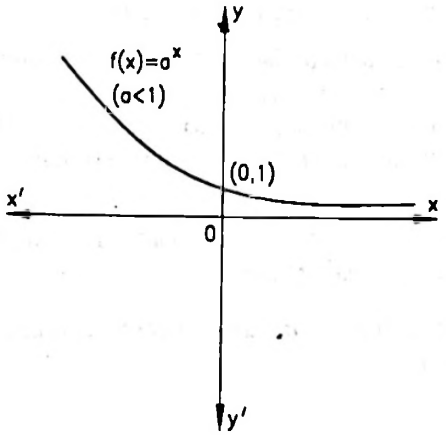


Fig. 4.6

(v) *Logarithmic function : If a is positive real number other than unity, then a function defined by*

$$f(x) = \log_a x$$

is called the logarithmic function.

Clearly its domain is the set of all positive real numbers and it is continuous on its domain. It should be noted that it is not everywhere continuous shown in Figs. 4.7 and 4.8.

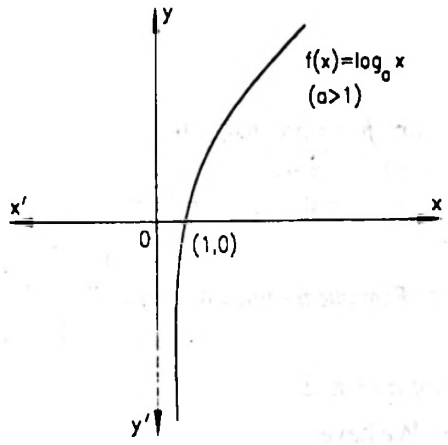


Fig. 4.7

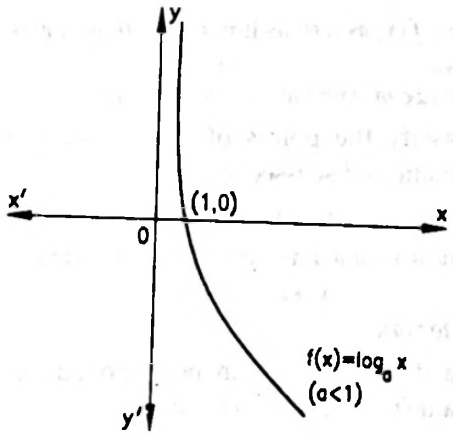


Fig. 4.8

(vi) *Polynomial function : A function of the form* $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, *where* $a_0, a_1, a_2, \ldots, a_n \in R$ *is called a polynomial function.*

This function is everywhere continuous.

(vii) *Rational function : If* $p(x)$ *and* $q(x)$ *are two polynomials, then a function,* $f(x)$ *of the form* $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ *is called a polynomial function.*

This function is continuous on its domain, i.e., it is everywhere continuous except at points where $q(x) = 0$.

(viii) *Trigonometric functions : All trigonometrical functions viz.* $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$ are continuous at each point of their respective domains.

(ix) *Inverse trigonometric functions :* We know that the graph of the inverse of a function is the mirror image of the graph of the given function in the line $y = x$. Therefore, graphs of $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \cot^{-1} x, \sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are the mirror images of those of the corresponding trigonometric functions in the line $y = x$. Since trigonometric functions are continuous in their domains, therefore each of the inverse trigonometric functions is continuous in its domain.

RESULT III *The composition of two continuous functions is a continuous function i.e. if* f *and* g *are two functions such that* g *is continuous at a point* a *and* f *is continuous at* $g(a)$, *then* fog *is continuous at* a.

4.3 DISCONTINUOUS FUNCTIONS

A function f *is said to be discontinuous at a point* a *of its domain* D *if it is not continuous thereat. The point* a *is then called a point of discontinuity of the function. The discontinuity may arise due to any of the following situations:*

(i) $\lim_{x \to a^+} f(x)$ *or* $\lim_{x \to a^-} f(x)$ *of both may not exist*

(ii) $\lim_{x \to a^-} f(x)$ *as well as* $\lim_{x \to a^-} f(x)$ *may exist, but are unequal.*

(iii) $\lim_{x\to a^+} f(x)$ *as well as* $\lim_{x\to a^-} f(x)$ *both may exist, but either of the two or both may not be equal to* $f(a)$.

We classify the points of discontinuity according to various situations discussed above.

(I) *Removable discontinuity: A function* f *is said to have removable discontinuity at* $x = a$ *if* $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ *but their common value is not equal to* $f(a)$.

Such a discontinuity can be removed by assigning a suitable value to the function f at $x = a$.

(II) *Discontinuity of the first kind: A function* f *is said to have a discontinuity of the first kind at* $x = a$ *if* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$ *both exist but are not equal.*

f is said to have a discontinuity of the first kind from the left at $x = a$ *if* $\lim_{x\to a^-} f(x)$ *exists but not equal to* $f(a)$. *Discontinuity of the first kind from the right is similarly defined.*

(III) *Discontinuity of second kind: A function* f *is said to have a discontinuity of the second kind at* $x = a$ *if neither* $\lim_{x\to a^-} f(x)$ *nor* $\lim_{x\to a^+} f(x)$ *exists.*

A function f if said to have discontinuity of the second kind from the left at $x = a$ if $\lim_{x\to a^-} f(x)$ does not exist.

Similarly, if $\lim_{x\to a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at $x = a$.

4.4 PROPERTIES OF CONTINUOUS FUNCTIONS

(I) If f, g *are two continuous functions at a point* a *of their common domain* D, *then* $f \pm g, fg$ *are continuous at* a *and if* $g(a) \neq 0$, *then* f/g *is also continuous at* a.

(II) If f *is continuous at* a *and* $f(a) \neq 0$, *then there exists an open interval* $(a - \delta, a + \delta)$ *such that for all* $x \in (a - \delta, a + \delta)$, $f(x)$ *has the same sign as* $f(a)$.

(III) If a function f *is continuous on a closed interval* $[a, b]$, *then it is bounded on* $[a, b]$ *i.e. there exist real numbers* k *and* K *such that*

$$k \leq f(x) \leq K \text{ for all } x \in [a, b].$$

(IV) If f *is a continuous function defined on* $[a, b]$ *such that* $f(a)$ *and* $f(b)$ *are of opposite signs, then there exists at least one solution of the equation* $f(x) = 0$ *in the open interval* (a, b).

(V) If f *is continuous on* $[a, b]$, *then* f *assumes atleast once, every value between its minimum and maximum values i.e. if* K *is any real number between minimum and maximum values of* $f(x)$, *then there exists at least one solution of the equation* $f(x) = K$ *in the open interval* (a, b).

(VI) If f *is continuous on* $[a, b]$ *and maps* $[a, b]$ *into* $[a, b]$, *then for some* $x \in [a, b]$ *we have* $f(x) = x$.

(VII) If g *is continuous at* a *and* f *is continuous at* $g(a)$, *the* fog *is continuous at* a.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the function* $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ *is continuous at* $x = 0$.

SOLUTION We have,

(LHL at $x = 0$)

$= \lim_{x\to 0^-} f(x)$

$= \lim_{h\to 0} f(0 - h)$

$= \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} -h \sin\left(\frac{1}{-h}\right)$

$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$

$= 0 \times$ (an oscillating number between -1 and 1)

$= 0$

(RHL at $x = 0$)

$= \lim_{x\to 0^+} f(x)$

$= \lim_{h\to 0} f(0 + h)$

$= \lim_{h\to 0} f(h)$

$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$

$= 0 \times$ (an oscillating number between -1 and 1)

$= 0$

and,

$f(0) = 0$.

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$.

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE 2 *Examine the function* $f(t) = \begin{cases} \frac{\cos t}{\pi/2 - t}, & t \neq \pi/2 \\ 1, & t = \pi/2 \end{cases}$ *for continuity at* $t = \pi/2$

SOLUTION We have,

(LHL at $t = \pi/2$)

$= \lim_{t \to \pi/2^-} f(t)$

$= \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$

$= \lim_{h \to 0} \frac{\cos(\pi/2 - h)}{\pi/2 - (\pi/2 - h)}$

$= \lim_{h \to 0} \frac{\sin h}{h} = 1$

and,

(RHL at $t = \pi/2$)

$= \lim_{t \to \pi/2^+} f(t)$

$= \lim_{h \to 0} f(\pi/2 + h)$

$= \lim_{h \to 0} \frac{\cos(\pi/2 + h)}{\pi/2 - (\pi/2 + h)}$

$= \lim_{h \to 0} \frac{-\sin h}{-h}$

$= \lim_{h \to 0} \frac{\sin h}{h} = 1$

Also, $f(\pi/2) = 1$.

Thus, $\lim_{t \to \pi/2^-} f(t) = \lim_{t \to \pi/2^+} f(t) = f(\pi/2)$.

So, $f(t)$ is continuous at $t = \pi/2$

EXAMPLE 3 *Show that the function*

$$f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is discontinuous at $x = 0$.

SOLUTION We have,

(LHL at $x = 0$)

$= \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} \frac{e^{-(1/h)} - 1}{e^{-(1/h)} + 1}$

$= \lim_{h \to 0} \frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1}$

$= \frac{0 - 1}{0 + 1} = -1$ $\left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = 0\right]$

and,

(RHL at $x = 0$)

$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$

$= \lim_{h \to 0} \frac{1 - 1/e^{1/h}}{1 + 1/e^{1/h}} = \frac{1 - 0}{1 + 0} = 1$

$\therefore$ (LHL at $x = 0$) $\neq$ (RHL at $x = 0$)

So, $f(x)$ is not continuous at $x = 0$ and has a discontinuity of first kind at $x = 0$.

EXAMPLE 4 *If the function* $f(x)$ *defined by*

$$f(x) = \begin{cases} \frac{\log(1 + ax) - \log(1 - bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$, *find* k.

SOLUTION Since $f(x)$ is continuous at $x = 0$. Therefore,

$\lim_{x \to 0} f(x) = f(0)$.

$\Rightarrow \lim_{x \to 0} \frac{\log(1 + ax) - \log(1 - bx)}{x} = k$ $[\because f(0) = k]$

$\Rightarrow \lim_{x \to 0} \left[\frac{\log(1 + ax)}{x} - \frac{\log(1 - bx)}{x}\right] = k$

$\Rightarrow \lim_{x \to 0} \frac{\log(1 + ax)}{x} - \lim_{x \to 0} \frac{\log(1 - bx)}{x} = k$

$\Rightarrow a \lim_{x \to 0} \frac{\log(1 + ax)}{ax} - (-b) \lim_{x \to 0} \frac{\log(1 - bx)}{(-b)x} = k$

$\Rightarrow a(1) - (-b)(1) = k$ $\left[\text{Using : } \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1\right]$

$\Rightarrow a + b = k$

Thus, $f(x)$ is continuous at $x = 0$, if $k = a + b$.

EXAMPLE 5 *Prove that the greatest integer function* $[x]$ *is continuous at all points except at integer points.*

SOLUTION Let $f(x) = [x]$ be the greatest integer function. Let k be any integer. Then,

$$f(x) = [x] = \begin{cases} k - 1, & \text{if } k - 1 \leq x < k \\ k, & \text{if } k \leq x < k + 1 \end{cases}$$ [By def.]

Now,

(LHL at $x = k$)

$= \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h)$

$= \lim_{h \to 0} [k-h]$

$= \lim_{h \to 0} (k-1)$ $\quad [\because\ k-1 \le k-h < k \ \therefore\ [k-h] = k-1]$

$= k-1$

and,

(RHL at $x = k$)

$= \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$

$= \lim_{h \to 0} k$ $\quad [\because k \le k+h < k+1 \ \therefore\ [k+h] = k]$

$= k$

$\therefore\ \lim_{x \to k^-} f(x) \ne \lim_{x \to k^+} f(x)$.

So, $f(x)$ is not continuous at $x = k$.

Since k is an arbitrary integer, therefore, $f(x)$ is not continuous at integer points.

Let a be any real number other than an integer. Then, there exists an integer k such that $k-1 < a < k$.

Now,

(LHL at $x = a$)

$= \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} [a-h]$

$= \lim_{x \to 0} k-1$ $\quad [\because\ k-1 < a-h < k \ \therefore\ [a-h] = k-1]$

$= k-1$

(RHL at $x = a$)

$= \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} [a+h]$

$= \lim_{h \to 0} k-1$ $\quad [\because\ k-1 < a+h < k \ \therefore\ [a+h] = k-1]$

$= k-1$

and,

$f(a) = k-1$ $\quad [\because\ k-1 < a < k \ \therefore\ [a] = k-1]$

Thus, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

So, $f(x)$ is continuous at $x = a$. Since a is an arbitrary real number, other than an integer, therefore $f(x)$ is continuous at all real points except integer points.

EXAMPLE 6 *A function is defined as follows:*

$$f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

What condition should be imposed on m so that $f(x)$ may be continuous at $x = 0$?

SOLUTION If $f(x)$ is continuous at $x = 0$, then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0.$$

Now,

$\lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} (-h)^m \sin\left(-\frac{1}{h}\right)$

$= -\lim_{h \to 0} (-h)^m \sin\left(\frac{1}{h}\right)$

$= 0$, only when $m > 0$

and,

$\lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{x \to 0} h^m \sin\left(\frac{1}{h}\right)$

$= 0$, only when $m > 0$

Hence, $f(x)$ is continuous at $x = 0$, if $m > 0$.

EXAMPLE 7 *A function $f(x)$ is defined by*

$$f(x) = \begin{cases} \frac{[x^2]-1}{x^2-1}, & \text{for } x^2 \ne 1 \\ 0, & \text{for } x^2 = 1 \end{cases}$$

Discuss the continuity of $f(x)$ at $x = 1$.

SOLUTION We have,

$$f(x) = \begin{cases} \frac{[x^2]-1}{x^2-1}, & \text{for } x^2 \ne 1 \\ 0, & \text{for } x^2 = 1 \end{cases}$$

$$= \begin{cases} \frac{-1}{x^2-1}, & \text{for } 0 < x^2 < 1 \\ 0, & \text{for } x^2 = 1 \\ \frac{1-1}{x^2-1} = 0, & \text{for } 1 \le x^2 < 2 \end{cases}$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1} \frac{-1}{x^2-1} \to -\infty$$

So, $\lim_{x\to 1^-} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x = 1$.

EXAMPLE 8 *If* $f(x) = \begin{cases} \dfrac{(4^x-1)^3}{\sin(x/4)\log(1+x^2/3)}, & x \neq 0 \\ k, & x = 0 \end{cases}$

is continuous at $x = 0$, *find* k.

SOLUTION Since $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0} \frac{(4^x-1)^3}{\sin\frac{x}{4}\log\left(1+\frac{x^2}{3}\right)} = k$$

$$\Rightarrow \lim_{x\to 0} \frac{12\left(\frac{4^x-1}{x}\right)^3}{\left(\frac{\sin\frac{x}{4}}{x/4}\right)\left(\frac{\log(1+x^2/3)}{x^2/3}\right)} = k$$

$$\Rightarrow 12(\log 4)^3 = k$$

$$\Rightarrow k = 12(\log 4)^3$$

EXAMPLE 9 *Discuss the continuity of the function*

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}} \text{ at } x = 1.$$

SOLUTION We have,

$$f(1) = \lim_{n\to\infty} \frac{\log 3 - \sin 1}{2} = \frac{1}{2}(\log 3 - \sin 1)$$

We know that

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } x^2 < 1 \\ \infty, & \text{if } x^2 > 1 \end{cases}$$

Therefore, for $x^2 < 1$, we have

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}} = \log(2+x)$$

For $x^2 > 1$, we have

$$f(x) = \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(2+x) - \sin x}{1+\frac{1}{x^{2n}}} = -\sin x$$

Thus,

$$f(x) = \begin{cases} \log(2+x), & \text{for } x^2 < 1 \\ \frac{1}{2}(\log 3 - \sin 1), & \text{for } x = 1 \\ -\sin x, & \text{for } x^2 > 1 \end{cases}$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1} \log(2+x) = \log 3$$

and,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} -\sin x = -\sin 1$$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

So, $f(x)$ is not continuous at $x = 1$.

EXAMPLE 10 *Let* $f(x) = \begin{cases} \dfrac{A+3\cos x}{x^2}, & x < 0 \\ B\tan\left(\dfrac{\pi}{[x+3]}\right), & x \geq 0 \end{cases}$

where $[\,]$ *represents the greatest integer function. Find* A *and* B *such that* $f(x)$ *is continuous at* $x = 0$.

SOLUTION We have,

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0} \frac{A+3\cos x}{x^2}$$

$$= \lim_{x\to 0} \frac{A+3\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.\right\}}{x^2}$$

$$= \lim_{x\to 0} \left\{\frac{A+3}{x^2} - \frac{3}{2!} + \frac{3}{4!}x^2 - \frac{3}{6!}x^4 + \ldots..\right\}$$

For this limit to exist, we must have $A + 3 = 0$ and in that case, we have

$$\lim_{x\to 0^-} f(x) = \frac{-3}{2}.$$

Now, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} B\tan\left(\frac{\pi}{[x+3]}\right) = B\tan\frac{\pi}{3} = B\sqrt{3}.$

Also, $f(0) = B\tan\frac{\pi}{3} = B\sqrt{3}.$

Thus, if $f(x)$ is continuous at $x = 0$, then

$$A + 3 = 0 \text{ and } \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow A + 3 = 0 \text{ and } \frac{-3}{2} = B\sqrt{3}$$

$$\Rightarrow A = -3 \text{ and } B = -\frac{\sqrt{3}}{2}.$$

EXAMPLE 11 *Discuss the continuity of the function*

$$f(x) = \lim_{n\to\infty} \frac{(1+\sin \pi x)^n - 1}{(1+\sin \pi x)^n + 1} \text{ at the point } x = 1.$$

SOLUTION We have,

$$f(1) = \lim_{n\to\infty} \frac{(1+\sin \pi)^n - 1}{(1+\sin \pi)^n + 1} = \frac{1-1}{1+1} = 0.$$

(RHL at $x = 1$)

$$= \lim_{x\to 1^+} f(x)$$

$$= \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \left\{ \lim_{n\to\infty} \frac{[1+\sin \pi (1+h)]^n - 1}{[1+\sin \pi (1+h)]^n + 1} \right\}$$

$$= \lim_{h\to 0} \left\{ \lim_{n\to\infty} \frac{[1-\sin \pi h]^n - 1}{[1-\sin \pi h]^n + 1} \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{0-1}{0+1} \right\} = -1$$

(LHL at $x = 1$)

$$= \lim_{x\to 1^-} f(x)$$

$$= \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} \left\{ \lim_{n\to\infty} \frac{[1+\sin \pi (1-h)]^n - 1}{[1+\sin \pi (1-h)]^n + 1} \right\}$$

$$= \lim_{h\to 0} \left\{ \lim_{n\to\infty} \frac{[1+\sin \pi h]^n - 1}{[1+\sin \pi h]^n + 1} \right\}$$

$$= \lim_{h\to 0} \left\{ \lim_{n\to\infty} \frac{1 - \frac{1}{[1+\sin \pi h]^n}}{1 + \frac{1}{[1+\sin \pi h]^n}} \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{1-0}{1+0} \right\} = 1$$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

So, $f(x)$ is not continuous at $x = 1$.

EXAMPLE 12 *Determine the values of a, b and c for which the function*

$$f(x) = \begin{cases} \dfrac{\sin (a+1)x + \sin x}{x}, & \text{for } x < 0 \\ c, & \text{for } x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{for } x > 0 \end{cases}$$

is continuous at $x = 0$. [IIT 1983]

SOLUTION We have,

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0} \frac{\sin (a+1)x + \sin x}{x}$$

$$= (a+1) \lim_{x\to 0} \frac{\sin (a+1)x}{(a+1)x} + \lim_{x\to 0} \frac{\sin x}{x}$$

$$= (a+1) + 1 = a + 2.$$

and,

$$\lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+bx} - 1}{bx}$$

$$= \lim_{x\to 0} \frac{1+bx-1}{(\sqrt{1+bx}+1)\,bx}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1+bx}+1}$$ [Assuming $b \neq 0$

$$= \frac{1}{2}.$$

For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad a + 2 = \frac{1}{2} = c$$

$$\Rightarrow \quad a + 2 = \frac{1}{2} \text{ and } c = \frac{1}{2} \Rightarrow a = -\frac{3}{2} \text{ and } c = \frac{1}{2}.$$

Clearly, b can take any non-zero real value.

EXAMPLE 13 *Discuss the continuity of the function* $f(x)$ *given b*

$$f(x) = \lim_{n\to\infty} \left| \frac{\cos \pi x - x^{2n} \sin (x-1)}{1 + x^{2n+1} - x^{2n}} \right|$$

at $x = 1$.

SOLUTION For $0 < x < 1$, we have

$$f(x) = \lim_{n\to\infty} \left| \frac{\cos \pi x - x^{2n} \sin (x-1)}{1 + x^{2n+1} - x^{2n}} \right|$$

$$= \frac{\cos \pi x - 0 \times \sin (x-1)}{1+0-0}$$

$$= \cos \pi x$$

At $x = 1$, we have

$$f(x) = \lim_{n\to\infty}\left|\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right|$$

$$= \frac{\cos\pi - \sin 0}{1+1-1}$$

$$= \cos\pi = -1$$

For $x > 1$, we have

$$f(x) = \lim_{n\to\infty}\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}$$

$$= \lim_{n\to\infty}\frac{\frac{\cos\pi x}{x^{2n+1}} - \frac{\sin(x-1)}{x}}{\frac{1}{x^{2n+1}}+1-\frac{1}{x}}$$

$$= \frac{0-\frac{\sin(x-1)}{x}}{0+1-\frac{1}{x}}$$

$$= -\frac{\sin(x-1)}{x-1}$$

Thus, we have

$$f(x) = \begin{cases}\cos\pi x & ,0<x<1\\ -1 & ,x=1\\ \dfrac{-\sin(x-1)}{x-1} & ,x>1\end{cases}$$

Now,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\cos\pi x = \cos\pi = -1$$

and, $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\frac{-\sin(x-1)}{x-1} = -1$$

Also, $f(1) = -1$.

Thus,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = -1.$$

Hence, $f(x)$ is continuous at $x = 1$.

EXAMPLE 14 *If* $f(x) = \begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & ,-\frac{\pi}{6}<x<0\\ b & ,x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & ,0<x<\frac{\pi}{6}\end{cases}$

is continuous at $x = 0$, *find the values of* a and b. **[IIT 1994]**

SOLUTION We have,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\{1+|\sin x|\}^{\frac{a}{|\sin x}}$$

$$= e^{\lim_{x\to 0}|\sin x|\frac{a}{|\sin x|}}$$

$$= e^a$$

and,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^{\frac{\tan 2x}{\tan 3x}}$$

$$= e^{\frac{2}{3}\lim_{x\to 0}\frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}}$$

$$= e^{2/3}.$$

For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad e^a = e^{2/3} = b$$

$$\Rightarrow \quad a = \frac{2}{3} \text{ and } b = e^{2/3}.$$

EXAMPLE 15 *If* $f(x) = \begin{cases}\dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5} & ,x\neq 0\\ C & ,x=0\end{cases}$

is continuous at $x = 0$, *find* A, B and C.

SOLUTION If $f(x)$ is continuous at $x = 0$, then

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin 3x + A\sin 2x + B\sin x}{x^5} = C, \text{ a finite number}$$

$$\Rightarrow \lim_{x\to 0}\frac{\left\{3x-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}\ldots\right\}+A\left\{2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}\ldots\right\}+B\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right\}}{x^5} = C$$

$$\Rightarrow \lim_{x\to 0}\frac{x(3+2A+B)+x^3\left\{-\frac{9}{2}-\frac{4A}{3}-\frac{B}{6}\right\}+x^5\left\{\frac{3^5}{5!}+\frac{2^5A}{5!}+\frac{B}{5!}\right\}-x^7(\ldots)+\ldots.}{x^5} = C$$

$$\Rightarrow \lim_{x\to 0}\frac{(3+2A+B)+x^2\left\{-\frac{9}{2}-\frac{4A}{3}-\frac{B}{6}\right\}+x^4\left\{\frac{3^5}{5!}+\frac{2^5A}{5!}+\frac{B}{5!}\right\}-x^6(\ldots)+\ldots}{x^4} = C$$

For this limit to exist and must be finite, we must have

$$3+2A+B = 0, \quad -\frac{9}{2}-\frac{4A}{3}-\frac{B}{6} = 0$$

In such a case, we have

$$\lim_{x\to 0}\left(\frac{3^5}{5!}+\frac{2^5A}{5!}+\frac{B}{5!}\right)-x^2(\ldots.) = C$$

$$\Rightarrow \frac{3^5}{5!}+\frac{2^5A}{5!}+\frac{B}{5!}=C$$

Solving these equations, we get

$A=-4, B=5$ and $C=1$.

EXAMPLE 16 *Find the values of a and b so that*

$$f(x)=\begin{cases}\dfrac{a\,e^{1/|x+2|}-1}{2-e^{1/|x+2|}}, & -3<x<-2\\ b, & x=-2\\ \sin\left(\dfrac{x^4-16}{x^5+32}\right), & -2<x<0\end{cases}$$

may be continuous at $x=-2$.

SOLUTION We have,

$$\lim_{x\to -2^-} f(x)$$

$$=\lim_{x\to -2^-}\frac{a\,e^{\frac{1}{|x+2|}}-1}{2-e^{1/|x+2|}}$$

$$=\lim_{x\to -2}\frac{a-e^{-\frac{1}{|x+2|}}}{2e^{-1/|x+2|}-1}$$

$$=\frac{a-0}{2\times 0-1}=-a$$

and,

$$\lim_{x\to -2^+} f(x)$$

$$=\lim_{x\to -2^+}\sin\left(\frac{x^4-16}{x^5+32}\right)$$

$$=\lim_{x\to -2^+}\sin\left(\frac{(x-2)(x^2+4)}{x^4-2x^3+4x^2-8x+16}\right)$$

$$=\sin\left(\frac{-2}{5}\right)$$

For $f(x)$ to be continous at $x=-2$, we must have

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^+} f(x)=f(-2)$$

$$\Rightarrow \quad -a=\sin\left(-\frac{2}{5}\right)=b$$

$$\Rightarrow \quad a=-\sin\left(\frac{-2}{5}\right) \text{ and } b=\sin\left(\frac{-2}{5}\right)$$

EXAMPLE 17 *Let f be a function satisfying*

$$f(x+y)=f(x)f(y)-\sqrt{4-f(y)} \quad \text{for all } x, y\in R.$$

and $\lim_{x\to 0} f(x)=4$. *Discuss the continuity of* $f(x)$.

SOLUTION We have,

$$f(x+y)=f(x)f(y)-\sqrt{4-f(y)} \quad \text{for all } x, y\in R$$

$$\Rightarrow f(x+h)=f(x)f(h)-\sqrt{4-f(h)}$$

$$\therefore \lim_{h\to 0} f(x+h)$$

$$=\lim_{h\to 0}[f(x)f(h)-\sqrt{4-f(h)}]$$

$$=\lim_{h\to 0} f(x)\,f(h)-\lim_{h\to 0}\sqrt{4-f(h)}$$

$$=f(x)\lim_{h\to 0} f(h)-\lim_{h\to 0}\sqrt{4-f(h)}$$

$$=4f(x)-\sqrt{4-4} \qquad \left[\lim_{h\to 0} f(h)=4\right]$$

$$=4f(x)$$

Thus, $\lim_{h\to 0} f(x+h)\neq f(x)$, if $f(x)\neq 0$. But, $f(x)=0$ doesnot satisfy the given condition. Therefore, $f(x)\neq 0$.

If $f(x)=0$, then $\lim_{h\to 0} f(x+h)=f(x)$.

Hence, $f(x)$ is discontinuous at x.

EXAMPLE 18 *Let* $f: R\to R$ *be a function satisfying*

$$f(x+y)=f(x)+f(y) \text{ for all } x, y\in R.$$

If the function $f(x)$ *is continuous at* $x=0$, *show that it is continuous for all* $x\in R$. **[IIT 1981]**

SOLUTION It is given that $f(x)$ is continuous at $x=0$. Therefore,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \quad \lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(0+h)=f(0)$$

$$\Rightarrow \quad \lim_{h\to 0}[f(0)+f(-h)]=\lim_{h\to 0}[f(0)+f(h)]=f(0)$$

$$\Rightarrow \quad f(0)+\lim_{h\to 0} f(-h)=\lim_{h\to 0} f(h)+f(0)=f(0)$$

$$\Rightarrow \quad \lim_{h\to 0} f(-h)=\lim_{h\to 0} f(h)=0 \qquad \ldots(i)$$

Now, let a be any real number. Then,

$$\lim_{x\to a^-} f(x)=\lim_{h\to 0} f(a-h)$$

$$=\lim_{h\to 0}[f(a)+f(-h)] \qquad \text{[Using the def. of } f\text{]}$$

$$=f(a)+\lim_{h\to 0} f(-h)$$

$$=f(a)+0 \qquad \text{[Using (i)]}$$

$$=f(a)$$

and,

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} [f(a)+f(h)] \qquad \text{[Using def. of } f\text{]}$$

$$= f(a) + \lim_{h\to 0} f(h)$$

$$= f(a) + 0 \qquad \text{[Using (i)]}$$

$$= f(a)$$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

So, $f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number.

Hence, $f(x)$ is continuous at all $x \in R$.

EXAMPLE 19 *Let $f: R \to R$ be a function satisfying $f(xy) = f(x)f(y)$ for all $x, y \in R$.*

If the function f is continuous at $x = 1$, show that it is continuous for all non-zero $x \in R$.

SOLUTION We have,

$$f(xy) = f(x)f(y) \text{ for all } x, y \in R$$

$$\Rightarrow f(1) = f(1)f(1) \qquad \text{[Replacing } x \text{ and } y \text{ both by 1]}$$

$$\Rightarrow f(1)\{f(1)-1\} = 0$$

$$\Rightarrow f(1) = 0 \text{ or, } f(1) = 1$$

If $f(1) = 0$, then for any $x \in R$, we have

$f(x) = f(x \cdot 1) = f(x)f(1) = f(x) \times 0 = 0$ for all $x \in R$, which is everywhere continuous.

So, let $f(1) = 1$.

It is given that f is continuous at $x = 1$. Therefore,

$$\lim_{x\to 1^+} f(x) = f(1) = \lim_{x\to 1^-} f(x)$$

$$\Rightarrow \lim_{h\to 0} f(1+h) = f(1) = \lim_{h\to 0} f(1-h)$$

$$\Rightarrow \lim_{h\to 0} f(1+h) = \lim_{h\to 0} f(1-h) = 1 \qquad \text{...(i)}$$

Now, let a be any non-zero real number. Then,

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} f\left\{a\left(1+\frac{h}{a}\right)\right\}$$

$$= \lim_{h\to 0} f(a)\, f\left(1+\frac{h}{a}\right) \qquad [\because f(xy) = f(x)f(y)]$$

$$= f(a) \lim_{h\to 0} f\left(1+\frac{h}{a}\right)$$

$$= f(a) \cdot 1 \qquad \text{[Using (i)]}$$

$$= f(a)$$

and,

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0} f\left\{a\left(1-\frac{h}{a}\right)\right\}$$

$$= \lim_{h\to 0} f(a)\, f\left(1-\frac{h}{a}\right) \qquad [\because f(xy) = f(x)f(y)]$$

$$= f(a) \lim_{h\to 0} f\left(1-\frac{h}{a}\right)$$

$$= f(a) \times 1 \qquad \text{[Using (i)]}$$

$$= f(a)$$

$$\therefore \lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

So, $f(x)$ is continuous at $x = a$. Since a is an arbitrary non-zero real number so, $f(x)$ is continuous for all non-zero $x \in R$.

EXAMPLE 20 *Let $f: R \to R$ be a function defined by*

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in R.$$

If $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous for all $x \in R$.

SOLUTION We have,

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in R$$

$$\Rightarrow f(0+0) = f(0)f(0)$$

$$\Rightarrow f(0)\{1-f(0)\} = 0$$

$$\Rightarrow f(0) = 0 \text{ or, } f(0) = 1$$

If $f(0) = 0$, then

$$f(x) = f(x+0) = f(x)f(0) = 0 \text{ for all } x \in R.$$

Hence, in this case $f(x)$ is the zero function, which is everywhere continuous.

So, let $f(0) = 1$.

It is given that $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0)f(-h) = \lim_{h\to 0} f(0)\, f(h) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 1 \qquad \text{...(i)} \qquad [\because f(0) = 1]$$

Now, let a be any real number. Then,

$$\lim_{x\to a^-} f(x)$$

$$= \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0} f(a)f(-h)$$

$$= f(a) \lim_{h\to 0} f(-h)$$

$$= f(a) \times 1 \qquad \text{[Using (i)]}$$

$$= f(a)$$

and,

$$\lim_{x\to a^+} f(x)$$

$$= \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} f(a)f(h)$$

$$= f(a)\lim_{h\to 0} f(h)$$

$$= f(a)\times 1 \qquad \text{[Using (i)]}$$

$$= f(a)$$

Thus, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

So, $f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. Therefore, $f(x)$ is continuous for all $x \in R$.

EXAMPLE 21 *Discuss the continuity of the function*

$$f(x) = \begin{cases} \dfrac{a^{[x]+x}-1}{[x]+x}, & x \neq 0 \\ \log_e a, & x = 0 \end{cases} \quad , \ a > 0$$

at $x = 0$.

SOLUTION We have,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{a^{[x]+x}-1}{[x]+x}$$

$$= \lim_{x\to 0^-} \frac{a^{x-1}-1}{x-1}$$

$$= 1-a^{-1}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{a^{[x]+x}-1}{[x]+x}$$

$$= \lim_{x\to 0} \frac{a^x-1}{x}$$

$$= \log_e a$$

Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

So, $f(x)$ is discontinuous at $x = 0$. In fact, $\lim_{x\to 0^+} f(x) = f(0)$ but $\lim_{x\to 0^-} f(x) \neq f(0)$. So, $f(x)$ is right continuous at $x = 0$ but it is not left continuous there at.

EXAMPLE 22 *Discuss the continuity of* $f(x)$ *in* $[0, 2]$ *given by*

$$f(x) = \begin{cases} |2x-3|\,[x], & \text{for } x > 1 \\ [\cos \pi x], & \text{for } x \leq 1 \end{cases}$$

where $[x]$ *is the greatest integer less than or equal to* x.

SOLUTION Simplifying the definition of $f(x)$, we have

$$f(x) = \begin{cases} 2, & \text{for } x = 2 \\ (2x-3), & \frac{3}{2} \leq x < 2 \\ -(2x-3), & 1 < x < \frac{3}{2} \\ -1, & \frac{1}{2} < x \leq 1 \\ 0, & 0 < x < \frac{1}{2} \\ 1, & x = 0 \end{cases}$$

Clearly, $f(x)$, being a polynomial, is continuous on $(3/2, 2) \cup (1, 3/2)$ and it is also continuous on $(1/2, 1) \cup (0, 1/2)$ as it is a constant function. Thus, possible points of discontinuity are 0, 1/2, 1, 3/2 and 2. We shall now discuss the continuity of $f(x)$ at these points one by one.

At $x = 0$: Since 0 is the left most point of [0, 2]. So, we will check whether $f(x)$ is right continuous at $x = 0$ or not.

We have,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 0 = 0.$$

Clearly, $\lim_{x\to 0^+} f(x) \neq f(0)$.

So, $f(x)$ is not continuous at $x = 0$.

At $x = 1/2$: We have,

$$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^-} 0 = 0$$

and, $\lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^+} (-1) = -1$

$$\therefore \lim_{x\to 1/2^-} f(x) \neq \lim_{x\to 1/2^+} f(x)$$

So, $f(x)$ is discontinuous at $x = 1/2$.

At $x = 1$: We have,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -1 = -1$$

and, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -(2x-3) = 1$

$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

So, $f(x)$ is not continuous at $x = 1$.

At $x = 3/2$: We have,

$$\lim_{x\to 3/2^-} f(x) = \lim_{x\to 3/2^-} -(2x-3) = 0$$

$$\lim_{x\to 3/2^+} f(x) = \lim_{x\to 3/2^+} (2x-3) = 0$$

and, $f\left(\frac{3}{2}\right) = \left(2\times\frac{3}{2}-3\right) = 0$

$\therefore \lim_{x \to 3/2^-} f(x) = \lim_{x \to 3/2^+} f(x) = f(3/2)$

So, $f(x)$ is continuous at $x = 3/2$.

At $x = 2$: Since $x = 2$ is the right most point of $[0, 2]$. So, we check the left continuity of $f(x)$ at $x = 2$.

We have,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 3) = 4 - 3 = 1$$

and, $f(2) = 2$

$\therefore \lim_{x \to 2^-} f(x) \neq f(2)$

So, $f(x)$ is not left continuous at $x = 2$.

Thus, the function $f(x)$ is continuous at every point on $[0, 2]$ except at $x = 0, 1/2, 1, 2$.

EXAMPLE 23 *If* $f(x) = \begin{cases} |x^2 - 1| - 1 & , x \leq 1 \\ |2x - 3| - |x - 2| & , x > 1 \end{cases}$

(a) Draw the graph of $f(x)$

(b) Discuss the continuity of $f(x)$.

SOLUTION We have,

$$f(x) = \begin{cases} |x^2 - 1| - 1 & , x \leq 1 \\ -(2x - 3) + (x - 2) = 1 - x & , 1 < x < \frac{3}{2} \\ (2x - 3) + (x - 2) = 3x - 5 & , \frac{3}{2} \leq x < 2 \\ (2x - 3) - (x - 2) = x - 1 & , x \geq 2 \end{cases}$$

(a) Graph of $f(x)$ is as shown in Fig. 4.9.

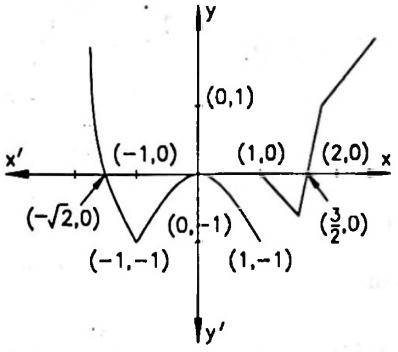


Fig. 4.9

(b) It is evident from the graph that $f(x)$ is not continuous at $x = 1$.

Alternatively : We have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} |x^2 - 1| - 1$$

$$= \lim_{x \to 1^-} (x^2 - 1 - 1)$$

$$= 1 - 1 - 1 = -1$$

and,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x)$$

$$= 1 - 1 = 0$$

Clearly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

So, $f(x)$ is discontinuous at $x = 1$.

It can be checked that $f(x)$ is continuous at all other points.

EXAMPLE 24 *If* $f(x) = \begin{cases} -1 & , x < 0 \\ 0 & , x = 0 \\ 1 & , x > 0 \end{cases}$ *and* $g(x) = x(1 - x^2)$, *then ;*

(i) Draw the graph of $f(g(x))$ *and discuss its continuity.*

(ii) Draw the graph of $g(f(x))$ *and discuss its continuity.*

SOLUTION We have,

$$g(x) = x(1 - x^2) = -x(x - 1)(x + 1)$$

The signs of $g(x)$ for different values of x are shown in Fig. 4.10.

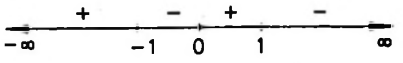


Fig. 4.10

Now,

$$f(g(x)) = \begin{cases} -1 & , \text{if } g(x) < 0 \\ 0 & , \text{if } g(x) = 0, -1, 1 \\ 1 & , \text{if } g(x) > 0 \end{cases}$$

$$= \begin{cases} -1 & , \text{if } -1 < x < 0 \text{ or } x > 1 \\ 0 & , \text{if } x = -1, 0, 1 \\ 1 & , \text{if } -\infty < x < -1 \text{ or } 0 < x < 1 \end{cases}$$

and,

$$g(f(x)) = \begin{cases} g(-1) & , \text{if } x < 0 \\ g(0) & , \text{if } x = 0 \\ g(1) & , \text{if } x > 0 \end{cases}$$

$$= \begin{cases} 0 & , \text{if } x < 0 \\ 0 & , \text{if } x = 0 \\ 0 & , \text{if } x > 0 \end{cases}$$

$= 0$ for all $x \in R$.

(i) The graph of $f(g(x))$ is shown in Fig. 4.11.

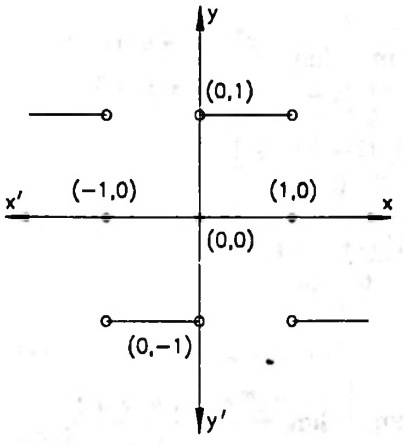


Fig. 4.11

It is evident from the graph of $f(g(x))$ that it is discontinuous at $x = 0, -1, 1$.

(ii) We have, $g(f(x)) = 0$ for all x. So, the graph of $gof(x)$ is represented by x-axis as shown in Fig. 4.12.

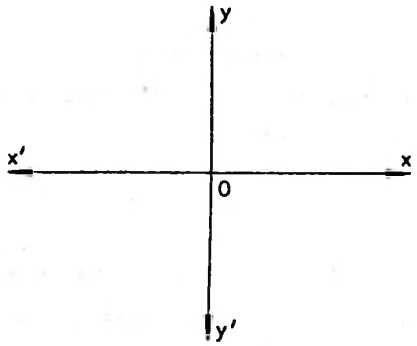


Fig. 4.12

Clearly, $g(f(x))$, being a constant function, is everywhere continuous.

EXAMPLE 25 *If* $g(x) = \lim_{m\to\infty} \dfrac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}$ *is continuous at* $x = 1$ *and* $g(1) = \lim_{x\to 1} \left\{\log_e (ex)\right\}^{\frac{2}{\log_e x}}$, *then find the value of* $2g(1) + 2f(1) - h(1)$. *Assume that* $f(x)$ *and* $h(x)$ *are continuous at* $x = 1$.

SOLUTION We have,

$$g(1) = \lim_{x\to 1} \left\{\log_e (ex)\right\}^{\frac{2}{\log_e x}}$$

$$= \lim_{x\to 1} \left\{1 + \log_e x\right\}^{\frac{2}{\log_e x}}$$

$$= e^{\lim\limits_{x\to 1} \log_e x \cdot \frac{2}{\log_e x}}$$

$$= e^2.$$

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \lim_{m\to\infty} \left\{\frac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \lim_{m\to\infty} \left\{\lim_{x\to 1^-} \frac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \frac{0 f(1) + h(1) + 1}{2 \times 0 + 3 + 3}$$

$$= \frac{h(1) + 1}{6}$$

and,

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \left\{\lim_{m\to\infty} \frac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \lim_{m\to\infty} \left\{\lim_{x\to 1^+} \frac{x^m f(x) + h(x) + 1}{2x^m + 3x + 3}\right\}$$

$$= \lim_{m\to\infty} \left[\lim_{x\to 1^+} \left\{\frac{f(x) + \frac{h(x)}{x^m} + \frac{1}{x^m}}{2 + \frac{3}{x^{m-1}} + \frac{3}{x^m}}\right\}\right]$$

$$= \frac{f(1) + h(1) \times 0 + 0}{2 + 0 + 0}$$

$$= \frac{f(1)}{2}$$

Since $g(x)$ is continuous at $x = 1$. Therefore,

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = g(1)$$

$$\Rightarrow \frac{h(1) + 1}{6} = e^2 \text{ and } \frac{f(1)}{2} = e^2$$

$$\Rightarrow h(1) = 6e^2 - 1 \text{ and } f(1) = 2e^2$$

$$\therefore 2g(1) + 2f(1) - h(1) = 2e^2 + 4e^2 - (6e^2 - 1) = 1.$$

EXAMPLE 26 *Given the function* $f(x) = \dfrac{1}{1-x}$. *Find the points of discontinuity of the composite function* $f\left\{f(f(x))\right\}$

SOLUTION We have,

$$f(x) = \frac{1}{1-x}$$

Clearly, $f(x)$ is not defined at $x = 1$. Also, it is a point of discontinuity of $f(x)$.

If $x \neq 1$, then

$$g(x) = fof(x) = f(f(x)) = f\left(\frac{1}{1-x}\right) = \frac{1}{1 - \frac{1}{1-x}} = \frac{x-1}{x}$$

We observe that $g(x)$ is discontinuous at $x = 0$. Also, $f(x)$ is discontinuous at $x = 1$. So, $g(x)$ is discontinuous at $x = 0$ and $x = 1$.

If $x \neq 0, 1$, then

$$h(x) = f\left\{f(f(x))\right\} = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \frac{x-1}{x}} = x,$$

which is everywhere continuous.

Thus, $h(x)$ is discontinuous at $x = 0, 1$.

EXAMPLE 27 *Let* $f(x)$ *and* $g(x)$ *be defined as :*

$$f(x) = \begin{cases} x^3 + 1 & \text{for } x < 0 \\ x^2 - 1 & \text{for } x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} \sqrt[3]{x-1} & \text{for } x < 1 \\ \sqrt{x+1} & \text{for } x \geq 1 \end{cases}$$

Examine the continuity of gof.

SOLUTION We have,

$$f(x) = \begin{cases} x^3+1, \text{ for } x<0 \\ x^2-1, \text{ for } 0 \le x < \sqrt{2} \\ x^2-1, \text{ for } \sqrt{2} \le x < \infty \end{cases}$$

and,

$$g(x) = \begin{cases} \sqrt[3]{x-1}, \text{ for } x<0 \\ \sqrt[3]{x-1}, \text{ for } 0 \le x < 1 \\ \sqrt{x+1}, \text{ for } x \ge 1 \end{cases}$$

For $x < 0$, we have

$$gof(x) = g(f(x)) = g(x^3+1) = \sqrt[3]{x^3+1-1} = x$$

For $0 \le x < \sqrt{2}$, we have

$$gof(x) = g(f(x)) = g(x^2-1) = \sqrt[3]{(x^2-1)-1} = \sqrt[3]{x^2-2}.$$

For $x \ge \sqrt{2}$, we have

$$gof(x) = g(f(x)) = g(x^2-1) = \sqrt{x^2-1+1} = |x| = x$$

Thus, we have

$$h(x) = gof(x) = \begin{cases} x & , x<0 \\ \sqrt[3]{x^2-2} & , 0 \le x < \sqrt{2} \\ x & , x \ge \sqrt{2} \end{cases}$$

Clearly, possible points of discontinuity are $x = 0, \sqrt{2}$.

Continuity at $x = 0$: We have,

$$\lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} x = 0$$

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} \sqrt[3]{x^2-2} = (-2)^{1/3}$$

and,

$$h(0) = (-2)^{1/3}$$

$$\therefore \lim_{x \to 0^-} h(x) \ne \lim_{x \to 0^+} h(x).$$

So, $h(x)$ is not continuous at $x = 0$.

Continuity at $x = \sqrt{2}$: We have,

$$\lim_{x \to \sqrt{2}^-} h(x) = \lim_{x \to \sqrt{2}^-} \sqrt[3]{x^2-2} = \sqrt[3]{2-2} = 0$$

$$\lim_{x \to \sqrt{2}^+} h(x) = \lim_{x \to \sqrt{2}^+} x = \sqrt{2}$$

$$\therefore \lim_{x \to \sqrt{2}^-} h(x) \ne \lim_{x \to \sqrt{2}^+} h(x)$$

So, $h(x)$ is discontinuous at $x = \sqrt{2}$.

Hence, points of discontinuity of $h(x)$ are 0 and $\sqrt{2}$.

EXAMPLE 28 *If* $f(x) = -1 + |x-2|, 0 \le x \le 4$

$$g(x) = 2 - |x| \quad , -1 \le x \le 3$$

then find $fog(x)$ *and* $gof(x)$. *Draw a rough sketch of the graphs of* $fog(x)$ *and* $gof(x)$. *Discuss the continuity of* $fog(x)$ *at* $x = 0$.

SOLUTION It should be noted that *fog* exists, if range of g is a subset of domain of f and in that case the domain of *fog* is same as that of g.

First we compute the range of g.

We have, $g(x) = 2 - |x|$, when $-1 \le x \le 3$

and, $-1 \le x \le 3$,

$\Rightarrow 1 \le |x| \le 3$

$\Rightarrow -1 \ge -|x| \ge -3$

$\Rightarrow 2-1 \ge 2-|x| \ge 2-3$

$\Rightarrow 1 \ge g(x) \ge -1$

$\Rightarrow -1 \le g(x) \le 1$

$\therefore$ Range of $g = [-1, 1]$

Clearly, range (g) is not a subset of domain of f. So, let us restrict the domain of g so that its range is contained in the domain of f. Thus,

If $0 \le g(x) \le 4$, then

$0 \le 2 - |x| \le 4$

$\Rightarrow -2 \le -|x| \le 2$

$\Rightarrow -2 \le |x| \le 2$

$\Rightarrow -2 \le x \le 2$

But, x should also be in the domain of g i.e. $-1 \le x \le 3$.

Thus, range of g is contained in the domain of f, if we take $[-1, 2]$ as the domain of g.

Now, for $x \in [-1, 2]$, we have

$$fog(x) = f(g(x))$$

$$= f(2 - |x|)$$

$$= -1 + |2 - |x| - 2|$$

$$= -1 + |x|$$

The graph of $fog(x)$ is as shown in Fig. 4.13.

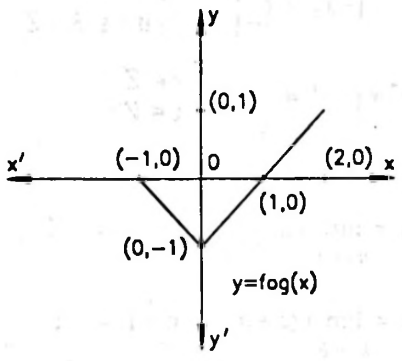


Fig. 4.13

It is evident from the graph that *fog* is continuous for all $x \in [-1, 2]$.

We know that *gof* exists if range of f is contained in the domain of g. Clearly,

$$-1 \le -1 + |x-2| \le 1 \quad \text{for all } x \in [0,4]$$

$$\Rightarrow -1 \le f(x) \le 1 \quad \text{for all } x \in [0,4]$$

$$\Rightarrow \text{Range of } f = [-1,1] \subset [-1,3]$$

$$\Rightarrow \text{Range of } f \subset \text{domain of } g.$$

So, gof exists for all $x \in [0, 4]$.

Now, for any $x \in [0, 4]$, we have

$$gof(x) = g(f(x))$$

$$= g(-1 + |x-2|)$$

$$= 2 - \left|-1 + |x-2|\right|$$

$$= \begin{cases} 2 - |-1-(x-2)|, & \text{for } 0 \le x < 2 \\ 2 - |-1+x-2|, & \text{for } 2 \le x \le 4 \end{cases}$$

$$= \begin{cases} 2 - |1-x|, & \text{for } 0 \le x < 2 \\ 2 - |x-3|, & \text{for } 2 \le x \le 4 \end{cases}$$

The graph of $y = gof(x)$ is as shown in Fig. 4.14. Clearly, it is continuous on [0, 4].

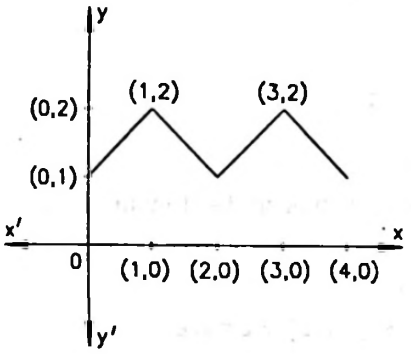


Fig. 4.14

EXAMPLE 29 *Discuss the continuity of the function $f(x)$ given by*

$$f(x) = [x] + [-x]$$

at integer points.

SOLUTION We know that

$$[-x] = \begin{cases} -[x], & \text{if } x \in Z \\ -[x]-1, & \text{if } x \in R - Z \end{cases}$$

$$\therefore \quad f(x) = [x] + [-x] = \begin{cases} 0, & \text{if } x \in Z \\ -1, & \text{if } x \notin Z \end{cases}$$

Now, let a be an arbitrary integer. Then,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} -1 = -1$$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} -1 = -1$$

and,

$$f(a) = 0$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$$

So, f is discontinuous at $x = a$.

Since a is an arbitrary integer. So, f is discontinuous at integer points.

EXAMPLE 30 *If $f(x+y) = f(x)f(y)$ for all $x, y \in R$ and $f(x) = 1 + g(x)\,G(x)$, where $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} G(x)$ exists. Prove that $f(x)$ is continuous for all $x \in R$.*

SOLUTION Let a be any real number. Then,

$$\lim_{x \to a^-} f(x)$$

$$= \lim_{h \to 0} f(a-h)$$

$$= \lim_{h \to 0} f(a) f(-h) \qquad [\because f(x+y) = f(x)f(y)]$$

$$= f(a) \lim_{h \to 0} f(-h)$$

$$= f(a) \lim_{h \to 0} \{1 + g(-h)\,G(-h)\} \qquad [\because f(x) = 1 + g(x)\,G(x)]$$

$$= f(a) \left\{1 + \lim_{h \to 0} g(h) \lim_{h \to 0} G(-h)\right\}$$

$$= f(a)\{1 + 0 \times \text{a finite number}\} \qquad \left[\because \lim_{h \to 0} G(-h) \text{ exists}\right]$$

$$= f(a)$$

Similarly,

$$\lim_{x \to a^+} f(x) = f(a)$$

$$\therefore \quad \lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

$\Rightarrow$ $f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. Therefore, $f(x)$ is continuous for all $x \in R$.

EXAMPLE 31 *If $f(x) = \lim_{n \to \infty} \left\{\sin \frac{\pi x}{2}\right\}^{2n}$, then discuss its continuity on [0, 2].*

SOLUTION We know that

$$\lim_{n \to \infty} x^{2n} = \begin{cases} 0, & \text{for } |x| < 1 \\ 1, & \text{for } |x| = 1 \end{cases}$$

$$\therefore \quad \lim_{n \to \infty} \left\{\sin \frac{\pi x}{2}\right\}^{2n} = \begin{cases} 0, & \text{for } \left|\sin \frac{\pi x}{2}\right| < 1 \\ 1, & \text{for } \left|\sin \frac{\pi x}{2}\right| = 1 \end{cases}$$

$$= \begin{cases} 0, & x \neq (2n+1) \\ 1, & x = (2n+1), \text{ where } n \in Z \end{cases}$$

$$\text{Thus, } f(x) = \begin{cases} 1, & x = (2n+1) \\ 0, & x \neq (2n+1), \text{ where } n \in Z \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = (2n + 1)$, where $n \in Z$.

EXERCISE 4.1

1. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Examine the continuity of $f(x)$ at $x = 0$.

2. If $f(x) = \begin{cases} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}, & x \neq 0 \\ A, & x = 0 \end{cases}$

Find A in order that $f(x)$ may be continuous at $x = 0$.

3. Prove that the function $f(x)$ given by

$$f(x) = \begin{cases} \left(2 - \sin\frac{1}{x}\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is everywhere continuous.

4. Let $f(x) = \begin{cases} -2\sin x & \text{for } x \leq -\pi/2 \\ a\sin x + b & \text{for } -\pi/2 < x < \pi/2 \\ \cos x & \text{for } x \geq \pi/2 \end{cases}$

Find a and b so that $f(x)$ is continuous for all x in its domain.

5. The function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} \frac{\sqrt{1+x} - \sqrt[3]{1+x}}{x}, & x \neq 0 \\ \frac{1}{6}, & x = 0 \end{cases}$$

Prove that $f(x)$ is continuous at $x = 0$.

6. Let

$$f(x) = \begin{cases} \{1 + |\cos x|\}^{\frac{ab}{|\cos x|}}, & n\pi < x < (2n+1)\frac{\pi}{2} \\ e^a e^b, & x = (2n+1)\frac{\pi}{2} \\ e^{\frac{\cot 2x}{\cot 8x}}, & (2n+1)\frac{\pi}{2} < x < (n+1)\pi \end{cases}$$

If $f(x)$ is continuous in $(n\pi, (n+1)\pi)$, then find the values of a and b.

7. Examine the continuity of

$$f(x) = \begin{cases} \frac{2(x-x^3) + |x-x^3|}{2(x-x^3) - |x-x^3|}, & x \neq 0, 1 \\ \frac{1}{3}, & x = 0 \\ 3, & x = 1 \end{cases}$$

at $x = 0, 1$,

8. Determine $f(0)$ so that

$$f(x) = \frac{(a^x - 1)^3}{\sin\{x \log_e a\} \log\{1 + x^2 \log_e a^2\}}$$

is everywhere continuous.

9. If $f(x) = -1 + |x-1|$, $g(x) = 2 - |x+1|$, find $fog(x)$ and $gof(x)$. Draw the graphs of $gof(x)$ and $fog(x)$ and discuss their continuity.

10. Show that the function $f(x) = \frac{x^3}{4} - \sin \pi x + 3$ takes the value $7/3$ within the interval $[-2, 2]$.

11. Let $f(x) = \begin{cases} 1 + x^3, & x < 0 \\ x^2 - 1, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{1/3}, & x < 0 \\ (x+1)^{1/2}, & x \geq 0 \end{cases}$

Discuss the continuity of $gof(x)$.

12. If

$$f(x) = \begin{cases} \lim_{n\to\infty}\left\{\frac{1}{2^n}(1+\cos x)\left(1+\cos\frac{x}{2}\right)\left(1+\cos\frac{x}{4}\right)\ldots\left(1+\cos\frac{x}{2^{n-1}}\right)\right\}^{1/2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Discuss the continuity of $f(x)$ at $x = 0$.

13. Let $f(x) = \begin{cases} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2}, & x < 0 \\ 3, & x = 0 \\ \left|1 + \left(\frac{cx + dx^3}{x^2}\right)\right|^{1/x}, & x > 0 \end{cases}$

If f is continuous at $x = 0$, find the values of a, b, c and d.

14. Discuss the continuity of the function

$$f(x) = \lim_{n\to\infty} \frac{(1+\sin x)^n + \log x}{2 + (1+\sin x)^n}$$

15. Discuss the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} [x] + \sqrt{x - [x]}, & x \geq 0 \\ \sin x, & x < 0 \end{cases}$$

16. Let $f(x) = \begin{cases} 1 + x, & 0 \leq x \leq 2 \\ 3 - x, & 2 < x \leq 3 \end{cases}$ and $g = fof$. Draw the graph of $g(x)$ and find the points of discontinuity of g, if any.

17. If $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$ then determine the function $g(x) = f(f(x))$ and check its continuity.

18. Discuss the continuity of

$$f(x) = \begin{cases} x^2\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$.

19. Let $f(x) = [x] \sin\left\{\frac{\pi}{[x+1]}\right\}$. Find the domain of $f(x)$ and the points of discontinuity of $f(x)$ in its domain. Here, $[\cdot]$ denotes the greatest integer function.

20. If $f(x) = \begin{cases} \frac{4 + \sin 2x + A \sin x + B \cos x}{x^2}, & x \neq 0 \\ C, & x = 0 \end{cases}$

is continuous at $x = 0$, find A, B and C.

21. Let $f(x) = \frac{1 - \sin x}{(\pi - 2x)^2} \cdot \frac{\log \sin x}{\log(1 + \pi^2 - 4\pi x + 4x^2)}, x \neq \pi/2$

What value be assigned to function $f(x)$ at $x = \pi/2$ in order that it may be continuous at $x = \pi/2$?

22. Show that $f(x) = [\tan^2 x]$ is continous at $x = 0$. $[\cdot]$ denotes the greatest integer function.

23. Find the points of discontinuity of $f(x) = \frac{1}{x - [x]}$.

24. If $f(x) = \cos x$ and

$$g(x) = \begin{cases} \min\{f(t) : 0 \le t \le x, 0 \le x \le \pi\} \\ \sin x - 1, \quad x > \pi \end{cases}$$

Discuss the continuity of $g(x)$ in $[0, \infty)$.

25. Draw the graph of the function

$$f(x) = x - |x - x^2|, \quad -1 \le x \le 1$$

and discuss its continuity in $[-1, 1]$.

26. Find the points of discontinuity of the composite function $y = \frac{1}{u^2 + u - 2}$, where $u = \frac{1}{x - 1}$

27. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} ax - b, & x \le 1 \\ 3x, & 1 < x < 2 \\ bx^2 - a, & x \ge 2 \end{cases}$$

If $f(x)$ is continuous at $x = 1$ but not at $x = 2$, prove that the locus of point (a, b) is a straight line excluding the point where it cuts the line $y = 3$.

28. $f(x) = \begin{cases} |x+1|, & x \le 0 \\ x, & x > 0 \end{cases}$ and $g(x) = \begin{cases} |x| + 1, & x \le 1 \\ -|x-2|, & x > 1 \end{cases}$.

Find $f(x) + g(x)$, draw its graph and discuss its continuity.

ANSWERS

1. $f(x)$ is continuous at $x = 0$
2. -4
4. $a = -1, b = 1$
6. $a = b = 2$
7. Discontinuous at $x = 0, 1$
8. $\frac{1}{2}\log_e a$
9. fog and gof are everywhere continuous
11. Not continuous at $x = -1, 0, 1$
12. Continuous at $x = 0$
13. $a = -1, b = -4, c = 0, d = \log_e 3$
14. Continuous for all $x \in R$ except at multiples of π.
15. Everywhere continuous.
16. $x = 1, 2$
17. Continuous on $[0, 2]$
18. Continous
19. Domain $= (-\infty, -1) \cup [0, \infty)$
    Set of point of discontinuity $= I - \{-1\}$
20. $A = -2, B = -4, C = 2$
21. $-\frac{1}{64}$
23. Integer points
24. Everywhere continuous.
25. Continuous on $[-1, 1]$
26. Discontinuous at $x = 1, 2, \frac{1}{2}$
28. $f(x) + g(x) = \begin{cases} -2x, & x < -1 \\ 2, & -1 \le x < 0 \\ 1 + 2x, & 0 \le x \le 1 \\ 2x - 2, & 1 < x < 2 \\ 2, & x \ge 2 \end{cases}$

Discontinuous at $x = 0, 1$.

HINTS TO SELECTED PROBLEMS

10. $f(x)$ is continuous and $7/3 \in [f(-2), f(2)]$. Since a continuous function attains every value between its maximum and minimum values. Therefore, there exists a point $c \in (-2, 2)$ such that $f(c) = 7/3$.

11. $gof(x) = \begin{cases} g(1 + x^3), & x < 0 \\ g(x^2 - 1), & x \ge 0 \end{cases}$

$$= \begin{cases} (1 + x^3 - 1)^{1/3}, & \text{when } x < -1 \\ (1 + x^3 + 1)^{1/2}, & \text{when } -1 \le x < 0 \\ (x^2 - 1 - 1)^{1/3}, & \text{when } 0 \le x < 1 \\ (x^2 - 1 + 1)^{1/2}, & \text{when } x \ge 1 \end{cases}$$

14. $f(x) = \begin{cases} 1 & \text{for } 2n\pi < x < (2n+1)\pi \\ \frac{1 + \log x}{3} & \text{for } x = (2n+1)\pi, 2n\pi) \\ \frac{1}{2}\log_e x & \text{for } (2n+1)\pi < x \le (2n+2)\pi \end{cases}$

16. $g(x) = f(f(x)) = \begin{cases} f(1 + x), & 0 \le x \le 2 \\ f(3 - x), & 2 < x \le 3 \end{cases}$

$$= \begin{cases} f(1 + x), & 0 \le x \le 1 \\ f(1 + x), & 1 < x \le 2 \\ f(3 - x), & 2 < x \le 3 \end{cases}$$

$$= \begin{cases} 1+(1+x), 0 \le x \le 1 \\ 3-(1+x), 1 < x \le 2 \\ 1+(3-x), 2 < x \le 3 \end{cases}$$

17. $g(x) = f\{f(x)\}$

$$= \begin{cases} f(x) & , 0 \le x \le 1 \\ f(2-x), 1 < x \le 2 \end{cases}$$

$$= \begin{cases} x & , 0 \le x \le 1 \\ 2-x, 1 < x \le 2 \end{cases}$$

26. The function $u = u(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$ and the function $y = y(u) = \dfrac{1}{u^2+u-2}$ is discontinuous at $u = 1, -2$.

When $u = 1$, we have $x = 2$ and $u = -2 \Rightarrow x = \dfrac{1}{2}$. Hence, the composite function is discontinuous at $x = 1, 2$ and $\dfrac{1}{2}$

27. ($f(x)$ is continuous at $x = 1$)

$$\Leftrightarrow \lim_{x \to 2^-} f(x) = f(1) = \lim_{x \to 2^+} f(x)$$

$$\Leftrightarrow a - b = 3$$

($f(x)$ is not continuous at $x = 2$)

$$\Leftrightarrow \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

$$\Leftrightarrow 6 \neq 4b - a$$

Hence, the locus of (a, b) is $x - y = 3$, given that $4y - x \neq 6$ i.e. $y \neq 3$.

— ❑❑❑ —

4.6 DIFFERENTIABILITY AT A POINT

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. *Then $f(x)$ is said to be differentiable or derivable at $x = c$, iff $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely.*

This limit is called the derivative or differential coefficient of the function $f(x)$ at $x = c$, and is denoted by $f'(c)$ or $Df(c)$ or $\left(\dfrac{d}{dx} f(x)\right)_{x=c}$

Thus, $f(x)$ is differentiable at $x = c$

$$\Leftrightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely}$$

$$\Leftrightarrow \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$$\Leftrightarrow \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

The limit $\lim_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$ or, $\lim_{h \to 0} \dfrac{f(c-h) - f(c)}{-h}$

is called the left hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or, $Lf'(c)$, while $\lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$ or, $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ *at* $x = c$ and is denoted by $f'(c^+)$ or, $Rf'(c)$.

Thus, $f(x)$ is differentiable at $x = c \Leftrightarrow Lf'(c) = Rf'(c)$. If $Lf'(c) \neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x = c$.

MEANING OF DIFFERENTIABILITY OF A FUNCTION AT A POINT

As we have seen that if a function $f(x)$ is continuous at a point $x = a$ (say), then its graph is an unbroken curve at $x = a$ and there are no holes and jumps in the graph of the function in the neighbourhood of point $x = a$. Now a natural question arises: What do we mean when we say that a function $f(x)$ is differentiable at a point $x = c$? In the following discussion we shall try to answer this question.

Consider the function $f(x)$ defined on an open interval (a, b). Let $P(c, f(c))$ be a point on the curve $y = f(x)$, and let $Q(c-h, f(c-h))$, and $R(c+h, f(c+h))$ be two neighbouring points on the left and right hand sides respectively of point P as shown in Fig. 4.15. Then

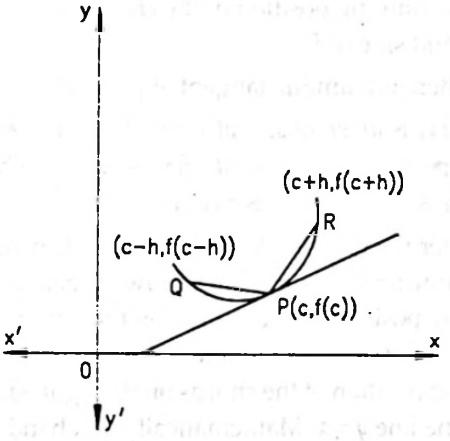


Fig. 4.15

Slope of chord $PQ = \dfrac{f(c-h) - f(c)}{-h}$,

and, Slope of chord $PR = \dfrac{f(c+h) - f(c)}{h}$

We know that tangent to a curve at a point P (say) is the limiting position of chord PQ when Q tends to P. Therefore as $h \to 0$, points Q and R both tend to P from left hand and right hand sides respectively. Consequently, chords PQ and PR become tangent(s) at point P.

Thus,

$$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

$= \lim_{h\to 0}$ (slope of chord PQ)

$= \lim_{Q\to P}$ (slope of chord PQ)

= Slope of the tangent at point P, which is the limiting position of the chords drawn on the left hand side of point P. (I)

and,

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$= \lim_{h\to 0}$ (Slope of chord PR)

$= \lim_{R\to P}$ (slope of chord PR)

= Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P (II)

Now, $f(x)$ is differentiable at $x = c$.

$$\Leftrightarrow \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$\Leftrightarrow$ Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of P is same as the slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of P

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point.

Consider the function $f(x) = |x|$. This function is not differentiable at $x = 0$, because if we draw tangent at the origin as the limiting position of the chords on the left hand side of the origin, it is the line $y = -x$ whereas the tangent at the origin as the limiting position of the chords on the right hand side of the origin is the line $y = x$. Mathematically, left hand derivative at the origin is -1 (slope of the line $y = -x$) and the right hand derivative at the origin is 1 (slope of the line $y = x$).

Let $f(x)$ be a differentiable function at a point P. Then the curve $y = f(x)$ has a unique tangent at P. Since tangent at P is the limiting position of the chord PQ when $Q \to P$. So, if $f(x)$ is differentiable at a point P, then chords exist on both sides of point P. This means that the curve exists on both sides of P. Consequently $f(x)$ is continuous at P.

It follows from the above discussion that, if a function is not differentiable at $x = c$, then either it has $(c, f(c))$ as a corner point or it is discontinuous at $x = c$.

Also, every differentiable function is continuous.

RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

In the above discussion, we have observed that if a function differentiable at a point, then it should be continuous at t point and a discontinuous function cannot be differentiab This fact is proved in the following theorem.

THEOREM *If a function is differentiable at a point, it is necessar continuous at that point. But the converse is not necessarily true.*

OR

$f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

PROOF *Let a function $f(x)$ be differentiable at $x = c$. Then,*

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ exists finitely.}$$

Let $$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$$

In order to prove that $f(x)$ is continuous at $x = c$, it is sufi cient to show that $\lim_{x\to c} f(x) = f(c)$.

Now,

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left\{\left(\frac{f(x)-f(c)}{x-c}\right)(x-c)+f(c)\right\}$$

$$= \lim_{x\to c}\left\{\left(\frac{f(x)-f(c)}{x-c}\right)(x-c)\right\}+f(c)$$

$$= \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right).\lim_{x\to c}(x-c)+f(c)$$

$$= f'(c)\times 0+f(c) \quad \text{[Using (i}$$

$$= f(c)$$

Hence, $f(x)$ is continuous at $x = c$.

Q.E.D

CONVERSE *The converse of the above theorem is not necessaril true i.e., a function may be continuous at a point but may not differentiable at that point because $f(x) = |x|$ is continuous $x = 0$ but it is not differentiable at $x = 0$ (See illustration 1 below..)*

ILLUSTRATION 1 *Show that $f(x) = |x|$ is not differentiable $x = 0$.*

SOLUTION We have,

(LHD at $x = 0$)

$$= \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{|h|-|0|}{-h}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1,$$

and,

(RHD at $x = 0$)

$$= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$).

So, $f(x)$ is not differentiable at $x = 0$.

ILLUSTRATION 2 *Show that the function*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$ *is differentiable at $x = 0$ and $f'(0) = 0$.*

SOLUTION We have,

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}.$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating number between -1 and 1)

$= 0$

(RHD at $x = 0$)

$$= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating number between -1 and 1)

$= 0$

$\therefore$ (LHD at $x = 0$) = (RHD at $x = 0$) = 0.

So, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

ILLUSTRATION 3 *Show that the function*

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$ *is continuous but not differentiable at $x = 0$.*

SOLUTION For the continuity of the function refer Example 1 on page 4.4.

Now,

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{f(-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-h \sin\left(\frac{1}{-h}\right)}{-h}$$

$$= -\lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

= (a number which oscillates between -1 and 1)

$\therefore$ (LHD at $x = 0$) does not exist.

Similarly, it can be shown that RHD at $x = 0$ does not exist. Hence, $f(x)$ is not differentiable at $x = 0$.

4.7 DIFFERENTIABILITY IN A SET

A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

A function $f(x)$ defined on $[a, b]$ is said to be differentiable or derivable at the end points a and b if it is differentiable from the right at a and from the left at b. In other words

$$\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \text{ and } \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b} \text{ both exist.}$$

If f is derivable in the open interval (a, b) and also at the end points a and b, then f is said to be derivable in the closed interval $[a, b]$.

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable.

For example, a constant function, a polynomial function, $\sin x$, $\cos x$ etc. are everywhere differentiable.

SOME STANDARD RESULTS ON DIFFERENTIABILITY

RESULT I *Every polynomial function is differentiable at each $x \in R$.*

RESULT II *The exponential function a^x, $a > 0$ is differentiable at each $x \in R$.*

RESULT III *Every constant function is differentiable at each $x \in R$.*

RESULT IV *The logarithmic function is differentiable at each point in its domian.*

RESULT V *Trigonometric and inverse-trigonometric functions are differentiable in their domains.*

RESULT VI *The sum, difference, product and quotient of two differentiable functions is differentiable.*

RESULT VII *The composition of differentiable function is a differentiable function.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Let* $f(x) = \begin{cases} x\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Show that $f(x)$ is not differentiable at $x = 0$.

SOLUTION We have,

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h \to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h \to 0} \frac{-h\left(\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right)-0}{-h}$$

$$= \lim_{h \to 0} \left(\frac{e^{-2/h}-1}{e^{-2/h}+1}\right) = -1.$$

and,

(RHD at $x = 0$)

$$= \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\left(\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right)-0}{h}$$

$$= \lim_{h \to 0} \left(\frac{1-e^{-2/h}}{1+e^{-2/h}}\right)$$

$$= \frac{1-0}{1+0} = 1$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

EXAMPLE 2 *Show that the function $f(x)$ defined by*

$$f(x) = \begin{cases} x & \text{for } x < 1 \\ 2-x & \text{for } 1 \leq x \leq 2 \\ -2+3x-x^2 & \text{for } x > 2 \end{cases}$$

is differentiable at $x = 2$ but not at $x = 1$.

SOLUTION *Differentiability at $x = 2$.*

We have,

$$\text{(LHD at } x = 2) = \left\{\frac{d}{dx}(2-x)\right\}_{x=2} = -1.$$

$$\text{(RHD at } x = 2) = \left\{\frac{d}{dx}(-2+3x-x^2)\right\}_{x=2}$$

$$= (3-2x)_{x=2}$$

$$= 3-4 = -1$$

$\therefore$ (LHD at $x = 2$) = (RHD at $x = 2$)

So, $f(x)$ is differentiable at $x = 2$.

Differentiability at $x = 1$.

We have,

$$\text{(LHD at } x = 1) = \left\{\frac{d}{dx}(x)\right\}_{x=1} = 1$$

$$(\text{RHD at } x=1) = \left\{\frac{d}{dx}(2-x)\right\}_{x=1} = -1$$

Clearly, (LHD at $x=1$) $\neq$ (RHD at $x=1$)

So, $f(x)$ is not differentiable at $x=1$.

EXAMPLE 3 *Let* $f(x) = \begin{cases} ax(x-1)+b & , x<1 \\ x-1 & , 1\le x\le 3 \\ px^2+qx+2 & , x>3 \end{cases}$

Find the values of the constants a, b, p and q so that

(i) $f(x)$ is continuous for all x.

(ii) $f(x)$ is not differentiable at $x=1$.

(iii) $f'(x)$ is continuous at $x=3$.

SOLUTION It is given that $f(x)$ is everywhere continuous. So, it is continuous at $x=1$ and $x=3$.

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) \text{ and } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$$\Rightarrow \lim_{x\to 1} ax(x-1)+b = \lim_{x\to 1} x-1$$

and,

$$\lim_{x\to 3} x-1 = \lim_{x\to 3} px^2+qx+2$$

$$\Rightarrow b = 0 \text{ and } 2 = 9p+3q+2$$

$$\Rightarrow b = 0 \text{ and } 9p+3q = 0$$

$$\Rightarrow b = 0 \text{ and } 3p+q = 0 \quad \ldots(i)$$

Now, $f(x)$ is not differentiable at $x=1$

$$\Rightarrow (\text{LHD at } x=1) \neq (\text{RHD at } x=1)$$

$$\Rightarrow \left[\frac{d}{dx}\{ax(x-1)+b\}\right]_{x=1} \neq \left[\frac{d}{dx}(x-1)\right]_{x=1}$$

$$\Rightarrow [a(2x-1)]_{x=1} \neq 1$$

$$\Rightarrow a \neq 1.$$

It is given that $f'(x)$ is continuous at $x=3$.

$$\therefore \lim_{x\to 3^-} 1 = \lim_{x\to 3^+} (2px+q)$$

$$\Rightarrow 1 = 6p+q \quad \ldots(ii)$$

Solving (i) and (ii), we get

$$p = \frac{1}{3} \text{ and } q = -1.$$

Hence, $a \neq 1, b = 0, p = \frac{1}{3}$ and $q = -1$.

EXAMPLE 4 *If* $f(x) = \begin{cases} (x-1)^2 \sin\left(\frac{1}{x-1}\right) - |x| & , x\neq 1 \\ -1 & , x = 1 \end{cases}$

Find the set of points where $f(x)$ is not differentiable.

SOLUTION We have,

$$f(x) = \begin{cases} (x-1)^2 \sin\left(\frac{1}{x-1}\right) - x & , \text{ if } x\ge 0, x\neq 1 \\ (x-1)^2 \sin\left(\frac{1}{x-1}\right) + x & , \text{ if } x<0 \\ -1 & , \text{ if } x=1 \end{cases}$$

Clearly, it is continuous and differentiable for all x except at $x=0$ and $x=1$. So, we check the differentiability at $x=0$ and 1.

Now,

(LHD at $x=0$)

$$= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h-1)^2 \sin\left(\frac{1}{-h-1}\right) - h - \sin(-1)}{-h}$$

$$= \lim_{h\to 0} \frac{-(h^2+2h+1)\sin\left(\frac{1}{h+1}\right) - h + \sin 1}{-h}$$

$$= \lim_{h\to 0} \frac{(h^2+2h+1)\sin\left(\frac{1}{h+1}\right) + h - \sin 1}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h+1}\right) + 2\sin\left(\frac{1}{h+1}\right) + 1 + \frac{\sin\left(\frac{1}{h+1}\right) - \sin 1}{h}$$

$$= 0\times \sin 1 + 2\sin 1 + 1$$

$$+ \lim_{h\to 0} \frac{2\sin\left(\frac{(h+1)^{-1}-1}{2}\right)\cos\left(\frac{(h+1)^{-1}+1}{2}\right)}{h}$$

$$= 2\sin 1 + 1 + 2 \lim_{h\to 0} \left\{\frac{\sin\left\{\frac{-h}{2(h+1)}\right\}\cos\left\{\frac{h+2}{2(h+1)}\right\}}{h}\right\}$$

$$= 2\sin 1 + 1 - 2 \lim_{h\to 0} \frac{\sin\left\{\frac{-h}{2(h+1)}\right\}\cos\left\{\frac{h+2}{2(h+1)}\right\}}{\frac{-h}{2(h+1)}\cdot 2(h+1)}$$

$$= 2\sin 1 + 1 - 2\left(\frac{\cos 1}{2}\right)$$

$$= 2\sin 1 + 1 - \cos 1.$$

(RHD at $x=0$)

$$= \lim_{h\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(h-1)^2 \sin\left(\frac{1}{h-1}\right) - h - \sin(-1)}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h-1}\right) - 2\sin\left(\frac{1}{h-1}\right) - 1 + \frac{\sin 1 + \sin\left(\frac{1}{h-1}\right)}{h}$$

$$= 2\sin 1 - 1 + \lim_{h\to 0} \frac{2\sin\left(\frac{1+\frac{1}{h-1}}{2}\right)\cos\left(\frac{1-\frac{1}{h-1}}{2}\right)}{h}$$

$$= 2\sin 1 - 1 + \lim_{h\to 0} 2\,\frac{\sin\left\{\frac{h}{2(h-1)}\right\}\cos\left\{\frac{h-2}{2(h-1)}\right\}}{\frac{h}{2(h-1)}\times 2(h-1)}$$

$$= 2\sin 1 - 1 - \cos 1.$$

Clearly, (LHD at $x = 0$) $\neq$ (RHD at $x = 0$).

So, $f(x)$ is not differentiable at $x = 0$.

Now,

(LHD at $x = 1$)

$$= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{1-h-1}$$

$$= \lim_{h\to 0} \frac{(1-h-1)^2 \sin\left(\frac{1}{1-h-1}\right) - (1-h) - (-1)}{-h}$$

$$= \lim_{h\to 0} \frac{-h^2 \sin\frac{1}{h} - 1 + h + 1}{-h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) - 1 = -1$$

and,

(RHD at $x = 1$)

$$= \lim_{h\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h-1)^2 \sin\left(\frac{1}{1+h-1}\right) - (1+h) - (-1)}{1+h-1}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - h}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) - 1 = -1$$

$\therefore$ (LHD at $x = 1$) = (RHD at $x = 1$)

So, $f(x)$ is differentiable at $x = 1$.

Hence, $f(x)$ is differentiable for all x, except at $x = 0$.

EXAMPLE 5 $f(x) = \begin{cases} xe^{-\left\{\frac{1}{|x|}+\frac{1}{x}\right\}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

test whether

(a) $f(x)$ is continuous at $x = 0$

(b) $f(x)$ is differentiable at $x = 0$. [IIT 1997]

SOLUTION (i) We have,

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} (-h)\, e^{-\left\{\frac{1}{|-h|}+\frac{1}{-h}\right\}}$$

$$= \lim_{h\to 0} (-h)\, e^0 = 0\times 1 = 0$$

$$\lim_{x\to 0^+} f(x)$$

$$= \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} h\, e^{-\left(\frac{1}{|h|}+\frac{1}{h}\right)}$$

$$= \lim_{h\to 0} h\, e^{-2h}$$

$$= 0\times 0 = 0$$

and, $f(0) = 0$ [Given]

$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

Thus, $f(x)$ is continuous at $x = 0$.

(ii) We have,

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{f(-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-h e^{\left(\frac{1}{|-h|} + \frac{1}{-h}\right)}}{-h}$$

$$= \lim_{h \to 0} e^{-\left(\frac{1}{h} - \frac{1}{h}\right)}$$

$$= \lim_{h \to 0} e^0 = 1$$

and,

(RHD at $x = 0$)

$$= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \frac{he^{-\left\{\frac{1}{|h|} + \frac{1}{h}\right\}}}{h}$$

$$= \lim_{h \to 0} e^{-2/h} = 0$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

EXAMPLE 6 Let $f(x) = \begin{cases} -2 & \text{for } -3 \le x \le 0 \\ x - 2 & \text{for } 0 < x \le 3 \end{cases}$

and, $g(x) = f(|x|) + |f(x)|$. *Test the differentiability of* $g(x)$ *in the interval* $[-3, 3]$.

SOLUTION We have,

$$f(x) = \begin{cases} -2 & \text{for } -3 \le x \le 0 \\ x - 2 & \text{for } 0 < x \le 3 \end{cases}$$

$$\therefore \quad |f(x)| = \begin{cases} |-2| & \text{for } -3 \le x \le 0 \\ |x - 2| & \text{for } 0 < x \le 3 \end{cases}$$

$$= \begin{cases} 2 & \text{for } -3 \le x \le 0 \\ -(x-2) & \text{for } 0 < x < 2 \\ x - 2 & \text{for } 2 \le x \le 3 \end{cases}$$

Since $x \in [-3, 3]$. Therefore, $|x| \in [0, 3]$.

Thus,

$$f(|x|) = |x| - 2 \text{ for } x \in [-3, 3]$$

$$= \begin{cases} -x - 2 & \text{for } -3 \le x \le 0 \\ x - 2 & \text{for } 0 < x \le 3 \end{cases}$$

$$\therefore \quad g(x) = f(|x|) + |f(x)|$$

$$= \begin{cases} -x - 2 + 2 = -x & \text{for } -3 \le x \le 0 \\ x - 2 + (-x + 2) = 0 & \text{for } 0 < x < 2 \\ x - 2 + x - 2 = 2x - 4 & \text{for } 2 \le x \le 3 \end{cases}$$

Clearly, $g(x)$, being a polynomial or a constant function, is differentiable on $(-3, 0) \cup (0, 2) \cup (2, 3)$. So, the only possible points where $g(x)$ may not be differentiable are $x = 0$ and $x = 2$.

Differentiability at $x = 0$ We have,

(LHD at $x = 0$)

$$= \lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-x - 0}{x} = -1$$

and,

(RHD at $x = 0$)

$$= \lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $g(x)$ is not differentiable at $x = 0$.

Differentiability at $x = 2$

(LHD at $x = 2$)

$$= \lim_{x \to 2^-} \frac{g(x) - g(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{0 - (2 \times 2 - 4)}{x - 2}$$

$$= 0$$

and,

(RHD at $x = 2$)

$$= \lim_{x \to 2^+} \frac{g(x) - g(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{2x - 4 - (2 \times 2 - 4)}{x - 2}$$

$$= \lim_{x \to 2} 2$$

$$= 2.$$

$\therefore$ (LHD at $x = 2$) $\neq$ (RHD at $x = 2$)

Hence, $g(x)$ is not differentiable at $x = 2$.

EXAMPLE 7 *Determine the values of x for which the following function fails to be continuous or differentiable*

$$f(x) = \begin{cases} 1-x & , x<1 \\ (1-x)(2-x) & , 1 \le x \le 2 \\ 3-x & , x>2 \end{cases} \quad \text{[IIT 1997]}$$

SOLUTION Clearly, $f(x)$ being a polynomial function, is continuous and differentiable on $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$. Thus, the possible points of non-continuity and non-differentiability of $f(x)$ are $x = 1, x = 2$.

Continuity and differentiability: at $x = 1$ We have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1-x = 1-1 = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1-x)(2-x) = (1-1)(2-1) = 0$$

and,

$$f(1) = (1-1)(2-1) = 0$$

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

Hence, $f(x)$ is continuous at $x = 1$.

Now,

(LHD at $x = 1$)

$$= \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1^-} \frac{(1-x)-0}{x-1} = -1$$

(RHD) at $x = 1$)

$$= \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1^+} \frac{(1-x)(2-x)-0}{x-1} = -1$$

$\therefore$ (LHD at $x = 1$) = (RHD at $x = 1$)

So, $f(x)$ is differentiable at $x = 1$.

Continuity and differentiability at $x = 2$: We have,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1-x)(2-x) = (1-2)(2-2) = 0$$

and,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3-x) = 3-2 = 1$$

$$\therefore \quad \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

Therefore, $f(x)$ is not continuous at $x = 2$ and hence not differentiable also.

EXAMPLE 8 *Let* $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

Examine the continuity and differentiability of $f(x)$ and $f'(x)$ at $x = 0$.

SOLUTION We have,

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} x^2 \sin\frac{1}{x}$$

$$= 0 \times (\text{an oscillating number}) = 0$$

$$= f(0)$$

So, $f(x)$ is continuous at $x = 0$.

Now,

$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x-0}$$

$$= \lim_{x \to 0} x \sin\frac{1}{x}$$

$$= 0 \times (\text{an oscillating number})$$

$$= 0$$

$$\therefore \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} \text{ exists finitely.}$$

Hence, $f(x)$ is differentiable at $x = 0$.

Now,

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\therefore \lim_{x \to 0} f'(x)$$

$$= \lim_{x \to 0} \left\{ 2x \sin\frac{1}{x} - \cos\frac{1}{x} \right\}$$

$$= \lim_{x \to 0} 2x \sin\frac{1}{x} - \lim_{x \to 0} \cos\frac{1}{x}$$

$= 0 -$ (An oscillating number which oscillates between -1 and 1)

$\Rightarrow \lim_{x \to 0} f'(x)$ does not exist.

Therefore, $f'(x)$ is not continuous at $x = 0$ and hence not differentiable also.

EXAMPLE 9 *Let* $f(x) = \begin{cases} x^m \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Find the set of values of m for which

(i) $f(x)$ is continuous at $x = 0$

(ii) $f(x)$ is differentiable at $x = 0$

(iii) $f(x)$ is continuous but not differentiable at $x = 0$.

SOLUTION (i) For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{n\to 0} x^m \sin\frac{1}{x} = 0$$

$$\Rightarrow m > 0$$

Thus, $f(x)$ is continuous at $x = 0$, if $m \in (0, \infty)$.

(ii) If $f(x)$ is differentiable at $x = 0$, then

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \text{ must exist finitely}$$

$$\Rightarrow \lim_{x\to 0} \frac{x^m \sin\frac{1}{x} - 0}{x} \text{ must exist finitely.}$$

$$\Rightarrow \lim_{x\to 0} x^{m-1} \sin\frac{1}{x} \text{ must exist finitely}$$

$$\Rightarrow m - 1 > 0$$

$$\Rightarrow m > 1.$$

Thus, $f(x)$ is differentiable at $x = 0$, if $m \in (1, \infty)$.

(iii) From (i) and (ii), we find that $f(x)$ will be continuous but not differntiable at $x = 0$, if

$$m > 0 \text{ and } m \leq 1 \Rightarrow m \in (0, 1]$$

EXAMPLE 10 *If* $f(x) = \begin{cases} \frac{\sin[x^2]\pi}{x^2 - 3x - 18} + ax^3 + b, & \text{for } 0 \leq x \leq 1 \\ 2\cos \pi x + \tan^{-1} x, & \text{for } 1 < x \leq 2 \end{cases}$

differentiable function in [0, 2], find a and b. Here, [x] denotes the greatest integer less than or equal to x.

SOLUTION Since $[x^2]\pi$ is an integral multiple of π for $0 \leq x \leq 1$. Therefore, $\sin[x^2]\pi = 0$. Hence,

$$f(x) = \begin{cases} ax^3 + b, & \text{for } 0 \leq x \leq 1 \\ 2\cos \pi x + \tan^{-1} x, & \text{for } 1 < x \leq 2 \end{cases}$$

Since $f(x)$ is differentiable in [0, 2]. So, it is continuous and differentiable at $x = 1$ also.

Now,

$f(x)$ is continuous at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-} ax^3 + b = \lim_{x\to 1^+} 2\cos \pi x + \tan^{-1} x = a + b$$

$$\Rightarrow a + b = 2\cos \pi + \tan^{-1} 1$$

$$\Rightarrow a + b = -2 + \frac{\pi}{4}. \quad \ldots(i)$$

Since $f(x)$ is differentiable at $x = 1$

$$\therefore \text{ (LHD at } x = 1) = \text{(RHD at } x = 1)$$

$$\Rightarrow \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \lim_{x\to 1^-} \frac{(ax^3 + b) - (a + b)}{x - 1} = \lim_{x\to 1^+} \frac{2\cos \pi x + \tan^{-1} x - (a + b)}{x - 1}$$

$$\Rightarrow \lim_{x\to 1^-} a\left(\frac{x^3 - 1}{x - 1}\right) = \lim_{x\to 1^+} \frac{2\cos \pi x + \tan^{-1} x + 2 - \pi/4}{x - 1}$$

[Using (i)]

$$\Rightarrow a \lim_{x\to 1^-} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1^+} \frac{2(1 + \cos \pi x) + (\tan^{-1} x - \pi/4)}{x - 1}$$

$$\Rightarrow 3a = \lim_{x\to 1^+} 2\left\{\frac{1 - \cos \pi(1 - x)}{1 - x}\right\} + \lim_{x\to 1^+} \frac{\tan^{-1} x - \tan^{-1} 1}{x - 1}$$

$$\Rightarrow 3a = 2\lim_{x\to 1^+} \frac{2\sin^2\left\{\frac{\pi(1 - x)}{2}\right\}}{1 - x} + \lim_{x\to 1^+} \frac{\tan^{-1}\left\{\frac{x - 1}{x + 1}\right\}}{x - 1}$$

$$\Rightarrow 3a = 2 \times 0 + \lim_{x\to 1^+} \frac{\tan^{-1}\left\{\frac{x - 1}{x + 1}\right\}}{\left\{\frac{x - 1}{x + 1}\right\}} \times \frac{1}{x + 1}$$

$$\Rightarrow 3a = \frac{1}{2} \quad \ldots(ii)$$

Solving (i) and (ii), we get

$$a = \frac{1}{6} \text{ and } b = \frac{\pi}{4} - \frac{13}{6}$$

EXAMPLE 11 *Let $f(x)$ be a function defined by*

$$f(x) = \begin{cases} x, & \text{for } x \leq 1 \\ -x^2 + 2x, & \text{for } x > 1 \end{cases}$$

Discuss the continuity of f and f'.

SOLUTION We have,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^2 + 2x = -1 + 2 = 1$$

and, $f(1) = 1$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

So, $f(x)$ is continuous at $x = 1$.

Now,

(LHD at $x = 1$)

$$= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{x-1}{x-1} = 1$$

(RHD at $x = 1$)

$$= \lim_{x\to 1^+} \frac{-x^2+2x-1}{x-1}$$

$$= \lim_{x\to 1^+} \frac{-(x-1)^2}{x-1}$$

$$= \lim_{x\to 1} (x-1) = 0$$

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$)

So, $f(x)$ is not differentiable at $x = 1$.

$$\therefore f'(x) = \begin{cases} 1 & , \text{for } x < 1 \\ -2x+2 & , \text{for } x > 1 \end{cases}$$

Now,

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 1 = 1$$

and,

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} (-2x+2) = -2+2 = 0$$

$$\therefore \lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$$

Hence, $f'(x)$ is not continuous at $x = 1$.

EXAMPLE 12 *Let* $f(x) = \begin{cases} |1-4x^2| & , 0 \le x < 1 \\ [x^2-2x] & , 1 \le x < 2 \end{cases}$

Then discuss the differentiability of $f(x)$ *in* $(0, 2)$. *Here,* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION We have,

$$f(x) = \begin{cases} |1-4x^2| & , 0 \le x < 1 \\ [x^2-2x] & , 1 \le x < 2 \end{cases}$$

$$= \begin{cases} 1-4x^2 & , 0 \le x < \frac{1}{2} \\ 4x^2-1 & , \frac{1}{2} \le x < 1 \\ -1 & , 1 \le x < 2 \end{cases}$$

Clearly, $f(x)$ being a polynomial function, is differentiable on $(0, 1/2) \cup (1/2, 1) \cup (1, 2)$. So, the possible points of non-differentiability are $x = 1/2, 1$.

Differentiability at $x = 1/2$:

(LHD at $x = 1/2$)

$$= \lim_{x\to 1/2^-} \frac{f(x)-f(1/2)}{x-\frac{1}{2}}$$

$$= \lim_{x\to 1/2^-} \frac{1-4x^2-0}{x-\frac{1}{2}}$$

$$= \lim_{x\to 1/2^-} -4\left(x+\frac{1}{2}\right) = -4$$

(RHD at $x = 1/2$)

$$= \lim_{x\to 1/2^+} \frac{f(x)-f(1/2)}{x-\frac{1}{2}}$$

$$= \lim_{x\to 1/2^+} \frac{4x^2-1}{x-\frac{1}{2}}$$

$$= \lim_{x\to 1/2^+} 4\left(x+\frac{1}{2}\right) = 4$$

$$\therefore \left(\text{LHD at } x = \frac{1}{2}\right) \neq \left(\text{RHD at } x = \frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x = 1/2$.

Similarly, $f(x)$ is not differentiable at $x = 1$.

Hence, $f(x)$ is differentiable in $(0, 2)$ except at $x = 1/2, 1$.

EXAMPLE 13 Let $f(x) = \begin{cases} \int_0^x \{5 + |1-t|\}\, dt & , \text{if } x > 2 \\ 5x+1 & , \text{if } x \le 2 \end{cases}$

Test $f(x)$ *for continuity and differentiability for all real* x.

SOLUTION For $x > 2$, we have

$$f(x) = \int_0^x \{5 + |1-t|\}\, dt$$

$$= \int_0^1 \{5 + (1-t)\}\, dt + \int_1^x \{5 - (1-t)\}\, dt$$

$$= \int_0^1 (6-t)\, dt + \int_1^x (4+t)\, dt$$

$$= \left[6t - \frac{t^2}{2}\right]_0^1 + \left[4t + \frac{t^2}{2}\right]_1^x$$

$$= \frac{11}{2} + 4x + \frac{x^2}{2} - \frac{9}{2}$$

$$= \frac{x^2}{2} + 4x + 1$$

$$\therefore \quad f(x) = \begin{cases} \frac{x^2}{2} + 4x + 1, & x > 2 \\ 5x + 1, & x \le 2 \end{cases}$$

Clearly, $f(x)$ being a polynomial function is continuous for all $x > 2$ and for all $x < 2$. So, we test $f(x)$ for continuity and differentiability at $x = 2$.

Continuity at $x = 2$: We have,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5x + 1 = 11$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2}{2} + 4x + 1 = 11$$

and,

$$f(2) = 5 \times 2 + 1 = 11.$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

Hence, $f(x)$ is continuous at $x = 2$.

Differentiability at $x = 2$: We have,

(LHD at $x = 2$)

$$= \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{5x + 1 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{5(x - 2)}{x - 2} = 5$$

and,

(RHD at $x = 2$)

$$= \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{\frac{x^2}{2} + 4x + 1 - 11}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{x^2 + 8x - 20}{2(x - 2)}$$

$$= \lim_{x \to 2^+} \frac{(x + 10)(x - 2)}{2(x - 2)}$$

$$= \lim_{x \to 2} \frac{x + 10}{2} = 6$$

$\therefore$ (LHD at $x = 2$) $\neq$ (RHD at $x = 2$)

Hence, $f(x)$ is not differentiable at $x = 2$.

EXAMPLE 14 *Let* $f(x) = x^3 - x^2 + x + 1$ *and*

$$g(x) = \begin{cases} \max\{f(t) : 0 \le t \le x ; \text{for } 0 \le x \le 1\} \\ 3 - x \quad \text{for } 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of the function $g(x)$ *in the interval* $(0, 2)$. [IIT 1985]

SOLUTION We have,

$$f(x) = x^3 - x^2 + x + 1$$

$$\therefore \quad f'(x) = 3x^2 - 2x + 1 > 0 \text{ for all } x$$

$$\Rightarrow \quad f(x) \text{ is increasing for all } x \in R \qquad \left[\because \text{Disc} < 0 \text{ and Coeff of } x^2 > 0\right]$$

$$\Rightarrow \quad \max\{f(t) \mid 0 \le t \le x\} = f(x)$$

$$\therefore \quad g(x) = \begin{cases} f(x) = x^3 - x^2 + x + 1, & \text{for } 0 \le x \le 1 \\ 3 - x, & \text{for } 1 < x \le 2 \end{cases}$$

Clearly, $g(x)$ being a polynomial function is continuous and differentiable on $(0, 1) \cup (1, 2)$. So, we check its continuity and differentiability at $x = 1$.

Continuity at $x = 1$: We have,

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^3 - x^2 + x + 1 = 1 - 1 + 1 + 1 = 2$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 3 - x = 3 - 1 = 2$$

and,

$$g(1) = 1 - 1 + 1 + 1 = 2$$

$$\therefore \quad \lim_{x \to 1^-} g(x) = g(1) = \lim_{x \to 1^+} g(x)$$

So, $g(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$: We have,

(LHD at $x = 1$)

$$= \lim_{x \to 1^-} \frac{g(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x^3 - x^2 + x + 1) - 2}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^3 - x^2 + x - 1}{x - 1}$$

$$= \lim_{x \to 1^-} x^2 + 1 = 2$$

and,

(RHD at $x = 1$

$$= \lim_{x \to 1^+} \frac{g(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{3 - x - 2}{x - 1}$$

$$= \lim_{x \to 1} -1 = -1.$$

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$)

Hence, $g(x)$ is not differentiable at $x = 1$.

EXAMPLE 15 *The function f is defined by $y = f(x)$, where*

$$x = 2t - |t|, y = t^2 + t|t|, t \in R$$

(i) Draw the graph of $f(x)$ in the interval $[-1, 1]$

(ii) Discuss the continuity and differentiability of $f(x)$.

SOLUTION We have,

$$x = 2t - |t| \text{ and } y = t^2 + t|t|, t \in R$$

CASE I When $t < 0$

In this case, we have

$$x = 3t \text{ and } y = t^2 - t^2 = 0$$

Also,

$t < 0$ and $x = 3t \Rightarrow x < 0$

Thus, for $x < 0$, we have $y = 0$.

CASE II *When $t \geq 0$*

In this case, we have,

$$x = 2t - t = t \text{ and } y = t^2 + t^2 = 2t^2$$

$\Rightarrow y = 2x^2$ [Eliminating t between x and y]

Also, $t \geq 0$ and $x = t \Rightarrow x \geq 0$

Thus, for $x \geq 0$; we have

$$y = 2x^2.$$

So, the function f is given by

$$f(x) = \begin{cases} 0 & , \text{for } x < 0 \\ 2x^2 & , \text{for } x \geq 0 \end{cases}$$

(i) For $x < 0$, the graph of $f(x)$ is x-axis and for $x \geq 0$, the graph of $f(x)$ is a parabola as shown in Fig. 4.16.

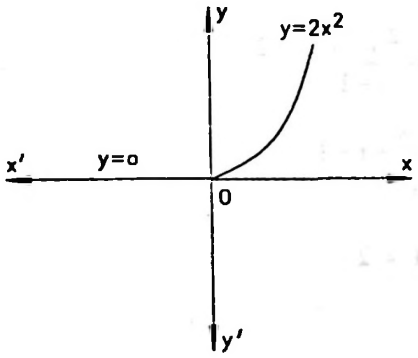


Fig. 4.16

(ii) Clearly, $f(x)$ is continuous and differentiable for all $x < 0$ and also for $x > 0$.

Now, we check these two at $x = 0$ only.

Continuity at $x = 0$: We have,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} = 0 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2 = 2 \times 0 = 0$$

Also, $f(0) = 2 \times 0 = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

Differentiability at $x = 0$: We have,

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$(\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{2x^2 - 0}{x} = 0$$

$\therefore$ (LHD at $x = 0$) = (RHD at $x = 0$)

So, $f(x)$ is differentiable at $x = 0$.

Hence, $f(x)$ is everywhere continuous and differentiable.

EXAMPLE 16 *If $f(x) = \min\{|x|, |x-2|, 2 - |x-1|\}$, draw the graph of $f(x)$ and discuss its continuity and differentiability.* [IIT 1995]

SOLUTION The graph of the function $f(x)$ is shown in Fig. 4.17. We observe from the graph of $f(x)$ that

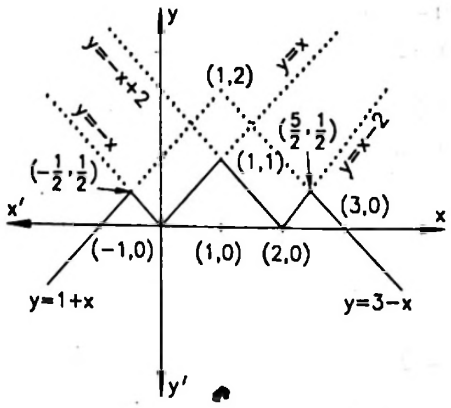


Fig. 4.17

$$f(x) = \begin{cases} 1 + x & , x \leq -\frac{1}{2} \\ -x & , -\frac{1}{2} < x \leq 0 \\ x & , 0 < x \leq 1 \\ -x + 2 & , 1 < x \leq 2 \\ x - 2 & , 2 < x \leq \frac{5}{2} \\ 3 - x & , x > \frac{5}{2} \end{cases}$$

It is evident from the graph that $f(x)$ is everywhere continuous. The continuity at $x = -1/2, 0, \infty, 1, 2$ and $5/2$ can also be verified algebraically also. We also observe from the graph

that $f(x)$ is not differentiable at $x = -1/2, 0, 1, 2$ and $5/2$. Students are advised to verify the same by computing the values of left hand and right hand derivatives at these points.

EXAMPLE 17 *If* $f(x) = x^2 - 2|x|$ and

$$g(x) = \begin{cases} \text{Min } \{f(t) : -2 \le t \le x, -2 \le x < 0\} \\ \text{Max } \{f(t) : 0 \le t \le x, 0 \le x \le 3\} \end{cases}$$

(i) Draw the graph of $f(x)$ *and discuss its continuity and differentiability.*

(ii) Find and draw the graph of $g(x)$*. Also, discuss the continuity and differentiability of* $g(x)$*.*

SOLUTION (i) The graph of $f(x)$ is shown in Fig. 4.18. It is evident from the curve $y = f(x)$ that $f(x)$ is everywhere continuous and differentiable except at $x = 0$ where it is continuous only. This fact can be checked by using the definitions of continuity and differentiability at $x = 0$.

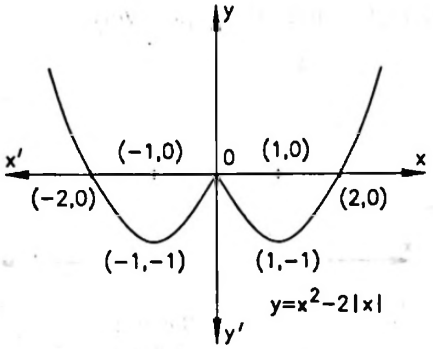


Fig. 4.18

(ii) We know that

If $f(x)$ is an increasing function on $[a, b]$ then,

$$\text{Max } \{f(t) : a \le t \le x, a \le x \le b\} = f(x)$$

$$\text{Min } \{f(t) : a \le t \le x, a \le x \le b\} = f(a)$$

If $f(x)$ is a decreasing function on $[a, b]$, then

$$\text{Max } \{f(t) : a \le t \le x, a \le x \le b\} = f(a)$$

$$\text{Min } \{f(t) : a \le t \le x, a \le x \le b\} = f(x)$$

It is evident from the graph of $f(x)$ that it is decreasing on $[-2, -1]$ and $[0, 1]$ and increasing on $[-1, 0]$ and $[1, 3]$.

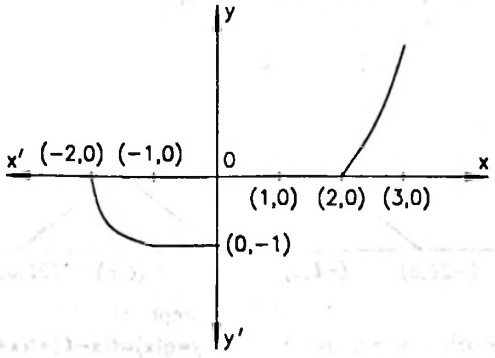


Fig. 4.19

$$\therefore \quad g(x) = \begin{cases} f(x), & \text{for } -2 \le x \le -1 \\ -1, & \text{for } -1 \le x < 0 \\ 0, & \text{for } 0 \le x < 2 \\ f(x), & \text{for } x \ge 2 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} x^2 + 2x, & \text{for } -1 \le x \le 1 \\ -1, & \text{for } -1 \le x < 0 \\ 0, & \text{for } 0 \le x < 2 \\ x^2 - 2x, & \text{for } x \ge 2 \end{cases}$$

It is evident from the graph of $g(x)$ that it is discontinuous at $x = 0$ and non-differentiable at $x = 0$ and $x = 2$. However, it is continuous and differentiable at all other points in $(-2, 3)$.

EXAMPLE 18 *If* $f(x) = \big||x| - 1\big|$*, then draw the graph of* $f(x)$ *and* $fof(x)$ *and also discuss their continuity and differentiability. Also, find the derivative of* $[fof(x)]^2$ *at* $x = 3/2$.

SOLUTION The graph of $f(x)$ is shown in Fig. 4.20.

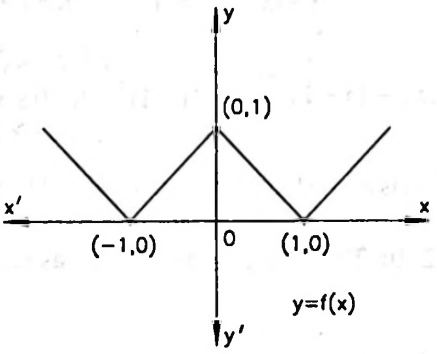


Fig. 4.20

It is evident from the graph of $f(x)$ that it is continuous everywhere but is not differentiable at $x = -1, 0, 1$.

Now,

$$fof(x) = f(f(x))$$

$$= \big||f(x)| - 1\big| \qquad \text{[Using def. of } f\text{]}$$

$$= \big|f(x) - 1\big| \qquad [\because f(x) \ge 0 \text{ for all } x \in R]$$

It follows from this that in order to draw the graph of $fog(x)$, the graph of $f(x)$ is translated along OY' by 1 unit and then the part of the graph of $y = f(x) - 1$ lying below x-axis is reflected above x-axis. Thus, the graph of $y = fof(x)$ is as shown in Fig. 4.21.

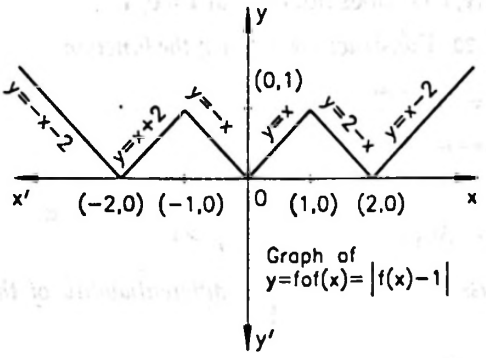


Fig. 4.21

From the graph, it is very clear that $fof(x)$ is every where continous but not differentiable at $x = -2, -1, 0, 1, 2$.

From the graph, we observe that

$$fof(x) = 2 - x \quad \text{for } 1 \le x \le 2$$

$$\therefore \quad |fof(x)|^2 = (2-x)^2$$

$$\Rightarrow \quad \frac{d}{dx}|fof(x)^2| = -2(2-x)$$

$$\Rightarrow \quad \left[\frac{d}{dx}|fof(x)|^2\right]_{x=3/2} = -2\left(2-\frac{3}{2}\right) = -1$$

EXAMPLE 19 *If $f(x) = ||x| - |x-1||^2$, draw the graph of $f(x)$ and discuss its continuity and differentiability. Also, find $f'(x)$.*

SOLUTION We have,

$$|x| - |x-1| = \begin{cases} -x + x - 1 = -1 & , \text{ for } x < 0 \\ x + x - 1 = 2x - 1 & , \text{ for } 0 \le x < 1 \\ x - (x-1) = 1 & , \text{ for } x \ge 1 \end{cases}$$

$$\therefore \quad f(x) = ||x| - |x-1||^2 = \begin{cases} 1 & , \text{ for } x < 0 \\ (2x-1)^2 & , \text{ for } 0 \le x < 1 \\ 1 & , \text{ for } x \ge 1 \end{cases}$$

Clearly, $f(x)$ is a constant function for all $x < 0$ and all $x \ge 1$. Between 0 and 1 the function represents a parabola having its vertex at $(1/2, 0)$. Thus the graph of $f(x)$ is as shown in Fig. 4.22.

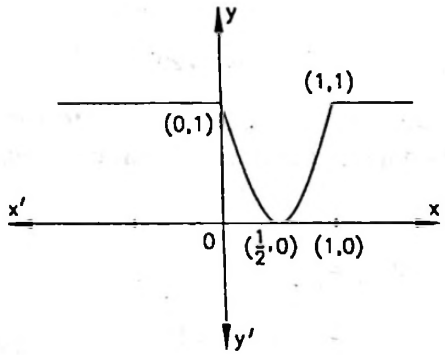


Fig. 4.22

From the graph of $y = f(x)$, we observe that it is everywhere continuous but not differentiable at $x = 0, 1$.

$$\therefore \quad f'(x) = \begin{cases} 0 & , \text{ for } x < 0 \\ 4(2x-1) & , \text{ for } 0 < x < 1 \\ 0 & , \text{ for } x > 1 \end{cases}$$

Clearly, $f'(x)$ does not exist at $x = 0, 1$.

EXAMPLE 20 *Construct the graph of the function*

$$g(x) = f(x+l) + f(x-l), \text{ where}$$

$$f(x) = \begin{cases} k\left\{1 - \dfrac{|x|}{l}\right\} & , \text{for } |x| \le l \\ 0 & , \text{ for } |x| > l \end{cases}$$

Also discuss the continuity and differentiability of the function $g(x)$.

SOLUTION We have,

$$f(x) = \begin{cases} k - \dfrac{k}{l}|x| & , \text{ for } |x| \le l \\ 0 & , \text{ for } |x| > l \end{cases}$$

The graph of $f(x)$ is shown in Fig. 4.23.

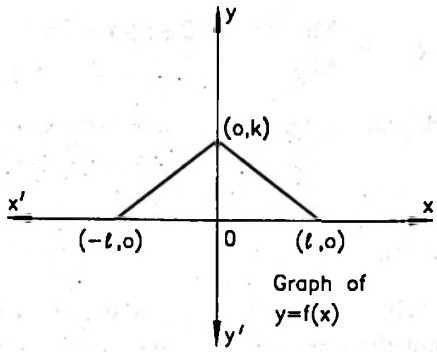


Fig. 4.23

In order to draw the graph of $g(x)$, we first draw the graphs of $f(x-l)$ and $f(x+l)$ by translating the graph of $f(x)$ by l units along OX and OX' respectively.

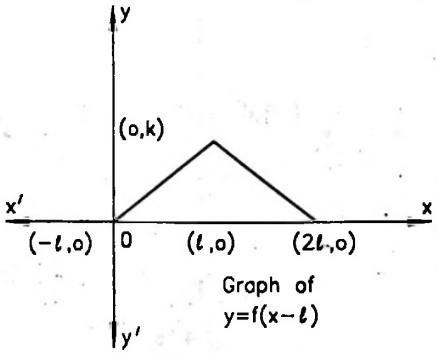


Fig. 4.24

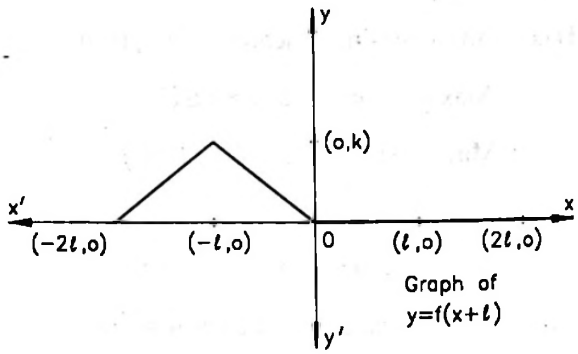


Fig. 4.25

The graph of $g(x)$ is obtaining by adding the graphs of $f(x-l)$ and $f(x+l)$ as shown in Fig. 4.26.

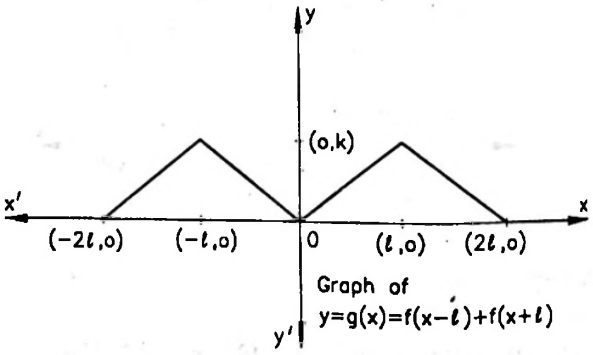


Fig. 4.26

It is evident from the graph of $y = g(x)$ that $g(x)$ is everywhere continuous but not differntiable at $x = \pm l, \pm 2l$ and $x = 0$.

EXAMPLE 21 *If* $f(x) = \begin{cases} 3-x^2, -1 \le x < 2 \\ 2x-4, 2 \le x \le 4 \end{cases}$, *then find* $fof(x)$ *and discuss its continuity and differentiability.*

SOLUTION We have,

$$f(x) = \begin{cases} 3-x^2, -1 \le x < 2 \\ 2x-4, 2 \le x \le 4 \end{cases}$$

$$\therefore \quad fof(x) = f(f(x))$$

$$= \begin{cases} f(3-x^2), -1 \le x < 2 \\ f(2x-4), 2 \le x \le 4 \end{cases}$$

$$= \begin{cases} 2(3-x^2)-4 & , \text{ for } -1 \le x \le 1 \\ 3-(3-x^2)^2 & , \text{ for } 1 < x < 2 \\ 3-(2x-4)^2 & , \text{ for } 2 \le x \le 3 \\ 2(2x-4)-4 & , \text{ for } 3 < x \le 4 \end{cases}$$

$$= \begin{cases} 2-2x^2 & , \text{ for } -1 \le x \le 1 \\ -6+6x^2-x^4 & , \text{ for } 1 < x < 2 \\ -13+16x-4x^2 & , \text{ for } 2 \le x \le 3 \\ 4x-12 & , \text{ for } 3 < x \le 4 \end{cases}$$

The possible points of discontinuity of $fof(x)$ are $x = 1, 2, 3$.

Continuity at $x = 1$: We have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 0^-} 2-2x^2 = 2-2 = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -6+6x^2-x^4 = -6+6-1 = -1$$

$$\therefore \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$$

So, $f(x)$ is not continuous at $x = 1$.

Continuity at $x = 2$: We have,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} -6+6x^2-x^4 = -6+24-16 = 2$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} -13+16x-4x^2 = -13+32-16 = 3$$

$$\therefore \lim_{x \to 2^-} f(x) \ne \lim_{x \to 2^+} f(x)$$

So, $f(x)$ is not continuous at $x = 2$.

Continuity at $x = 3$ We have,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -13+16x-4x^2 = -13+48-36 = -1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (4x-12) = 4 \times 3 - 12 = 0$$

$$\therefore \lim_{x \to 3^-} f(x) \ne \lim_{x \to 3^+} f(x)$$

So, $f(x)$ is discontinuous at $x = 3$.

Hence, $f(x)$ is discontinuous and hence non-differentiable at $x = 1, 2, 3$.

EXAMPLE 22 *If* $f(x) = \begin{cases} 2x^2+12x+16, -4 \le x \le -2 \\ 2-|x| \quad\quad\quad , -2 < x \le 1 \\ 4x-x^2-2 \quad , 1 < x \le 3 \end{cases}$

then draw the graphs of $|f(x)|$ *and* $[f(x)]$ *and also discuss their continuity and differentiability. Here,* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION The graphs of $y = f(x)$ is shown in Fig. 4.27. To drawn the graphs of $y = |f(x)|$, the portion lying below x-axis in the graphs of $y = f(x)$ is reflected above x-axis.

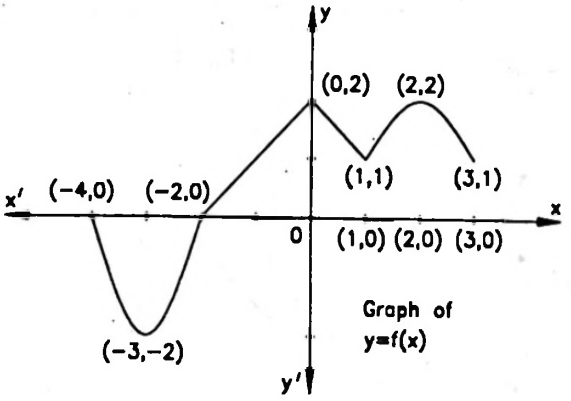


Fig. 4.27

Fig. 4.28 shows the graphs of $y = |f(x)|$. Clearly, it is continuous on $[-4, 3]$ but it is not differentiable at $x = -2, 0, 1$.

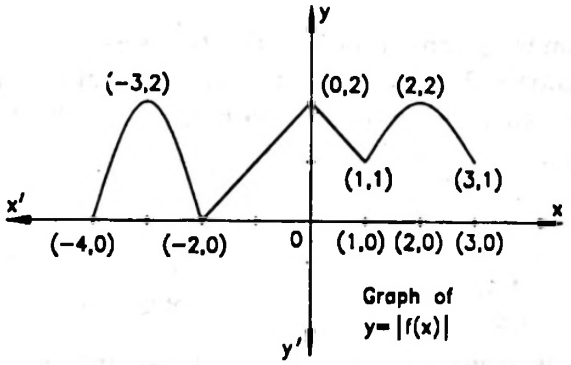


Fig. 4.28

The graph of $y = [f(x)]$ is shown in Fig. 4.29. Clearly, $[f(x)]$ is discontinuous and hence non-differentiable at $x = -3 \pm 1/\sqrt{2}, -1, 0, 2$. At all other points in $(-4, 3)$ it is continuous and differentiable.

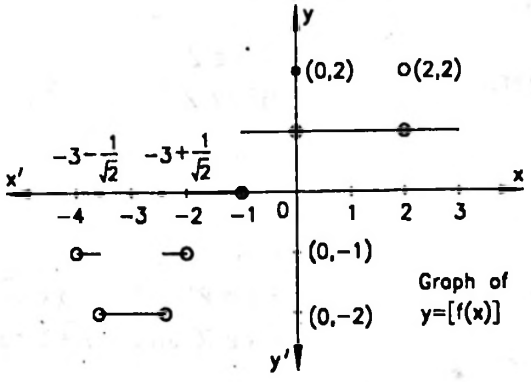


Fig. 4.29

EXAMPLE 23 *If* $f(x) = |x+1|\{|x| + |x-1|\}$, *then draw the graph of* $f(x)$ *in the interval* $[-2, 2]$ *and discuss its continuity and differentiability in* $(-2, 2)$.

SOLUTION We have,

$$f(x) = |x+1|\{|x| + |x-1|\}$$

$$= \begin{cases} (x+1)(2x-1) & , -2 \le x < -1 \\ -(x+1)(2x-1) & , -1 \le x < 0 \\ x+1 & , 0 < x < 1 \\ (x+1)(2x-1) & , 1 \le x \le 2 \end{cases}$$

The graph of $f(x)$ is as shown in Fig. 4.30.

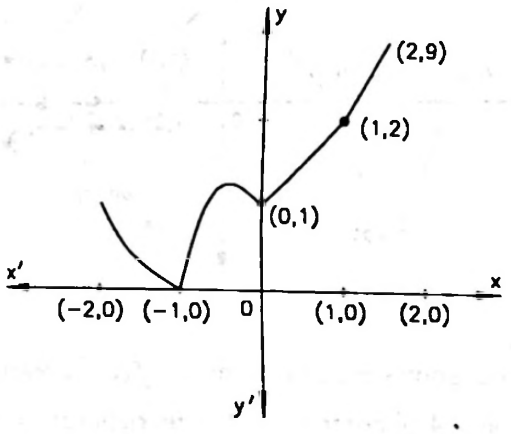


Fig. 4.30

From the graph it is evident that $f(x)$ is everywhere continuous on $(-2, 2)$ but it is not differentiable at $x = -1, 0, 1$. Reader is advised to check the same by using the definitions of continuity and differentiability also.

EXAMPLE 24 *Let* $f(x)$ *and* $g(x)$ *be defined by*

$$f(x) = [x] \text{ and } g(x) = \begin{cases} 0 & , x \in Z \\ x^2 & , \text{otherwise} \end{cases}$$

Discuss the continuity and differentiability of gof and *fog*.

SOLUTION We have,

$$gof(x) = g(f(x)) = g([x]) = 0 \text{ for all } x.$$

Thus, *gof* is a constant function, which is everywhere continuous and differentiable.

Now,

$$fog(x) = f(g(x)) = \begin{cases} f(0) & , x \in Z \\ f(x^2) & , x \notin Z \end{cases}$$

$$= \begin{cases} [0] & , x \in Z \\ [x^2] & , x \notin Z \end{cases}$$

$$= \begin{cases} 0 \text{ if } x \in Z \\ n \text{ if } x \in R^+ \text{ and } \sqrt{n} < x < \sqrt{n+1} \\ n \text{ if } x \in R^- \text{ and } -\sqrt{n+1} < x < -\sqrt{n} \end{cases}$$

Here, R^+ and R^- denote the sets of positive and negative real numbers respectively.

Clearly, *fog* is continuous and differentiable at all x exce when $x = \pm\sqrt{n}$, where $n = 0, 1, 2, 3, \ldots$.

EXAMPLE 25 *Let* $f(x+y) = f(x)f(y)$ *for all* $x, y \in R$ *and supp that* f *is differentiable at* $x = 0$. *Show that* f *is everywhere differentia* [IIT 19

SOLUTION We have,

$f(x+y) = f(x)f(y)$ for all $x, y \in R$.

$\Rightarrow f(0) = f(0)f(0)$

$\Rightarrow f(0)(1 - f(0)) = 0$

$\Rightarrow f(0) = 0$ or, $f(0) = 1$

If $f(0) = 0$, then by replacing y by 0 in $f(x+y) = f(x)f($ we get $f(x+0) = f(x)f(0)$ for all $x \in R$

$\Rightarrow f(x) = 0$ for all $x \in R$

Clearly, $f(x)$ is everywhere differentiable.

So, let $f(0) = 1$.

Since $f(x)$ is differentiable at $x = 0$. Therefore,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) \text{ exists finitely}$$

Let a be any real number. Then,

$f'(a)$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a)f(h) - f(a)}{h} \qquad [\because f(x+y) = f(x)f$$

$$= f(a) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(a) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \qquad [\because f(0) =$$

$$= f(a)f'(0) \qquad [\text{Using}$$

Thus, $f(x)$ is differentiable at $x = a$. Since a is an orbitr real number. Therefore, $f(x)$ is everywhere differentiable a $f'(x)$ is given by $f'(x) = f(x)f'(0)$.

EXAMPLE 26 *Let a function* $f: R \to R$ *be satisfying*

$$|f(x) - f(y)| \le |x - y|^3 \text{ for all } x, y \in R.$$

Prove that $f(x)$ *is a constant function.* [IIT 19

SOLUTION We have,

$$|f(x) - f(y)| \le |x - y|^3 \qquad \text{for all } x, y \in R$$

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \le |x - y|^2 \qquad \text{for all } x, y \in R$$

$$\Rightarrow \lim_{y \to x} \left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{y \to x} |x - y|^2 \qquad \text{for all } x$$

$\Rightarrow \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le 0$ for all x

$\Rightarrow |f'(x)| \le 0$ for all x

$\Rightarrow f'(x) = 0$ for all x

Hence, $f(x)$ is a constant function on R.

EXAMPLE 27 *If g is the inverse function of function f and $f'(x)=\frac{1}{1+x^n}$, prove that $g'(x)=1+\{g(x)\}^n$.*

SOLUTION Since g is the inverse of function f. Therefore,

$fog(x) = x$, for all x

$\Rightarrow \frac{d}{dx}\{fog(x)\} = 1$, for all x

$\Rightarrow f'\{g(x)\}g'(x) = 1$, for all x [By Chain Rule]

$\Rightarrow g'(x) = \frac{1}{f'\{g(x)\}}$, for all x

$\Rightarrow g'(x) = \frac{1}{1+\{g(x)\}^n}$ [By def. of f]

$\Rightarrow g'(x) = 1+\{g(x)\}^n$.

EXAMPLE 28 *Let a function $f: R \to R$ be given by*

$f(x+y) = f(x)f(y)$ for all $x, y \in R$

and $f(x) \neq 0$ for any $x \in R$. If the function $f(x)$ is differentiable at $x=0$, show that $f'(x)=f'(0)\,f(x)$ for all $x \in R$. Also, determine $f(x)$.

SOLUTION We have,

$f(x+y) = f(x)f(y)$ for all $x, y \in R$

$\Rightarrow f(0+0) = f(0)f(0)$ [Putting $x=y=0$]

$\Rightarrow f(0) = f(0)f(0)$

$\Rightarrow f(0)\{1-f(0)\} = 0$

$\Rightarrow f(0) = 1$ [$\because f(x)\neq 0$ for any $x \in R$]

Since $f(x)$ is differentiable at $x=0$. Therefore,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) \text{ exists finitely} \quad ...(i)$$

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} \quad \text{[By def. of } f\text{]}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= f(x)\lim_{x\to 0}\frac{f(h)-f(0)}{h} \quad [\because f(0)=1]$$

$$= f(x).f'(0) \quad \text{[Using (i)]}$$

Hence, $f'(x)=f(x).f'(0)$ for all $x \in R$.

Now,

$f'(x) = f(x)f'(0)$

$\Rightarrow \frac{f'(x)}{f(x)} = f'(0)$

$\Rightarrow \log|f(x)| = xf'(0) + \log A$ [On integration]

$\Rightarrow |f(x)| = Ae^{xf'(0)}$

$\Rightarrow f(x) = \pm Ae^{xf'(0)}$ for all $x \in R$.

EXAMPLE 29 *A function $f(x)$ is defined for all $x>0$, and satisfies $f(x^2)=x^3$ for all $x>0$. Show that f is differentiable at $x=4$ and find $f'(4)$.*

SOLUTION We have,

$$\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$$

$$= \lim_{x\to 4}\frac{f\{(\sqrt{x})^2\}-f(2^2)}{x-4}$$

$$= \lim_{x\to 4}\frac{(\sqrt{x})^3-2^3}{x-4}$$

$$= \lim_{x\to 4}\frac{x^{3/2}-4^{3/2}}{x-4}$$

$$= \frac{3}{2}\times 4^{3/2-1} \quad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 3$$

Thus, $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$ exists finitely and is equal to 3.

Hence, $f(x)$ is differentiable at $x=4$ and $f'(4)=3$.

EXAMPLE 30 *Let $f: R \to R$ be a function satisfying*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \text{for all } x, y \in R.$$

If $f'(0)$ exists and equals to -1 and $f(0)=1$, find $f(2)$. **[IIT 1995]**

SOLUTION We have,

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \text{for all } x, y \in R$$

$$\therefore f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} \quad \text{for all } x \in R \quad \text{[Putting } y=0\text{]}$$

$$\Rightarrow f\left(\frac{x}{2}\right) = \frac{f(x)+1}{2} \quad \text{for all } x \in R \quad [\because f(0)=1 \text{ (given)}]$$

$$\Rightarrow f(x) = 2f\left(\frac{x}{2}\right)-1 \quad \text{for all } x \in R \quad ...(i)$$

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\left\{2f\left(\frac{x+h}{2}\right)-1\right\}-\left\{2f\left(\frac{x}{2}\right)-1\right\}}{h}$$

$$\Rightarrow f'(x) = 2\lim_{h\to 0} \frac{f\left(\frac{x+h}{2}\right)-f\left(\frac{x}{2}\right)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\frac{f(x)+f(h)}{2}-f\left(\frac{x}{2}\right)}{h} \quad \left[\because f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}\right]$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x)+f(h)-2f(x/2)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h)-1}{h} \qquad \text{[Using (i)]}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} \qquad [\because f(0) = 1]$$

$\Rightarrow f'(x) = f'(0)$

$\Rightarrow f'(x) = -1 \qquad [\because f'(0) = -1]$

$\Rightarrow f(x) = -x + C \quad$ for all $x \in R \qquad$ [On integration]

$\Rightarrow f(0) = C \qquad [\because f(0) = 1]$

$\Rightarrow C = 1$

$\therefore f(x) = -x + 1$

$\Rightarrow f(2) = -2 + 1 = -1$

ALITER We have,

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \text{for all } x, y \in R \qquad \text{...(i)}$$

$$\Rightarrow f\left(\frac{x+0}{2}\right) = \frac{f(x)+f(0)}{2} \quad \text{for all } x \in R$$

$$\Rightarrow f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} \quad \text{for all } x \in R$$

$$\therefore f\left(\frac{x+y}{2}\right) = \frac{f(x+y)+f(0)}{2} \quad \text{for all } x, y \in R \qquad \text{...(ii)}$$

For (i) and (ii), we get

$$\frac{f(x)+f(y)}{2} = \frac{f(x+y)+f(0)}{2} \quad \text{for all } x, y \in R$$

$\Rightarrow f(x+y) = f(x)+f(y)-f(0) \quad$ for all $x, y \in R$

$\Rightarrow f(x+y)-1 = \{f(x)-1\}+\{f(y)-1\}$ for all $x, y \in R$

$[\because f(0) = -1]$

Let $g(x) = f(x) - 1$. Then the above relation becomes

$g(x+y) = g(x)+g(y) \quad$ for all $x, y \in R$

$\Rightarrow g(x) = xg(1) \quad$ for all $x \in R \qquad$ [See chapter 1]

$\Rightarrow f(x)-1 = x\{f(1)-1\} \quad$ for all $x \in R \qquad$ ...(iii)

$\Rightarrow f'(x) = f(1)-1 \quad$ for $x \in R$

$\Rightarrow f'(0) = f(1)-1$

$\Rightarrow -1 = f(1)-1$

$\Rightarrow f(1) = 0$

$\therefore f(x)-1 = -x \qquad$ [Putting $f(1) = 0$ in (iii)]

$\Rightarrow f(x) = 1-x$

$\therefore f(2) = 1-2 = -1$

EXAMPLE 31 *If $f: R \to R$ is given by*

$$f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \text{ for all } x, y \in R$$

and $f'(0) = 1$. Prove that $f(x)$ is continuous for all $x \in R$.

SOLUTION We have,

$$f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \text{for all } x, y \in R \qquad \text{...(i}$$

$$\Rightarrow f\left(\frac{3x+0}{3}\right) = \frac{f(3x)+2f(0)}{3} \quad \text{[Replacing } x \text{ by } 3x \text{ and } y \text{ by } 0$$

$$\Rightarrow f(x) = \frac{f(3x)+2f(0)}{3} \quad \text{for all } x \in R$$

$$\Rightarrow f(3x) = 3f(x)-2f(0) \quad \text{for all } x \in R \qquad \text{...(ii}$$

Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x+2(3h/2)}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+2f(3h/2)}{3}-f(x)}{h} \qquad \text{[Using (ii}$$

$$= \lim_{h\to 0} \frac{f(3x)+2f(3h/2)-3f(x)}{3h}$$

$$= \lim_{h\to 0} \frac{2f(3h/2)-2f(0)}{3h} \qquad \text{[Using (iii}$$

$$= \lim_{h\to 0} \frac{f(3h/2)-f(0)}{\left(\frac{3h}{2}\right)}$$

$= f'(0) \quad$ for all $x \in R$

$= 1 \quad$ for all $x \in R$

$\therefore f(x) = x + C \quad$ for all $x \in R$

Clearly, $f(x)$ is a polynomial function which is everywhere continuous.

EXAMPLE 32 *If* $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ *for all* $x, y \in R$ *and* $xy \neq 1$ *and*, $\lim_{x\to 0}\frac{f(x)}{x}=2$, *find* $f\left(\frac{1}{\sqrt{3}}\right)$ and $f'(1)$.

SOLUTION We have,

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right) \text{ for all } x, y \in R \qquad ...(i)$$

Putting $x = y = 0$, we get

$f(0) + f(0) = f(0) \Rightarrow f(0) = 0$

Again putting $y = -x$, we get

$f(x)+f(-x) = f(0)$ for all $x \in R$

$\Rightarrow f(x)+f(-x) = 0$ for all $x \in R$ $[\because f(0) = 0]$

$\Rightarrow f(-x) = -f(x)$ for all $x \in R$

Hence, $f(x)$ is an odd function.

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)+f(-x)}{h} \qquad [\because -f(x) = f(-x)]$$

$$= \lim_{h\to 0}\frac{f\left(\frac{x+h+(-x)}{1-(x+h)(-x)}\right)}{h} \qquad \text{[Using (i)]}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{h}{1+x^2+hx}\right)}{\frac{h}{1+x^2+hx}}\times\frac{1}{1+x^2+hx}$$

$$= \frac{2}{1+x^2} \qquad \left[\text{Using} : \lim_{x\to 0}\frac{f(x)}{x} = 2\right]$$

$\Rightarrow f(x) = 2\tan^{-1}x + C$ [On integration]

$\Rightarrow f(0) = 2\tan^{-1}0 + C$ [Putting $x = 0$]

$\Rightarrow 0 = 0 + C$

$\Rightarrow C = 0$

$\therefore f(x) = 2\tan^{-1}x.$

$$\therefore f\left(\frac{1}{\sqrt{3}}\right) = 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2\times\frac{\pi}{6} = \frac{\pi}{3}$$

Also,

$$f'(x) = \frac{2}{1+x^2}$$

$$\Rightarrow f'(1) = \frac{2}{1+1} = 1$$

Hence, $f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$ and $f'(1) = 1$.

EXAMPLE 33 *If* $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ *for all* $x, y \in R$ *and* $f'(2) = 2$, *then determine* $f(x)$.

SOLUTION We have,

$$f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3} \quad \text{for all } x, y \in R.$$

Putting $y = 0$, we obtain

$$f\left(\frac{x}{3}\right) = \frac{2+f(x)+f(0)}{3}$$

$$\Rightarrow 3f\left(\frac{x}{3}\right) = 2+f(x)+f(0)$$

Replacing x by $3x$, we obtain

$$3f(x) = 2+f(3x)+f(0) \qquad ...(i)$$

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3}-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{2+f(3x)+f(3h)-3f(x)}{3h}$$

$$= \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} \qquad \text{[Using (i)]}$$

$$= f'(0)$$

Thus, $f'(x) = f'(0)$ for all $x \in R$

$\Rightarrow f'(2) = f'(0)$

$\Rightarrow f'(0) = 2$ $[\because f'(2) = 2 \text{ (given)}]$

$\therefore f'(x) = 2$ for all $x \in R$

$\Rightarrow f(x) = 2x + C$ for all $x \in R$

Putting $x = 0$ in (i), we get

$$f(0) = 2$$

Now,

$f(x) = 2x + C$

$\Rightarrow f(0) = C$

$\Rightarrow 2 = C.$

$\therefore f(x) = 2x + 2$

EXAMPLE 34 *Let* $f(x+y)=f(x)-f(y)+2xy+1$ *for all* $x, y \in R$. *If* $f(x)$ *is differentiable and* $f'(0)=\sqrt{3+a-a^2}$, *prove that* $f(x) > 0$, *for all* $x \in R$.

SOLUTION We have,

$f(x+y) = f(x) - f(y) + 2xy + 1$ for all $x, y \in R$ ...(i)

Putting $x = y = 0$, we get

$f(0) = f(0) - f(0) + 0 + 1$

$\Rightarrow f(0) = 1$

Now,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) - f(h) + 2xh + 1 - f(x)}{h} \quad \text{[By using (i)]}$$

$$= \lim_{h \to 0} 2x - \left\{\frac{f(h) - 1}{h}\right\}$$

$$= 2x - \lim_{h \to 0} \frac{f(h) - f(0)}{h} \quad [\because f(0) = 1]$$

$$= 2x - f'(0)$$

$$= 2x - \sqrt{3 + a - a^2}$$

$\therefore f(x) = x^2 - \sqrt{3 + a - a^2}\ x + C$ [On integration]

Putting $x = 0$, we get

$f(0) = C \Rightarrow C = 1$ $[\because f(0) = 1]$

$\therefore f(x) = x^2 - \sqrt{3 + a - a^2}\ x + 1$

Clearly, the expression for $f(x)$ is a quadratic expression.

The discriminant D of this is given by

$$D = 3 + a - a^2 - 4 = -a^2 + a - 1 = -(a^2 - a + 1)$$

Clearly, $a^2 - a + 1 > 0$ for all real values of a, because its discriminant is negative and coefficient of a^2 is positive.

$\therefore D < 0$

Thus, discriminant of $f(x)$ is negative and coefficient of x^2 is positive.

Hence, $f(x) > 0$ for all $x \in R$.

EXAMPLE 35 *Let f be a real function satisfying $f(x+y+z) = f(x) f(y) f(z)$ for all real x, y, z. If $f(2) = 4$ and $f'(0) = 3$, then find $f(0)$ and $f'(2)$.* **[IIT 1982]**

SOLUTION We have,

$f(x+y+z) = f(x) f(y) f(z)$ for all $x, y, z \in R$ ...(i)

Putting $x = y = z = 0$, we get

$f(0) = [f(0)]^3$

$\Rightarrow f(0)[1 - (f(0))^2] = 0$

$\Rightarrow f(0) = 0$ or, $f(0) = \pm 1$

Putting $y = z = -1$ in (i), we obtain

$f(x-2) = f(x)\,[f(-1)]^2$ for all $x \in R$

$\Rightarrow f(0) = f(2)\,[f(-1)]^2$ [Putting $x = 2$]

$\Rightarrow f(0) = 4\,[f(-1)]^2$

$\Rightarrow f(0) > 0$

$\therefore f(0) = 1$

Now, putting $y = 2$ and $z = 0$ in (i), we get

$f(x+2) = f(x) f(2) f(0)$

$\Rightarrow f(x+2) = 4 f(x)$

$\Rightarrow f'(x+2) = 4 f'(x)$ for all $x \in R$

$\Rightarrow f'(2) = 4 f'(0)$ [Putting $x =$

$\Rightarrow f'(2) = 4 \times 3 = 12.$

EXAMPLE 36 *Let $f(x)$ be a real valued function not identically zero such that*

$$f(x + y^n) = f(x) + [f(y)]^n \text{ for all } x, y \in R$$

(where n is an odd integer greater than 1) and $f'(0) \geq 0$. Find the values of $f'(0)$ and $f(5)$.

SOLUTION We have,

$f(x + y^n) = f(x) + [f(y)]^n$ for all $x, y \in R$ ...(

Putting $x = y = 0$ in (i), we obtain

$f(0) = f(0) + [f(0)]^n$

$\Rightarrow [f(0)]^n = 0$

$\Rightarrow f(0) = 0$

Now,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{f(h)}{h} \quad \text{...(ii}$$

Again,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(0 + (h^{1/n})^n\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{f(0) + [f(h^{1/n})]^n}{h} \quad \text{[Using (i)}$$

$$= \lim_{h \to 0} \left|\frac{f(h^{1/n})}{h^{1/n}}\right|^n$$

$$= [f'(0)]^n \quad \text{[Using (iii)}$$

$\therefore f'(0) = [f'(0)]^n$

$\Rightarrow f'(0)\left\{[f'(0)]^{n-1} - 1\right\} = 0$

$\Rightarrow f'(0) = 0$ or, $[f'(0)]^{n-1} = 1$

$\Rightarrow f'(0) = 0$ or, $f'(0) = \pm 1$ $[\because n$ is odd $\therefore (n-1)$ is even

$\Rightarrow f'(0) = 0, 1$ $[\because f'(0) \geq$

Now,

$f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\{x+(h^{1/n})^n\}-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)+\{f(h^{1/n})\}^n - f(x)}{h} \quad \text{[Using (i)]}$$

$$= \lim_{h\to 0} \left|\frac{f(h^{1/n})}{h^{1/n}}\right|^n$$

$$= |f'(0)|^n \quad \text{...(iii)}$$

If $f'(0) = 0$, then from (iii), we have

$f'(x) = 0$ for all $x \in R$

$\Rightarrow$ $f(x)$ is a constant function

But, $f(0) = 0$

$\therefore$ $f(x) = 0$ for all $x \in R$

But it is given that $f(x)$ is not identically equal to zero. Therefore, $f'(0) \neq 0$

Hence, $f'(0) = 1$.

Putting, $f'(0) = 1$ in (iii), we get

$f'(x) = 1$

$\Rightarrow$ $f(x) = x + C$

But, $f(0) = 0$. Therefore, $C = 0$

Hence, $f(x) = x$ for all $x \in R$

$\therefore$ $f(5) = 5$.

Hence, $f'(0) = 1$ and $f(5) = 5$.

EXAMPLE 37 *A function $f(x)$ is defined for all $x \in R$ and satisfies $f(x+y) = f(x) + 2y^2 + kxy$ for all $x, y \in R$, where k is a given constant.*

If $f(1) = 2$ and $f(2) = 8$, find $f(x)$ and show that $f(x+y)f\left(\frac{1}{x+y}\right) = k$, $x + y \neq 0$.

SOLUTION We have,

$$f(x+y) = f(x) + 2y^2 + kxy \quad \text{for all } x, y \in R$$

$$\Rightarrow \frac{f(x+y)-f(x)}{y} = 2y + kx \quad \text{for all } x, y \in R$$

$$\Rightarrow \lim_{y\to 0} \frac{f(x+y)-f(x)}{(x+y)-x} = \lim_{y\to 0} (2y + kx) \quad \text{for all } x \in R$$

$$\Rightarrow f'(x) = kx \quad \text{for all } x \in R$$

$$\Rightarrow f(x) = \frac{kx^2}{2} + C \quad \text{...(i)}$$

$$\Rightarrow f(1) = \frac{k}{2} + C \text{ and } f(2) = 2k + C$$

$$\Rightarrow 2 = \frac{k}{2} + C \quad \text{and } 8 = 2k + C \quad [\because f(1) = 2 \text{ and } f(2) = 8]$$

$$\Rightarrow k = 4, C = 0$$

$\therefore$ $f(x) = 2x^2$ for all $x \in R$

Putting $C = 0$ in (i), we get

$$f(x) = \frac{k}{2}x^2$$

$$\Rightarrow f(x+y) = \frac{k}{2}(x+y)^2 \text{ and, } f\left(\frac{1}{x+y}\right) = \frac{k}{2}\left(\frac{1}{x+y}\right)^2$$

$$\therefore f(x+y)\, f\left(\frac{1}{x+y}\right) = \frac{k}{2}(x+y)^2 \times \frac{k}{2}\left(\frac{1}{x+y}\right)^2$$

$$= \frac{k^2}{4} = \frac{16}{4} = 4 \quad [\because k = 4]$$

$$= k.$$

EXAMPLE 38 *A function $f: R \to R$, where 'R' is the set of all real numbers satisfies the equation*

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3} \quad \textit{for all } x, y \in R$$

If $f(x)$ is differentiable at $x = 0$, then show that $f'(x)$ exists for all $x \in R$.

SOLUTION We have,

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3} \quad \text{for all } x, y \in R \quad \text{...(i)}$$

Putting $y = 0$ and replacing x by $3x$, we get

$$f(x) = \frac{f(3x)+f(0)+f(0)}{3} \text{ for all } x \in R$$

$$\Rightarrow \quad 3f(x) = f(3x) + 2f(0) \quad \text{for all } x \in R \quad \text{...(ii)}$$

Since $f(x)$ is differentiable at $x = 0$. Therefore,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) \text{ exists finitely.}$$

Now,

$f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+f(3h)+f(0)}{3} - f(x)}{h} \quad \text{[Using (i)]}$$

$$= \lim_{h\to 0} \frac{f(3x)+f(3h)+f(0)-3f(x)}{3h} \quad \text{[Using (ii)]}$$

$$= \lim_{h\to 0} \frac{f(3h)-f(0)}{3h}$$

$$= f'(0)$$

$\therefore\ f'(x) = f'(0)$ for all $x \in R$.

Hence, $f(x)$ is differentiable for all $x \in R$.

EXAMPLE 39 *Let $[x]$ stands for the greatest integer function. Find the derivative of $f(x) = \{x + [x^3+1]\}^{x^2+\sin x}$, wherever it exists in $(1, 3/2)$. Indicate the points where it does not exist.*

SOLUTION We observe that $[x^3+1]$ takes a jump from 2 to 3 at $2^{1/3}$ and again from 3 to 4 at $3^{1/3}$ in $(1, 3/2)$. Therefore, it is discontinuous at $x = 2^{1/3}$ and $x = 3^{1/3}$. Hence, it is not differentiable at $x = 2^{1/3}$ and $x = 3^{1/3}$.

Now,

$$f(x) = \{x + [x^3+1]\}^{x^2+\sin x}$$

$$= \begin{cases} (x+2)^{x^2+\sin x}, & 1 < x < 2^{1/3} \\ (x+3)^{x^2+\sin x}, & 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2+\sin x}, & 3^{1/3} < x < 1.5 \end{cases}$$

$$\therefore\ f'(x) = \begin{cases} (x+2)^{x^2+\sin x}\left\{(2x+\cos x)\log(x+2) + \dfrac{x^2+\sin x}{x+2}\right\}, & 1 < x < 2^{1/3} \\ (x+3)^{(x^2+\sin x)}\left\{(2x+\cos x)\log(x+3) + \dfrac{x^2+\sin x}{x+3}\right\}, & 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2+\sin x}\left\{(2x+\cos x)\log(x+4) + \dfrac{x^2+\sin x}{x+4}\right\}, & 3^{1/3} < x < 1.5 \end{cases}$$

EXAMPLE 40 *Given a function $g(x)$ which has derivative $g'(x)$ for every real x and which satisfies the following equation :*

$$g(x+y) = e^y g(x) + e^x g(y) \quad \text{for all } x \text{ and } y.$$

and $g'(0) = 2$. Show that $g'(x) = g(x) + C e^x$. Find C also.

SOLUTION We have,

$$g(x+y) = e^y g(x) + e^x g(y) \quad \text{for all } x \text{ and } y$$

Putting $x = y = 0$, we obtain

$$g(0) = e^0 g(0) + e^0 g(0)$$

$$\Rightarrow\ g(0) = 0$$

Now,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^h g(x) + e^x g(h) - g(x)}{h}$$

$$= \lim_{h\to 0} \left[g(x)\left(\frac{e^h-1}{h}\right) + e^x \cdot \frac{g(h)}{h}\right]$$

$$= g(x) \lim_{h\to 0} \frac{e^h-1}{h} + e^x \lim_{h\to 0} \frac{g(h)-g(0)}{h} \quad [\because g(0) =$$

$$= g(x) + e^x g'(0)$$

$$= g(x) + 2e^x$$

Thus, $g'(x) = g(x) + C e^x$, where $C = 2$.

EXAMPLE 41 *Let f be a differentiable function satisfying the condition :*

$$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)} \quad \text{for all } x, y\ (\neq 0) \in R \text{ and } f(y) \neq 0.$$

If $f'(1) = 2$, find $f(x)$.

SOLUTION We have,

$$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)} \quad \text{for all } x, y\ (\neq 0) \in R.$$

Putting $x = y = 1$, we get

$$f(1) = \frac{f(1)}{f(1)} = 1$$

Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{\dfrac{f(x+h)}{f(x)} - 1}{h}$$

$$= f(x) \lim_{h\to 0} \frac{f\left(\dfrac{x+h}{x}\right) - 1}{h} \quad \left[\because \frac{f(x)}{f(y)} = f\left(\frac{x}{y}\right.\right.$$

$$= \frac{f(x)}{x} \lim_{h\to 0} \frac{f\left(1+\dfrac{h}{x}\right) - f(1)}{\dfrac{h}{x}} \quad [\because f(1) =$$

$$= \frac{f(x)}{x} \cdot f'(1)$$

$$= \frac{2f(x)}{x} \quad [\because f'(1) =$$

Thus,

$$f'(x) = \frac{2f(x)}{x}$$

$$\Rightarrow\ \frac{f'(x)}{f(x)} = \frac{2}{x}.$$

Integrating both sides, we get

$$\log f(x) = 2\log x + \log C$$

$$\Rightarrow\ f(x) = Cx^2$$

$$\Rightarrow\ f(1) = C \times (1)^2$$

$$\Rightarrow\ 1 = C \quad [\because f(1) =$$

Hence, $f(x) = x^2$

EXAMPLE 42 *If $f(xy)=f(x)+f(y)$ for all $x, y>0$ and $f(x)$ is differentiable at $x=1$, then prove that*

(i) *$f(x)$ is differentiable for all $x>0$*

(ii) $\lim_{x\to 0}\frac{f(1+x)}{x}=f(e)$

(iii) $f\left(\frac{x}{y}\right)=f(x)-f(y)$ **[IIT 1996]**

SOLUTION We have,

$$f(xy)=f(x)+f(y) \text{ for all } x, y>0.$$

Putting $x=y=1$, we get

$$f(1)=f(1)+f(1)$$

$$\Rightarrow f(1)=0$$

Now,

$f(x)$ is differentiable at $x=1$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=f'(1) \text{ exists finitely.}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)}{h}=f'(1) \quad ...(i) \quad [\because f(1)=0]$$

(i) We have,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)+f\left(1+\frac{h}{x}\right)-f(x)}{h} \quad \text{[Using def. of } f\text{]}$$

$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}$$

$$=\frac{f'(1)}{x}$$

Hence, $f'(x)$ exists for all $x>0$.

Now,

$$f'(x)=\frac{f'(1)}{x}$$

$$f(x)=f'(1)\log_e x+C \quad \text{[On integration]}$$

Putting $x=1$, we get

$$f(1)=f'(1)\log_e 1+C \Rightarrow 0=C$$

$$\therefore f(x)=f'(1)\log_e x$$

(ii) $\lim_{x\to 0}\frac{f(1+x)}{x}$

$$=\lim_{x\to 0}\frac{f'(1)\log_e(1+x)}{x}$$

$$=f'(1)\lim_{x\to 0}\frac{\log_e(1+x)}{x}$$

$$=f'(1)$$

$$=f(e) \quad [\because f(e)=f'(1)\log_e e=f'(1)]$$

(iii) $f\left(\frac{x}{y}\right)$

$$=f'(1)\log_e\left(\frac{x}{y}\right)$$

$$=f'(1)\{\log_e x-\log_e y\}$$

$$=f'(1)\log_e x-f'(1)\log_e y$$

$$=f(x)-f(y).$$

EXAMPLE 43 *If $f(xy)=f(x)f(y)$ for all real x and y. If $f(1)=1$ and $f'(1)=2$, find $f'(10)$.*

SOLUTION We have,

$$f'(1)=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)-1}{h}=2 \quad ...(i)$$

Now,

$$f'(x)$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-1}{h}$$

$$=\frac{f(x)}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-1}{\frac{h}{x}}$$

$$=\frac{2f(x)}{x} \quad \text{[Using (i)]}$$

Hence, $f(x)$ is differentiable for $x\neq 0$ and $f'(x)$ is given by

$$f'(x) = \frac{2f(x)}{x}$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{2}{x}$$

$\Rightarrow \log f(x) = 2\log x + \log C$ [On integration]

$\Rightarrow f(x) = Cx^2$

Putting $x = 1$, we get

$f(1) = C \Rightarrow C = 1$ $[\because f(1) = 1]$

$\therefore f(x) = x^2$, for all x

$\Rightarrow f'(x) = 2x$, for all x

$\Rightarrow f'(10) = 20$

EXAMPLE 44 *Let $f(x^m y^n) = m f(x) + n f(y)$ for all $x, y \in R^+$ and for all $m, n \in R$. If $f'(x)$ exists and has the value $\frac{e}{x}$, then find* $\lim_{x\to 0} \frac{f(1+x)}{x}$.

SOLUTION For any $x \in R^+$, we have

$$f(x^m y^n) = m f(x) + n f(y) \quad \ldots(i)$$

$\therefore f(1) = f(1) + f(1)$ [Putting $x = y = m = n = 1$]

$\Rightarrow f(1) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left[(x^{1/m})^m \left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}^n\right] - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{m f(x^{1/m}) + n f\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{m f(x^{1/m}) + n f\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\} - f\{(x^{1/m})^m\}}{h}$$

Putting $y = 1$ in (i), we get

$f(x^m) = m f(x)$

$\Rightarrow f\{(x^{1/m})^m\} = m f(x^{1/m})$

$$\therefore f(x) = \lim_{h\to 0} \frac{m f(x^{1/m}) + n f\left(1+\frac{h}{x}\right)^{1/n} - m f(x^{1/m})}{h}$$

$$= \lim_{h\to 0} \frac{n f\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{h} \quad [\because m f(x) = f(x^m)]$$

$$= \frac{1}{x} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

But, $f'(x) = \frac{e}{x}$.

$$\therefore \frac{e}{x} = \frac{1}{x} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$\Rightarrow \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = e$$

$$\Rightarrow \lim_{\theta\to 0} \frac{f(1+\theta)}{\theta} = e, \text{ where } \theta = \frac{h}{x}.$$

EXAMPLE 45 *Let $f(xy) = x f(y) + y f(x)$ for all $x, y \in R^+$. If $f(e) = e$ and $f(x)$ is differentiable for all $x > 0$, prove that* $\lim_{x\to 0} \frac{f(1+x)}{x} = l$

SOLUTION We have,

$f(xy) = x f(y) + y f(x)$ for all $x, y > 0$.

$\Rightarrow f(e) = e f(1) + f(e)$ [Putting $x = e, y = 1$]

$\Rightarrow e = e f(1) + e$ $[\because f(e) = e]$

$\Rightarrow f(1) = 0$

Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x f\left(1+\frac{h}{x}\right) + \left(1+\frac{h}{x}\right) f(x) - f(x)}{h} \quad \left[\begin{matrix}\text{Using : } f(xy) \\ = x f(y) + y f(x)\end{matrix}\right]$$

$$f'(x) = x \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{h} + \frac{f(x)}{x}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} + \frac{f(x)}{x}$$

$$\Rightarrow f'(x) = l + \frac{f(x)}{x}, \text{ where } l = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$\Rightarrow \frac{df(x)}{dx} - \frac{f(x)}{x} = l$$

$$\Rightarrow \frac{x\,df(x) - f(x)\,dx}{x} = l\,dx$$

$$\Rightarrow \frac{x\,df(x) - f(x)\,dx}{x^2} = \frac{l}{x}dx$$

$$\Rightarrow d\left\{\frac{f(x)}{x}\right\} = l\,d(\log x)$$

$$\Rightarrow \frac{f(x)}{x} = l\log x + C \qquad \text{[Integrating both sides w.r.to } x\text{]}$$

Putting $x = 1$, we get

$f(1) = l \cdot \log 1 + C$

$\Rightarrow 0 = C$

$\therefore f(x) = l \cdot x \log x$

$\Rightarrow f(e) = l \cdot e \log e$

$\Rightarrow e = l \cdot e \qquad [\because f(e) = e]$

$\Rightarrow l = 1$

$$\Rightarrow \lim_{x\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = 1 \qquad \left[\because l = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right]$$

$$\Rightarrow \lim_{z\to 0} \frac{f(1+z)}{z} = 1, \text{ where } z = \frac{h}{x}.$$

EXAMPLE 46 *If the functional relationship* $f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x}$ *holds for all real* x *and* y *greater than 0 and* $f(x)$ *is a differentiable function for all* $x > 0$ *such that* $f(e) = \frac{1}{e}$, *then find the maximum value of* $f(x)$.

SOLUTION We have,

$$f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x} \qquad \text{for all } x, y > 0.$$

$\Rightarrow f(1) = f(1) + f(1) \qquad$ [Putting $x = y = 1$]

$\Rightarrow f(1) = 0$

Now, $f(x)$ is differentiable for all $x > 0$. Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x)}{1+\frac{h}{x}} + \frac{f\left(1+\frac{h}{x}\right)}{x} - f(x)}{h} \qquad \left[\text{Using : } f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x}\right]$$

$$= \lim_{h\to 0} \frac{\frac{-\frac{h}{x}f(x)}{1+\frac{h}{x}} + \frac{f\left(1+\frac{h}{x}\right)}{x}}{h}$$

$$= -\frac{f(x)}{x} + \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{hx}$$

$$= -\frac{f(x)}{x} + \frac{1}{x^2} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$f'(x) = \frac{-f(x)}{x} + \frac{A}{x^2}, \text{ where } A = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$\Rightarrow \frac{d}{dx}f(x) = -\frac{f(x)}{x} + \frac{A}{x^2}$$

$$\Rightarrow \frac{d}{dx}f(x) + \frac{f(x)}{x} = \frac{A}{x^2}$$

$$\Rightarrow x\frac{d}{dx}f(x) + f(x) = \frac{A}{x}$$

$$\Rightarrow \frac{d}{dx}\{x f(x)\} = \frac{A}{x}$$

$\Rightarrow x f(x) = A \log_e x + \log C \qquad$ [On integration]

Putting $x = 1$, we get

$f(1) = A \log_e 1 + \log C$

$\Rightarrow 0 = \log C \qquad [\because f(1) = 0]$

$\therefore x f(x) = A \log_e x$

Putting $x = e$, we get

$e f(e) = A \log_e e$

$$\Rightarrow A = 1 \qquad \left[\because f(e) = \frac{1}{e}\right]$$

$$\therefore x f(x) = \log_e x$$

$$\Rightarrow f(x) = \frac{\log_e x}{x}$$

$$\Rightarrow f'(x) = \frac{1-\log_e x}{x^2}$$

For maximum or minimum, we have

$f'(x) = 0 \Rightarrow 1 - \log_e x = 0 \Rightarrow \log_e x = 1 \Rightarrow x = e$

Clearly, $f''(e) < 0$

Hence, $f(x)$ is maximum for $x = e$. The maximum value of $f(x)$ is given by $f(e) = \frac{\log_e e}{e} = \frac{1}{e}$.

EXAMPLE 47 *Let $\alpha \in R$. Prove that a function $f: R \to R$ is differentiable at α if and only if there exists a function $g: R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x-\alpha)$ for all $x \in R$.* [IIT 2001]

SOLUTION Let $f: R \to R$ be a function such that there is a function $g: R \to R$ which is continuous at $x = \alpha$ and satisfies the relation

$$f(x) - f(\alpha) = g(x)(x-\alpha) \text{ for all } x \in R.$$

Then we have to prove that $f(x)$ is differentiable at $x = \alpha$.

We have,

$$f(x) - f(\alpha) = g(x)(x-\alpha)$$

$$\Rightarrow \frac{f(x)-f(\alpha)}{x-\alpha} = g(x)$$

$$\Rightarrow \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha} = \lim_{x\to\alpha} g(x)$$

$$\Rightarrow \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha} = g(\alpha) \quad [\because g(x) \text{ is continuous at } x = \alpha]$$

$\Rightarrow f(x)$ is differentiable at $x = \alpha$ and $f'(\alpha) = g(\alpha)$.

Conversely, let $f: R \to R$ be differentiable at $x = \alpha$. Then, we have to prove that there is a function $g(x)$ continuous at $x = \alpha$ such that

$$f(x) - f(\alpha) = g(x)(x-\alpha) \quad \text{for all } x \in R.$$

Now, $f(x)$ is differentiable at $x = \alpha$.

$$\Rightarrow \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha} \text{ exists finitely}$$

Let $\lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha)$

$$\Rightarrow \frac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha) + \phi(x), \text{ where } \phi(x) \to 0 \text{ as } x \to \alpha. \quad \ldots(i)$$

Since $f(x)$ is differentiable at $x = \alpha$. So, $f(x)$ is continuous at $x = \alpha$.

Consequently, $\phi(x)$ is continuous at $x = \alpha$ and $\lim_{x\to\alpha} \phi(x) = f'(\alpha) - f'(\alpha) = 0$.

$$\Rightarrow \phi(\alpha) = 0$$

From (i), we have

$$f(x) - f(\alpha) = (x-\alpha)\{f'(\alpha) + \phi(x)\}$$

$$= (x-\alpha)\, g(x), \text{ where } g(x) = f'(\alpha) + \phi(x).$$

Clearly, $g(x)$ is continuous at $x = \alpha$.

EXAMPLE 48 *Let $F(x) = f(x)\, g(x)\, h(x)$ for all $x \in R$, where $f(x), g(x), h(x)$ are differentiable functions. At some point x_0 if $F'(x_0) = 21 F(x_0), f'(x_0) = 4 f(x_0), \quad g'(x_0) = -7 g(x_0)$ and $h'(x_0) = k\, h(x_0)$, find the value of k.* [IIT 1997]

SOLUTION We have,

$$F(x) = f(x)\, g(x)\, h(x)$$

$$\Rightarrow F'(x) = f'(x)\, g(x)\, h(x) + f(x)\, g'(x)\, h(x) + f(x)\, g(x)\, h'(x)$$

$$\Rightarrow F'(x_0) = f'(x_0)\, g(x_0)\, h(x_0) + f(x_0)\, g'(x_0)\, h(x_0) + f(x_0)\, g(x_0)\, h'(x_0)$$

$$\Rightarrow 21 F(x_0) = 4 f(x_0)\, g(x_0)\, h(x_0) - 7 f(x_0)\, g(x_0)\, h(x_0) + k f(x_0)\, g(x_0)\, h(x_0)$$

$$\Rightarrow 21 F(x_0) = (k-3) F(x_0)$$

$$\Rightarrow k = 24$$

EXAMPLE 49 *Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that*

$$|a_1 + 2a_2 + \ldots + n a_n| \le 1$$ [IIT 2000]

SOLUTION We have,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$\Rightarrow p'(x) = a_1 + 2a_2 x + 3 a_3 x^2 + \ldots + n a_n x^{n-1}$$

$$\Rightarrow p'(1) = a_1 + 2a_2 + 3 a_3 + \ldots + n a_n$$

Thus, we have to prove that

$$|p'(1)| \le 1$$

i.e., $\left| \lim_{h\to 0} \frac{p(1+h)-p(1)}{h} \right| \le 1$

Now,

$$|p(x)| \le |e^{x-1} - 1| \quad \text{for all } x \ge 0$$

$$\Rightarrow |p(1)| \le |e^0 - 1|$$

$$\Rightarrow |p(1)| \le 0$$

$$\Rightarrow p(1) = 0$$

Again,

$$|p(x)| \le |e^{x-1} - 1| \quad \text{for all } x \ge 0$$

$$\Rightarrow |p(1+h)| \le |e^h - 1| \quad \text{for all } h \ge -1$$

$$\Rightarrow |p(1+h) - p(1)| \le |e^h - 1| \quad \text{for all } h \ge -1 \ [\because p(1) = 0]$$

$$\Rightarrow \quad \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{e^h-1}{h}\right| \text{ for all } h \ge -1, h \ne 0$$

$$\Rightarrow \quad \left|\lim_{h\to 0}\frac{p(1+h)-p(1)}{h}\right| \le \left|\lim_{h\to 0}\frac{e^h-1}{h}\right|$$

$$\Rightarrow \quad |p'(1)| \le 1 \qquad \left[\because \lim_{h\to 0}\frac{e^h-1}{h} = 1\right]$$

$$\Rightarrow \quad |a_1 + 2a_2 + 3a_3 + ... + n\,a_n| \le 1$$

EXAMPLE 50 *Let*

$$f(x) = \begin{cases} x+a & , \text{ if } x<0 \\ |x-1| & , \text{ if } x \ge 0 \end{cases}$$

and,

$$g(x) = \begin{cases} x+1 & \text{if } , x<0 \\ (x-1)^2+b & \text{if } , x \ge 0 \end{cases}$$

where a and *b non-negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the value of a* and *b. Further, for these values of a and b, is gof differentiable at x = 0 ? Justify your answer.* **[IIT 2002]**

SOLUTION We have,

$$f(x) = \begin{cases} x+a & , \text{ if } x<0 \\ |x-1| & , \text{ if } x \ge 0 \end{cases}$$

and,

$$g(x) = \begin{cases} x+1 & , \text{ if } x<0 \\ (x-1)^2+b & , \text{ if } x \ge 0 \end{cases}$$

$$\therefore \quad gof(x) = g\{f(x)\}$$

$$= \begin{cases} g(x+a) & , x<0 \\ g(|x-1|) & , x \ge 0 \end{cases}$$

$$= \begin{cases} x+a+1 & , x+a<0 \\ (x+a-1)^2+b & , x+a \ge 0 \\ \{|x-1-1|\}^2+b & , x \ge 0 \end{cases}$$

$$= \begin{cases} x+a+1 & , x<-a \\ (x+a-1)^2+b & , 0>x \ge -a \\ x^2+b & , 0 \le x<1 \\ (x-2)^2+b & , x \ge 1 \end{cases}$$

It is given that $gof(x)$ is continuous for all real x. Therefore, it is continuous at $x = -a, 0, 1$.

Now,

gof is continuous at $x = -a$.

$$\Rightarrow \lim_{x\to -a^-} gof(x) = \lim_{x\to -a^+} gof(x) = gof(-a)$$

$$\Rightarrow \lim_{x\to -a^-} x+a+1 = \lim_{x\to -a^+} (x+a-1)^2+b = (-a+a-1)^2+b$$

$$\Rightarrow \quad -a+a+1 = (-a+a-1)^2+b$$

$$\Rightarrow \quad 1 = 1+b$$

$$\Rightarrow \quad b = 0$$

And,

$gof(x)$ is contnuous at $x = 0$

$$\Rightarrow \lim_{x\to 0^-} gof(x) = \lim_{x\to 0^+} gof(x) = gof(0)$$

$$\Rightarrow \lim_{x\to 0^-} (x+a-1)^2+b = \lim_{x\to 0^+} x^2+b = 0+b$$

$$\Rightarrow \quad (a-1)^2+b = b$$

$$\Rightarrow \quad (a-1)^2 = 0$$

$$\Rightarrow \quad a = 1$$

Now,

(LHD at $x = 0$)

$$= \left\{\frac{d}{dx}\{(x+a-1)^2+b\}\right\}_{\text{at } x=0}$$

$$= \{2(x+a-1)\}_{\text{at } x=0}$$

$$= 2(a-1)$$

$$= 0 \qquad [\because a = 1]$$

and,

(RHD at $x = 0$)

$$= \left\{\frac{d}{dx}(x^2+b)\right\}_{\text{at } x=0}$$

$$= \{2x\}_{\text{at } x=0} = 2\times 0 = 0$$

$$\therefore \quad (\text{LHD at } x = 0) = (\text{RHD at } x = 0)$$

So, $gof(x)$ is differentiable at $x = 0$.

EXAMPLE 51 *If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at $x = a$ is 0, then find the left hand derivative at $x = -a$.*

[IIT 2003]

SOLUTION It is given that the left hand derivative at $x = a$ is 0.

Therefore,

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a-h)-f(a)}{h} = 0 \qquad ...(i)$$

Now,

(LHD at $x = -a$)

$$= \lim_{h\to 0}\frac{f(-a-h)-f(-a)}{-h}$$

$$= \lim_{h\to 0}\frac{-f(a+h)+f(a)}{-h} \qquad [\because f(x) \text{ is odd function}]$$

$= \lim_{h\to 0} \frac{f(a-h)+f(a)}{h}$

$= \lim_{h\to 0} \frac{f(2a-(a+h))-f(a)}{h}$ [$\because f(2a-x)=f(x)$ for $x \in [a, 2a]$]

$= \lim_{h\to 0} \frac{f(a-h)-f(a)}{h}$

$= 0$ [Using (i)]

EXERCISE 4.2

1. A function $f(x)$ is defined by

$$f(x) = \begin{cases} -x & \text{for } x<0 \\ x^2 & \text{for } 0\le x\le 1 \\ x^3-x^2+x & \text{for } x>1 \end{cases}$$

Prove that $f(x)$ is not differentiable at $x=0$ but differentiable at $x=1$.

2. A function $f(x)$ is defined by

$$f(x) = \begin{cases} -x^2 & \text{for } x\le 0 \\ 5x-4 & \text{for } 0<x\le 1 \\ 4x^2-3x & \text{for } 1<x\le 2 \\ 3x+5 & \text{for } x\ge 2 \end{cases}$$

Discuss the continuity and differentiability of $f(x)$.

3. Let $f(x)$ be a function and c be a real number such that

$f'(c)=2$ and $f(c)=4$. Find $\lim_{x\to C} \frac{xf(c)-cf(x)}{x-c}$

4. Let $f(x) = \begin{cases} -1, & -2\le x\le 0 \\ x-1, & 0<x\le 2 \end{cases}$ and,

$g(x) = f(|x|)+|f(x)|$

Test $g(x)$ for continuity and differentiability in $[-2, 2]$.

5. Let $f(x) = \lim_{n\to\infty} \frac{\log(x+2)-x^{2n}\sin x}{1+x^{2n}}$. Examine the differentiability of $f(x)$ at $x=1$.

6. Let $f(x) = \begin{cases} x+2, & 0\le x<2 \\ 6-x, & x\ge 2 \end{cases}$, $g(x) = \begin{cases} 1+\tan x, & 0\le x<\pi/4 \\ 3-\cot x, & \pi/4\le x<\pi \end{cases}$

Find fog and discuss its continuity and differentiability.

7. $f(x) = \begin{cases} \frac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x\ne 0 \\ 0, & x=0 \end{cases}$

Discuss the continuity and differentiability of $f(x)$ at $x=0$.

8. Let $f(x) = \begin{cases} \frac{x^2}{2}, & 0\le x\le 1 \\ 2x^2-3x+\frac{3}{2}, & 1<x\le 2 \end{cases}$

Discuss the continuity of f, f' and f'' on $(0, 2)$.

9. Let $f(x) = \begin{cases} x^4+x^2-x+1, & x\le 1 \\ 2x^3-x^2+x, & x>1 \end{cases}$

Show that $f(x)$ is continuous and possess a continuous first order derivative at $x=1$ but the second order derivative does not exist at this point.

10. Draw the graph of the function $f(x)$ given by

$$f(x) = [x]+|1-x|, 1\le x\le 3$$

and determine the points, if any, where this function is not differentiable.

11. If $f(x) = \begin{cases} |x-1|\{[x]-x\}, & x\ne 1 \\ 0, & x=1 \end{cases}$

Test the differentiability of $f(x)$ at $x=1$.

12. Let $f(x) = \begin{cases} ax^2-bx+2, & x<3 \\ bx^2-3, & x\ge 3 \end{cases}$

Find the values of a and b so that $f(x)$ is differentiable everywhere.

13. If $f(x) = \begin{cases} \frac{5e^{1/x}+2}{3-e^{1/x}}, & x\ne 0 \\ 0, & x=0 \end{cases}$

Examine $f(x)$, $xf(x)$ and $x^2f(x)$ for continuity and differentiability at $x=0$.

14. Draw the graph and discuss the continuity and differentiability of the following functions:

(i) $f(x) = [|x|\sin \pi x]$ in $(-1,$

(ii) $f(x) = |x|x|-1|$

15. If $f(x) = x^3-3x$

and,

$$g(x) = \begin{cases} \min\{f(t): 0\le t\le x\} \\ (2x-5) \\ (x-2)^2 \end{cases}$$

Draw the graph of $g(x)$ and differentiability in $(0, 4)$.

16. If $f(x) = \begin{cases} x, & x\le 1 \\ -x^2+2x, & x>1 \end{cases}$ then dis

f and f'.

17. If $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all $x, y \in$

$f'(0)=1$, show that

$$|f(0)|^2 + |f(1)|^2 + |f(2)|^2 + \ldots + |f(n)|^2 = 1 + \frac{n(n+1)(2n-1)}{6}$$

18. Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in R$. If $f(x)$ is differentiable and $f'(0) = \sin\phi$, then prove that $f(x) > 0$ for all $x \in R$.

19. If $f(x) = \begin{cases} |x| & , x \le 1 \\ 2 - x & , x > 1 \end{cases}$

then discuss the continuity and differentiability of the function $fof(x)$. Also give a rough sketch of the function.

20. If $f(x) = \sqrt{|x-1|}$ and $g(x) = \sin x$, then discuss the continuity and differentiability of gof at $x = 1$.

21. If $f(x) = \begin{cases} x[x] & , 0 \le x < 3 \\ (x-1)[x] & , 3 \le x \le 4 \end{cases}$

then examine the continuity and differentiability of $f(x)$ at $x = 3$.

22. Let $g(x)$ be a polynomial of degree one and $f(x)$ is defined by

$$f(x) = \begin{cases} g(x) & , x \le 0 \\ \left(\frac{x+1}{x+2}\right)^{1/x} & , x > 0 \end{cases}$$

Find $g(x)$ such that $f(x)$ is continuous and $f'(1) = f(-1)$.

23. Examine the continuous function

$$f(x) = \begin{cases} x^2 + 1 & , x \le 1 \\ x^3 + ax^2 + bx & , |x| > 1 \end{cases}$$ for differentiability.

24. If $f(x) = \begin{cases} 2x^2 \sin \pi x & , x \le 1 \\ x^3 + ax^2 + b & , x > 1 \end{cases}$ be a differentiable function. Examine whether it is twice differentiable in R.

25. If $f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3}$ for all real x and y and $f'(2) = 2$, then prove that $f(x)$ is differentiable for all $x \in R$.

26. Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$.
If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that

$$|a_1 + 2a_2 + \ldots + n a_n| \le 1.$$

27. If $f(x) = \min\{|x+1|, |x|, |x-1|\}$, discuss the continuity and differentiability of $f(x)$.

28. Draw the graph of $f(x) = \begin{cases} x+1 & , -2 \le x < 0 \\ 0 & , x = 0 \\ x-1 & , 0 < x \le 2 \end{cases}$

and discuss the continuity and differentiability of $fof(x)$.

29. Let $f(x)$ be defined in the interval $[-2, 2]$ such that

$$f(x) = \begin{cases} [x] & , -2 \le x \le -\frac{1}{2} \\ 2x^2 - 1 & , -\frac{1}{2} < x \le 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$. Discuss the differentiability of $g(x)$ in $[-2, 2]$.

30. Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all real x and y and $f(x)$ be a differentiable function. If $f'(0) = \cos\alpha$, then prove that $f(x) > 0$ for all $x \in R$.

ANSWERS

'ifferentiable at $x = 0, 2$

$x = 1$

'e at $x = 1, 0$

14. (i) Discontinuous and non-differentiable at $x = 0$.
(ii) Everywhere continuous but non-differentiable at $x = 1$

15. Discontinuous and non-differentiable at $x = 2$ only.

16. f is everywhere continuous
f' is also continuous at all x except $x = 1$

17. f and f' are everywhere continuous but f' is discontinuous at $x = 1$.

19. Everywhere continuous but not differentiable at $x = -1, 0, 1, 2$.

20. Not differentiale

21. Not differentiable.

$\varphi(x) = \frac{1}{9}\left(6\log\frac{2}{3} - 1\right)x$

ot differentiable at $x = 1$ only

differentiable in R except at $x = 1$

HINTS TO SELECTED PROBLEMS

5. $f(x)$ is not continuous at $x = 1$. So, it is not differentiable thereat.

6. We have,

$$fog(x) = f(g(x)) = \begin{cases} f(1+\tan x), 0 \le x < \pi/4 \\ f(3-\cot x), \pi/4 \le x < \pi \end{cases}$$

$$= \begin{cases} 1+\tan x+2 & , 0 \le x < \pi/4 \\ 6-(3-\cot x), & \pi/4 \le x < \pi \end{cases}$$

10. We have,

$$f(x) = \begin{cases} -1+1-x = -x, & -1 \le x < 0 \\ 1-x & , 0 \le x < 1 \\ x & , 1 \le x < 2 \\ 1+x & , 2 \le x < 3 \\ 5 & , x = 3 \end{cases}$$

11. We have,

$$f(x) = \begin{cases} (x-1)x & , 0 \le x < 1 \\ 0 & , x = 1 \\ -(x-1)^2, & 1 < x < 2 \end{cases}$$

14. (i) For $x \in (0, 1)$, we have

$0 < |x| \sin \pi x < 1$

$\Rightarrow [|x| \sin \pi x] = 0$

For $x \in (-1, 0)$, we have

$-1 < |x| \sin \pi x < 0$

$\Rightarrow [|x| \sin \pi x] = -1$

At $x = 0, [|x| \sin \pi x] = 0$

Thus, $f(x) = \begin{cases} 0, 0 \le x \le 1 \\ -1, -1 < x < 0 \end{cases}$

(ii) $f(x) = \begin{cases} x^2+1 & , x \le 0 \\ 1-x^2 & , 0 < x \le 1 \\ x^2-1 & , 1 < x \end{cases}$

15. We have,

$$g(x) = \begin{cases} x^3-3x & , 0 \le x \le 1 \\ -2 & , 1 < x \le 2 \\ (2x-5)^2, & 2 < x < 3 \\ (x-2)^2 & , x \ge 3 \end{cases}$$

17. From Example 30, we have $f(x) = -x + 1$ for all $x \in R$.

$$\therefore \sum_{r=0}^{n} \{f(r)\}^2 = 1 + 1^2 + 2^2 + \ldots + (n-1)^2$$

$$= 1 + \frac{n(n-1)(2n-1)}{6}$$

—❑❑❑—

CHAPTER

5

DIFFERENTIATION

5.1 DIFFERENTIATION

Let $f(x)$ be a differentiable or derivable function on $[a, b]$. Then corresponding to each point $c \in [a, b]$ we obtain a unique real number equal to the derivative $f'(c)$ of $f(x)$ at $x = c$. This correspondence between the points in $[a, b]$ and derivatives at these points defines a new real valued function with domain $[a, b]$ and range a subset of R, set of real numbers, such that the image of x in $[a, b]$ is the value of the derivative of f at x *i.e.*, $f'(x)$ or $Df(x)$. This function is called the derivative or differentiation of $f(x)$ with respect to x or simply differentiation of $f(x)$ and is denoted by $f'(x)$ or $Df(x)$ or $\frac{d}{dx}(f(x))$.

Thus, $$\frac{d}{dx}\left(f(x)\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad ...(i)$$

or $$\frac{d}{dx}\left(f(x)\right) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \quad ...(ii)$$

The differentiation or derivative of a function $f(x)$ is also called the differential coefficient of $f(x)$. But we shall be using the words differentiation or derivative only.

5.2 GEOMETRICAL MEANING OF DERIVATIVE AT A POINT

Consider the curve $y = f(x)$. Let $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and let $Q(x, f(x))$ be a neighbouring point on the curve. Then,

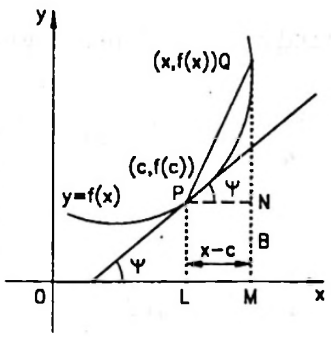


Fig. 5.1

Slope of the chord $PQ = \frac{f(x) - f(c)}{x - c}$.

Taking limit as $Q \to P$ *i.e.*, $x \to c$, we get

$$\lim_{Q \to P} (\text{Slope of the chord } PQ) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad ...(i)$$

As $Q \to P$, chord PQ becomes tangent at P. Therefore, from (i), we have

$$\text{Slope of the tangent at } P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \left(\frac{df(x)}{dx}\right)_{x=c} \text{ or } f'(c)$$

Thus, the derivative of a function at a point $x = c$ gives the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$. Also, if a function is not differentiable at $x = c$ only if the point $(c, f(c))$ is a corner point of the curve $y = f(x)$ *i.e.* the curve suddenly changes its direction at point $(c, f(c))$.

5.3 DIFFERENTIATION OF SOME STANDARD FUNCTIONS

The following is the list of derivatives of some standard functions :

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log_e a$. (iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ (vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$ (viii) $\frac{d}{dx}(\tan x) = sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$ (x) $\frac{d}{dx}(sec\ x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(xii) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}, -1<x<1$

(xiii) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}, -1<x<1$

(xiv) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}, -\infty<x<\infty$

(xv) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}, |x|>1$

(xvii) $\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}, |x|>1$

5.4 DIFFERENTIATION OF A FUNCTION

FUNDAMENTAL RULES FOR DIFFERENTIATION This section is mainly devoted to develop several rules that allow us to find derivatives without using definition directly.

RULE (I) *Differentiation of a constant function is zero i.e.,*

$$\frac{d}{dx}(c)=0.$$

PROOF Let $f(x)=c$, be a constant function. Then,

$$\frac{d}{dx}\left(f(x)\right)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{c-c}{h}=0.$$

Hence, $\frac{d}{dx}(c)=0$, where c is a constant.

Geometrically, graph of a constant function is a straight line parallel to x-axis. So, tangent at every point is parallel to x-axis. Consequently, slope of the tangent is zero, *i.e.*, $\frac{dy}{dx}=0$.

RULE (II) *Let $f(x)$ be a differentiable function and let c be a constant. Then $c\cdot f(x)$ is also differentiable such that*

$$\frac{d}{dx}\{c\cdot f(x)\}=c\cdot\frac{d}{dx}(f(x))$$

That is the derivative of a constant times a function is the constant times the derivative of the function.

PROOF Since $f(x)$ is differentiable. Therefore,

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists finitely and is equal to $\frac{d}{dx}\left(f(x)\right)$

i.e. $\frac{d}{dx}\left(f(x)\right)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ...(i)

Let $g(x)=c\cdot f(x)$. Then

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{c\cdot f(x+h)-c\cdot f(x)}{h}$$

$$=c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=c\cdot\frac{d}{dx}(f(x)) \quad \text{[From (i)}$$

So, $g(x)=cf(x)$ is differentiable such that

$$\frac{d}{dx}\left(c\cdot f(x)\right)=c\cdot\frac{d}{dx}\left(f(x)\right).$$

RULE (III) *If $f(x)$ and $g(x)$ are differentiable functions, then show that $f(x)\pm g(x)$ are also differentiable such that*

$$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x).$$

That is the derivative of the sum or difference of two functions is the sum or difference of their derivatives.

PROOF Since $f(x)$ and $g(x)$ both are differentiable functions Therefore,

$$\left.\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{d}{dx}(f(x))\\ &\text{and,}\\ &\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\frac{d}{dx}(g(x))\end{aligned}\right\} \quad ...(i$$

both exist finitely

Now,

$$\lim_{h\to 0}\frac{[f(x+h)\pm g(x+h)]-[f(x)\pm g(x)]}{h}$$

$$=\lim_{h\to 0}\frac{[f(x+h)-f(x)]\pm[g(x+h)-g(x)]}{h}$$

$$=\lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}\pm\lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\right\}$$

$$=\frac{d}{dx}\left(f(x)\right)\pm\frac{d}{dx}\left(g(x)\right) \quad \text{[From (i)}$$

So, $f(x)\pm g(x)$ is differentiable such that

$$\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}\left(f(x)\right)\pm\frac{d}{dx}\left(g(x)\right).$$

RULE (IV) (PRODUCT RULE). *If $f(x)$ and $g(x)$ are two differentiable functions, show that $f(x)\cdot g(x)$ is also differentiable such that*

$$\frac{d}{dx}[f(x)g(x)]=f(x)\frac{d}{dx}[g(x)]+\frac{d}{dx}[f(x)]\cdot g(x).$$

PROOF Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\left.\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{d}{dx}(f(x))\\ &\text{and,}\\ &\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\frac{d}{dx}(g(x))\end{aligned}\right\} \quad ...($$

both exist finitely

Let $\phi(x) = f(x)\,g(x)$. Then,

$$\frac{d}{dx}\left(\phi(x)\right)$$

$$= \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x+h)\,g(x)+f(x+h)\,g(x)-f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,\{g(x+h)-g(x)\}+g(x)\,\{f(x+h)-f(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,\{g(x+h)-g(x)\}}{h}+\lim_{h\to 0}\frac{g(x)\{f(x+h)-f(x)\}}{h}$$

$$= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\frac{d}{dx}\{g(x)\}+g(x)\frac{d}{dx}\{f(x)\}$$

Thus, $\phi'(x) = f(x)\,g(x)$ is differentiable such that

$$\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}.$$

RULE (V) (QUOTIENT RULE) *If $f(x)$ and $g(x)$ are two differentiable functions and $g(x)\neq 0$, then show that $\frac{f(x)}{g(x)}$ is also differentiable such that*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\frac{d}{dx}\{f(x)\}-f(x)\cdot\frac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

5.5 RELATION BETWEEN $\frac{dy}{dx}$ AND $\frac{dx}{dy}$

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$. Let Δx be a small change in x and let Δy be the corresponding change in y. Then

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \text{ and } \frac{dx}{dy} = \lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}.$$

Now,

$$\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y} = 1$$

$$\Rightarrow \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y}\right) = 1$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\cdot\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y} = 1 \qquad [\because \Delta x\to 0 \Leftrightarrow \Delta y\to 0]$$

$$\Rightarrow \frac{dy}{dx}\cdot\frac{dx}{dy} = 1.$$

Hence,

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

5.6 MORE ON DIFFERENTIATION

DIFFERENTIATION OF IMPLICIT FUNCTIONS

If the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function of x in the form $y = \phi(x)$, then y is said to be an *implicit* function of x. To find $\frac{dy}{dx}$ in such a case, we differentiate both sides of the given relation with respect to x, keeping in mind that the derivative of $\phi(y)$ w.r.t. x is $\frac{d\phi}{dy}\cdot\frac{dy}{dx}$. For example,

$$\frac{d}{dx}(\sin y) = \cos y\cdot\frac{dy}{dx}, \frac{d}{dx}(y^2) = 2y\frac{dy}{dx}.$$

LOGARITHMIC DIFFERENTIATION

In this section, we will be mainly discussing derivatives of the functions of the form $[f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x. To find the derivative of this type of functions we proceed as follows :

Let $y = [f(x)]^{g(x)}$.

Taking logarithm of both the sides, we have

$$\log y = g(x)\cdot\log\{f(x)\}$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\frac{df(x)}{dx}+\log\{f(x)\}\cdot\frac{dg(x)}{dx}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}\cdot\frac{df(x)}{dx}+\log\{f(x)\}\cdot\frac{dg(x)}{dx}\right].$$

ALITER We have,

$$y = [f(x)]^{g(x)}$$

$$= e^{g(x)\cdot\log\{f(x)\}}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{g(x)\log\{f(x)\}}\left[g(x)\cdot\frac{1}{f(x)}\frac{df(x)}{dx}+\log\{f(x)\}\cdot\frac{dg(x)}{dx}\right]$$

$$= [f(x)]^{g(x)}\left[\frac{g(x)}{f(x)}\cdot\frac{df(x)}{dx}+\log\{f(x)\}\cdot\frac{dg(x)}{dx}\right]$$

DIFFERENTIATION OF PARAMETRIC FUNCTIONS

Sometimes x and y are given as functions of a single variable, e.g., $x = \phi(t)$, $y = \psi(t)$ are two functions and t is a variable. In such a case x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain the relationship between x and y by eliminating the parameter t and then we differentiate it with respect to x. But every time it is not convenient to eliminate the parameter. Therefore, $\frac{dy}{dx}$ can also be obtained by the following formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

So far we have discussed derivative of one variable, say, y with respect to other variable, say, x. In this section we will discuss derivative of a function with respect to another function. Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then to find the derivative of $f(x)$ w.r.t. $g(x)$ i.e., to find $\frac{du}{dv}$ we use the following formula

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}.$$

Thus, to find the derivative of $f(x)$ w.r.t. $g(x)$, we first differentiate both w.r.t. x and then divide the derivative of $f(x)$ w.r.t. x by the derivative of $g(x)$ w.r.t. x.

HIGHER ORDER DERIVATIVES

If $y = f(x)$, then $\frac{dy}{dx}$, the derivative of y with respect to x, is itself, in general, a function of x and can be differentiated again. To fix up the idea, we shall call $\frac{dy}{dx}$ as the first order derivative of y with respect to x and the derivative of $\frac{dy}{dx}$ w.r.t. x as the second order derivative of y w.r.t x and will be denoted by $\frac{d^2y}{dx^2}$. Similarly the derivative of $\frac{d^2y}{dx^2}$ w.r.t. x will be termed as the third order derivative of y w.r.t. x and will be denoted by $\frac{d^3y}{dx^3}$ and so on. The nth order derivative of y w.r.t. x will be denoted by $\frac{d^ny}{dx^n}$.

If $y = f(x)$, then the other alternative notations for

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ are

$y_1, y_2, y_3, \ldots, y_n$

$y', y'', y''', \ldots, y^{(n)}$

$Dy, D^2y, D^3y, \ldots, D^ny$

$f'(x), f''(x), f'''(x), \ldots, f^n(x)$

The values of these derivatives at $x = a$ are denoted by

$$y_n(a),\ y^n(a),\ D^n y(a),\ f^n(a) \text{ or } \left(\frac{d^ny}{dx^n}\right)_{x=a}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the derivative of x^x $(x > 0)$ with respect to x, from first principles.*

SOLUTION Let $f(x) = x^x$. Then, $f(x+h) = (x+h)^{x+h}$

Now,

$$\frac{d}{dx}\left(f(x)\right)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^{(x+h)} - x^x}{h}$$

$$= \lim_{h\to 0} \frac{e^{(x+h)\log(x+h)} - e^{x\log x}}{h}$$

$$= \lim_{h\to 0} e^{x\log x}\left\{\frac{e^{(x+h)\log(x+h) - x\log x} - 1}{h}\right\}$$

$$= e^{x\log x} \lim_{h\to 0}\left\{\frac{e^{(x+h)\log(x+h) - x\log x} - 1}{h}\right\}$$

$$= e^{x\log x} \lim_{h\to 0} \frac{e^{x\log(x+h) - x\log x + h\log(x+h)} - 1}{h}$$

$$= x^x \lim_{h\to 0}\left\{\frac{e^{x\log(1+h/x) + h\log(x+h)} - 1}{x\log(1+h/x) + h\log(x+h)}\right\}$$

$$\times \left\{\frac{x\log\left(1+\frac{h}{x}\right) + h\log(x+h)}{h}\right\}$$

$$= x^x \lim_{h\to 0}\left\{\frac{e^{x\log(1+h/x) + h\log(x+h)} - 1}{x\log(1+h/x) + h\log(x+h)}\right\}$$

$$\times \left\{\frac{\log\left(1+\frac{h}{x}\right)}{h/x} + \log(x+h)\right\}$$

$$= x^x \times 1 \times (1 + \log x)$$

$$= x^x(1 + \log x).$$

EXAMPLE 2 *Differentiate $f(x) = \sqrt[3]{\sin x}$ from first principles.*

SOLUTION Let $f(x) = \sqrt[3]{\sin x}$. Then,

$$\frac{d}{dx}\left(f(x)\right)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{\sin(x+h)} - \sqrt[3]{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\{\sqrt[3]{\sin(x+h)}\}^3 - \{\sqrt[3]{\sin x}\}^3}{h\,[\sin^{2/3}(x+h) + \sin^{2/3}x + \sin^{1/3}(x+h)\sin^{1/3}x]}$$

$$\left[\because a-b = \frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \times \frac{1}{\sin^{2/3}(x+h) + \sin^{2/3}x + \sin^{1/3}(x+h)\sin^{1/3}(x)}$$

$$= \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \times \frac{1}{\sin^{2/3}(x+h) + \sin^{2/3}x + \sin^{1/3}(x+h)\sin^{1/3}x}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h\to 0} \frac{1}{\sin^{2/3}(x+h) + \sin^{2/3}x + \sin^{1/3}(x+h)\sin^{1/3}x}$$

$$= \cos x \cdot \frac{1}{\sin^{2/3}x + \sin^{2/3}x + \sin^{1/3}x \cdot \sin^{1/3}x}$$

$$= \frac{\cos x}{3\sin^{2/3}x}$$

EXAMPLE 3 *Differentiate* $\log \sin x$ *by first principles.*

SOLUTION Let $f(x) = \log \sin x$. Then,

$$\frac{d}{dx}\left(f(x)\right)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log\sin(x+h) - \log\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left\{\frac{\sin(x+h)}{\sin x}\right\}}{h}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)}{\sin x}-1\right\}}{h}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \times \left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \times \frac{\sin(x+h)-\sin x}{h} \cdot \frac{1}{\sin x}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \cdot \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos\left\{x+\frac{h}{2}\right\}}{h} \cdot \frac{1}{\sin x}$$

$$= \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \cdot \lim_{h\to 0} \frac{\sin\left\{\frac{h}{2}\right\}\cos\left\{x+\frac{h}{2}\right\}}{\frac{h}{2}} \cdot \frac{1}{\sin x}$$

$$= 1 \times \cos x \times \frac{1}{\sin x} = \cot x.$$

EXAMPLE 4 *If* $f(x) = x\tan^{-1}x$, *find* $f'(\sqrt{3})$ *by first priniciples.*

SOLUTION We have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\therefore\ f'(\sqrt{3})$$

$$= \lim_{h\to 0} \frac{f(\sqrt{3}+h)-f(\sqrt{3})}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{3}+h)\tan^{-1}(\sqrt{3}+h) - \sqrt{3}\tan^{-1}\sqrt{3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3}\,[\tan^{-1}(\sqrt{3}+h) - \tan^{-1}\sqrt{3}] + h\tan^{-1}(\sqrt{3}+h)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3}}{h}\tan^{-1}\left\{\frac{\sqrt{3}+h-\sqrt{3}}{1+\sqrt{3}(\sqrt{3}+h)}\right\} + \lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$= \sqrt{3}\lim_{h\to 0}\left\{\frac{\tan^{-1}\left\{\frac{h}{4+\sqrt{3}h}\right\}}{\frac{h}{4+\sqrt{3}h}}\right\} \cdot \frac{1}{4+\sqrt{3}h} + \lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$= \sqrt{3}\times 1\times\frac{1}{4} + \tan^{-1}\sqrt{3}$$

$$= \frac{\sqrt{3}}{4} + \tan^{-1}\sqrt{3} = \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

EXAMPLE 5 *If* $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, *find* $\frac{dy}{dx}$.

SOLUTION We have,

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$= \frac{1}{\sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}} \times \frac{(1+x^2)\times 2 - 2x \times 2x}{(1+x^2)^2}$$

$$= \frac{|1+x^2|}{\sqrt{(1-x^2)^2}} \times \frac{2(1-x^2)}{(1+x^2)^2}$$

$$= \frac{1+x^2}{|1-x^2|} \times \frac{2(1-x^2)}{(1+x^2)^2}$$

$$= \frac{1-x^2}{|1-x^2|} \cdot \frac{2}{1+x^2}$$

$$= \begin{cases} \dfrac{2}{1+x^2} & \text{if } , -1 < x < 1 \\ \text{does not of exist} & \text{if } , x = \pm 1 \\ \dfrac{-2}{1+x^2} & \text{if } , |x| > 1 \end{cases}$$

ALITER Putting $x = \tan\theta$, we have

$$y = \sin^{-1}(\sin 2\theta)$$

$$= \begin{cases} 2\theta & , \text{if } -\pi/2 \le 2\theta \le \pi/2 \\ \pi - 2\theta & , \text{if } \pi/2 < 2\theta < \pi \\ -\pi - 2\theta & , \text{if } -\pi < 2\theta < -\pi/2 \end{cases}$$

$$= \begin{cases} 2\tan^{-1} x & , \text{if } |x| \le 1 \\ \pi - 2\tan^{-1} x & , \text{if } x > 1 \\ -\pi - 2\tan^{-1} x & , \text{if } x < -1 \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & , \text{if } -1 < x < 1 \\ -\dfrac{2}{1+x^2} & , \text{if } |x| > 1 \end{cases}$$

EXAMPLE 6 *Differentiate* $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *with respect to x.*

SOLUTION Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. Then,

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}} \times \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-(1+x^2)}{\sqrt{4x^2}} \times \frac{-4x}{(1+x^2)^2}$$

$$= \frac{x}{|x|} \times \frac{2}{1+x^2}$$

$$= \begin{cases} \dfrac{2}{1+x^2} & , x > 0 \\ \text{does not exist} & , x = 0 \\ \dfrac{-2}{1+x^2} & , x < 0 \end{cases}$$

ALITER On putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= \begin{cases} 2\theta & , \text{if } 0 \le 2\theta < \pi \\ -2\theta & , \text{if } -\pi < 2\theta < 0 \end{cases}$$

$$= \begin{cases} 2\tan^{-1} x & , \text{if } x \ge 0 \\ -2\tan^{-1} x & , \text{if } x \le 0 \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & , x > 0 \\ \text{does not exist} & , x = 0 \\ \dfrac{-2}{1+x^2} & , x < 0 \end{cases}$$

EXAMPLE 7 *If*

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1, \text{ prove that } \frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}. \quad \text{[IIT 1998]}$$

SOLUTION We have,

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c+x-c}{x-c}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx+x(x-b)}{(x-b)(x-c)}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$\Rightarrow y = \frac{ax^2+x^2(x-a)}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow \log y = \log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\}$$

$$\Rightarrow \log y = 3\log x - [\log(x-a) + \log(x-b) + \log(x-c)]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \left\{\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right\}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right\}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{-a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{x-c}\right\}$$

EXAMPLE 8 *Prove that the derivative of an even function is an odd function and that of an odd function is an even function.*

SOLUTION Let $f(x)$ be an even function. Then,

$$f(-x) = f(x)$$

$$\Rightarrow \quad \frac{d}{dx}\{f(-x)\} = \frac{d}{dx}\{f(x)\}$$

$$\Rightarrow \quad f'(-x).\frac{d}{dx}(-x) = f'(x)$$

$$\Rightarrow \quad -f'(-x) = f'(x)$$

$$\Rightarrow \quad f'(-x) = -f'(x)$$

$\Rightarrow f'(x)$ is an odd function.

Let $f(x)$ be an odd function. Then,

$$f(-x) = -f(x)$$

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$$\Rightarrow \quad f'(-x)\frac{d}{dx}(-x) = -f'(x)$$

$$\Rightarrow \quad -f'(-x) = -f'(x)$$

$$\Rightarrow \quad f'(-x) = f'(x)$$

$\Rightarrow f'(x)$ is an even function.

EXAMPLE 9 *If* $y = f\left(\frac{2x-1}{x^2+1}\right)$ *and* $f'(x) = \sin x^2$, *find* $\frac{dy}{dx}$.

SOLUTION Let $z = \frac{2x-1}{x^2+1}$. Then,

$$y = f(z)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\{f(z)\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dz}\{f(z)\}.\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = f'(z).\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = f'(z).\left|\frac{2(x^2+1)-(2x-1)2x}{(x^2+1)^2}\right|$$

$$\Rightarrow \frac{dy}{dx} = \sin z^2.\frac{2(x^2+1)-(4x^2-2x)}{(x^2+1)^2} \qquad \left[\because f'(x) = \sin x^2 \Rightarrow f'(z) = \sin z^2\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\sin\left(\frac{2x-1}{x^2+1}\right)^2.\left|\frac{1+x-x^2}{(x^2+1)^2}\right|$$

EXAMPLE 10 *If* $\sqrt{1-x^6}+\sqrt{1-y^6} = a(x^3-y^3)$, *prove that*

$$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}.$$

SOLUTION Putting $x^3 = \sin A$ and $y^3 = \sin B$ in the given relation, we get

$$\sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \quad \cos A + \cos B = a(\sin A - \sin B)$$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A-B}{2}\right) = a$$

$$\Rightarrow \quad \frac{A-B}{2} = \cot^{-1}(a)$$

$$\Rightarrow \quad A - B = 2\cot^{-1}(a)$$

$$\Rightarrow \sin^{-1}x^3 - \sin^{-1}y^3 = 2\cot^{-1}(a).$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^6}}\frac{d}{dx}(x^3) - \frac{1}{\sqrt{1-y^6}}\frac{d}{dx}(y^3) = 0.$$

$$\Rightarrow \frac{1}{\sqrt{1-x^6}}3x^2 - \frac{1}{\sqrt{1-y^6}}3y^2\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}.$$

EXAMPLE 11 *If* $x^2+y^2 = t-\frac{1}{t}$ *and* $x^4+y^4 = t^2+\frac{1}{t^2}$, *then prove that* $\frac{dy}{dx} = \frac{1}{x^3y}$.

SOLUTION We have,

$$x^2+y^2 = t-\frac{1}{t}$$

$$\Rightarrow (x^2+y^2)^2 = \left(t-\frac{1}{t}\right)^2$$

$$\Rightarrow x^4+y^4+2x^2y^2 = t^2+\frac{1}{t^2}-2$$

$$\Rightarrow x^4+y^4+2x^2y^2 = x^4+y^4-2 \qquad \left[\because x^4+y^4 = t^2+\frac{1}{t^2}\right]$$

$$\Rightarrow 2x^2y^2 = -2$$

$$\Rightarrow \quad y^2 = -x^{-2}$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = -(-2)x^{-3}$$

$$\Rightarrow \quad y\frac{dy}{dx} = \frac{1}{x^3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^3y}$$

EXAMPLE 12 *Prove that the function*

$$y = \frac{x^2}{2} + \frac{1}{2}\, x\sqrt{x^2+1} + \log \sqrt{x+\sqrt{x^2+1}}$$

satisfies the relationship $2y = x\frac{dy}{dx} + \log \frac{dy}{dx}$.

SOLUTION We have,

$$y = \frac{x^2}{2} + \frac{1}{2}\, x\sqrt{x^2+1} + \log \sqrt{x+\sqrt{x^2+1}}$$

$$\Rightarrow y = \frac{x^2}{2} + \frac{1}{2}\, x\sqrt{x^2+1} + \frac{1}{2}\log (x+\sqrt{x^2+1})$$

$$\Rightarrow 2y = x^2 + x\sqrt{x^2+1} + \log (x+\sqrt{x^2+1}) \qquad ...(i)$$

Differentiating both sides w.r.t. x, we obtain

$$2\frac{dy}{dx} = 2x + \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow 2y\frac{dy}{dx} = 2x + \sqrt{x^2+1} + \sqrt{x^2+1}$$

$$\Rightarrow 2y\frac{dy}{dx} = 2x + 2\sqrt{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = x + \sqrt{x^2+1} \qquad ...(ii)$$

From (i), we have

$$2y = x(x+\sqrt{x^2+1}) + \log (x+\sqrt{x^2+1})$$

$$\Rightarrow 2y = x\frac{dy}{dx} + \log \frac{dy}{dx} \qquad \text{[Using (ii)]}$$

EXAMPLE 13 *If* $y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, *then show that*

$$\frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

SOLUTION We have,

$$y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\therefore \quad \cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\Rightarrow \quad \cos y = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$$

$$\Rightarrow \quad \cos y = \sqrt{4 - 3\sec^2 x}$$

$$\Rightarrow \quad \cos^2 y = 4 - 3(1+\tan^2 x)$$

$$\Rightarrow 1 - \cos^2 y = 3\tan^2 x$$

$$\Rightarrow \quad \sin^2 y = 3\tan^2 x$$

$$\Rightarrow \quad \sin y = \sqrt{3}\tan x$$

Differentiating both sides with respect to x, we get

$$\cos y\frac{dy}{dx} = \sqrt{3}\sec^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos^2 x}\sqrt{\frac{\cos^3 x}{\cos 3x}} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

EXAMPLE 14 *If* $y^2 = a^2\cos^2 x + b^2\sin^2 x$, *then prove that*

$$\frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}.$$

SOLUTION We have,

$$y^2 = a^2\cos^2 x + b^2\sin^2 x$$

$$\Rightarrow 2y^2 = a^2(2\cos^2 x) + b^2(2\sin^2 x)$$

$$\Rightarrow 2y^2 = a^2(1+\cos 2x) + b^2(1-\cos 2x)$$

$$\Rightarrow 2y^2 = (a^2+b^2) + (a^2-b^2)\cos 2x \qquad ...(i)$$

Differentiating with respect to x, we get

$$4y\frac{dy}{dx} = -2(a^2-b^2)\sin 2x$$

$$\Rightarrow 2y\frac{dy}{dx} = -(a^2-b^2)\sin 2x \qquad ...(ii)$$

From (i), we have

$$2y^2 - (a^2+b^2) = (a^2-b^2)\cos 2x \qquad ...(iii)$$

Squaring (i) and (ii) and adding, we get

$$4y^2\left(\frac{dy}{dx}\right)^2 + \left\{2y^2 - (a^2+b^2)\right\}^2 = (a^2-b^2)^2$$

$$\Rightarrow 4y^2\left[\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2)\right] = (a^2-b^2)^2 - (a^2+b^2)^2$$

$$\Rightarrow 4y^2\left[\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2)\right] = -4a^2b^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2) = -\frac{a^2b^2}{y^2}$$

Differentiating both sides w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = \frac{2a^2b^2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}$$

EXAMPLE 15 *If* $y = a(x+\sqrt{x^2+1})^n + b(x-\sqrt{x^2-1})^{-n}$, *prove that*

$$(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2 y = 0$$

SOLUTION We have,

$$y = a\left(x+\sqrt{x^2-1}\right)^n + b\left(x-\sqrt{x^2-1}\right)^{-n}$$

$$\therefore \frac{dy}{dx} = na\,[x+\sqrt{x^2-1}]^{n-1}\left\{1+\frac{x}{\sqrt{x^2-1}}\right\}$$

$$-bn\,(x-\sqrt{x^2-1})^{-n-1}\times\left\{1-\frac{x}{\sqrt{x^2-1}}\right\}$$

$$\Rightarrow \frac{dy}{dx}\sqrt{x^2-1} = n\left[a\,(x+\sqrt{x^2-1})^n + b\,(x-\sqrt{x^2-1})^{-n}\right]$$

$$\Rightarrow \sqrt{x^2-1}\,\frac{dy}{dx} = n\,y$$

$$\Rightarrow (x^2-1)\left(\frac{dy}{dx}\right)^2 = n^2 y^2$$

Differentiating w.r.t. x, we get

$$2\,(x^2-1)\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = n^2\,2y\frac{dy}{dx}$$

$$\Rightarrow \quad (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2 y = 0$$

EXAMPLE 16 *If $x = \sin\theta$, $y = \cos p\,\theta$, prove that*

$$(1-x^2)\,y_2 - xy_1 + p^2 y = 0, \text{ where } y_2 = \frac{d^2y}{dx^2} \text{ and } y_1 = \frac{dy}{dx}.$$

SOLUTION We have,

$$x = \sin\theta \text{ and } y = \cos p\,\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-p\sin p\theta}{\cos\theta}$$

$$= \frac{-p\sqrt{1-\cos^2 p\,\theta}}{\sqrt{1-\sin^2\theta}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{p^2(1-y^2)}{(1-x^2)}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

Differentiating both sides w.r.t. x, we get

$$(1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = p^2\left(0-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

EXAMPLE 17 *If $x^2+y^2+z^2-2xyz = 1$, then show that*

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$$

SOLUTION We have,

$$x^2+y^2+z^2-2xyz = 1 \qquad \ldots(i)$$

$$\Rightarrow d\,(x^2+y^2+z^2-2xyz) = 0$$

$$\Rightarrow 2x\,dx + 2y\,dy + 2\,zdz - 2\,(xydz + yz\,dx + zx\,dy) = 0$$

$$\Rightarrow (x-yz)\,dx + (y-zx)\,dy + (z-xy)\,dz = 0$$

$$\Rightarrow \frac{dx}{(y-zx)(z-xy)} + \frac{dy}{(x-yz)(z-xy)} + \frac{dz}{(x-yz)(y-zx)} = 0 \quad \ldots(ii)$$

Now,

$$(y-zx)^2\,(z-xy)^2$$

$$= (y^2 - 2xyz + z^2x^2)\,(z^2 - 2xyz + x^2y^2)$$

$$= (1-x^2-z^2+z^2x^2)\,(1-x^2-y^2+x^2y^2) \qquad \text{[Using (i)]}$$

$$= (1-x^2)\,(1-z^2)\,(1-x^2)\,(1-y^2)$$

$$= (1-x^2)^2\,(1-y^2)\,(1-z^2)$$

$$\therefore (y-zx)\,(z-xy) = (1-x^2)\sqrt{(1-y^2)\,(1-z^2)}$$

Similarly,

$$(x-yz)\,(z-xy) = (1-y^2)\sqrt{(1-x^2)\,(1-z^2)}$$

and, $(x-yz)\,(y-zx) = (1-z^2)\sqrt{(1-x^2)\,(1-y^2)}$

Substituting these values in (ii), we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$$

EXAMPLE 18 *If y is a twice differentiable function of x, transform the expression $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$ by means of the transformation $x = \sin t$ in terms of the independent variable t.*

SOLUTION We have, $x = \sin t$ or $t = \sin^{-1} x$.

Now,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt}\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = \frac{dy}{dt}$$

Differentiating w.r.t. x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{d^2y}{dt^2}\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{d^2y}{dt^2} + y \quad \text{[Adding } y \text{ on both sides]}$$

EXAMPLE 19 *If* $x = k\,(1 + e^{(y/x)^2})$, *prove that*

$$\frac{dy}{dx} = \frac{x^2 + x^2\,e^{(y/x)^2} + 2y^2\,e^{(y/x)^2}}{2xye^{(y/x)^2}}$$

SOLUTION We have,

$$x = k\left(1 + e^{(y/x)^2}\right)$$

$$\Rightarrow \quad e^{(y/x)^2} = \frac{x}{k} - 1 \qquad \ldots(i)$$

Differentiating both sides w.r.t. x

$$e^{(y/x)^2} \cdot \frac{2y}{x^3}\left\{x\frac{dy}{dx} - y\right\} = \frac{1}{k}$$

$$\Rightarrow e^{(y/x)^2} \cdot \frac{2y}{x^3}\left\{x\frac{dy}{dx} - y\right\} = \frac{1 + e^{(y/x)^2}}{x} \qquad \text{[Using (i)]}$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{x^2}{2y}\left\{\frac{1 + e^{(y/x)^2}}{e^{(y/x)^2}}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + x^2\,e^{(y/x)^2} + 2y^2\,e^{(y/x)^2}}{2xy\,e^{(y/x)^2}}$$

EXAMPLE 20 *Prove that the expression* $\dfrac{y'''}{y'} - \dfrac{3}{2}\left(\dfrac{y''}{y'}\right)^2 + \dfrac{1}{2}\left(\dfrac{y'}{y}\right)^2$ *remains unchanged if* y *is replaced by* $\dfrac{1}{y^2}$.

SOLUTION Let $y_1 = \dfrac{1}{y^2}$. Then,

$$y_1' = \frac{-2}{y^3}y' \qquad \ldots(i)$$

$$\Rightarrow \frac{y_1'}{y_1} = -2\frac{y'}{y}$$

Differentiating both sides with respect to x, we get

$$\frac{y_1''}{y_1} - \frac{y_1'^2}{y_1^2} = -2\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right) \qquad \ldots(ii)$$

Differentiating w.r.t. x, we get

$$\frac{y_1'''}{y_1} - \frac{3y_1'y_1''}{y_1^2} + \frac{2y_1'^3}{y_1^3} = -2\left(\frac{y'''}{y} - \frac{3y'y''}{y^2} + \frac{2y^3}{y^3}\right) \qquad \ldots(iii)$$

Using (i), (ii) and (iii) can be written as

$$\frac{y_1''}{y_1'} - \frac{y_1'}{y_1} = \frac{y''}{y'} - \frac{y'}{y} \qquad \ldots(iv)$$

$$\frac{y'''}{y_1'} - \frac{3y_1'''}{y_1} + \frac{2y_1'^2}{y_1^2} = \frac{y'''}{y'} - \frac{3y''}{y} + \frac{2y'^2}{y^2} \qquad \ldots(v)$$

Subtracting 3/2 times the square of (iv) from (v), we get

$$\frac{y_1'''}{y_1'} - \frac{3}{2}\left(\frac{y_1''}{y_1'}\right)^2 + \frac{1}{2}\left(\frac{y_1'}{y_1}\right)^2 = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y'}{y'}\right)^2 + \frac{1}{2}\left(\frac{y'}{y}\right)^2$$

This shows that the given expression does not change if y is replaced by $\dfrac{1}{y^2}$.

EXAMPLE 21 *For a function* $f(x)$, *we have*

$$f'(x) = \frac{1}{1+x^2} \text{ and } f(0) = 0$$

prove that

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

SOLUTION We have,

$$f'(x) = \frac{1}{1+x^2}$$

$$f'\left(\frac{x+y}{1-xy}\right) = \frac{1}{1 + \left(\dfrac{x+y}{1-xy}\right)^2}$$

$$= \frac{(1-xy)^2}{(1-xy)^2 + (x+y)^2}$$

$$= \frac{(1-xy)^2}{(1+x^2)(1+y^2)}$$

$$\Rightarrow \quad \frac{1+y^2}{(1-xy)^2} f'\left(\frac{x+y}{1-xy}\right) = \frac{1}{1+x^2}$$

$$\Rightarrow f'\left(\frac{x+y}{1-xy}\right)\frac{d}{dx}\left(\frac{x+y}{1-xy}\right) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left\{f\left(\frac{x+y}{1-xy}\right)\right\} = \frac{d}{dx}(f(x)) \qquad \left[\because f'(x) = \frac{1}{1+x^2} \text{ (given)}\right]$$

On integration, we get

$$f\left(\frac{x+y}{1-xy}\right) = f(x) + C \qquad \ldots(i)$$

Putting $x = 0$, we get

$$f(y) = f(0) + C$$

$$\Rightarrow f(y) = 0 + C \qquad [\because f(0) = 0]$$

Substituting the value of C in (i), we get

$$f\left(\frac{x+y}{1-xy}\right) = f(x) + f(y)$$

EXAMPLE 22 *If $f(x)$ is a real function such that $f(0)=0$ and $f'(x)=\frac{1}{\sqrt{1-x^2}}$ for $-1<x<1$, then without using integration show that*

$$f(x)+f(a)=f\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)$$

SOLUTION We have,

$$f'(x)=\frac{1}{\sqrt{1-x^2}}$$

Therefore,

$$f'\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)$$

$$=\frac{1}{\sqrt{1-[x\sqrt{1-a^2}+a\sqrt{1-x^2}]^2}}$$

$$=\frac{1}{\sqrt{[\sqrt{(1-a^2)(1-x^2)}-ax]^2}}$$

$$=\frac{1}{\sqrt{(1-a^2)(1-x^2)}-ax}$$

$$\Rightarrow \frac{\sqrt{(1-a^2)(1-x^2)}-ax}{\sqrt{1-x^2}}f'\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left\{f\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right\}=\frac{d}{dx}\{f(x)\}\left[\because f'(x)=\frac{1}{\sqrt{1-x^2}}\right]\right.$$

On integration, we get

$$\Rightarrow f\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)=f(x)+C \qquad ...(i)$$

Putting $x=0$, we get

$$f(a)=f(0)+C$$

$$\Rightarrow \quad C=f(a) \qquad [\because f(0)=0]$$

$$\therefore f\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)=f(x)+f(a)$$

EXAMPLE 23 *Show that transformation $x=\cos\theta$ reduces the differential equation*

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \text{ into } \frac{d^2y}{d\theta^2}+y=0$$

SOLUTION We have,

$$x=\cos\theta \quad \Rightarrow \quad \frac{dx}{d\theta}=-\sin\theta$$

Now,

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\operatorname{cosec}\theta\cdot\frac{dy}{d\theta}$$

and,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$=\frac{d}{d\theta}\left\{-\operatorname{cosec}\theta\frac{dy}{d\theta}\right\}\cdot\frac{d\theta}{dx}$$

$$=\left|\operatorname{cosec}\theta\cot\theta\frac{dy}{d\theta}-\operatorname{cosec}\theta\frac{d^2y}{d\theta^2}\right|\times-\operatorname{cosec}\theta$$

$$=-\operatorname{cosec}^2\theta\cot\theta\frac{dy}{d\theta}+\operatorname{cosec}^2\theta\frac{d^2y}{d\theta^2}$$

Substituting the values of x, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in

$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$, we get

$$(1-\cos^2\theta)\left|-\operatorname{cosec}^2\theta\cot\theta\frac{dy}{d\theta}+\operatorname{cosec}^2\theta\frac{d^2y}{d\theta^2}\right|$$

$$-\cos\theta\left\{-\operatorname{cosec}\theta\frac{dy}{d\theta}\right\}+y=0$$

$$\Rightarrow \quad -\cot\theta\frac{dy}{d\theta}+\frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}+y=0$$

$$\Rightarrow \frac{d^2y}{d\theta^2}+y=0$$

EXAMPLE 24 *Show that the transformation $z=\log\tan(x/2)$ reduces the differential equation*

$$\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y\operatorname{cosec}^2x=0 \text{ into } \frac{d^2y}{dz^2}+4y=0$$

SOLUTION We have,

$$z=\log\tan\frac{x}{2}$$

$$\Rightarrow \frac{dz}{dx}=\frac{1}{\tan\frac{x}{2}}\times\sec^2\frac{x}{2}\times\frac{1}{2}=\operatorname{cosec}x$$

Now,

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\operatorname{cosec}x\frac{dy}{dz}$$

and,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}\left\{\operatorname{cosec}x\frac{dy}{dz}\right\}$$

$$=-\operatorname{cosec}x\cot x\frac{dy}{dz}+\operatorname{cosec}x\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$=-\operatorname{cosec}x\cot x\frac{dy}{dz}+\operatorname{cosec}^2x\frac{d^2y}{dz^2}$$

$$\therefore \frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

$$\Rightarrow \left| -\operatorname{cosec} x \cot x \frac{dy}{dx} + \operatorname{cosec}^2 x \frac{d^2y}{dz^2} \right| + \cot x \operatorname{cosec} x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

$$\Rightarrow \operatorname{cosec}^2 x \frac{d^2y}{dz^2} + 4y \operatorname{cosec}^2 x = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} + 4y = 0$$

EXAMPLE 25 *Given* $F(x) = f(x)\,\phi(x)$ *and* $f'(x)\,\phi'(x) = C$, *prove that*

$$\frac{F''}{F} = \frac{f''}{f} + \frac{\phi''}{\phi} + \frac{2C}{f\phi}.$$

SOLUTION We have,

$$F(x) = f(x)\,\phi(x) \qquad \ldots(i)$$

and,

$$f'(x)\,\phi'(x) = C \qquad \ldots(ii)$$

Differentiating (i) w.r.t. x, we get

$$F'(x) = f'(x)\,\phi(x) + f(x)\,\phi'(x)$$

Again differentiating w.r.t. x, we get

$$F''(x) = f''(x)\,\phi(x) + 2f'(x)\,\phi'(x) + f(x)\,\phi''(x)$$

$$\Rightarrow F''(x) = f''(x)\,\phi(x) + f(x)\,\phi''(x) + 2C \qquad \text{[Using (ii)]}$$

Dividing both sides by F, we get

$$\frac{F''(x)}{F(x)} = \frac{f''(x)\,\phi(x) + f(x)\,\phi''(x) + 2C}{f(x)\,\phi(x)}$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{f''(x)}{f(x)} + \frac{\phi''(x)}{\phi(x)} + 2 \qquad [\because C = \phi(x) f(x)]$$

EXAMPLE 26 *If* $(a + bx)\,e^{y/x} = x$, *prove that* $x^3 y_2 = (xy_1 - y)^2$

If $y = x \log\left(\frac{x}{a+bx}\right)$, *prove that* $x^3 y_2 = (x\,y_1 - y)^2$.

SOLUTION We have,

$$y = x \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow y = x\{\log x - \log(a+bx)\}$$

$$\Rightarrow \frac{y}{x} = \log x - \log(a+bx)$$

On differentiating w.r.t. x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\frac{d}{dx}(a+bx)$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2\left\{\frac{1}{x} - \frac{b}{a+bx}\right\}$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad \ldots(i)$$

Differentiating both sides of (i) w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx).a - ax(0+b)}{(a+bx)^2}$$

$$\Rightarrow x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} \qquad \text{[Multiplying both sides by } x^2\text{]}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 \qquad \ldots(ii)$$

From (i) and (ii), we have

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

EXAMPLE 27 $y = x^x$, *prove that* $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$

SOLUTION We have,

$$y = x^x.$$

$$\Rightarrow y = e^{x \log x}$$

$$\Rightarrow \frac{dy}{dx} = e^{x\log x}\frac{d}{dx}(x \log x)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x) \qquad \ldots(i)$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y \cdot \frac{d}{dx}(1 + \log x) + \frac{dy}{dx} \cdot (1 + \log x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = y \cdot \frac{1}{x} + \frac{dy}{dx} \cdot (1 + \log x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{dy}{dx}\frac{1}{y}\frac{dy}{dx} \qquad \text{[Using (i)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

EXAMPLE 28 *If* $x = a\cos\theta + b\sin\theta$ *and* $y = a\sin\theta - b\cos\theta$, *prove that*

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

SOLUTION We have,

$$x^2+y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$

$$\Rightarrow \quad x^2+y^2 = a^2(\cos^2\theta+\sin^2\theta) + b^2(\sin^2\theta+\cos^2\theta)$$

$$\Rightarrow \quad x^2+y^2 = a^2+b^2.$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \qquad ...(i)$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left\{\frac{y\cdot 1 - x\frac{dy}{dx}}{y^2}\right\}$$

$$= -\left\{\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right\} \qquad \text{[Using (i)]}$$

$$= -\frac{(x^2+y^2)}{y^3} \qquad ...(ii)$$

Now,

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$$

$$= -y^2\left(\frac{x^2+y^2}{y^3}\right) - x\left(-\frac{x}{y}\right) + y = 0 \qquad \text{[Using (i) and (ii)]}$$

EXAMPLE 29 *If* $(\sin y)^{\sin\frac{\pi x}{2}} + \frac{\sqrt{3}}{2}\sec^{-1}2x + 2^x\tan\{\log(x+2)\} = 0$, *find* $\frac{dy}{dx}$ *at* $x=-1$. **[IIT 1992]**

SOLUTION We have,

$$(\sin y)^{\sin\frac{\pi x}{2}} + \frac{\sqrt{3}}{2}\sec^{-1}2x + 2^x\tan\{\log(x+2)\} = 0 \qquad ...(i)$$

$$\text{or, } e^{\sin\frac{\pi x}{2}\log\sin y} + \frac{\sqrt{3}}{2}\sec^{-1}2x + 2^x\tan\{\log(x+2)\} = 0$$

Differentiating w.r.t. x, we get

$$e^{\sin\frac{\pi x}{2}\log\sin y}\frac{d}{dx}\left\{\sin\frac{\pi x}{2}\log\sin y\right\} + \frac{\sqrt{3}}{2}\frac{1}{|2x|\sqrt{4x^2-1}}\times 2$$

$$+ 2^x\log 2\tan\{\log(x+2)\} + 2^x\cdot\sec^2\{\log(x+2)\}\cdot\frac{1}{x+2} = 0$$

$$\Rightarrow (\sin y)^{\sin\frac{\pi x}{2}}\left\{\frac{\pi}{2}\cos\frac{\pi x}{2}\log\sin y + \sin\frac{\pi x}{2}\cot y\frac{dy}{dx}\right\}$$

$$+ \frac{\sqrt{3}}{2|x|}\times\frac{1}{\sqrt{4x^2-1}}$$

$$+ 2^x\left\{\log 2\cdot\tan\{\log(x+2)\} + \frac{\sec^2\{\log(x+2)\}}{x+2}\right\} = 0 \qquad ...(ii)$$

Putting $x=-1$ in (i), we get

$$(\sin y)^{-1} + \frac{\sqrt{3}}{2}\times\frac{2\pi}{3} + \frac{1}{2}\times 0 = 0$$

$$\Rightarrow \frac{1}{\sin y} = -\frac{\pi}{\sqrt{3}}$$

$$\Rightarrow \sin y = \frac{-\sqrt{3}}{\pi}$$

$$\therefore \tan y = \frac{\sqrt{3}}{\sqrt{\pi^2-3}}$$

$$\Rightarrow \cot y = \sqrt{\frac{\pi^2-3}{3}}$$

Now, putting $x=-1$ and $\cot y = \sqrt{\frac{\pi^2-3}{3}}$ in (ii), we get

$$\left(\frac{-\sqrt{3}}{\pi}\right)^{-1}\left\{0 - \sqrt{\frac{\pi^2-3}{3}}\cdot\frac{dy}{dx}\right\} + \frac{\sqrt{3}}{2\times\sqrt{3}} + \{2^{-1}(0+1)\} = 0$$

$$\Rightarrow \frac{-\pi}{\sqrt{3}}\times -\sqrt{\frac{\pi^2-3}{3}}\frac{dy}{dx} = -\frac{1}{2} - \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3}{\sqrt{\pi^2-3}}$$

EXAMPLE 30 *If* $y^3 - 3ax^2 + x^3 = 0$, *then prove that* $y_2 + 2a^2x^2y^{-5} = 0$.

SOLUTION We have,

$$y^3 = 3ax^2 - x^3$$

Differntiating both sides w.r.t. x, we get

$$3y^2y_1 = 6ax - 3x^2$$

$$\Rightarrow \quad y^2y_1 = 2ax - x^2 \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$y^2y_2 + 2yy_1^2 = 2a - 2x$$

$$\Rightarrow y_2 + 2\frac{y_1^2}{y} = \frac{2(a-x)}{y^2}$$

$$\Rightarrow y_2 + \frac{2}{y}\left(\frac{2ax-x^2}{y^2}\right)^2 = \frac{2(a-x)}{y^2} \qquad \text{[Using (i)]}$$

$$\Rightarrow y_2 + \frac{2}{y^5}[(2ax-x^2)^2 - y^3(a-x)] = 0$$

$$\Rightarrow y_2 + \frac{2}{y^5}[(2ax-x^2)^2 - (3ax^2-x^3)(a-x)] = 0 \ [\because y^3 = 3ax^2 - x^3]$$

$$\Rightarrow y_2 + \frac{2}{y^5}a^2x^2 = 0$$

$$\Rightarrow y_2 + 2y^{-5}a^2x^2 = 0$$

EXAMPLE 31 If $f(x) = \log_{\sec 2x} |\cos 4x| + |\sin x|$, then find $\frac{df(x)}{dx}$ at $x = -\frac{\pi}{6}$.

SOLUTION We find that in the neighbourhood of $x = -\pi/6$,

$\cos 4x < 0$, $\sec 2x > 0$ and $\sin x < 0$

$\therefore |\cos 4x| = -\cos 4x$, $|\sin x| = -\sin x$
and, $|\sec 2x| = \sec 2x$

Thus,

$$f(x) = \log_{\sec 2x} |\cos 4x| + |\sin x|$$
$$= \log_{\sec 2x} (-\cos 4x) - \sin x$$
$$= \frac{\log(-\cos 4x)}{\log \sec 2x} - \sin x$$

$$\therefore \frac{df(x)}{dx}$$
$$= \frac{(\log \sec 2x)(-4\tan 4x) - \log(-\cos 4x) \cdot 2\tan 2x}{(\log \sec 2x)^2} - \cos x$$

$$\Rightarrow \left(\frac{df(x)}{dx}\right)_{x=-\pi/6} = \frac{(\log 2)(-4\sqrt{3}) - (\log 2) \times (-2\sqrt{3})}{(\log 2)^2} - \frac{\sqrt{3}}{2}$$
$$= \frac{-6\sqrt{3}}{\log 2} - \frac{\sqrt{3}}{2}$$

EXAMPLE 32 *If $f(x)$ is a real valued polynomial and $f(x) = 0$ has real and distinct roots, show that the equation $\{f'(x)\}^2 - f(x) f''(x)$ cannot have real roots.*

SOLUTION Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the real roots of $f(x)$. Then,

$$f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$\Rightarrow \log\{f(x)\} = \sum_{i=1}^{n} \log(x-\alpha_i)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \sum_{i=1}^{n} \frac{1}{(x-\alpha_i)} \quad \ldots(i) \qquad \text{[Diff. both sides w.r.t. } x\text{]}$$

$$\Rightarrow f'(x) = \left\{\sum_{i=1}^{n} \frac{1}{x-\alpha_i}\right\} f(x)$$

$$\Rightarrow f''(x) = \left\{\sum_{i=1}^{n} \frac{1}{x-\alpha_i}\right\} f'(x) - \left\{\sum_{i=1}^{n} \frac{1}{(x-\alpha_i)^2}\right\} f(x)$$

[Diff. w.r.t. x]

$$\Rightarrow f''(x) = \frac{(f'(x))^2}{f(x)} - \left\{\sum_{i=1}^{n} \frac{1}{(x-\alpha_i)^2}\right\} f(x) \qquad \text{[Using (i)]}$$

$$\Rightarrow (f'(x))^2 - f(x) f''(x) = \left\{\sum_{i=1}^{n} \frac{1}{(x-\alpha_i)^2}\right\} \{f(x)\}^2 > 0 \text{ for all } x \in R$$

$\Rightarrow \{f'(x)\}^2 - f(x) f''(x) > 0$ for all $x \in R$ $x \notin \alpha_i$

$\Rightarrow y = \{f'(x)\}^2 - f(x) f''(x)$ does not cross x-axis.

$\Rightarrow \{f'(x)\}^2 - f(x) f''(x)$ does not have real roots.

—❑❑❑—

5.7 DIFFERENTIATION OF DETERMINANTS

In the previous sections, we have studied differentiation in detail. In this section, we shall discuss the differentiation of determinants.

To differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

For example, if

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}, \text{ then}$$

$$\frac{d}{dx}\{\Delta(x)\} = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$$

Also,

$$\frac{d}{dx}\{\Delta(x)\} = \begin{vmatrix} f'(x) & g(x) \\ u'(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) \\ u(x) & v'(x) \end{vmatrix}$$

Similar results hold for the differentiation of determinants of higher order.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ab & bc & x+c^2 \end{vmatrix}$, *find* $f'(x)$.

SOLUTION We have,

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} x+b^2 & bc \\ bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ac \\ ac & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab \\ ab & x+b^2 \end{vmatrix}$$

$$\Rightarrow f'(x) = \left[(x+b^2)(x+c^2) - b^2c^2\right] + \left[(x+a^2)(x+c^2) - a^2c^2\right] + \left[(x+a)^2(x+b^2) - a^2b^2\right]$$

$$\Rightarrow f'(x) = x^2 + x(b^2+c^2) + x^2 + x(a^2+c^2) + x^2 + x(a^2+b^2)$$

$$\Rightarrow f'(x) = 3x^2 + 2x(a^2+b^2+c^2).$$

EXAMPLE 2 *If $f_r(x), g_r(x)$ and $h_r(x)$; $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$; $r = 1, 2, 3$ and*

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}, \text{ find } F'(x) \text{ at } x = a.$$

SOLUTION We have,

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$\therefore F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(& g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$$\Rightarrow F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$$\Rightarrow F'(a) = \begin{vmatrix} f'_1(a) & f'_2(a) & f'_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ g_1(a) & g_2(a) & g_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g'_1(a) & g'_2(a) & g'_3(a) \\ f_1(a) & f_2(a) & f_3(a) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(a) & f_2(a) & f_3(x) \\ f_1(a) & f_2(a) & f_3(a) \\ g_1'(x) & g_2'(x) & g_3'(x) \end{vmatrix} \quad \left[\begin{array}{l}\text{Using : } f_r(a) = g_r(a) \\ = h_r(a)\,;\, r = 1, 2, 3\end{array}\right]$$

$$\Rightarrow F'(a) = 0 + 0 + 0 = 0 \quad \left[\begin{array}{l}\because \text{Two rows are identical} \\ \text{in each of the determinants}\end{array}\right]$$

EXAMPLE 3 *If $f(x), g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that*

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

is a constant polynomial.

SOLUTION Let $f(x) = a_1 x^2 + a_2 x + a_3, g(x) = b_1 x^2 + b_2 x + b_3$ and $h(x) = c_1 x^2 + c_2 x + c_3$. Then,

$f'(x) = 2a_1 x + a_2, g'(x) = 2b_1 x + b_2$ and, $h'(x) = 2c_1 x + c_2$

$f''(x) = 2a_1, g'' = 2b_1, h''(x) = 2c_1$

and, $f'''(x) = g'''(x) = h'''(x) = 0$ ...(i)

In order to prove that $\phi(x)$ is a constant polynomial, it is sufficient to show that $\phi''(x) = 0$ for all x.

Now,

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$\Rightarrow \phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$$\Rightarrow \phi'(x) = 0 + 0 + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix} \quad \text{[Using (i)]}$$

$\Rightarrow \phi'(x) = 0 + 0 + 0 = 0$ for all x.

$\Rightarrow \phi(x) =$ constant for all.

Hecne, $\phi(x)$ is a constant polynomial.

EXAMPLE 4 *If f, g, h are differentiable functions of x and*

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}.$$

prove that

$$\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

SOLUTION We have,

$$(xf)' = xf' + f, (xg)' = xg' + g, (xh)' = xh' + h$$

$$(x^2 f)' = x^2 f' + 2xf, (x^2 g)' = x^2 g' + 2xg$$

$$(x^2 h)' = x^2 h' + 2xh$$

$$(x^2 f)'' = x^2 f'' + 4xf' + 2f,$$

$$(x^2 g)'' = x^2 g'' + 4xg' + 2g$$

and $(x^2 h)'' = x^2 h'' + 4xh' + 2h$

$$\therefore \Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' + 4xf' + 2f & x^2 g'' + 4xg' + 2g & x^2 h'' + 4xh' + 2h \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - 2R_1$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' + 4xf' & x^2 g'' + 4xg' & x^2 h'' + 4xh' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$ Applying $R_3 \to R_3 - 4R_2$

$$= x\begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix} \quad [\because \text{Taking } x \text{ common from } R_2]$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} \quad [\text{Multiplying } R_3 \text{ by } x]$$

$$\therefore \Delta' = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$\Rightarrow \Delta' = 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

EXERCISE 5.1

1. Differentiate the following functions with respect to x from first principles :
   (i) $\sin(\log x)$ (ii) $\log \sin x$
   (iii) $e^{\sin x}\cos x$ (iv) $\sqrt[3]{\cos x}$

2. If $f(x) = x\tan^{-1} x$, find $f'(\sqrt{3})$ by first principles.

3. If $f(x) = \sin^{-1} x$, find $f'(1/\sqrt{2})$ by first principles.

4. Prove that the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ does not exist for $|x| = 1$.

5. Prove that the derivative of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ does not exist for $x = 0$.

6. Prove that the function $y = f(x)$ defined parametrically by $x = 2t - |t|$ and $y = t^2 + t\,|t|$ is differentiable at $t = 0$. Also, find its derivative at $t = 0$

7. Prove that $\frac{d}{dx}\left\{\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)\right\} = \frac{1}{a + b\cos x}$

8. If $f(x) = \begin{vmatrix} 1+x^2\cos x & 2-\sin^2 x \\ 1-x^2\cos x & 2+\sin^2 x \end{vmatrix}$, find $f'(x)$

9. If $y = \begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos n(x+1) & \cos n(x+2) \\ \sin nx & \sin n(x+1) & \sin n(x+2) \end{vmatrix}$, find $\frac{dy}{dx}$

10. If $y = \frac{1}{2ab}\sin^{-1}\left(\frac{(a^2+b^2)x^2 - a^4 - b^4}{(b^2-a^2)x^2 + a^4 - b^4}\right)$, prove that
$$\frac{dy}{dx} = \frac{x}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}}$$

11. If $y = \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots\ldots}}}$, $x \geq 0$, find $y''\left(\frac{1}{2}\right)$.

12. If $y^3 - y = 2x$, then prove that
$$\frac{d^2y}{dx^2} = \frac{-24y}{(3y^2-1)^3}$$
Hence, show that $9\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} = y$.

13. If $y = \log_{g(x)} f(x)$, where $f(x)$ and $g(x)$ and derivable functions and $f(x) > 0, g(x) > 0, g(x) \neq 1$, then find $\frac{dy}{dx}$.

14. Prove that the function represented parametrically by $x = 3t^2$ and $y = 3t - t^3$ satisfies the relation
$$36\frac{d^2y}{dx^2}(y - \sqrt{3x}) = x + 3$$

15. Show that the function represented by
$$x = \frac{1+t}{t^3} \text{ and } y = \frac{3}{2t^2} + \frac{2}{t}$$
satisfies the relation $x\left(\frac{dy}{dx}\right)^3 = 1 + \frac{dy}{dx}$.

16. If $y = (f(x))^{g(x)}$ and $f(x) > 0$, then prove that
$$\frac{dy}{dx} = g(x)(f(x))^{g(x)-1}\frac{df(x)}{dx} + (f(x))^{g(x)}\log f(x)\cdot\frac{dg(x)}{dx}$$

17. If $y = x^n[a\cos(\log x) + b\sin(\log x)]$, prove that
$$x^2 y_2 + (1-2n)xy_1 + (1+n^2)y = 0$$

18. If $x = \sin t, y = \sin kt$, prove that
$$(1-x^2)y_2 - xy_1 + k^2 y = 0$$

19. If $x = \tan\frac{y}{2} - \log\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan y/2}\right]$, find $\frac{dy}{dx}$.

20. If $u = \sin(m\cos^{-1} x)$, $v = \cos(m\sin^{-1} x)$, prove that
$$\frac{du}{dv} = \sqrt{\frac{1-u^2}{1-v^2}}.$$

21. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that
$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4).$$

22. If $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$, then find the value of
$$\lim_{x\to a}\frac{g(x)f(a) - f(x)g(a)}{x-a}.$$

23. If $y = |x|^2 - 3|x| + |x-1|$, find $\frac{dy}{dx}$.

24. If $(a - b\cos y)(a + b\cos x) = a^2 - b^2$, prove that
$$\frac{dy}{dx} = \frac{a^2 - b^2}{a + b\cos x}.$$

25. If $y = 2\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right) + \log\left(\frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2}\right)$, show that
$$\frac{dy}{dx} = \frac{4\sqrt{2}}{1 + x^4}.$$

26. If $f(x) = \log_x(\ln x)$, then find $f'(x)$ at $x = e$. Here, in x means natural logarithm of x i.e., $\log_e x$.

27. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that
$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

28. Find the derivative of $f(x) = \log_x \sin x^2 + (\sin x^2)^{\log_e^x}$ with respect to $\log_e x$.

29. Find the derivative of $\log_{(1-\sqrt{x})}(\sin^{-1}(1 - \sqrt{x}))$ with respect to $2^{2(1-\sqrt{x})}$.

30. If $y = f(x)\,\phi(x)$ and $z = f'(x)\,\phi'(x)$, prove that
$$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{1}{f}\cdot\frac{d^2f}{dx^2} + \frac{1}{\phi}\frac{d^2\phi}{dx^2} + \frac{2z}{f\phi}$$

31. If $u = f(x^2)$, $v = g(x^3)$, $f'(x) = \sin x$ and $g'(x) = \cos x$, then find $\frac{du}{dv}$.

ANSWERS

9. 0

11. $\frac{3\sqrt{3}}{2\sqrt{2}}$

13. $\frac{1}{\log g(x)}\left[\frac{f'(x)}{f(x)} - \frac{y\,g'(x)}{g(x)}\right]$

19. $\frac{1}{2}\sin y\,(1 + \sin y + \cos y)$

22. 5

23. $\frac{dy}{dx} = \begin{cases} 2(x+1), & x < 0 \\ 2(x-2), & 0 < x < 1 \\ 2(x-1), & x > 1 \end{cases}$

26. $\frac{1}{e}$

28. $\frac{2x^2}{\log x}\cot x^2 - \frac{\log \sin x^2}{(\log x)^2}$
$+(\sin x^2)^{\log_e^x}\left\{\log \sin x^2 + 2x^2 \cot x^2 \cdot \log x\right\}$

29. $\dfrac{-\frac{\log(1-\sqrt{x})}{2\sqrt{x}\sqrt{2\sqrt{x} - x}}\cdot\frac{1}{\sin^{-1}(1-\sqrt{x})} + \frac{\log \sin^{-1}(1-\sqrt{x})}{2\sqrt{x}(1-\sqrt{x})}}{\left[\log 4\left(\frac{-4^{1-\sqrt{x}}}{2\sqrt{x}}\right)\{\log(1-\sqrt{x})\}^2\right]}$

31. $\frac{2}{3x}\cdot\frac{\sin x^2}{\cos x^3}$

CHAPTER

6

TANGENTS AND NORMALS

6.1 SLOPES OF TANGENT AND NORMAL

In chapter 4, we have learnt that the slope of the tangent to a continuous curve $y=f(x)$ at a point $P(x_1, y_1)$ is equal to $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$. Thus, if the tangent to the curve $y=f(x)$ at point (x_1, y_1) makes an angle θ with x-axis, then

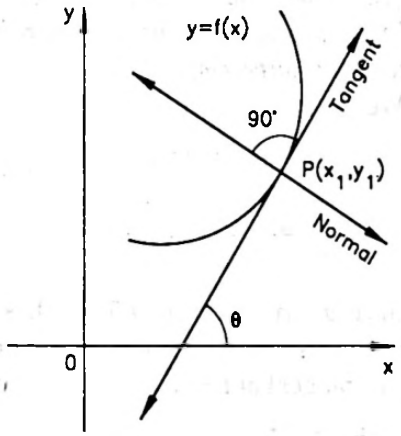


Fig. 6.1

$$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

If the tangent at P is parallel to x-axis, then

$$\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

If the tangent at P is parallel to y-axis, then

$$\theta = \frac{\pi}{2} \Rightarrow \cot\theta = 0 \Rightarrow \left(\frac{dx}{dy}\right)_{x_1, y_1} = 0.$$

We know that the normal to a curve at a point $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P.

Therefore,

$$\text{Slope of the normal at } P = -\frac{1}{\text{Slope of the tangent at } P}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P.$$

6.2 EQUATIONS OF TANGENT AND NORMAL

We know that the equation of a straight line passing through (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

Since tangent and normal to the curve $y = f(x)$ at point (x_1, y_1) pass through P and have slopes $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ and $\frac{-1}{\left(\frac{dy}{dx}\right)_{x_1, y_1}}$ respectively. Therefore, the equation of the tangent at (x_1, y_1) to $y=f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

The equation of the normal to $y=f(x)$ at (x_1, y_1) is

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

or,
$$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1)$$

REMARK 1 *If* $\left(\frac{dy}{dx}\right)_P = \infty$, *then the tangent at* $P(x_1, y_1)$ *is parallel to* y*-axis and its equation is* $x = x_1$.

REMARK 2 *If* $\left(\frac{dy}{dx}\right)_P = 0$, *then the normal at* $P(x_1, y_1)$ *is parallel to y-axis and its equation is* $x = x_1$.

REMARK 3 *The equations of tangent and normal to the curve having its paramaetric equations* $x = f(t)$ *and* $y = g(t)$ *are given by*

$$y - g(t) = \frac{g'(t)}{f'(t)}\{x - f(t)\}$$ (Equation of tangent

and, $$y - g(t) = \frac{-f'(t)}{g'(t)}\{x - f(t)\}$$ (Equation of normal

respectively.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *The slope of the curve* $2y^2 = ax^2 + b$ *at* $(1, -1)$ *is* -1. *Find* a, b

SOLUTION The equation of the curve is

$$2y^2 = ax^2 + b \quad ...(i)$$

Differentiating w.r.t. x, we get

$$4y\frac{dy}{dx} = 2ax$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ax}{2y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,-1)} = \frac{-a}{2}.$$

It is given that the slope of the tangent at $(1, -1)$ is -1. Therefore,

$$-\frac{a}{2} = -1 \Rightarrow a = 2.$$

Since the point $(1, -1)$ lies on (i). Therefore

$$2(-1)^2 = a(1)^2 + b$$

$$\Rightarrow \quad a + b = 2.$$

Putting $a = 2$ in $a + b = 2$, we obtain $b = 0$.

Hence, $a = 2$ and $b = 0$.

EXAMPLE 2 *Find the point on the curve* $y = x^3 - 11x + 5$ *at which the tangent has the equation* $y = x - 11$.

SOLUTION Let the required point be $P(x_1, y_1)$. Since (x_1, y_1) lies on $y = x^3 - 11x + 5$. Therefore,

$$y_1 = x_1^3 - 11x_1 + 5 \quad ...(i)$$

Now,

$$y = x^3 - 11x + 5$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 11$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 11$$

Since the line $y = x - 11$ is tangent at the point (x_1, y_1). Therefore,

Slope of the tangent at (x_1, y_1) = Slope of the line $y = x - 11$.

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \text{Slope of the line } x - y - 11 = 0$$

$$\Rightarrow \quad 3x_1^2 - 11 = \frac{-1}{-1}$$

$$\Rightarrow \quad 3x_1^2 = 12$$

$$\Rightarrow \quad x_1 = \pm 2$$

Now, $x_1 = 2$

$$\Rightarrow \quad y_1 = 2^3 - 22 + 5 = -9$$ [Using (i)

$x_1 = -2$

$$y = (-2)^3 - 11(-2) + 5 = 19$$ [Using (i

So, two points are $(2, -9)$ and $(-2, 19)$. Of these two point $(-2, 19)$ does not lie on $y = x - 11$. Therefore, the required poir is $(2, -9)$.

EXAMPLE 3 *The curve* $y = ax^3 + bx^2 + cx + 5$ *touches the x-axis* $P(-2, 0)$ *and cuts the y-axis at the point Q where its gradient is Find the equation of the curve completely.* [IIT 199

SOLUTION We have,

$$y = ax^3 + bx^2 + cx + 5$$

$$\therefore \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$$

Since the curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis $P(-2, 0)$. This means that the curve passes through $P(-2,$ and x-axis is the tangent at $P(-2, 0)$.

$$\therefore \quad 0 = -8a + 4b - 2c + 5$$

$$\Rightarrow \quad 8a - 4b + 2c = 5 \quad ...$$

and, $$\left(\frac{dy}{dx}\right)_P = 0$$

$$\Rightarrow \quad 3a(-2)^2 + 2b \times (-2) + c = 0$$

$$\Rightarrow \quad 12a - 4b + c = 0 \quad ...$$

The curve $y = ax^3 + bx^2 + cx + 5$ meets y-axis at Q.

Putting $x = 0$ in $y = ax^3 + bx^2 + cx + 5$, we get $y = 5$

Thus, the coordinates of Q are $(0, 5)$.

It is given that the gradient of the curve at Q is 3.

$$\therefore \quad \left(\frac{dy}{dx}\right)_Q = 3$$

$\Rightarrow 3a \times 0 + 2b \times 0 + c = 3$

$\Rightarrow \quad c = 3$

Putting $c = 3$ in (i) and (ii), we get

$8a - 4b = -1$ and $12a - 4b = -3$

Solving these two equations, we get $a = -\frac{1}{2}$ and $b = -\frac{3}{4}$.

Substituting the values of a, b and c in the equation of the curve, we obtain

$$y = -\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$$

as the equation of the curve.

EXAMPLE 4 *Find the equation of the normal to the curve*

$$y = (1+x)^y + \sin^{-1}(\sin^2 x) \text{ at } x = 0. \quad \text{[IIT 1993]}$$

SOLUTION We have,

$$y = (1+x)^y + \sin^{-1}(\sin^2 x) \quad \ldots(i)$$

Putting $x = 0$, we get

$$y = (1+0)^y + \sin^{-1}(\sin^2 0)$$

$\Rightarrow y = 1.$

Thus, we have to write the equation of the normal to (i) at $P(0, 1)$.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = e^{y\log(1+x)} \cdot \frac{d}{dx}\{y\log(1+x)\} + \frac{1}{\sqrt{1-\sin^4 x}}\frac{d}{dx}(\sin^2 x)$$

$$\Rightarrow \frac{dy}{dx} = (1+x)^y \left\{\frac{dy}{dx}\cdot\log(1+x) + \frac{y}{1+x}\right\} + \frac{2\sin x\cos x}{|\cos x|\sqrt{1+\sin^2 x}}$$

Putting $x = 0$ and $y = 1$, we obtain

$$\left(\frac{dy}{dx}\right)_P = \left\{\left(\frac{dy}{dx}\right)_P \times 0 + 1\right\} + 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = 1$$

Hence, the equation of the normal at $P(0, 1)$ is

$$y - 1 = -1(x - 0) \quad \text{or,} \quad x + y = 1$$

EXAMPLE 5 *Find the equation of the tangent to the curve* $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ *at the point* (x_1, y_1).

SOLUTION We have,

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \quad \ldots(i)$$

Differentiating both sides with respect to x, we get

$$m\frac{x^{m-1}}{a^m} + m\frac{y^{m-1}}{b^m}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-b^m}{a^m}\left(\frac{x_1}{y_1}\right)^{m-1}$$

The equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

or, $$y - y_1 = \frac{-b^m}{a^m}\left(\frac{x_1}{y_1}\right)^{m-1}(x - x_1)$$

or, $a^m y\, y_1^{m-1} - a^m y_1^m = -b^m x_1^{m-1} x + b^m x_1^m$

or, $(b^m x_1^{m-1})x + (a^m y_1^{m-1})y = b^m x_1^m + a^m y_1^m$

or, $$\left(\frac{x_1^{m-1}}{a^m}\right)x + \left(\frac{y_1^{m-1}}{b^m}\right)y = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} \quad \left[\text{Dividing both sides by } a^m b^m\right]$$

or, $$\left(\frac{x_1^{m-1}}{a^m}\right)x + \left(\frac{y_1^{m-1}}{b^m}\right)y = 1 \quad \left[\because (x_1, y_1) \text{ lies on (i)}, \therefore \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1\right]$$

Thus, the equation of the tangent to (i) at (x_1, y_1) is

$$\left(\frac{x_1}{a}\right)^{m-1}\frac{x}{a} + \left(\frac{y_1}{b}\right)^{m-1}\frac{y}{b} = 1$$

EXAMPLE 6 *Find the equation of tangent to the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ *at* (x_1, y_1). *Hence, prove that the length of the portion of tangent intercepted between the axes is constant.*

SOLUTION We have,

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \ldots(i)$$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x^{-1/3}}{y_1^{-1/3}} = -\left(\frac{y_1}{x_1}\right)^{1/3}$$

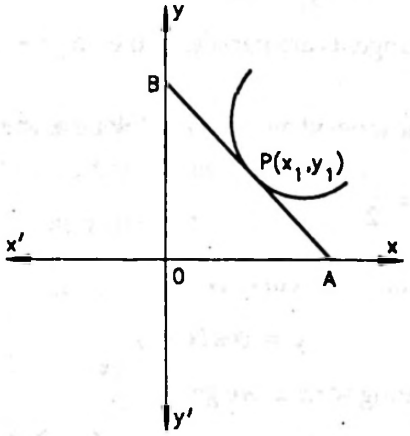


Fig. 6.2

The equation of the tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

or, $y - y_1 = -\left(\frac{y_1}{x_1}\right)^{1/3} (x - x_1)$

or, $\frac{y - y_1}{y_1^{1/3}} = -\frac{x - x_1}{x_1^{1/3}}$

or, $yy_1^{-1/3} - y_1^{2/3} = -xx_1^{-1/3} + x_1^{2/3}$

or, $xx_1^{-1/3} + yy_1^{-1/3} = x_1^{2/3} + y_1^{2/3}$

or, $xx_1^{-1/3} + yy_1^{-1/3} = a^{2/3}$ ...(ii)

$[\because (x_1, y_1)$ lies on (i) $\therefore x_1^{2/3} + y_1^{2/3} = a^{2/3}]$

Suppose the tangent at $P(x_1, y_1)$ meets the coordinate axes at A and B respectively.

Putting $y = 0$ in (ii), we get

$$xx_1^{-1/3} = a^{2/3} \Rightarrow x = a^{2/3} x_1^{1/3}$$

So, the coordinates of A are $(a^{2/3} x_1^{1/3}, 0)$

Now, putting $x = 0$ in (ii), we get

$$yy_1^{-1/3} = a^{2/3} \Rightarrow y = a^{2/3} y_1^{1/3}$$

So, the coordinates of B are $(0, a^{2/3} y_1^{1/3})$

$\therefore AB = \sqrt{(a^{2/3} x_1^{1/3})^2 + (a^{2/3} y_1)^2}$

$= \sqrt{a^{4/3}(x_1^{2/3} + y_1^{2/3})}$

$= \sqrt{a^{4/3} \times a^{2/3}}$ $[\because x_1^{2/3} + y_1^{2/3} = a^{2/3}]$

$= a$, which is a constant.

Hence, the length of the portion of the tangent intercepted between the axes is constant.

EXAMPLE 7 *Find all the tangents to the curve* $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ *that are parallel to the line* $x + 2y = 0$. [IIT 1985]

SOLUTION Let the point of contact of one of the tangents be (x_1, y_1). Then, (x_1, y_1) lies on $y = \cos(x + y)$

$\therefore$ $y_1 = \cos(x_1 + y_1)$ ...(i)

Since the tangents are parallel to the line $x + 2y = 0$. Therefore,

Slope of the tangent at (x_1, y_1) = (Slope of the line $x + 2y = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-1}{2}$

The equation of the curve is

$$y = \cos(x + y)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(x + y) \cdot \left(1 + \frac{dy}{dx}\right)$$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\sin(x_1 + y_1)\left\{1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right\}$

$\Rightarrow -\frac{1}{2} = -\sin(x_1 + y_1)\left(1 - \frac{1}{2}\right)$

$\Rightarrow \sin(x_1 + y_1) = 1$ ...(ii

Squaring (i) and (ii) and then adding, we get

$\cos^2(x_1 + y_1) + \sin^2(x_1 + y_1) = y_1^2 + 1$

$\Rightarrow 1 = y_1^2 + 1$

$\Rightarrow y_1 = 0$

Putting $y_1 = 0$ in (i) and (ii), we get

$\cos x_1 = 0$ and $\sin x_1 = 1$

$\Rightarrow x_1 = \frac{\pi}{2}, \frac{-3\pi}{2}$ $[\because -2\pi \le x_1 \le 2\pi$

Hence, the points of contact are $(\pi/2, 0)$ and $(-3\pi/2, 0)$

The slope of the tangent is $(-1/2)$. Therefore, equations of tangents are

$$y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right) \text{ and } y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$\Rightarrow 2x + 4y - \pi = 0$ and $2x + 4y + 3\pi = 0$

EXAMPLE 8 *Determine the quadratic curve* $y = f(x)$ *if it touches the line* $y = x$ *at the point* $x = 1$ *and passes through the point* $(-1, 0)$.

SOLUTION Let the required quadratic curve be

$$y = ax^2 + bx + c$$ ...(i)

It passes through $(-1, 0)$. Therefore,

$$0 = a - b + c$$ ...(ii

Since the line $y = x$ touches (i) at $x = 1$. Therefore,

(Slope of the tantent at $x = 1$) = (Slope of the line $y = x$)

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 1$

$\Rightarrow 2a + b = 1$ ...(iii) $\left[\because y = ax^2 + bx + c \therefore \frac{dy}{dx} = 2ax + b\right.$

Putting $x = 1$ in $y = x$, we get $y = 1$

Thus, passes through $(1, 1)$.

$\Rightarrow 1 = a + b + c$ ...(iv

Solving (ii), (iii) and (iv), we get

$$a = \frac{1}{4}, b = \frac{1}{2} \text{ and } c = \frac{1}{4}$$

Substituting these values in (i), we get

$y = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}$ as the required quadratic curve.

EXAMPLE 9 *Determine the constant c such that the straight line joining the points (0, 3) and (5, −2) is tangent to the curve* $y = \frac{c}{x+1}$.

SOLUTION The equation of the line joining (0, 3) and (5, −2) is

$$y - 3 = \frac{-2-3}{5-0}(x-0)$$

or, $y - 3 = -x$

or, $x + y = 3$.

Suppose it touches the curve $y = \frac{c}{x+1}$ at (x_1, y_1).

Now,

$$y = \frac{c}{x+1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-c}{(x_1+1)^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-c}{(x_1+1)^2}$$

But, $x + y = 3$ touches the curve at (x_1, y_1). Therefore,

Slope of the tangent at $(x_1, y_1) = -1$.

$$\therefore \quad -1 = \frac{-c}{(x+1)^2}$$

$$\Rightarrow \quad (x_1+1)^2 = c \qquad \ldots(i)$$

Since (x_1, y_1) lies on the line and also on the curve. Therefore,

$$x_1 + y_1 = 3 \qquad \ldots(ii)$$

and,

$$y_1 = \frac{c}{x_1+1} \qquad \ldots(iii)$$

From (i) and (iii), we obtain

$$x_1 = \pm\sqrt{c} - 1 \text{ and } y_1 = \pm\sqrt{c}$$

Substituting these values in (ii), we get

$$\pm\sqrt{c} - 1 \pm \sqrt{c} = 3$$

$$\Rightarrow \pm 2\sqrt{c} = 4$$

$$\Rightarrow \pm\sqrt{c} = 2$$

$$\Rightarrow c = 4$$

ALITER The line $x + y = 3$ touches the curve $y = \frac{c}{x+1}$. Therefore,

$$x + \frac{c}{x+1} = 3 \text{ or, } x^2 - 2x + c - 3 = 0 \text{ must have equal roots}$$

$$\therefore \; 4 - 4(c-3) = 0 \Rightarrow c = 4.$$

EXAMPLE 10 *Prove that all normals to the curve*

$$x = a\cos t + at\sin t,\ y = a\sin t - at\cos t$$

are at a distance a from the origin. **[IIT 1983]**

SOLUTION We have,

$$x = a\cos t + at\sin t \text{ and } y = a\sin t - at\cos t$$

$$\therefore \quad \frac{dx}{dt} = at\cos t \text{ and } \frac{dy}{dt} = at\sin t$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$$

The equation of the normal at any point t is given by

$$y - (a\sin t - at\cos t) = -\frac{1}{\tan t}\{x - (a\cos t + at\sin t)\}$$

$$\Rightarrow \quad y - (a\sin t - at\cos t) = -\frac{\cos t}{\sin t}\{x - (a\cos t + at\sin t)\}$$

$$\Rightarrow \quad y\sin t - (a\sin^2 t - at\sin t\cos t) = -x\cos t + a\cos^2 t + at\sin t\cos t$$

$$\Rightarrow \quad x\cos t + y\sin t = a \qquad \ldots(i)$$

$$\therefore \quad \text{Length of the perpendicular from the origin to (i)} = \frac{|0\cos t + 0\sin t - a|}{\sqrt{\cos^2 t + \sin^2 t}} = a$$

Hence, all normals to the given curve are at a distance a from the orgin.

EXAMPLE 11 *Find the points on the curve* $9y^2 = x^3$ *where normal to the curve makes equal intercepts with the axes.*

SOLUTION Let the required point be (x_1, y_1).

The equation of the curve is $9y^2 = x^3$.

Since (x_1, y_1) lies on the curve. Therefore,

$$9y_1^2 = x_1^3 \qquad \ldots(i)$$

Now, $\quad 9y^2 = x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{6y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1^2}{6y_1}.$$

Since the normal to the curve at (x_1, y_1) make equal intercepts with the coordinate axes. Therefore,

Slope of the normal $= \pm 1$

$$\Rightarrow \quad -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = \pm 1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1$$

$$\Rightarrow \quad \frac{x_1^2}{6y_1} = \pm 1$$

$$\Rightarrow \quad x_1^2 = \pm 6\, y_1$$

$$\Rightarrow \quad x_1^4 = 36\, y_1^2$$

$$\Rightarrow \quad x_1^4 = 36\left(\frac{x_1^3}{9}\right) \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad x_1^4 = 4x_1^3$$

$$\Rightarrow \quad x_1^3 (x_1 - 4) = 0$$

$$\Rightarrow \quad x_1 = 0, 4$$

Putting $x_1 = 0$ in (i), we get

$$9\, y_1^2 = 0 \Rightarrow y_1 = 0$$

Putting $x_1 = 4$ in (i), we get

$$9\, y_1^2 = 4^3 \Rightarrow y_1 = \pm\frac{8}{3}$$

But, the line making equal intercepts with the coordinate axes cannot pass through the origin.

Hence, the required points are $(4, 8/3)$ and $(4, -8/3)$.

EXAMPLE 12 *Prove that the segment of the normal to the curve* $x = 2a \sin t + a \sin t \cos^2 t, y = -a \cos^3 t$ *contained between the coordinate axes is equal to* $2a$.

SOLUTION We have,

$$x = 2a \sin t + a \sin t \cos^2 t \text{ and } y = -a \cos^3 t$$

$$\therefore \quad \frac{dx}{dt} = 2a \cos t + a \cos^3 t - 2a \cos t \sin^2 t$$

$$= 2a \cos t (1 - \sin^2 t) + a \cos^3 t$$

$$= 3a \cos^3 t$$

and,

$$\frac{dy}{dt} = 3a \cos^2 t \sin t$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3a \cos^2 t \sin t}{3a \cos^3 t} = \tan t$$

The equation of the normal at t is given by

$$y + a \cos^3 t = -\frac{1}{dy/dx}(x - 2a \sin t - a \sin t \cos^2 t)$$

$$\Rightarrow y + a \cos^3 t = -\cot t\,(x - 2a \sin t - a \sin t \cos^2 t)$$

$$\Rightarrow y + a \cos^3 t = -\frac{\cos t}{\sin t}(x - 2a \sin t - a \sin t \cos^2 t)$$

$$\Rightarrow x \cos t + y \sin t = 2a \sin t \cos t$$

Suppose this normal meets the coordinate axes at A and B respectively. Then, the coordinates of A and B are $A(2a \sin t, 0)$ and $B(0, 2a \cos t)$ respectively.

$$\therefore \quad AB = \sqrt{4a^2 \sin^2 t + 4a^2 \cos^2 t} = \sqrt{4a^2 (\sin^2 t + \cos^2 t)} = 2a.$$

Hence, the length of the segment intercepted between the coordinate axes is constant.

EXAMPLE 13 *Find the points at which the tangents to the curves* $y = x^3 - x - 1$ *and* $y = 3x^2 - 4x + 1$ *are parallel. Also, find the equations of tangents.*

SOLUTION Let $y = mx + c$ be tangent to $y = 3x^2 - 4x + 1$ at point (x_1, y_1).

Then,

$3x^2 - 4x + 1 - (mx + c) = 0$ must have equal roots.

$\Rightarrow 3x^2 - x(4 + m) + 1 - c = 0$ must have equal roots

$$\Rightarrow (4 + m)^2 - 12(1 - c) = 0$$

$$\Rightarrow m^2 + 8m + 4 + 12c = 0$$

$$\Rightarrow c = -\frac{(m^2 + 8m + 4)}{12}$$

Since $y = mx + c$ touches $y = 3x^2 - 4x + 1$ at (x_1, y_1). Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m$$

$$\Rightarrow \quad 6x_1 - 4 = m$$

$$\Rightarrow \quad x_1 = \frac{m + 4}{6}$$

Since, (x_1, y_1) lies on $y = 3x^2 - 4x + 1$. Therefore,

$$\Rightarrow y_1 = 3x_1^2 - 4x_1 + 1$$

$$\Rightarrow y_1 = 3\left(\frac{m+4}{6}\right)^2 - 4\left(\frac{m+4}{6}\right) + 1$$

$$\Rightarrow y_1 = \frac{m^2 - 4}{12}$$

Thus, $y = mx + c$ or, $y = mx - \left(\frac{m^2 + 8m + 4}{12}\right)$ is tangent to the curve $y = 3x^2 - 4x + 1$ at the point $\left(\frac{m+4}{6}, \frac{m^2 - 4}{12}\right)$.

It is given that the two curves have parallel tangents. So, slope of the tangent to the curve $y = x^3 - x - 1$ is also m. Let the point of contact of the tangent of slope m be (x_2, y_2).

Then,

$$\left(\frac{dy}{dx}\right)_{(x_2, y_2)} = m$$

$\Rightarrow 3x_2^2 - 1 = m$

$\Rightarrow \quad x_2 = \pm\sqrt{\frac{m+1}{3}}$ and $m \geq -1$

Now,

(x_2, y_2) lies on $y = x^3 - x - 1$.

$\therefore \quad y_2 = x_2^3 - x_2 - 1$

$\Rightarrow y_2 = \pm\left(\frac{m+1}{3}\right)^{3/2} \mp \sqrt{\frac{m+1}{3}} - 1$

$\Rightarrow y_2 = -1 \pm \sqrt{\frac{m+1}{3}}\left(\frac{m+1}{3} - 1\right)$

$\Rightarrow y_2 = -1 \pm \sqrt{\frac{m+1}{3}}\left(\frac{m-2}{3}\right)$

$\Rightarrow y_2 = -1 \pm \frac{m-2}{9}\sqrt{3(m+1)}$

Thus, the points of contact of tangent of slope m to the curve $y = x^3 - x - 1$ are $\left(\pm\sqrt{\frac{m+1}{3}}, -1 \pm \frac{m-2}{9}\sqrt{3(m+1)}\right)$, where $m \geq -1$. The equation of the tangent at this point is

$$y - y_2 = \left(\frac{dy}{dx}\right)_{(x_2, y_2)}(x - x_2)$$

$\Rightarrow y + 1 \mp \frac{m-2}{9}\sqrt{3(m+1)} = m\left(x \mp \sqrt{\frac{m+1}{3}}\right)$

$\Rightarrow y = mx \pm 2\sqrt{\left(\frac{1+m}{3}\right)^3}$

EXAMPLE 14 *Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on the curve $x^2y^2 = x^2 - y^2$.*

SOLUTION Let (h, k) be a point of contact of tangents from the origin to the curve $y = \sin x$.

Now, $y = \sin x$

$\Rightarrow \quad \frac{dy}{dx} = \cos x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(h,k)} = \cos h$

The equation of the tangent at (h, k) is

$$y - k = \left(\frac{dy}{dx}\right)_{(h,k)}(x - h)$$

or, $y - k = \cos h\,(x - h)$

It passes through $(0, 0)$. Therefore,

$-k = -h\cos h$

$\Rightarrow \quad k = h\cos h$ ...(i)

Also, (h, k) lies on the curve. Therefore,

$k = \sin h$ ...(ii)

From (i) and (ii), we get

$$k^2 + \left(\frac{k}{h}\right)^2 = \sin^2 h + \cos^2 h$$

$\Rightarrow \quad k^2 + \frac{k^2}{h^2} = 1$

$\Rightarrow k^2h^2 + k^2 = h^2$

$\Rightarrow \quad k^2h^2 = h^2 - k^2$

Hence, (h, k) lies on the curve $x^2y^2 = x^2 - y^2$.

EXAMPLE 15 *Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point $(1, 2)$.*

SOLUTION Suppose the tangent drawn from $(1, 2)$ to the curve $y^2 - 2x^3 - 4y + 8 = 0$ touches the curve at (h, k).

We have,

$$y^2 - 2x^3 - 4y + 8 = 0 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{y-2}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(h,k)} = \frac{3h^2}{k-2}$

The equation of the tangent at (h, k) is

$$y - k = \frac{3h^2}{k-2}(x - h) \quad \text{...(ii)}$$

It passes through $(1, 2)$. Therefore,

$$2 - k = \frac{3h^2}{k-2}(1 - h)$$

$\Rightarrow -(k-2)^2 = 3h^2(1-h)$

$\Rightarrow 3h^3 - 3h^2 - k^2 + 4k - 4 = 0$ ...(iii)

Also, (h, k) lies on (i). Therefore,

$k^2 - 2h^3 - 4k + 8 = 0$ ...(iv)

Adding (iii) and (iv), we get

$h^3 - 3h^2 + 4 = 0$

$\Rightarrow (h-2)^2(h+1) = 0$

$\Rightarrow h = -1, 2.$

Putting $h = 2$ in (iii), we get

$24 - 12 - k^2 + 4k - 4 = 0$

$\Rightarrow k^2 - 4k - 8 = 0$

$\Rightarrow k = 2 \pm 2\sqrt{3}$

$h = -1$ gives imaginary values of k.

Thus, the points are $(2, 2 \pm 2\sqrt{3})$.

Putting the values of h and k in (ii), we obtain the following equations of the tangent

$y-(2+2\sqrt{3}) = 2\sqrt{3}(x-2)$ and $y-(2-\sqrt{3}) = -2\sqrt{3}(x-2)$.

EXAMPLE 16 *Find the equation of the tangent to $x^3 = ay^2$ at the point $A(at^2, at^3)$. Find also the point where this tangent meets the curve again.*

SOLUTION The equation of the curve is

$$x^3 = ay^2 \qquad ...(i)$$

Differentrating w.r.t. x, we get

$$3x^2 = 2ay\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(at^2, at^3)} = \frac{3a^2t^4}{2a^2t^3} = \frac{3t}{2}$$

The equation of the tangent at (at^2, at^3) is

$$y-at^3 = \left(\frac{dy}{dx}\right)_{(at^2, at^3)}(x-at^2)$$

$$\Rightarrow \quad y-at^3 = \frac{3t}{2}(x-at^2)$$

$$\Rightarrow 3tx-2y-at^3 = 0 \qquad ...(ii)$$

Let $B(at_1^2, at_1^3)$ be the point where this tangent again meets the curve.

$$\therefore \quad 3t \times at_1^2 - 2at_1^3 - at^3 = 0 \qquad [\because \text{Point } B \text{ lies on (ii)}]$$

$$\Rightarrow 2t_1^3 - 3t_1^2 t + t^3 = 0$$

$$\Rightarrow (t_1-t)^2(2t_1+t) = 0$$

$$\Rightarrow \quad t_1 = t \text{ or, } t_1 = -t/2$$

Since $t_1 = t$ gives the point A itself. Therefore, $t_1 = -t/2$.

Hence, the required point is $B\left(\frac{at^2}{4}, \frac{-at^3}{8}\right)$.

EXAMPLE 17 *If the tangent at (a, b) to the curve $x^3+y^3 = c^3$ meet the curve again in (a_1, b_1), prove that $\frac{a_1}{a}+\frac{b_1}{b} = -1$.*

SOLUTION The equation of the curve is $x^3+y^3 = c^3$.

Differntiating w.r.t. x, we get

$$3x^2+3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(a, b)} = -\frac{a^2}{b^2}.$$

If the tangent at (a, b) cuts the curve again at (a_1, b_1), then

$$a^3+b^3 = c^3 \qquad ...(ii)$$

$$a_1^3+b_1^3 = c^3 \qquad ...(ii)$$

and,

Slope of the tangent = Slope of the line joining (a, b) and (a_1, b_1)

$$\Rightarrow \quad -\frac{a^2}{b^2} = \frac{b_1-b}{a_1-a} \qquad ...(iii)$$

From (i) and (ii), we get

$$a^3+b^3 = a_1^3+b_1^3$$

$$\Rightarrow \quad a_1^3-a^3 = -(b_1^3-b^3)$$

$$\Rightarrow -\frac{a_1-a}{b_1-b} = \frac{b_1^2+bb_1+b^2}{a_1^2+aa_1+a^2}$$

$$\Rightarrow -\frac{b_1-b}{a_1-a} = \frac{a_1^2+aa_1+a^2}{b_1^2+bb_1+b^2} \qquad ...(iv)$$

From (iii) and (iv), we get

$$\frac{a_1^2+aa_1+a^2}{b_1^2+bb_1+b^2} = \frac{a^2}{b^2}$$

$$\Rightarrow a_1^2b^2+aa_1b^2+a^2b^2 = a^2b_1^2+a^2bb_1+a^2b^2$$

$$\Rightarrow a_1^2b^2-a^2b_1^2 = a^2bb_1-b^2aa_1$$

$$\Rightarrow (a_1b+ab_1)(a_1b-ab_1) = ab(ab_1-ba_1)$$

$$\Rightarrow a_1b+ab_1 = -ab$$

$$\Rightarrow \frac{a_1}{a}+\frac{b_1}{b} = -1$$

EXAMPLE 18 *Prove that the portion of the tangent intercepted between the axes at any point on the curve $x^my^n = a^{m+n}$ is divided at its point of contact in a constant ratio.*

SOLUTION Let $P(x_1, y_1)$ be an arbitrary point on the curve $x^my^n = a^{m+n}$.

Now,

$$x^my^n = a^{m+n}$$

$$\Rightarrow m\log x+n\log y = (m+n)\log a$$

$$\Rightarrow \quad \frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = 0 \qquad [\text{Differentiating w.r.t. } x]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{my}{nx}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-my_1}{nx_1}$$

The equation of the tangent at (x_1, y_1) is

$$y-y_1 = \frac{-my_1}{nx_1}(x-x_1)$$

$$\Rightarrow \frac{n(y-y_1)}{y_1} = -\frac{m(x-x_1)}{x_1}$$

$$\Rightarrow \frac{mx}{x_1} + \frac{ny}{y_1} = m+n$$

Suppose it meets the coordinate axes at A and B respectively. Then, the coordinates of A and B are given by

$$A\left(\frac{(m+n)x_1}{m}, 0\right), B\left(0, \frac{(m+n)y_1}{n}\right)$$

$$\therefore \quad PA = \sqrt{\left\{\frac{(m+n)}{m}.x_1 - x_1\right\}^2 + (y_1-0)^2}$$

$$= \frac{\sqrt{n^2 x_1^2 + m^2 y_1^2}}{m}$$

and,

$$PB = \sqrt{(0-x_1)^2 + \left\{y_1 - \frac{(m+n)y_1)}{n}\right\}^2}$$

$$= \frac{\sqrt{n^2 x_1^2 + m^2 y_1^2}}{n}$$

$\Rightarrow PA : PB = n : m.$

Hence, P divides AB in a constant ratio.

EXAMPLE 19 *Tangent at point P_1 (other than (0, 0)) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. Show that the abscissae of $P_1, P_2, P_3, \ldots, P_n$ form a GP. Also, find the ratio*

$$\frac{\text{area } \Delta P_1 P_2 P_3}{\text{area } \Delta P_2 P_3 P_4}.$$ [IIT 1993]

SOLUTION Let $P_1(t, t^3)$ be any point on the curve $y = x^3$.

Now, $y = x^3$

$$\Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(t, t^3)} = 3t^2$$

The equation of the tangent at (t, t^3) is

$$y - t^3 = 3t^2(x-t)$$

$$\Rightarrow 3t^2 x - y = 2t^3 \quad \ldots(i)$$

This meets the curve $y = x^3$ again at P_2 whose x-coordinate is given by

$$3t^2 x - x^3 = 2t^3 \qquad [\text{Putting } y = x^3 \text{ in (i)}]$$

$$\Rightarrow x^3 - 3t^2 x + 2t^3 = 0$$

$$\Rightarrow (x-t)^2(x+2t) = 0$$

$$\Rightarrow \quad x = -2t \qquad [\because x = t \text{ corresponds to point } P_1]$$

Putting $x = -2t$ in $y = x^3$, we obtain

$$y = (-2t)^3 = -8t^3.$$

So, the coordinates of P_2 are $(-2t, -8t^3)$

Now,

$$\left(\frac{dy}{dx}\right)_{(-2t, -8t^3)} = 3(-2t)^2 = 12t^2$$

The equation of the tangnet at $P_2(-2t, -8t^3)$ is

$$y + 8t^3 = 12t^2(x+2t)$$

The x-coordinates of the point where this tangent meets with the curve $y = x^3$ is given by

$$x^3 + 8t^3 = 12t^2(x+2t)$$

$$\Rightarrow x^3 - 12t^2 x - 16t^3 = 0$$

$$\Rightarrow (x+2t)^2(x-4t) = 0$$

$$\Rightarrow x = 4t \qquad [\because x = -2t \text{ corresponds to point } P_2]$$

Putting $x = 4t$ in $y = x^3$, we get

$$y = 64t^3$$

Thus, the coordinates of P_3 are $(4t, 64t^3)$.

Continuing like this, we obtain the coordinates of P_4 as $(-8t, -512t^3)$.

The abscissae of $P_1, P_2, P_3, P_4 \ldots\ldots$ are

$$t, -2t, 4t, -8t \ldots\ldots$$

Clearly, they form a GP with common ratio -2.

Now,

$$\Delta_1 = \text{Area of } \Delta P_1 P_2 P_3 = \frac{1}{2}\begin{vmatrix} t & t^3 & 1 \\ -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \end{vmatrix}$$

and,

$$\Delta_2 = \text{Area of } \Delta P_2 P_3 P_4 = \frac{1}{2}\begin{vmatrix} -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \\ -8t & -512t^3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times -2 \times -8 \begin{vmatrix} t & t^3 & 1 \\ -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \end{vmatrix}$$

$$= 16\,\Delta_1.$$

$$\therefore \frac{\Delta_1}{\Delta_2} = \frac{1}{16}$$

$$\text{or, } \frac{\text{Area of } \Delta P_1 P_2 P_3}{\text{Area of } \Delta P_2 P_3 P_4} = \frac{1}{16}$$

EXAMPLE 20 *Find the condition that the line*

$x\cos\alpha + y\sin\alpha = p$ *may touch the curve* $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1.$

SOLUTION We have,

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1 \qquad ...(i)$$

Differentiating w.r.t. x, we get

$$\frac{m\,x^{m-1}}{a^m}+\frac{m\,y^{m-1}}{b^m}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$$

Suppose the line $x\cos\alpha+y\sin\alpha=p$ touches (i) at (x_1, y_1).

$$\therefore \quad \frac{x_1^{\,m}}{a^m}+\frac{y_1^{\,m}}{b^m}=1 \qquad ...(ii)$$

and,

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^m}{a^m}\left(\frac{x_1}{y_1}\right)^{m-1}.$$

The equation of the tangent to (i) at (x_1, y_1) is

$$y-y_1=\frac{-b^m}{a^m}\left(\frac{x_1}{y_1}\right)^{m-1}(x-x_1)$$

$$\Rightarrow \quad (y-y_1)\frac{y_1^{m-1}}{b^m}=\frac{-x_1^{m-1}}{a^m}(x-x_1)$$

$$\Rightarrow \quad \frac{y\,y_1^{m-1}}{b^m}-\frac{y_1^{\,m}}{b^m}=\frac{-x\,x_1^{m-1}}{a^m}+\frac{x_1^m}{a^m}$$

$$\Rightarrow \quad \frac{x\,x_1^{m-1}}{a^m}+\frac{y\,y_1^{m-1}}{b^m}=\frac{x_1^{\,m}}{a^m}+\frac{y_1^{\,m}}{b^m}$$

$$\Rightarrow \left(\frac{x}{a}\right)\left(\frac{x_1}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1}=1 \quad ...(iii) \qquad \text{[Using (i)]}$$

If the line $x\cos\alpha+y\sin\alpha=p$ touches (i) at (x_1, y_1), then this line and (iii) must coincide.

$$\therefore \quad \frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{m-1}}{\cos\alpha}=\frac{\frac{1}{b}\left(\frac{y_1}{b}\right)^{m-1}}{\sin\alpha}=\frac{1}{p}$$

$$\Rightarrow \left(\frac{x_1}{a}\right)^{m-1}=\frac{a\cos\alpha}{p} \text{ and } \left(\frac{y_1}{b}\right)^{m-1}=\frac{b\sin\alpha}{p}$$

$$\Rightarrow \frac{x_1}{a}=\left(\frac{a\cos\alpha}{p}\right)^{1/m-1} \text{ and } \frac{y_1}{b}=\left(\frac{b\sin\alpha}{p}\right)^{1/m-1}$$

Subsituting the values of $\frac{x_1}{a}$ and $\frac{y_1}{b}$ in (ii), we get

$$\left(\frac{a\cos\alpha}{p}\right)^{m/(m-1)}+\left(\frac{b\sin\alpha}{p}\right)^{m/(m-1)}=1$$

$(a\cos\alpha)^{m/(m-1)}+(b\sin\alpha)^{m/(m-1)}=p^{m/(m-1)}$, which is the required condition.

EXAMPLE 21 *Find the condition that the line* $x\cos\alpha+y\sin\alpha=p$ *may touch the curve* $x^m y^n=a^{m+n}$.

SOLUTION Suppose the line $x\cos\alpha+y\sin\alpha=p$ touches the curve $x^m y^n=a^{m+n}$ at (x_1, y_1). Therefore,

$$x_1\cos\alpha+y_1\sin\alpha=p \qquad ...(ii)$$

and

$$x_1^m y_1^n=a^{m+n} \qquad ...(iii)$$

Now,

$$x^m y^n=a^{m+n}$$

$$\Rightarrow m\log x+n\log y=(m+n)\log a$$

$$\Rightarrow \quad \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{m}{n}\frac{y}{x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-\frac{m}{n}\frac{y_1}{x_1}.$$

The equation of the tangent to $x^m y^n=a^{m+n}$ at (x_1, y_1) is

$$y-y_1=-\frac{m}{n}\frac{y_1}{x_1}(x-x_1)$$

or, $(my_1)x+(nx_1)y=(m+n)x_1y_1$

or, $\left(\frac{m}{x_1}\right)x+\left(\frac{n}{y_1}\right)y=m+n.$

If the line $x\cos\alpha+y\sin\alpha=p$ touches $x^m y^n=a^{m+n}$ at (x_1, y_1), then this should be same as $x\cos\alpha+y\sin\alpha=p$.

$$\therefore \quad \frac{\frac{m}{x_1}}{\cos\alpha}=\frac{\frac{n}{y_1}}{\sin\alpha}=\frac{m+n}{p}$$

$$\Rightarrow \frac{m}{x_1\cos\alpha}=\frac{n}{y_1\sin\alpha}=\frac{m+n}{p}$$

$$\Rightarrow \quad x_1=\frac{mp}{(m+n)\cos\alpha}, y_1=\frac{np}{(m+n)\sin\alpha}$$

Substituting the values of m and n in (ii), we get

$$\left\{\frac{mp}{(m+n)\cos\alpha}\right\}^m\left\{\frac{np}{(m+n)\sin\alpha}\right\}^n=a^{m+n}$$

$$\Rightarrow m^m n^n p^{m+n}=(m+n)^{m+n}a^{m+n}(\cos\alpha)^m(\sin\alpha)^n.$$

This is the required condition.

EXAMPLE 22 *Find the condition that the line* $Ax+By=1$ *may be a normal to the curve* $a^{n-1}y=x^n$.

SOLUTION Let the line $Ax + By = 1$ be normal to the curve $a^{n-1}y = x^n$ at point (x_1, y_1). Then, (x_1, y_1) lies on the curve as well as on the line and the slope of the normal at (x_1, y_1) is same as the slope of the line.

Now,

$$a^{n-1}y = x^n$$

$$\Rightarrow a^{n-1}\frac{dy}{dx} = n x^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{n x^{n-1}}{a^{n-1}} = \frac{n x^{n-1}.y}{x^n} \qquad \left[\because a^{n-1}y = x^n,\ \therefore a^{n-1} = \frac{x^n}{y}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{ny}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{ny_1}{x_1}$$

The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{x_1}{ny_1}(x - x_1)$$

$$\Rightarrow xx_1 + nyy_1 = ny_1^2 + x_1^2$$

But the normal at (x_1, y_1) is given to be $Ax + By = 1$. Therefore, comparing these two, we obtain

$$\frac{x_1}{A} = \frac{ny_1}{B} = \frac{ny_1^2 + x_1^2}{1} = \lambda, \text{ say}$$

$$\Rightarrow x_1 = A\lambda, y_1 = \frac{B\lambda}{n} \text{ and } ny_1^2 + x_1^2 = \lambda$$

$$\Rightarrow \frac{n B^2\lambda^2}{n^2} + A^2\lambda^2 = \lambda$$

$$\Rightarrow (B^2 + A^2 n)\lambda = n$$

$$\Rightarrow \lambda = \frac{n}{B^2 + A^2 n}$$

$$\therefore x_1 = \frac{An}{B^2 + A^2 n} \text{ and } y_1 = \frac{B}{B^2 + A^2 n}$$

Since (x_1, y_1) lies on the curve $a^{n-1}y = x^n$. Therefore,

$$a^{n-1}y_1 = x_1^n$$

$$\Rightarrow \frac{a^{n-1}B}{B^2 + A^2 n} = \left(\frac{An}{B^2 + A^2 n}\right)^n$$

$\Rightarrow a^{n-1}B(B^2 + A^2 n)^{n-1} = (An)^n$, which is the required condition.

EXAMPLE 23 *Prove that the curves $y = f(x), (f(x) > 0)$ and $y = f(x)\sin x$, where $f(x)$ is differentiable function, have common tangents at common points.*

SOLUTION The equations of two curves are

$$y = f(x) \quad \ldots(i)$$

and,

$$y = f(x)\sin x \quad \ldots(ii)$$

Solving these two equations, we get

$$f(x) = f(x).\sin x$$

$$\Rightarrow \sin x = 1 \qquad [\because f(x) > 0]$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2}, n \in Z$$

$$\Rightarrow x = (4n+1)\frac{\pi}{2}, n \in Z.$$

Differentiating (i) and (ii) w.r.t. x, we get

$$\frac{dy}{dx} = f'(x) \text{ and } \frac{dy}{dx} = f'(x)\sin x + f(x)\cos x$$

$$\therefore m_1 = \text{Slope of the tangent to (i) at } x = (4n+1)\frac{\pi}{2}$$

$$= f'\left\{(4n+1)\frac{\pi}{2}\right\}$$

and,

$$m_2 = \text{Slope of the tangent to (ii) at } x = (4n+1)\frac{\pi}{2}$$

$$= f'\left\{(4n+1)\frac{\pi}{2}\right\}\sin\left\{(4n+1)\frac{\pi}{2}\right\} + f\left\{(4n+1)\frac{\pi}{2}\right\}\cos(4n+1)\frac{\pi}{2}$$

$$= f'\left\{(4n+1)\frac{\pi}{2}\right\}$$

Clearly, $m_1 = m_2$. Hence, the two curves have common tangents at common points.

EXAMPLE 24 *Find the equation of a straight line which is tangent to one point and normal to the other point on the curve $x = 4t^2 + 3, y = 8t^3 - 1$.*

SOLUTION The parametric equations of the given curve are

$$x = 4t^2 + 3 \text{ and } y = 8t^3 - 1$$

$$\therefore \frac{dx}{dt} = 8t \text{ and } \frac{dy}{dt} = 24t^2$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{24t^2}{8t} = 3t \quad \ldots(i)$$

The equation of the tangent to the given curve at any point t is given by

$$y - (8t^3 - 1) = 3t(x - 4t^2 - 3) \quad \ldots(ii)$$

If this tangent meets the curve again at t_1, then $(4t_1^2 + 3, 8t_1^3 - 1)$ must satisfy it.

$$\therefore 8t_1^3 - 1 - (8t^3 - 1) = 3t(4t_1^2 + 3 - 4t^2 - 3)$$

$$\Rightarrow 8(t_1^3 - t^3) = 12t(t_1^2 - t^2)$$

$$\Rightarrow 4(t_1 - t)\{2(t_1^2 + t^2 + tt_1) - 3t(t_1 + t)\} = 0$$

$\Rightarrow 4(t_1 - t)\{2t_1^2 - tt_1 - t^2\} = 0$

$\Rightarrow 4(t_1 - t)^2 (2t_1 + t) = 0$

$\Rightarrow t_1 = t$ or $t_1 = -\frac{t}{2}$.

Since $t_1 = t$ gives the point where tangent is drawn. Therefore, $t_1 = -\frac{t}{2}$.

Now, (Slope of the normal at t_1) $= \frac{-1}{\left(\frac{dy}{dx}\right)_{(t_1)}}$

$= \frac{-1}{3t_1}$ [Using (i)]

Since line given in (ii) is normal at t_1. Therefore, slope of normal at t_1 = slope of (ii)

$\therefore -\frac{1}{3t_1} = 3t$

$\Rightarrow 9t\,t_1 = -1 \Rightarrow -\frac{9t^2}{2} = -1$ $[\because t_1 = -t/2]$

$\Rightarrow t = \pm\sqrt{\frac{2}{9}} = \pm\frac{\sqrt{2}}{3}$.

Putting the values of t in (i), we obtain the following equations of the required straight lines.

$$\sqrt{2}x - y = \frac{89\sqrt{2}}{27} + 1 \text{ and } \sqrt{2}x + y = \frac{89\sqrt{2}}{27} - 1$$

EXAMPLE 25 *Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real points for $x > 1$. If the tangent at the point t is inclined at an angle θ with x-axis, prove that $\tan\theta + \sec\theta = 3t$. If the tangent at $P(-2, 2)$ meets the curve again at Q, prove that the tangents at P and Q are at right angles.*

SOLUTION We observe that the expression for x does not change when t is replaced by $-t$. So, the curve is symmetrical about x-axis. Also, for $x > 1$, the equation $x = 1 - 3t^2$ gives imaginary values of t. So, the curve has no real points for $x > 1$.

Now, $x = 1 - 3t^2$ and $y = t - 3t^3$

$\Rightarrow \frac{dx}{dt} = -6t$ and $\frac{dy}{dt} = 1 - 9t^2$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{\frac{dx}{dt}} = \frac{1 - 9t^2}{-6t}$ ...(i)

$\Rightarrow \tan\theta = \frac{9t^2 - 1}{6t}$ $\left[\because \text{The tangent makes an angle } \theta \text{ with } x\text{-axis}\right]$

$\Rightarrow 1 + \tan^2\theta = 1 + \left(\frac{9t^2 - 1}{6t}\right)^2$.

$\Rightarrow \sec^2\theta = \left(\frac{9t^2 + 1}{6t}\right)^2$

$\Rightarrow \sec\theta = \frac{9t^2 + 1}{6t}$

$\therefore \tan\theta + \sec\theta = \frac{9t^2 - 1}{6t} + \frac{9t^2 + 1}{6t} = 3t$.

When $x = -2$ and $y = 2$, we have

$1 - 3t^2 = -2$ and $t - 3t^3 = 2$

$\Rightarrow t = -1$

Putting $t = -1$ in (i), we have

$\left(\frac{dy}{dx}\right)_P = \frac{1 - 9}{6} = -\frac{4}{3}$

So, the equation of the tangent at $P(-2, 2)$,is

$$y - 2 = -\frac{4}{3}(x + 2) \text{ or } 4x + 3y + 2 = 0$$

Suppose this tangent meets the curve at t_1 i.e. at

$Q(1 - 3t_1^2, t_1 - 3t_1^3)$

$\therefore 4(1 - 3t_1^2) + 3(t_1 - 3t_1^3) + 2 = 0$

$\Rightarrow -9t_1^3 - 12t_1^2 + 3t_1 + 6 = 0$

$\Rightarrow 3t_1^3 + 4t_1^2 - t_1 - 2 = 0$

$\Rightarrow (t_1 + 1)^2 (3t_1 - 2) = 0$

$\Rightarrow t_1 = -1, t_1 = \frac{2}{3}$.

Clearly, $t_1 = -1$ corresponds to point P. So, $t_1 = \frac{2}{3}$.

Now,

m_1 = Slope of the tangent to the given curve at P

$= \frac{9 \times (-1)^2 - 1}{6 \times -1} = \frac{-4}{3}$ [Putting $t = -1$]

and,

m_2 = Slope of the tangent to the given curve at Q

$= \frac{9 \times (2/3)^2 - 1}{6 \times 2/3}$ [Putting $t = t_1 = \frac{2}{3}$]

$= \frac{4 - 1}{4} = \frac{3}{4}$.

Clearly, $m_1 m_2 = -1$

Hence, the tangents at P and Q are at right angles.

EXAMPLE 26 *Show that the curve $y = x^2 \sin 3x$ touches the parabolas $y = x^2$ and $y = -x^2$ alternately at infinite number of points.*

SOLUTION We know that two curves touch each other at a point of intersection iff their slopes at the point are equal.

First we find the points of intersection of $y = x^2 \sin 3x$ and $y = x^2$. At the points of intersection, we have

$$x^2 \sin 3x = x^2$$

$$\Rightarrow \quad \sin 3x = 1$$

$$\Rightarrow \quad 3x = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{2n\pi}{3} + \frac{\pi}{6}, n \in Z$$

Now,

$$y = x^2 \sin 3x$$

$$\Rightarrow \frac{dy}{dx} = 2x \sin 3x + 3x^2 \cos 3x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{at\ x = \frac{2n\pi}{3} + \frac{\pi}{6}} = 2\left(\frac{2n\pi}{3} + \frac{\pi}{6}\right) = \frac{4n\pi}{3} + \frac{\pi}{3} \quad ...(i)$$

and,

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{at\ x = \frac{2n\pi}{3} + \frac{\pi}{6}} = 2\left(\frac{2n\pi}{3} + \frac{\pi}{6}\right) = \frac{4n\pi}{3} + \frac{\pi}{3} \quad ...(ii)$$

From (i) and (ii), we find that the curves $y = x^2 \sin 3x$ and $y = x^2$ have the same slope at their points of intersection. So, they touch each other.

Let us now find the points of intersection of $y = x^2 \sin 3x$ and $y = -x^2$. At the points of intersection, we have

$$x^2 \sin 3x = -x^2$$

$$\Rightarrow \quad \sin 3x = -1$$

$$\Rightarrow \quad 3x = 2n\pi - \frac{\pi}{2}, n \in Z$$

$$\Rightarrow \quad x = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$$

Now,

$$y = x^2 \sin 3x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{at\ x = \frac{2n\pi}{3} - \frac{\pi}{6}} = -2\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) = -\frac{4n\pi}{3} + \frac{\pi}{3} \quad ...(iii)$$

and,

$$y = -x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{at\ x = \frac{2n\pi}{3} - \frac{\pi}{6}} = -2\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) = \frac{-4n\pi}{3} + \frac{\pi}{3} \quad ...(iv)$$

From (iii), (iv), we observe that the curves $y = x^2 \sin 3x$ and $y = -x^2$ have the same slopes at their points of intersection. Hence, they touch each other.

It is obvious that the sets of points given by $x = \frac{2n\pi}{3} + \frac{\pi}{6}$ and $x = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$ occur alternately.

Hence, $y = x^2 \sin 3x$ touches $y = x^2$ and $y = -x^2$ alternately at infinite number of points.

EXAMPLE 27 *For the function* $F(x) = \int_0^x 2\,|t|\,dt$, *find the tangent lines which are parallel to the bisector of the angle in the first quadrant.* **[IIT 1995]**

SOLUTION We have,

$$F(x) = \int_0^x 2\,|t|\,dt$$

$$F(x) = \begin{cases} \int_0^x 2t\,dt & , \text{if } x \geq 0 \\ -\int_0^x 2t\,dt & , \text{if } x < 0 \end{cases}$$

Thus,

$$F(x) = \begin{cases} x^2 & , \text{if } x \geq 0 \\ -x^2 & , \text{if } x < 0 \end{cases}$$

$$\therefore F'(x) = \begin{cases} 2x & , x \geq 0 \\ -2x & , \text{if } x < 0 \end{cases}$$

CASE I *When* $x \geq 0$

In this case, we have

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x.$$

It is given that the tangent lines are parallel to the bisector of the angle in the first quadrant.

$$\therefore \frac{dy}{dx} = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}.$$

Putting $x = \frac{1}{2}$ in $y = x^2$, we get $y = \frac{1}{4}$.

So, the equation of the tangent at $(1/2, 1/4)$ is

$$y - \frac{1}{4} = \left(x - \frac{1}{2}\right) \Rightarrow y = x - \frac{1}{4}.$$

CASE II *When* $x < 0$.

In this case, we have

$$y = -x^2 \Rightarrow \frac{dy}{dx} = -2x.$$

Since the tangent lines are parallel to the bisector of the angle in the first quadrant.

$$\therefore \frac{dy}{dx} = 1 \Rightarrow -2x = 1 \Rightarrow x = -\frac{1}{2}$$

Putting $x = -\frac{1}{2}$ in $y = -x^2$, we get $y = -\frac{1}{4}$.

So, the equation of the tangent at $(-1/2, -1/4)$ is

$$y + \frac{1}{4} = \left(x + \frac{1}{2}\right) \text{ or, } x - y + \frac{1}{4} = 0.$$

EXAMPLE 28 *Tangent at the point $P_1 \equiv (a, a^3+1), (a \neq 0)$ on the curve $y = x^3 + 1$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on. Find $\sum_{i=1}^{2n} x(P_i)$ and $\sum_{i=1}^{2n} y(P_i)$, where $x(P_i)$ and $y(P_i)$ are the abscissa and ordinate of P_i respectively.*

SOLUTION We have,

$$y = x^3 + 1$$

$$\Rightarrow \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a, a^3+1)} = 3a^2.$$

The equation of the tangent at $P_1(a, a^3+1)$ is

$$y - (a^3+1) = 3a^2(x - a) \quad \ldots(i)$$

This meets the curve at P_2. So, let the coordinates of P_2 be $(a_2, a_2^3 + 1)$.

As P_2 lies on (i), therefore

$$a_2^3 + 1 - (a^3+1) = 3a^2(a_2 - a)$$

$$\Rightarrow a_2^3 - a^3 = 3a^2(a_2 - a)$$

$$\Rightarrow (a_2 - a)^2 (a_2 + 2a) = 0$$

$$\Rightarrow a_2 = -2a \qquad [\because a_2 \neq a]$$

So, the coordinates of P_2 are $(-2a, -8a^3 + 1)$

Comparing this with the coordinates of P_1, we find that the coordinates of P_2 can be obtained from the coordinates of P_1 if we replace a by $-2a$.

Similarly, coordinates of P_3, P_4, ... are

$P_3(4a, (4a)^3 + 1)$, $P_4(-8a, (-8a)^3 + 1)$ and so on.

Now,

$$\sum_{i=1}^{2n} x(P_i) = a - 2a + 4a - 8a + \ldots \text{ upto } 2n \text{ terms}$$

$$= a\left\{\frac{1 - (-2)^{2n}}{1 - 2}\right\}$$

$$= \frac{a}{3}(1 - 4^n)$$

and,

$$\sum_{i=1}^{2n} y(P_i) = (a^3+1) + (-8a^3 + 1 + (64a^3 + 1) + \ldots 2n \text{ terms}$$

$$= a^3(1 - 8 + 64 + \ldots) + 2n$$

$$= a^3\left\{\frac{1 - (8)^{2n}}{1 - 8}\right\} + 2n$$

$$= \frac{a^3}{9}(1 - 64^n) + 2n.$$

EXERCISE 6.1

1. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

2. Find the tangents to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

3. Find the equations of the tangent and normal to the curve $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

4. Find the equations of the tangent and normal to the curve $2y = 3 - x^2$ at the point $(1, 1)$.

5. Find the equation of tangent and normal to the following curves at the points where x-coordinates are a :

(i) $y^2(2a - x) = x^3$ (ii) $y^2(a + x) = x^2(3a - x)$

6. Find the equations of the tangent and normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the axis of x.

7. Find the equations of the tangent and normal to the curve $y = x^3 - 2x^2 + 4$ at the point $x = 2$.

8. Find the equations of tangents to the curve $3x^2 + y^2 + x + 2y = 0$ which are perpendicular to the line $4x - 2y = 1$.

9. Find the equations of tangent and normal to the curve whose parametric equations are

$$x = \frac{2at^2}{1+t^2}, y = \frac{2at^3}{1+t^2} \text{ at the point } t = \frac{1}{2}$$

10. Show that the sum of the intercepts made on the axes of coordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is equal to a.

11. Find the point on the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ where the tangent is parallel to the line $y = 2x$. Show that two of these points have the same tangent.

12. Find the equation of the normal to the curve whose parametric equations are $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ at any point θ. Also, prove that the normal at a point $\theta = \pi/4$ passes through the origin.

13. If the tangent at any point $(1, 2)$ on the curve $y = ax^2 + bx + \frac{7}{2}$ be parallel to the normal at $(-2, 2)$ on the curve $y = x^2 + 6x + 10$, find the values of a and b.

14. Prove that the tangent at any point on the rectangular hyperbola $xy = c^2$, makes a triangle of constant area with the coordinate axes.

15. Prove that the segment of the tangent to $xy = c^2$ intercepted between the axes of coordinates is bisected at the point of contact.

16. Find the equation of the normal to the curve $x^3 + y^3 = 8xy$ at the point other than the origin where it meets the curve $y^2 = 4x$.

17. If α, β are the intercepts made on the axes by the tangent at any point of the curve $x = a\cos^3\theta, y = b\sin^3\theta$, prove that

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$$

18. Show that the tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$.

19. Find the abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts off equal intercepts from the axes.

20. Find the equations of tangents to the curve $y = x^4$ which are drawn from the point (2, 0).

21. Show that two tangents can be drawn from the point $A(2a, 3a)$ to the parabola $y^2 = 4ax$. Find the equations of these tangents.

22. Find the condition that the line $x\cos\alpha + y\sin\alpha = p$ may touch the curve

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\left(\frac{x}{a}\right)^{n/n-1} + \left(\frac{y}{b}\right)^{n/n-1} = 1$

(iii) $\left(\frac{x-a}{a}\right)^{n/n-1} + \left(\frac{y-b}{b}\right)^{n/n-1} = 1$

23. Find the abscissa of the point on the curve $xy = (x + c)^2$, the normal at which cuts off numerically equal intercepts from the axes of coordinates.

24. For the curve whose parametric equations are $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$ and $y = a\sin t$, show that the portion of the tangent intercepted between the point of contact and the x-axis is of constant length.

25. Prove that the portion of the tangent to the curve

$$x = \sqrt{a^2 - y^2} + \frac{a}{2}\log\frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$$

between the point of contact and the x-axis is of length a.

26. If $ax + by = 1$ is a normal to the parabola $y^2 = 4cx$, prove that

$$ca^3 + 2ab^3c = b^2$$

27. If x_1 and y_1 be the intercepts on the axes of X an Y cut off by the tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, then prove that

$$\left(\frac{a}{x_1}\right)^{n/n-1} + \left(\frac{b}{y_1}\right)^{n/n-1} = 1.$$

28. If (x_1, y_1) be the parts of the axes intercepted by the tangent at any point on the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, prove that

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 = 1.$$

29. The tangent at any point of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ meets the axes in P and Q. Prove that the locus of the mid-point of PQ is a circle.

30. If p_1 and p_2 be the lengths of perpendiculars from the origin on the tangent and normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ respectively, prove that $4p_1^2 + p_2^2 = a^2$.

31. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(ct_1, \frac{c}{t_1}\right)$ meets the curve again at the point $Q\left(ct_2, \frac{c}{t_2}\right)$, if $t_1^3 t_2 = -1$.

32. Show that the normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at $P(0, -3)$ meets the curve again at two points. Find the equations of tangents at these points.

33. If the tangent at the point (at^2, at^3) on the curve $ay^2 = x^3$ meets the curve again at Q, then prove that the coordinates of Q are $\left(\frac{at^2}{4}, -\frac{at^3}{8}\right)$

34. Find the equation of the straight line which is tangent at one point and normal at another point of the curve $x = 3t^2, y = 2t^3$.

ANSWERS

2. $3x + y = 3, 7x - y = 14$
3. Tangent : $x + y = 6$, Normal $x - y = 0$
4. Tangent $x + y = 2$, Normal $x - y = 0$
5. (i) At (a, a) Tangent : $2x - y - a = 0$, Normal : $x + 2y - 3a = 0$
   At $(a, -a)$ Tangent : $2x + y - a = 0, x - 2y - 3a = 0$
   (ii) At (a, a)
   Tangent : $x - 2y + a = 0$, Normal : $2x + y - 3a = 0$
   At $(a, -a)$
   Tangent : $x + 2y + a = 0, 2x - y - 3a = 0$
6. Tangent : $x - 20y - 7 = 0$, Normal $20x + y - 140 = 0$
7. Tangent : $4x - y = 4$, Normal : $x + 4y = 18$
8. $3x + 6y + 13 = 0, x + 2y = 0$
9. Tangent : $13x - 16y = 2a$, Normal : $16x + 13y = 9a$
11. $(1, 3), (2, 5)\ (3/2, 65/6)$
13. $a = 1, b = -\frac{5}{2}$
16. $y - x = 0$
19. $\frac{4a}{9}$
20. $y = 0, y - \left(\frac{8}{3}\right)^4 = 4\left(\frac{8}{3}\right)^3\left(x - \frac{8}{3}\right)$

21. $x - y + a = 0, x - 2y + 4a = 0$

22. (i) $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$

(ii) $(a \cos \alpha)^n + (b \sin \alpha)^n = p^n$

(iii) $(p - a \cos \alpha - b \sin \alpha)^n = (a \cos \alpha)^n + (b \sin \alpha)^n$

23. $x = \pm \frac{c}{\sqrt{2}}$

34. $y = \sqrt{2}x - 2\sqrt{2}, y = -\sqrt{2}x + 2\sqrt{2}$

HINTS TO SELECTED PROBLEMS

30. Write tangent and normal at any point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$

31. Euqate the s.ope of the normal at P to the slope of line PQ

32. The equation of the normal at $P(0, -3)$ is $x + 2y + 6 = 0$. The points of intersection of this normal and the curve are given by $5x^5 - 10x^3 = 0$ or, $x = 0, \pm \sqrt{2}$. $x = 0$ gives the point P. Hence, the normal meets the curve again at two points.

—❑ ❑ ❑ —

6.4 ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$ respectively. Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 respectively at their common point of intersection. Then the angle ϕ between PT_1 and PT_2 is the angle of intersection of C_1 and C_2. Let ψ_1 and ψ_2 be the angles made by PT_1 and

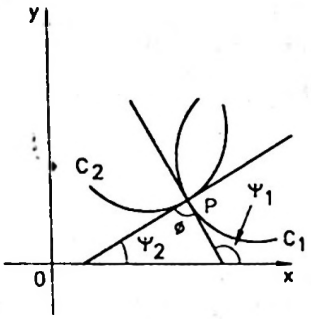


Fig. 6.3

PT_2 with the positive direction of x-axis in anticlockwise sense. Then

$m_1 = \tan \psi_1$ = Slope of the tangent to $y = f(x)$ at P

$$= \left(\frac{dy}{dx}\right)_{C_1}.$$

and,

$m_2 = \tan \psi_2$ = Slope of the tangent to $y = g(x)$ at P

$$= \left(\frac{dy}{dx}\right)_{C_2}$$

From Fig. 6.3, it follows that

$$\phi = \psi_1 - \psi_2.$$

$$\Rightarrow \tan \phi = \tan(\psi_1 - \psi_2)$$

$$\Rightarrow \tan \phi = \frac{\tan \psi_1 - \tan \psi_2}{1 + \tan \psi_1 \tan \psi_2}$$

$$\Rightarrow \tan \phi = \frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}$$

The other angle between the tangents is $180° - \phi$. *Generally the smaller of these two angles is taken to be the angle of intersection.*

ORTHOGONAL CURVES *If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.*

If the curves C_1 and C_2 are orthogonal, then $\phi = \pi/2$.

$$\therefore m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{C_1} \cdot \left(\frac{dy}{dx}\right)_{C_2} = -1$$

REMARK *The two curves* C_1 and C_2 *will touch each other at a point* P, *if* $\left(\frac{dy}{dx}\right)_{C_1} = \left(\frac{dy}{dx}\right)_{C_2}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the angle of intersection of the following curves:*

(i) $xy = 6$ and $x^2 y = 12$ (ii) $y^2 = 4x$ and $x^2 = 4y$

SOLUTION (i) The equations of the two curves are

$$xy = 6 \quad ...(i) \quad \text{and} \quad x^2 y = 12 \quad ...(ii)$$

From (i), we obtain $y = \frac{6}{x}$. Putting this value of y in (ii), we obtain

$$x^2\left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$$

Putting $x = 2$ in (i) or (ii), we get $y = 3$. Thus, the two curves intersect at $P(2, 3)$

Differentiating (i) w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(2,3)} = -\frac{3}{2}$$

Differentiating (ii) w.r.t x, we get

$$x^2\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(2,3)} = -3$$

Let θ be the angle of intersection of (i) and (ii), then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-(3/2) + 3}{1 + (-3/2)(-3)} = \frac{3}{11}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{11}\right)$$

(ii) The equations of the two curves are

$$y^2 = 4x \quad ...(i) \qquad \text{and} \qquad x^2 = 4y \quad ...(ii)$$

From (i), we obtain $x = \frac{y^2}{4}$. Putting $x = \frac{y^2}{4}$ in (ii), we get

$$\left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0 \Rightarrow y = 0, y = 4$$

From (i), when $y = 0$, we get $x = 0$ and when $y = 4$, we get $x = 4$.

Thus the two curves intersect at (0, 0) and (4, 4).

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y} \quad ...(iii)$$

Differentiating (ii) w.r.t. x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \quad ...(iv)$$

Angle of Intersection at (0, 0) :

From (iii), we have,

$$m_1 = \left(\frac{dy}{dx}\right)_{(0,0)} = \infty.$$

Therefore the tangent to (i) curve at (0,0) is parallel to y-axis.

From (iv), we have

$$m_2 = \left(\frac{dy}{dx}\right)_{(0,0)} = 0$$

Therefore the tangent to (ii) curve at (0, 0) is parallel to x-axis.

Hence the angle between the tangents to two curves at (0, 0) is a right angle. Consequently the two curves intersect at right angle at (0, 0).

Angle of Intersection at (4, 4) :

From (iii), we have

$$m_1 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{2}{4} = \frac{1}{2}.$$

From (iv), we have,

$$m_2 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{4}{2} = 2.$$

Let θ be the angle of intersection of the two curves. Then

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2 - (1/2)}{1 + 2 \times (1/2)}\right| = \frac{3}{4}.$$

EXAMPLE 2 *Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.*

SOLUTION Graphs of the two curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ are shown in Fig. 6.4

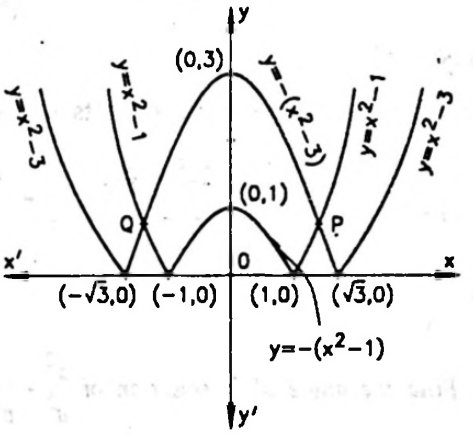


Fig. 6.4

Clearly, two curves intersect at P and Q. Point P and Q are the intersection point of $y = x^2 - 1$ and $y = -(x^2 - 3)$. Solving these two equations simultaneously, we get

$$x = \pm\sqrt{2} \text{ and } y = \pm 1.$$

Thus, the coordinates of P and Q are $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$ respectively.

Angle of intersection at $(\sqrt{2}, 1)$:

The equations of the two curves are

$$y = x^2 - 1 \quad ...(i)$$

and $$y = -(x^2 - 3) \quad ...(ii)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x \quad \text{[For curve (i)]}$$

and, $$\left(\frac{dy}{dx}\right) = -2x \quad \text{[For curve (ii)]}$$

Let m_1 and m_2 be the slopes of the tangents at point P to curve (i) and (ii) respectively. Then,

$$m_1 = \left(\frac{dy}{dx}\right)_P = 2\sqrt{2} \text{ and } m_2 = \left(\frac{dy}{dx}\right)_P = -2\sqrt{2}$$

Let θ be the acute angle of intersection. Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 + 2\sqrt{2} \times -2\sqrt{2}}\right| = \frac{4\sqrt{2}}{7}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Angle of intersection at $Q(-\sqrt{2}, 1)$:

In this case, we have

$$m_1 = \text{Slope of the tangent to (i) curve at } Q = \left(\frac{dy}{dx}\right)_Q$$

$$= -2\sqrt{2}$$

and,

$$m_2 = \text{slope of the tangent to (ii) curve at } Q = \left(\frac{dy}{dx}\right)_Q$$

$$= 2\sqrt{2}$$

So, the acute angle θ_1 between the tangents is given by

$$\tan\theta_1 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 + (-2\sqrt{2}) \times (2\sqrt{2})}\right| = \frac{4\sqrt{2}}{7}$$

$$\Rightarrow \theta_1 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

EXAMPLE 3 *Find the angle of intersection of* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and* $x^2 + y^2 = ab$.

SOLUTION The equations of the two curves are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

and,

$$x^2 + y^2 = ab \qquad \ldots(ii)$$

Solving (i) and (ii) by cross-multiplication, we obtain

$$\frac{x^2}{\frac{-ab}{b^2} + 1} = \frac{y^2}{-1 + \frac{ab}{a^2}} = \frac{1}{\frac{1}{a^2} - \frac{1}{b^2}}$$

$$\Rightarrow \frac{x^2}{\left(\frac{b-a}{b}\right)} = \frac{y^2}{\left(\frac{b-a}{a}\right)} = \frac{a^2 b^2}{b^2 - a^2}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{a^2 b}{a+b}} \text{ and } y = \pm\sqrt{\frac{ab^2}{a+b}}$$

Thus, the two curves intersect at four points given by

$$\left(\pm\sqrt{\frac{a^2 b}{a+b}}, \pm\sqrt{\frac{ab^2}{a+b}}\right)$$

Differentiating (i) and (ii) w.r.t. x, we get

$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{-b^2 x}{a^2 y} \text{ and } \left(\frac{dy}{dx}\right)_{C_2} = \frac{-x}{y}$$

Angle of intersection at $P\left(\sqrt{\frac{a^2 b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$:

We have,

m_1 = Slope of the tangent at P to (i)

$$= \frac{-b^2}{a^2}\sqrt{\frac{a}{b}} = \frac{-b^{3/2}}{a^{3/2}}$$

and, m_2 = Slope of the tangent at P to (ii)

$$= -\sqrt{\frac{a}{b}}$$

Let θ be the acute angle of intersection at P. Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{-b^{3/2}}{a^{3/2}} + \sqrt{\frac{a}{b}}}{1 + \frac{b}{a}}\right|$$

$$= \left|\frac{(-b^2 + a^2)}{(a+b)\sqrt{ab}}\right|$$

$$= \frac{b-a}{\sqrt{ab}} \qquad [\because b > a]$$

Similarly, we can find the angles of intersection at the remaining points also.

EXAMPLE 4 *Show that the condition that the curves*

$$ax^2 + by^2 = 1 \qquad \ldots(i)$$

and

$$a' x^2 + b' y^2 = 1 \qquad \ldots(iii)$$

should intersect orthogonally is that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

SOLUTION Let (x_1, y_1) be the point of intersection of the curves. Then,

$$ax_1^2 + by_1^2 = 1 \qquad \ldots(iii)$$

$$a' x_1^2 + b' y_1^2 = 1 \qquad \ldots(iv)$$

Differentiating (i) w.r.t. x, we get

$$2ax + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax}{by}$$

$$\Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{ax_1}{by_1} \qquad \ldots(v)$$

Differentiating (ii) w.r.t. x, we obtain

$$2a' x + 2b' y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a' x}{b' y}$$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a' x_1}{b' y_1} \qquad \ldots(vi)$$

The two curves will intersect orthogonally, if

$m_1 m_2 = -1$

$\Rightarrow \frac{ax_1}{by_1} \times -\frac{a' x_1}{b' y_1} = -1$

$\Rightarrow aa' x_1^2 = -bb' y_1^2$ ...(vii)

Subtracting (iv) from (iii), we obtain

$$(a - a') x_1^2 = -(b - b') y_1^2 \quad \text{...(viii)}$$

Dividing (viii) by (vii), we get

$\frac{a - a'}{aa'} = \frac{b - b'}{bb'}$

$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

EXAMPLE 5 *If the curves* $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ *and* $y^3 = 16x$ *intersect at right angles, show that* $a^2 = \frac{4}{3}$.

SOLUTION Let (x_1, y_1) be the point of intersection of the two curves.

Equations of two curves are $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$

Differentiating these w.r.t. x, we get

$\frac{2x}{a^2} + \frac{2y}{4}\frac{dy}{dx} = 0$ and $3y^2 \frac{dy}{dx} = 16$

$\Rightarrow \left(\frac{dy}{dx}\right)_{C_1} = \frac{-4x}{a^2 y}$ and $\left(\frac{dy}{dx}\right)_{C_2} = \frac{16}{3y^2}$

Let m_1 and m_2 be the slopes of the tangents to the two curves at their point of intersection. Then,

$m_1 = \left(\frac{dy}{dx}\right)_{C_1}$ at (x_1, y_1)

$= \frac{-4x_1}{a^2 y_1}$

$m_2 = \left(\frac{dy}{dx}\right)_{C_2}$ at (x_1, y_1)

$= \frac{16}{3y_1^2}$

If the two curves intersect at right angles, then

$m_1 m_2 = -1$

$\Rightarrow \frac{-4x_1}{a^2 y_1} \times \frac{16}{3y_1^2} = -1$

$\Rightarrow 64 x_1 = 3a^2 y_1^3$

$\Rightarrow 64 x_1 = 3a^2 \times 16x_1$ $[\because (x_1, y_1)$ lies on $y^3 = 16x \therefore y_1^3 = 16x_1]$

$\Rightarrow 4 = 3a^2$

$\Rightarrow a^2 = 4/3.$

EXAMPLE 6 *Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve*

$$\log (x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right)$$

SOLUTION We have,

$$\log x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right)$$

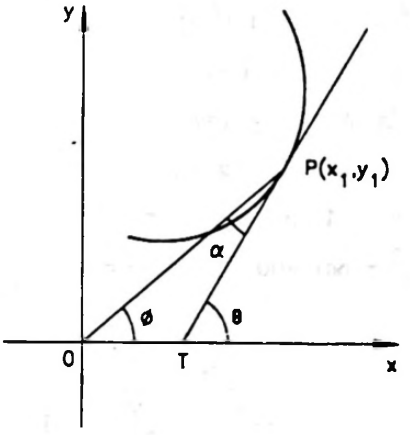


Fig. 6.5

Differentiating w.r.t. x, we get

$$\frac{1}{x^2 + y^2}\left\{2x + 2y\frac{dy}{dx}\right\} = k \frac{1}{1 + \frac{y^2}{x^2}}\left\{\frac{x\frac{dy}{dx} - y}{x^2}\right\}$$

$\Rightarrow 2\left\{x + y\frac{dy}{dx}\right\} = k\left\{x\frac{dy}{dx} - y\right\}$

$\Rightarrow 2x + ky = (kx - 2y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x + ky}{kx - 2y}$

Let the coordinates of P be (x_1, y_1). Then,

$$\left(\frac{dy}{dx}\right)_P = \frac{2x_1 + ky_1}{kx_1 - 2y_1}$$

If the tangent at P makes an angle θ with x-axis, then

$$\tan\theta = \frac{2x_1 + ky_1}{kx_1 - 2y_1}$$

Suppose OP makes an angle ϕ with x-axis. Then,

$\tan\phi$ = Slope of $OP = \frac{y_1}{x_1}$

Let α be the angle between OP and PT. Then,

$\theta = \alpha + \phi$

$\Rightarrow \quad \alpha = \theta - \phi$

$\Rightarrow \tan\alpha = \tan(\theta - \phi)$

$$\Rightarrow \tan\alpha = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

$$\Rightarrow \tan\alpha = \frac{\frac{2x_1 + ky_1}{kx_1 - 2y_1} - \frac{y_1}{x_1}}{1 + \frac{2x_1 + ky_1}{kx_1 - 2y_1} \times \frac{y_1}{x_1}}$$

$$\Rightarrow \tan\alpha = \frac{2x_1^2 + kx_1y_1 - kx_1y_1 + 2y_1^2}{kx_1^2 - 2x_1y_1 + 2x_1y_1 + ky_1^2} = \frac{2}{k}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{2}{k}\right) = \text{constant.}$$

EXERCISE 6.2

1. Find the angle of intersection of the following curves :
   (i) $x^2 + y^2 = \sqrt{2}a^2$ and $x^2 - y^2 = a^2$
   (ii) $x^2 = y$ and $x^2 = 4 - y$
   (iii) $y^2 = 2x$ and $x^2 = 16y$
   (iv) $x^2 + y^2 = 2a^2$ and $xy = a^2$
   (v) $2x^2 + y^2 = 4$ and $y^2 = 16x$
   (vi) $x^2 + y^2 = 8ax$ and $y^2(4a - x) = x^3$

2. Show that the curves $\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1$ and, $\frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1$ intersect orthogonally.

3. Find the condition so that the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ may intersect orthogonally.

4. Show that the curves $x^3 - 3xy^2 = -2$ and $3x^2y - y^3 = 2$ cut orthogonally.

5. Find the acute angles between the curves
$$y = |2x^2 - 4| \text{ and } y = |x^2 - 5|.$$

6. Show that the curves $y^2 = 4ax$ and $ay^2 = 4x^3$ intersect each other at an angle of $\tan^{-1}\frac{1}{2}$ and also if PG_1 and PG_2 be the normals to two curves at common point of intersection (other than the origin) meeting the axis of X in G_1 and G_2, then $G_1G_2 = 4a$.

7. Prove that the curves $y = e^{-ax}$ and $y = e^{-ax}\sin bx$ touch at the points $x = \frac{(4n+1)\pi}{2b}$.

8. Find the angle of intersection of curves
$y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where $[\cdot]$ denotes the greatest integer function.

ANSWERS

1. (i) $\theta = \frac{\pi}{4}$ (ii) $\pm\tan^{-1}\frac{4\sqrt{2}}{7}$ (iii) $\frac{\pi}{2}$ and $\pm\tan^{-1}\frac{7}{6}$
   (iv) 0 (v) $\frac{\pi}{2}$ (vi) $\tan^{-1}\sqrt{2}$

3. $a^2 - b^2 = A^2 - B^2$

5. $\pm\tan^{-1}\left(\frac{6\sqrt{3}}{23}\right)$

8. $\tan^{-1}2$

HINTS TO SELECTED PROBLEMS

4. Let (x_1, y_1) be the point of intersection of the two curves. Then,
$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{x_1^2 - y_1^2}{2x_1y_1} \text{ and } \left(\frac{dy}{dx}\right)_{C_2} = \frac{-2x_1y_1}{x_1^2 - y_1^2}.$$
Clearly, $\left(\frac{dy}{dx}\right)_{C_1} \times \left(\frac{dy}{dx}\right)_{C_2} = -1$

7. Show that the values of $\frac{dy}{dx}$ obtained from the two curves at $x = \frac{(4n+1)\pi}{2b}$ are equal.

8. We know that $1 \le |\sin x| + |\cos x| \le \sqrt{2}$ for all $x \in R$
$\therefore \; y = [|\sin x| + |\cos x|] = 1$
Thus, the two curves are $y = 1$ and $x^2 + y^2 = 5$

—❑ ❑ ❑ —

6.5 LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

Let the tangent and normal at a point $P(x, y)$ on the curve $y = f(x)$, meet the x-axis at T and N respectively. If G is the foot of the ordinate at P, then TG and GN are called the cartesian subtangent and subnormal, while the lengths PT and PN are called the lengths of the tangent and normal respectively.

If PT makes angle ψ with x-axis, then $\tan\psi = \frac{dy}{dx}$. From Fig.6.6, we find that

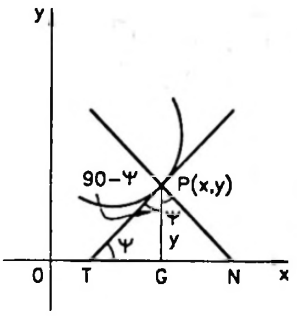


Fig. 6.6

$$\text{Subtangent} = TG = y\cot\psi = \frac{y}{\left(\frac{dy}{dx}\right)}$$

$$\text{Subnormal} = GN = y\tan\psi = y\frac{dy}{dx}$$

$$\text{Length of the tangent} = PT = y\,\text{cosec}\,\psi = y\sqrt{1+\cot^2\psi} = \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$$

$$\text{Length of the normal} = PN = y\sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the lengths of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.*

SOLUTION We have,

$$y^2 = 4ax$$

$$\Rightarrow 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{at^2,\,2at} = \frac{2a}{2at} = \frac{1}{t}.$$

Now,

$$\text{Length of the tangent at } (at^2, 2at) = y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} = \frac{2at\sqrt{1+1/t^2}}{1/t} = 2at^2\cdot\frac{\sqrt{t^2+1}}{t} = 2at\sqrt{t^2+1}$$

$$\text{Length of the normal at } (at^2, 2at) = y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2at\sqrt{1+\frac{1}{t^2}} = 2a\sqrt{t^2+1}$$

$$\text{Length of the subtangent at } (at^2, 2at) = \frac{y}{dy/dx} = \frac{2at}{1/t} = 2at^2$$

$$\text{Length of the subnormal at } (at^2, 2at) = y\frac{dy}{dx} = 2at\times\frac{1}{t} = 2a.$$

EXAMPLE 2 *Prove that mth power of subtangent at any point on the curve $x^{m+n} = a^{m-n}y^{2n}$ varies as the nth power of subnormal at the same point.*

SOLUTION We have,

$$x^{m+n} = a^{m-n}y^{2n}$$

$$\Rightarrow (m+n)\log x = (m-n)\log a + 2n\log y$$

$$\Rightarrow \frac{m+n}{x} = 0 + \frac{2n}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{m+n}{2n}\left(\frac{y}{x}\right)$$

$\therefore\ l_1$ = length of the subtangent at (x, y)

$$= \frac{y}{\frac{dy}{dx}} = \frac{2nx}{(m+n)}$$

and,

l_2 = length of the subnormal at (x, y)

$$= y\frac{dy}{dx} = \frac{(m+n)y^2}{2nx}$$

Now,

$$l_1^m = \left(\frac{2nx}{(m+n)}\right)^m = \frac{(2n)^m x^m}{(m+n)^m}$$

and,

$$l_2^n = \left(\frac{(m+n)y^2}{2nx}\right)^n = \frac{(m+n)^n y^{2n}}{(2n)^n x^n}$$

$$\therefore \frac{l_1^m}{l_2^n} = \frac{(2n)^{m+n} x^{m+n}}{(m+n)^{m+n} y^{2n}}$$

$$= \frac{(2n)^{m+n}}{(m+n)^{m+n}} a^{m-n} \qquad [\because x^{m+n} = a^{m-n} y^{2n}]$$

$$= \text{constant}$$

Hence, l_1^m varies as l_2^n.

EXERCISE 6.3

1. If the subnormal of the curve $xy^n = a^{n+1}$ is of constant length, prove that $n = -2$.

2. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that the length of the normal at any point varies inversely as perpendicular from origin on any tangent at the same point.

3. For the curve $x^m y^n = a^{m+n}$, prove that the subtangent at any point varies as the abscissa of the point.

4. Find the lengths of tangent and normal at point $t = \pi/3$ on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

5. Prove that for the curve $y = a \log (x^2 - a^2)$, sum of the tangent and subtangent varies as the product of the coordinates of the point of contact.

ANSWERS

4. $a, a/\sqrt{3}$

CHAPTER

7

MEAN VALUE THEOREMS AND SOME OTHER APPLICATIONS OF DERIVATIVES

7.1 ROLLE'S THEOREM

STATEMENT *Let f be a real valued function defined on the closed interval $[a, b]$ such that*

(i) *it is continuous on the closed interval $[a, b]$,*

(ii) *it is differentiable on the open interval (a, b), and*

(iii) $f(a) = f(b)$.

Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical interpretation of Rolle's theorem Let $f(x)$ be a real valued function defined on $[a, b]$ such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ and it is possible to draw a unique tangent at every point on the curve $y = f(x)$ between points A and B. Also, the ordinates at the end points of the interval $[a, b]$ are equal. Then, there exists at least one point $(c, f(c))$ lying between A and B on the curve $y = f(x)$ where tangent is parallel to x-axis.

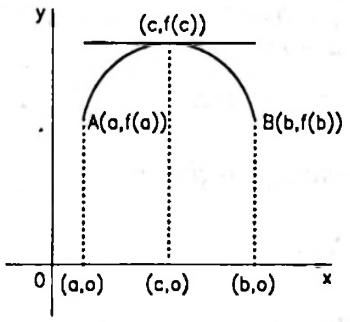


Fig. 7.1

Algebraic interpretation of Rolle's theorem : Let $f(x)$ be a polynomial with a and b as its roots. Since a polynomial function is everywhere continuous and differentiable and a and b are roots of $f(x)$, therefore $f(a) = f(b) = 0$. So, $f(x)$ satisfies conditions of Rolle's theorem. Consequently there exists $c \in (a, b)$ such that $f'(c) = 0$ i.e. $f'(x) = 0$ at $x = c$. In other words $x = c$ is a root of $f'(x)$. Thus, algebraically Rolle's theorem can be interpreted as follows :

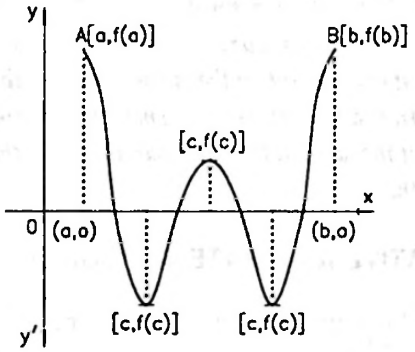


Fig. 7.2

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

7.2 LAGRANGE'S MEAN VALUE THEOREM

STATEMENT *Let $f(x)$ be a function defined on $[a, b]$ such that*

(i) *it is continuous on $[a, b]$,*

(ii) *it is differentiable on (a, b).*

Then there exists a real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation : Let $f(x)$ be a function defined on $[a, b]$, and let APB be the curve represented by $y = f(x)$. Then coordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively. Suppose the chord AB makes an angle ψ with the axis of x. Then from the triangle ARB, we have

$$\tan \psi = \frac{BR}{AR}$$

$\Rightarrow \tan\psi = \dfrac{f(b)-f(a)}{b-a}$

By Lagrange's Mean Value theorem, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\therefore \quad \tan\psi = f'(c)$

$\Rightarrow$ Slope of the chord AB = Slope of the tangent at $(c, f(c))$

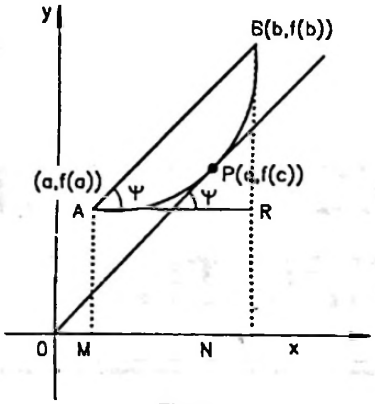


Fig. 7.3

Thus, we arrive at the following geometrical interpretation of Lagrange's mean value theorem:

Let $f(x)$ be a function defined on $[a, b]$ such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ and at every point on the curve, except at the end points, it is possible to draw a unique tangent. Then there exists a point on the curve such that the tangent thereat is parallel to the chord joining the end points of the curve.

7.3 DERIVATIVE AS A RATE MEASURER

Let $y = f(x)$ be a function of x. Let Δy be the change in y corresponding to a small change Δx in x. Then, $\dfrac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. In other words, $\dfrac{\Delta y}{\Delta x}$ represents the average rate of change of y w.r.t. x as x changes from x to $x + \Delta x$.

As $\Delta x \to 0$, the limiting value of this average rate of change of y with respect to x in the interval $[x, x + \Delta x]$ becomes the instantaneous rate of change of y w.r.t. x.

Thus,

$\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ = Instantaneous rate of change of y w.r.t. x

$\Rightarrow \quad \dfrac{dy}{dx}$ = Rate of change of y w.r.t. x $\quad \left[\because \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}\right]$

The word "instantaneous" is often dropped.

Hence, $\dfrac{dy}{dx}$ represents the rate of change of y w.r.t. x for a definite value of x.

7.4 DIFFERENTIALS, ERRORS AND APPROXIMATIONS

In the chapter on differentiation, we have defined derivative o y w.r.t. x i.e. $\dfrac{dy}{dx}$ as the limit of the ratio $\dfrac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ anc considered $\dfrac{dy}{dx}$ as a symbol not as a quotient of two separate quantities dy and dx. In this chapter, we shall give a meaning to the symbols dx and dy in such a way that the origina meaning of the symbol dy/dx coincides with the quotient wher dy is divided by dx.

Let $y = f(x)$ be a function of x, and let Δx be a small change in x. Let Δy be the corresponding change in y. Then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$\Rightarrow \dfrac{\Delta y}{\Delta x} = f'(x) + \varepsilon$, where $\varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow \quad \Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x \Rightarrow \quad \Delta y = f'(x)\,\Delta x$, approximately

or $\Delta y = \dfrac{dy}{dx}\Delta x$, approximately $\quad \left[\because f'(x) = \dfrac{dy}{dx}\right.$

This formula is very useful in the calculation of small changes (or errors) in dependent variable corresponding to small changes (or errors) in the independent variable and is of great importance in the theory of errors in Engineering, Physics, Statistics and several other branches of the science.

SOME IMPORTANT TERMS

Absolute error : The error Δx in x is called the absolute error in x.

Relative error : If Δx is an error in x, then $\dfrac{\Delta x}{x}$ is called the relative error in x

Percentage error : If Δx is an error in x, then $\dfrac{\Delta x}{x} \times 100$ is called percentage error in x.

REMARK 1 *We have $\Delta y = f'(x) \cdot \Delta x + \varepsilon \cdot \Delta x$. Since $\varepsilon \cdot \Delta x$ is very small, therefore principal value of Δy is $f'(x)\,\Delta x$ which is called differential of y and is denoted by dy i.e.*

$$dy = f'(x)\,\Delta x \text{ or } dy = \frac{dy}{dx} \cdot \Delta x$$

So, the differential of x is given by

$$dx = \frac{dx}{dx} \cdot \Delta x = 1 \cdot \Delta x = \Delta x$$

$$\therefore \quad dy = \frac{dy}{dx}\,dx$$

REMARK 2 *Let $y = f(x)$ be a function of x, and let Δx be a small change in x. Let the corresponding change in y be Δy. Then*

$$y + \Delta y = f(x + \Delta x)$$

But, $\Delta y = \dfrac{dy}{dx} \cdot \Delta x = f'(x)\,\Delta x$, approximately

$$\therefore \quad f(x + \Delta x) = y + \Delta y = y + f'(x) \cdot \Delta x, \text{ approximately}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $a_0, a_1, a_2, \sim\ldots, a_n$ are real numbers such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0$, $(n \neq -1)$, then prove that the equation $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0$ has a real root lying between 0 and 1.*

SOLUTION Consider the polynomial

$$f(x) = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots + a_n \frac{x^{n+1}}{n+1} \quad \ldots(i)$$

We have,

$f(0) = 0$,

and,

$$f(1) = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0 \quad \text{[By hypothesis]}$$

Therefore, 0 and 1 are roots of $f(x)$.

Hence, by the algebraic interpretation of Rolle's theorem

$$f'(x) = 0 \text{ i.e. } a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0$$

must have a root between 0 and 1.

EXAMPLE 2 *Prove that there is no value of λ for which the equation $x^3 - 3x + \lambda = 0$ has two distinct roots lying between 0 and 1.*

SOLUTION If possible, let there be a value of λ for which the equation $x^3 - 3x + \lambda = 0$ has two distinct roots α and β lying between 0 and 1 such that $\alpha < \beta$.

Let $f(x) = x^3 - 3x + \lambda$.

Since α and β are roots of $f(x) = 0$. So, by the algebraic interpretation of Rolle's theorem, the equation $f'(x) = 0$ must have a root γ (say) between α and β. But,

$$f'(x) = 3x^2 - 3$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

We have, $0 < \alpha < \beta < 1$. Therefore, none of the roots of $f'(x) = 0$ lie between α and β.

This is a contradiction to the fact that $f'(x) = 0$ must have a root between any two roots of $f(x) = 0$.

Thus, our supposition is wrong. Hence, there is no value of λ for which the equation $x^3 - 3x + \lambda = 0$ has two distinct roots lying between 0 and 1.

ALITER Consider $f(x) = x^3 - 3x$ and $g(x) = -\lambda$

Solving the equation $x^3 - 3x + \lambda = 0$ means, finding the coordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$. For this, let us draw the graphs of $y = f(x)$ and $y = g(x)$.

It is evident from the graphs of $y = x^3 - 3x$ and $y = -\lambda$ that :

(i) Two curves will intersect at exactly one point if $|-\lambda| > 2$ i.e. $\lambda < -2$ or $\lambda > 2$ and in this case the equation $x^3 - 3x + \lambda = 0$ will have exactly one root which will be either greater than $\sqrt{3}$ or less than $\sqrt{3}$.

(ii) If $0 < \lambda < 2$; then the two curves will intersect at three points which will mean that $x^3 - 3x + \lambda = 0$ will have three real roots. One root will be less than $\sqrt{3}$, one will be between 0 and 1 and the third will be greater than 1 but less than $\sqrt{3}$.

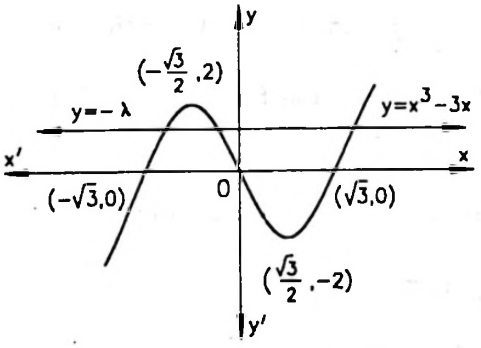


Fig. 7.4

(iii) If $-2 < \lambda < 0$, then also the two curves intersect at three points. This means that the given equation has three roots two of which lie between $-\sqrt{3}$ and 0 and the third will be greater than $\sqrt{3}$.

It follows from the above discussion that there is no value of λ for which the equation has exactly two roots lying between 0 and 1.

EXAMPLE 3 *Use Rolle's theorem to find the condition for the polynomial equation $f(x) = 0$ to have a repeated real root. Hence, or otherwise prove that the equation*

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} = 0$$

cannot have repeated roots.

SOLUTION By the algebraic interpretation of Rolle's theorem, we can say that between any two roots of a polynomial there is always a root of its derivative. Thus, if α is a repeated root of a polynomial $f(x)$, then there must be a root of $f'(x)$ in the interval (α, α). This means that α is a root of $f'(x) = 0$

$$\therefore \quad f'(\alpha) = 0$$

Thus, if α is a repeated root of a polynomial $f(x) = 0$, then $f(\alpha) = 0$ and $f'(\alpha) = 0$.

If possible, let $\phi(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!}$ have a repeated root α.

Then,

$$\phi(\alpha) = 0 \text{ and } \phi'(\alpha) = 0$$

$$\Rightarrow 1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} = 0 \text{ and } 1 + \alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^{n-1}}{(n-1)!} = 0$$

$$\Rightarrow \frac{\alpha^n}{n!} = 0 \quad \text{[Subtracting the two equations]}$$

$$\Rightarrow \alpha = 0$$

Thus, 0 is a repeated root of $\phi(x) = 0$.

But, 0 does not satisfy $\phi(x) = 0$ i.e. it is not a root of $\phi(x) = 0$.

Hence, $1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} = 0$ cannot have a repeated root.

EXAMPLE 5 *Using Lagrange's theorem prove that*

$$\frac{x}{1+x} < \log(1+x) < x \quad \text{for all } x > 0$$

SOLUTION Consider the function $f(t) = \log(1+t)$ for all $t \in [0, x]$, where $x > 0$.

Clearly, $f(t)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. So, by Lagrange's theorem there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \quad \frac{1}{1+c} = \frac{\log(1+x) - 0}{x - 0}$$

$$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+c}$$

Clearly, $\frac{1}{1+c} < 1$ for $c > 0$.

$$\therefore \quad \frac{\log(1+x)}{x} < 1 \Rightarrow \log(1+x) < x \qquad ...(i)$$

Also, $c \in (0, x)$

$$\Rightarrow \quad c < x$$

$$\Rightarrow \quad \frac{1}{1+c} > \frac{1}{1+x}$$

$$\Rightarrow \quad \frac{\log(1+x)}{x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \log(1+x) > \frac{x}{1+x} \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{x}{1+x} < \log(1+x) < x \quad \text{for all } x > 0.$$

EXAMPLE 6 *Show that the square roots of two successive natural numbers greater than N^2 differ by less than $\frac{1}{2N}$.*

SOLUTION Let n be a natural number greater than N^2.

Consider the function $f(x) = \sqrt{x}$ defined on the interval $[n, n+1]$. Clearly, $f(x)$ satisfies all the conditions of Lagrange's mean value theorem on $[n, n+1]$. So, there exists $c \in (n, n+1)$ such that

$$f'(c) = \frac{f(n+1) - f(n)}{(n+1) - n}$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \sqrt{n+1} - \sqrt{n}$$

Now, $n > N^2$ and $c \in (n, n+1)$

$$\therefore \quad c > N^2$$

$$\sqrt{c} > N$$

$$\Rightarrow \quad 2\sqrt{c} > 2N$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}} < \frac{1}{2N}$$

$$\Rightarrow \quad \sqrt{n+1} - \sqrt{n} < \frac{1}{2N}$$

EXAMPLE 7 *If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$ and they are differentiable in (a, b), then prove that there exists $c \in (a, b)$ such that*

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

SOLUTION Consider the function $\phi(x) = f(a)\,g(x) - f(x)\,g(a)$ for all $x \in [a, b]$.

As $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b) therefore $\phi(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Consequently, there exists $c \in (a, b)$ such that

$$\phi'(c) = \frac{\phi(b) - \phi(a)}{b - a}$$

Now,

$$\phi(x) = f(a)\,g(x) - f(x)\,g(a)$$

$$\Rightarrow \phi'(x) = f(a)\,g'(x) - f'(x)\,g(a)$$

$$\Rightarrow \phi'(c) = f(a)\,g'(c) - f'(c)\,g(a) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Also,

$$\phi(b) = f(a)\,g(b) - f(b)\,g(a) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

and,

$$\phi(a) = f(a)\,g(a) - f(a)\,g(a) = 0.$$

$$\therefore \quad \phi'(c) = \frac{\phi(b) - \phi(a)}{b - a}$$

$$\Rightarrow \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} = \frac{1}{b-a} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

EXAMPLE 8 *A ladder 15 m long, leans againsnt a wall 7 m high, and a portion of the ladder protrudes over the wall such that its projection along the vertical is 3 m. How fast does the bottom start to slip away from the wall if the ladder slides down along the top edge of the wall at 2 m/sec.*

SOLUTION Let OC be the wall and AB be the ladder of which the portion BC is protruded over the wall OC. The projection of BC on the wall is $CD = 3$ m (given).

Let the foot A of the ladder be x metres away from the wall and the protruded portion of the ladder be y m in length. Further, let $\angle OCA = \theta$.

Now, $AB = 15$ m and $BC = y$ m

$$\therefore \quad AC = (15 - y) \text{ m}.$$

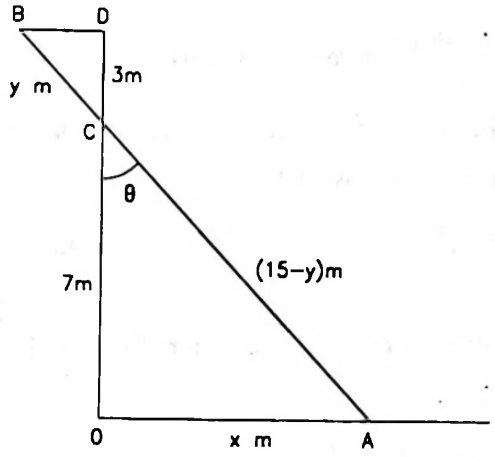


Fig. 7.5

It is given that $\frac{dy}{dt} = -2$ m/sec.

We have to find $\frac{dx}{dt}$.

In ΔOCA, we have

$\tan\theta = \frac{x}{7}$ and, $\cos\theta = \frac{7}{15-y}$

$\Rightarrow x = 7\tan\theta$

and, $15 - y = 7\sec\theta$.

In ΔBCD, we have

$\cos\theta = \frac{3}{y}$

$\Rightarrow y = 3\sec\theta$.

$\therefore 15 - 3\sec\theta = 7\sec\theta$

$\Rightarrow \sec\theta = \frac{15}{10}$

$\Rightarrow \sec\theta = \frac{3}{2}$

$\Rightarrow \cos\theta = \frac{2}{3}$

Now,

$15 - y = 7\sec\theta$

$\Rightarrow y = 15 - 7\sec\theta$

$\Rightarrow \frac{dy}{dt} = -7\sec\theta\tan\theta\frac{d\theta}{dt}$ $\left[\because \frac{dy}{dt} = -2 \text{ (given)}\right]$

$\Rightarrow -2 = -7\sec\theta\tan\theta\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \frac{2}{7\sec\theta\tan\theta}$ ...(i)

Now,

$x = 7\tan\theta$

$\Rightarrow \frac{dx}{dt} = 7\sec^2\theta\frac{d\theta}{dt}$

$\Rightarrow \frac{dx}{dt} = 7\sec^2\theta \times \frac{2}{7\sec\theta\tan\theta}$ [Using (i)]

$\Rightarrow \frac{dx}{dt} = \frac{2}{\sin\theta}$

$\Rightarrow \frac{dx}{dt} = \frac{2}{\sqrt{1-\frac{4}{9}}}$ $\left[\because \cos\theta = \frac{2}{3}\right]$

$\Rightarrow \frac{dx}{dt} = \frac{6}{\sqrt{5}}$

Hence, the bottom starts to slip away from the wall at the rate of $\frac{6}{\sqrt{5}}$ m/sec.

EXAMPLE 9 *If the function of $f: [0, 4] \to R$ is differentiable, then show that*

(i) $(f(4))^4 - (f(0))^2 = 8f'(a)f(b)$ for $a, b \in (0, 4)$

(ii) $\int_0^4 f(t)\,dt = 2\left[\alpha f(\alpha^2) + \beta f(\beta)^2\right]$ for all $0 < \alpha, \beta < 2$.

[IIT 2003]

SOLUTION (i) It is given that f is differentiable on $[0, \pi/4]$. So, it is continuous also. Consequently, by Lagrange's mean value theorem there exists $a \in (0, 4)$ such that

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} \quad ...(i)$$

Also, by intermediate mean value theorem there exists $b \in (0, 4)$ such that

$$f(b) = \frac{f(4) + f(0)}{2} \quad ...(ii)$$

From (i) and (ii), we have

$$f'(a)\,f(b) = \frac{f(4) - f(0)}{4} \times \frac{f(4) + f(0)}{2}$$

$$\Rightarrow f'(a)f(b) = \frac{(f(4))^2 - (f(0))^2}{8}$$

$\Rightarrow (f(4))^2 - (f(0))^2 = 8f'(a)f(b)$ for same $a, b \in (0, 4)$.

(ii) Putting $t = z^2$, we have

$$\int_0^4 f(t)\,dt = \int_0^2 2z f(z^2)\,dz.$$

Consider the function of $\phi(t)$ given by

$$\Phi(t) = \int_0^t 2z f(z^2)\,dz, \; t \in [0, 2].$$

Clearly, $\Phi(t)$, being an integral function of a continuous function, is continuous and differentiable on $[0, 2]$. Therefore, by Lagrange's mean value theorem there exists $c \in (0, 2)$ such that

$$\frac{\Phi(2) - \Phi(0)}{2 - 0} = \Phi'(c)$$

$$\Rightarrow \frac{\int_0^2 2z f(z^2)\,dz - \int_0^0 2z f(z^2)\,dz}{2-0} = 2cf(c^2) \quad [\because \Phi'(t) = 2t f(t^2)]$$

$$\Rightarrow \int_0^2 2z f(z^2)dz = 2(2cf(c^2)) \quad ...(i)$$

Also, by intermediate mean value theorem for $c \in (0, 2)$ there exist $\alpha, \beta \in (0, 2)$ such that

$$\frac{\Phi'(\alpha) + \Phi'(\beta)}{2} = \Phi'(c), \text{ where } 0 < \alpha < c < \beta < 2.$$

$$\Rightarrow 2\alpha f(\alpha^2) + 2\beta f\beta^2) = 2|2cf(c^2)| \quad ...(ii) \quad [\because \Phi'(t) = 2tf(t^2)$$

From (i) and (ii), we get

$$\int_0^2 2z f(z^2)\,dz = 2\alpha f(\alpha^2) + 2\beta f(\beta^2) \text{ for all } 0 < \alpha, \beta < 2$$

$$\Rightarrow \int_0^4 f(t)\,dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2)) \text{ for all } 0 < \alpha, \beta < 2.$$

EXERCISE 7.1

1. If $2a + 3b + 6c = 0$, then prove that the equation $ax^2 + bx + c = 0$ has at least one real root in (0, 1).
2. Prove that the equation $4x^3 - 9x^2 + 6x - 3 = 0$ has a real root lying between 1 and 2.
3. Prove that the equation $x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120 = 0$ does not have a repeated root.
4. If $a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x = 0$ has a real positive root α, then prove that the equation $n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + ... + a_{n-1} = 0$ has a positive real root less than α.
5. Prove that $\frac{b-a}{b} < \log \frac{b}{a} < \frac{b-a}{a}$, where $0 < a < b$.
6. Prove that

   $(b-a)\sec^2 a < \tan b - \tan a < (b-a)\sec^2 b$,

   where $0 < a < b < \frac{\pi}{2}$.
7. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6, g(1) = 2$, then show that there exists $c \in (0, 1)$ such that $f'(c) = 2g'(c)$.
8. Show that there is no value of k for which the equation $x^3 - x = k$ has two roots lying between 0 and $1/\sqrt{3}$. Also, find the values of k so that it has exactly one root between 0 and $1/\sqrt{3}$ and also no root between 0 and $1/\sqrt{2/3}$
9. A point is moving along the curve $y^3 = 27x$. Find the interval of values of x in which the ordinate changes faster than abscissa.
10. Find the approximate value of $f(x) = \left(\frac{2-x}{2+x}\right)^{1/5}$ at $x = 0.15$
11. Use differentials to find the approximate value of

    $(8.01)^{4/3} + (8.01)^2$.

ANSWERS

8. $\frac{-2}{3\sqrt{3}} > k < 0, k < \frac{-2}{3\sqrt{3}}$ 9. $(-1, 1)$ 10. 0.97 11. 80.1867

HINTS TO SELECTED PROBLEMS

6. Apply LMVT on $f(x) = \tan x$ defined on $[a, b]$
7. Apply LMVT on

   $\phi(x) = f(x)\{g(1) - g(0)\} - g(x)\{f(1) - f(0)\}$
9. We have, $y^3 = 27x$

$$\Rightarrow 3y^2 \frac{dy}{dt} = 27 \frac{dx}{dt}$$

$$y^2 \frac{dy}{dt} = 9 \frac{dx}{dt}.$$

Now,

$$\frac{dy}{dt} > \frac{dx}{dt}$$

$$\Rightarrow y^2 \frac{dy}{dt} > y^2 \frac{dx}{dt}$$

$$\Rightarrow 9\frac{dx}{dt} > y^2 \frac{dx}{dt}$$

$$\Rightarrow y^2 - 9 < 0$$

$$\Rightarrow -3 < y < 3 \Rightarrow -1 < x < 1.$$

10. Take $x = 0$ and $x + \Delta x = 0.15$. When $x = 0$, we have $y = 1$

Find Δy by using $\Delta y = \frac{dy}{dx}\Delta x$.

CHAPTER

8

MONOTONIC FUNCTIONS

8.1 INTRODUCTION

In this chapter, we shall study monotonicity of functions. A function $f(x)$ is said to be a monotonically increasing function on $[a, b]$, if the values of $f(x)$ increase or decrease with the increase or decrease in the values of x. If the values of $f(x)$ decrease with the increase in the values of x, then $f(x)$ is said to be a monotically decreasing function. The monotonicity of a function is strongly connected to the injectivity of the function and also with the sign of its derivative in $[a, b]$. In this chapter, we shall discuss the same.

8.2 SOME DEFINITIONS

STRICTLY INCREASING FUNCTION *A function $f(x)$ is said to be a strictly increasing function on (a, b) if*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

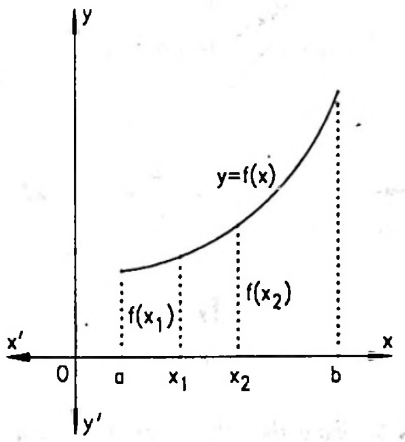


Fig. 8.1

Thus, $f(x)$ is strictly increasing on (a, b) *if* the values of $f(x)$ increase with the increase in the values of x. Graphically, $f(x)$ is increasing on (a, b) if the graph $y = f(x)$ moves up as x moves to the right. The graph shown in Fig. 8.1 is the graph of a strictly increasing function on (a, b).

ILLUSTRATION 1 *Show that the function $f(x) = 2x + 3$ is strictly increasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ and let $x_1 < x_2$. Then,

$$x_1 < x_2$$
$$\Rightarrow \quad 2x_1 < 2x_2$$
$$\Rightarrow \quad 2x_1 + 3 < 2x_2 + 3$$
$$\Rightarrow \quad f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$.

So, $f(x)$ is strictly increasing function on R.

This result is also evident from the graph of the function shown in Fig. 8.2

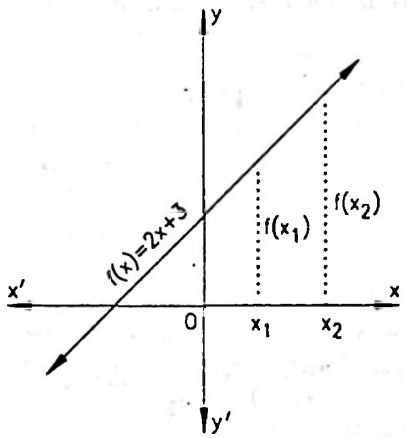


Fig. 8.2

ILLUSTRATION 2 *Show that the function $f(x) = x^2$ is strictly increasing function on $[0, \infty)$.*

SOLUTION Let $x_1, x_2 \in [0, \infty)$ such that $x_1 < x_2$. Then

$$x_1 < x_2 \Rightarrow x_1^2 < x_1 x_2 \quad \ldots(i)$$

Also,
$$x_1 < x_2 \Rightarrow x_1 x_2 < x_2^2 \quad \ldots(ii)$$

From (i) and (ii), we get

$$x_1 < x_2$$
$$\Rightarrow \quad x_1^2 < x_2^2$$
$$\Rightarrow \quad f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in [0, \infty)$.

Hence, $f(x)$ is strictly increasing function on $[0, \infty)$ which is evident from the graph shown in Fig. 8.3

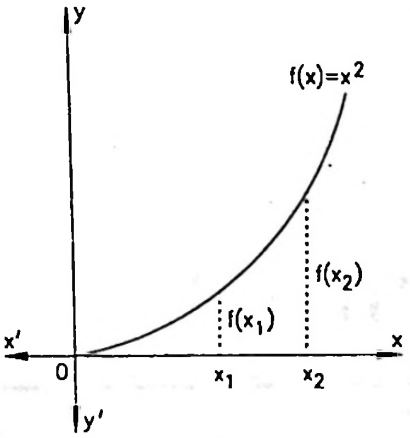


Fig. 8.3

ILLUSTRATION 3 *Show that the function $f(x) = a^x$, $a > 1$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2$$

$$\Rightarrow \quad a^{x_1} < a^{x_2}$$

$$\Rightarrow \quad f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly increasing function on R. This fact is also exhibited in the graph of this function as shown in Fig. 8.4.

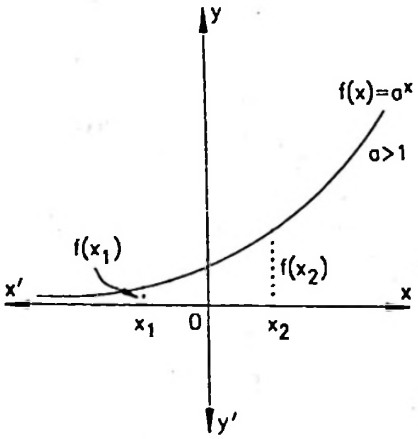


Fig. 8.4

REMARK *In the above example if we replace a by e (≈ 2.71), then we find that $f(x) = e^x$ is also increasing on R.*

STRICTLY DECREASING FUNCTION *A function $f(x)$ is said to be a strictly decreasing function on (a, b) if*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

Thus, $f(x)$ is strictly decreasing on (a, b) if the values of $f(x)$ decrease with the increase in the values of x. Graphically it means that $f(x)$ is a decreasing function on (a, b) if its graph moves down as x moves to the right. The graph shown in Fig. 8.5 is the graph of a strictly decreasing function.

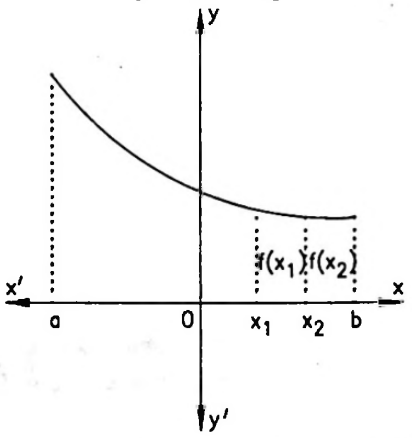


Fig. 8.5

ILLUSTRATION 4 *Show that the function $f(x) = -3x + 12$ is strictly decreasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2$$

$$\Rightarrow \quad -3x_1 > -3x_2$$

$$\Rightarrow \quad -3x_1 + 12 > -3x_2 + 12$$

$$\Rightarrow \quad f(x_1) > f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

So, $f(x)$ is strictly decreasing function on R.

This fact can also be observed from the graph of the function as shown in Fig. 8.6

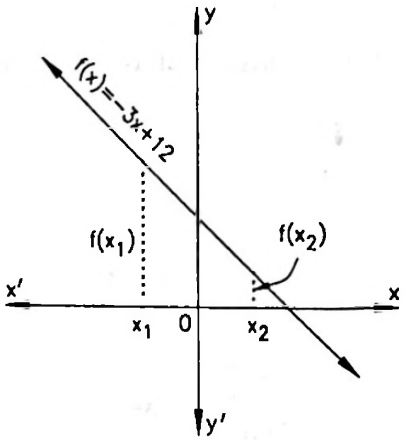


Fig. 8.6

ILLUSTRATION 5 *Show that the function $f(x) = a^x$, $0 < a < 1$ is strictly decreasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2$$

$$\Rightarrow \quad a^{x_1} > a^{x_2}$$

$$\Rightarrow \quad f(x_1) > f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2, \in R$.

Hence, $f(x)$ is strictly decreasing function on R. This is also evident from the graph of $f(x)$ as shown in Fig. 8.7.

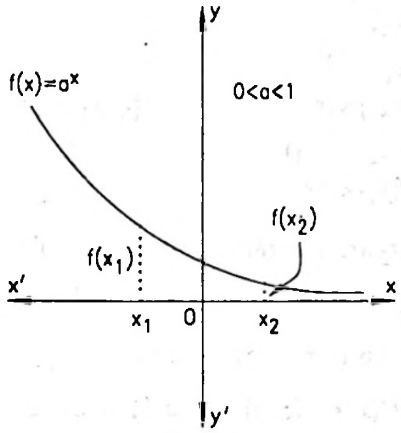


Fig. 8.7

REMARK *Since* $0 < e^{-1} = \frac{1}{e} < 1$, *therefore* $f(x) = (e^{-1})^x = e^{-x}$ *is also a strictly decreasing function on R.*

ILLUSTRATION 6 *Show that the function* $f(x) = x^2$ *is a strictly decreasing function on* $(-\infty, 0]$.

SOLUTION Let $x_1, x_2 \in (-\infty, 0]$, and let $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow x_1^2 > x_1 x_2 \quad \text{...(i)}$$

Also, $$x_1 < x_2 \Rightarrow x_1 x_2 > x_2^2 \quad \text{...(ii)}$$

From (i) and (ii), we get

$$x_1 < x_2 \Rightarrow x_1^2 > x_2^2 \Rightarrow f(x_1) > f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (-\infty, 0]$.

Hence, $f(x)$ is strictly decreasing on $(-\infty, 0]$. See also Fig. 8.8.

So far we have discussed about a strictly increasing or strictly decreasing function. But it is possible that a function may neither be strictly increasing nor strictly decreasing on a given interval. For example, $f(x)$ in Fig. 8.9 is neither strictly increasing nor strictly decreasing on (a, b). However, it is increasing on the sub-intervals (a, a_1), (a_2, a_3) and (a_4, b) and decreasing on the intervals (a_1, a_2) and (a_3, a_4).

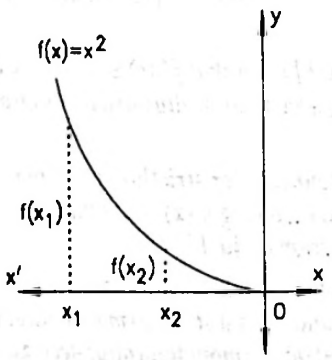


Fig. 8.8

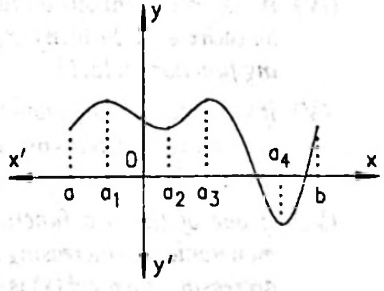


Fig. 8.9

ILLUSTRATION 7 *Show that the function* $f(x) = x^2$ *is neither strictly increasing nor strictly decreasing on R.*

SOLUTION In illustrations 3 and 6, we have seen that $f(x) = x^2$ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$. Hence, it is neither strictly increasing nor strictly decreasing on R i.e. $(-\infty, \infty)$.

Uptill now we were talking about strictly increasing and strictly decreasing functions. But there can be functions which are increasing (decreasing) but not strictly increasing (decreasing). For example, consider the function whose graph is shown in Fig. 8.10. Clearly, $f(x)$ is increasing on (a, b) but it is strictly increasing only in the intervals (a, a_1) and (a_2, b). In this chapter we shall be studying only strictly increasing and strictly decreasing functions.

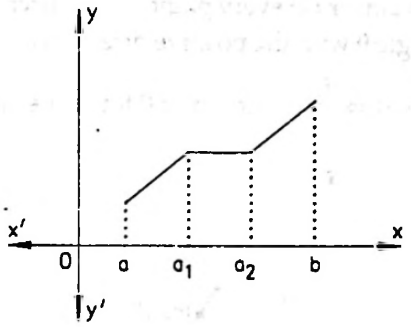


Fig. 8.10

NOTE *From now onwards, by an increasing or a decreasing function we shall mean a strictly increasing or a strictly decreasing function.*

MONOTONIC FUNCTION *A function* $f(x)$ *is said to be monotonic on an interval* (a, b) *if it is either increasing or decreasing on* (a, b).

DEFINITION 1 *A function* $f(x)$ *is said to be increasing (decreasing) at a point* x_0 *if there is an interval* $(x_0 - h, x_0 + h)$ *containing* x_0 *such that* $f(x)$ *is increasing (decreasing) on* $(x_0 - h, x_0 + h)$.

DEFINITION 2 *A function* $f(x)$ *is said to be increasing (decreasing) on* $[a, b]$ *if it is increasing (decreasing) on* (a, b) *and it is also increasing (decreasing) at* $x = a$ *and* $x = b$.

8.3 NECESSARY AND SUFFICIENT CONDITIONS FOR MONOTONICITY OF FUNCTIONS

In this section, we intend to see how we can use derivative of a function to determine where it is increasing and where it is decreasing.

NECESSARY CONDITION *Let* $f(x)$ *be a differentiable function defined on* (a, b). *Then,* $f'(x) > 0$ *or* < 0 *according as* $f(x)$ *is increasing or decreasing on* (a, b).

From Fig. 8.11, we observe that if $f(x)$ is an increasing function on (a, b), then tangent at every point on the curve $y = f(x)$ makes an acute angle θ with the positive direction of x-axis.

$$\therefore \tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0 \text{ or } f'(x) > 0 \text{ for all } x \in (a, b)$$

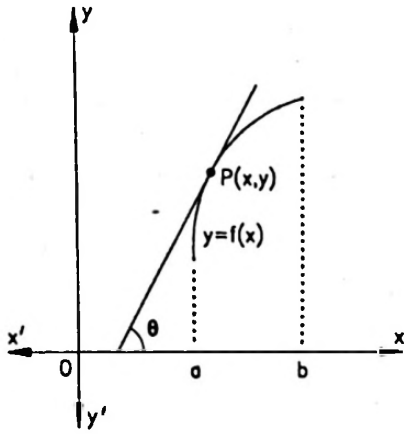


Fig. 8.11

It is evident from Fig. 8.12 that if $f(x)$ is a decreasing function on (a, b), then tangent at every point on the curve $y = f(x)$ makes an obtuse angle θ with the positive direction of x-axis.

$\therefore \tan\theta < 0 \Rightarrow \frac{dy}{dx} < 0$ or $f'(x) < 0$ for all $x \in (a, b)$.

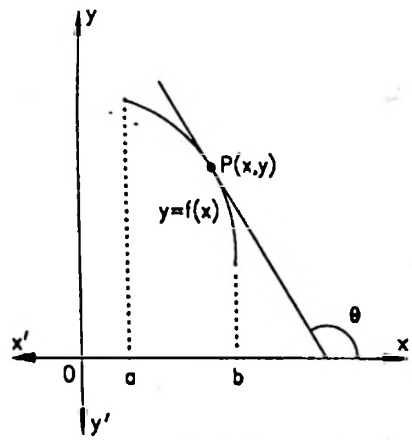


Fig. 8.12

Thus, $f'(x) > 0\ (< 0)$ for all $x \in (a, b)$ is the necessary condition for a function $f(x)$ to be increasing (decreasing) on a given interval (a, b). In other words, if it is given that $f(x)$ is increasing (decreasing) on (a, b), then we can say that $f'(x) > 0\ (< 0)$.

SUFFICIENT CONDITION

THEOREM *Let f be a differentiable real function defined on an open interval (a, b).*

(a) *If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).*

(b) *If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).*

PROOF Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. Consider the sub-interval $[x_1, x_2]$. Since $f(x)$ is differentiable on (a, b) and $[x_1, x_2] \subset (a, b)$. Therefore $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Lagrange's mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \ldots(i)$$

(a) Since $f'(x) > 0$ for all $x \in (a, b)$, so in particular, $f'(c) > 0$

Now,

$$f'(c) > 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \qquad \text{[Using (i)]}$$

$$\Rightarrow f(x_2) - f(x_1) > 0 \qquad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$

$$\Rightarrow f(x_2) > f(x_1)$$

$$\Rightarrow f(x_1) < f(x_2)$$

Since x_1, x_2 are arbitrary points in (a, b). Therefore,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

Hence, $f(x)$ is increasing on (a, b).

(b) Since $f'(x) < 0$ for all $x \in (a, b)$, so in particular, $f'(c) < 0$.

Now,

$$f'(c) < 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0 \qquad \text{[Using (ii)]}$$

$$\Rightarrow f(x_2) - f(x_1) < 0 \qquad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$

$$\Rightarrow f(x_2) < f(x_1)$$

$$\Rightarrow f(x_1) > f(x_2)$$

Since x_1, x_2 are arbitrary points in (a, b). Therefore,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \quad \text{for all } x_1, x_2 \in (a, b).$$

Hence, $f(x)$ is decreasing on (a, b).

COROLLARY *Let $f(x)$ be a function defined on (a, b).*

(a) If $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is decreasing on (a, b).

SOME USEFUL PROPERTIES OF MONOTONIC FUNCTIONS

(I) If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.

(II) If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

(III) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ $(f'(c) > 0)$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.

(IV) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ $(f'(c) < 0)$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) decreasing function on $[a, b]$.

(V) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing function on $[a, b]$.

(VI) If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Determine the intervals in which the function* $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ *is decreasing or increasing.*

SOLUTION We have,

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21.$$

$$\Rightarrow f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x^3 - 6x^2 + 11x - 6)$$

$$= 4(x-1)(x^2 - 5x + 6).$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow 4(x-1)(x^2 - 5x + 6) > 0$$

$$\Rightarrow (x-1)(x^2 - 5x + 6) > 0 \qquad [\because 4 > 0]$$

$$\Rightarrow (x-1)(x-2)(x-3) > 0$$

$$\Rightarrow 1 < x < 2 \text{ or } 3 < x < \infty \qquad \text{[See Fig. 8.13]}$$

$$\Rightarrow x \in (1, 2) \cup (3, \infty)$$

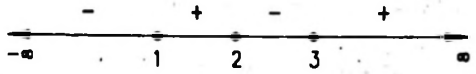


Fig. 8.13

Thus, $f(x)$ is increasing on $(1, 2) \cup (3, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow 4(x-1)(x^2 - 5x + 6) < 0$$

$$\Rightarrow (x-1)(x^2 - 5x + 6) < 0 \qquad [\because 4 > 0]$$

$$\Rightarrow (x-1)(x-2)(x-3) < 0$$

$$\Rightarrow 2 < x < 3 \text{ or } x < 1 \qquad \text{[See Fig. 8.14]}$$

$$\Rightarrow x \in (2, 3) \cup (-\infty, 1)$$

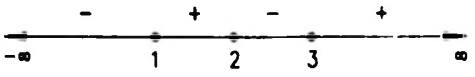


Fig. 8.14

Thus, $f(x)$ is decreasing on $(2, 3) \cup (-\infty, 1)$.

EXAMPLE 2 *Determine the values of* x *for which* $f(x) = x^x$, $x > 0$ *is increasing or decreasing.*

SOLUTION Clearly, $f(x) = x^x$ is defined for $x > 0$. So, the domain of $f(x) = x^x$ is $(0, \infty)$.

Now,

$$f(x) = x^x = e^{x \log x}$$

$$\Rightarrow f'(x) = e^{x \log x} \frac{d}{dx}(x \log_e x)$$

$$\Rightarrow f'(x) = x^x (1 + \log_e x)$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow x^x (1 + \log_e x) > 0$$

$$\Rightarrow 1 + \log_e x > 0 \qquad [\because x^x > 0 \text{ for } x > 0]$$

$$\Rightarrow \log_e x > -1$$

$$\Rightarrow x > e^{-1} \qquad \left[\because \log_a x > N \Rightarrow x > a^N \text{ for } a > 1\right]$$

$$\Rightarrow x \in (1/e, \infty)$$

Thus, $f(x)$ is increasing on $(1/e, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow x^x (1 + \log_e x) < 0$$

$$\Rightarrow 1 + \log_e x < 0 \qquad [\because x^x > 0 \text{ for } x > 0]$$

$$\Rightarrow \log_e x < -1$$

$$\Rightarrow x < e^{-1}$$

$$\Rightarrow x \in (0, 1/e)$$

Thus, $f(x)$ is decreasing on $(0, 1/e)$.

Hence, $f(x)$ is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

EXAMPLE 3 *Find the intervals in which* $f(x) = \dfrac{x}{\log x}$ *is increasing or decreasing.*

SOLUTION Note that the domain of $f(x)$ is the set of all positive real numbers i.e. $f(x)$ is defined for all $x > 0$ and $x \neq 1$.

Now,

$$f(x) = \frac{x}{\log x}$$

$$\Rightarrow f'(x) = \frac{\log x - 1}{(\log x)^2}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow \frac{\log x - 1}{(\log x)^2} > 0$$

$$\Rightarrow \log x - 1 > 0 \qquad [\because (\log x)^2 > 0 \text{ for } x > 0]$$

$$\Rightarrow \log x > 1$$

$$\Rightarrow x > e^1 \qquad \left[\because \log_a x > N \Rightarrow x > a^N \text{ for } a > 1\right]$$

$$\Rightarrow x \in (e, \infty).$$

So, $f(x)$ is increasing on (e, ∞).

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow \frac{\log x - 1}{(\log x)^2} < 0$$

$$\Rightarrow \log x - 1 < 0 \qquad [\because (\log x)^2 > 0 \text{ for } x > 0]$$

$$\Rightarrow \log x < 1$$

$$\Rightarrow x < e^1$$

$$\Rightarrow x \in (0, e) \qquad [\because f(x) \text{ is defined for } x > 0]$$

So, $f(x)$ is decreasing on $(0, 1) \cup (1, e)$.

EXAMPLE 4 *Find the intervals in which* $f(x) = 2\log(x-2) - x^2 + 4x + 1$ *is increasing or decreasing.*

SOLUTION Clearly, $f(x)$ is defined for all $x > 2$.

We have,

$$f(x) = 2\log(x-2) - x^2 + 4x + 1.$$

$$\Rightarrow f'(x) = \frac{2}{x-2} - 2x + 4$$

$$\Rightarrow f'(x) = \frac{2 - 2x(x-2) + 4(x-2)}{x-2}$$

$$\Rightarrow f'(x) = \frac{-2x^2 + 8x - 6}{x-2}$$

$$\Rightarrow f'(x) = \frac{-2(x^2 + 4x - 3)}{x-2}$$

$$\Rightarrow f'(x) = \frac{-2(x-1)(x-3)}{x-2}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \frac{-2(x-1)(x-3)}{x-2} > 0$$

$\Rightarrow (x-3) < 0$ [$\because x-1 > 0, x-2 > 0$ for $x > 2$]

$\Rightarrow x < 3$ [See Fig. 8.15]

$\Rightarrow x \in (2, 3)$



Fig. 8.15

So, $f(x)$ is increasing on $(2, 3)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \frac{-2(x-1)(x-3)}{x-2} < 0$$

$\Rightarrow (x-3) > 0$ [$\because x-1 > 0,\ x-2 > 0$ for $x > 2$]

$\Rightarrow x > 3$ [See Fig. 8.16]

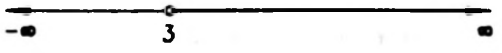


Fig. 8.16

$\Rightarrow x \in (3, \infty)$.

So, $f(x)$ is decreasing on $(1, 2) \cup (3, \infty)$.

EXAMPLE 5 *If* a, b, c *are real numbers, then find the intervals in which*

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

is increasing or decreasing.

SOLUTION We have,

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow f'(x) = (x+b^2)(x+c^2) - b^2c^2 + (x+a^2)(x+c^2) - a^2c^2 + (x+a^2)(x+b^2) - a^2b^2$$

$$\Rightarrow f'(x) = 3x^2 + 2x(a^2 + b^2 + c^2).$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow 3x^2 + 2x(a^2 + b^2 + c^2) > 0$$

$$\Rightarrow x[3x + 2(a^2 + b^2 + c^2)] > 0$$

$$\Rightarrow x < -\frac{2}{3}(a^2 + b^2 + c^2) \text{ or } x > 0$$

$$\Rightarrow x \in \left(-\infty, -\frac{2}{3}(a^2 + b^2 + c^2)\right) \cup (0, \infty).$$

So, $f(x)$ is increasing on $\left(-\infty, -\frac{2}{3}(a^2 + b^2 + c^2)\right) \cup (0, \infty)$

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

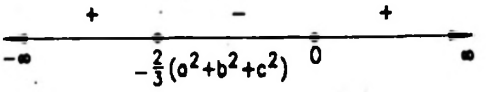


Fig. 8.17

$$\Rightarrow 3x^2 + 2x(a^2 + b^2 + c^2) < 0$$

$$\Rightarrow x\left(3x + 2(a^2 + b^2 + c^2)\right) < 0$$

$$\Rightarrow -\frac{2}{3}(a^2 + b^2 + c^2) < x < 0$$ [See Fig. 8.18]

$$\Rightarrow x \in \left(-\frac{2}{3}(a^2 + b^2 + c^2), 0\right)$$

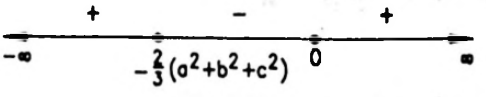


Fig. 8.18

So, $f(x)$ is decreasing on $\left(-\frac{2}{3}(a^2 + b^2 + c^2), 0\right)$.

Hence, $f(x)$ is increasing on $\left(-\infty, -\frac{2}{3}(a^2 + b^2 + c^2)\right) \cup (0, \infty)$ and decreasing on $\left(-\frac{2}{3}(a^2 + b^2 + c^2), 0\right)$.

EXAMPLE 6 *Separate the interval* $[0, \pi/2]$ *into sub-intervals in which* $f(x) = \sin^4 x + \cos^4 x$ *is increasing or decreasing.* **[IIT 1999]**

SOLUTION We have,

$$f(x) = \sin^4 x + \cos^4 x$$

$$\Rightarrow f'(x) = 4\sin^3 x \cos 4 - 4\cos^3 x \sin x$$

$$\Rightarrow f'(x) = -4 \sin x \cos x (\cos^2 x - \sin^2 x)$$

$$\Rightarrow f'(x) = -2 (2 \sin x \cos x) (\cos 2x)$$

$$\Rightarrow f'(x) = -2 \sin 2x \cos 2x$$

$$\Rightarrow f'(x) = -\sin 4x$$

We have, $0 < x < \pi/2 \Rightarrow 0 < 4x < 2\pi$.

Since sine function is positive in the first and second quadrants and negative in the third and fourth quadrants. So, we consider the following :

CASE I When $0 < 4x < \pi$

In this case, we have

$\Rightarrow \sin 4x > 0$

$\Rightarrow -\sin 4x < 0$

$\Rightarrow f'(x) < 0$

$\therefore f'(x) < 0$ for $0 < 4x < \pi$ i.e. $0 < x < \pi/4$.

So, $f(x)$ is decreasing on $[0, \pi/4]$.

CASE II When, $\pi < 4x < 2\pi$

In this case, we have

$\Rightarrow \sin 4x < 0$

$\Rightarrow -\sin 4x > 0$

$\Rightarrow f'(x) > 0$

$\therefore f'(x) > 0$ for $\pi < 4x < 2\pi$ i.e. $\pi/4 < x < \pi/2$.

So, $f(x)$ is increasing on $[\pi/4, \pi/2]$.

EXAMPLE 7 *Find the intervals in which the function*

$$f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3, 0 \le x \le \pi$$

is monotonically increasing or decreasing.

SOLUTION We have,

$$f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$$

$$\Rightarrow f'(x) = -(12\cos^3 x + 30\cos^2 x + 12\cos x)\sin x$$

$$= -2 \sin x \cos x (6\cos^2 x + 15\cos x + 6)$$

$$= -3 \sin 2x (2\cos^2 x + 5\cos x + 2)$$

$$= -3 \sin 2x (2\cos x + 1)(\cos x + 2) \quad ...(i)$$

Now,

$$f'(x) = 0$$

$$\Rightarrow -3 \sin 2x (2\cos x + 1)(\cos x + 2) = 0$$

$$\Rightarrow \sin 2x = 0 \text{ or } 2\cos x + 1 = 0$$

$$\Rightarrow \sin 2x = 0 \text{ or } \cos x = -1/2$$

$$\Rightarrow 2x = 0, \pi, 2\pi \text{ or } x = 2\pi/3 \quad [\because 0 \le x \le \pi]$$

$$\Rightarrow x = 0, \pi/2, 2\pi/3, \pi$$

Now, three cases arise :

CASE I *When* $0 < x < \pi/2$:

In this case, we have

$\sin 2x > 0, 2\cos x + 1 > 0$ and $\cos x + 2 > 0$

$\therefore f'(x) = -3 \sin 2x (2\cos x + 1)(\cos x + 2) < 0$.

So, $f(x)$ is monotonically decreasing for $0 \le x \le \pi/2$.

CASE II *When* $\pi/2 < x < 2\pi/3$:

In this case, we have

$\sin 2x < 0, 2\cos x + 1 > 0$ and $\cos x + 2 > 0$

$\therefore f'(x) = -3 \sin 2x (2\cos x + 1)(\cos x + 2) > 0$

So, $f(x)$ is monotonically increasing for $\pi/2 \le x \le 2\pi/3$.

CASE III *When* $2\pi/3 < x < \pi$:

In this case, we have

$\sin 2x < 0, 2\cos x + 1 < 0$ and $\cos x + 2 > 0$

$\therefore f'(x) = -3 \sin 2x (2\cos x + 1)(\cos x + 2) < 0$.

So, $f(x)$ is monotonically decreasing for $2\pi/3 \le x \le \pi$.

Hence, $f(x)$ is increasing on $[\pi/2, 2\pi/3]$ and decreasing on $[0, \pi/2] \cup [2\pi/3, \pi]$.

EXAMPLE 8 *Let* $f(x) = \begin{cases} x e^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$, *where* a *is a positive constant.* **[IIT 1996]**

Find the intervals in which $f'(x)$ *is increasing.*

SOLUTION We have,

$$f(x) = \begin{cases} x e^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$$

Clearly, $f(x)$ is continuous and differentiable at $x = 0$.

$$\therefore f'(x) = \begin{cases} (1 + ax)\, e^{ax}, & x \le 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$$

$f'(x)$ is also continuous and differentiable at $x = 0$.

$$\therefore f''(x) = \begin{cases} (2 + ax)\, a e^{ax}, & x \le 0 \\ 2a - 6x, & x > 0 \end{cases}$$

For $f'(x)$ to be increasing, we must have $f''(x) > 0$.

Now two cases arise :

CASE I When $x \le 0$:

In this case,

$$f''(x) > 0$$

$$\Rightarrow (2 + ax)\, a\, e^{ax} > 0$$

$$\Rightarrow 2 + ax > 0 \quad [\because a > 0 \text{ and } e^{ax} > 0]$$

$$\Rightarrow x > \frac{-2}{a}$$

$$\therefore x \in (-2/a, 0] \quad ...(i)$$

CASE II When $x > 0$:

In this case,

$$f''(x) > 0$$

$$\Rightarrow 2a - 6x > 0$$

$$\Rightarrow 2a > 6x$$

$$\Rightarrow x < \frac{a}{3}$$

$$\therefore x \in (0, a/3) \quad \text{...(ii)}$$

From (i) and (ii), we find that $f'(x)$ is increasing for

$$x \in (-2/a \cdot a/3)$$

EXAMPLE 9 *Find the intervals of monotonicity of the function* $f(x) = \frac{|x-1|}{x^2}$.

SOLUTION The given function can be written as

$$f(x) = \begin{cases} \frac{1-x}{x^2}, x < 1, x \neq 0 \\ \frac{x-1}{x^2}, x \geq 1, x \neq 0 \end{cases}$$

CASE I When $x < 1$

In this case,

$$f'(x) = -\frac{2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \frac{x-2}{x^3} > 0$$

$$\Rightarrow \frac{x(x-2)}{x^4} > 0$$

$$\Rightarrow x(x-2) > 0$$

$$\Rightarrow x \in (-\infty, 0) \cup (2, \infty)$$

But, $x < 1$. Therefore, $f(x)$ is increasing on $(-\infty, 0)$ and hence decreasing on $(0, 1)$.

CASE II When $x \geq 1$

In this case,

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \frac{2-x}{x^3} > 0$$

$$\Rightarrow 2 - x > 0 \quad [\because x \geq 1]$$

$$\Rightarrow x < 2$$

But, $x \geq 1$. Therefore, $f(x)$ is increasing on $[1, 2)$ and decreasing on $(2, \infty)$.

From case I and II, we conclude that $f(x)$ is increasing on $(-\infty, 0) \cup [1, 2)$ and decreasing on $(0, 1) \cup (2, \infty)$.

EXAMPLE 10 *Find the intervals in which the function* $f(x) = \sin(\log_e x) + \cos(\log_e x)$ *is decreasing.*

SOLUTION We have,

$$f(x) = \sin(\log_e x) + \cos(\log_e x).$$

Clearly, $f(x)$ is defined for all $x > 0$

We have,

$$f'(x) = \frac{\cos(\log_e x) - \sin(\log_e x)}{x}$$

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \frac{\cos(\log_e x) - \sin(\log_e x)}{x} < 0$$

$$\Rightarrow \cos(\log_e x) - \sin(\log_e x) < 0 \quad [\because x > 0]$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos(\log_e x) - \frac{1}{\sqrt{2}}\sin(\log_e x) < 0$$

$$\Rightarrow \cos\frac{\pi}{4}\cos(\log_e x) - \sin\frac{\pi}{4}\sin(\log_e x) < 0$$

$$\Rightarrow \cos\left\{\log_e x + \frac{\pi}{4}\right\} < 0$$

$$\Rightarrow 2n\pi + \frac{\pi}{2} < \log_e x + \frac{\pi}{4} < 2n\pi + \frac{3\pi}{2}, n \in Z$$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < \log_e x < 2n\pi + \frac{5\pi}{4}$$

$$\Rightarrow e^{2n\pi + \pi/4} < x < e^{2n\pi + 5\pi/4}$$

$$\Rightarrow x \in \left(e^{2n\pi + \pi/4}, e^{2n\pi + 5\pi/4}\right), n \in Z.$$

Hence, $f(x)$ is decreasing on $(e^{2n\pi + \pi/4}, e^{2n\pi + 5\pi/4})$

EXAMPLE 11 *If* $\phi(x) = f(x) + f(1-x)$ *and* $f''(x) < 0$ *for all* $x \in [0, 1]$. *Prove that* $\phi(x)$ *is increasing in* $[0, 1/2)$ *and decreasing in* $(1/2, 1]$.

SOLUTION We have,

$$f''(x) < 0 \text{ for all } [0, 1]$$

$$\Rightarrow f'(x) \text{ is decreasing for all } x \in [0, 1]$$

$$\Rightarrow f'(x) > f'(1-x), \text{ if } x < 1 - x \text{ and } x \in [0, 1]$$

$$\Rightarrow f'(x) > f'(1-x), \text{ if } x < \frac{1}{2} \text{ and } x \in [0, 1]$$

$$\Rightarrow f'(x) > f'(1-x), \text{ if } x \in [0, 1/2) \quad \text{...(i)}$$

Again,

$f'(x)$ is decreasing for all $x \in [0, 1]$

$\Rightarrow f'(x) < f'(1-x)$, if $x > 1-x$ and $x \in [0, 1]$

$\Rightarrow f'(x) < f'(1-x)$, if $x > \frac{1}{2}$ and $x \in [0, 1]$

$\Rightarrow f'(x) < f'(1-x)$, if $x \in (1/2, 1]$ ...(ii)

Now,

$\phi(x) = f(x) + f(1-x)$

$\Rightarrow \phi'(x) = f'(x) - f'(1-x)$

$\Rightarrow \phi'(x) > 0$ for $x \in [0, 1/2)$ [Using (i)]

< 0 for $x \in (1/2, 1]$ [Using (ii)]

Hence, $\phi(x)$ is increasing on $[0, 1/2)$ and decreasing on $(1/2, 1]$.

EXAMPLE 12 *Let $g(x) = 2f(x/2) + f(2-x)$ and $f''(x) < 0$ for all $x \in (0, 2)$. Find the intervals of increase and decrease of $g(x)$.*

SOLUTION We have,

$f''(x) < 0$

$\Rightarrow f'(x)$ is decreasing on $(0, 2)$.

$\Rightarrow f'\left(\frac{x}{2}\right) > f'(2-x)$, if $\frac{x}{2} < 2-x$

and,

$f'\left(\frac{x}{2}\right) < f'(2-x)$, if $\frac{x}{2} > 2-x$

$\Rightarrow f'\left(\frac{x}{2}\right) > f'(2-x)$, if $x < \frac{4}{3}$

and,

$f'\left(\frac{x}{2}\right) < f'(2-x)$, if $x > \frac{4}{3}$

$\Rightarrow f'\left(\frac{x}{2}\right) > f'(2-x)$ if $x \in (0, 4/3)$

and

$f'\left(\frac{x}{2}\right) < f'(2-x)$ if $x \in (4/3, 2)$

$\Rightarrow f'\left(\frac{x}{2}\right) - f'(2-x) > 0$, if $x \in (0, 4/3)$

and,

$f'\left(\frac{x}{2}\right) - f'(2-x) < 0$ if $x \in (4/3, 2)$ ...(i)

Now,

$g(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$

$\Rightarrow g'(x) = f'\left(\frac{x}{2}\right) - f'(2-x)$

$\Rightarrow g'(x) > 0$ if $x \in (0, 4/3)$ [Using (i)]

and,

$g'(x) < 0$ if $x \in (4/3, 2)$

$\Rightarrow g(x)$ is increasing on $(0, 4/3)$ and decreasing on $(4/3, 2)$

EXAMPLE 13 *Discuss the monotonicity of $g(x)$, where*

$$g(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2) \text{ for all } x \in R$$

It is given that $f''(x) > 0$ for all $x \in R$.

SOLUTION We have,

$f''(x) > 0$ for all $x \in R$

$\Rightarrow f'(x)$ is increasing for all $x \in R$

$\Rightarrow f'\left(\frac{x^2}{2}\right) > f'(6-x^2)$ if $\frac{x^2}{2} > 6 - x^2$

and,

$f'\left(\frac{x^2}{2}\right) < f'(6-x^2)$ if $\frac{x^2}{2} < 6 - x^2$

$\Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6-x^2) > 0$ if $x^2 - 4 > 0$

and,

$f'\left(\frac{x^2}{2}\right) - f'(6-x^2) < 0$ if $x^2 - 4 < 0$

$\Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6-x^2) > 0$ if $x \in (-\infty, -2) \in (2, \infty)$

and, ...(i)

$f'\left(\frac{x^2}{2}\right) - f'(6-x^2) < 0$ if $x \in (-2, 2)$.

Now,

$g(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2)$

$\Rightarrow g'(x) = 2f'\left(\frac{x^2}{2}\right) \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right) + f'(6-x^2) \cdot \frac{d}{dx}(6-x^2)$

$\Rightarrow g'(x) = 2x\left[f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right]$ ...(ii)

From (i) and (ii), we observe that

$g'(x) < 0$ if $x \in (-\infty, -2) \cup (0, 2)$

and,

$g'(x) > 0$ if $x \in (2, \infty) \cup (-2, 0)$

Hence, $g(x)$ is decreasing on $(-\infty, -2) \cup (0, 2)$ and increasing on $(-2, 0) \cup (2, \infty)$.

EXAMPLE 14 *Let $f(x)$ be a positive and increasing function in an interval I. Given that $\phi(x) = 3f^4 - 4f^3 - 36f^2 - 8$ is decreasing in I, examine whether $G(x) = f^4 + f^2 - 20$ maintains the same sign in I.*

SOLUTION Since $f(x)$ is positive and increasing function in I. Therefore,

$f(x) > 0$ and $f'(x) > 0$ for all $x \in I$

Also, $\phi(x)$ is decreasing in I. Therefore,

$\phi'(x) < 0$ for all $x \in I$

Now,

$\phi(x) = 3f^4 - 4f^3 - 36f^2 - 8$

$\Rightarrow \phi'(x) = (12f^3 - 12f^2 - 72f)f'$

But, $\phi'(x) < 0$ for all $x \in I$. Therefore,

$(12f^3 - 12f^2 - 72f)f' < 0$ for all $x \in I$.

$\Rightarrow 12f(f^2 - f - 6) < 0$ $[\because f' > 0]$

$\Rightarrow f^2 - f - 6 < 0$ $[\because f > 0]$

$\Rightarrow (f-3)(f+2) < 0$

$\Rightarrow -2 < f < 3$

$\Rightarrow 0 \le f^2 < 9$

Now,

$G(x) = f^4 + f^2 - 20$

$= (f^2+5)(f^2-4)$

Since $0 \le f^2 < 9$. Therefore, $f^2 - 4$ can take both positive and negative values.

Hence, $G(x)$ may not maintain the same sign in I.

EXAMPLE 15 *Prove that*

$$f(x) = \int_1^{e^x} \left(\sin^2(2\log t) - 3\sin(2\log t) + 2\right) dt$$

cannot be decreasing any where.

SOLUTION We have,

$$f(x) = \int_1^{e^x} \left(\sin^2(2\log t) - 3\sin(2\log t) + 2\right) dt$$

Using Leibnitz's rule, we have

$$f'(x) = \int_1^{e^x} 0\, dt + \frac{d}{dx}(e^x) \cdot \left\{\sin^2(2\log e^x) - 3\sin(2\log e^x) + 2\right\}$$

$$- \frac{d}{dx}(1) \cdot \left\{\sin^2(2\log 1) - 3\sin(2\log 1) + 2\right\}$$

$\Rightarrow f'(x) = e^x \{\sin^2 2x - 3\sin 2x + 2\}$

$\Rightarrow f'(x) = e^x (\sin 2x - 1)(\sin 2x - 2)$

$\because \sin 2x - 1 < 0$ and $\sin 2x - 2 < 0$ for all x

$\therefore f'(x) > 0$ for all x

$\Rightarrow f(x)$ is increasing for all x.

EXAMPLE 16 *Determine the values of a for which the function*

$$f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$$

is invertible.

SOLUTION We know that a function is invertible iff it is a bijection and for $f(x)$ to be a bijection, it must be either strictly increasing or strictly decreasing.

Clearly, domain of $f = R$. Therefore,

For $f(x)$ to be invertible, we must have

$f'(x) > 0$ or, $f'(x) < 0$ for all $x \in R$

$\Rightarrow 3(a+2)x^2 - 6ax + 9a > 0$ or, $3(a+2)x^2 - 6ax + 9a < 0$ for all $x \in R$

$\Rightarrow (a+2)x^2 - 2ax + 3a > 0$ or, $(a+2)x^2 - 2ax + 3a < 0$ for all $x \in R$

Now,

$(a+2)x^2 - 2ax + 3a > 0$ for all $x \in R$

$\Rightarrow a + 2 > 0$

and,

$4a^2 - 12a(a+2) < 0$ $\left[\because ax^2 + bx + c > 0 \text{ for all } x \in R \Rightarrow a > 0 \text{ and Disc} < 0\right]$

$\Rightarrow a + 2 > 0$ and $a(a+3) > 0$

$\Rightarrow a > -2$ and $a < -3$ or $a > 0$

$\Rightarrow a \in (0, \infty)$

And,

$(a+2)x^2 - 2ax + 3a < 0$ for all $x \in R$

$\Rightarrow (a+2) < 0$ and $4a^2 - 12a(a+2) < 0$

$\Rightarrow a < -2$ and $a(a+3) > 0$

$\Rightarrow a < -2$ and $a < -3$ or $a > 0$

$\Rightarrow a \in (-\infty, -3)$

For $a = 0$, we have

$f(x) = 2x^3 - 1$ which is clearly a bijection.

Also, for $a = -3$, we have,

$f(x) = -x^3 + 9x^2 - 27x - 1$ which is also a bijection.

Hence, $a \in [0, \infty) \cup (-\infty, -3]$

EXAMPLE 17 *Find the set of all values of a for which* $f(x) = x^3 + (a+2)x^2 + 3ax + 5$ *is invertible.*

SOLUTION Clearly, dom $(f) = R$.

Now,

$f: R \to R$ is invertible

$\Rightarrow f: R \to R$ is a bijection

$\Rightarrow f(x)$ is either strictly increasing or strictly decreasing on R

$\Rightarrow f'(x) > 0$ or, $f'(x) < 0$ for all $x \in R$

$\Rightarrow 3x^2 + 2x(a+2) + 3a > 0$ or < 0 for all $x \in R$

But, $3x^2 + 2x(a+2) + 3a$ cannot be less than zero for all $x \in R$. Because the curve $y = 3x^2 + 2x(a+2) + 3a$ is a parabola opening upward. Therefore, $y > 0$ for same $x \in R$. Consequently

$3x^2 + 2x(a+2) + 3a > 0$ for all $x \in R$

$\Rightarrow 4(a+2)^2 - 36a < 0$ $[\because \text{Disc} < 0]$

$\Rightarrow a^2 + 4a + 4 - 9a < 0$

$\Rightarrow a^2 - 5a + 4 < 0$

$\Rightarrow (a-1)(a-4)<0$

$\Rightarrow 1<a<4$

$\Rightarrow a\in(1,4)$.

Hence, $f(x)$ is invertible, if $a\in(1,4)$.

EXAMPLE 18 *If the function $f:R\to R$ given by $f(x)=x^3+ax^2+bx+5\sin^2 x$ is an increasing function, find the condition that must be satisfied by a and b.*

SOLUTION Since $f:R\to R$ given by

$$f(x)=x^3+ax^2+bx+5\sin^2 x$$

is an increasing function on R. Therefore,

$f'(x)>0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b+5\sin 2x>0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b-5>0$ for all $x\in R$ $\left[\because 3x^2+2ax+b+5\sin 2x \geq 3x^2+2ax+b-5\right]$

$\Rightarrow (2a)^2-4\times 3(b-5)<0$

$\Rightarrow 4(a^2-3b+15)<0$

$\Rightarrow a^2-3b+15<0$

$\Rightarrow a^2<3b-15$

Hence, the required condition is $a^2<3b-15$.

EXAMPLE 19 *Find the condition so that the function $f:R\to R$ given by*

$$f(x)=ax^3+bx^2+cx+d\sin x$$

is an injection.

SOLUTION If $f:R\to R$ is an injection, it must be either strictly increasing or strictly decreasing on R.

Thus,

$f'(x)>0$ or, $f'(x)<0$ for all $x\in R$

CASE I When $f'(x)>0$ for all $x\in R$

In this case,

$f'(x)>0$ for all $x\in R$

$\Rightarrow 3ax^2+2bx+c+d\cos x>0$ for all $x\in R$

We know that the minimum value of $d\cos x$ is $-|d|$. Therefore,

$3ax^2+2bx+c+d\cos x>0$ for all $x\in R$

$\Rightarrow 3ax^2+2bx+c-|d|>0$ for all $x\in R$

$\Rightarrow 3a>0$ and $(2b)^2-4\times 3a\times(c-|d|)<0$

$\Rightarrow a>0$ and $4\{b^2-3a(c-|d|)\}<0$

$\Rightarrow a>0$ and $b^2-3a(c-|d|)<0$

$\Rightarrow a>0$ and $b^2<3a(c-|d|)$

$\Rightarrow a>0$, $b^2<3a(c-|d|)$ and $c>|d|$ ...(i)

CASE II When $f'(x)<0$ for all $x\in R$

In this case,

$f'(x)<0$ for all $x\in R$

$\Rightarrow 3ax^2+2bx+c+d\cos x<0$ for all $x\in R$

Since the maximum value of $d\cos x$ is $|d|$. Therefore,

$3ax^2+2bx+c+d\cos x<0$ for all $x\in R$

$\Rightarrow 3ax^2+2bx+c+|d|<0$ for all $x\in R$

$\Rightarrow 3a<0$ and $(2b)^2-4\times 3a\times(c+|d|)<0$

$\Rightarrow a<0$ and $4\{b^2-3a(c+|d|)\}<0$

$\Rightarrow a<0$ and $b^2<3a(c+|d|)$

$\Rightarrow a<0$, $b^2<3a(c+|d|)$ and $c+|d|>0$

Hence, the required conditions are :

$b^2<3a(c-|d|)$, $c>|d|$, if $a>0$

$b^2<3a(c+|d|)$, $c>-|d|$, if $a<0$.

EXAMPLE 20 *Consider the function $f:R\to R$ given by $f(x)=x^3+ax^2+bx+c\sin x$ where $a,b,c\in R$. Find the condition that should be imposed on a, b, c so that the given function becomes invertible.*

SOLUTION We know that a function is invertible iff it is a bijection and a function is a bijection iff it is either strictly increasing or stritly decreasing.

Thus,

f is invertible

$\Rightarrow f$ is strictly increasing or strictly decreasing

$\Rightarrow f'(x)>0$ or, $f'(x)<0$ for all $x\in R$

CASE I When $f'(x)>0$:

In this case, we have

$f'(x)>0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b+c\cos x>0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b-|c|>0$ for all $x\in R$

$\Rightarrow (2a)^2-4\times 3\times(b-|c|)<0$

$\Rightarrow 4\{a^2-3(b-|c|)\}>0$

$\Rightarrow a^2-3(b-|c|)<0$

$\Rightarrow a^2<3(b-|c|)$

CASE II When $f'(x)<0$

In this case, we have

$f'(x)<0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b+c\cos x<0$ for all $x\in R$

$\Rightarrow 3x^2+2ax+b+|c|<0$ for all $x\in R$

But this is not possible, as $y=3x^2+2ax+b+|c|$ represents parabola opening upwards. Consequently, $y>0$ for some x.

Hence, the required condition is $a^2<3(b-|c|)$.

EXAMPLE 21 *Let $f(x)$ be a real function and $g(x)$ be a function given by*

$$g(x) = f(x) - (f(x))^2 + (f(x))^3 \text{ for all } x \in R.$$

prove that $f(x)$ and $g(x)$ increase or decrease together. **[IIT 1998]**

SOLUTION We have,

$$g(x) = f(x) - (f(x))^2 + (f(x))^3.$$

$$\therefore\ g'(x) = f'(x) - 2f(x).f'(x) + 3(f(x))^2 f'(x)$$

$$= f'(x)\{1 - 2f(x) + 3(f(x))^2\}$$

$$= f'(x)(3y^2 - 2y + 1), \text{ where } y = f(x).$$

Consider the quadratic expression $3y^2 - 2y + 1$. Clearly, discriminant of this quadratic expression is less than zero. So, its sign is always same as that of coefficient of y^2 i.e. positive.

$\therefore\ g'(x) = f'(x) \times A$ positive real number.

$\Rightarrow$ Sign of $g'(x)$ is same as that of $f'(x)$

$\Rightarrow$ either $g'(x) > 0$ and $f'(x) > 0$ or, $g'(x) < 0$ and $f'(x) < 0$

$\Rightarrow$ $g(x)$ and $f(x)$ increase and decrease together.

XAMPLE 22 *Let a function $f(x)$ be given by $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$.*

ove that $f(x)$ is decreasing on $[0, \infty)$. **[IIT 1995]**

LUTION We have,

$$f(x) = \frac{\log(\pi + x)}{\log(e + x)}$$

Clearly, domain of $f(x)$ is $(-e, \infty) - \{1 - e\}$.

Now,

$$f'(x) = \frac{\dfrac{\log(e+x)}{\pi + x} - \dfrac{\log(\pi + x)}{e + x}}{\{\log(e + x)\}^2}$$

$$\Rightarrow f'(x) = \frac{(e+x)\log(e+x) - (\pi + x)\log(\pi + x)}{(\pi + x)(e + x)\{\log(e + x)\}^2} \quad ...(i)$$

Let $g(x) = x \log x$ for $x > 0$. Then,

$$g'(x) = (1 + \log x)$$

Now,

$g'(x) > 0$

$\Rightarrow 1 + \log x > 0$

$\Rightarrow \log x > -1$

$\Rightarrow x > e^{-1} = \dfrac{1}{e}$

Thus,

$g(x)$ is increasing on $(1/e, \infty)$

$\Rightarrow g(\pi + x) > g(e + x)$ for all $x > 0$ $\quad \left[\because \pi + x > e + x > 1/e \text{ for all } x > 0\right]$

$\Rightarrow (\pi + x)\log(\pi + x) > (e + x)\log(e + x)$ for all $x > 0$ ...(ii)

From (i) and (ii), we find that

$$f'(x) < 0 \text{ for all } x \in [0, \infty)$$

Hence, $f(x)$ is decreasing on $[0, \infty)$.

EXAMPLE 23 *Find the set of values of b so that the function $f: R \to R$ given by*

$$f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right)x^3 + 5x + a,\ a, b \in R$$

is increasing at every point in its domain.

SOLUTION Clearly, $f(x)$ will assume real values if

$$21 - 4b - b^2 \geq 0 \text{ and } b + 1 \neq 0$$

$$\Rightarrow b^2 + 4b - 21 \leq 0 \text{ and } b \neq -1$$

$$\Rightarrow -7 \leq b \leq 3 \quad \text{and } b \neq -1$$

$$\Rightarrow b \in [-7, -1) \cup (-1, 3] \quad ...(i)$$

Now,

$f(x)$ is increasing on R

$$\Rightarrow f'(x) > 0 \quad \text{for all } x \in R$$

$$\Rightarrow 3\left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right)x^2 + 5 > 0 \text{ for all } x \in R$$

$$\Rightarrow 1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1} > 0$$

Clearly, this inequation is true for all $b \in [-7, -1)$ ...(ii)

So, let $b \in (-1, 3]$

In this case,

$$1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1} > 0$$

$$\Rightarrow \frac{\sqrt{21 - 4b - b^2}}{b + 1} < 1$$

$$\Rightarrow \sqrt{21 - 4b - b^2} < b + 1 \quad [\because b + 1 > 0 \text{ for all } b \in (-1, 3)]$$

$$\Rightarrow 21 - 4b - b^2 < (b + 1)^2$$

$$\Rightarrow b^2 + 3b - 10 > 0$$

$$\Rightarrow (b + 5)(b - 2) > 0$$

$$\Rightarrow b < -5 \text{ or } b > 2$$

But $b \in (-1, 3)$. Therefore, $b \in (2, 3]$ ...(iii)

From (ii) and (iii), we obtain that $f(x)$ is increasing on R, if $b \in [-7, -1) \cup (2, 3]$.

EXAMPLE 24 *Let*

$$f(x) = \{c^2 + (b - 1)c + 3\}x + \int (\sin^2 x + \cos^4 x)\,dx,$$

where $b, c \in R$. Find all possible values of b, if $f(x)$ is an increasing function for all $x \in R$ and for all $c \in R$

SOLUTION We have,

$$f(x) = \{c^2 + (b - 1)c + 3\}x + \int (\sin^2 x + \cos^4 x)\,dx$$

$\therefore f'(x) = \{c^2 + (b-1)c + 3\} + \sin^2 x + \cos^4 x$

Since $f(x)$ is increasing for all $x \in R$. Therefore,

$f'(x) > 0$ for all $x \in R$ and for all $c \in R$

$\Rightarrow \{c^2 + (b-1)c + 3\} + \sin^2 x + \cos^4 x > 0$ for all $x \in R$ and for all $c \in R$

$\Rightarrow \{c^2 + (b-1)c + 3\} + \frac{3}{4} > 0$ for all $c \in R$

$\left[\because \text{Min value of } \sin^2 x + \cos^4 x \text{ is } \frac{3}{4}\right]$

$\Rightarrow 4c^2 + 4(b-1)c + 15 > 0$ for all $c \in R$

$\Rightarrow 16(b-1)^2 - 16 \times 15 < 0$

$\Rightarrow (b-1)^2 - 15 < 0$

$\Rightarrow -\sqrt{15} < b-1 < \sqrt{15}$

$\Rightarrow 1-\sqrt{15} < b < 1+\sqrt{15}$

Hence, $b \in (1-\sqrt{15}, 1+\sqrt{15})$.

EXAMPLE 25 *Let $a+b=4$, where $a<2$, and let $g(x)$ be differentiable function. If $\frac{dg(x)}{dx} > 0$ for all x. Then, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b-a)$ increases.* **[IIT 1997]**

SOLUTION We have,

$a+b = 4.$

$\Rightarrow b-a = 4-a-a = 4-2a$

Now,

$$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx = \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx$$

Let $\phi(a) = \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx$

By Leibnitz's Rule, we have

$$\frac{d\phi(a)}{da} = g(a) - g(4-a) \qquad ...(i)$$

Since $\frac{d}{dx}\{g(x)\} > 0$ for all x. Therefore, $g(x)$ is increasing for all x.

Also,

$a+b = 4$ and $a<2$.

$\Rightarrow \quad a < 4-a$

$\Rightarrow \quad g(a) < g(4-a)$

$\Rightarrow \frac{d\phi(a)}{da} < 0 \qquad ...(ii)$ [Using (i)]

$\Rightarrow \frac{d(\phi(a))}{d(b-a)} \times \frac{d(b-a)}{da} < 0 \qquad \left[\because \frac{d\phi(a)}{d(b-a)} \times \frac{d(b-a)}{da} = \frac{d\phi(a)}{da}\right]$

$\Rightarrow \frac{d(\phi(a))}{d(b-a)} \times -2 < 0$

$\Rightarrow \frac{d(\phi(a))}{d(b-a)} > 0$

$\Rightarrow \phi(a)$ increases as $(b-a)$ increases.

ALITER Let $b-a = t$

$\therefore \quad a+b = 4$ and $b-a = t$

$\Rightarrow a = \frac{4-t}{2}$ and $b = \frac{4+t}{2}$

$$\therefore \int_0^a g(x) + \int_0^b g(x)\,dx = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx$$

Clearly, it is a function of t.

Let $\phi(t) = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx$

In order to prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases with $b-a$, it is sufficient to prove that $\phi(t)$ increases with t.

Now,

$\frac{dg(x)}{dx} > 0$ for all $x \in R$

$\Rightarrow g(x)$ is increasing for all $x \in R$

$\Rightarrow g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right) \qquad ...(i)$ $\left[\because a<2 \Rightarrow \frac{4-t}{2} < 2 \Rightarrow t > 0 \Rightarrow 4+\frac{t}{2} > 4-\frac{t}{2}\right]$

Now,

$$\phi(t) = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx$$

$\Rightarrow \phi'(t) = -\frac{1}{2}g\left(\frac{4-t}{2}\right) + \frac{1}{2}g\left(\frac{4+t}{2}\right)$

$\Rightarrow \phi'(t) = \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$

$\Rightarrow \phi'(t) > 0$ [Using (i)]

Hence, $\phi(t)$ increases with t. In other words,

$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases with $(b-a)$.

EXAMPLE 26 *Let* $f(x)=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)-\log_e(x^2+x+1)+(b^2-5b+3)x+c$. *If* $f(x)$ *is a decreasing function of* x *for all* $x\in R$, *find the permissible values of parameter* b.

SOLUTION We have,

$$f(x)=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)-\log_e(x^2+x+1)+(b^2-5b+3)x+C$$

$$\therefore\ f'(x)=\frac{1}{x^2+x+1}-\frac{2x+1}{x^2+x+1}+(b^2-5b+3)$$

$$\Rightarrow f'(x)=\frac{-2x}{x^2+x+1}+(b^2-5b+3)$$

$$\Rightarrow f'(x)=\frac{-2x}{x^2+x+1}+b^2-5b+3$$

Now, for $f(x)$ to be decreasing for all $x\in R$, we must have

$f'(x)<0$ for all $x\in R$

$$\Rightarrow \frac{-2x}{x^2+x+1}+b^2-5b+3<0 \text{ for all } x\in R$$

$$\Rightarrow 2+b^2-5b+3<0 \qquad \left[\text{Max. value of } \frac{-2x}{x^2+x+1} \text{ is } 2\right]$$

$$\Rightarrow b^2-5b+5<0$$

$$\Rightarrow \frac{5-\sqrt{5}}{2}<b<\frac{5+\sqrt{5}}{2}$$

$$\Rightarrow b\in\left(\frac{5-\sqrt{5}}{2},\frac{5+\sqrt{5}}{2}\right)$$

EXAMPLE 27 *Prove that the function* $f(x)=\frac{x}{\sin x}$ *is strictly increasing on* $(0,\pi/2)$. [IIT 1997]

SOLUTION We have,

$$f(x)=\frac{x}{\sin x}$$

$$\therefore\ f'(x)=\frac{\sin x-x\cos x}{\sin^2 x}$$

$$=\frac{g(x)}{\sin^2 x}, \text{ where } g(x)=\sin x-x\cos x$$

Now,

$$g(x)=\sin x-x\cos x$$

$$\Rightarrow g'(x)=\cos x-\cos x+x\sin x$$

$$=x\sin x$$

$\Rightarrow g'(x)>0$ for all $x\in(0,\pi/2)$ $[\because \sin x>0 \text{ for } x\in(0,\pi/2)]$

$\Rightarrow g(x)$ is an increasing function on $(0,\pi/2)$

$\Rightarrow g(x)>g(0)$ for all $x\in(0,\pi/2)$

$\Rightarrow g(x)>0$ for all $x\in(0,\pi/2)$

$\Rightarrow \frac{g(x)}{\sin^2 x}>0$ for all $x\in(0,\pi/2)$ $[\because \sin^2 x>0 \text{ for } x\in(0,\pi/2)]$

$\Rightarrow f'(x)>0$ for all $x\in(0,\pi/2)$

Hence, $f(x)$ is increasing on $(0,\pi/2)$

EXAMPLE 28 *Prove the following inequalities :*

(i) $x-\frac{x^3}{6}<\sin x<x$ for all $x\in(0,\pi/2)$

(ii) $x+\frac{x^3}{6}>\tan^{-1}x$ for all $x\in(0,1)$

(iii) $\sqrt{1+x^2}\le 1+x\log_e(x+\sqrt{x^2+1})$ for all $x>0$.

SOLUTION (i) Let $f(x)=x-\sin x$ for all $x\in(0,\pi/2)$

Then,

$f'(x)=1-\cos x$

$\Rightarrow f'(x)>0$ for all $x\in(0,\pi/2)$ $[\because \cos x\le 1]$

$\Rightarrow f(x)$ is an increasing function on $(0,\pi/2)$

$\Rightarrow f(x)>f(0)$ for all $x\in(0,\pi/2)$

$\Rightarrow x-\sin x>0$ for all $x\in(0,\pi/2)$

$\Rightarrow \sin x<x$ for all $x\in(0,\pi/2)$ ...(i)

Let $g(x)=x-\frac{x^3}{6}-\sin x$ for all $x\in(0,\pi/2)$. Then,

$$g'(x)=1-\frac{x^2}{2}-\cos x$$

$$\Rightarrow g''(x)=-x+\sin x$$

$\Rightarrow g''(x)<0$ for all $x\in(0,\pi/2)$ [Using (i)]

$\Rightarrow g'(x)$ is decreasing on $(0,\pi/2)$

$\Rightarrow g'(x)<g'(0)$ for all $x\in(0,\pi/2)$

$\Rightarrow g'(x)<0$ for all $x\in(0,\pi/2)$

$\Rightarrow g(x)$ is decreasing on $(0,\pi/2)$

$\Rightarrow g(x)<g(0)$ for all $x\in(0,\pi/2)$

$\Rightarrow x-\frac{x^3}{6}-\sin x<0$ for all $x\in(0,\pi/2)$

$\Rightarrow x-\frac{x^3}{6}<\sin x$ for all $x\in(0,\pi/2)$ ...(ii)

From (i) and (ii), we get

$$x-\frac{x^3}{6}<\sin x<x \text{ for all } x\in(0,\pi/2)$$

(ii) Let $f(x)=x+\frac{x^3}{6}-\tan^{-1}x$ for all $x\in(0,1)$. Then,

$$f'(x)=1+\frac{x^2}{2}-\frac{1}{1+x^2}=\frac{x^2}{1+x^2}+\frac{x^2}{2}>0 \text{ for all } x\in(0,1)$$

$\Rightarrow f(x)$ is an increasing function on $(0,1)$

$\Rightarrow f(x)>f(0)$ for all $x\in(0,1)$

$\Rightarrow x+\frac{x^3}{6}-\tan^{-1}x>0$ for all $x\in(0,1)$

$\Rightarrow x+\frac{x^3}{6}>\tan^{-1}x$ for all $x\in(0,1)$.

(iii) Let $f(x)=\sqrt{1+x^2}-1-x\log_e(x+\sqrt{x^2+1})$ for all $x>0$

Then,

$$f'(x)=\frac{x}{\sqrt{x^2+1}}-\log_e(x+\sqrt{x^2+1})-\frac{x}{\sqrt{x^2+1}} \text{ for all } x>0$$

$\Rightarrow f'(x)=-\log(x+\sqrt{x^2+1})$ for all $x>0$

$\Rightarrow f'(x)<0$ for all $x>0$ $[\because x+\sqrt{x^2+1}>1$ for all $x>0]$

$\Rightarrow f(x)$ is decreasing for all $x>0$

$\Rightarrow f(x)<f(0)$ for all $x>0$

$\Rightarrow \sqrt{1+x^2}-1-x\log_e(x+\sqrt{x^2+1})<0$ for all $x>0$

$\Rightarrow \sqrt{1+x^2}<1+x\log_e(x+\sqrt{x^2+1})$ for all $x>0$.

EXAMPLE 29 *Prove that :*

$$\left(\tan^{-1}\frac{1}{e}\right)^2+\frac{2e}{\sqrt{e^2+1}}<(\tan^{-1}e)^2+\frac{2}{\sqrt{e^2+1}}$$

SOLUTION Consider the function $f(x)$ given by

$$f(x)=(\tan^{-1}x)^2+\frac{2}{\sqrt{x^2+1}} \text{ for all } x\in R$$

We have,

$$f'(x)=\frac{2\tan^{-1}x}{1+x^2}-\frac{2x}{(x^2+1)^{3/2}}$$

$$=\frac{2}{1+x^2}\left\{\tan^{-1}x-\frac{x}{\sqrt{x^2+1}}\right\}$$

$$=\frac{2}{1+x^2}\cdot g(x), \text{ where } g(x)=\tan^{-1}x-\frac{x}{\sqrt{x^2+1}}$$

Now,

$$g'(x)=\frac{1}{1+x^2}-\frac{1}{\sqrt{x^2+1}}+\frac{x^2}{(x^2+1)^{3/2}}$$

$$=\frac{1}{1+x^2}-\frac{1}{(x^2+1)^{3/2}}$$

$$=\frac{1}{1+x^2}\left\{1-\frac{1}{\sqrt{x^2+1}}\right\}>0 \text{ for all } x\in R$$

$\Rightarrow g(x)$ is increasing for all $x\in R$

$\Rightarrow g(x)>g(0)$ for all $x\in R$

$\Rightarrow \tan^{-1}x-\frac{x}{\sqrt{x^2+1}}>0$ for all $x>0$

$\Rightarrow f'(x)>0$ for all $x>0$.

$\Rightarrow f(x)$ is increasing for all $x>0$.

$\Rightarrow f(1/e)<f(e)$ $[\because 1/e<e]$

$$\Rightarrow \left\{\tan^{-1}\left(\frac{1}{e}\right)\right\}^2+\frac{2}{\sqrt{\frac{1}{e^2}+1}}<(\tan^{-1}e)+\frac{2}{\sqrt{e^2+1}}$$

$$\Rightarrow \left\{\tan^{-1}\frac{1}{e}\right\}^2+\frac{2e}{\sqrt{e^2+1}}<(\tan^{-1}e)^2+\frac{2}{\sqrt{e^2+1}}$$

EXAMPLE 30 *Use the function $f(x)=x^{1/x}$, $x>0$ to determine the bigger of the two numbers e^π and π^e.*

SOLUTION We have,

$$f(x)=x^{1/x}, x>0$$

$$\Rightarrow f(x)=e^{(1/x)\log x}$$

$$\Rightarrow f'(x)=e^{\frac{1}{x}\log x}\frac{d}{dx}\left(\frac{1}{x}\log x\right)$$

$$\Rightarrow f'(x)=x^{1/x}\left(\frac{1-\log x}{x^2}\right)$$

$\Rightarrow f'(x)>0$ if $\log x<1$

<0 if $\log x>1$

$\Rightarrow f'(x)>0$ if $x<e$

<0 if $x>e$

Thus, $f(x)$ is increasing for all $x\in(0,e)$ and decreasing for all $x\in(e,\infty)$.

Since $\pi>e$ and $f(x)$ is decreasing for all $x\in(e,\infty)$.

Therefore,

$$f(\pi)<f(e)$$

$$\Rightarrow \pi^{1/\pi}<e^{1/e}$$

$$\Rightarrow \pi^e<e^\pi$$

EXAMPLE 31 *Prove that $(a+b)^n\le a^n+b^n$ for all $a,b>0$ and $0<n<1$.*

SOLUTION We have to prove that

$(a+b)^n\le a^n+b^n$ for all $a, b>0$ and $0<n<1$.

i.e. $\left(\frac{a}{b}+1\right)^n\le\left(\frac{a}{b}\right)^n+1$ for all $a, b>0$ and $0<n<1$

So, let us consider the function $f(x)$ given by

$$f(x)=(1+x)^n-x^n-1 \text{ for } x>0.$$

Differentiating w.r.t. x, we get

$$f'(x)=n(1+x)^{n-1}-nx^{n-1}$$

$$f'(x)=n\left(\frac{1}{(1+x)^{1-n}}-\frac{1}{x^{1-n}}\right)$$

$$\Rightarrow f'(x)<0 \qquad \left[\begin{array}{l}\because 1+x>x \text{ and } 1-n>0\\ \therefore \frac{1}{(1+x)^{1-n}}<\frac{1}{x^{1-n}}\end{array}\right]$$

$\Rightarrow f(x)$ is decreasing for all $x > 0$.

$\Rightarrow f(x) \le f(0)$ for all $x \ge 0$

$\Rightarrow (1+x)^n - x^n - 1 < 0$ for all $x \ge 0$

$\Rightarrow (1+x)^n < 1 + x^n$ for all $x \ge 0$

$\Rightarrow \left(1+\frac{a}{b}\right)^n < 1 + \left(\frac{a}{b}\right)^n$ [Replacing x by a/b]

$\Rightarrow (a+b)^n < a^n + b^n$

Hence, $(a+b)^n \le a^n + b^n$ for all $a, b > 0$ and $0 < n < 1$.

EXAMPLE 32 *Let* $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f (x) has the smallest value at x = 1. [IIT 1996]

SOLUTION We have,

$$f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$$

Clearly, $f(x)$ is not differentiable at $x = 1$.

$$\therefore \quad f'(x) = \begin{cases} -3x^2, & 0 < x < 1 \\ 2, & 1 < x < 3 \end{cases}$$

We find that $f'(x) < 0$ for $x \in (0, 1)$ and $f'(x) > 0$ for all $x \in (1, 3)$. Therefore, $f(x)$ is decreasing in $(0, 1)$ and increasing in $(1, 3)$. Hence,

$f(x) \ge f(1)$ for all $x \in [1, 3]$

$\Rightarrow f(x)$ has the smallest value at $x = 1$ in $[1, 3]$

For $f(x)$ to have the smallest value at $x = 1$ for all $x \in [0, 3]$, we must have

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

$$\Rightarrow \lim_{x \to 1^-} -x^3 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge -1 \qquad [\because f(1) = -1]$$

$$\Rightarrow -1 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge -1$$

$$\Rightarrow \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge 0$$

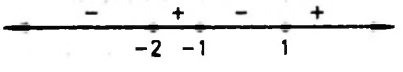


Fig. 8.19

$$\Rightarrow \frac{(b-1)(b^2+1)}{(b+2)(b+1)} \ge 0$$

$$\Rightarrow \frac{(b-1)}{(b+2)(b+1)} \ge 0 \qquad [\because b^2 + 1 > 0]$$

$\Rightarrow b \in (-2, -1) \cup [1, \infty)$

EXAMPLE 33 *Let* $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2 (b^2 - 2), & x > 1 \end{cases}$

Find all possible real values of b such that f (x) has the greatest value at x = 1.

SOLUTION We have,

$$f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2 (b^2 - 2), & x > 1 \end{cases}$$

Clearly, $f(x)$ is not differentiable at $x = 1$.

$$\therefore \quad f'(x) = \begin{cases} 3x^2 - 2x + 10, & x < 1 \\ -2, & x > 1 \end{cases}$$

Now,

$$f'(x) = 3x^2 - 2x + 10 \qquad \text{for } x < 1$$

$$\Rightarrow \quad f'(x) = 3\left(x - \frac{1}{3}\right)^2 + \frac{29}{3} > 0 \quad \text{for } x < 1$$

$\Rightarrow f(x)$ is increasing for $x < 1$

So, $f(1)$ is the greatest value of $f(x)$.

Clearly,

$f'(x) = -2 < 0$ for all $x > 1$.

$\Rightarrow f(x)$ is decreasing for all $x > 1$.

$\Rightarrow f(x) < f(1)$ for all $x > 1$.

Thus, $f(x)$ will have greatest value at $x = 1$, if

$$\lim_{x \to 1^+} f(x) < f(1)$$

$$\Rightarrow \lim_{x \to 1^+} -2x + \log_2 (b^2 - 2) < 1 - 1 + 10 - 5$$

$$\Rightarrow -2 + \log_2 (b^2 - 2) < 5$$

$$\Rightarrow \log_2 (b^2 - 2) < 7$$

$$\Rightarrow b^2 - 2 < 2^7$$

$$\Rightarrow b^2 - 130 < 0$$

$$\Rightarrow -\sqrt{130} < b < \sqrt{130}.$$

But, $\log_2 (b^2 - 2)$ exists if $b^2 - 2 > 0$ i.e.

$b \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

$\therefore b \in (-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130})$

Hence, $f(x)$ will have the greatest value at $x = 1$ if $b \in (-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130})$.

EXAMPLE 34 *For what real values of a does the function* $f(x) = 2e^x - a e^{-x} + (2a + 1)x - 3$ *increases for all x ?*

SOLUTION For $f(x)$ to be increasing for all x, we must have

$f'(x) > 0$ for all x

$\Rightarrow 2e^x + a e^{-x} + (2a + 1) > 0$ for all x

$\Rightarrow 2(e^x)^2 + (2a + 1) e^x + a > 0$ for all x

$\Rightarrow 2y^2 + (2a+1)y + a > 0$ for all $y = e^x > 0$

We know that $ax^2 + bx + c > 0$ for all $x > 0$, if

either $a > 0$ and disc < 0

or,

$a > 0$, disc ≥ 0, $\frac{-b}{2a} \leq 0$ and $f(0) > 0$

Now,

Disc $= (2a+1)^2 - 8a = (2a-1)^2$

Clearly, disc $\not< 0$.

Therefore,

$2y^2 + (2a+1)y + a > 0$ for all $y > 0$

$\Rightarrow (2a-1)^2 \geq 0, -\frac{(2a+1)}{4} \leq 0$ and $a > 0$

$\Rightarrow a \in R, \quad a \geq -\frac{1}{2}$ and $a > 0$

$\Rightarrow a > 0$

$\Rightarrow a \in (0, \infty)$

EXAMPLE 35 *Let $f(x), x \geq 0$ be a non-negative continuous function and let $F(x) = \int_0^x f(t)\,dt, x \geq 0$. If for some $c > 0, f(x) \leq c\,F(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$.* **[IIT 2001]**

SOLUTION We have,

$f(x) \geq 0$ for all $x \geq 0$

and,

$F(x) = \int_0^x f(t)\,dt$

$\Rightarrow F'(x) = f(x)$ for all $x \geq 0$

$\Rightarrow F'(x) \geq 0$ for all $x \geq 0$

Also, $F(x) = \int_0^x f(t)\,dt$ and $f(x) \geq 0$ for all $x \geq 0$. Therefore, $F(x) \geq 0$ for all $x \geq 0$.

Now,

$f(x) \leq c\,F(x)$ for all $x \geq 0$ and for some $c > 0$

$\Rightarrow F'(x) - c\,F(x) \leq 0$ for all $x \geq 0$ $[\because F'(x) = f(x)]$

$\Rightarrow F'(x) - c\,F(x) \leq 0$ for all $x \geq 0$

$\Rightarrow e^{-cx} F'(x) - c\,e^{-cx} F(x) \leq 0$ for all $x \geq 0$

$[\because e^{-Cx} > 0$ for all $x \geq 0]$

$\Rightarrow \frac{d}{dx}(F(x) \cdot e^{-cx}) \leq 0$ for all $x \geq 0$

$\Rightarrow F(x)\,e^{-cx}$ is monotonically decreasing for all $x \geq 0$

$\Rightarrow F(x)\,e^{-cx} \leq F(0)\,e^0$ for all $x \geq 0$

$\Rightarrow F(x)\,e^{-cx} \leq 0$ for all $x \geq 0$ $[\because F(x) = \int_0^x f(t)\,dt \Rightarrow F(0) = 0]$

$\Rightarrow F(x) \leq 0$ for all $x \geq 0$ $[\because e^{-Cx} > 0$ for all $x]$

But, $F(x) \geq 0$ for all $x \geq 0$

$\therefore F(x) = 0$ for all $x \geq 0$

$\Rightarrow \int_0^x f(t)\,dt = 0$ for all $x \geq 0$ and $f(x) \geq 0$ for all $x \geq 0$ [Given]

$\Rightarrow f(x) = 0$ for all $x \geq 0$

EXAMPLE 36 *If $f(x)$ is a monotonically increasing function for all $x \in R$ such that $f''(x) > 0$ and $f^{-1}(x)$ exists, then prove that*

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

SOLUTION Let $g(x) = f^{-1}(x)$.

Since g is the inverse of f. Therefore,

$fog(x) = f(g(x)) = x$

$\Rightarrow f'(g(x)) \cdot g'(x) = 1$ [Diff. both sides w.r.t. x]

$\Rightarrow g'(x) = \frac{1}{f'(g(x))}$ for all $x \in R$

$\Rightarrow g'(x) > 0$ for all $x \in R$ $\left[\begin{array}{r}\because f(x) \text{ is increasing for all } x \in R \\ \therefore f'(x) > 0 \text{ for all } x \in R \\ \Rightarrow f'(g(x)) > 0 \text{ for all } x \in R\end{array}\right]$

$\Rightarrow g(x)$ is increasing for all $x \in R$

$\Rightarrow f^{-1}(x)$ is increasing for all $x \in R$.

Now,

$g'(x) = \frac{1}{f'(g(x))}$ for all $x \in R$

$\Rightarrow g''(x) = -\frac{1}{[f'(g(x))]^2} f''(g(x)) g'(x)$ for all $x \in R$

$\Rightarrow g''(x) < 0$ for all $x \in R$

$[\because g'(x) > 0, \ \forall x \in R$ and $f''(x) > 0$ (given)]

$\Rightarrow g'(x)$ is decreasing for all $x \in R$.

Thus, $f^{-1}(x)$ is an increasing function and $\frac{d}{dx}(f^{-1}(x))$ is a decreasing function of x for all $x \in R$.

Now, consider the curve $y = f^{-1}(x)$. Let P, Q, R be any three points on the curve $y = f(x)$ as shown in Fig. 8.20. Let the abscissae of P, Q, R be x_1, x_2, x_3 respectively. Then,

$OS = x_1, OU = x_2$ and $OW = x_3$.

Clearly, $\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}$ is the ordinate of the centroid of ΔPQR. Let N be the centroid of ΔPQR and M be the

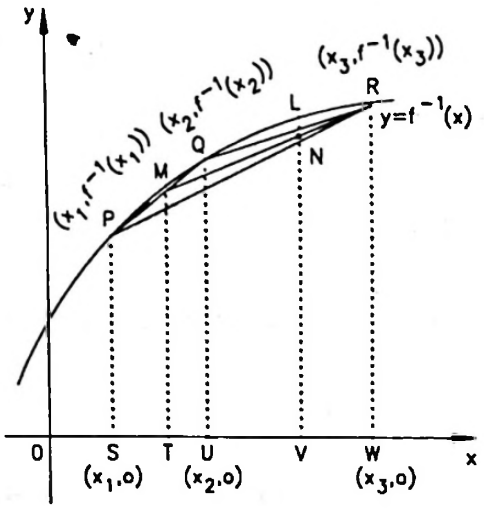


Fig. 8.20

mid-point of PQ. Further, let T and V be the foot of the ordinates of M and N respectively.

Since M is the mid-point of PQ. Therefore, T is the mid-point of SU. Also, N (being centroid of ΔPQR) divides RM in the ratio 1 : 2. Therefore, V will divide WT in the ratio 2 : 1.

Hence,

$$OT = \frac{x_1 + x_2}{2} \text{ and } OV = \frac{2\left(\frac{x_1 + x_2}{2}\right) + 1 \cdot x_3}{2+1} = \frac{x_1 + x_2 + x_3}{3}$$

$$\therefore LV = f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

Since N is the centroid of ΔPQR. So, the coordinates of N are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}\right)$$

$$\therefore NV = \frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}$$

Clearly, $NV < LV$

$$\Rightarrow \frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

EXAMPLE 37 *Using the relation* $2(1 - \cos x) < x^2$, $x \neq 0$ *or otherwise, prove that* $\sin(\tan x) \geq x$ *for all* $x \in \left[0, \frac{\pi}{4}\right]$. **[IIT 2003]**

SOLUTION Let $f(x) = \sin(\tan x) - x$. Then,

$$f'(x) = \cos(\tan x)\sec^2 x - 1$$

$$f'(x) = \cos(\tan x)(1 + \tan^2 x) - 1$$

$$\Rightarrow f'(x) = \cos(\tan x) + \tan^2 x \cos(\tan x) - 1$$

$$\Rightarrow f'(x) = \tan^2 x \cos(\tan x) - |1 - \cos(\tan x)|$$

$$\Rightarrow f'(x) > \tan^2 x \cos(\tan x) - \frac{1}{2}\tan^2 x$$

$$\Rightarrow f'(x) > \frac{1}{2}\tan^2 x\,|2\cos(\tan x) - 1| \quad \left[\because 1 - \cos x < \frac{x^2}{2} \text{ (given)}, \ \therefore 1 - \cos(\tan x) > \frac{\tan^2 x}{2}\right]$$

$$\Rightarrow f'(x) > \frac{1}{2}\tan^2 x(1 - \tan^2 x) \quad [\because 2(1 - \cos x) < x^2, \ \therefore 2\cos x - 1 > 1 - x^2]$$

$$\Rightarrow f'(x) \geq 0 \text{ for all } x \in [0, \pi/4] \quad [\because \tan x \leq 1 \text{ for all } x \in (0, \pi/4)]$$

$\Rightarrow f(x)$ is an increasing function for all $x \in [0, \pi/4]$

$\Rightarrow f(x) \geq f(0)$ for all $x \in [0, \pi/4]$

$\Rightarrow \sin(\tan x) - x \geq 0$ for all $x \in [0, \pi/4]$

$\Rightarrow \sin(\tan x) \geq x$ for all $x \in [0, \pi/4]$.

EXAMPLE 38 *If* $P(1) = 0$ *and* $\frac{dP(x)}{dx} > P(x)$ *for all* $x \geq 1$, *then prove that* $P(x) > 0$ *for all* $x > 1$. **[IIT 2003]**

SOLUTION We have,

$$\frac{dP(x)}{dx} > P(x) \text{ for all } x \geq 1$$

$$\Rightarrow \frac{dP(x)}{dx}e^{-x} > P(x)e^{-x} \text{ for all } x \geq 1$$

$$\Rightarrow \frac{dP(x)}{dx}e^{-x} - P(x)e^{-x} > 0 \text{ for all } x \geq 1$$

$$\Rightarrow \frac{d}{dx}\left(P(x)e^{-x}\right) > 0 \text{ for all } x \geq 1$$

$\Rightarrow P(x)e^{-x}$ is an increasing function for all $x \geq 1$

$\Rightarrow P(x)e^{-x} > P(1)e^{-1}$ for all $x > 1$.

$\Rightarrow P(x)e^{-x} > 0$ for all $x > 1$. $[\because P(1) = 0]$

EXERICSE 8.1

1. Find the intervals of monotonicity of the following functions :

(i) $f(x) = x^4 - 14x^2 + 24x - 25$

(ii) $f(x) = x - e^x$

(iii) $f(x) = x - 2\sin x$, $0 \leq x \leq 2\pi$

(iv) $f(x) = x\sqrt{ax - x^2}$, $a > 0$

(v) $f(x) = \frac{1 - x + x^2}{1 + x + x^2}$

(vi) $f(x) = \log(x + \sqrt{x^2 + 1})$

(vii) $f(x) = 2x^2 - \log |x|$

(viii) $f(x) = \dfrac{10}{4x^3 - 9x^2 + 6x}$

(ix) $f(x) = \left(1 + \dfrac{1}{x}\right)^x$

(x) $f(x) = \cos\left(\dfrac{\pi}{x}\right)$

(xi) $f(x) = (2^x - 1)(2^x - 2)^2$

(xii) $f(x) = 2\sin x + \cos 2x,\ 0 \le x \le 2\pi$

2. Find the values of m for which the function $f: R \to R$ given by $f(x) = x^3 - mx^2 + 3x - 11$ is invertible.

3. For what values of λ does the function

$f(x) = (\lambda + 2)x^3 - 3\lambda x^2 + 9\lambda x - 1$ decreases for all x ?

4. For what values of m does the function

$f(x) = 2e^x - me^{-x} + (2m + 1)x - 3$ increases for all x ?

5. Prove the following in equalities :

(i) $\log_e(1 + x) < x$ for all $x > 0$

(ii) $\sin x + \tan x \ge 2x$ for all $x \in (0, \pi/2)$

(iii) $2x > 3\sin x - x\cos x$ for all $x \in (0, \pi/2)$

(iv) $\sin x < x < \tan x$ for $0 < x < \dfrac{\pi}{2}$

(v) $\dfrac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$

(vi) $2x\tan^{-1} x \ge \log(1 + x^2)$ for all $x \in R$

(vii) $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$ for all $x > 0$

(viii) $x - \dfrac{x^3}{3} < \tan^{-1} x < x - \dfrac{x^3}{6}$ for $0 < x < 1$

(ix) $1 - x < e^{-x} < 1 - x + \dfrac{x^2}{2}$ for all $x \ge 0$

(x) $\cos(\sin x) > \sin(\cos x)$ for $0 < x < \dfrac{\pi}{2}$

(xi) $e^{\tan^{-1} x}(x + \sqrt{x^2 + 1}) > e^{2x}$ for $x > 0$

(xii) $2\sin x + \tan x \ge 3x$ for $0 \le x < \dfrac{\pi}{2}$

6. Prove that

$$\frac{\tan x}{\tan y} > \frac{y}{x} \text{ for all } 0 < y < x < \frac{\pi}{2}$$

7. Given that $f'(x) > g'(x)$ for all real x and $f(0) = g(0)$, prove that $f(x) > g(x)$ for all $x \in (0, \infty)$ and $f(x) < g(x)$ for all $x \in (-\infty, 0)$.

8. If $ad - bc \ne 0$, then prove that the function

$$f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$$

is either increasing everywhere or decreasing everywhere in its domain.

9. If $f(x) = x^3 + bx^2 + cx + d$ is invertible, prove that $b^2 \le 3c$.

10. Let $f(x)$ be a monotonically increasing function for all $x \in R$, $f''(x) < 0$ and $f^{-1}(x)$ exists. Prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

11. Determine the interval in which

$f(x) = 2x + \cot^{-1} x - \log(x + \sqrt{x^2 + 1})$ is increasing.

12. Find the values of parameter b, if the function $f(x)$ given by

$$f(x) = [-c^2 + (b - 1)c - 2]x + \int (\sin^2 x + \cos^4 x)\,dx$$

decreases for all $x \in R$ and for all $c \in R$.

13. For what values of b, the function $f(x) = \sin x - bx + c$ decreases for all $x \in R$?

14. Let $f(x) = \begin{cases} 4x - x^3 + \log_e(b^2 - 3b + 3), & 2 \le x < 3 \\ x - 18, & x \ge 3 \end{cases}$

Find all possible real values of b such that $f(x)$ has the smallest value at $x = 3$.

15. Let $g(x) = \int_a^x f(t)\,dt$ and $f(x)$ satisfies the following conditions :

$f(x + y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in R$

and $f'(0) = \sqrt{3 + a - a^2}$.

Prove that $g(x)$ is increasing for all $x \in R$.

16. Find the set of possible real values of a such that the function

$$f(x) = \{b^2 + (a - 1)b + 2\}x + \int_0^x (\sin^4 x + \cos^4 x)\,dx$$

is an increasing function of x for all $x \in R$ and $b \in R$.

17. Let $a + b = 4$, $a < 2$ and $g(x)$ be monotonically increasing function of x. Prove that $f(x) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases with increase in $(b - a)$.

18. Find the values of a for which the following functions increase on R :

(i) $f(x) = \dfrac{a^2 - 1}{3}x^3 + (a - 1)x^2 + 2x + 1$

(ii) $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$

(iii) $f(x) = \sin x - a\sin 2x - \dfrac{1}{3}\sin 3x + 2ax$

ANSWERS

	Increasing	*Decreasing*
(i)	$(-3,1)\cup(2,\infty)$	$(-\infty,-3)\cup(1,2)$
(ii)	$(-\infty,0)$	$(0,\infty)$
(iii)	$\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$	$\left[0,\frac{\pi}{3}\right)\cup\left(\frac{5\pi}{3},2\pi\right]$
(iv)	$\left(0,\frac{3a}{4}\right)$	$\left(\frac{3a}{4},a\right)$
(v)	$(-\infty,-1)\cup(1,\infty)$	$(-1,1)$
(vi)	R	
(vii)	$\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$	$\left(-\infty,-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)$
(viii)	$\left(\frac{1}{2},1\right)$	$(-\infty,0)\cup\left(0,\frac{1}{2}\right)\cup(1,\infty)$
(ix)	$(0,\infty)$	

(x) $\left(\frac{1}{2n+1},\frac{1}{2n}\right)$ when n is non-negative; $\left(\frac{1}{2n+2},\frac{1}{2n+1}\right)$ when n is a non-negative integer

(xi) $\left(\frac{1}{2n},\frac{1}{2n+1}\right)$, when n is negative integer; $\left(\frac{1}{2n+1},\frac{1}{2n+2}\right)$ when n is negative integer

(xii) $\left(-\infty,\log_2(4/3)\right)\cup(1,\infty)$ $\left(\log_2 4/3,1\right)$

2. $m\in(-\infty,-3)$ **3.** $\lambda\in(-\infty,-3)$

11. R

12. $b\in(-1,3)$ **13.** $b\in[1,\infty)$

14. $b\in(-\infty,1]\cup[2,\infty)$ **16.** $a\in[1-\sqrt{10},1+\sqrt{10}]$

18. (i) $a\in(-\infty,-3][1,\infty)$ (ii) $a\in[0,\infty)$

(iii) $a\in[1,\infty)$

CHAPTER

9

MAXIMA AND MINIMA

9.1 MAXIMUM AND MINIMUM VALUES OF A FUNCTION IN ITS DOMAIN

MAXIMUM *Let $f(x)$ be a function with domain $D \subset R$. Then, $f(x)$ is said to attain the maximum value at a point $a \in D$ if*

$$f(x) \leq f(a) \text{ for all } x \in D.$$

In such a case, a is called the point of maximum and $f(a)$ is known as the maximum value or the greatest value or the absolute maximum value of $f(x)$.

Consider the function

$$f(x) = -(x-1)^2 + 10 \quad \text{for all} \quad x \in R.$$

We have, $-(x-1)^2 \leq 0 \quad \text{for all} \quad x \in R$

$\Rightarrow \quad -(x-1)^2 + 10 \leq 10 \quad \text{for all} \quad x \in R$

$\Rightarrow \quad f(x) \leq 10 \quad \text{for all} \quad x \in R.$

Thus, 10 is the maximum value of $f(x)$. Clearly $f(x)$ attains this value at $x = 1$. So $x = 1$ is the point of maximum or the point of absolute maximum.

MINIMUM *Let $f(x)$ be a function with domain $D \subset R$. Then, $f(x)$ is said to attain the minimum value at a point $a \in D$ if*

$$f(x) \geq f(a) \text{ for all } x \in D.$$

In such a case, the point a is called the point of minimum and $f(a)$ is known as the minimum value or the least value or the absolute minimum value of $f(x)$.

Consider the function $f(x) = (3x-1)^2 + 5 \quad \text{for all} \quad x \in R.$

We have,

$$(3x-1)^2 \geq 0 \quad \text{for all} \quad x \in R$$

$\Rightarrow \quad (3x-1)^2 + 5 \geq 5 \quad \text{for all} \quad x \in R$

Thus, 5 is the minimum value or the least value or the absolute minimum value of $f(x)$ in its domain. Clearly, $f(x)$ attains this value at $x = 1/3$. So, $x = 1/3$ is the point of minimum or the point of absolute minimum.

ILLUSTRATION 1 *Find the maximum and the minimum values, if any, of the following functions*

(i) $f(x) = 3x^2 + 6x + 8, x \in R$

(ii) $f(x) = -|x-1| + 5$ for all $x \in R$

(iii) $f(x) = \sin 3x + 4, x \in R$

(iv) $f(x) = x^3 + 1, x \in R$

(v) $f(x) = \sin(\sin x)$

(vi) $f(x) = |x+3|$.

SOLUTION (i) We have,

$$\begin{aligned} f(x) &= 3x^2 + 6x + 8 \\ &= 3(x^2 + 2x + 1) + 5 \\ &= 3(x+1)^2 + 5. \end{aligned}$$

We have,

$$3(x+1)^2 \geq 0 \quad \text{for all} \quad x \in R$$

$\Rightarrow \quad 3(x+1)^2 + 5 \geq 5 \quad \text{for all} \quad x \in R$

$\Rightarrow \quad f(x) \geq 5 \quad \text{for all} \quad x \in R.$

Thus, 5 is the minimum value of $f(x)$ which it attains at $x = -1$.

Since $f(x)$ can be made as large as we please, therefore the maximum value does not exist.

(ii) We have,

$$f(x) = -|x-1| + 5 \quad \text{for all} \quad x \in R$$

We have,

$$|x-1| \geq 0 \quad \text{for all} \quad x \in R$$

$\Rightarrow \quad -|x-1| \leq 0 \quad \text{for all} \quad x \in R$

$\Rightarrow -|x-1| + 5 \leq 5 \quad \text{for all} \quad x \in R$

$\Rightarrow \quad f(x) \leq 5 \quad \text{for all} \quad x \in R.$

So, 5 is the maximum value of $f(x)$.

Now, $f(x) = 5$

$\Rightarrow \quad -|x-1| + 5 = 5$

$\Rightarrow |x-1| = 0 \Rightarrow x = 1.$

Thus, $f(x)$ attains the maximum value 5 at $x = 1$.

Since $f(x)$ can be made as small as we please, therefore the minimum value of $f(x)$ does not exist.

(iii) We have,

$$f(x) = \sin 3x + 4 \quad \text{for all} \quad x \in R$$

We have,

$$-1 \leq \sin 3x \leq 1 \quad \text{for all} \quad x \in R$$

$\Rightarrow -1+4 \le \sin 3x+4 \le 1+4$ for all $x \in R$

$\Rightarrow 3 \le \sin 3x+4 \le 5$ for all $x \in R$

$\Rightarrow 3 \le f(x) \le 5$ for all $x \in R$.

Thus, the maximum value of $f(x)$ is 5 and the minimum value is 3.

Now,

$f(x) = 5$

$\Rightarrow \sin 3x+4 = 5$

$\Rightarrow \sin 3x = 1$

$\Rightarrow 3x = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{6}$.

So, $f(x)$ attains its maximum value 5 at $x = \pi/6$.

And,

$f(x) = 3$

$\Rightarrow \sin 3x+4 = 3$

$\Rightarrow \sin 3x = -1$

$\Rightarrow 3x = -\frac{\pi}{2}$

$\Rightarrow x = -\frac{\pi}{6}$.

So, $f(x)$ attains the minimum value 3 at $x = -\pi/6$.

(iv) We have,

$$f(x) = x^3+1,\ x \in R.$$

Here, we observe that the values of $f(x)$ increase when the values of x are increased and $f(x)$ can be made as large as we please by giving large values to x. So $f(x)$ does not have the maximum value. Similarly, $f(x)$ can be made as small as we please by giving smaller values to x. So, $f(x)$ does not have the minimum value also.

(v) We have,

$f(x) = \sin(\sin x),\ x \in R$.

$\because -1 \le \sin x \le 1$ for all $x \in R$

$\therefore \sin(-1) \le \sin(\sin x) \le \sin 1$ for all $x \in R$

[$\because \sin x$ is an increasing function on $[-1, 1]$]

$\Rightarrow -\sin 1 \le f(x) \le \sin 1$ for all $x \in R$

This shows that the maximum value of $f(x)$ is $\sin 1$ and the minimum value is $-\sin 1$.

(vi) We have,

$f(x) = |x+3|$ for all $x \in R$

$\because |x+3| \ge 0$ for all $x \in R$

$\Rightarrow f(x) \ge 0$ for all $x \in R$.

So the minimum value of $f(x)$ is 0, which it attains at $x = -3$.

Clearly, $f(x) = |x+3|$ does not have the maximum value.

9.2 LOCAL MAXIMA AND LOCAL MINIMA

In the previous section, we have talked about the greatest (maximum) and the least (minimum) values of a function in its domain. But, there may be points in the domain of a function where the function does not attain the greatest (or the least) value but the values at these points are greater than or less than the values of the function at the neighbouring points. Such points are known as the points of local minimum or local maximum and we will be mainly discussing about the local maximum and local minimum values of a function.

LOCAL MAXIMUM *A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that*

$$f(x) < f(a) \text{ for all } x \in (a-\delta, a+\delta),\ x \ne a$$

or, $$f(x)-f(a) < 0 \text{ for all } x \in (a-\delta, a+\delta),\ x \ne a.$$

In such a case $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.

LOCAL MINIMUM *A function $f(x)$ is said to attain a local minimum at $x = a$ if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that*

$$f(x) > f(a) \quad \text{for all } x \in (a-\delta, a+\delta),\ x \ne a$$

or, $$f(x)-f(a) > 0 \quad \text{for all } x \in (a-\delta, a+\delta),\ x \ne a.$$

The value of the function at $x = a$ i.e., $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$. Thus, a function attains an extreme value at $x = a$ if $f(a)$ is either a local maximum value or a local minimum value. Consequently at an extreme point 'a', $f(x)-f(a)$ keeps the same sign for all values of x in a deleted nbd of a.

In Fig. 9.1, we observe that the x-coordinates of the points A, C, E are points of local maximum and the values at these points i.e., their y-coordinates are the local maximum values of $f(x)$. The x-coordinates of points B and D are points of local minimum and their y-coordinates are the local minimum values of $f(x)$.

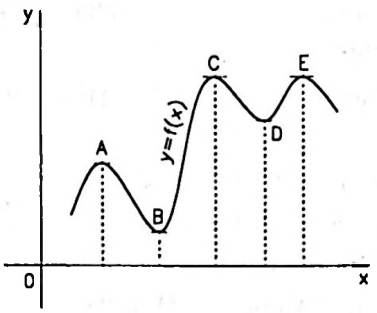


Fig. 9.1

NOTE *By a local maximum (or local minimum) value of a function at a point $x = a$ we mean the greatest (or the least) value in the neighbourhood of point $x = a$ and not the absolute maximum (or the absolute minimum). In fact a function may have any number of points of local maximum (or local minimum) and even a local minimum value may be greater than a local maximum value. In Fig. 9.1 the*

minimum value at D is greater than the maximum value at A. Thus, a local maximum value may not be the greatest value and a local minimum value may not be the least value of the function in its domain.

NECESSARY CONDITION FOR EXTREME VALUES

We have the following theorem which we state without proof.

THEOREM *A necessary condition for $f(a)$ to be an extreme value of a function $f(x)$ is that $f'(a) = 0$, in case it exists.*

REMARK 1 *This result states that if the derivative exists, it must be zero at the extreme points.*

A function may however attain an extreme value at a point without being derivable thereat. For example, the function $f(x) = |x|$ attains the minimum value at the origin even- though it is not derivable at $x = 0$.

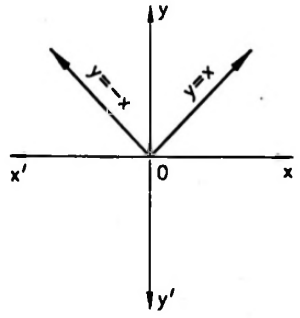


Fig. 9.2

REMARK 2 *This condition is only a necessary condition for the point $x = a$ to be an extreme point. It is not sufficient i.e., $f'(a) = 0$ does not necessarily imply that $x = a$ is an extreme point. There are functions for which the derivatives vanish at a point but do not have an extreme value thereat. For example, for the function $f(x) = x^3$, $f'(0) = 0$ but at $x = 0$ the function does not attain an extreme value.*

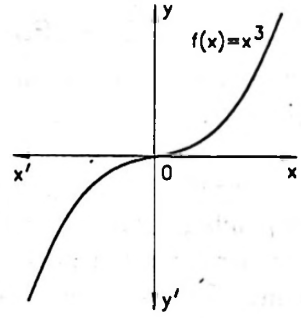


Fig. 9.3

REMARK 3 *Geometrically the above condition means that the tangent to the curve $y = f(x)$ at a point where the ordinate is maximum or minimum is parallel to the x-axis.*

REMARK 4 *As discussed in Remark 2 that all x, for which $f'(x) = 0$, do not give us the extreme values. The values of x for which $f'(x) = 0$ are called **stationary values or critical values** of x and the corresponding values of $f(x)$ are called **stationary or turning values** of $f(x)$.*

As we have seen in Remark 2 that $f'(a) = 0$ is not the sufficient condition for $x = a$ to be an extreme point. The following theorem provides the sufficient conditions for $x = a$ to be an extreme point. This is known as the first derivative test and is stated without any proof.

FIRST DERIVATIVE TEST FOR LOCAL MAXIMA AND MINIMA

Let $f(x)$ be a function differentiable at $x = a$. Then,

(a) $x = a$ is a point of local maximum of $f(x)$ if

(i) $f'(a) = 0$ *and*

(ii) *$f'(x)$ changes sign from positive to negative as x passes through a i.e., $f'(x) > 0$ at every point in the left nbd $(a - \delta, a)$ of a and $f'(x) < 0$ at every point in the right nbd $(a, a + \delta)$ of a.*

(b) $x = a$ is a point of local minimum of $f(x)$ if

(i) $f'(a) = 0$ *and*

(ii) *$f'(x)$ changes sign from negative to positive as x passes through a i.e., $f'(x) < 0$ at every point in the left nbd $(a - \delta, a)$ of a and $f'(x) > 0$ at every point in the right nbd $(a, a + \delta)$ of a.*

(c) *If $f'(a) = 0$, but $f'(x)$ does not change sign, that is, $f'(a)$ has the same sign in the complete nbd of a, then a is neither a point of local maximum nor a point of local minimum.*

The proof of this result is beyond the scope of this book.

In order to find the extreme values of a function, we may use the following algorithm :

ALGORITHM

STEP I *Put $y = f(x)$*

STEP II *Find $\frac{dy}{dx}$.*

STEP III *Put $\frac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2, c_3, \ldots,$ c_n be the roots of this equation. $c_1, c_2, c_3, \ldots, c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum. So we test the function at each one of these points.*

STEP IV *Consider $x = c_1$. If $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a local maximum at $x = c_1$.*

If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a local minimum at $x = c_1$.

If $\frac{dy}{dx}$ does not change sign as x increases through c_1, then $x = c_1$ is neither a point of local maximum nor a point of local minimum. In this case $x = c_1$ is a point of inflexion.

Similarly we may deal with other values of x.

ILLUSTRATION 1 *Find all the points of local maxima and minima of the function $f(x) = x^3 - 6x^2 + 9x - 8$.*

SOLUTION Let $y = f(x) = x^3 - 6x^2 + 9x - 8$. Then,

$$\frac{dy}{dx} = f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3).$$

For a local maximum or local minimum, we have

$$\frac{dy}{dx} = 0$$

$\Rightarrow 3(x^2-4x+3)=0 \Rightarrow x=1, 3$

We have to examine whether these points are points of local maximum or local minimum or neither of them.

The signs of $\frac{dy}{dx}$ for different values of x are shown in Fig. 9.4.

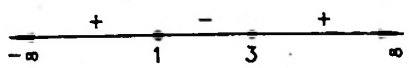


Fig. 9.4

Clearly, $\frac{dy}{dx}$ changes sign from positive to negative as it increases through 1. Therefore, $x=1$ is a point of local maximum.

Also, $\frac{dy}{dx}$ changes sign from negative to positive as x passes through 3. So, $x=3$ is a point of local minimum.

ILLUSTRATION 2 *Find the local maxima or local minima, if any, of the function*

$$f(x)=\sin^4 x+\cos^4 x,\ 0<x<\frac{\pi}{2}$$

using the first derivative test.

SOLUTION We have,

$$y=f(x)=\sin^4 x+\cos^4 x$$

$$\therefore \frac{dy}{dx}=4\sin^3 x\cos x-4\cos^3 x\sin x$$

$$=-4\cos x\sin x(\cos^2 x-\sin^2 x)$$

$$=-2\sin 2x\cos 2x$$

$$=-\sin 4x$$

For a local maximum or a local minimum, we have

$$\frac{dy}{dx}=0$$

$$\Rightarrow -\sin 4x=0$$

$$\Rightarrow \sin 4x=0$$

$$\Rightarrow 4x=\pi \qquad \left[\because 0<x<\frac{\pi}{2} \therefore 0<4x<2\pi\right]$$

$$\Rightarrow x=\frac{\pi}{4}$$

In the left nbd of $x=\frac{\pi}{4}$, we have

$$x<\frac{\pi}{4} \Rightarrow 4x<\pi \Rightarrow \sin 4x>0 \Rightarrow -\sin 4x<0 \Rightarrow \frac{dy}{dx}<0$$

In the right nbd of $x=\frac{\pi}{4}$, we have

$$x>\frac{\pi}{4} \Rightarrow 4x>\pi \Rightarrow \sin 4x<0 \Rightarrow -\sin 4x>0 \Rightarrow \frac{dy}{dx}>0$$

Thus, $\frac{dy}{dx}$ changes sign from negative to positive as x increases through $\frac{\pi}{4}$. So, $x=\frac{\pi}{4}$ is a point of local minimum.

The local minimum value is

$$f\left(\frac{\pi}{4}\right)=\left(\sin\frac{\pi}{4}\right)^4+\left(\cos\frac{\pi}{4}\right)^4=\frac{1}{2}.$$

ILLUSTRATION 3 *Let*

$$f(x)=\begin{cases}x^3+x^2+10x, & x<0\\ -3\sin x, & x\geq 0\end{cases}$$

Investigate $x=0$ for local maximum/minimum.

SOLUTION We have,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

So, $f(x)$ is continuous at $x=0$.

Now,

$$(\text{LHD at } x=0)=\left(\frac{d}{dx}(x^3+x^2+10)\right)_{x=0}$$

$$=(3x^2+2x+10)_{at\ x=0}$$

$$=10>0$$

and,

$$(\text{RHD at } x=0)=\left(\frac{d}{dx}(-3\sin x)\right)_{at\ x=0}$$

$$=(-3\cos x)_{at\ x=0}$$

$$=-3<0$$

Thus, $f(x)$ is changing its sign from positive to negative in the neighbourhood of $x=0$.

Hence, $x=0$ is a point of local maximum.

9.3 HIGHER ORDER DERIVATIVE TEST

As we have seen in an above section that finding the local maximum or local minimum by first derivative test is very time consuming and of course tedious for beginners because it is slightly difficult to determine the change in the sign of $f'(x)$ as x passes through the points given by $f'(x)=0$. We have another test known as the Higher order derivative test which enables us to find the local maxima or local minima more easily and more quickly.

HIGHER ORDER DERIVATIVE TEST

Let f be a differentiable function on an interval I and let c be an interior point of I such that

(i) $f'(c)=f''(c)=f'''(c)=\ldots=f^{n-1}(c)=0$, *and*

(ii) $f^n(c)$ *exists and is non-zero.*

Then,

if n is even and $f^n(c)<0 \Rightarrow x=c$ is a point of local maximum

if n is even and $f^n(c)>0 \Rightarrow x=c$ is a point of local minimum

if n is odd $\Rightarrow$ $x=c$ is neither a point of local maximum nor a point of local minimum.

From the above test criteria we obtain the following rule for determining maxima and minima of $f(x)$:

ALGORITHM

STEP I *Find $f'(x)$*

STEP II *Put $f'(x) = 0$ and solve this equation for x. Let $c_1, c_2, \ldots, c_n$ be the roots of this equation. $c_1, c_2, \ldots, c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum. So we test the function at each one of these points.*

STEP III *Find $f''(x)$. Consider $x = c_1$.*

If $f''(c_1) < 0$, then $x = c_1$ is a point of local maximum.

If $f''(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f''(c_1) = 0$, we must find $f'''(x)$ and substitute in it c_1 for x.

If $f'''(c_1) \neq 0$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is called the point of inflection.

If $f'''(c_1) = 0$, we must find $f^{IV}(x)$ and substitute in it c_1 for x.

If $f^{IV}(c_1) < 0$, then $x = c_1$ is a point of local maximum and if $f^{IV}(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f^{IV}(c_1) = 0$, we must find $f^{V}(x)$, and so on. Similarly the values of $c_2, c_3, \ldots$, may be tested.

POINT OF INFLECTION An arc of a curve $y = f(x)$ is called *concave upward* if, at each of its points, the arc lies above the tangent at the point (see Fig. 9.5).

If $y = f(x)$ is a concave upward curve, then as x increases, $f'(x)$ either is of the same sign and increasing (see Fig. 9.5) or changes sign from negative to positive (see Fig. 9.6). In either case $f'(x)$ is increasing and so $f''(x) > 0$.

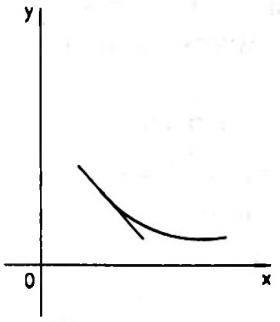


Fig. 9.5

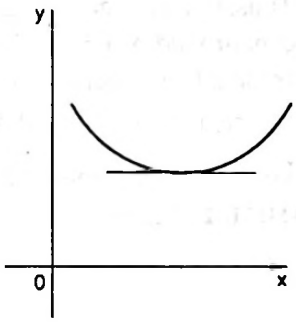


Fig. 9.6

Thus, for a concave upward curve $f''(x) > 0$.

An arc of a curve $y = f(x)$ is called *concave downward* if, at each of its points, the arc lies below the tangent at the point.

If an arc of a curve $y = f(x)$ is *concave downward*, then as x increases, $f'(x)$ either is of the same sign and decreasing (see Fig. 9.7) or changes sign from positive to negative (see Fig. 9.8). In either case $f'(x)$ is decreasing and so $f''(x) < 0$.

Thus, for a concave downward curve $f''(x) < 0$.

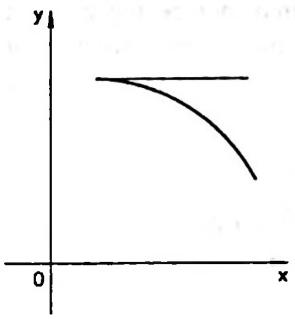


Fig. 9.7

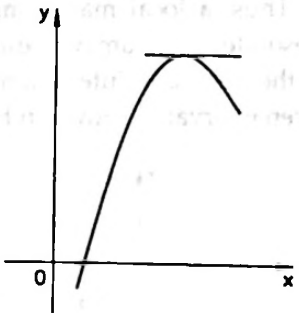


Fig. 9.8

DEFINITION *A point of inflection is a point at which a curve is changing concave upward to concave downward, or vice-versa.*

A curve $y = f(x)$ has one of its points $x = c$ as an inflection point

If $f''(c) = 0$ or is not defined and

If $f''(x)$ changes sign as x increases through $x = c$.

The later condition may be replaced by $f'''(c) \neq 0$ when $f'''(c)$ exists.

Thus, $x = c$ is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.

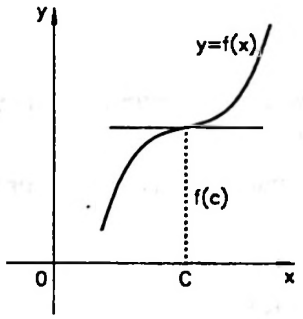


Fig. 9.9

PROPERTIES OF MAXIMA AND MINIMA

(I) *If $f(x)$ is continuous function in its domain, then at least one maxima and one minima must lie between two equal values of x.*

(II) *Maxima and Minima occur alternately, that is, between two maxima there is one minimum and vice-versa.*

(III) *If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.*

If $f(x) \to -\infty$ as $x \to a$ or b, then $f(c)$ is necessarily the maximum and the greatest value.

9.4 MAXIMUM AND MINIMUM VALUES IN A CLOSED INTERVAL

Let $y = f(x)$ be a function defined on [a, b]. By a local maximum (or local minimum) value of a function at a point $c \in [a, b]$ we mean the greatest (or the least) value in the immediate neighbourhood of $x = c$. It does not mean the greatest or absolute maximum (or the least or absolute minimum) of $f(x)$ in the interval $[a, b]$. A function may have a number of local maxima or local minima in a given interval and even a local minimum may be greater than a relative maximum.

Thus, a local maximum value may not be the greatest (absolute maximum) value and a local minimum value may not be the least (absolute minimum) value of the function in any given interval as shown in Fig. 9.10.

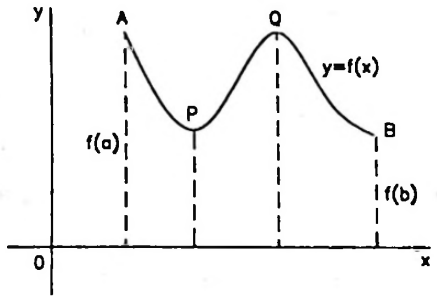


Fig. 9.10

However, if a function $f(x)$ is differentiable and consequently continuous on a closed interval $[a, b]$, then it attains the absolute maximum (absolute minimum) at stationary points (points where $f'(x) = 0$) or at the end points of the interval $[a, b]$. Thus, to find the absolute maximum (absolute minimum) value of the function, we choose the largest and the smallest amongst the numbers $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$ where $x = c_1, c_2, \ldots, c_n$ are the stationary points.

The following is the working rule for finding the maximum (absolute maximum) and the minimum (absolute minimum) of a function defined on a closed interval.

ALGORITHM

Let $y = f(x)$ be a function defined on $[a, b]$.

STEP I *Find* $dy/dx = f'(x)$

STEP II *Put* $f'(x) = 0$ *and find values of* x. *Let* $c_1, c_2, \ldots, c_n$ *be the values of* x.

STEP III *Take the maximum and minimum values out of the values* $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$.

The maximum and minimum values obtained in step III are respectively the largest or absolute maximum and the smallest or absolute minimum values of the function.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the local maximum and local minimum values of the function* $f(x) = x^x$.

SOLUTION Clearly, the domain of $f(x)$ is the set of all positive real numbers.

Now,

$$f(x) = x^x = e^{x \log_e x}$$

$$\Rightarrow f'(x) = e^{x \log x} \frac{d}{dx}(x \log_e x)$$

$$\Rightarrow f'(x) = x^x (1 + \log_e x)$$

For local maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow x^x (1 + \log_e x) = 0$$

$$\Rightarrow \log_e x = -1$$

$$\Rightarrow x = e^{-1}.$$

Now,

$$f'(x) = x^x (1 + \log_e x)$$

$$\Rightarrow f''(x) = x^x (1 + \log_e x)^2 + x^{x-1}$$

$$\therefore f''(e^{-1}) = \left(\frac{1}{e}\right)^{1/e - 1} > 0$$

Hence, $x = \frac{1}{e}$ is a point of local minimum of $f(x)$.

$$\text{Local minimum value} = f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} = e^{-1/e}$$

ALITER We have,

$$f'(x) = x^x (1 + \log_e x)$$

$$\therefore \quad f'(x) = 0 \Rightarrow x = 1/e$$

We observe that

$$f'(x) < 0 \text{ for } x < \frac{1}{e}$$

and,

$$f'(x) > 0 \text{ for } x > \frac{1}{e}.$$

Thus, $f'(x)$ changes its sign from negative to positive in the neighbourhood of $x = 1/e$. In other words, $f(x)$ changes from decreasing to increasing at $x = 1/e$.

Hence, $x = 1/e$ is a point of local minimum.

Local minimum value $= f(1/e) = (1/e)^{1/e} = e^{-1/e}$.

EXAMPLE 2 *Determine the points of maxima and minima of the function* $f(x) = \frac{1}{8} \log x - bx + x^2$, $x > 0$, *where* $b \geq 0$ *is a constant.*

[IIT 1996]

SOLUTION We have,

$$f(x) = \frac{1}{8} \log x - bx + x^2$$

$$\Rightarrow \quad f'(x) = \frac{1}{8x} - b + 2x$$

$$\Rightarrow \quad f'(x) = \frac{16x^2 - 8bx + 1}{8x}$$

For points of local maximum and local minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow \frac{16x^2 - 8bx + 1}{8x} = 0$$

$$\Rightarrow 16x^2 - 8bx + 1 = 0$$

$$\Rightarrow x = \frac{8b \pm \sqrt{64b^2 - 64}}{32}$$

$$\Rightarrow x = \frac{b \pm \sqrt{b^2 - 1}}{4}$$

Now three cases arise :

CASE I *When* $0 \le b < 1$:

In this case,

$$f'(x) = \frac{16x^2 - 8bx + 1}{8x} > 0, \text{ for all } x > 0 \quad \left[\because \text{Disc. of } 16x^2 - 8bx + 1 = 0 \text{ is less than zero}\right]$$

$\Rightarrow f(x)$ is strictly increasing for all $x > 0$

$\Rightarrow f(x)$ has no local maximum or local minimum.

CASE II *When* $b = 1$:

In this case,

$$f'(x) = \frac{16x^2 - 8x + 1}{8x}$$

$$\therefore f'(x) = 0$$

$$\Rightarrow 16x^2 - 8x + 1 = 0$$

$$\Rightarrow (4x - 1)^2 = 0$$

$$\Rightarrow x = \frac{1}{4}$$

Now,

$$f'(x) = \frac{16x^2 - 8x + 1}{8x} = 2x - 1 + \frac{1}{8x}$$

$$\Rightarrow f''(x) = 2 - \frac{1}{8x^2} \text{ and } f'''(x) = \frac{1}{4x^3}$$

Clearly, $f''\left(\frac{1}{4}\right) = 0$ and $f'''\left(\frac{1}{4}\right) = 16 \neq 0$.

So, $x = \frac{1}{4}$ is a point of inflexion.

Hence, in this case also there is no local maximum or local minimum.

CASE III *When* $b > 1$:

In this case, we have

$$f'(x) = \frac{16x^2 - 8bx + 1}{8x}$$

and $f'(x) = 0$ at $x = \frac{b \pm \sqrt{b^2 - 1}}{4}$

Let $x_1 = \frac{b + \sqrt{b^2 - 1}}{4}$ and $x_2 = \frac{b - \sqrt{b^2 - 1}}{4}$.

Clearly, $x_1 > x_2 > 0$

Now,

$$f'(x) = \frac{16x^2 - 8bx + 1}{8x}$$

$$= \frac{16}{8x}\left\{x^2 - \frac{1}{2}bx + \frac{1}{16}\right\}$$

$$= \frac{2}{x}(x - x_1)(x - x_2) \quad \left[\because x_1, x_2 \text{ are roots of } x^2 - \frac{1}{2}bx + \frac{1}{16} = 0\right]$$

The changes in signs of $x^2 - \frac{1}{2}bx + \frac{1}{16} = (x - x_1)(x - x_2)$ are shown in Fig. 9.11.

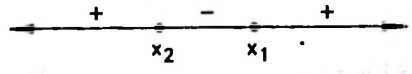


Fig. 9.11

We find that $f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = x_2$ and negative to positive in the neighbourhood of $x = x_1$.

Hence, $x = x_2$ is a point of local maximum and $x = x_1$ is a point of local minimum.

EXAMPLE 3 *Suppose* $f(x)$ *is a function satisfying the following conditions :*

(i) $f(0) = 2, f(1) = 1$

(ii) f *has a minimum value at* $x = \frac{5}{2}$,

and, (iii) *for all* x

$$f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$$

where a, b *are constants.*

Determine the constants a, b and *the function* $f(x)$. [IIT 1998]

SOLUTION We have,

$$f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Applying } R_3 \to R_3 - R_1 - 2R_2$$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & 2ax - 1 \\ b & b + 1 \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad \text{Applying } C_2 \to C_2 - C_1$$

$$\Rightarrow f'(x) = 2ax + b \quad \ldots(i)$$

$\Rightarrow f(x) = ax^2 + bx + c$ ...(ii) [On integration]

Now,

$f(0) = 2$ and $f(1) = 1$

$\Rightarrow \quad c = 2$ and $a + b + c = 1$

$\Rightarrow \quad c = 2$ and $a + b = -1$ ...(iii)

Since $f(x)$ has a minimum at $x = \frac{5}{2}$. Therefore,

$f'(x) = 0$ at $x = \frac{5}{2}$

$\Rightarrow 2a \times \frac{5}{2} + b = 0$ [Using (i)]

$\Rightarrow 5a + b = 0$ ...(iv)

Solving (iii) and (iv), we get

$a = \frac{1}{4}, b = \frac{-5}{4}$

Substituting the values of a, b and c in (ii), we get

$$f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$$

EXAMPLE 4 *Let* $f(x) = \begin{cases} x^2 + 2x + \frac{a^2-1}{a^4-5} & ; x < 0 \\ x + 2 & ; x > 0 \end{cases}$

Find all possible real values of a such that $f(x)$ possesses the smallest value at $x = 0$.

SOLUTION Clearly, $f(x)$ is not continuous at $x = 0$.

Also, $f'(x) > 0$ for all $x > 0$.

So, $f(x)$ is increasing for all $x > 0$.

For $x < 0$, the curve $y = f(x)$ is a parabola opening upwards. Therefore, $f(x)$ will attain its smallest value at $x = 0$, if the absolute minimum value of $f(x)$ is greater than $f(0)$.

For $x < 0$, we have

$$f(x) = x^2 + 2x + \frac{a^2-1}{a^4-5}$$

$\Rightarrow f'(x) = 2x + 2$

$\Rightarrow f'(x) = 0$ at $x = -1$.

Clearly, $f''(x) = 2 > 0$ for all $x < 0$.

So, $f(x)$ attains local minimum at $x = -1$.

local minimum value of $f(x)$

$= f(-1)$

$= 1 - 2 + \frac{a^2-1}{a^4-5}$

$= \frac{a^2-1}{a^4-5} - 1.$

Now,

$f(-1) > f(0)$

$\Rightarrow \frac{a^2-1}{a^4-5} - 1 > 2$

$\Rightarrow \frac{a^2-1}{a^4-5} - 3 > 0$

$\Rightarrow \frac{a^2 - 3a^4 + 14}{a^4-5} > 0$

$\Rightarrow \frac{3a^4 - a^2 - 14}{a^4-5} < 0$

$\Rightarrow \frac{(a^2+2)(3a^2-7)}{(a^2-5)(a^2+5)} < 0$

$\Rightarrow \frac{3a^2-7}{a^2-5} < 0$

$\Rightarrow a \in \left(-\sqrt{\frac{7}{3}}, -\sqrt{5}\right) \cup \left(\sqrt{5}, \sqrt{\frac{7}{3}}\right)$

EXAMPLE 5 *Find the polynomial $f(x)$ of degree 6, which satisfies* $\lim_{x \to 0} \left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e^2$ *and has local maximum at $x = 1$ and local minimum at $x = 0$ and 2.*

SOLUTION Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

be the required polynomial.

Now,

$$\lim_{x \to 0} \left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e^2$$

$\Rightarrow \lim_{x \to 0} \frac{f(x)}{x^3} = 0$ and $e^{\lim_{x \to 0} \frac{f(x)}{x^4}} = e^2$

$\Rightarrow \lim_{x \to 0} \frac{f(x)}{x^3} = 0$ and $\lim_{x \to 0} \frac{f(x)}{x^4} = 2$

$\Rightarrow \lim_{x \to 0} \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6}{x^3} = 0$

and,

$$\lim_{x \to 0} \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6}{x^4} = 2$$

$\Rightarrow a_0 = a_1 = a_2 = a_3 = 0$

and,

$$\lim_{x \to 0} \frac{a_4 x^4 + a_5 x^5 + a_6 x^6}{x^4} = 2$$

$\Rightarrow a_0 = a_1 = a_2 = a_3 = 0$ and $a_4 = 2$

$\therefore f(x) = 2x^4 + a_5 x^5 + a_6 x^6$

$\Rightarrow f'(x) = 8x^3 + 5a_5 x^4 + 6a_6 x^5$

Since $f(x)$ has local maximum at $x = 1$ and minimum at $x = 0$ and 2.

Therefore,

$f'(0) = 0, f'(1) = 0$ and, $f'(2) = 0$

$\Rightarrow 8 + 5a_5 + 6a_6 = 0$ and, $64 + 80a_5 + 192a_6 = 0$

$\Rightarrow a_5 = -\frac{12}{5}$ and, $a_6 = \frac{2}{3}$

Hence, $f(x) = 2x^4 - \frac{12}{5}x^5 + \frac{2}{3}x^6$.

EXAMPLE 6 *A cubic polynomial $f(x)$ vanishes at $x = -2$ and has a relative minimum/maximum at $x = -1$ and $x = 1/3$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find the cubic $f(x)$.* [IIT 1992]

SOLUTION Since $f(x)$ is a cubic polynomial. Therefore, $f'(x)$ is a quadratic polynomial. It is given that $f(x)$ has relative minimum/maximum at $x = -1$ and $x = 1/3$. Therefore,

$f'(-1) = 0$ and $f'(1/3) = 0$

$\Rightarrow x + 1$ and $x - \frac{1}{3}$ are factors of $f'(x)$.

So, let

$f'(x) = a(x+1)\left(x - \frac{1}{3}\right)$, where a is a constant.

$\Rightarrow f'(x) = a\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$

$\Rightarrow f(x) = a\left(\frac{x^3}{3} + \frac{1}{3}x^2 - \frac{1}{3}x\right) + b,$

where b is the constant of integration.

Now,

$f(-2) = 0$ [Given]

$\Rightarrow \frac{-8a}{3} + \frac{4}{3}a + \frac{2}{3}a + b = 0$

$\Rightarrow b = \frac{2}{3}a$

$\therefore f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3} + \frac{2}{3}\right)$

Now,

$\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$

$\Rightarrow \int_{-1}^{1} a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3} + \frac{2}{3}\right)dx = \frac{14}{3}$

$\Rightarrow a\int_{-1}^{1} (x^3 + x^2 - x + 2) = 14$

$\Rightarrow a\int_{-1}^{1} (x^2 + 2)\,dx = 14$

$\Rightarrow 2a\int_{0}^{1} (x^2 + 2)\,dx = 14$

$\Rightarrow 2a\left[\frac{x^3}{3} + 2x\right]_0^1 = 14$

$\Rightarrow \frac{14a}{3} = 14$

$\Rightarrow a = 3$

Hence, $f(x) = x^3 + x^2 - x + 2$.

EXAMPLE 7 *Let $f(x) = 2x^3 - 3(a+b)x^2 + 6abx$. Find the relative maximum/minimum of $f(x)$ if*

(i) $a < b$ (ii) $a > b$. (iii) $a = b$

SOLUTION We have,

$f(x) = 2x^3 - 3(a+b)x^2 + 6abx$

$\Rightarrow f'(x) = 6x^2 - 6(a+b)x + 6ab$

$\Rightarrow f'(x) = 6(x-a)(x-b)$

For points of local maximum or minimum, we must have

$f'(x) = 0$

$\Rightarrow 6(x-a)(x-b) = 0$

$\Rightarrow x = a, b.$

CASE I *When $a < b$*

In this case, the changes in signs of $f'(x)$ for different values of x are shown in Fig. 9.12

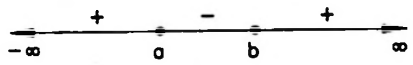


Fig. 9.12

Clearly, $f'(x)$ changes its sign from positive to negative in the neighbourhood of 'a' and negative to positive in the neighbourhood of 'b'. Therefore, $x = a$ is a point of local maximum and $x = b$ is a piont of local minimum.

CASE II *When $a > b$*

In this case, the changes in signs of $f'(x)$ for different values of x are shown in Fig. 9.13

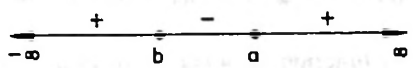


Fig. 9.13

Since $f'(x)$ changes its sign from positive to negative in the neighbourhood of b and negative to positive in the neighbourhood of a. Therefore, $x = b$ and $x = a$ are points of local maximum and local minimum respectively.

CASE III *When* $a = b$

In this case,

$$f'(x) = 6(x-a)^2$$

Clearly, there is no change in the sign of $f'(x)$ in the neighbourhood of a.

Hence, a is the point of inflexion.

EXAMPLE 8 *Find a polynomial $f(x)$ of degree 4 such that $f(0) = 1$ and it increases in the intervals $(-\infty, 1)$ and $(2, 3)$ and decreases in the intervals $(1, 2)$ and $(3, \infty)$.*

SOLUTION It is given that $f(x)$ is increasing for all $x \in (-\infty, 1) \cup (2, 3)$ and decreasing for all $x \in (1, 2) \cup (3, \infty)$. Therefore,

$$f'(x) > 0 \text{ for all } x \in (-\infty, 1) \cup (2, 3)$$

and, $$f'(x) < 0 \text{ for all } x \in (1, 2) \cup (3, \infty).$$

The changes in signs of $f'(x)$ for different values of x are shown in Fig. 9.14

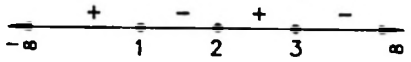


Fig. 9.14

We find that $f'(x)$ is changing its sign from positive to negative in the neighbourhood of $x = 1$. Therefore, $x = 1$ is a point of local maximum. Similarly, $x = 3$ is a point of local maximum and $x = 2$ is a point of local minimum. Thus, $x = 1, 2, 3$ are roots of $f'(x)$. So, let

$$f'(x) = \lambda(x-1)(x-2)(x-3)$$

$$\Rightarrow f'(x) = \lambda(x^3 - 6x^2 + 11x - 6)$$

$$\Rightarrow f(x) = \lambda\left\{\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right\} + c$$

Now,

$$f(0) = 1$$

$$\Rightarrow \quad 1 = c$$

$$\text{Hence, } f(x) = \lambda\left\{\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right\} + 1$$

$$= A(x^4 - 8x^3 + 22x^2 - 24x) + 1, \text{ where } A = \frac{\lambda}{4}.$$

EXAMPLE 9 *For what real values of 'a' are all the extrema for the function*

$$f(x) = \frac{5}{3}a^2x^3 + 2ax^2 - 9x + b$$

are positive and the function has a maximum at the point $x = -5/9$.

SOLUTION We have

$$f(x) = \frac{5}{3}a^2x^3 + 2ax^2 - 9x + b$$

$$\therefore \quad f'(x) = 5a^2x^2 + 4ax - 9$$

$$\text{and, } f''(x) = 10a^2x + 4a$$

Since $x = -\frac{5}{9}$ is a point of local maximum. Therefore,

$$f'\left(\frac{-5}{9}\right) = 0 \text{ and } f''\left(\frac{-5}{9}\right) < 0$$

$$\Rightarrow 5a^2 \times \frac{25}{81} + 4a \times \frac{-5}{9} - 9 = 0 \text{ and, } 10a^2 \times \frac{-5}{9} + 4a < 0$$

$$\Rightarrow 125a^2 - 180a - 729 = 0 \quad \text{and, } -25a^2 + 18a < 0$$

$$\Rightarrow a = \frac{81}{25}, \frac{-9}{5} \quad \text{and, } \left(a < 0 \text{ or } a > \frac{18}{25}\right)$$

$$\Rightarrow a = \frac{81}{25}, \frac{-9}{5}.$$

CASE I *When* $a = 81/25$

We have,

$$f'(x) = \frac{9}{125}(729x^2 + 180x - 125)$$

Now,

$$f'(x) = 0$$

$$\Rightarrow 729x^2 + 180x - 125 = 0$$

$$\Rightarrow (9x + 5)(81x - 25) = 0$$

$$\Rightarrow x = -\frac{5}{9}, \frac{25}{81}$$

Thus, the points of extremum are $x = \frac{-5}{9}, \frac{25}{81}$.

For the function to have positive extremum, we must have

$$f\left(-\frac{5}{9}\right) > 0 \text{ and, } f\left(\frac{25}{81}\right) > 0$$

$$\Rightarrow \frac{5}{3} \times \left(\frac{81}{25}\right)^2 \times \left(-\frac{5}{9}\right)^3 + 2 \times \frac{81}{25} \times \left(\frac{-5}{9}\right)^2 - 9 \times \frac{-5}{9} + b > 0$$

and,

$$\frac{5}{3} \times \left(\frac{81}{25}\right)^2 \times \left(\frac{25}{81}\right)^3 + 2 \times \frac{81}{25} \times \left(\frac{25}{81}\right)^2 - 9 \times \frac{25}{81} + b > 0$$

$$\Rightarrow -3 + 2 + 5 + b > 0 \text{ and } \frac{125}{243} + \frac{50}{81} - \frac{225}{81} + b > 0$$

$$\Rightarrow b + 4 > 0 \quad \text{and } 243b - 400 > 0$$

$$\Rightarrow b > -4 \quad \text{and } b > \frac{400}{243}$$

$$\Rightarrow b > \frac{400}{243}$$

$\Rightarrow b \in (400/243, \infty)$.

CASE II *When* $a = -9/5$

In this case, we have

$$f'(x) = \frac{9}{5}(9x^2 - 4x - 5)$$

$\therefore f'(x) = 0$

$\Rightarrow x = 1, \frac{-5}{9}$

Thus, the points of extremum are $x = 1, x = \frac{-5}{9}$.

For the function $f(x)$ to have positive extremum, we must have

$$f(1) > 0 \text{ and } f\left(\frac{-5}{9}\right) > 0$$

$$\Rightarrow \frac{5}{3} \times \left(\frac{-9}{5}\right)^2 \times 1^3 + 2 \times \left(\frac{-9}{5}\right) \times 1^2 - 9 + b > 0 \text{ and } b > \frac{400}{243}$$

$$\Rightarrow \frac{27}{5} - \frac{18}{5} - 9 + b > 0 \text{ and } b > \frac{400}{243}$$

$$\Rightarrow b > \frac{36}{5} \text{ and } b > \frac{400}{243}$$

$$\Rightarrow b > \frac{36}{5}$$

$\Rightarrow b \in (36/5, \infty)$

Thus, if $a = \frac{81}{25}$, then $b \in (400/243, \infty)$

if $a = -\frac{9}{5}$, then $b \in (36/5, \infty)$

EXAMPLE 10 *For what values of the parameter 'a' does the function*

$$f(x) = x^3 + 3(a-7)x^2 + 3(a^2 - 9)x - 1$$

have a positive point of maximum.

SOLUTION We have,

$$f(x) = x^3 + 3(a-7)x^2 + 3(a^2 - 9)x - 1$$

$$\therefore f'(x) = 3x^2 + 6(a-7)x + 3(a^2 - 9)$$

$$= 3\{x^2 + 2(a-7)x + (a^2 - 9)\}$$

For local maximum or minimum, we must have

$f'(x) = 0$

$$\Rightarrow x^2 + 2(a-7)x + a^2 - 9 = 0$$

$$\Rightarrow x = -(a-7) \pm \sqrt{58 - 14a}$$

Let $x_1 = -(a-7) + \sqrt{58 - 14a}$ and $x_2 = -(a-7) - \sqrt{58 - 14a}$

For x_1, x_2 to be real, we must have

$$58 - 14a > 0 \Rightarrow a < \frac{58}{14} = \frac{29}{7}$$

Now,

$$f''(x) = 3\{2x + 2(a-7)\}$$

$$= 6(x + a - 7)$$

Clearly, $f''(x_2) = 6(x_2 + a - 7) = -6\sqrt{58 - 14a} < 0$

So, $x_2 = -(a-7) - \sqrt{58 - 14a}$ is a point of local maximum.

Now,

$x_2 > 0$

$$\Rightarrow -(a-7) - \sqrt{58 - 14a} > 0$$

$$\Rightarrow 7 - a > \sqrt{58 - 14a}$$

$$\Rightarrow (7-a)^2 > 58 - 14a$$

$$\Rightarrow a^2 - 9 > 0$$

$\Rightarrow a < -3$ or $a > 3$

Hence, $a \in (-\infty, -3) \cup (3, 29/7)$

EXAMPLE 11 *Investigate for maxima and minima the function*

$$f(x) = \int_1^x \left\{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\right\} dt \quad \textbf{[IIT 1998]}$$

SOLUTION We have,

$$f(x) = \int_1^x \left\{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\right\} dt$$

$$\therefore f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$$

$$= (x-1)(x-2)^2\{2(x-2) + 3(x-1)\}$$

$$= (x-1)(x-2)^2(5x-7)$$

For points of local maximum and minimum, we must have

$f'(x) = 0$

$$\Rightarrow (x-1)(x-2)^2(5x-7) = 0$$

$$\Rightarrow x = 1, 2, 7/5$$

Now,

$$f''(x) = (x-2)^2(5x-7) + 5(x-1)(x-2)^2 + 2(x-1)(x-2)(5x-7)$$

$\therefore f''(1) = -2 < 0$

So, $x = 1$ is a point of local maximum.

$$f''\left(\frac{7}{5}\right) = 5\left(\frac{2}{5}\right)\left(\frac{-3}{5}\right)^2 > 0$$

So, $x = 7/5$ is a point of local minimum.

$f''(2) = 0$

It can be easily checked that $f'''(2) \neq 0$

Hence, $x = 2$ is a point of inflexion.

EXAMPLE 12 *Let* $f(x) = \sin^3 x + \lambda \sin^2 x$ *for* $-\pi/2 < x < \pi/2$. *Find the interval in which* λ *should lie in order that* $f(x)$ *has exactly one minimum and one maximum.* **[IIT 1985]**

SOLUTION We have,

$f(x) = \sin^3 x + \lambda \sin^2 x.$

$\therefore f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x$

$= \frac{3}{2}\sin 2x \sin x + \lambda \sin 2x$

$= \left(\frac{3}{2}\sin x + \lambda\right)\sin 2x$

For local maximum or minimum, we must have

$f'(x) = 0$

$\Rightarrow \left(\frac{3}{2}\sin x + \lambda\right)\sin 2x = 0$

$\Rightarrow \sin 2x = 0$ or, $\frac{3}{2}\sin x + \lambda = 0$

$\Rightarrow x = 0$ or, $\sin x = \frac{-2\lambda}{3}$

$\Rightarrow x = 0$ or, $x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$

Now,

$f''(x) = \frac{3}{2}\cos x \sin 2x + 2\left\{\frac{3}{2}\sin x + \lambda\right\}\cos 2x$

$= \frac{3}{2}\cos x \sin 2x + (3\sin x + 2\lambda)\cos 2x$

$\therefore f''(0) = 2\lambda$

It is given that

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore -1 < \sin x < 1$

$\Rightarrow -1 < -\frac{2\lambda}{3} < 1 \qquad \left[\because \sin x = \frac{-2\lambda}{3}\right]$

$\Rightarrow 3 > 2\lambda > -3$

$\Rightarrow -\frac{3}{2} < \lambda < \frac{3}{2}$

CASE I *When* $-3/2 < \lambda < 0$

In this case, x lies between 0 and $\pi/2$

Also, $f''(0) = 2\lambda < 0$

$\therefore$ $x = 0$ is a point of local minimum.

For $x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$, we have

$f''(x) = \frac{3}{2}\cos x \sin 2x + 0 > 0 \qquad [\because x \in (0, \pi/2)]$

$\therefore x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ is a point of local minimum.

Thus, $f(x)$ has exactly one maximum and one minimum.

CASE II *When* $0 < \lambda < 3/2$

In this case, x lies between $-\pi/2$ and 0.

Also, $f''(0) = 2\lambda > 0$

So, $x = 0$ is a point of local minimum.

For $x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$, we have

$f''(x) = \frac{3}{2}\cos x \sin 2x < 0 \qquad [\because x \in (-\pi/2, 0) \therefore \sin 2x < 0]$

$\therefore x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ is a point of local maximum.

Thus, $f(x)$ has exactly one minimum and one maximum.

Hence, λ must lie in the interval $(-3/2, 0) \cup (0, 3/2)$.

EXAMPLE 13 *Find the greatest value of the function* $f(x) = \frac{1}{2bx^2 - x^4 - 3b^2}$ *in the interval* $[-2, 1]$ *depending on the parameter b.*

SOLUTION We have,

$f(x) = \frac{1}{2bx^2 - x^4 - 3b^2}$

$\Rightarrow f'(x) = \frac{4x(x^2 - b)}{(2bx^2 - x^4 - 3b^2)^2}$

CASE I *When* $b < 0$

In this case,

$f'(x) = 0$

$\Rightarrow \frac{4x(x^2 - b)}{(2bx^2 - x^4 - 3b^2)^2} = 0$

$\Rightarrow x = 0 \qquad [\because x^2 - b > 0 \text{ for } b < 0]$

Clearly, $f'(x)$ changes its sign from negative to positive in the neighbourhood of 0. Therefore, $x = 0$ is a point of local minimum.

Clearly, $f'(x) < 0$ for $x \in [-2, 0]$. So, $f(x)$ is a decreasing function on $[-2, 0]$. Also, $f(-x) = f(x)$. So, $f(x)$ is an even function. So, it is symmetric about $x = 0$ i.e. y-axis. Consequently the grestest value of $f(x)$ exists at $x = -2$ and is given by

$f(-2) = \frac{1}{8b - 16 - 3b^2}$

CASE II *When* $b = 0$

In this case, we have

$f(x) = -\frac{1}{x^4}$

Clearly, $f(x)$, being an even function, is symmetric about y-axis.

We have,

$f'(x) = \frac{4}{x^5}$

$\therefore$ $f'(x) < 0$ for $x \in [-2, 0)$

and,

$f'(x) > 0$ for $x \in (0, 1]$

So, $f(x)$ is decreasing on $[-2, 0)$ and increasing on $(0, 1]$.

As $f(x)$ is symmetric about y-axis. So, the greatest value of $f(x)$ occurs at $x = -2$ and is given by

$$f(-2) = -\frac{1}{16}$$

CASE III *When* $0 < b < 1$

In this case,

$f'(x) = 0 \Rightarrow x = 0, x = \pm\sqrt{b}$

The changes in the signs of $f'(x)$ for different values of x are shown in Fig. 9.15

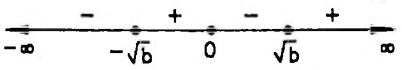


Fig. 9.15

From the variations in signs of $f'(x)$, we find that $f(x)$ attains a local maxima at $x = 0$ and local minima at $x = \pm\sqrt{b}$. Since $f(x)$ is an even function, therefore either $f(-2)$ or $f(0)$ is the greatest value.

We have,

$$f(-2) = \frac{1}{8b - 16 - 3b^2} \text{ and } f(0) = \frac{-1}{3b^2}$$

$$\therefore \quad f(-2) - f(0) = \frac{1}{8b - 16 - 3b^2} + \frac{1}{3b^2}$$

$$= \frac{8b - 16}{3b^2(8b - 16 - 3b^2)}$$

$$= \frac{16 - 8b}{3b^2(3b^2 - 8b + 16)} > 0 \text{ for } 0 < b < 1$$

$\Rightarrow f(-2)$ is the greatest value of $f(x)$.

Thus,

$$\text{Greatest value} = f(-2) = \frac{1}{8b - 16 - 3b^2}$$

CASE IV *When* $1 \le b < 4$

In this case, we have

$$f'(x) = 0 \text{ at } x = 0, -\sqrt{b}$$

Note that $x = \sqrt{b} \notin (-2, 1)$

From the changes of signs of $f'(x)$, we observe that $f(x)$ attains a local minimum at $x = -\sqrt{b}$ and maximum at $x = 0$. As the graph of $f(x)$ is symmetric about y-axis. Therefore,

$$\text{Greatest value} = f(-2) = \frac{1}{8b - 16 - 3b^2}$$

CASE V *When* $b \ge 4$

In this case, $f'(x) = 0$ at $x = 0$ only and $f(x)$ changes its sign from positive to negative. So, $x = 0$ is a point of local maximum. Since $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$. Therefore,

$$\text{Greatest value} = f(0) = -\frac{1}{3b^4}$$

EXAMPLE 13 *Find all the values of the parameter a for which the point of minimum of the function $f(x) = 1 + a^2 x - x^3$ satisfy the inequality* $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} < 0$.

SOLUTION We have,

$$\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$$

$$\Rightarrow x^2 + 5x + 6 < 0 \qquad [\because x^2 + x + 2 > 0 \text{ for all } x]$$

$$\Rightarrow -3 < x < -2 \qquad \ldots(i)$$

Now,

$$f(x) = 1 + a^2 x - x^3$$

$$\Rightarrow f'(x) = a^2 - 3x^2 \text{ and } f''(x) = -6x$$

For points of local maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow a^2 - 3x^2 = 0$$

$$\Rightarrow x = \pm\frac{a}{\sqrt{3}}$$

Now two cases arise :

CASE I *When* $a > 0$

In this case, we have

$$f''\left(\frac{-a}{\sqrt{3}}\right) = \frac{6a}{\sqrt{3}} > 0$$

So, $x = \dfrac{-a}{\sqrt{3}}$ is the point of minimum.

For this point to satisfy (i), we must have

$$-3 < \frac{-a}{\sqrt{3}} < -2$$

$$\Rightarrow 3\sqrt{3} > a > 2\sqrt{3}$$

$$\Rightarrow a \in (2\sqrt{3}, 3\sqrt{3})$$

CASE II When $a < 0$

In this case, we have

$$f''\left(\frac{a}{\sqrt{3}}\right) = \frac{-6a}{\sqrt{3}} > 0$$

So, $x = \dfrac{a}{\sqrt{3}}$ is a point of minimum.

This point will satisfy (i), if

$$-3 < \frac{a}{\sqrt{3}} < -2$$

$$\Rightarrow -3\sqrt{3} < a < -2\sqrt{3}$$

$$\Rightarrow a \in (-3\sqrt{3}, -2\sqrt{3})$$

Hence, $a \in (2\sqrt{3}, 3\sqrt{3}) \cup (-3\sqrt{3}, -2\sqrt{3})$

EXAMPLE 14 *A function $y = f(x)$ is represented parametrically as follows :*

$$x = \phi(t) = t^5 - 5t^3 - 20t + 7$$

$$y = \psi(t) = 4t^3 - 3t^2 - 18t + 3, \; -2 < t < 2.$$

Find the extremum of this function.

SOLUTION We have,

$$x = t^5 - 5t^3 - 20t + 7$$

and, $y = 4t^3 - 3t^2 - 18t + 3$

Therefore,

$$\frac{dx}{dt} = 5t^4 - 15t^2 - 20 \quad \text{and} \quad \frac{dy}{dt} = 12t^2 - 6t - 18$$

$$\Rightarrow \frac{dx}{dt} = 5(t^2 - 4)(t^2 + 1) \text{ and } \frac{dy}{dt} = 6(2t - 3)(t + 1) \qquad ...(i)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6(2t - 3)(t + 1)}{5(t^2 - 4)(t^2 + 1)}$$

Now, $\frac{dy}{dx} = 0 \Rightarrow t = \frac{3}{2}, -1$

Clearly, these values satisfy $-2 < t < 2$.

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\cdot\frac{dt}{dx}$$

$$= \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{(5t^4 - 15t^2 - 20)(24t - 6) - (12t^2 - 6t - 18)(20t^3 - 30t)}{(5t^4 - 15t^2 - 20)^3}$$

When $t = -1$, we have

$$\frac{d^2y}{dx^2} = \frac{-30 \times -30}{(-30)^3} < 0, x = 31 \text{ and } y = 14$$

When $t = \frac{3}{2}$, we have

$$\frac{d^2y}{dx^2} > 0, x = \frac{-1033}{32} \text{ and } y = \frac{-69}{4}$$

Hence, the function $y = f(x)$ attains a local maximum at $(31, 14)$ and a local minimum at $(-1033/32, -69/4)$

EXAMPLE 15 *Show that* $\left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$ *has just one maximum and one minimum value. Show also that the difference between them is* $\frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$. *What is the least value of this difference.*

SOLUTION Let $f(x) = \left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$

$$\Rightarrow \quad f'(x) = 9x^2 - 6\left(\alpha - \frac{1}{\alpha}\right)x - 4$$

For maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow 9x^2 - 6\left(\alpha - \frac{1}{\alpha}\right)x - 4 = 0$$

$$\Rightarrow x = \frac{6\left(\alpha - \frac{1}{\alpha}\right) \pm \sqrt{36\left(\alpha - \frac{1}{\alpha}\right)^2 + 144}}{18}$$

$$\Rightarrow x = \frac{2\alpha}{3}, \frac{-2}{3\alpha}$$

Let $x_1 = \frac{2\alpha}{3}$ and $x_2 = -\frac{2}{3\alpha}$

Thus, $f'(x) = 9\left(x - \frac{2\alpha}{3}\right)\left(x + \frac{2}{3\alpha}\right)$

If $\alpha > 0$, then $\frac{2\alpha}{3} > -\frac{2}{3\alpha}$ and the changes of signs of $f'(x)$ are as shown in Fig. 9.16.

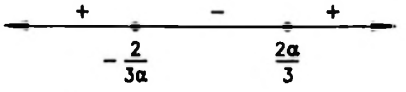


Fig. 9.16

Clearly, $x_2 = -\frac{2}{3\alpha}$ is a point of local maximum and $x_1 = \frac{2\alpha}{3}$ is a point of local minimum.

If $\alpha < 0$, then $\frac{-2}{3\alpha} > \frac{2\alpha}{3}$ and the changes in the signs of $f'(x)$ are as shown in Fig. 9.17.



Fig. 9.17

In this case $x_1 = \frac{2\alpha}{3}$ is a point of local maximum and $x_2 = -\frac{2}{3\alpha}$ is a point of local minimum.

The difference between the local maximum and local minimum values

$$= |f(x_1) - f(x_2)|$$

$$= \left|\left(\alpha - \frac{1}{\alpha} - \frac{2\alpha}{3}\right)\left(4 - \frac{4\alpha^2}{3}\right) - \left(\alpha - \frac{1}{\alpha} + \frac{2}{3\alpha}\right)\left(4 - \frac{4}{3\alpha^2}\right)\right|$$

$$= \left|\left(\frac{\alpha}{3} - \frac{1}{\alpha}\right)\left(4 - \frac{4\alpha^2}{3}\right) - \left(\alpha - \frac{1}{3\alpha}\right)\left(4 - \frac{4}{3\alpha^2}\right)^2\right|$$

$$= \frac{4}{9}\left|\left(\alpha - \frac{3}{\alpha}\right)(3 - \alpha^2) - \left(3\alpha - \frac{1}{\alpha}\right)\left(3 - \frac{1}{\alpha^2}\right)\right|$$

$$= \frac{4}{9}\left|\left(3\alpha - \alpha^3 - \frac{9}{\alpha} + 3\alpha\right) - \left(9\alpha - \frac{3}{\alpha} - \frac{3}{\alpha} + \frac{1}{\alpha^3}\right)\right|$$

$$= \frac{4}{9}\left|6\alpha - \alpha^3 - \frac{9}{\alpha} - 9\alpha + \frac{6}{\alpha} - \frac{1}{\alpha^3}\right|$$

$$= \frac{4}{9}\left|-\alpha^3 - \frac{1}{\alpha^3} - 3\alpha - \frac{3}{\alpha}\right|$$

$$= \frac{4}{9}\left|\left(\alpha + \frac{1}{\alpha}\right)^3\right|$$

Clearly, it will be least when $\alpha = 1$ and the least value is 32/9.

EXAMPLE 16 *Find the points on the curve* $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, *whose distance from the origin is minimum.*

SOLUTION Let $P(x, y)$ be a point on the curve $ax^2 + 2bxy + ay^2 = c$, whose distance from the origin is r i.e. $OP = r$. If OP makes an angle θ with X-axis, then

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Now, $P(r\cos\theta, r\sin\theta)$ lies on $ax^2 + 2bxy + ay^2 = c$

$$\therefore (a + b\sin 2\theta)\, r^2 = c$$

$$\Rightarrow r^2 = \frac{c}{a + b\sin 2\theta}$$

Clearly, r will be minimum, if $a + b\sin 2\theta$ is maximum i.e. $\sin 2\theta$ is maximum.

We know that $\sin 2\theta$ is maximum, when

$$2\theta = \frac{\pi}{2} \text{ or } \frac{5\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

For these values of θ, the maximum value of $\sin 2\theta$ is 1 and the value of r is $\sqrt{\frac{c}{a+b}}$.

When $\theta = \frac{\pi}{4}$, coordinates of P are

$$\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) \text{ i.e. } \left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$$

When $\theta = \frac{5\pi}{4}$, coordinates of P are

$$\left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right) \text{ i.e. } \left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$$

EXAMPLE 17 *Prove that the minimum radius vector of the curve* $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ *is of length* $a + b$.

SOLUTION Let r be the radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$.

Suppose it makes an angle θ with X-axis. Then, the coordinates of any point on the curve are $(r\cos\theta, r\sin\theta)$.

$$\therefore \frac{a^2}{r^2\cos^2\theta} + \frac{b^2}{r^2\sin^2\theta} = 1$$

$$\Rightarrow r^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$$

Clearly, r will be minimum when $a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$ is minimum.

Let $Z = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$. Then,

$$\frac{dZ}{d\theta} = 2a^2\sec^2\theta\tan\theta - 2b^2\operatorname{cosec}^2\theta\cot\theta$$

For maximum or minimum values of Z, we must have

$$\frac{dZ}{d\theta} = 0$$

$$\Rightarrow 2a^2\sec^2\theta\tan\theta - 2b^2\operatorname{cosec}^2\theta\cot\theta = 0$$

$$\Rightarrow \tan^4\theta = \frac{b^2}{a^2}$$

$$\Rightarrow \tan^2\theta = \frac{b}{a}$$

$$\Rightarrow \tan\theta = \pm\sqrt{\frac{b}{a}}$$

Now,

$$\frac{d^2Z}{d\theta^2} = 2a^2\sec^4\theta + 4a^2\sec^2\theta\tan^2\theta + 2b^2\operatorname{cosec}^4\theta + 4b^2\operatorname{cosec}^2\theta\cot^2\theta$$

Clearly, $\frac{d^2Z}{d\theta^2} > 0$ for $\tan\theta = \pm\sqrt{b/a}$

Hence, Z is minimum when $\tan\theta = \pm\sqrt{\frac{b}{a}}$. The minimum value of Z is

$$Z = a^2(1 + \tan^2\theta) + b^2(1 + \cot^2\theta)$$

$$\Rightarrow Z = a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right)$$

$$\Rightarrow Z = a^2 + b^2 + 2ab$$

$$\Rightarrow Z = (a+b)^2$$

$$\Rightarrow r^2 = (a+b)^2$$

$$\Rightarrow r = a + b \qquad [\because Z = r^2]$$

Hence, the minimum radius vector of the curve is $a + b$.

EXAMPLE 18 *Prove that the minimum intercept made by the axes on any tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $(a + b)$. *Also, find the ratio in which the point of contact divides the intercept.*

SOLUTION Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

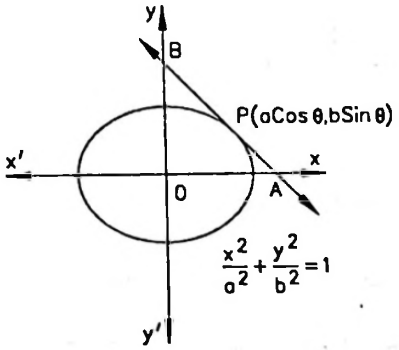


Fig. 9.18

$$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-b^2}{a^2}\frac{x}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P=\frac{-b^2}{a^2}\times\frac{a\cos\theta}{b\sin\theta}=-\frac{b}{a}\cot\theta$$

The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is

$$y-b\sin\theta=-\frac{b}{a}\cot\theta\,(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$\Rightarrow bx\cos\theta+ay\sin\theta=ab$$

The coordinates of A and B are : $A(a\sec\theta, 0)$ and $B(0, b\,\mathrm{cosec}\,\theta)$ respectively.

$$\therefore AB=\sqrt{a^2\sec^2\theta+b^2\,\mathrm{cosec}^2\theta}$$

Clearly, AB will be maximum or minimum if AB^2 is maximum or minimum.

Let $Z=AB^2$. Then,

$$Z=a^2\sec^2\theta+b^2\,\mathrm{cosec}^2\theta.$$

$$\Rightarrow \frac{dZ}{d\theta}=2a^2\sec^2\theta\tan\theta-2b^2\,\mathrm{cosec}^2\theta\cot\theta$$

For maximum or minimum, we must have

$$\frac{dZ}{d\theta}=0$$

$$\Rightarrow 2a^2\sec^2\theta\tan\theta-2b^2\,\mathrm{cosec}^2\theta\cot\theta=0$$

$$\Rightarrow \tan^4\theta=\frac{b^2}{a^2}$$

$$\Rightarrow \tan^2\theta=\frac{b}{a}$$

$$\Rightarrow \tan\theta=\pm\sqrt{\frac{b}{a}}$$

Now,

$$\frac{d^2Z}{d\theta^2}=2a^2\sec^4\theta+4a^2\sec^2\theta\tan^2\theta+2b^2\,\mathrm{cosec}^4\theta+2b^2\,\mathrm{cosec}^2\theta\cot^2\theta$$

which is always positive.

Thus, Z or AB is minimum when $\tan\theta=\pm\sqrt{\frac{b}{a}}$

The minimum value of AB is given by

$$AB=\sqrt{a^2(1+\tan^2\theta)+b^2(1+\cot^2\theta)}$$

$$=\sqrt{a^2\left(1+\frac{b}{a}\right)+b^2\left(1+\frac{a}{b}\right)}$$

$$=a+b$$

Now,

$$PA=\sqrt{(a\cos\theta-a\sec\theta)^2+b^2\sin^2\theta}$$

$$=\sqrt{a^2\frac{\sin^4\theta}{\cos^2\theta}+b^2\sin^2\theta}$$

$$=\sqrt{a^2\tan^2\theta+b^2}\,.\sin\theta$$

$$=\sqrt{a^2\times\frac{b}{a}+b^2}\,\frac{\sqrt{b}}{\sqrt{a+b}} \qquad \left[\because \tan\theta=\sqrt{b/a} \Rightarrow \sin\theta=\sqrt{\frac{b}{a+b}}\right]$$

$$=\sqrt{b(a+b)\times\frac{b}{a+b}}$$

$$=b$$

$$\therefore AB=a+b$$

$$\Rightarrow PB=a+b-b=a$$

$$\therefore \frac{PA}{PB}=\frac{b}{a}$$

Hence, P divides AB in the ratio $b:a$.

EXAMPLE 19 *Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.*

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let ABC be an isosceles triangle having one vertex at $A(a, 0)$. Let the coordinates of B be $(a\cos\theta, b\sin\theta)$. Then the coordinates of C are $(a\cos\theta, -b\sin\theta)$

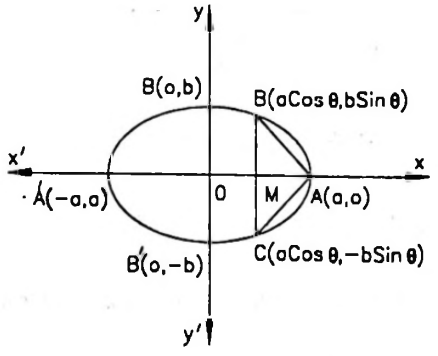


Fig. 9.19

Let A be the area of ΔABC. Then,

$$A=\frac{1}{2}(2b\sin\theta)(a-a\cos\theta)$$

$\Rightarrow A = ab\ (\sin\theta - \sin\theta\cos\theta)$

$\Rightarrow \frac{dA}{d\theta} = ab\ (\cos\theta - \cos^2\theta + \sin^2\theta)$

$\Rightarrow \frac{dA}{d\theta} = ab\ (\cos\theta - \cos 2\theta)$

For maximum and minimum values of A, we must have

$\frac{dA}{d\theta} = 0$

$\Rightarrow ab(\cos\theta - \cos 2\theta) = 0$

$\Rightarrow \cos\theta = \cos 2\theta$

$\Rightarrow \theta = 2\pi - 2\theta$

$\Rightarrow \theta = \frac{2\pi}{3}$

Now,

$\frac{d^2A}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$

For $\theta = \frac{2\pi}{3}$, we have

$$\frac{d^2A}{d\theta^2} = ab\left(-\sin\frac{2\pi}{3} + 2\sin\frac{4\pi}{3}\right)$$

$$= ab\left(-\frac{\sqrt{3}}{2} - 2\times\frac{\sqrt{3}}{2}\right) < 0$$

Hence, A is maximum when $\theta = 2\pi/3$

$$\text{Maximum area} = ab\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{3}\cos\frac{2\pi}{3}\right)$$

$$= ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\times\frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}}{4}ab.$$

EXAMPLE 20 *From a variable point of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *normal is drawn to the ellipse. Find the maximum distance of the normal from the centre of the ellipse.*

SOLUTION Let $P(a\cos\theta, b\sin\theta)$ be a variable point on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2}{a^2}\frac{x}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = -\frac{b}{a}\cot\theta$$

So, the equation of the normal at $P(a\cos\theta, b\sin\theta)$ is

$$y - b\sin\theta = -\frac{1}{-\frac{b}{a}\cot\theta}(x - a\cos\theta)$$

$$\Rightarrow y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$\Rightarrow ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

$$\Rightarrow ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$$

Let O be the centre of the ellipse and l be the length of the perpendicular from O on the normal at P. Then,

$$l = \frac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta}}$$

Now, l will be maximum, if $a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta$ is minimum. As discussed in Ex. 17, the minimum value of $a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta$ is $a + b$.

Hence, the maximum distance of the normal from the centre of the ellipse is given by

$$l = \frac{a^2 - b^2}{a + b} = a - b$$

EXAMPLE 21 *P is a point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and Q is the corresponding point on the auxiliary circle* $x^2 + y^2 = a^2$. *If θ is the greatest angle between the tangents at P and Q, show that* $\tan\theta = \left(\frac{a-b}{2\sqrt{ab}}\right)$.

SOLUTION Let $P(a\cos\alpha, b\sin\alpha)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Q is the corresponding point on the auxiliary circle. So, the coordinates of Q are $(a\cos\alpha, a\sin\alpha)$.

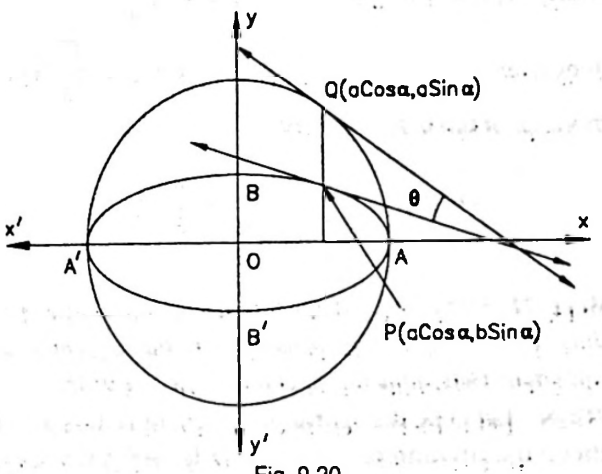


Fig. 9.20

The equations of tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = a^2$ at P and Q are respectively

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \qquad \ldots(i)$$

and, $x\cos\alpha + y\sin\alpha = a$ ...(ii)

Let m_1 and m_2 be the slopes of (i) and (ii) respectively. Then,

$$m_1 = \frac{-b}{a}\cot\alpha \text{ and } m_2 = -\cot\alpha$$

The angle θ between the tangents (i) and (ii) is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \tan\theta = \frac{-\frac{b}{a}\cot\alpha + \cot\alpha}{1 + \frac{b}{a}\cot^2\alpha} = \frac{a-b}{a\tan\alpha + b\cot\alpha}$$

Clearly, $\tan\theta$ will be maximum if $a\tan\alpha + b\cot\alpha$ is minimum.

Let $f(\alpha) = a\tan\alpha + b\cot\alpha$. Then,

$$\frac{df(\alpha)}{d\alpha} = a\sec^2\alpha - b\,\text{cosec}^2\alpha$$

For maximum or minimum $f(\alpha)$, we must have

$$\frac{df(\alpha)}{d\alpha} = 0$$

$$\Rightarrow a\sec^2\alpha - b\,\text{cosec}^2\alpha = 0$$

$$\Rightarrow \tan^2\alpha = \frac{b}{a}$$

$$\Rightarrow \tan\alpha = \sqrt{\frac{b}{a}}$$

Now,

$$\frac{d^2f(\alpha)}{d\alpha^2} = 2a\sec^2\alpha\tan\alpha + 2b\,\text{cosec}^2\alpha\cot\alpha$$

$$\therefore \frac{d^2f(\alpha)}{d\alpha^2} = 2a\sqrt{\frac{b}{a}}\left(1+\frac{b}{a}\right) + 2b\sqrt{\frac{a}{b}}\left(1+\frac{a}{b}\right) > 0$$

Thus, $f(\alpha)$ is minimum when $\tan\alpha = \sqrt{\frac{b}{a}}$.

Hence, $\tan\theta$ will be maximum when $\tan\alpha = \sqrt{\frac{b}{a}}$. The maximum value of $\tan\theta$ is given by

$$\tan\theta = \frac{a-b}{a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}}} = \frac{a-b}{2\sqrt{ab}}$$

EXAMPLE 22 *From a fixed point P in the circumference of a circle of radius 'a', PN is drawn perpendicular to the tangent at a point M of the circle. Determine the maximum area of* ΔPMN.

SOLUTION Let O be the centre of a circle of radius a. Let P be a point on the circumference of the circle. MT is the tangent at M to the circle. PN is the perpendicular from P to the tangent MT. Draw OL perpendicular to PN. Let $\angle POL = \theta$.

Then, $MN = OL = a\cos\theta$.

$LP = a\sin\theta$ and hence $NP = a + a\sin\theta$.

Let A denote the area of ΔPMN. Then,

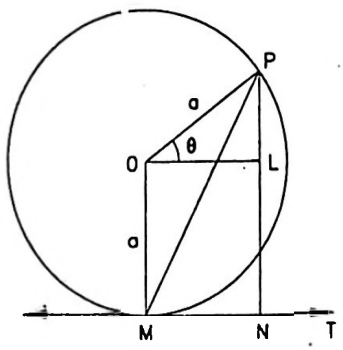


Fig. 9.21

$$A = \frac{1}{2}MN \times PN$$

$$= \frac{1}{2}a\cos\theta \times (a + a\sin\theta)$$

$$= \frac{1}{2}a^2(\cos\theta + \sin\theta\cos\theta)$$

$$\Rightarrow \frac{dA}{d\theta} = \frac{1}{2}a^2(-\sin\theta + \cos^2\theta - \sin^2\theta)$$

For maximum or minimum values of θ, we must have

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow -\sin\theta + \cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow \sin\theta = \cos 2\theta$$

$$\Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow 2\theta = \frac{\pi}{2} - \theta$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Now,

$$\frac{dA}{d\theta} = \frac{1}{2}a^2(-\sin\theta + \cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \frac{d^2A}{d\theta^2} = \frac{1}{2}a^2(-\cos\theta - 2\sin 2\theta)$$

At $\theta = \frac{\pi}{6}$, we have

$$\frac{d^2A}{d\theta^2} = \frac{1}{2}a^2\left(-\cos\frac{\pi}{6} - 2\sin\frac{\pi}{3}\right) < 0$$

So, A is maximum when $\theta = \frac{\pi}{6}$. The maximum area A is given by

$$A = \frac{1}{2}a^2\left(\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos\frac{\pi}{6}\right) = \frac{1}{2}a^2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}}{8}a^2.$$

EXAMPLE 23 *Find a point* (α, β) *on the ellipse* $4x^2 + 3y^2 = 12$ *in the first quadrant, so that the area enclosed by the lines* $y = x$, $y = \beta$, $x = \alpha$ *and the x-axis is maximum.*

SOLUTION Let the lines $y = \beta$ and $x = \alpha$ meet $y = x$ at Q and R respectively. The coordinates of Q and R are (β, β) and (α, α) respectively.

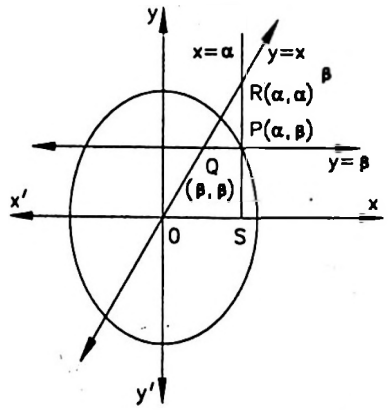


Fig. 9.22

Let A denote the area of trapezium $PQOS$. Then,

$$A = \frac{1}{2}(PQ + OS) \times PS$$

$$\Rightarrow A = \frac{1}{2}(\alpha - \beta + \alpha) \times \beta$$

$$\Rightarrow A = \frac{1}{2}(2\alpha - \beta)\beta.$$

Since (α, β) lies on $4x^2 + 3y^2 = 12$ or, $\frac{x^2}{3} + \frac{y^2}{4} = 1$. So, let $\alpha = \sqrt{3}\cos\theta$ and $\beta = 2\sin\theta$.

$$\therefore A = \frac{1}{2}(2\sqrt{3}\cos\theta - 2\sin\theta) \cdot 2\sin\theta$$

$$\Rightarrow A = 2(\sqrt{3}\sin\theta\cos\theta - \sin^2\theta)$$

$$\Rightarrow \frac{dA}{d\theta} = 2(\sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta - 2\sin\theta\cos\theta)$$

$$\Rightarrow \frac{dA}{d\theta} = 2(\sqrt{3}\cos 2\theta - \sin 2\theta)$$

For maximum or minimum values of A, we must have

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \sqrt{3}\cos 2\theta - \sin 2\theta = 0$$

$$\Rightarrow \tan 2\theta = \sqrt{3}$$

$$\Rightarrow 2\theta = \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Now,

$$\frac{d^2A}{d\theta^2} = 2(-2\sqrt{3}\sin 2\theta - 2\cos 2\theta)$$

Clearly, $\frac{d^2A}{d\theta^2} < 0$ for $\theta = \frac{\pi}{6}$.

Hence, A is maximum when $\theta = \frac{\pi}{6}$. For this value of θ, we have

$$\alpha = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2} \text{ and } \beta = 2\sin\frac{\pi}{6} = 1$$

Hence, the required point is (3/2, 1).

EXAMPLE 24 *A point P is given on the circumference of a circle of radius r. The chord QR is parallel to the tangent line at P. Find the maximum area of* ΔPQR. **[IIT 1990]**

SOLUTION Since the chord QR is parallel to the tangent at P. Therefore, $ON \perp QR$.

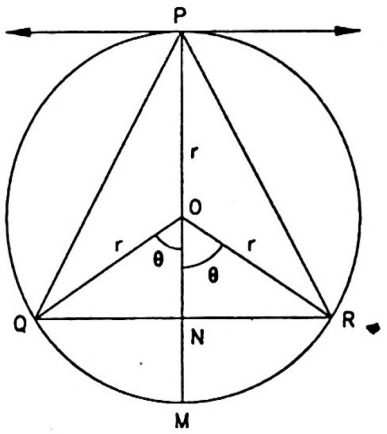


Fig. 9.23

Consequently, N is the mid-point of chord QR.

$$\therefore \quad QR = 2QN = 2r\sin\theta.$$

Also, $$ON = r\cos\theta$$

$$\therefore \quad PN = r + r\cos\theta$$

Let A denote the area of ΔPQR. Then,

$$A = \frac{1}{2}(QR \times PN)$$

$$\Rightarrow A = \frac{1}{2} \times 2r\sin\theta \times (r + r\cos\theta)$$

$$\Rightarrow A = r^2(\sin\theta + \sin\theta\cos\theta)$$

$$\Rightarrow A = r^2\left\{\sin\theta + \frac{1}{2}\sin 2\theta\right\}$$

$$\Rightarrow \frac{dA}{d\theta} = r^2(\cos\theta + \cos 2\theta) \text{ and } \frac{d^2A}{d\theta^2} = r^2(-\sin\theta - 2\sin 2\theta)$$

For maximum and minimum values of θ, we must have

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \cos\theta + \cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = -\cos\theta$$

$$\Rightarrow \cos 2\theta = \cos(\pi - \theta)$$

$$\Rightarrow 2\theta = \pi - \theta \Rightarrow \theta = \frac{\pi}{3}$$

Clearly, $\frac{d^2A}{d\theta^2} < 0$ for $\theta = \frac{\pi}{3}$.

Hence, area of ΔPQR is maximum when $\theta = \frac{\pi}{3}$. The maximum area of ΔPQR is given by

$$A = r^2\left\{\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right\} = r^2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}r^2}{4}$$

EXAMPLE 25 *Let (h, k) be a fixed point, where $h > 0, k > 0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of ΔOPQ, O being the origin.* **[IIT 1995]**

SOLUTION Let m be the slope of the line passing through $A(h, k)$. Then, its equation is

$$y - k = m(x - h)$$

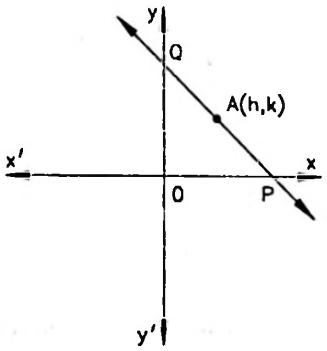


Fig. 9.24

Since the line cuts the positive direction of the coordinate axes. Therefore, $m < 0$.

The line cuts X-axis at $P\left(h - \frac{k}{m}, 0\right)$ and Y-axis at $Q(0, k - mh)$.

Let A denote the area of ΔOPQ. Then,

$$A = \frac{1}{2}(OP \times OQ)$$

$$\Rightarrow A = \frac{1}{2}\left(h - \frac{k}{m}\right) \times (k - mh)$$

$$\Rightarrow A = -\frac{1}{2m}(k - mh)^2$$

$$\Rightarrow \frac{dA}{dm} = \frac{1}{2m^2}(k - mh)^2 + \frac{h}{m}(k - mh)$$

$$= \frac{k - mh}{2m^2}(k - mh + 2mh)$$

$$= \frac{1}{2m^2}(k^2 - m^2h^2)$$

For maximum or minimum values of A, we must have

$$\frac{dA}{dm} = 0$$

$$\Rightarrow k^2 - m^2h^2 = 0$$

$$\Rightarrow m = \pm\frac{k}{h}$$

$$\Rightarrow m = -\frac{k}{h} \qquad [\because m < 0]$$

Now,

$$\frac{d^2A}{dm^2} = -\frac{1}{m^3}(k^2 - m^2h^2) - \frac{1}{m}h^2$$

When $m = -\frac{k}{h}$, we have

$$\frac{d^2A}{dm^2} = \frac{h^3}{k} > 0$$

Hence, A is minimum when $m = -\frac{k}{h}$. The minimum area of ΔOPQ is given by

$$A = \frac{h}{2k}(k + k)^2 = 2hk$$

EXAMPLE 26 *The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of ΔQSR.* **[IIT 1994]**

SOLUTION Let r be the variable radius of the circle with centre at $Q(1, 0)$. Since the two circles intersect each other, therefore,

$$r - 1 < OQ < r + 1$$

$$\Rightarrow r - 1 < 1 < r + 1$$

$$\Rightarrow 0 < r < 2$$

The equation of the circle with centre at Q and radius r is

$$(x - 1)^2 + (y - 0)^2 = r^2 \qquad ...(i)$$

The equation of the circle with centre at O is

$$x^2 + y^2 = 1 \qquad ...(ii)$$

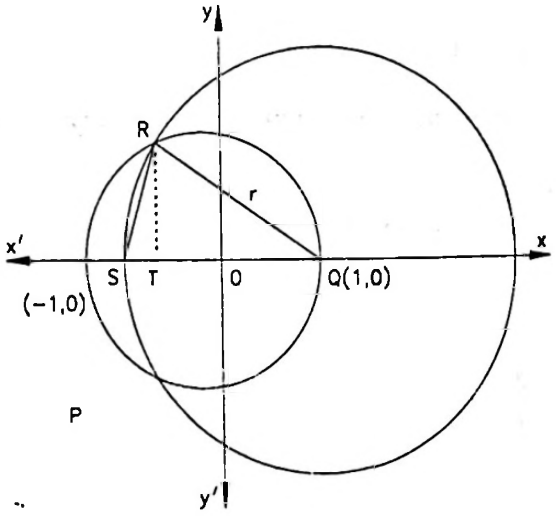


Fig. 9.25

Solving these two equations, we get the coordinates of R as

$$\left(\frac{2 - r^2}{2}, \frac{r\sqrt{4 - r^2}}{2}\right)$$

$$\therefore RT = \frac{r\sqrt{4 - r^2}}{2}$$

Circle (i) meets X-axis at $y = 0$.

Putting $y = 0$ in (i), we get

$(x-1)^2 = r^2$

$\Rightarrow \quad x - 1 = \pm r$

$\Rightarrow \quad x = 1 \pm r$

Thus, the coordinates of S are $(1-r, 0)$

$\therefore \; QS = OS + OQ$

$= |1-r| + 1$

$= -(1-r) + 1 \qquad [\because r > 1]$

$= r$

Let A denote the area of ΔQSR. Then,

$$A = \frac{1}{2}(QS \times RT)$$

$$\Rightarrow \quad A = \frac{1}{2} \times r \times r\frac{\sqrt{4-r^2}}{2}$$

$$\Rightarrow \quad A = \frac{1}{4} r^2 \sqrt{4-r^2}$$

$$\Rightarrow \frac{dA}{dr} = \frac{1}{4}\left[2r\sqrt{4-r^2} - \frac{r^3}{\sqrt{4-r^2}}\right]$$

$$\Rightarrow \frac{dA}{dr} = \frac{1}{4}\frac{\{2r(4-r^2) - r^3\}}{\sqrt{4-r^2}} = \frac{8r - 3r^3}{4\sqrt{4-r^2}}$$

For maximum or minimum values of A, we must have

$$\frac{dA}{dr} = 0$$

$$\Rightarrow 8r - 3r^3 = 0 \Rightarrow r = \frac{2\sqrt{2}}{\sqrt{3}}$$

Now,

$$\frac{d^2A}{dr^2} = \frac{1}{4}\left[\frac{\sqrt{4-r^2}(8-9r^2) - (8r-3r^3)\left\{\frac{-r}{\sqrt{4-r^2}}\right\}}{4-r^2}\right]$$

When $r = 2\sqrt{\frac{2}{3}}$, we have

$$\frac{d^2A}{dr^2} = \frac{1}{4}\left[\frac{\sqrt{4-\frac{8}{3}}\left(8 - 9\times\frac{8}{3}\right)}{4-\frac{8}{3}}\right] < 0$$

Hence, A is maximum when $r = 2\sqrt{\frac{2}{3}}$

The maximum area of ΔQSR is given by

$$A = \frac{1}{4}\times\frac{8}{3}\times\sqrt{4-\frac{8}{3}} = \frac{4\sqrt{3}}{9}$$

EXAMPLE 27 *What normal to the curve $y = x^2$ forms the shortest chord ?* **[IIT 1992]**

SOLUTION Let $P(t, t^2)$ be any point on the parabola $y = x^2$.

Now,

$$y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = 2t$$

Equation of the normal at $P(t, t^2)$ is

$$y - t^2 = -\frac{1}{2t}(x - t)$$

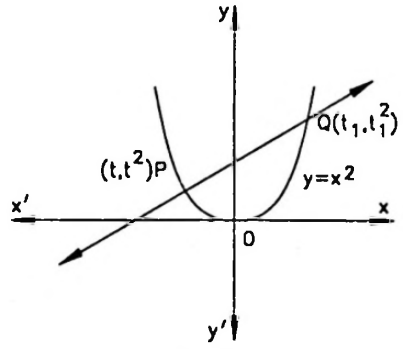


Fig. 9.26

Suppose it meets the curve $y = x^2$ again at $Q(t_1, t_1^2)$. Then,

$$t_1^2 - t^2 = -\frac{1}{2t}(t_1 - t)$$

$$\Rightarrow \quad t_1 + t = -\frac{1}{2t}$$

$$\Rightarrow \quad t_1 = -t - \frac{1}{2t}$$

Now,

$$PQ^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$$

$$= (t - t_1)^2\left\{(t + t_1)^2 + 1\right\}$$

$$= \left(2t + \frac{1}{2t}\right)^2\left(\frac{1}{4t^2} + 1\right) \qquad \left[\because t_1 = -t - \frac{1}{2t}\right]$$

$$= 4t^2\left(1 + \frac{1}{4t^2}\right)^3$$

$$\therefore \frac{d}{dt}\{(PQ)^2\}$$

$$= 8t\left(1 + \frac{1}{4t^2}\right)^3 + 4t^2 \times 3\left(1 + \frac{1}{4t^2}\right)^2 \times \frac{-2}{4t^3}$$

$$= 2\left(1 + \frac{1}{4t^2}\right)^2\left\{4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right\}$$

$$= 4\left(1+\frac{1}{4t^2}\right)^2\left(2t-\frac{1}{t}\right)$$

$$= 4\left(1+\frac{1}{4t^2}\right)\left(\frac{2t^2-1}{t}\right)$$

For maximum or minimum, we must have

$$\frac{d}{dt}(PQ^2)\ 0$$

$$\Rightarrow 4\left(1+\frac{1}{4t^2}\right)\left(\frac{2t^2-1}{t}\right) = 0$$

$$\Rightarrow 2t^2-1 = 0$$

$$\Rightarrow t = \pm\frac{1}{\sqrt{2}}$$

Now,

$$\frac{d^2}{dt^2}(PQ^2) = -2\frac{(2t^2-1)}{t^3}+4\left(1+\frac{1}{4t^2}\right)\left(2+\frac{1}{t^2}\right)$$

$$\Rightarrow \left|\frac{d^2}{dt^2}(PQ^2)\right|_{t=\pm\frac{1}{\sqrt{2}}} = 4\times2\times4>0$$

Thus, PQ is minimum when $t=\pm\frac{1}{\sqrt{2}}$.

For $t=\frac{1}{\sqrt{2}}$, coordinates of P and Q are $(1/\sqrt{2},1/2)$ and $(-\sqrt{2},2)$ respectively and $PQ^2=\frac{27}{4}$. The equation of PQ is $\sqrt{2}x+2y-2=0$.

For $t=-\frac{1}{\sqrt{2}}$, coordinates of P and Q are $(-1/\sqrt{2},1/2)$ and $(\sqrt{2},2)$ respectively and $PQ^2=27/4$. The equation of PQ is $\sqrt{2}x-2y+2=0$.

EXAMPLE 28 *Find the points on the curve* $5x^2+5y^2-6xy=4$ *nearest to the origin.*

SOLUTION Let $P(x,y)$ be any point on the curve $5x^2+5y^2-6xy=4$ and O be the origin.

Let $x=r\cos\theta$ and $y=r\sin\theta$. Then, $OP=r$.

Putting $x=r\cos\theta, y=r\sin\theta$ in $5x^2+5y^2-6xy=4$, we get

$$5r^2-3r^2\sin2\theta = 4$$

$$\Rightarrow r^2 = \frac{4}{5-3\sin2\theta}$$

Clearly, r^2 will be minimum, when $5-3\sin2\theta$ is maximum. But, $5-3\sin2\theta$ will be maximum, when

$$\sin2\theta = -1$$

$$\Rightarrow 2\theta = \frac{3\pi}{2}, \frac{-\pi}{2}$$

$$\Rightarrow \theta = \frac{-\pi}{4}, \frac{3\pi}{4}$$

For these values of θ, we have

$$r^2 = \frac{4}{5+3} = \frac{1}{2} \Rightarrow r = \frac{1}{\sqrt{2}}$$

Hence, required points are :

$$\left(\frac{1}{\sqrt{2}}\cos(-\pi/4), \frac{1}{\sqrt{2}}\sin(-\pi/4)\right) i.e. \left(\frac{1}{2}, \frac{-1}{2}\right)$$

and, $\left(\frac{1}{\sqrt{2}}\cos\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\sin\frac{3\pi}{4}\right)$ i.e. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

EXAMPLE 29 *A perpendicular is let fall from the centre on a tangent to an ellipse. Prove that the maximum value of the intercept between the point of contact and the foot of the perpendicular is* $a-b$, *where* a *and* b *are the semi-axes of the ellipse.*

SOLUTION Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad ...(i)$$

The equation of the tangent at P to (i) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1$$

Let O be the centre of the ellipse and let ON be the perpendicular from O on the tangent at P. Then,

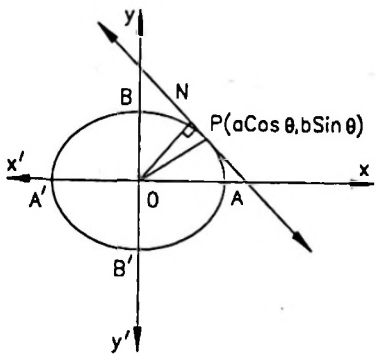


Fig. 9.27

$$ON = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Also,

$$OP = \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}.$$

$$\therefore PN^2 = OP^2-ON^2$$

$$= (a^2\cos^2\theta+b^2\sin^2\theta)-\frac{a^2b^2}{(a^2\sin^2\theta+b^2\cos^2\theta)}$$

$$= (a^2+b^2)-(a^2\sin^2\theta+b^2\cos^2\theta)-\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$= (a^2+b^2)-u-\frac{a^2b^2}{u}, \text{ where } u = a^2\sin^2\theta+b^2\cos^2\theta$$

$$\therefore \frac{d}{du}(PN^2) = -1+\frac{a^2b^2}{u^2} \text{ and } \frac{d^2}{du^2}(PN^2) = \frac{-2a^2b^2}{u^3}$$

For maximum or minimum values of PN, we must have

$$\frac{d}{du}(PN^2) = 0$$

$$\Rightarrow -1 + \frac{a^2b^2}{u^2} = 0$$

$$\Rightarrow u^2 = a^2b^2$$

$$\Rightarrow u = ab$$

For this value of u, we have

$$\frac{d^2}{du^2}(PN^2) = \frac{-2}{ab} < 0$$

Hence, PN is maximum, when $u = ab$.

Now,

$$u = ab \Rightarrow PN^2 = a^2 + b^2 - ab - \frac{a^2b^2}{ab} = (a-b)^2$$

$$\Rightarrow PN = a - b$$

Hence, the maximum value of PN is $(a - b)$.

EXAMPLE 30 *P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose centre is O and N is the foot of the perpendicular from O upon the tangent at P. Find the maximum area of ΔOPN and the coordinates of P.*

[IIT 1999]

SOLUTION Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The equation of the tangent at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad ...(i)$$

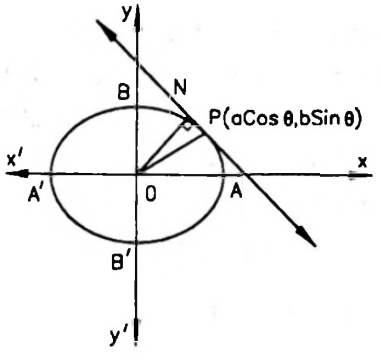


Fig. 9.28

Since N is the foot of the perpendicular from O on the tangent at P. Therefore, equation of ON is

$$y - 0 = \frac{a}{b}\tan\theta\,(x - 0)$$

or, $$ax\sin\theta - by\cos\theta = 0 \qquad ...(ii)$$

$\therefore$ PN = length of the perpendicular from P on ON

$$= \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

Also,

ON = length of the perpendicular from O on (i)

$$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

Let A denote the area of ΔOPN. Then,

$$A = \frac{1}{2}\,ON \times PN$$

$$\Rightarrow A = \frac{1}{2}\,\frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \times \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\Rightarrow A = \frac{ab(a^2 - b^2)\sin\theta\cos\theta}{2(a^2\sin^2\theta + b^2\cos^2\theta)} = \frac{ab(a^2 - b^2)}{2(a^2\tan\theta + b^2\cot\theta)}$$

Let $u = a^2\tan\theta + b^2\cot\theta$. Clearly, A will be maximum, if u is minimum.

Now,

$$u = a^2\tan\theta + b^2\cot\theta$$

$$\frac{du}{d\theta} = a^2\sec^2\theta - b^2\operatorname{cosec}^2\theta$$

For maximum or minimum values of u, we must have

$$\frac{du}{d\theta} = 0$$

$$\Rightarrow a^2\sec^2\theta - b^2\operatorname{cosec}^2\theta = 0$$

$$\Rightarrow \tan^2\theta = \frac{b^2}{a^2}$$

$$\Rightarrow \tan\theta = \pm\frac{b}{a}$$

Now,

$$\frac{d^2u}{d\theta^2} = 2(a^2\sec^2\theta\tan\theta + b^2\operatorname{cosec}^2\theta\cot\theta)$$

Clearly, $\frac{d^2u}{d\theta^2} > 0$ for $\tan\theta = \frac{b}{a}$.

Thus, u is minimum and hence A is maximum when $\tan\theta = \frac{b}{a}$. The maximum value of A is given by

$$A = \frac{ab(a^2 - b^2)}{2\left(a^2 \times \frac{b}{a} + b^2 \times \frac{a}{b}\right)} = \frac{a^2 - b^2}{4}$$

The coordinates of point P are $(a\cos\theta, b\sin\theta)$

i.e. $$\left(\pm\frac{a^2}{\sqrt{a^2 + b^2}}, \pm\frac{b^2}{\sqrt{a^2 + b^2}}\right)$$

EXAMPLE 31 *Suppose a, b, c, d are fixed real numbers such that a quadrilateral can be formed with sides a, b, c, d in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle.*

SOLUTION Let $PQRS$ be a quadrilateral such that $a = PQ$, $b = QR$, $c = RS$ and $d = PS$.

Let $\angle PQR = \alpha$ and $\angle RSP = \beta$. In order to prove that $PQRS$ in a cyclic quadrilateral it is sufficient to prove that $\alpha + \beta = \pi$.

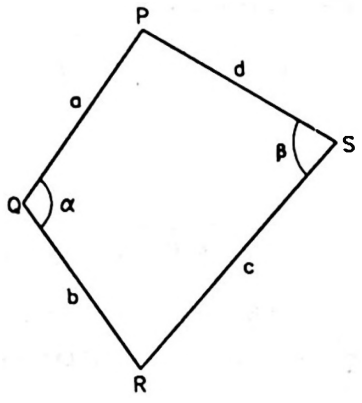


Fig. 9.29

In triangles PQR and PSR, we have

$PR^2 = a^2 + b^2 - 2ab\cos\alpha$ and $PR^2 = c^2 + d^2 - 2cd\cos\beta$.

$\therefore\ a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta$ ...(i)

Let A denote the area of quadrilateral $PQRS$. Then,

A = Area of ΔPQR + Area of ΔRSP

$$\Rightarrow\ A = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta$$

$$\Rightarrow\ \frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\frac{d\beta}{d\alpha} \quad ...(ii)$$

Differentiating both sides of (i) with respect to α, we get

$$2ab\sin\alpha = 2cd\sin\beta\frac{d\beta}{d\alpha}$$

$$\Rightarrow\ \frac{d\beta}{d\alpha} = \frac{ab}{cd}\frac{\sin\alpha}{\sin\beta}$$

Substituting the value of $\frac{d\beta}{d\alpha}$ in (i), we get

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\times\frac{ab\sin\alpha}{cd\sin\beta}$$

$$= \frac{1}{2}ab\left\{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\beta}\right\}$$

$$= \frac{1}{2}\frac{ab\sin(\alpha+\beta)}{\sin\beta}$$

For maximum or minimum values of A, we must have

$$\frac{dA}{d\alpha} = 0$$

$$\Rightarrow\ \frac{1}{2}ab\frac{(\sin(\alpha+\beta)}{\sin\beta} = 0$$

$$\Rightarrow\ \sin(\alpha+\beta) = 0$$

$$\Rightarrow\ \alpha+\beta = \pi$$

$$\Rightarrow\ 1 + \frac{d\beta}{d\alpha} = 0$$

$$\Rightarrow\ \frac{d\beta}{d\alpha} = -1 \text{ and } \frac{d^2\beta}{d\alpha^2} = 0$$

Now, differantiating (ii) w.r. to α, we get

$$\frac{d^2A}{d\alpha^2} = -\frac{1}{2}ab\sin\alpha - \frac{1}{2}cd\sin\beta\left(\frac{d\beta}{d\alpha}\right)^2 + \frac{1}{2}cd\cos\beta\frac{d^2\beta}{d\alpha^2}$$

For $\alpha + \beta = \pi$, we have

$$\frac{d^2A}{d\alpha^2} = -\frac{1}{2}ab\sin\alpha - \frac{1}{2}cd\sin\alpha < 0$$

Hence, A is maximum when $\alpha + \beta = \pi$ i.e. the quadrilateral is a cyclic quadrilateral.

EXAMPLE 32 *A circle of radius 1 unit touches positive x-axis and positive y-axis at P and Q respectively. A variable line l passing through origin intersects circle C in two points M and N. Find the equation of line l for which area of triangle MNQ is maximum.*

SOLUTION The equation of the circle C is

$$(x-1)^2 + (y-1)^2 = 1$$

or, $$x^2 + y^2 - 2x - 2y + 1 = 0 \quad ...(i)$$

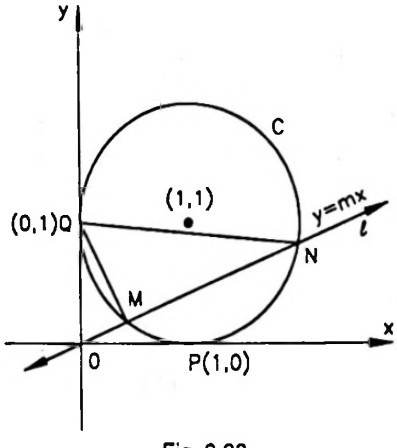


Fig. 9.30

Let the equation of the line l be $y = mx$.

Line l intersects the circle C in two points M and N. Let the coordinates of M and N be (x_1, y_1) and (x_2, y_2) respectively.

Putting $y = mx$ in (i), we find that the x-coordinates of M and N are the roots of the equation

$$(x-1)^2 + (mx-1)^2 = 1$$

or, $x^2(1+m^2) - 2x(1+m) + 1 = 0$

$$\therefore\ x_1 + x_2 = \frac{2(1+m)}{1+m^2} \text{ and } x_1x_2 = \frac{1}{1+m^2}$$

$$\Rightarrow\ (x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2$$

$$= \frac{4(1+m)^2}{(1+m^2)^2} - \frac{4}{1+m^2} = \frac{8m}{(1+m^2)^2}$$

Since M and N lie on $y = mx$. Therefore,

$$y_1 = mx_1 \text{ and } y_2 = mx_2$$

$$\Rightarrow\ y_2 - y_1 = m(x_2 - x_1)$$

$$\Rightarrow\ (y_2 - y_1)^2 = m^2(x_2 - x_1)^2$$

$$\therefore MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= (x_2 - x_1)\sqrt{1 + m^2}$$

$$= \sqrt{\frac{8m}{(1 + m^2)}}$$

Let A denote the area of ΔMNQ. Then,

$$A = \frac{1}{2} MN \times (\text{Length of perpendicular from } Q \text{ on } MN)$$

$$= \frac{1}{2} MN \times (\text{Length of perpendicular from } (0, 1) \text{ on } y = mx)$$

$$= \frac{1}{2} \sqrt{\frac{8m}{\sqrt{1 + m^2}}} \times \frac{1}{\sqrt{1 + m^2}}$$

$$\therefore A^2 = \frac{2m}{(1 + m^2)^2}$$

$$\Rightarrow \frac{d(A^2)}{dm} = \frac{2(1 + m^2)^2 - 8m^2(1 + m^2)}{(1 + m^2)^4} = \frac{(2 - 6m^2)}{(1 + m^2)^3}$$

For maximum or minimum values of A, we must have

$$\frac{d}{dm}(A^2) = 0$$

$$\Rightarrow 2 - 6m^2 = 0$$

$$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

We have,

$$\frac{d}{dm}(A^2) = \frac{-2(3m^2 - 1)}{(1 + m^2)^3}$$

The changes in the signs of $\frac{d(A^2)}{dm}$ for different values of m are shown in Fig. 9.31

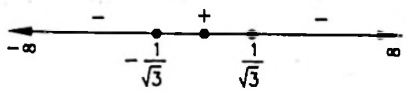


Fig. 9.31

Clearly, $\frac{d}{dm}(A^2)$ changes its sign from positive to negative in the neighbourhood of $m = \frac{1}{\sqrt{3}}$.

So, A^2 i.e. A is maximum when $m = \frac{1}{\sqrt{3}}$.

Hence, the equation of the line is $y = \frac{1}{\sqrt{3}} x$.

EXAMPLE 33 *Find the greatest rectangle which can be described so as to have two of its corners on the latusrectum and the other two on the portion of the curve cut off by the latusrectum of the parabola* $y^2 = 4ax$.

SOLUTION Let LSL' be the latusrectum of the parabola $y^2 = 4ax$. The equation of LSL' is $x = a$. Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$ and let $PQRS$ be a rectangle such that P and S are on the parabola and Q and R on the latusrectum LSL'. Clearly, coordinates of Q are $(a, 2at)$. From the symmetry, the coordinates of R and S are $(a, -2at)$ and $(at^2, -2at)$ respectively.

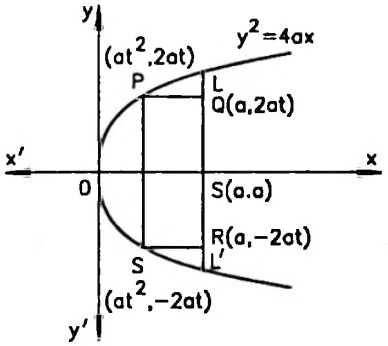


Fig. 9.32

$\therefore PQ = a - at^2$ and $QR = 4at$.

Let A denote the area of the rectangle $PQRS$. Then,

$$A = PQ \times QR$$

$$\Rightarrow A = (a - at^2)\,4at$$

$$\Rightarrow A = 4a^2(t - t^3)$$

$$\Rightarrow \frac{dA}{dt} = 4a^2(1 - 3t^2) \text{ and } \frac{d^2A}{dt^2} = -24a^2t$$

For maximum or minimum values of A, we must have

$$\frac{dA}{dt} = 0$$

$$\Rightarrow 1 - 3t^2 = 0$$

$$\Rightarrow t = \pm\frac{1}{\sqrt{3}}.$$

Clearly, $\frac{d^2A}{dt^2} < 0$ for $t = \frac{1}{\sqrt{3}}$.

Hence, area A is maximum when $t = 1/\sqrt{3}$. The maximum value of area A is given by

$$A = 4a^2\left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) = \frac{8a^2}{3\sqrt{3}}$$

EXAMPLE 34 *Let* (x, y) *be a point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and* $A > 0, B > 0$. *Show that* $Ax^3 + By^3$ *has a minimal value which is less than both the maximal values that it attains.*

SOLUTION Let $x = a\cos\theta$ and $y = b\sin\theta$.

Let $Z = Ax^3 + By^3$. Then,

$$Z = A a^3 \cos^3\theta + B b^3 \sin^3\theta.$$

$$\therefore \frac{dZ}{d\theta} = 3\cos\theta\sin\theta\,(Bb^3\sin\theta - A a^3\cos\theta)$$

For maximum or minimum values of Z, we must have

$$\frac{dZ}{d\theta} = 0$$

$$\Rightarrow 3\cos\theta\sin\theta\,(Bb^3\sin\theta - A\,a^3\cos\theta)$$

$$\Rightarrow \cos\theta = 0 \text{ or, } \sin\theta = 0 \text{ or, } \tan\theta = \frac{Aa^3}{Bb^3}$$

$$\Rightarrow \theta = 0, \frac{\pi}{2}, \tan^{-1}\left(\frac{Aa^3}{Bb^3}\right)$$

Now,

$$\frac{d^2Z}{d\theta^2} = 3\cos 2\theta\,(Bb^3\sin\theta - Aa^3\cos\theta) + 3\cos\theta\sin\theta\,(Bb^3\cos\theta + Aa^3\sin\theta)$$

CASE I *When* $\theta = 0$:

In this case, we have

$$\frac{d^2Z}{d\theta^2} = -3\,Aa^3 < 0$$

$\therefore$ $\theta = 0$ is a point of local maximum of Z.

The local maximum value of Z is given by

$$Z_1 = Aa^3$$

CASE II *When* $\theta = \frac{\pi}{2}$:

In this case, we have,

$$\frac{d^2Z}{d\theta^2} = -3Bb^3 < 0$$

$\therefore$ $\theta = \frac{\pi}{2}$ is a point of local maximum.

The local maximum value of Z is given by

$$Z_2 = Bb^3$$

CASE III *When* $\theta = \tan^{-1}\left(\frac{Aa^3}{Bb^3}\right)$ *i.e.* $\tan\theta = \frac{Aa^3}{Bb^3}$

In this case, we have,

$$\frac{d^2Z}{d\theta^2} = \frac{3Bb^3 . Aa^3}{\sqrt{A^2a^6 + B^2b^6}} > 0$$

So, $\theta = \tan^{-1}\left(\frac{Aa^3}{Bb^3}\right)$ is a point of local minimum.

The local minimum value of Z is given by

$$Z_3 = Aa^3\left(\frac{Bb^3}{\sqrt{A^2a^6 + B^2b^6}}\right)^3 + Bb^3\left(\frac{Aa^3}{\sqrt{A^2a^6 + B^2b^6}}\right)^3$$

$$= \frac{ABa^3b^3}{\sqrt{A^2a^6 + B^2b^6}}$$

Clearly, $Z_3 < Z_1$ and $Z_3 < Z_2$.

Hence, the minimal value of $Ax^3 + By^3$ is less than both the minimal values attained by it.

EXAMPLE 35 *Find the point on the curve* $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$, *that is farthest from the point* $(0, -2)$. [IIT 1987]

SOLUTION Let $P(a\cos\theta, 2\sin\theta)$ be a point on the ellipse

$$4x^2 + a^2y^2 = 4a^2 \text{ i.e. } \frac{x^2}{a^2} + \frac{y^2}{4} = 1.$$

Let $A(0, -2)$ be the given point. Then,

$$AP^2 = a^2\cos^2\theta + 4(1 + \sin\theta)^2$$

$$\Rightarrow \frac{d}{d\theta}(AP)^2 = -a^2\sin 2\theta + 8(1 + \sin\theta)\cos\theta$$

$$\Rightarrow \frac{d}{d\theta}(AP^2) = [(8 - 2a^2)\sin\theta + 8]\cos\theta$$

For maximum or minimum, we must have

$$\frac{d}{d\theta}(AP^2) = 0$$

$$\Rightarrow \{(8 - 2a^2)\sin\theta + 8\}\cos\theta = 0$$

$$\Rightarrow \cos\theta = 0 \text{ or, } \sin\theta = \frac{4}{a^2 - 4}$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} \qquad \left[\because 4 < a^2 < 8 \Rightarrow \frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1, \text{ which is impossible}\right]$$

Now,

$$\frac{d^2}{d\theta^2}(AP^2) = -[(8 - 2a^2)\sin\theta + 8]\sin\theta + (8 - 2a^2)\cos^2\theta$$

For $\theta = \frac{\pi}{2}$, we have

$$\frac{d^2}{d\theta^2}(AP^2) = -(16 - 2a^2) < 0$$

Thus, AP^2 *i.e.* AP is maximum when $\theta = \frac{\pi}{2}$. The point on the curve $4x^2 + a^2y^2 = 4a^2$ that is farthest from the point $A(0, -2)$ is $\left(a\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}\right) = (0, 2)$.

EXAMPLE 36 *Find the shortest distance between the line* $y - x = 1$ *and the curve* $x = y^2$.

SOLUTION Let $P(t^2, t)$ be any point on the curve $x = y^2$. The distance S of P from the given line is

$$S = \left|\frac{t - t^2 - 1}{\sqrt{2}}\right| = \left|\frac{t^2 - t + 1}{\sqrt{2}}\right| = \frac{t^2 - t + 1}{\sqrt{2}} \qquad [\because t^2 - t + 1 > 0 \text{ for all } t \in R]$$

$$= \frac{f(t)}{\sqrt{2}}, \text{ where } f(t) = t^2 - t + 1.$$

Clearly, S will be maximum or minimum according as $f(t)$ is maximum or minimum.

Now,

$$f(t) = t^2 - t + 1$$

$$\Rightarrow \quad f'(t) = 2t - 1 \text{ and } f''(t) = 2$$

For maximum or minimum, we must have

$$f'(t) = 0$$

$$\Rightarrow 2t - 1 = 0 \Rightarrow t = \frac{1}{2}$$

For $t = \frac{1}{2}$, we have

$$f''(t) > 0.$$

Hence, $f(t)$ is minimum when $t = \frac{1}{2}$.

So, S is minimum when $t = \frac{1}{2}$. The minimum value of S is given by

$$S = \frac{\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1}{\sqrt{2}} = \frac{3\sqrt{2}}{8}.$$

EXAMPLE 37 *Let $A(p^2, -p)$, $B(q^2, q)$ and $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segments BC, CA and AB respectively. Using calculus show that the maximum area of such a parallelogram is*

$$\frac{1}{4}(p+q)(q+r)(p-r) \qquad \textbf{[IIT 1985]}$$

SOLUTION Let $AF = x$ and $AE = y$. Triangles ABC and EDC are similar. Therefore,

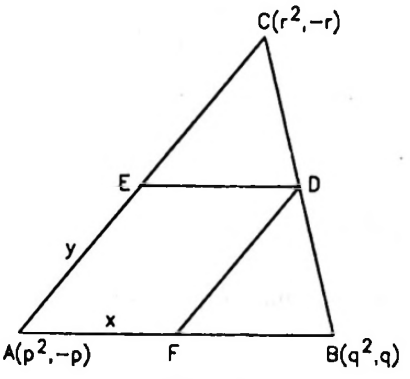


Fig. 9.33

$$\frac{AB}{ED} = \frac{AC}{CE}$$

$$\Rightarrow \quad \frac{c}{x} = \frac{b}{b-y}$$

$$\Rightarrow \quad bx = c(b-y)$$

$$\Rightarrow \quad x = \frac{c}{b}(b-y) \qquad \text{...(i)}$$

Let Z denote the area of parallelogram $AFDE$. Then,

$$Z = xy \sin A$$

$$\Rightarrow Z = \frac{c}{b}(by - y^2) \qquad \text{[Using (i)]}$$

$$\Rightarrow \frac{dZ}{dy} = \frac{c}{b}(b - 2y) \text{ and } \frac{d^2Z}{dy^2} = \frac{-2c}{b}$$

For maximum or minimum values of Z, we must have

$$\frac{dZ}{dy} = 0$$

$$\Rightarrow \frac{c}{b}(b - 2y) = 0 \Rightarrow y = \frac{b}{2}$$

Clearly, $\frac{d^2Z}{dy^2} = \frac{-2c}{b} < 0$ for all y.

Hence, Z is maximum when $y = b/2$. Putting $y = b/2$ in (i), we get $x = c/2$.

The maximum area is given by

$$Z = \frac{c}{2} \times \frac{b}{2} \sin A$$

$$= \frac{1}{4} bc \sin A$$

$$= \frac{1}{2} \text{ Area of } \Delta ABC$$

$$= \frac{1}{2} \times \frac{1}{2} \begin{vmatrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

$$= \frac{1}{4} \begin{vmatrix} p^2 & -p & 1 \\ q^2 - p^2 & q+p & 0 \\ r^2 - p^2 & -r+p & 0 \end{vmatrix} \qquad \begin{bmatrix} \text{Applying : } R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= \frac{1}{4}(p+q)(r-p) \begin{vmatrix} p^2 & -p & 1 \\ q-p & 1 & 0 \\ r+p & -1 & 0 \end{vmatrix}$$

$$= \frac{1}{4}(p+q)(r-p)(-r-q)$$

$$= \frac{1}{4}(p+q)(q+r)(p-r)$$

EXAMPLE 38 *Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c, d denote the lengths of the sides of the quadrilateral, prove that*

$$2 \le a^2 + b^2 + c^2 + d^2 \le 4 \qquad \textbf{[IIT 1997]}$$

SOLUTION Let $OABC$ be the square S of unit area and $PQRS$ be a quadrilateral whose vertices P, Q, R and S are respectively on the sides OA, AB, BC and OC. Let the coordinates of P, Q, R and S be $(p, 0)$, $(1, q)$, $(r, 1)$ and $(0, s)$ respectively.

We have,

$$(1-p)^2 + q^2 = a^2$$

$$(1-q)^2 + (1-r)^2 = b^2$$

$$r^2 + (1-s)^2 = c^2$$

and, $$p^2 + s^2 = d^2$$

$\therefore\ a^2+b^2+c^2+d^2$

$= \{p^2+(1-p)^2\}+q^2+(1-q)^2+\{r^2+(1-r)^2\}+\{s^2+(1-s)^2\}$

where p, q, r, s all vary between 0 and 1.

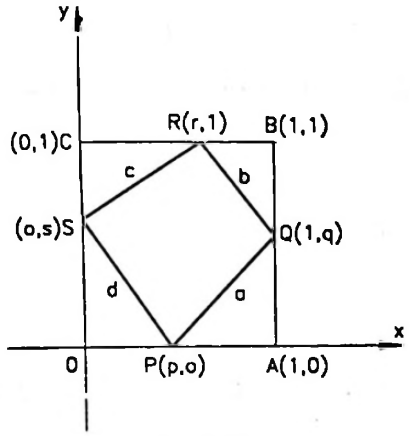


Fig. 9.34

We shall now find the maximum and minimum values of $a^2+b^2+c^2+d^2$. From the RHS of $a^2+b^2+c^2+d^2$, we observe that the maximum and minimum values of $a^2+b^2+c^2+d^2$ depend upon the maximum and minimum values of $x^2+(1-x)^2$. So, let us consider the function $f(x)=x^2+(1-x)^2$, where $0\le x\le 1$

Now,

$f(x) = x^2+(1-x)^2$

$f'(x) = 2x-2(1-x)$ and $f''(x) = 4$

For local maximum or minimum, we must have

$f'(x) = 0$

$\Rightarrow \quad x = \frac{1}{2}$

Clearly, $f''(x) > 0$ for all x.

Hence, $f(x)$ is minimum at $x = 1/2$ and the minimum value of $f(x)$ is given by

$$f\left(\frac{1}{2}\right) = \frac{1}{4}+\left(1-\frac{1}{2}\right)^2 = \frac{1}{2}$$

Clearly, $f(x)$ is maximum at $x = 0, 1$ and the maximum value of $f(x)$ is 1.

$\therefore\ \frac{1}{2}\le f(x)\le 1 \qquad$ for all $x\in[0,1]$

$\Rightarrow \frac{1}{2}\le x^2+(1-x)^2\le 1$ for all $x\in[0,1]$

$\Rightarrow \frac{1}{2}\le p^2+(1-p)^2\le 1$

$\frac{1}{2}\le q^2+(1-q)^2\le 1$

$\frac{1}{2}\le r^2+(1-r)^2\le 1$

$\frac{1}{2}\le s^2+(1-s)^2\le 1$

$\Rightarrow \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\le \{p^2+(1-p)^2\}+\{q^2+(1-q)^2\}$

$+\{r^2+(1-r)^2\}+\{s^2+(1-s)^2\}\le 1+1+1+1$

$\Rightarrow\ 2\le a^2+b^2+c^2+d^2\le 4.$

EXAMPLE 39 *Find the shortest distance between two points, one of which lies on the curve $y^2=4ax$, and the other on the circle $x^2+y^2-24ay+128a^2=0$.*

SOLUTION We have,

$$y^2 = 4ax \qquad \text{...(i)}$$

and, $x^2+y^2-24ay+128a^2 = 0 \qquad$ ...(ii)

or, $(x-0)^2+(y-12a)^2 = (4a)^2$

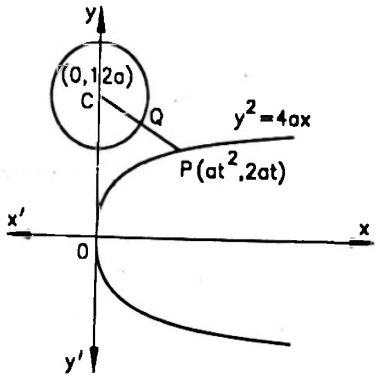


Fig. 9.35

Let $P(at^2, 2at)$ be a point on the parabola $y^2=4ax$.

Clearly, $(x-0)^2+(y-12a)^2 = (4a)^2$ is a circle with centre at $C(0, 12a)$ and radius $4a$.

Let Q be a point on the circle (ii) such that PQ is minimum. Then, CP will also be minimum.

Now,

$$CP^2 = (at^2-0)^2+(2at-12a)^2$$

$\Rightarrow \qquad CP^2 = a^2(t^4+4t^2-48t+144)$

$\therefore \qquad \frac{d}{dt}(CP^2) = a^2(4t^3+8t-48)$

and,

$$\frac{d^2}{dt^2}(CP^2) = a^2(12t^2+8)$$

For maximum or minimum, we must have

$\frac{d}{dt}(CP^2) = 0$

$\Rightarrow a^2(4t^3+8t-48) = 0$

$\Rightarrow t^3+2t-12 = 0$

$\Rightarrow (t-2)(t^2+2t+6) = 0$

$\Rightarrow t = 2 \qquad [\because\ t^2+2t+6\ne 0]$

Clearly, $\frac{d^2}{dt^2}(CP^2) > 0$ for $t = 2$.

Hence, CP^2 i.e. CP is minimum when $t = 2$. The minimum value of CP^2 is given by

$$CP^2 = a^2(16 + 16 - 96 + 144) = 80a^2$$

$\Rightarrow CP = 4\sqrt{5}a.$

$\therefore$ Minimum value of $PQ = CP - CQ$

$$= 4\sqrt{5}a - 4a = 4(\sqrt{5} - 1)a.$$

EXAMPLE 40 *Four points A, B, C and D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B and D are known as: A (−2, 3), B (−1, 1), D (2, 7). Find the coordinates of C for which the area of the quadrilateral ABCD is the greatest.*

SOLUTION Let the coordinates of C be (x, y). Since points A, B and D lie on $y = ax^2 + bx + c$. Therefore,

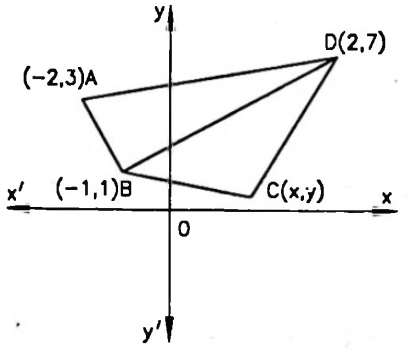


Fig. 9.36

$$3 = 4a - 2b + c$$

$$1 = a - b + c$$

$$7 = 4a + 2b + c$$

Solving these equations, we get

$a = 1, b = 1, c = 1$

$\therefore y = x^2 + x + 1$ ...(i)

Clearly, area of quadrilateral $ABCD$ will be maximum, if the area of ΔBCD is maximum.

Let P denote the area of ΔBCD. Then,

$$P = \frac{1}{2}\begin{vmatrix} -1 & 1 & 1 \\ x & y & 1 \\ 2 & 7 & 1 \end{vmatrix}$$

$$= \frac{3}{2}(2x - y + 3)$$

$$= \frac{3}{2}(2x - x^2 - x - 1 + 3) \quad \text{[Using (i)]}$$

$$= \frac{3}{2}(-x^2 + x + 2)$$

$$\therefore \frac{dP}{dx} = \frac{3}{2}(-2x + 1) \text{ and, } \frac{d^2P}{dx^2} = -3 < 0$$

For maximum or minimum values of P, we must have

$$\frac{dP}{dx} = 0 \Rightarrow -2x + 1 = 0 \Rightarrow x = \frac{1}{2}.$$

Clearly, $\frac{d^2P}{dx^2} < 0$ for all x.

Hence, P is maximum when $x = \frac{1}{2}$.

Putting $x = \frac{1}{2}$ in (i), we get $y = \frac{7}{4}$.

Hence, the coordinates of C are (1/2 , 7/4).

EXAMPLE 41 *A chord of length 2a divides a circular area of radius r into two segments. Find the sides of the rectangle with the largest area that can be inscribed in the smaller segment.*

SOLUTION Let PQ be a chord of length $2a$ of a circle of radius

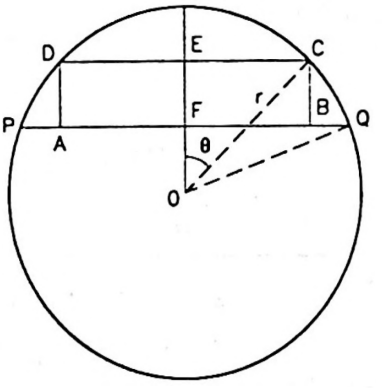


Fig. 9.37

r having its centre at O. Let $ABCD$ be the rectangle inscribed in the smaller segment of the circle. Let E and F be the mid-points of DC and AB respectively. Also, let $\angle COE = \theta$.

In ΔOQF, we have

$$OF = \sqrt{OQ^2 - QF^2} = \sqrt{r^2 - a^2}$$

In ΔOCE, we have

$$OE = r\cos\theta \text{ and } CE = r\sin\theta$$

$$\therefore \quad EF = OE - OF = r\cos\theta - \sqrt{r^2 - a^2}$$

and, $AB = CD = 2CE = 2r\sin\theta$

Let A denote the area of rectangle $ABCD$. Then,

$$A = AB \times BC$$

$$= AB \times EF$$

$$= 2r\sin\theta \times (r\cos\theta - \sqrt{r^2 - a^2})$$

$$= 2r^2\sin\theta\cos\theta - 2r\sqrt{r^2 - a^2}\sin\theta$$

$$A = r^2\sin 2\theta - 2r\sqrt{r^2 - a^2}\sin\theta$$

$$\therefore \frac{dA}{d\theta} = 2r^2\cos 2\theta - 2r\sqrt{r^2 - a^2}\cos\theta$$

For maximum and minimum values of A, we must have

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow 2r^2\cos 2\theta - 2r\sqrt{r^2 - a^2}\cos\theta = 0$$

$$\Rightarrow r\cos 2\theta - \sqrt{r^2 - a^2}\cos\theta = 0$$

$$\Rightarrow r(2\cos^2\theta - 1) - \sqrt{r^2 - a^2}\cos\theta = 0$$

$$\Rightarrow 2r\cos^2\theta - \sqrt{r^2 - a^2}\cos\theta - r = 0$$

$$\Rightarrow \cos\theta = \frac{\sqrt{r^2 - a^2} \pm \sqrt{r^2 - a^2 + 8r^2}}{4r}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{r^2-a^2}+\sqrt{9r^2-a^2}}{4r} \qquad [\because \theta \text{ is acute}]$$

For this value of θ, it can be checked that $\frac{d^2A}{d\theta^2} < 0$

Hence, A is maximum when θ is given by

$$\cos\theta = \frac{\sqrt{r^2-a^2}+\sqrt{9r^2-a^2}}{4r}$$

For this value of θ, we have

$$AB = CD$$
$$= 2CE$$
$$= 2r\sin\theta$$
$$= 2r\sqrt{1-\left\{\frac{\sqrt{r^2-a^2}+\sqrt{9r^2-a^2}}{4r}\right\}^2}$$
$$= \frac{1}{2}\sqrt{6r^2+2a^2-2\sqrt{(r^2-a^2)(9r^2-a^2)}}$$

and, $BC = r\cos\theta - \sqrt{r^2-a^2} = \frac{\sqrt{9r^2-a^2}-3\sqrt{r^2-a^2}}{4}$

EXAMPLE 42 *The lower corner of a page in a book is folded over so as to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is minimum is 2/3.*

SOLUTION Let the width of the page be a units.

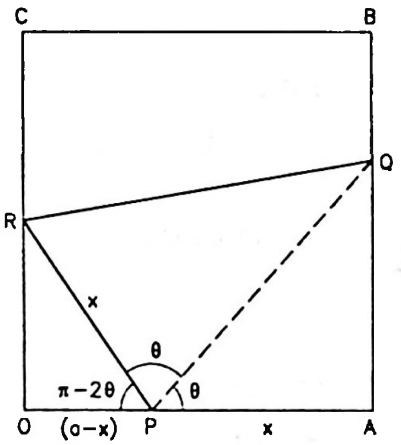


Fig. 9.38

The lower corner A of the page $OABC$ is folded over so that it reaches to R on the inner edge OC.

Let $\angle APQ = \theta$ and $AP = x$. Then,

$PR = AP = x$

In ΔOPR, we have

$$OP = PR\cos(\pi - 2\theta)$$
$$\Rightarrow a - x = -x\cos 2\theta$$
$$\Rightarrow x = \frac{a}{1-\cos 2\theta} = \frac{a}{2}\operatorname{cosec}^2\theta.$$

In ΔAPQ, we have

$$\tan\theta = \frac{AQ}{AP}$$
$$\Rightarrow AQ = AP\tan\theta = x\tan\theta$$
$$\Rightarrow AQ = \frac{a}{2}\operatorname{cosec}^2\theta\tan\theta = \frac{a}{2}\operatorname{cosec}\theta\sec\theta.$$

Let A denote the area of the folded part. Then,

$$A = \frac{1}{2}(AP \times AQ)$$
$$\Rightarrow A = \frac{1}{2}\left\{\frac{a}{2}\operatorname{cosec}^2\theta \times \frac{a}{2}\operatorname{cosec}\theta\sec\theta\right\}$$
$$\Rightarrow A = \frac{a^2}{8}\operatorname{cosec}^3\theta\sec\theta.$$
$$\therefore \frac{dA}{d\theta} = \frac{a^2}{8}[-3\operatorname{cosec}^3\theta\cot\theta\sec\theta + \operatorname{cosec}^3\theta\sec\theta\tan\theta]$$
$$= \frac{a^2}{8}[\operatorname{cosec}^3\theta\sec\theta(-3\cot\theta+\tan\theta)]$$

For maximum or minimum values of A, we must have

$$\frac{dA}{d\theta} = 0$$
$$\Rightarrow \frac{a^2}{8}\operatorname{cosec}^3\theta\sec\theta(-3\cot\theta+\tan\theta) = 0$$
$$\Rightarrow -3\cot\theta+\tan\theta = 0$$
$$\Rightarrow \tan^2\theta = 3$$
$$\Rightarrow \tan\theta = \sqrt{3} \qquad [\because \theta \text{ is acute}]$$
$$\Rightarrow \theta = \frac{\pi}{3}.$$

Now,

$$\frac{d^2A}{d\theta^2} = \frac{a^2}{8}[(-3\operatorname{cosec}^3\theta\cot\theta\sec\theta + \operatorname{cosec}^3\theta\sec\theta\tan\theta)$$
$$(-3\cot\theta+\tan\theta) + \operatorname{cosec}^3\theta\sec\theta(3\operatorname{cosec}^2\theta+\sec^2\theta)]$$

When $\theta = \frac{\pi}{3}$, we have

$$\frac{d^2A}{d\theta^2} = \frac{a^2}{8}\times\frac{100}{3\sqrt{3}} > 0$$

Hence, A is maximum when $\theta = \frac{\pi}{3}$.

$$\therefore \quad x = \frac{a}{2}\operatorname{cosec}^2\theta$$
$$\Rightarrow \quad x = \frac{a}{2}\operatorname{cosec}^2\frac{\pi}{3} = \frac{2a}{3}.$$
$$\therefore \quad \frac{AP}{OA} = \frac{2a/3}{a} = \frac{2}{3}.$$

Hence, the fraction of the width folded over when the area of the folded part is minimum is $\frac{2}{3}$.

EXAMPLE 43 *One corner of a long rectangular sheet of paper of width a units is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.*

SOLUTION Let $OABC$ be the sheet of paper as shown in Fig. 9.37

The corner A of the rectangular sheet $OABC$ is folded over along PQ so as to reach the opposite edge OC at R.

Let the crease PQ be of length x.

Let $\angle APQ = \theta$. Then, $\angle PQR = \theta$ and $\angle OPR = \pi - 2\theta$.

In ΔAPQ, we have

$$\cos\theta = \frac{AP}{PQ}$$

$\Rightarrow AP = x\cos\theta$.

In ΔOPR, we have

$$\cos(\pi - 2\theta) = \frac{OP}{RP}$$

$\Rightarrow -\cos 2\theta = \frac{OP}{AP}$ $[\because AP = RP]$

$\Rightarrow OP = -AP\cos 2\theta = -x\cos\theta\cos 2\theta$.

Now,

$a = OA = OP + AP$

$\Rightarrow a = x\cos\theta - x\cos 2\theta\cos\theta$

$$\Rightarrow x = \frac{a}{\cos\theta - \cos\theta\cos 2\theta} \quad ...(i)$$

$$\Rightarrow \frac{a}{x} = \cos\theta - \cos\theta\cos 2\theta$$

Let $y = \frac{a}{x}$. Then, y is maximum when x is minimum.

Now,

$y = \cos\theta - \cos\theta\cos 2\theta$

$$\Rightarrow \frac{dy}{d\theta} = -\sin\theta + \sin\theta\cos 2\theta + 2\cos\theta\sin 2\theta$$

For maximum or minimum values of y, we must have

$$\frac{dy}{d\theta} = 0$$

$\Rightarrow -\sin\theta + \sin\theta\cos 2\theta + 4\sin\theta\cos^2\theta = 0$

$\Rightarrow -\sin\theta(1 - \cos 2\theta) + 4\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow -2\sin^3\theta + 4\sin\theta - 4\sin^3\theta = 0$

$\Rightarrow 4\sin\theta = 6\sin^3\theta$

$\Rightarrow \sin^2\theta = \frac{2}{3}$ or, $\sin\theta = 0$

$\Rightarrow \sin\theta = \sqrt{\frac{2}{3}}$ or $\theta = 0$.

Now,

$$\frac{d^2y}{d\theta^2} = -\cos\theta + \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 4\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta$$

$$\Rightarrow \frac{d^2y}{d\theta^2} = -\cos\theta + 5\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta$$

For $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{1}{3}}$, we have

$$\frac{d^2y}{d\theta^2} = -\frac{1}{\sqrt{3}} + 5\times\sqrt{\frac{2}{3}}\times\left(\frac{2}{3} - 1\right) - 4\times\sqrt{\frac{2}{3}}\times 2\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{3}} < 0.$$

So, y is maximum when $\sin\theta = \sqrt{\frac{2}{3}}$.

Hence, x is minimum when $\sin\theta = \sqrt{\frac{2}{3}}$.

Putting $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{1}{3}}$ in (i), we get

$$\text{Length of the crease} = x = \frac{a}{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\left(1 - 2\times\frac{2}{3}\right)} = \frac{3\sqrt{3}a}{4}$$

EXAMPLE 44 *Assuming that the stiffness of a beam of rectangular cross-section varies as the breadth and as the cube of the depth, prove that for the stiffest beam, breadth must be equal to half the diameter of the log.*

SOLUTION Let the diameter of the log be $2a$ and the breadth and depth of the beam be x and y units respectively. Then,

$$x^2 + y^2 = (2a)^2 \quad ...(i)$$

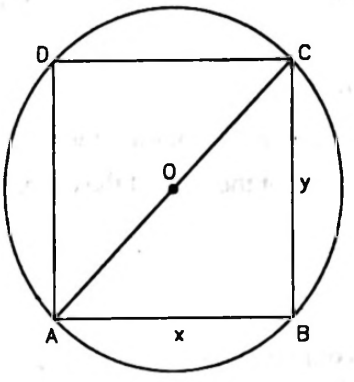


Fig. 9.39

Let S denote the stiffness of the beam. Then, it is given that

$S \propto xy^3$

$\Rightarrow S = \lambda xy^3$, λ is the constant of proportionality.

$\Rightarrow S = \lambda x(4a^2 - x^2)^{3/2}$ [Using (i)]

$\Rightarrow S^2 = \lambda^2 x^2(4a^2 - x^2)^3$

$$\Rightarrow \frac{d}{dx}(S^2) = \lambda^2\left[2x(4a^2 - x^2)^3 - 6x^3(4a^2 - x^2)^2\right]$$

$$= 2x\lambda^2(4a^2 - x^2)^2[4a^2 - x^2 - 3x^2]$$

$$= 8\lambda^2 x(4a^2 - x^2)^2(a^2 - x^2)$$

For maximum or minimum values of S or S^2, we must have

$$\frac{d}{dx}(S^2) = 0 \Rightarrow x = a \quad [\because x \neq 0, x \neq 2a]$$

Clearly, $\frac{d}{dx}(S^2)$ changes its sign from positive to negative in the neighbourhood of $x = a$.

So, S or S^2 is maximum when $x = a$ i.e. breadth of the beam is half the diameter of the log.

EXAMPLE 45 *A conical vessel is to be prepared out of a circular sheet of gold of unit radius. How much sectorial area is to be removed from the sheet so that the vessel has the maximum volume ?*

SOLUTION Let the angle AOB of the sectorial area removed be α. Then,

$$\alpha = \frac{AB}{1} \qquad \left[\text{Using} : \theta = \frac{\text{arc}}{\text{radius}}\right]$$

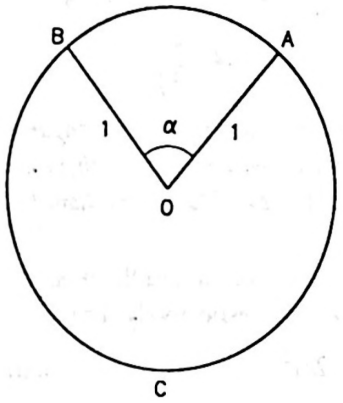


Fig. 9.40

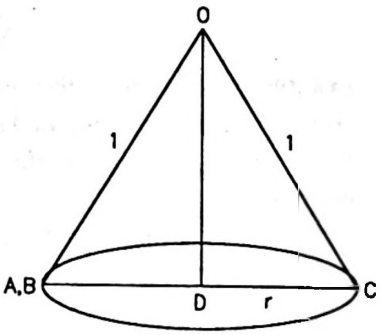


Fig. 9.41

$\Rightarrow AB = \alpha$

$\Rightarrow ACB = 2\pi - \alpha$

$\Rightarrow$ Circumference of the base of the cone $= 2\pi - \alpha$

Let r be the radius of the base of the cone. Then,

$2\pi r = 2\pi - \alpha$

$$\Rightarrow \quad r = 1 - \frac{\alpha}{2\pi} \qquad \ldots(i)$$

Let V be the volume of the vessel. Then,

$$V = \frac{1}{3}\pi r^2 \times OD$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 \sqrt{1 - r^2}$$

$$\Rightarrow \frac{dV}{d\alpha} = \frac{\pi}{3}\left[2r\frac{dr}{d\alpha}\sqrt{1-r^2} - \frac{r^3}{\sqrt{1-r^2}}\frac{dr}{d\alpha}\right]$$

$$\Rightarrow \frac{dV}{d\alpha} = \frac{\pi}{3}\left[\frac{2r(1-r^2) - r^3}{\sqrt{1-r^2}}\right]\frac{dr}{d\alpha}$$

$$\Rightarrow \frac{dV}{d\alpha} = -\frac{1}{6}\left[\frac{2r - 3r^3}{\sqrt{1-r^2}}\right] \qquad \left[\text{Using (i)} : \frac{dr}{d\alpha} = \frac{-1}{2\pi}\right]$$

For maximum or minimum values of V, we must have

$$\frac{dV}{d\alpha} = 0$$

$$\Rightarrow 2r - 3r^3 = 0$$

$$\Rightarrow r = \sqrt{\frac{2}{3}}$$

Now,

$$\frac{d^2V}{d\alpha^2} = -\frac{1}{6}\left[\frac{\sqrt{1-r^2}(2-9r^2)\frac{dr}{d\alpha} + (2r-3r^3)\frac{r}{\sqrt{1-r^2}}\frac{dr}{d\alpha}}{(1-r^2)}\right]$$

Putting $r = \sqrt{\frac{2}{3}}$ and $\frac{dr}{d\alpha} = -\frac{1}{2\pi}$, we get

$$\frac{d^2V}{d\alpha^2} = -\frac{1}{6} \times \frac{1}{\left(1 - \frac{2}{3}\right)}(2-6) \times \frac{-1}{2\pi} < 0$$

Hence, V is maximum when $r = \sqrt{\frac{2}{3}}$.

Putting $r = \sqrt{\frac{2}{3}}$ in (i), we get

$$\alpha = 2\pi - \frac{2\sqrt{2}\,\pi}{\sqrt{3}}.$$

Hence,

$$\text{required sectorial area} = \frac{\pi(1)^2}{2\pi}.\alpha$$

$$= \frac{\alpha}{2}$$

$$= \pi - \sqrt{\frac{2}{3}}\pi$$

$$= \pi\left(\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}\right)$$

EXAMPLE 46 *$(at_i^2, 2at_i)$; $i = 1, 2, 3$ are the vertices of a triangle inscribed in the parabola $y^2 = 4ax$. A parallellogram $AFDE$ is drawn with D, E, F on the line segments BC, CA and AB respectively. Show that the maximum area of such a parallelogram is $\frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$.*

SOLUTION Let $DE = x$ and $DF = y$. Draw FN perpendicular to AE. Triangles ABC and DEC are similar.

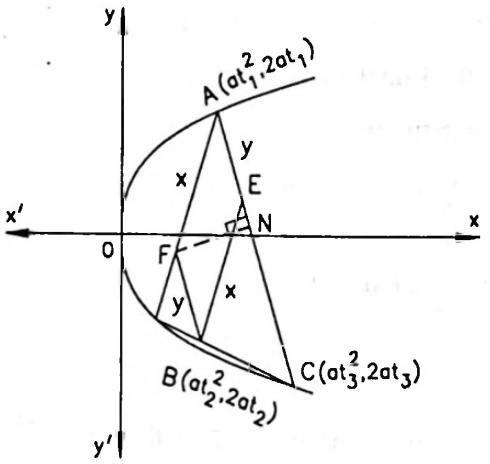


Fig. 9.42

$$\therefore \frac{AB}{DE} = \frac{AC}{EC}$$

$$\Rightarrow \frac{c}{x} = \frac{b}{b-y}$$

$$\Rightarrow x = \frac{(b-y)\,c}{b} \qquad ...(i)$$

Let S denote the area of parallelogram $DFAE$. Then,

$$S = AE \,.\, FN$$

$$\Rightarrow S = xy \sin A \qquad [\because FN = FA \sin A = x \sin A]$$

$$\Rightarrow S = \frac{(b-y)\,c}{b}\, y \sin A$$

$$\Rightarrow S = \frac{c}{b}(by - y^2) \sin A$$

$$\Rightarrow \frac{dS}{dy} = \frac{c}{b}(b - 2y) \sin A$$

and, $$\frac{d^2S}{dy^2} = -\frac{2c}{b} \sin A$$

For maximum or minimum, we must have

$$\frac{dS}{dy} = 0$$

$$\Rightarrow \frac{c}{b}(b-2y) \sin A = 0$$

$$\Rightarrow y = \frac{b}{2}.$$

Clearly, $\frac{d^2S}{dy^2} = \frac{-2c}{b} \sin A < 0$ for all y.

Hence, S is maximum when $y = \frac{b}{2}$.

The maximum value of S is given by

$$S = \frac{c}{b}\left(\frac{b^2}{2} - \frac{b^2}{4}\right) \sin A$$

$$\Rightarrow S = \frac{1}{4} bc \sin A$$

$$\Rightarrow S = \frac{1}{2}(\text{Area of } \Delta ABC) \qquad \left[\because \text{Area of } \Delta ABC = \frac{1}{2} bc \sin A\right]$$

$$\Rightarrow S = \frac{1}{2} \times \frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$\Rightarrow S = \frac{a^2}{2} \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix} = \frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$$

EXAMPLE 47 *A tree trunk l metre long is in the shape of a frustum of a cone, the radii of its ends being a and b metres (a > b). It is required to cut from it a beam of uniform square cross section. Prove that the beam of greatest volume that can be cut is* $\frac{al}{3(a-b)}$ *metre long.*

SOLUTION Let $2x$ metre be the length of the diagonal of the square cross section of the beam and let h metre be its height.

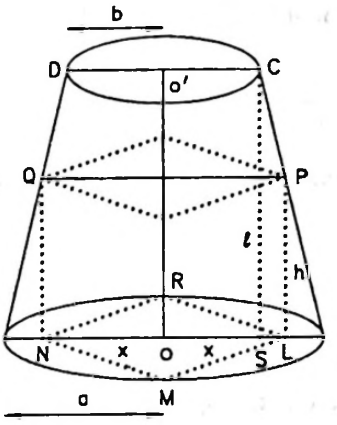


Fig. 9.43

We have,

$OA = OB = a$, $O'C = O'D = b$ and $LN = 2x$.

Let V be the volume of the beam. Then,

$$V = \text{Area of base} \times \text{height}$$

$$= \frac{1}{2}(\text{Diagonal})^2 \times \text{height}$$

$$= \frac{1}{2}(LN)^2 \times PL$$

$$= \frac{1}{2} \times (2x)^2 \times h$$

$$= 2x^2 h \qquad ...(i)$$

Draw $CS \perp AB$.

Triangles BCS and BPL are similar. Therefore,

$$\frac{CS}{PL} = \frac{BS}{BL}$$

$$\Rightarrow \frac{l}{h} = \frac{a-b}{a-x}$$

$$\Rightarrow h = \frac{l(a-x)}{a-b}$$

Putting the value of h in (i), we get

$$V = \frac{2x^2\, l\,(a-x)}{a-b} = \frac{2l}{a-b}(ax^2 - x^3)$$

$$\therefore \frac{dV}{dx} = \frac{2l}{a-b}(2ax - 3x^2) \text{ and } \frac{d^2V}{dx^2} = \frac{2l}{a-b}(2a - bx)$$

For maximum or minimum value of V, we must have

$$\frac{dV}{dx} = 0 \Rightarrow x = \frac{2a}{3}, 0.$$

For $x = \frac{2a}{3}$, we have

$$\frac{d^2V}{dx^2} = \frac{2l}{a-b}\left(2a - 6 \times \frac{2a}{3}\right) = \frac{-4al}{a-b} < 0$$

Hence, V is maximum when $x = \frac{2a}{3}$.

Putting $x = \frac{2a}{3}$ in $h = \frac{l(a-x)}{a-b}$, we get

$$h = \frac{al}{3(a-b)}$$

Hence, the height of the beam of maximum volume is $\frac{al}{3(a-b)}$.

EXAMPLE 48 *A regular square pyramid is inscribed in a sphere of given radius R so that all vertices of the pyramid belong to the sphere. Find the greatest value of the volume of the pyramid.*

SOLUTION If we cut the sphere from the centre along the diagonal of the square base, we get the view shown in Fig. 9.44

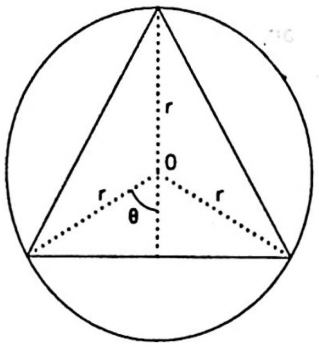


Fig. 9.44

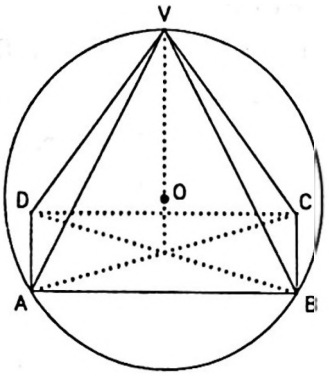


Fig. 9.45

Let r be the radius of the sphere. Then,

Height of the pyramid $= r + r\cos\theta$

Diagonal of the square base $= 2r\sin\theta$

$\therefore$ Side of the square base $= \frac{2r\sin\theta}{\sqrt{2}}$ [$\because$ Diagonal $= \sqrt{2}$ (side)]

Let V be the volume of the pyramid. Then,

$$V = \frac{1}{3}(\text{Area of the base} \times \text{Height})$$

$$\Rightarrow V = \frac{1}{3}\left(\frac{2r\sin\theta}{\sqrt{2}}\right)^2 \times (r + r\cos\theta)$$

$$\Rightarrow V = \frac{2}{3}r^3\sin^2\theta(1+\cos\theta)$$

$$\Rightarrow \frac{dV}{d\theta} = \frac{2}{3}r^3[\sin 2\theta(1+\cos\theta) - \sin^3\theta]$$

$$= \frac{2}{3}r^3[2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta]$$

$$= \frac{2}{3}r^3 \times \sin\theta(2\cos\theta + 2\cos^2\theta - \sin^2\theta)$$

$$= \frac{2}{3}r^3\sin\theta(3\cos^2\theta + 2\cos\theta - 1)$$

$$= \frac{2}{3}r^3\sin\theta(\cos\theta+1)(3\cos\theta-1)$$

For maximum or minimum values of V, we must have

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin\theta(\cos\theta+1)(3\cos\theta-1) = 0$$

$$\Rightarrow 3\cos\theta - 1 = 0 \qquad [\because \sin\theta = 0 \text{ and } \cos\theta + 1 = 0 \text{ are not possible}]$$

$$\Rightarrow \cos\theta = \frac{1}{3}.$$

Now,

$$\frac{d^2V}{d\theta^2} = \frac{2}{3}r^3\cos\theta(3\cos^2\theta + 2\cos\theta - 1) + \frac{2}{3}r^3\sin\theta(-6\sin\theta\cos\theta - 2\sin\theta)$$

For $\cos\theta = \frac{1}{3}$ and $\sin\theta = \frac{2\sqrt{2}}{3}$, we have

$$\frac{d^2V}{d\theta^2} = \frac{-64}{27}r^3 < 0$$

Hence, V is maximum when $\cos\theta = \frac{1}{3}$ and $\sin\theta = \frac{2\sqrt{2}}{3}$.

The maximum value of V is given by

$$V = \frac{2}{3}r^3 \times \frac{8}{9}\left(1 + \frac{1}{3}\right) = \frac{64}{81}r^3.$$

EXAMPLE 49 *A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/hr and walk at a speed of v km/hr. ($v > u$). At what point on the shore should he land so t hat he reaches his house in the shortest possible time ?*

SOLUTION Suppose the swimmer lands at a point P on the shore such that $AP = x$ km. Then,

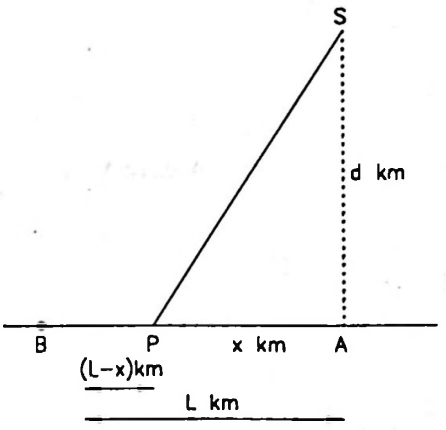


Fig. 9.46

$$SP = \sqrt{d^2 + x^2}$$

Since the swimmer swims at a speed of u km/hr. So, time taken by him to cover the distance SP is

$$t_1 = \frac{\sqrt{d^2+x^2}}{u}$$

The swimmer has to walk distance BP to arrive at his house at the speed of v km/hr. So,

Time taken by him to cover the distance $BP = \dfrac{L-x}{v}$

Let T be the total time taken by the swimmer to arrive at his house. Then,

$$T = t_1 + t_2$$

$$\Rightarrow \quad T = \frac{\sqrt{d^2+x^2}}{u} + \frac{L-x}{v}$$

$$\Rightarrow \quad \frac{dT}{dx} = \frac{x}{u\sqrt{d^2+x^2}} - \frac{1}{v}$$

$$\text{and,} \quad \frac{d^2T}{dx^2} = \frac{1}{u}\left[\frac{1}{\sqrt{d^2+x^2}} - \frac{x^2}{(d^2+x^2)^{3/2}}\right]$$

$$= \frac{d^2}{u(d^2+x^2)^{3/2}}$$

For maximum or minimum values of T, we must have

$$\frac{dT}{dx} = 0$$

$$\Rightarrow \frac{x}{u\sqrt{d^2+x^2}} = \frac{1}{v}$$

$$\Rightarrow v^2x^2 = u^2d^2 + u^2x^2$$

$$\Rightarrow x = \frac{ud}{\sqrt{v^2-u^2}}$$

For $x = \dfrac{ud}{\sqrt{v^2-u^2}}$, we have

$$\frac{d^2T}{dx^2} = \frac{d^2(v^2-u^2)^{3/2}}{uv^3d^3} > 0.$$

Hence, T is minimum when $x = \dfrac{ud}{\sqrt{v^2-u^2}}$.

Thus, the swimmer should land at a distance $\dfrac{ud}{\sqrt{v^2-u^2}}$ from point A.

EXAMPLE 50 *A lane runs at right angles out of a road a metres wide. Find how many metres wide the lane must be if it is just possible to carry a pole b metres long ($b > a$) from the road into the lane keeping it horizontal.*

SOLUTION Let PQ be the given pole of length b metres. When this pole is carried from the road into the lane, its end P will rest on the side of the wall and the end Q will rest in contact with the side of the lane.

A point R on the pole will be at the turning R as shown in Fig. 9.47. Let the width of the lane be x metres. If it is just possible to carry the pole from the road into the lane, keeping it horizontal, then the width RS ($= x$) of the lane shoule not exceed its maximum vlaue. Thus, we have to find the maximum value of x.

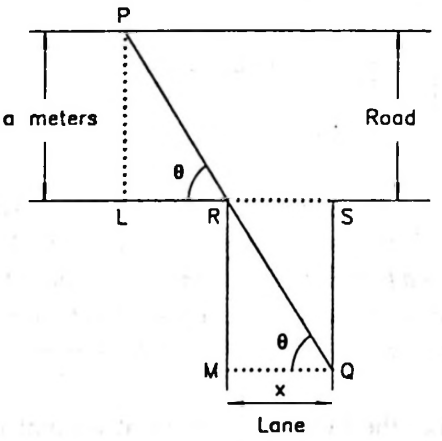


Fig. 9.47

In triangle PLR and QRM, we have

$$\sin\theta = \frac{PL}{PR} \qquad \text{and,} \quad \cos\theta = \frac{MQ}{QR}$$

$$\Rightarrow \sin\theta = \frac{a}{PR} \qquad \text{and,} \quad \cos\theta = \frac{x}{QR}$$

$$\Rightarrow \quad PR = a\,\text{cosec}\,\theta \quad \text{and,} \quad QR = x\sec\theta$$

But, $\quad b = PR + RQ$

$$\Rightarrow \quad b = a\,\text{cosec}\,\theta + x\sec\theta$$

$$\Rightarrow \quad x = b\cos\theta - a\cot\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -b\sin\theta + a\,\text{cosec}^2\theta$$

and,

$$\frac{d^2x}{d\theta^2} = -b\cos\theta - 2a\,\text{cosec}^2\theta\cot\theta$$

For maximum and minimum values of x, we must have

$$\frac{dx}{d\theta} = 0 \Rightarrow -b\sin\theta + a\,\text{cosec}^2\theta = 0$$

$$\Rightarrow \sin^3\theta = \frac{a}{b}$$

$$\Rightarrow \sin\theta = \left(\frac{a}{b}\right)^{1/3}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{b^{2/3}-a^{2/3}}}{b^{1/3}}$$

Putting the values of $\sin\theta$ and $\cos\theta$ in $\dfrac{d^2x}{d\theta^2}$, we get,

$$\frac{d^2x}{d\theta^2} = -b\left(\frac{\sqrt{b^{2/3}-a^{2/3}}}{b^{1/3}}\right) - 2a \times \frac{b^{2/3}}{a^{2/3}} \times \frac{\sqrt{b^{2/3}-a^{2/3}}}{a^{1/3}}$$

$$= -3b^{2/3}\sqrt{b^{2/3}-a^{2/3}} < 0.$$

Hence, x is maximum when $\sin\theta = (a/b)^{1/3}$. Consequently,

Required width of the lane is given by

$$x = b\cos\theta - a\cot\theta$$

$$= b\left(\frac{\sqrt{b^{2/3}-a^{2/3}}}{b^{1/3}}\right) - a\frac{\sqrt{b^{2/3}-a^{2/3}}}{a^{1/3}}$$

$$= (b^{2/3}-a^{2/3})^{3/2}$$

EXAMPLE 51 *Towns A and B are situated on the same side of a straight road at distances a and b respectively from it. Perpendiculars drawn from A and B meet the road at the points C and D respectively. The distance between C and D is c. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find the position of P.*

SOLUTION Let the hospital be built at a point P such that $CP = x$. Then, $DP = c - x$.

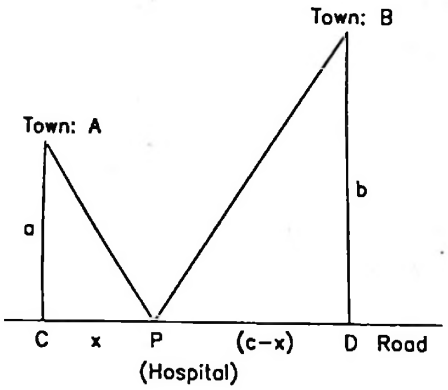


Fig. 9.48

In right triangles ACP and BDP, we have

$$AP^2 = a^2 + x^2 \text{ and } BP^2 = b^2 + (c-x)^2$$

Let $S = AP + BP$. Then,

$$S = \sqrt{a^2+x^2} + \sqrt{b^2+(c-x)^2}$$

$$\therefore \quad \frac{dS}{dx} = \frac{x}{\sqrt{a^2+x^2}} - \frac{(c-x)}{\sqrt{b^2+(c-x)^2}}$$

For maximum or minimum values of S, we must have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{b^2+(c-x)^2}}$$

$$\Rightarrow x^2\{b^2+(c-x)^2\} = (a^2+x^2)(c-x)^2$$

$$\Rightarrow x^2b^2 = (c-x)^2(a^2+x^2-x^2)$$

$$\Rightarrow x^2b^2 = a^2(c-x)^2$$

$$\Rightarrow bx = \pm a(c-x)$$

$$\Rightarrow bx \pm ax = \pm ac$$

$$\Rightarrow x = \frac{ac}{(b \pm a)} \qquad [\because x > 0]$$

$$\Rightarrow x = \frac{ac}{b+a} \qquad \left[\because x = \frac{ac}{b-a} > c, \text{ which is not possible}\right]$$

It can be checked that $\frac{d^2S}{dx^2} > 0$ for $x = \frac{ac}{b+a}$.

Hence, $S = APB$ is minimum when the hospital is made at a distance $\frac{ac}{a+b}$ from point C.

EXAMPLE 52 *Assuming the petrol burnt in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3c}{2}$ miles per hour.*

SOLUTION Let the motor boat be going against the current at the speed of v miles per hour. Suppose the motor boat is to cover a distance of s miles. Since the motor boat is moving against the current. So, it will cover the distance of s miles with effective speed of $(v-c)$ miles per hour in time

$$t = \frac{s}{v-c} \text{ hours}$$

Let P be the quantity of petrol burnt in time t. Then,

$$P = \lambda v^3 \times t, \lambda \text{ is a constant} \quad [\because \text{Petrol burnt} \propto (\text{velocity})^3]$$

$$\Rightarrow P = \lambda\frac{v^3 s}{v-c}$$

$$\Rightarrow P = \lambda s\left(\frac{v^3}{v-c}\right)$$

$$\Rightarrow \frac{dP}{dv} = \lambda s\left|\frac{3v^2(v-c)-v^3}{(v-c)^2}\right|$$

$$\Rightarrow \frac{dP}{dv} = \lambda s\left|\frac{2v^3-3v^2c}{(v-c)^2}\right|$$

For maximum or minimum values of P, we must have

$$\frac{dP}{dv} = 0$$

$$\Rightarrow \lambda s\left|\frac{2v^3-3v^2c}{(v-c)^2}\right| = 0$$

$$\Rightarrow 2v^3 - 3v^2c = 0$$

$$\Rightarrow 2v = 3c$$

$$\Rightarrow v = \frac{3}{2}c$$

Now,

$$\frac{d^2P}{dv^2} = \lambda s\left|\frac{(v-c)^2(6v^2-6vc)-2(2v^3-3v^2c)(v-c)}{(v-c)^4}\right|$$

For $v = \frac{3c}{2}$, we have

$$\frac{d^2P}{dv^2} = 18\lambda s > 0$$

Hence, P is minimum when $v = \frac{3c}{2}$.

Hence, the most economical speed when going against the current is $\frac{3}{2}c$ miles per hour.

EXAMPLE 53 *The amount of fuel consumed per hour by a certain steamer varies with the cube of its speed. When the speed is 15 kms/hour, the fuel consumed is 4.5 tonnes of coal per hour at Rs 40 per ton. The other expenses total Rs 100 per hour. Find the most economical speed and the cost of a voyage of 1980 kms.*

SOLUTION Let F tons per hour be the consumptionof the fuel when the speed of the steamer is v km/hr. Then,

$F \propto v^3$ [Given]

$\Rightarrow F = \lambda v^3$

It is given that $F = 4.5$ when $v = 15$.

$\therefore 4.5 = \lambda \times 15^3$

$\Rightarrow \lambda = \frac{1}{750}$.

$\therefore F = \frac{v^3}{750}$

Total time taken by the steamer to cover a distance of 1980 kms at the speed of v km/hr is $t = \frac{1980}{v}$ hours.

Let C be the total cost of voyage. Then,

$$C = \frac{1980}{v}\left\{100 + \frac{v^3}{750} \times 40\right\}$$

$$\Rightarrow C = \frac{198000}{v} + \frac{528}{5}v^2$$

$$\Rightarrow \frac{dC}{dv} = -\frac{198000}{v^2} + \frac{528}{5} \times 2v$$

and, $\frac{d^2C}{dv^2} = \frac{396000}{v^3} + \frac{1056}{5}$

For maximum or minimum, we must have

$$\frac{dC}{dv} = 0$$

$$\Rightarrow \frac{-198000}{v^2} + \frac{528}{5} \times 2v = 0$$

$$\Rightarrow v^3 = \frac{198000 \times 5}{1056} = \frac{1875}{2}$$

$$\Rightarrow v = \left(\frac{1875}{2}\right)^{1/3}$$

Clearly, $\frac{d^2C}{dv^2} > 0$ for $v = \left(\frac{1875}{2}\right)^{1/3}$

Hence, the most economical speed is

$$\left(\frac{1875}{2}\right)^{1/3} = 9.78 \text{ km/hr}$$

and, the cost of voyage is

$$C = \text{Rs.}\left\{\frac{198000}{9.78} + \frac{528}{5} \times (9.78)^2\right\} = \text{Rs. } 30345.86.$$

EXAMPLE 54 *A long wire of length l cm is bent to form a triangle with one of its angle as 60°. Find the sides of the triangle for which area is the largest.*

SOLUTION Let ABC be the triangle formed by folding the wire of length l cm such that $\angle A = 60°$, $BC = a$, $CA = b$ and $AB = c$.

Then, $a + b + c = l$.

Now,

$\angle A = 60°$

$\Rightarrow \cos A = \frac{1}{2}$

$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}$

$\Rightarrow b^2 + c^2 - a^2 = bc$

$\Rightarrow (b + c)^2 - a^2 = 3bc$

$\Rightarrow (l - a)^2 - a^2 = 3bc$ $[\because a + b + c = l \therefore b + c = l - a]$

$\Rightarrow l^2 - 2al = 3bc$ ...(i)

Now,

$bc = \frac{1}{3}l(l - 2a)$ and $b + c = l - a$

$\therefore (b - c)^2 = (b + c)^2 - 4bc$

$\Rightarrow (b - c)^2 = (l - a)^2 - \frac{4}{3}l(l - 2a)$

$\Rightarrow (b - c)^2 = \frac{1}{3}(-l^2 + 2al + 3a^2)$ ...(ii)

$\because (b - c)^2 \geq 0$

$\therefore \frac{1}{3}(-l^2 + 2al + 3a^2) \geq 0$

$\Rightarrow l^2 - 2al - 3a^2 \leq 0$

$\Rightarrow (l - 3a)(l + a) \leq 0$

$\Rightarrow l - 3a \leq 0$ $[\because l + a > 0]$

$\Rightarrow l \leq 3a$

$\Rightarrow a \geq l/3$

$\Rightarrow a \in [l/3, l)$ ...(iii)

Let Δ be the area of ΔABC. Then,

$$\Delta = \frac{1}{2} bc \sin 60°$$

$$\Rightarrow \Delta = \frac{1}{2} bc \left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \Delta = \frac{\sqrt{3}}{4} bc$$

$$\Rightarrow \Delta = \frac{\sqrt{3}}{12} (l^2 - 2al) \qquad \text{[Using (i)]}$$

$$\Rightarrow \frac{d\Delta}{da} = \frac{1}{4\sqrt{3}} (-2l) < 0$$

$\Rightarrow \Delta$ is a decreasing function of a.

$\Rightarrow \Delta$ is maximum for $a = \frac{l}{3}$ $\qquad [\because a \in [l/3, l)$ From (iii)]

Putting $a = \frac{l}{3}$ in (ii), we get $b = c$.

Substituting $a = \frac{l}{3}$ and $b = c$ in $a + b + c = l$, we get

$$a = b = c = \frac{l}{3}$$

Hence, area is maximum when sides of the triangle are equal and each equal to $l/3$.

EXAMPLE55 *For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum.* **[IIT 2003]**

SOLUTION We have,

In ΔOMQ, we have

$$\cos\theta = \frac{MQ}{OQ} \text{ and } \sin\theta = \frac{OM}{OQ}$$

$$\Rightarrow \cos\theta = \frac{MQ}{r} \text{ and } \sin\theta = \frac{OM}{r}$$

$$\Rightarrow MQ = r\cos\theta \text{ and } OM = r\sin\theta$$

$$\therefore \quad QR = 2MQ = 2r\cos\theta$$

and,

$$PM = OP - OM = 10 - r\sin\theta$$

Let A denote the area of ΔPQR. Thus,

$$A = \frac{1}{2} QR \times PM$$

$$\Rightarrow A = \frac{1}{2}(2r\cos\theta) \times (10 - r\sin\theta) \left[\because \sin\theta = \frac{OQ}{OP} = \frac{r}{10} \Rightarrow r = 10\sin\theta\right]$$

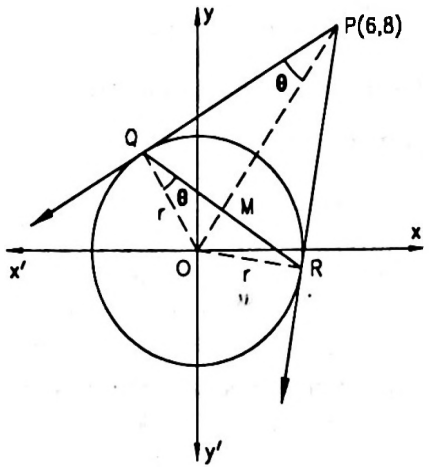


Fig. 9.49

$$\Rightarrow A = 10\sin\theta\cos\theta(10 - 10\sin^2\theta)$$

$$\Rightarrow A = 100\sin\theta\cos^3\theta.$$

$$\therefore \frac{dA}{d\theta} = 100(\cos^4\theta - 3\cos^2\theta\sin^2\theta)$$

and,

$$\frac{d^2A}{d\theta^2} = 100(-10\cos^3\theta\sin\theta + 6\sin^3\theta\cos\theta)$$

Now,

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \cos^4\theta - 3\cos^2\theta\sin^2\theta = 0$$

$$\Rightarrow \tan^2\theta = \frac{1}{3}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \qquad [\because 0 < \theta C < \pi/2]$$

$$\Rightarrow \theta = \frac{\pi}{6}.$$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta = \pi/6} = 100\left(-\frac{15\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right) < 0.$$

Thus, A is maximum when $\theta = \pi/6$ and hence $r = 10\sin\pi/6 = 5$.

ALITER We know that the area of the triangle formed by the tangents drawn from a point $P(h, k)$ to the circle $x^2 + y^2 = a^2$ and their chord contact is given by

$$A = \frac{a(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}.$$ [See page 30.79]

In this case, we have

$a = r$ and $(h, k) \equiv (6, 8)$.

$$\therefore \quad A = \frac{r}{100}(10 - r^2)^{3/2}$$

$$\Rightarrow \frac{dA}{dr} = \frac{1}{100}\left[(100 - r^2)^{3/2} - 3r^2(100 - r^2)^{1/2}\right]$$

$$\Rightarrow \frac{dA}{dr} = \frac{(100-r^2)^{1/2}}{100}(100-4r^2)$$

$$\Rightarrow \frac{dA}{dr} = \frac{1}{25}(100-r^2)^{1/2}\ (25-r^2)$$

For maximum or minimum, we have

$$\frac{dA}{dr} = 0$$

$$\Rightarrow (100-r^2)^{1/2}(25-r^2) = 0$$

$$\Rightarrow r = 5 \qquad [\because r \neq 10]$$

Clearly, $\frac{dA}{dr}$ changes its sign from positive to negative in the neighbourhood of $r = 5$.

Thus, A is maximum when $r = 5$.

EXERCISE 9.1

1. Find the values of a and b, if $f(x) = a \log |x| + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$.

2. If $f(x) = x^3 + ax^2 + bx + c$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. Determine the constants a, b, c.

3. If the function $f(x) = \frac{ax+b}{(x-1)(x-4)}$ has turning point at $P(2,-1)$. Find the values of a and b and show that y is maximum at P.

4. Investigate for maxima and minima the function

$$f(x) = \int_1^x \left\{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\right\} dt$$

5. A biquadratic polynomial $f(x)$ has a local maximum/minimum at $x = 1, 2$ and 3 and $f(-1) = 0$. If $\int_{-2}^{2} f(x)\,dx = -\frac{1348}{15}$, then find $f(x)$.

6. A biquadratic polynomial has a relative maximum/minimum at $x = 0, \pm 1$ and $f(0) = 3$. If $\int_{-2}^{2} f(x)\,dx = \frac{556}{45}$, prove that $f(x) = \frac{1}{6}(x^4 - 2x^2 + 18)$.

7. Determine the constants A and B so that $f(x) = \frac{A}{x} + Bx$ has a minimum at $(1, 6)$.

8. Find the maximum and minimum radii vectors of the curve

$$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}$$

9. Show that a triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.

10. The coordinates of a point P on a curve are $x = a\cos^5\theta$, $y = a\sin^5\theta$, where θ is a parameter. Prove that the least distance of P from the origin is $a/4$.

11. If $x > 0$, find the least value of the function

$$f(x) = 4x + \frac{9\pi^2}{x} + \sin x.$$

12. Show that the maximum and minimum values of $x^2 + y^2$, where $ax^2 + 2hxy + by^2 = 1$ are given by the roots of the quadratic

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2$$

13. Find the area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

14. A rectangle is inscribed in the ellipse $\frac{x^2}{400} + \frac{y^2}{225} = 1$ with its sides parallel to the axes of the ellipse. Find the dimensions of the rectangle if its perimeter is maximum.

15. From a fixed point P on the circumference of a circle of radius a, the perpendicular PM is let fall on the tangent at point Q. Prove that the maximum area of ΔPQM is $\frac{3\sqrt{3}\,a^2}{8}$.

16. Find the point on the curve $y = \frac{x}{1+x^2}$, where the tangent to the curve has its greatest slope.

17. Find the maximum value of

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

18. Evaluate the expression $\frac{[1+(dy/dx)^2]^{3/2}}{\frac{d^2y}{dx^2}}$ at a point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and show that it attains a maximum of a^2/b and a minimum of b^2/a.

19. Find the points on the curve $ax^2 + 2bxy + ay^2 = c$, $c > b > a > 0$, whose distance from the origin is minimum.

20. Find the normals to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which are farthest from its centre.

21. Find the coordinates of a point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$.

22. Find the point on the hyperbola $18x^2 - 24y^2 = 432$ which is nearest to the line $3x + 2y + 1 = 0$. Compute the distance between the point and the line.

23. Find the dimensions of the rectangle of greatest area that can be inscribed in a semi-circle of radius r.

24. Find the point on the curve $y = x^2$ which is closest to the point $A(0, a)$.

25. A printed page is required to contain k square units of printed matter. Side margins of width a and top and bottom margins of width b are required. Find the length of the printed lines when the page is designed to use the least paper.

26. Show that for a given total area of two boxes, one in the form of a rectangular parallelopiped with its sides in the ratio 6 : 3 : 1 and another in the form of a cube, the total volume is minimum when the ratio of their surface areas equals that of their volumes.

27. The strength of a beam varies as the product of its breadth and square of its depth. Find the dimensions of the strongest beam which can be cut from a circular log of radius a.

28. A straight line is drawn through the point $A(2, 4)$. Its segment forms a triangle with the segments $(x > 0, y > 0)$ of the coordinate axes. What must be the length of the larger leg for the area of the triangle to be least?

29. It is required to construct an open cylindrical reservoir of capacity π cubic units. The thickness of the material is t. What dimensions should the reservoir have to have the least possible cost ?

30. A cone is circumscribed to a sphere of radius r. Show that the volume of the cone is minimum if its altitude is $4r$ and its semi-vertical angle is $\sin^{-1}\left(\frac{1}{3}\right)$

31. Find the vertical angle of a right circular cone of minimum curved surface that circumscribes a given sphere.

32. For a given curved surface of a right circular cone, show that the volume is maximum when semi-vertical angle of the cone is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

33. Prove that the least perimeter of an isosceles triangle in which a circle of radius a can be inscribed is $6a\sqrt{3}$.

34. Prove that the greatest volume which can be inscribed in a given sphere has an altitude equal to (2/3)rd the diameter of the sphere. Prove also that the curved surface area of the cone is maximum for the same value of altitude.

35. Three sides of a trapezium are equal, each being 6 cm long. Find the area of the trapezium when it is maximum.

36. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved about one of its sides.

37. A rectangular sheet of metal has four equal square portions removed at the corner and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is maximum the depth will be

$$\frac{1}{6}\left[(a+b) - (a^2 - ab + b^2)^{1/2}\right]$$

where a and b $(a > b)$ are the sides of the original rectangle. Find the corresponding result if the given metal sheet be a square.

38. A window of fixed perimeter (including the base of the arc) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light ?

39. A thin closed rectangular box is to have one rectangular edge of n-times the length of another edge and the volume of the box is given to be V. Prove that the least surface S is given by

$$nS^3 = 54(n+1)^2 V^2$$

40. A given quantity of metal is to be cast into a half cylinder, i.e. with a rectangular base and semi-circular ends. Show that in order that the total surface area may be minimum the ratio of the height of the cylinder to the diameter of the semi-circular ends is $\pi : \pi + 2$.

41. If the sum of the areas of a cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum.

42. A sector is cut out of a circular sheet such that when made into a funnel it will have the greatest capacity. Find the central angle of the sector.

43. Show that the greatest value of $ax + by$ where x and y are positive and $x^2 + xy + y^2 = 3k^2$ is $2k\sqrt{a^2 - ab + b^2}$.

44. A bell-tent consists of a conical portion above and a cylindrical portion near the ground. For a given volume and a circular base of a given radius, prove that the amount of canvas used is minimum when the semi-vertical angle of the cone is $\cos^{-1}\left(\frac{2}{3}\right)$.

45. A flower bed is to be in the shape of a circular sector of radius r and the central angle θ. Find r and θ if the area is fixed and the perimeter is minimum.

46. LL' is the latusrectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate between the vertex and the latusrectum. Show that the area of the trapezium $PP'LL'$ is maximum when the distance of PP' from the vertex is $\frac{a}{9}$.

47. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $(b - a\cot\alpha)$ from the place to be reached.

48. A ladder is to be carried in a horizontal position round a corner formed by two streets a metres and b metres wide at right angles. Prove that the length of the longest ladder that will pass round the corner without jamming is $(a^{2/3} + b^{2/3})^{3/2}$.

49. Two towns are to get their water supply from a river. Both towns are on the same side of the river bank. If the distance between the points on the river bank nearest to two towns be 10 km, find :

(a) where a single pumping station may be located requiring the least amount of pipe,

(b) how much pipe be needed ?

50. A firm has a branch store in each of the three cities A, B and C. A and B are 3 km apart and C is 200 km from each of them. A godown is to be built equidistant from A and B. In order to minimize the time of transportation it should be located so that sum of the distances from the godown to each of the cities is minimum. Where should the godown be built ?

51. Two roads OA and OB intersect at an angle of 60°. A car driver approaches O from A where $OA = 600$ metres at a uniform speed of 20 m/sec. Simultaneously another car moves from O towards B at a uniform speed of 25 m/sec. Find the time when the two cars are closest.

52. A private telephone company serving a small community makes a profit of Rs 12 per subscriber, if it has 725 subscribers. It decides to reduce the rate by a fixed amount for each subscriber over 725, thereby reducing the profit by one paisa per subscriber. Thus there will be profit of Rs 11.99 on each of 726 subscribers, Rs 11.98 on each of 727 subscribers etc. What is the number of subscribers which will give the company the maximum profit ?

53. The fuel charges for running a train are proportional to the square of the speed generated in kms per hour and costs Rs 48 per hour at 16 km/hr. What is the most economical speed if the fixed charges i.e. salaries etc. amount to Rs 300 per hour?

54. When travelling x km/hr, a truck burns diesel at the rate of $\frac{1}{300}\left(\frac{900}{x}+x\right)$ litres per km. If the diesel oil costs 40 paise per litre and driver is paid Rs 1.50 per hour, find the steady speed that will minimize the total cost of the trip of 500 km.

55. A biquadratic function $f(x)$ has a realtive maximum/minimum at $x=0, x=1$ and $x=-1, f(0)=3$ and $\int_{-2}^{2} f(x)\,dx = \frac{552}{45}$. Find $f(x)$.

56. Find the values of p for which

$$f(x) = x^3 + 6(p-3)x^2 + 3(p^2-4)x + 10$$

has positive pointof maximum.

57. Find the absolute maximum and absolute minimum of the function

$$f(x) = \log_e\left\{(\sin x)^{\sin x} + 1\right\}, \text{ where } 0 < x \le \frac{\pi}{2}$$

58. P is any point on the curve $y = f(x)$ and C is a fixed point on the curve. If the length PC is extremum, prove that PC is perpendicular to the tangent at P.

59. Find the condition that $f(x) = x^3 + ax^2 + bx + c$ has

(i) a local minimum at a certain $x \in R^+$

(ii) a local maximum at a certain $x \in R^-$

(iii) a local maximum at certain $x \in R^-$ and minimum at certain $x \in R^+$.

60. If $f(x) = \cos^3 x + \lambda \cos^2 x, x \in (0, \pi)$. Find the range of λ so that $f(x)$ has exactly one maximum and exactly one minimum.

61. If 'a' is the lateral edge of a rectangular pyramid and α is the angle made by a lateral edge with the base. Prove that the volume is maximum when $\alpha = \cot^{-1}\sqrt{2}$.

ANSWERS

1. $a = 2, b = -\frac{1}{2}$

2. $a = -3, b = -9, c \in R$

3. $a = 1, b = 0$

5. $f(x) = x^4 - 8x^3 + 22x^2 - 24x - 55$

7. $A = B = 3$

8. Max. radius vector $= \frac{c}{\sqrt{a+b}}$ and Min. radius vector does not exist

13. $2ab$

14. 18, 32

15. (0, 0)

17. 6

19. $\left(\pm\sqrt{\frac{c}{2(a+b)}}, \pm\sqrt{\frac{c}{2\sqrt{(a+b)}}}\right)$

20. $\sqrt{3}x - \sqrt{2}y = \pm\sqrt{5}, \sqrt{3}x + \sqrt{2}y \pm \sqrt{5}$

21. $(-2, -8)$

22. $(6, -3), (-6, 3)$

23. Length $= \sqrt{2}r$, Breadth $= \frac{r}{\sqrt{2}}$

24. (0, 0) if $a < \frac{1}{2}$; $\left(\sqrt{a-\frac{1}{2}}, \frac{2a-1}{2}\right), \left(-\sqrt{a-\frac{1}{2}}, \frac{2a-1}{2}\right)$ if $a > \frac{1}{2}$

25. $\sqrt{\frac{ak}{b}}$ units

27. $\frac{2a}{\sqrt{3}}, 2a\sqrt{\frac{2}{3}}$

28. 8 units

29. Height = radius of the bar = 1 unit

31. $\alpha = \sin^{-1}(\sqrt{2}-1)$ **35.** $27\sqrt{3}$ cm^2

36. 12 cm, 6 cm **38.** $6 : 6 + \pi$ **41.** 1 : 1

42. $2\pi\sqrt{\frac{2}{3}}$ **45.** $r = \sqrt{\text{area}}$, $\theta = 2$ radians **57.** Absolute Max. $= \log_e 2$

Absolute Min. $= \log_e\left\{\left(\frac{1}{e}\right)^{1/e} + 1\right\}$

51. After $\frac{840}{51}$ sec. **52.** 963 **53.** 40 km/hr

54. 45 km/hr **56.** $p \in (-\infty, -2) \cup (2, 13/6)$

HINTS TO SELECTED PROBLEMS

11. We have,

$$f(x) = \left(2\sqrt{x} - \frac{3\pi}{\sqrt{x}}\right)^2 + 12\pi + \sin x$$

$\therefore$ Min. value of $f(x)$ is $12\pi - 1$.

17. First apply : $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$. Then, apply $C_1 \to C_1 + C_2$ to obtain $f(x) = 2 + 4\sin^2 x$.

22. Take the point as $(2\sqrt{6}\sec\theta, 3\sqrt{2}\tan\theta)$.

25. Let the length and breadth of the page be x and y units. Then, we have to minimize $A = xy$ subjected to $(x - 2b)(y - 2a) = k$.

26. Let the edges of I box be $6x, 3x, x$ that of II box be y.

Then, $54x^2 + 6y^2 = \lambda$ (const) (given). Let V be the total volume. Then,

$$V = 18x^3 + y^3$$

$$\therefore \frac{dV}{dx} = 54x^2 + 3y^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dV}{dx} = 54x^2 + 3y^2\left(\frac{-9x}{y}\right)\left[\because 54x^2 + 6y^2 = \lambda \Rightarrow 108x + 12y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{9x}{y}\right]$$

$$\Rightarrow \frac{dV}{dx} = 27x(2x - y)$$

For max or min, we must have

$$\frac{dV}{dx} = 0 \Rightarrow y = 2x.$$

Clearly, $\frac{dV}{dx}$ changes its sign from negative to positive at $x = y/2$. Therefore, V is minimum for $x = y/2$.

28. Let the line be $y - 4 = m(x - 2)$ and let A be the area of triangle formed by the line with the coordinate axes.

Then, $A = \frac{1}{2}\left(2 - \frac{4}{m}\right)(4 - 2m) = 2\left(1 - \frac{2}{m}\right)(2 - m)$

CHAPTER

10

INDEFINITE INTEGRALS

10.1 PRIMITIVE OR ANT-DERIVATIVE OF A FUNCTION

A function $\phi(x)$ is called a primitive or an antiderivative of a function $f(x)$ if

$$\phi'(x) = f(x).$$

Let $\phi(x)$ be a primitive of a function $f(x)$ and let C be any constant. Then,

$$\frac{d}{dx}\{\phi(x) + C\} = \phi'(x) = f(x) \qquad [\because \phi'(x) = f(x)]$$

$\Rightarrow$ $\phi(x) + C$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x) + C$, where C is a constant.

For example, $\frac{x^4}{4}, \frac{x^4}{4} + 2, \frac{x^4}{4} - 1$ etc. are primitives of x^3.

10.2 INDEFINITE INTEGRAL AND INDEFINITE INTEGRATION

Let $f(x)$ be a function. Then the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)\,dx$.

Thus, $\frac{d}{dx}\{\phi(x) + C\} = f(x) \Leftrightarrow \int f(x)\,dx = \phi(x) + C$ ...(i)

where $\phi(x)$ is primitive of $f(x)$ and C is an arbitrary constant known as the *constant of integration.*

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration or differential of x.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}(\phi(x)) = f(x)$.

10.3 FUNDAMENTAL FORMULAS ON INTEGRATION

Since $\frac{d}{dx}\{\phi(x)\} = f(x) \Leftrightarrow \int f(x)\,dx = \phi(x) + C$, therefore based upon this definition and various standard differentiation formulas, we obtain the following integration formulas:

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \quad \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\frac{d}{dx}(\log x) = \frac{1}{x} \quad \Rightarrow \int \frac{1}{x}\,dx = \log|x| + C$

(iii) $\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \int e^x\,dx = e^x + C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1 \Rightarrow \int a^x\,dx = \frac{a^x}{\log a} + C$

(v) $\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \int \sin x\,dx = -\cos x + C$

(vi) $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \int \cos x\,dx = \sin x + C$

vii) $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \int \sec^2 x\,dx = \tan x + C$

(viii) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \quad \Rightarrow \int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \int \sec x \tan x\,dx = \sec x + C$

(x) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$

$$\Rightarrow \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

(xi) $\frac{d}{dx}(\log \sin x) = \cot x \Rightarrow \int \cot x\,dx = \log|\sin x| + C$

(xii) $\frac{d}{dx}(-\log \cos x) = \tan x$

$$\Rightarrow \int \tan x\,dx = -\log|\cos x| + C$$

(xiii) $\frac{d}{dx}(\log(\sec x + \tan x)) = \sec x$

$\Rightarrow \int \sec x\,dx = \log|\sec x + \tan x| + C$

(xiv) $\frac{d}{dx}(\log(\operatorname{cosec} x - \cot x)) = \operatorname{cosec} x$

$\Rightarrow \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

(xv) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$

$\Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}$

$\Rightarrow \int -\frac{1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2}$

$\Rightarrow \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xviii) $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right) = -\frac{1}{a^2 + x^2}$

$\Rightarrow \int -\frac{1}{a^2 + x^2}dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}}$

$\Rightarrow \int \frac{1}{x\sqrt{x^2 - a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xx) $\frac{d}{dx}\left(\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}\right) = -\frac{1}{x\sqrt{x^2 - a^2}}$

$\Rightarrow \int -\frac{1}{x\sqrt{x^2 - a^2}}dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

10.4 SOME STANDARD RESULTS ON INTEGRATION

(i) $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$

i.e., the differentiation of an integral is the integrand itself or differentiation and integration are inverse operations.

(ii) $\int k f(x)\,dx = k\int f(x)\,dx$, *where k is a constant*

i.e., the integral of the product of a constant and a function = the constant × integral of the function.

(iii) $\int |f(x) \pm g(x)|\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

i.e., the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

The above results can be generalized to the form

$$\int |k_1 \cdot f(x) \pm k_2 f_2(x) \pm k_3 f_3(x) \pm \ldots \pm k_n f_n(x)|\,dx$$
$$= k_1 \cdot \int f_1(x)\,dx \pm k_2 \cdot \int f_2(x)\,dx \pm \ldots \pm k_n \int f_n(x)\,dx$$

i.e., the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals.

GENERALIZATIONS OF FUNDAMENTAL RESULTS ON INTEGRATION

We have,

(i) $\int (ax + b)^n\,dx$

$= \frac{1}{a}\int (ax + b)^n\,a\,dx$

$= \frac{1}{a}\int (ax + b)^n\,d(ax + b)$ $\left[d(ax + b) = \frac{d}{dx}(ax + b)\,.\,dx = a\,dx\right]$

$= \frac{1}{a}\frac{(ax + b)^{n+1}}{n + 1} + C, n \neq -1$ $\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + C\right]$

(ii) $\int \frac{1}{(ax + b)}\,dx$

$= \frac{1}{a}\int \frac{1}{ax + b}\,a\,dx$

$= \frac{1}{a}\int \frac{1}{(ax + b)}\,d(ax + b)$

$= \frac{1}{a}\log|ax + b| + C$ $\left[\because \int \frac{1}{x}dx = \log|x| + C\right]$

(iii) $\int e^{ax+b}\,dx$

$= \frac{1}{a}\int e^{ax+b}\,.\,a\,dx$

$= \frac{1}{a}\int e^{ax+b}\,d(ax + b)$

$= \frac{1}{a}e^{ax+b} + C$ $[\because \int e^x\,dx = e^x + C]$

(iv) $\int a^{cx+d}\,dx$

$= \frac{1}{c}\int a^{cx+d}\,.\,c\,dx$

$= \frac{1}{c}\int a^{cx+d}\,d(cx + d)$

$= \frac{1}{c}\frac{a^{cx+d}}{\log_e a} + C$ $\left[\because \int a^x\,dx = \frac{a^x}{\log_e a} - C\right]$

(v) $\int \sin(ax + b)\,dx$

$= \frac{1}{a}\int \sin(ax + b)\,.\,a\,dx$

$= \frac{1}{a}\int \sin(ax + b)\,d(ax + b)$

$= \frac{-1}{a}\cos(ax+b)+C \qquad \left[\because \int \sin x\,dx = -\cos x + C\right]$

(vi) $\int \cos(ax+b)\,dx$

$= \frac{1}{a}\int \cos(ax+b)\,a\,dx$

$= \frac{1}{a}\int \cos(ax+b)\,d(ax+b)$

$= \frac{1}{a}\sin(ax+b)+C \qquad \left[\because \int \cos x\,dx = \sin x + C\right]$

(vii) $\int \sec^2(ax+b)\,dx$

$= \frac{1}{a}\int \sec^2(ax+b)\,a\,dx$

$= \frac{1}{a}\int \sec^2(ax+b)\,d(ax+b)$

$= \frac{1}{a}\tan(ax+b)+C \qquad \left[\because \int \sec^2 x\,dx = \tan x + C\right]$

(viii) $\int \operatorname{cosec}^2(ax+b)\,dx$

$= \frac{1}{a}\int \operatorname{cosec}^2(ax+b)\,.\,a\,dx$

$= \frac{1}{a}\int \operatorname{cosec}^2(ax+b)\,d(ax+b)$

$= -\frac{1}{a}\cot(ax+b)+C \qquad \left[\because \int \operatorname{cosec}^2 x\,dx = -\cot x + C\right]$

(viii) $\int \sec(ax+b)\tan(ax+b)\,dx$

$= \frac{1}{a}\int \sec(ax+b)\tan(ax+b)\,a\,dx$

$= \frac{1}{a}\int \sec(ax+b)\tan(ax+b)\,d(ax+b)$

$= \frac{1}{a}\sec(ax+b)+C \qquad [\because \int \sec x\tan x\,dx = \sec x + C]$

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx$

$= \frac{1}{a}\int \operatorname{cosec}(ax+b)\cot(ax+b)\,a\,dx$

$= \frac{1}{a}\int \operatorname{cosec}(ax+b)\cot(ax+b)\,d(ax+b)$

$= -\frac{1}{a}\operatorname{cosec}(ax+b)+C \qquad \left[\because \int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + C\right]$

(xi) $\int \tan(ax+b)\,dx$

$= \frac{1}{a}\int \tan(ax+b)\,a\,dx$

$= \frac{1}{a}\int \tan(ax+b)\,d(ax+b)$

$= -\frac{1}{a}\log|\cos(ax+b)|+C \qquad \left[\because \int \tan x\,dx = -\log|\cos x| + C\right]$

(xii) $\int \cot(ax+b)\,dx$

$= \frac{1}{a}\int \cot(ax+b)\,a\,dx$

$= \frac{1}{a}\int \cot(ax+b)\,a\,dx$

$= \frac{1}{a}\int \cot(ax+b)\,d(ax+b)$

$= \frac{1}{a}\log|\sin(ax+b)|+C \qquad \left[\because \int \cot x\,dx = \log|\sin x| + C\right]$

(xiii) $\int \sec(ax+b)\,dx$

$= \frac{1}{a}\int \sec(ax+b)\,a\,dx$

$= \frac{1}{a}\int \sec(ax+b)\,d(ax+b)$

$= \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)|+C$

$[\because \int \sec x\,dx = \log|\sec x+\tan x|+C]$

(xiv) $\int \operatorname{cosec}(ax+b)\,dx$

$= \frac{1}{a}\int \operatorname{cosec}(ax+b)\,a\,dx$

$= \frac{1}{a}\int \operatorname{cosec}(ax+b)\,d(ax+b)$

$= \frac{1}{a}\log|\operatorname{cosec}(ax+b)-\cot(ax+b)|+C$

$[\because \int \operatorname{cosec} dx = \log|\operatorname{cosec} x-\cot x|+C]$

(xv) $\int \frac{1}{\sqrt{a^2-(bx+c)^2}}\,dx$

$= \frac{1}{b}\int \frac{b}{\sqrt{a^2-(bx+c)^2}}\,dx$

$= \frac{1}{b}\int \frac{1}{\sqrt{a^2-(bx+c)^2}}\,d(bx+c)$

$= \frac{1}{b}\sin^{-1}\left(\frac{bx+c}{a}\right)+C \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C\right]$

(xvi) $\int \frac{1}{a^2+(bx+c)^2}\,dx$

$= \frac{1}{b}\int \frac{1}{a^2+(bx+c)^2}\,b\,dx$

$= \frac{1}{b}\int \frac{1}{a^2+(bx+c)^2}\,d(bx+c)$

$= \frac{1}{ab}\tan^{-1}\left(\frac{bx+c}{a}\right)+C \qquad \left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right]$

(xvii) $\int \frac{1}{(bx+c)\sqrt{(bx+c)^2-a^2}}dx$

$= \frac{1}{b}\int \frac{1}{(bx+c)\sqrt{(bx+c)^2-a^2}}d(bx+c)$

$= \frac{1}{b}\times\frac{1}{a}\sec^{-1}\left(\frac{bx+c}{a}\right)+C \quad \left[\because \int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+C\right]$

$= \frac{1}{ab}\sec^{-1}\left(\frac{bx+c}{a}\right)+C$

(xviii) $\int \frac{-1}{(bx+c)\sqrt{(bx+c)^2-a^2}}dx$

$= \frac{1}{b}\int \frac{-1}{(bx+c)\sqrt{(bx+c)^2-a^2}}d(bx+c)$

$= \frac{1}{ab}\operatorname{cosec}^{-1}\left(\frac{bx+c}{a}\right)+C$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int \frac{2^x+3^x}{5^x}dx$ (ii) $\int \frac{(a^x+b^x)^2}{a^x b^x}dx$

(iii) $\int \frac{e^{5\log_e x}-e^{4\log_e x}}{e^{3\log_e x}-e^{2\log_e x}}dx$

SOLUTION We have,

(i) $\int \frac{2^x+3^x}{5^x}dx$

$= \int \left(\frac{2^x}{5^x}+\frac{3^x}{5^x}\right)dx$

$= \int \left[\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x\right]dx$

$= \frac{(2/5)^x}{\log_e(2/5)}+\frac{(3/5)^x}{\log_e(3/5)}+C$

(ii) $\int \frac{(a^x+b^x)^2}{a^x b^x}dx$

$= \int \frac{a^{2x}+b^{2x}+2a^x b^x}{a^x b^x}dx$

$= \int \left(\frac{a^x}{b^x}+\frac{b^x}{a^x}+2\right)dx$

$= \int \left\{\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x+2\right\}dx$

$= \frac{(a/b)^x}{\log_e(a/b)}+\frac{(b/a)^x}{\log_e(b/a)}+2x+C$

(iii) $\int \frac{e^{5\log_e x}-e^{4\log_e x}}{e^{3\log_e x}-e^{2\log_e x}}$

$= \int \frac{x^5-x^4}{x^3-x^2}dx \qquad [\because e^{a\log_e x}=x^a]$

$= \int \frac{x^4(x-1)}{x^2(x-1)}dx$

$= \int x^2\,dx = \frac{x^3}{3}+C$

EXAMPLE 2 *Evaluate :*

(i) $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$ (ii) $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

SOLUTION We have,

(i) $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$

$= \int \frac{\cos x-(2\cos^2 x-1)}{1-\cos x}dx$

$= \int \frac{-(2\cos^2 x-\cos x-1)}{-(\cos x-1)}dx$

$= \int \frac{(2\cos x+1)(\cos x-1)}{(\cos x-1)}dx$

$= \int (2\cos x+1)\,dx$

$= 2\sin x+x+C$

(ii) We have,

$\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

$= \int \frac{(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}$

$= \int \frac{1-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx$

$= \int \left(\frac{1}{\sin^2 x\cos^2 x}-3\right)dx$

$= \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-3\,dx$

$= \int (\sec^2 x+\operatorname{cosec}^2 x-3)\,dx$

$= \tan x-\cot x-3x+C$

EXAMPLE 3 *Evaluate :* $\int \sec^4 x \operatorname{cosec}^2 x \, dx$.

SOLUTION We have,

$$\int \sec^4 x \operatorname{cosec}^2 x \, dx$$

$$= \int \frac{1}{\cos^4 x \sin^2 x}$$

$$= \int \frac{(\sin^2 x + \cos^2 x)^2}{\cos^4 x \sin^2 x} dx$$

$$= \int \frac{\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x}{\cos^4 x \sin^2 x} dx$$

$$= \int \tan^2 x \sec^2 x \, dx + 2\int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$$

$$= \int \tan^2 x \, d(\tan x) + 2\int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$$

$$= \frac{\tan^3 x}{3} + 2\tan x - \cot x + C$$

EXAMPLE 4 *Evaluate :* $\int \frac{\cos x + x \sin x}{x(x + \cos x)} dx$

SOLUTION We have,

$$\int \frac{\cos x + x \sin x}{x(x + \cos x)} dx$$

$$= \int \frac{(x + \cos x) - x + x \sin x}{x(x + \cos x)} dx$$

$$= \int \left(\frac{1}{x} - \frac{1 - \sin x}{x + \cos x}\right) dx$$

$$= \int \frac{1}{x} dx - \int \frac{1 - \sin x}{x + \cos x} dx$$

$$= \int \frac{1}{x} dx - \int \frac{1}{x + \cos x} d(x + \cos x)$$

$$= \log|x| - \log|x + \cos x| + C.$$

EXAMPLE 5 *Evaluate the following integrals :*

(i) $\int \frac{(\sin^{-1} x)^3}{\sqrt{1 - x^2}} dx$ (ii) $\int \sec^n x \tan x \, dx$

(iii) $\int \sin^7 x \cos x \, dx$ (iv) $\int \frac{(\log x)^n}{x} dx$

SOLUTION We have,

(i) $\int \frac{(\sin^{-1} x)^3}{\sqrt{1 - x^2}} dx$

$$= \int (\sin^{-1} x)^3 \, d(\sin^{-1} x)$$

$$= \frac{(\sin^{-1} x)^4}{4} + C$$

(ii) $\int \sec^n x \tan x \, dx$

$$= \int \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int \sec^{n-1} x \, d(\sec x)$$

$$= \int (\sec x)^{n-1} \, d(\sec x)$$

$$= \frac{\sec^n x}{n} + C$$

(iii) $\int \sin^7 x \cos x \, dx$

$$= \int (\sin x)^7 \cos x \, dx$$

$$= \int (\sin x)^7 \, d(\sin x)$$

$$= \frac{(\sin x)^8}{8} + C$$

(iv) $\int \frac{(\log x)^n}{x} dx$

$$= \int (\log x)^n \frac{1}{x} dx$$

$$= \int (\log x)^n \, d(\log x)$$

$$= \frac{(\log x)^{n+1}}{n + 1} + C$$

EXAMPLE 6 *Evaluate :*

(i) $\int \frac{1}{x^2 (x^4 + 1)^{3/4}} dx$ (ii) $\int \frac{1}{\sqrt{e^{5x}} \sqrt[4]{(e^{2x} + e^{-2x})^3}} dx$

(iii) $\int \left(x + \frac{1}{x}\right)^{3/2} \left(\frac{x^2 - 1}{x^2}\right) dx$ (iv) $\int \frac{\sqrt{5 + x^{10}}}{x^{16}} dx$

(v) $\int \sqrt[3]{\frac{\sin^n x}{\cos^{n+6} x}} dx, \; n \in N$

SOLUTION We have,

(i) $\int \frac{1}{x^2 (x^4 + 1)^{3/4}} dx$

$$= \int \frac{1}{x^2 \left\{x^4 \left(1 + \frac{1}{x^4}\right)\right\}^{3/4}} dx$$

$$= \int \frac{1}{x^5 \left(1 + \frac{1}{x^4}\right)^{3/4}} dx$$

$$= -\frac{1}{4} \int \frac{1}{\left(1 + \frac{1}{x^4}\right)^{3/4}} \times \frac{-4}{x^5} dx$$

$$= -\frac{1}{4} \int \left(1 + \frac{1}{x^4}\right)^{-3/4} d\left(1 + \frac{1}{x^4}\right)$$

$$= -\frac{1}{4} \frac{\left(1 + \frac{1}{x^4}\right)^{-(3/4)+1}}{\left(-\frac{3}{4} + 1\right)} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4} + C$$

(ii) $\int \frac{1}{\sqrt{e^{5x}}\,(e^{2x}+e^{-2x})^{3/4}}\,dx$

$$= \int \frac{1}{\sqrt{e^{5x}}\,e^{3x/2}\,(1+e^{-4x})^{3/4}}\,dx$$

$$= \int \frac{1}{e^{4x}\,(1+e^{-4x})^{3/4}}\,dx$$

$$= \int (1+e^{-4x})^{-3/4}\,e^{-4x}\,dx$$

$$= -\frac{1}{4}\int (1+e^{-4x})^{-3/4}\,d\,(1+e^{-4x})$$

$$= -\frac{1}{4}\left\{\frac{(1+e^{-4x})^{1/4}}{1/4}\right\} + C$$

$$= -(1+e^{-4x})^{1/4} + C$$

(iii) $\int \left(x+\frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx$

$$= \int \left(x+\frac{1}{x}\right)^{3/2}\left(1-\frac{1}{x^2}\right)dx$$

$$= \int \left(x+\frac{1}{x}\right)^{3/2} d\left(x+\frac{1}{x}\right)$$

$$= \frac{\left(x+\frac{1}{x}\right)^{5/2}}{\frac{5}{2}} + C$$

$$= \frac{2}{5}\left(x+\frac{1}{x}\right)^{5/2} + C$$

(iv) $\int \frac{\sqrt{5+x^{10}}}{x^{16}}\,dx$

$$= \int \frac{\sqrt{5+x^{10}}}{x^{10}}\cdot\frac{1}{x^{11}}\,dx$$

$$= \int \sqrt{\frac{5}{x^{10}}+1}\cdot\frac{1}{x^{11}}\,dx$$

$$= -\frac{1}{50}\int \sqrt{1+\frac{5}{x^{10}}}\cdot\left(\frac{-50}{x^{11}}\right)dx$$

$$= -\frac{1}{50}\int \sqrt{1+\frac{5}{x^{10}}}\;d\left(1+\frac{5}{x^{10}}\right)$$

$$= -\frac{1}{50}\left\{\frac{\left(1+\frac{5}{x^{10}}\right)^{3/2}}{3/2}\right\} + C$$

$$= -\frac{1}{75}\left(1+\frac{5}{x^{10}}\right)^{3/2} + C$$

(v) $\int \sqrt[3]{\frac{\sin^n x}{\cos^{n+6} x}}\,dx$

$$= \int \sqrt[3]{\frac{\sin^n x}{\cos^n x\,.\cos^6 x}}\,dx$$

$$= \int \sqrt[3]{\tan^n x}\,\sec^2 x\,dx$$

$$= \int (\tan x)^{n/3}\,d\,(\tan x)$$

$$= \frac{(\tan x)^{\frac{n}{3}+1}}{\frac{n}{3}+1} + C$$

$$= \frac{3}{3+n}(\tan x)^{\frac{n}{3}+1} + C$$

EXAMPLE 7 *Evaluate* $\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

SOLUTION We have,

$$\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx$$

$$= \int \frac{x\,(\sqrt{x+a}-\sqrt{x+b})}{(\sqrt{x+a}+\sqrt{x+b})\,(\sqrt{x+a}-\sqrt{x+b})}\,dx$$

$$= \int \frac{x\,(\sqrt{x+a}-\sqrt{x+b})}{a-b}\,dx$$

$$= \frac{1}{a-b}\int \left\{x\sqrt{x+a}-x\sqrt{x+b}\right\}dx$$

$$= \frac{1}{a-b}\int \left\{(x+a-a)\sqrt{x+a}-(x+b-b)\sqrt{x+b}\right\}dx$$

$$= \frac{1}{a-b}\int \left\{(x+a)^{3/2}-a\sqrt{x+a}-(x+b)^{3/2}+b\sqrt{x+b}\right\}dx$$

$$= \frac{1}{a-b}\left[\frac{2}{5}(x+a)^{5/2}-\frac{2a}{3}(x+a)^{3/2}\right.$$

$$\left.-\frac{2}{5}(x+b)^{5/2}+\frac{2b}{3}(x+b)^{3/2}\right] + C$$

EXAMPLE 8 *Evaluate :*

(i) $\int \frac{x^7}{x+1}\,dx$ *(ii)* $\int \frac{x^3}{(x+1)^2}\,dx$ *(iii)* $\int \frac{ax+b}{cx+d}\,dx$

SOLUTION (i) We have,

$$\int \frac{x^7}{x+1}\,dx$$

$$= \int \frac{x^7+1}{x+1}\,dx$$

$$= \int \frac{x^7+1}{x+1}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \int (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)\, dx - \int \frac{1}{x+1}\, dx$$

$$= \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

(ii) $\int \frac{x^3}{(x+1)^2}\, dx$

$$= \int \frac{x^3+1-1}{(x+1)^2}\, dx$$

$$= \int \frac{x^3+1}{(x+1)^2}\, dx - \int \frac{1}{(x+1)^2}\, dx$$

$$= \int \frac{(x+1)(x^2-x+1)}{(x+1)^2}\, dx - \int \frac{1}{(x+1)^2}\, dx$$

$$= \int \frac{x^2-x+1}{x+1}\, dx - \int \frac{1}{(x+1)^2}\, dx$$

$$= \int \left(x - 2 + \frac{3}{x+1}\right) dx - \int \frac{1}{(x+1)^2}\, dx$$

$$= \frac{x^2}{2} - 2x + 3\log(x+1) + \frac{1}{x+1} + C$$

(iii) $\int \frac{ax+b}{cx+d}\, dx$

$$= \frac{a}{c} \int \frac{(cx+d) - \left(d - \frac{bc}{a}\right)}{cx+d}\, dx$$

$$= \frac{a}{c} \int dx - \frac{ad-bc}{c} \int \frac{1}{cx+d}\, dx$$

$$= \frac{a}{c}x - \left(\frac{ab-bc}{c^2}\right) \log|cx+d| + C$$

EXAMPLE 9 *Evaluate :* $\int \tan 2x \tan 3x \tan 5x\, dx$

SOLUTION We have,

$$\tan 5x = \tan(2x+3x)$$

$$= \frac{\tan 2x + \tan 3x}{1 - \tan 2x \tan 3x}$$

$$\Rightarrow \tan 5x - \tan 2x \tan 3x \tan 5x = \tan 2x + \tan 3x$$

$$\Rightarrow \tan 2x \tan 3x \tan 5x = \tan 5x - \tan 2x - \tan 3x$$

$$\therefore \int \tan 2x \tan 3x \tan 5x\, dx$$

$$= \int \tan 5x - \tan 2x - \tan 3x$$

$$= \int (\tan 5x - \tan 2x - \tan 3x)\, dx$$

$$= \frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| + C$$

EXAMPLE 10 *Evaluate :*

(i) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

(ii) $\int \frac{1}{\sin(x-a)\cos(x-b)}$

(iii) $\int \frac{1}{\cos(x-a)\cos(x-b)}\, dx$

SOLUTION (i) We have,

$$\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \{\cot(x-a) - \cot(x-b)\}\, dx$$

$$= \frac{1}{\sin(a-b)} \{\log|\sin(x-a)| - \log|\sin(x-b)|\} + C$$

$$= \operatorname{cosec}(a-b) \cdot \log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C.$$

(ii) We have,

$$\int \frac{1}{\sin(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\cos(a-b)} \int \{\cot(x-a) + \tan(x-b)\}\, dx$$

$$= \frac{1}{\cos(a-b)} [\log_e|\sin(x-a)| - \log_e|\cos(x-b)|] + C$$

$$= \frac{1}{\cos(a-b)} \log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$

(iii) We have,

$$\int \frac{1}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)]\, dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log_e|\cos(x-b)| + \log_e|\cos(x-a)|\right] + C$$

$$= \frac{1}{\sin(a-b)}\log_e\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C$$

EXAMPLE 11 *Evaluate :* $\int \frac{\sin(x+a)}{\sin(x+b)}\,dx$

SOLUTION (i) Let $I = \int \frac{\sin(x+a)}{\sin(x+b)}\,dx$ and $x+b=t$. Then, $dx = dt$.

Also,

$$\sin(x+a)$$
$$= \sin(t-b+a)$$
$$= \sin t\cos(a-b) + \cos t\sin(a-b)$$

$$\therefore\ I = \int \frac{\sin(x+a)}{\sin(x+b)}\,dx$$
$$= \int \frac{\sin t\cos(a-b) + \cos t\sin(a-b)}{\sin t}\,dt$$
$$= \int [\cos(a-b) + \cot t\sin(a-b)]\,dt$$
$$= \cos(a-b)\int 1\cdot dt + \sin(a-b)\int \cot t\,dt$$
$$= t\cos(a-b) + \sin(a-b)\log|\sin t| + C$$
$$= (x+b)\cos(a-b) + \sin(a-b)\log|\sin(x+b)| + C$$

EXAMPLE 12 *Evaluate :* $\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}\,dx$

SOLUTION We have,

$$\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}\,dx$$
$$= \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - 2\sin 3x\cos 3x}\,dx$$
$$= \int \frac{\left(2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{\sin 3x - \sin 6x}$$
$$= 4\int \frac{\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}\,dx$$
$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\,dx$$
$$= -\int(\cos 2x + \cos x)\,dx$$
$$= -\left(\frac{\sin 2x}{2} + \sin x\right) + C$$

EXAMPLE 13 *Evaluate :* $\int \frac{\cos 7x - \cos 8x}{1+2\cos 5x}\,dx$

SOLUTION We have,

$$\int \frac{\cos 7x - \cos 8x}{1+2\cos 5x}\,dx$$
$$= \frac{1}{2}\int \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{\sin\frac{5x}{2} + 2\cos 5x\ \sin\frac{5x}{2}}\,dx$$
$$= \frac{1}{2}\int \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}}\,dx$$
$$= \frac{1}{2}\int \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{21x}{2} + \sin\frac{9x}{2}\right)}{\sin\frac{15x}{2}}\,dx$$
$$= \frac{1}{2}\int \frac{2\sin\frac{15x}{2}\cos 2x - 2\sin\frac{15x}{2}\cos 3x}{\sin\frac{15x}{2}}\,dx$$
$$= \int(\cos 2x - \cos 3x)\,dx$$
$$= \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + C$$

EXAMPLE 14 *Evaluate* $\int \frac{1+\cos 4x}{\cot x - \tan x}\,dx$

SOLUTION We have,

$$\int \frac{1+\cos 4x}{\cot x - \tan x}\,dx$$
$$= \int \frac{2\cos^2 2x\cos x\sin x}{\cos^2 x - \sin^2 x}\,dx$$
$$= \int \cos 2x\sin 2x\,dx$$
$$= \frac{1}{2}\int \sin 4x\,dx$$
$$= -\frac{1}{8}\cos 8x + C$$

EXAMPLE 15 *Evaluate :* $\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\frac{x}{2}}{x-\sin x} + 3x\right\}dx$

SOLUTION We have,

$$\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\frac{x}{2}}{x-\sin x} + 3x\right\}dx$$
$$= \int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{3x^2(1-\cos x) + 3x(x-\sin x)}{x-\sin x}\right\}dx$$
$$= \int \frac{\sqrt{x-\sin x}}{\sqrt{x}}\left\{3x^2(1-\cos x) + 3x(x-\sin x)\right\}dx$$
$$= 3\int \sqrt{x^2 - x\sin x}\ (2x - x\cos x - \sin x)\,dx$$

$= 3\int \sqrt{x^2 - x\sin x}\; d(x^2 - x\sin x)$

$= 3 \times \frac{2}{3}(x^2 - x\sin x)^{3/2} + C$

$= 2(x^2 - x\sin x)^{3/2} + C$

EXAMPLE 16 *Evaluate :* $\int \frac{(1+\log x)^2}{1+\log x^{x+1} + (\log x^{\sqrt{x}})^2}\, dx$

SOLUTION We have,

$$\int \frac{(1+\log x)^2}{1+\log x^{x+1} + (\log x^{\sqrt{x}})^2}\, dx$$

$$= \int \frac{(1+\log x)^2}{1+(x+1)\log x + (\sqrt{x}\log x)^2}\, dx$$

$$= \int \frac{(1+\log x)^2}{1+(x+1)\log x + x(\log x)^2}\, dx$$

$$= \int \frac{(1+\log x)^2}{(1+\log x) + x\log x\,.\,(1+\log x)}\, dx$$

$$= \int \frac{(1+\log x)^2}{(1+x\log x)(1+\log x)}\, dx$$

$$= \int \frac{1+\log x}{(1+x\log x)}\, dx$$

$$= \int \frac{1}{(1+x\log x)}\, d(1+x\log x) = \log|1+x\log x| + C.$$

EXERCISE 10.1

Evaluate the following integrals :

1. $\int \cos^2 x\, dx$
2. $\int \cos^3 x\, dx$
3. $\int \cos^4 x\, dx$
4. $\int \sin^4 x \cos^4 x\, dx$
5. $\int \sin 2x \cos 4x \cos 5x\, dx$
6. $\int \sin^2 x \cos^2 x\, dx$
7. $\int \sin^3 x \cos^3 x\, dx$
8. $\int \frac{\sin 4x}{\sin x}\, dx$
9. $\int x^x (1+\log x)\, dx$
10. $\int x^{2x}(1+\log x)\, dx$
11. $\int \left(x - \frac{1}{x}\right)^{3/2} \left(\frac{x^2+1}{x^2}\right) dx$
12. $\int \sec^5 x \tan x\, dx$
13. $\int \sin^5 x \cos x\, dx$
14. $\int \frac{(\cos^{-1} x)^{3/2}}{\sqrt{1-x^2}}\, dx$
15. $\int \frac{1}{\sqrt{x}+\sqrt{x-1}}\, dx$
16. $\int \frac{x}{\sqrt{x+a}-\sqrt{x+b}}\, dx$
17. $\int \frac{x-1}{\sqrt{2x+1}}\, dx$
18. $\int \{1+2\tan x(\tan x + \sec x)\}^{1/2}\, dx$
19. $\int \tan x \tan 2x \tan 3x\, dx$
20. $\int \frac{\sin(x-a)}{\sin(x-b)}\, dx$
21. $\int \frac{1}{\sin(x+a)\cos(x-b)}$
22. $\int \frac{\sin 2x}{\sin(x-\pi/3)\sin(x+\pi/3)}\, dx$
23. $\int \{1+\tan x \tan(x+\theta)\}\, dx$
24. $\int \tan(x-\theta)\tan(x+\theta)\tan 2x\, dx$
25. $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right) dx$
26. $\int \frac{1+x+x^2}{x^3(1+x)} + \frac{2x-1}{(x+1)^2} + \frac{4-5\sin x}{\cos^2 x}\, dx$
27. $\int \frac{\sqrt{x^2+1}}{x^4}\, dx$
28. $\int \frac{1}{(e^x+e^{-x})^2}\, dx$
29. $\int \frac{1+x\sin x+\cos x}{x(1+\cos x)}\, dx$
30. $\int \frac{1+2x\sin x-\cos x}{x(1-\cos x)}$
31. $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\, dx$

ANSWERS

1. $\frac{x}{2}+\frac{\sin 2x}{4}+C$
2. $\frac{\sin 3x}{12}+\frac{3}{4}\sin x+C$
3. $\frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$
4. $\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$
5. $-\frac{1}{44}\cos 11x+\frac{1}{28}\cos 7x-\frac{1}{12}\cos 3x-\frac{1}{4}\cos x+C$
6. $\frac{1}{8}x-\frac{1}{32}\sin 4x+C$
7. $\frac{-3}{64}\cos 2x+\frac{1}{192}\cos 6x+C$
8. $\frac{2}{3}\sin 3x+2\sin x+C$
9. x^x+C
10. $\frac{x^{2x}}{2}+C$
11. $\frac{2}{5}\left(x-\frac{1}{x}\right)^{5/2}+C$
12. $\frac{\sec^5 x}{5}+C$
13. $\frac{\sin^6 x}{6}+C$
14. $-\frac{2}{5}\left(\cos^{-1}x\right)^{5/2}+C$
15. $\frac{2}{3}x^{3/2}-\frac{2}{3}(x-1)^{3/2}+C$
16. $\frac{1}{a-b}\left[\frac{2}{5}(x+a)^{5/2}-\frac{2a}{3}(x+a)^{3/2}+\frac{2}{5}(x+b)^{5/2}-\frac{2b}{3}(x+b)^{3/2}\right]+C$
17. $\frac{1}{6}(2x+1)^{3/2}-\frac{3}{2}(2x+1)^{1/2}+C$
18. $\log|\sec x+\tan x|+\log|\sec x|+C$
19. $\frac{1}{3}\log|\sec 3x|-\frac{1}{2}\log|\sec 2x|-\log|\sec x|+C$
20. $x\cos(b-a)+\sin(b-a)\log|\sin(x-b)|+C$
21. $\frac{1}{\cos(a+b)}\left\{\log|\sin(x+a)|+\log|\sec(x-b)|\right\}+C$
22. $\log\left|\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)\right|+C$
23. $\cot\theta\log\left|\frac{\cos x}{\cos(x+\theta)}\right|+C$
24. $\frac{1}{2}\log|\sec 2x|-\log\sec(x+\theta)-\log\sec(x-\theta)+C$
25. $-\frac{x^2}{2}+C$
26. $-\frac{1}{2x^2}+\log\left|\frac{x}{x+1}\right|+2\log|x+1|+\frac{3}{x+1}+4\tan x-5\sec x+C$
27. $-\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^3}+C$
28. $-\frac{1}{2(e^{2x}+1)}+C$
29. $\log|x|+2\log\left|\sec\frac{x}{2}\right|+C$
30. $\log|x|+2\log|1-\cos x|+C$
31. $\frac{1}{4}\log\left|\frac{1+\tan x}{1-\tan x}\right|+\frac{1}{4}\sin 2x+C$

—❑❑❑—

10.4 METHODS OF INTEGRATION

We have the following methods of integration :

(i) Integration by substitution or change of independent variable.

(ii) Integration by parts

(iii) Integration of rational algebraic function by using partial fractions.

In the following sections, we shall learn about these methods.

10.5 INTEGRATION BY SUBSTITUTION

Integrations of certain functions cannot be obtained directly if they are not in one of the standard ferms discussed in previous sections, but they may be reduced to standard forms by proper substitution. The process of evaluating an integral by reducing it to standard form by a proper substitution is called integration by substitution.

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form

$$\int f(\phi(x))\,\phi'(x)\,dx,$$

we substitute $\phi(x) = t$ and $\phi'(x)\,dx = dt$

This substitution reduces the integral $\int f(\phi(x))\,\phi'(x)\,dx$ to the form

$$\int f(t)\,dt$$

This integral can be evaluated by using the standard results.

After evaluating this integral, we substitute back the value of t as $\phi(x)$.

The above substitution can also be performed in the following manner :

$$\int f(\phi(x))\,\phi'(x)\,dx = \int f(\phi(x))\,d(\phi(x)) \quad [\because d\phi(x) = \phi'(x)\,dx]$$

$$= \int f(t)\,dt, \text{ where } t = \phi(x).$$

Follwoing examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following integrals :*

(i) $\int e^x (x+1)\cos^2(x e^x)\,dx$ (ii) $\int \dfrac{\tan x}{a + b\tan^2 x}\,dx$

SOLUTION (i) We observe that $e^x(x+1)$ occurs in the derivative of $x e^x$. So, let us substitute

$$x e^x = t$$

$$\Rightarrow d(x e^x) = dt$$

$$\Rightarrow \frac{d}{dx}(x e^x)\,.\,dx = dt \qquad \left[\because dt = \frac{df}{dx}\,.\,dx\right]$$

$$\Rightarrow (x+1)e^x\,dx = dt$$

$$\Rightarrow dx = \frac{dt}{(x+1)e^x}$$

$$\therefore \int e^x(x+1)\cos^2(x e^x)\,dx$$

$$= \int e^x(x+1)\cos^2 t\,\frac{dt}{(x+1)e^x}$$

$$= \int \cos^2 t\,dt$$

$$= \frac{1}{2}\int (1+\cos 2t)\,dt$$

$$= \frac{1}{2}\left\{t + \frac{\sin 2t}{2}\right\} + C$$

$$= \frac{1}{2}t + \frac{1}{4}\sin 2t + C$$

$$= \frac{1}{2}x e^x + \frac{1}{4}\sin 2(x e^x) + C$$

ALITER We have,

$$\int e^x(x+1)\cos^2(x e^x)\,dx$$

$$= \int \cos^2(x e^x)(x+1)e^x\,dx$$

$$= \int \cos^2(x e^x)\,d(x e^x)$$

$$= \int \cos^2 t\,dt, \text{ where } t = x e^x$$

$$= \frac{1}{2}\int (1+\cos 2t)\,dt$$

$$= \frac{1}{2}\left\{t + \frac{\sin 2t}{2}\right\} + C$$

$$= \frac{1}{2}x e^x + \frac{1}{4}\sin 2(x e^x) + C$$

(ii) We have,

$$\int \frac{\tan x}{a + b\tan^2 x}\,dx = \int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx$$

We observe that $\sin x \cos x$ appears in the differentiation of $a\cos^2 x + b\sin^2 x$, so we substitute

$$a\cos^2 x + b\sin^2 x = t$$

$$\Rightarrow d(a\cos^2 x + b\sin^2 x) = dt$$

$$\Rightarrow \frac{d}{dx}(a\cos^2 x + b\sin^2 x)\,.\,dx = dt$$

$$\Rightarrow (-2a\cos x\sin x + 2b\sin x\cos x)\,dx = dt$$

$$\Rightarrow dx = \frac{dt}{2(b-a)\sin x\cos x}$$

$$\therefore \int \frac{\tan x}{a + b\tan^2 x}\,dx$$

$$= \int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx$$

$$= \int \frac{\sin x\cos x}{t}\cdot\frac{dt}{2(b-a)\sin x\cos x}$$

$$= \frac{1}{2(b-a)}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2(b-a)}\log|t| + C$$

$$= \frac{1}{2(b-a)}\log\left|a\cos^2 x + b\sin^2 x\right| + C$$

ALITER We have,

$$\int \frac{\tan x}{a + b\tan^2 x}\,dx$$

$$= \int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx$$

$$= \frac{1}{2(b-a)}\int \frac{2(b-a)\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx$$

$$= \frac{1}{2(b-a)}\int \frac{1}{a\cos^2 x + b\sin^2 x}\,d(a\cos^2 x + b\sin^2 x)$$

$$= \frac{1}{2(b-a)} \int \frac{1}{t}\, dt, \text{ where } t = a\cos^2 x + b\sin^2 x$$

$$= \frac{1}{2(b-a)} \log |t| + C$$

$$= \frac{1}{2(b-a)} \log \left| a\cos^2 x + b\sin^2 x \right| + C$$

EXAMPLE 2 *Evaluate the following integrals :*

(i) $\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\, dx$

(ii) $\int \frac{\sec x}{\sqrt{\sin(2x+\theta) + \sin\theta}}\, dx$

(iii) $\int \frac{2a\sin x + b\sin 2x}{(b + a\cos x)^3}\, dx$

(iv) $\int \frac{\sin^{3/2} x + \cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}}\, dx$

SOLUTION (i) We have,

$$\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\, dx$$

$$= \int \frac{1}{\sqrt{\sin^3 x (\sin x \cos a + \cos x \sin a)}}\, dx$$

$$= \int \frac{1}{\sqrt{\sin^4 x (\cos a + \cot x \sin a)}}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos a + \cot x \sin a}}\, dx$$

$$= \frac{-1}{\sin a} \int \frac{-\sin a \operatorname{cosec}^2 x}{\sqrt{\cos a + \cot x \sin a}}\, dx$$

$$= -\frac{1}{\sin a} \int \frac{1}{\sqrt{\cos a + \cot x \sin a}}\, d(\cos a + \cot x \sin a)$$

$$= -\frac{1}{\sin a} \int \frac{1}{\sqrt{t}}\, dt, \text{ where } t = \cos a + \cot x \sin a$$

$$= -\frac{2}{\sin a} \sqrt{t} + C$$

$$= -\frac{2}{\sin a} \sqrt{\cos a + \cot x \sin a} + C$$

(ii) We have,

$$\int \frac{\sec x}{\sqrt{\sin(2x+\theta) + \sin\theta}}\, dx$$

$$= \int \frac{\sec x}{\sqrt{\sin 2x \cos\theta + \cos 2x \sin\theta + \sin\theta}}\, dx$$

$$= \int \frac{\sec x}{\sqrt{2\sin x \cos x \cos\theta + \sin\theta (1 + \cos 2x)}}\, dx$$

$$= \int \frac{\sec x}{\sqrt{2\sin x \cos x \cos\theta + 2\cos^2 x \sin\theta}}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{2\tan x \cos\theta + 2\sin\theta}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\tan x \cos\theta + \sin\theta}} \sec^2 x\, dx$$

$$= \frac{1}{\sqrt{2}\cos\theta} \int \frac{1}{\sqrt{\tan x \cos\theta + \sin\theta}} (\cos\theta \sec^2 x)\, dx$$

$$= \frac{1}{\sqrt{2}\cos\theta} \int \frac{1}{\sqrt{\tan x \cos\theta + \sin\theta}}\, d(\tan x \cos\theta + \sin\theta)$$

$$= \frac{1}{\sqrt{2}\cos\theta} \int \frac{dt}{\sqrt{t}}, \quad t = \tan x \cos\theta + \sin\theta$$

$$= \frac{1}{\sqrt{2}\cos\theta} \left(\frac{t^{1/2}}{1/2} \right) + C$$

$$= \frac{\sqrt{2}}{\cos\theta} \sqrt{\tan x \cos\theta + \sin\theta} + C$$

(iii) We have,

$$\int \frac{2a\sin x + b\sin 2x}{(b + a\cos x)^3}\, dx$$

$$= 2\int \frac{\sin x (a + b\cos x)}{(b + a\cos x)^3}\, dx$$

$$= -\frac{2}{a} \int \frac{a + b\left(\frac{t-b}{a}\right)}{t^3}\, dt \qquad \left[\begin{array}{l}\text{Putting } b + a\cos x = t \\ \text{and } -a\sin x\, dx = dt\end{array}\right]$$

$$= -\frac{2}{a^2} \int \frac{(a^2 - b^2) + bt}{t^3}\, dt$$

$$= \frac{-2}{a^2}(a^2 - b^2) \int \frac{1}{t^3}\, dt - \frac{2b}{a^2} \int \frac{1}{t^2}\, dt$$

$$= \frac{1}{a^2} \frac{(a^2 - b^2)}{t^2} + \frac{2b}{a^2 t} + C$$

(iii) $\int \frac{\sin^{3/2} x + \cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}}\, dx$

$$= \int \frac{\sin^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}}\, dx$$

$$+ \int \frac{\cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}}\, dx$$

$$= \int \frac{1}{\sqrt{\cos^3 x \sin(x+\theta)}}\, dx + \int \frac{1}{\sqrt{\sin^3 x \sin(x+\theta)}}\, dx$$

$$= \int \frac{1}{\sqrt{\cos^4 x (\tan x \cos\theta + \sin\theta)}}\, dx$$

$$+ \int \frac{1}{\sqrt{\sin^4 x (\cos\theta + \cot x \sin\theta)}}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x \cos\theta + \sin\theta}}\, dx + \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\theta + \cot x \sin\theta}}\, dx$$

$$= \frac{1}{\cos\theta}\int \frac{\cos\theta \sec^2 x\, dx}{\sqrt{\tan x \cos\theta + \sin\theta}} - \frac{1}{\sin\theta}\int \frac{-\operatorname{cosec}^2 x \sin\theta}{\sqrt{\cos\theta + \cot x \sin\theta}}\, dx$$

$$= \frac{1}{\cos\theta}\int \frac{1}{\sqrt{\tan x \cos\theta + \sin\theta}}\, d(\tan x \cos\theta + \sin\theta)$$

$$- \frac{1}{\sin\theta}\int \frac{1}{\sqrt{\cos\theta + \cot x \sin\theta}}\, d(\cos\theta + \cot x \sin\theta)$$

$$= \frac{1}{\cos\theta}\int \frac{du}{\sqrt{u}} - \frac{1}{\sin\theta}\int \frac{dv}{\sqrt{v}}, \text{ where}$$

$u = \tan x \cos\theta + \sin\theta,\ v = \cos\theta + \cot x \sin\theta$

$$= \frac{2}{\cos\theta}\sqrt{u} - \frac{2}{\sin\theta}\sqrt{v} + C$$

$$= \frac{2}{\cos\theta}\sqrt{\tan x \cos\theta + \sin\theta} - \frac{2}{\sin\theta}\sqrt{\cos\theta + \cot x \sin\theta} + C$$

EXAMPLE 3 *Evaluate the following integral :*

$$\int x^{13/2}\,(1 + x^{5/2})^{1/2}\, dx$$

SOLUTION We have,

$$\int x^{13/2}\,(1 + x^{5/2})^{1/2}\, dx$$

$$= \int x^5 (1 + x^{5/2})^{1/2} x^{3/2}\, dx$$

$$= \int (x^{5/2})^2\,(1 + x^{5/2})^{1/2}\, x^{3/2}\, dx$$

$$= \frac{2}{5}\int (t-1)^2\, t^{1/2}\, dt, \text{ where } t = 1 + x^{5/2}$$

$$= \frac{2}{5}\int (t^2 - 2t + 1)\sqrt{t}\, dt$$

$$= \frac{2}{5}\int (t^{5/2} - 2t^{3/2} + t^{1/2})\, dt$$

$$= \frac{2}{5}\left[\frac{2}{7}t^{7/2} - 2\times\frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2}\right] + C$$

$$= \frac{2}{5}\left[\frac{2}{7}(1 + x^{5/2})^{7/2} - \frac{4}{5}(1 + x^{5/2})^{5/2} + \frac{2}{3}(1 + x^{5/2})^{3/2}\right] + C$$

EXAMPLE 4 *Evaluate :*

(i) $\displaystyle\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\,\tan^{-1}\left(\frac{x^2+1}{x}\right)}\, dx$

(ii) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x + \sin x}\, dx$

(iii) $\displaystyle\int \frac{x^2 - 2}{(x^4 + 5x^2 + 4)\,\tan^{-1}\left(\frac{x^2+2}{x}\right)}\, dx$

SOLUTION (i) We have,

$$\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\,\tan^{-1}\left(\frac{x^2+1}{x}\right)}\, dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} + 3\right)\tan^{-1}\left(x + \frac{1}{x}\right)}\, dx \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } x^2]$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left\{\left(x + \frac{1}{x}\right)^2 + 1\right\}\tan^{-1}\left(x + \frac{1}{x}\right)}\, dx$$

$$= \int \frac{1}{\left\{\left(x + \frac{1}{x}\right)^2 + 1\right\}\tan^{-1}\left(x + \frac{1}{x}\right)}\, d\left(x + \frac{1}{x}\right)$$

$$= \int \frac{1}{(t^2+1)\tan^{-1} t}\, dt, \quad \text{where } t = x + \frac{1}{x}$$

$$= \int \frac{1}{\tan^{-1} t}\, d(\tan^{-1} t)$$

$$= \int \frac{1}{u}\, du, \quad \text{where } u = \tan^{-1} t$$

$$= \log|u| + C$$

$$= \log\left|\tan^{-1}\left(x + \frac{1}{x}\right)\right| + C$$

(ii) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x + \sin x}\, dx$

$$= \int \frac{(1 - \sin^2 x)\cos x}{\sin x\,(\sin x + 1)}\, dx$$

$$= \int \frac{1 - \sin x}{\sin x}\cos x\, dx$$

$$= \int \frac{1 - \sin x}{\sin x}\, d(\sin x)$$

$$= \int \frac{1 - t}{t}\, dt, \quad t = \sin x$$

$$= \int \left(\frac{1}{t} - 1\right) dt$$

$$= \log|t| - t + C$$

$$= \log|\sin x| - \sin x + C$$

(iii) We have,

$$\int \frac{x^2 - 2}{(x^4 + 5x^2 + 4)\,\tan^{-1}\left(\frac{x^2+2}{x}\right)}\, dx$$

$$= \int \frac{1 - 2/x^2}{\left(x^2 + 5 + \frac{4}{x^2}\right) \tan^{-1}\left(x + \frac{2}{x}\right)} dx$$

$$= \int \frac{1}{\left\{\left(x + \frac{2}{x}\right)^2 + 1\right\} \tan^{-1}\left(x + \frac{2}{x}\right)} \; d\left(x + \frac{2}{x}\right)$$

$$= \int \frac{1}{(t^2 + 1) \tan^{-1} t} dt, \quad \text{where } t = \frac{x+2}{x}$$

$$= \int \frac{1}{\tan^{-1} t} d(\tan^{-1} t)$$

$$= \log \left|\tan^{-1} t\right| + C$$

$$= \log \left|\tan^{-1}\left(x + \frac{2}{x}\right)\right| + C$$

EXAMPLE 5 *Evaluate :*

$$\int \frac{1}{x\, l(x)\, l^2(x)\, l^3(x) \ldots l^{n-1}(x)\, l^n(x)} dx,$$

where $l^n(x) = \log_e \left(\log_e (\log_e \ldots (\log_e x) \ldots..)\right)$

$\longleftarrow n\text{-times} \longrightarrow$

SOLUTION We have,

$$d\{l(x)\} = d(\log x) = \frac{1}{x} dx$$

$$d\{l^2(x)\} = d\{\log_e (\log_e x)\} = \frac{1}{x \log_e x} dx = \frac{1}{x\, l(x)} dx$$

$$d\{l^3(x)\} = d\{\log (\log (\log_e x)\}$$

$$= \frac{1}{x \log_e x \log_e (\log_e x)}$$

$$= \frac{1}{x\, l(x)\, l^2(x)} dx \text{ and so on.}$$

$$\therefore \; d\{l^n(x)\} = \frac{1}{x\, l(x)\, l^2(x)\, l^3(x) \ldots l^{n-1}(x)}$$

Thus,

$$\int \frac{1}{x\, l(x)\, l^2(x) \ldots.. l^{n-1}(x)\, l^n(x)} dx$$

$$= \int \frac{1}{t} dt, \quad \text{where } t = l^n(x)$$

$$= \log_e (t) + C$$

$$= \log_e \{l^n(x)\} + C$$

$$= l^{n+1}(x) + C$$

EXAMPLE 6 *Evaluate :*

$$\int \{(x+a)^{-3} (x+b)^{-5}\}^{1/4} dx$$

SOLUTION We have,

$$I = \int \left\{(x+a)^{-3} (x+b)^{-5}\right\}^{1/4} dx$$

$$= \int \left\{\left(\frac{x+a}{x+b}\right)^{-3} (x+b)^{-8}\right\}^{1/4} dx$$

$$= \int \left(\frac{x+a}{x+b}\right)^{-3/4} \frac{1}{(x+b)^2} dx$$

Let $\frac{x+a}{x+b} = t$. Then,

$$d\left(\frac{x+a}{x+b}\right) = dt$$

$$\Rightarrow \frac{b-a}{(x+b)^2} dx = dt$$

$$\Rightarrow \frac{1}{(x+b)^2} dx = \frac{1}{b-a} dt$$

$$\therefore \; I = \frac{1}{b-a} \int t^{-3/4} dt$$

$$= \frac{1}{b-a} \times 4\, t^{1/4} + C$$

$$= \frac{4}{b-a} \left(\frac{x+a}{x+b}\right)^{1/4} + C$$

EXAMPLE 7 *Evaluate :* $\int \left(\frac{x-1}{x+1}\right)^4 dx$

SOLUTION Putting $x + 1 = t$ and $dx = dt$, we get

$$\int \left(\frac{x-1}{x+1}\right)^4 dx$$

$$= \int \frac{(t-2)^4}{t^4} dt$$

$$= \int \frac{t^4 - 8t^3 + 24t^2 - 32t + 16}{t^4} dt$$

$$= \int \left\{1 - \frac{8}{t} + \frac{24}{t^2} - \frac{32}{t^3} + \frac{16}{t^4}\right\} dt$$

$$= t - 8 \log |t| - \frac{24}{t} + \frac{16}{t^2} - \frac{16}{3t^3} + C$$

$$= (x+1) - 8 \log |x+1| - \frac{24}{x+1} + \frac{16}{(x+1)^2} - \frac{16}{3(x+1)^3} + C$$

EXAMPLE 8 *Evaluate* : $\int \frac{1}{x^4\sqrt{x^2+4}}\,dx$

SOLUTION Putting $x = 2\tan\theta$ and $dx = 2\sec^2\theta\,d\theta$, we get

$$I = \int \frac{1}{x^4\sqrt{x^2+4}}dx$$

$$= \int \frac{1}{16\tan^4\theta\sqrt{4\tan^2\theta+4}}2\sec^2\theta\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos^3\theta}{\sin^4\theta}d\theta$$

$$= \frac{1}{16}\int \frac{(1-\sin^2\theta)}{\sin^4\theta}\,d(\sin\theta)$$

$$= \frac{1}{16}\int \frac{1-t^2}{t^4}dt$$

$$= \frac{1}{16}\int \left\{\frac{1}{t^4}-\frac{1}{t^2}\right\}dx$$

$$= \frac{1}{16}\left\{-\frac{1}{3t^3}+\frac{1}{t}\right\}+C$$

$$= \frac{1}{16}\left\{-\frac{1}{3\sin^3\theta}+\frac{1}{\sin\theta}\right\}+C$$

$$= -\frac{1}{48}\operatorname{cosec}^3\theta+\frac{1}{16}\operatorname{cosec}\theta+C$$

$$= -\frac{1}{48}(1+\cot^2\theta)^{3/2}+\frac{1}{16}\sqrt{1+\cot^2\theta}+C$$

$$= -\frac{1}{48}\left(1+\frac{4}{x^2}\right)^{3/2}+\frac{1}{16}\sqrt{1+\frac{4}{x^2}}+C$$

$$= -\frac{1}{48}\frac{(x^2+4)^{3/2}}{x^3}+\frac{\sqrt{x^2+4}}{16x}+C$$

EXAMPLE 9 *Evaluate* : $\int \frac{\sqrt{(x^2-a^2)^5}}{x}\,dx$

SOLUTION Let $I = \int \frac{\sqrt{(x^2-a^2)^5}}{x}\,dx$

Putting $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$, we get

$$I = \int \frac{\sqrt{(a^2\sec^2\theta-a^2)^5}}{a\sec\theta}a\sec\theta\tan\theta\,d\theta$$

$$= a^5\int \tan^6\theta\,d\theta$$

$$= a^5\int \tan^4\theta(\sec^2\theta-1)\,d\theta$$

$$= a^5\left[\int \tan^4\theta\sec^2\theta\,d\theta-\int \tan^4\theta\,d\theta\right]$$

$$= a^5\left[\int \tan^4\theta\sec^2\theta\,d\theta-\int \tan^2\theta(\sec^2\theta-1)\,d\theta\right]$$

$$= a^5\left[\int \tan^4\theta\sec^2\theta\,d\theta-\int \tan^2\theta\sec^2\theta\,d\theta+\int \tan^2\theta\,d\theta\right]$$

$$= a^5\left[\int \tan^4\theta\,d(\tan\theta)-\int \tan^2\theta\,d(\tan\theta)+\int(\sec^2\theta-1)\,d\theta\right]$$

$$= a^5\left[\frac{\tan^5\theta}{5}-\frac{\tan^3\theta}{3}+\tan\theta-\theta\right]+C$$

$$= \frac{(x^2-a^2)^{5/2}}{5}-a^2\frac{(x^2-a^2)^{3/2}}{3}+a^4\sqrt{x^2-a^2}-a^5\sec^{-1}\frac{x}{a}+C$$

EXAMPLE 10 *Evaluate* : $\int \frac{1}{x^6+x^4}\,dx$

SOLUTION Let $I = \int \frac{1}{x^6+x^4}\,dx = \int \frac{1}{x^4(x^2+1)}\,dx$

Putting $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, we get

$$I = \int \frac{1}{\tan^4\theta(1+\tan^2\theta)}\sec^2\theta\,d\theta$$

$$= \int \cot^4\theta\,d\theta$$

$$= \int \cot^2\theta(\operatorname{cosec}^2\theta-1)\,dx$$

$$= \int \cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int \cot^2\theta\,d\theta$$

$$= -\int \cot^2\theta\,d(\cot\theta)-\int(\operatorname{cosec}^2\theta-1)\,d\theta$$

$$= -\frac{\cot^3\theta}{3}+\cot\theta+\theta+C$$

$$= -\frac{1}{3x^3}+\frac{1}{x}+\tan^{-1}x+C$$

EXAMPLE 11 *Evaluate* : $\int \frac{x^7}{(1-x^2)^5}\,dx$

SOLUTION Let $I = \int \frac{x^7}{(1-x^2)^5}\,dx.$

Putting $x = \sin\theta$ and $dx = \cos\theta\,d\theta$, we get

$$I = \int \frac{\sin^7\theta}{(1-\sin^2\theta)^5}\cos\theta\,d\theta$$

$$= \int \tan^7\theta\sec^2\theta\,d\theta$$

$$= \int \tan^7\theta\,d(\tan\theta)$$

$$= \frac{\tan^8\theta}{8}+C$$

$$= \frac{1}{8}\frac{x^8}{(1-x^2)^4}+C$$

EXAMPLE 12 *Evaluate* : $\int \frac{\sqrt{1-x^3}}{x^2\sqrt{x}}\,dx$

SOLUTION Let $I = \int \frac{\sqrt{1-x^3}}{x^{5/2}}\,dx$

Putting $x^3 = \sin^2\theta$ and $dx = \frac{2}{3}(\sin\theta)^{-1/3}\cos\theta\, d\theta$, we get

$$I = \int \frac{\sqrt{1-\sin^2\theta}}{\sin^{5/3}\theta}\frac{2}{3}(\sin\theta)^{-1/3}\cos\theta\, d\theta$$

$$= \frac{2}{3}\int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{2}{3}\int (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= \frac{2}{3}(-\cot\theta - \theta) + C$$

$$= -\frac{2}{3}\sqrt{\frac{1-x^3}{x^3}} - \frac{2}{3}\sin^{-1}\left(x^{3/2}\right) + C$$

EXAMPLE 13 *Evaluate* : $\int \frac{x}{(1-x^4)^{3/2}}\, dx$

SOLUTION Let $I = \int \frac{x}{(1-x^4)^{3/2}}\, dx.$

Putting $x^2 = \sin\theta$ and $2x$ and $dx = \cos\theta\, d\theta$, we get

$$I = \frac{1}{2}\int \frac{\cos\theta}{(1-\sin^2\theta)^{3/2}}\, d\theta$$

$$= \frac{1}{2}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\tan\theta + C$$

$$= \frac{1}{2}\frac{x^2}{\sqrt{1-x^4}} + C$$

EXAMPLE 14 *Evaluate* : $\int \frac{1}{x^4\sqrt{x^2-3}}\, dx$

SOLUTION Let $I = \int \frac{1}{x^4\sqrt{x^2-3}}\, dx.$

Putting $x = \sqrt{3}\sec\theta$ and $dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$, wer get

$$I = \int \frac{1}{9\sec^4\theta\sqrt{3\sec^2\theta - 3}}\sqrt{3}\sec\theta\tan\theta\, d\theta$$

$$= \frac{1}{9}\int \cos^3\theta\, d\theta$$

$$= \frac{1}{9}\int \frac{\cos 3\theta + 3\cos\theta}{4}\, d\theta$$

$$= \frac{1}{36}\left(\frac{\sin 3\theta}{3} + 3\sin\theta\right) + C$$

$$= \frac{1}{108}(12\sin\theta - 4\sin^3\theta) + C$$

$$= \frac{1}{27}(3\sin\theta - \sin^3\theta) + C$$

$$= \frac{\sqrt{x^2-3}}{27x^3}(2x^2+3) + C$$

EXAMPLE 15 *Evaluate* : $\int \frac{1}{x^3\sqrt{x-1}}\, dx$

SOLUTION Putting $x = \sec^2\theta$ and $dx = 2\sec^2\theta\tan\theta\, d\theta$, we get

$$I = \int \frac{1}{\sec^6\theta\sqrt{\sec^2\theta - 1}} 2\sec^2\theta\tan\theta\, d\theta$$

$$= 2\int \cos^4\theta\, d\theta$$

$$= 2\int \left\{\frac{1+\cos 2\theta}{2}\right\}^2 d\theta$$

$$= \frac{1}{2}\int (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \frac{1}{4}\int (3 + 4\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left\{3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right\} + C$$

$$= \frac{3}{4}\theta + \sin\theta\cos\theta + \frac{1}{16}\sin 4\theta + C$$

$$= \frac{3}{4}\tan^{-1}\sqrt{x-1} + \frac{\sqrt{x-1}}{x} + \frac{(2-x)\sqrt{x-1}}{4x^2} + C$$

$$= \frac{3}{4}\tan^{-1}\sqrt{x-1} + \frac{\sqrt{x-1}}{4x^2}(3x+2) + C$$

EXAMPLE 16 *Evaluate* : $\int \frac{1}{(x^2+2x+2)^2}\, dx$

SOLUTION We have,

$$I = \int \frac{1}{(x^2+2x+2)^2}\, dx$$

$$= \int \frac{1}{[(x+1)^2+1^2]^2}\, dx$$

Putting $x + 1 = \tan\theta$ and $dx = \sec^2\theta\, d\theta$, we get

$$I = \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\left\{\tan^{-1}(\sqrt{x+1}) + \frac{x+1}{x^2+2x+2}\right\} + C$$

EXAMPLE 17 *Evaluate :* $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\,dx$

SOLUTION Let $I = \int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\,dx$

Putting $x = \sin^2 t$ and $dx = 2\sin t \cos t\, dt$, we get

$$I = \int \frac{2\sin t\cos t\, dt}{(1+\sin t)\sqrt{\sin^2 t - \sin^4 t}}$$

$$\Rightarrow I = 2\int \frac{1}{1+\sin t}\,dt$$

$$\Rightarrow I = 2\int \frac{1-\sin t}{\cos^2 t}\,dt$$

$$\Rightarrow I = 2\int (\sec^2 t - \sec t\tan t)\,dt$$

$$\Rightarrow I = 2(\tan t - \sec t) + C$$

$$\Rightarrow I = 2\left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}}\right) + C$$

EXAMPLE 18 *Evaluate :* $\int (x+\sqrt{x^2+1})^n\,dx$

SOLUTION Putting $x+\sqrt{x^2+1} = t$ and

$$\left(1+\frac{x}{\sqrt{x^2+1}}\right)dx = dt, \text{ or } \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\,dx = dt$$

or, $\frac{t}{\sqrt{x^2+1}}dx = dt$, we get

$$I = \int (x+\sqrt{x^2+1})^n\,dx$$

$$= \int t^n \frac{\sqrt{x^2+1}}{t}\,dt$$

$$= \int t^{n-1}\sqrt{x^2+1}\,dt$$

$$= \int t^{n-1}\sqrt{1+\left(\frac{t^2-1}{2t}\right)^2}\,dt$$

$$\left[\because x+\sqrt{x^2+1} = t \Rightarrow t-x = \sqrt{x^2+1} \Rightarrow (t-x)^2 = x^2+1 \Rightarrow t^2 - 2tx = 1 \Rightarrow x = \frac{t^2-1}{2t}\right]$$

$$I = \frac{1}{2}\int t^{n-2}(t^2+1)\,dt$$

$$= \frac{1}{2}\int (t^n + t^{n-2})\,dt$$

$$= \frac{1}{2}\left(\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}\right) + C$$

$$= \frac{1}{2}\left[\frac{(x+\sqrt{x^2+1})^{n+1}}{n+1} + \frac{(x+\sqrt{x^2+1})^{n-1}}{n-1}\right] + C$$

EXAMPLE 19 *Evaluate :*

(i) $\int \frac{\sqrt{1-x}}{1+x}\cdot\frac{1}{x}\,dx$

(ii) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

(iii) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,\frac{1}{x}\,dx$ **[IIT1997]**

SOLUTION (i) Putting $x = \cos 2\theta$, and $dx = -2\sin 2\theta\, d\theta$, we get

$$I = \int \sqrt{\frac{1-x}{1+x}}\,\frac{1}{x}\,dx$$

$$= \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times \frac{1}{\cos 2\theta} \times -2\sin 2\theta\, d\theta$$

$$= \int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times \frac{-4\sin\theta\cos\theta}{\cos 2\theta}\,d\theta$$

$$= -\int \frac{4\sin^2\theta}{\cos 2\theta}\,d\theta$$

$$= -2\int \frac{1-\cos 2\theta}{\cos 2\theta}\,d\theta$$

$$= -2\int (\sec 2\theta - 1)\,d\theta$$

$$= -\log|\sec 2\theta + \tan 2\theta| + 2\theta + C$$

$$= -\log\left|\frac{1+\sin 2\theta}{\cos 2\theta}\right| + 2\theta + C$$

$$= -\log\left|\frac{1+\sin 2\theta}{\sqrt{1-\sin^2 2\theta}}\right| + 2\theta + C$$

$$= -\log\left|\sqrt{\frac{1+\sin 2\theta}{1-\sin 2\theta}}\right| + 2\theta + C$$

$$= -\log\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| + \cos^{-1}x + C$$

$$= -\log\left|\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}\right| + \cos^{-1}x + C$$

(ii) Putting $x = \cos^2\theta$ and $dx = -2\sin\theta\cos\theta\, d\theta$, we get

$$I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times -2\sin\theta\cos\theta\, d\theta$$

$$= -\int 2\tan\frac{\theta}{2}\sin\theta\cos\theta\,d\theta$$

$$= -4\int \sin^2\frac{\theta}{2}\cos\theta\,d\theta$$

$$= -2\int (1-\cos\theta)\cos\theta\,d\theta$$

$$= -2\int (\cos\theta - \cos^2\theta)\,d\theta$$

$$= -2\int \cos\theta\,d\theta + \int (1+\cos 2\theta)\,d\theta$$

$$= -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= -2\sin\theta + \theta + \sin\theta\cos\theta + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$

(iii) Putting $x = \cos^2\theta$ and $dx = -2\sin\theta\cos\theta\, d\theta$, we get

$$I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,\frac{1}{x}\,dx$$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times \frac{1}{\cos^2\theta} \times -2\sin\theta\cos\theta\, d\theta$$

$$= \int \frac{\sin\theta/2}{\cos\theta/2} \times -\frac{2\sin\theta}{\cos\theta}\, d\theta$$

$$= -4\int \frac{\sin^2\theta/2}{\cos\theta}\, d\theta$$

$$= -2\int \frac{1-\cos\theta}{\cos\theta}\, d\theta$$

$$= -2\int (\sec\theta - 1)\, d\theta$$

$$= -2\log|\sec\theta + \tan\theta| + 2\theta + C$$

$$= -2\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + 2\theta + C$$

$$= -2\log\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| + 2\cos^{-1}\sqrt{x} + C$$

EXAMPLE 20 *Evalaute :* $\int \sqrt{\frac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}}\cdot\frac{1}{x}\,dx$

SOLUTION Putting $x = \cos^3\theta$ and $dx = -3\cos^2\theta\sin\theta\, d\theta$, we get

$$I = \int \sqrt{\frac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}}\cdot\frac{1}{x}\,dx$$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times \frac{1}{\cos^3\theta} \times -3\cos^2\theta\sin\theta\, d\theta$$

$$= -3\int \frac{\sin\theta/2}{\cos\theta/2} \times \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= -6\int \frac{\sin^2\theta/2}{\cos\theta}\, d\theta$$

$$= -3\int \frac{1-\cos\theta}{\cos\theta}\, d\theta$$

$$= -3\int (\sec\theta - 1)\, d\theta$$

$$= -3\log|\sec\theta + \tan\theta| + 3\theta + C$$

$$= -3\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + 3\theta + C$$

$$= -3\log\left|\frac{1+\sqrt{1-x^{2/3}}}{x^{1/3}}\right| + 3\cos^{-1}(x^{1/3}) + C.$$

EXAMPLE 21 *Evaluate :* $\int \frac{1}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}\,dx$

SOLUTION We have,

$$I = \int \frac{1}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}\,dx$$

$$= \int \frac{1}{(x+1)^3\sqrt{(x+1)^2-2^2}}\,dx$$

Putting $x+1 = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$, we get

$$I = \int \frac{1}{(2\sec\theta)^3\sqrt{4\sec^2\theta - 4}}\, 2\sec\theta\tan\theta\, d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{16}\left[\sec^{-1}\left(\frac{x+1}{2}\right) + \frac{2\sqrt{x^2+2x-3}}{(x+1)^2}\right] + C$$

EXAMPLE 22 *Evaluate :*

(i) $\int \frac{1}{(x^2+2x+10)^3}\,dx$ (ii) $\int \frac{x+2}{(x^2+2x+2)^3}\,dx$

SOLUTION (i) We have,

$$I = \int \frac{1}{(x^2+2x+10)^3}\,dx = \int \frac{1}{[(x+1)^2+3^2]^3}\,dx$$

Putting $x+1 = 3\tan\theta$ and $dx = 3\sec^2\theta\, d\theta$, we get

$$I = \int \frac{1}{(9\tan^2\theta+9)^3}\, 3\sec^2\theta\, d\theta$$

$$= \frac{1}{243}\int \cos^4\theta\, d\theta$$

$$= \frac{1}{243}\times\frac{1}{8}\int (3 + 4\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= \frac{1}{1944}\left\{3\theta + 2\sin 2\theta + \frac{\sin 4\theta}{4}\right\} + C$$

$$= \frac{1}{1944}\left[3\tan^{-1}\frac{x+1}{3} + 2\sin 2\theta + \frac{1}{2}\sin 2\theta\cos 2\theta\right] + C$$

$$= \frac{1}{1944}\left[3\tan^{-1}\frac{x+1}{3} + \frac{1}{2}\sin 2\theta\,(4+\cos 2\theta)\right] + C$$

$$= \frac{1}{1944}\left[3\tan^{-1}\frac{x+1}{3} + \frac{\tan\theta}{1+\tan^2\theta}\times\frac{5+3\tan^2\theta}{1+\tan^2\theta}\right] + C$$

$$= \frac{1}{1944}\left[3\tan^{-1}\frac{x+1}{3} + \frac{3(x+1)}{9+(x+1)^2}\times\frac{3\{15+(x+1)^2\}}{9+(x+1)^2}\right] + C$$

$$= \frac{1}{1944}\left[3\tan^{-1}\frac{x+1}{3} + \frac{9(x+1)(x^2+2x+16)}{(x^2+2x+10)^2}\right] + C$$

(ii) We have,

$$I = \int \frac{x+2}{(x^2+2x+2)^3}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+2)+2}{(x^2+2x+2)^3}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{(x^2+2x+2)^3}\,dx + \int \frac{1}{(x^2+2x+2)^3}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x^2+2x+2)^3}\,d(x^2+2x+2) + \int \frac{1}{\{(x+1)^2+1^2\}^3}\,dx$$

$$= -\frac{1}{4(x^2+2x+2)^2} + I_1, \text{ where } I_1 = \int \frac{1}{\{(x+1)^2+1^2\}^3}\,dx$$

Putting $x+1 = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, we get

$$I_1 = \int \frac{1}{(\tan^2\theta+1^2)^3}\sec^2\theta\,d\theta$$

$$= \int \cos^4\theta\,d\theta$$

$$= \frac{1}{8}\int (3+4\cos 2\theta+\cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\left\{3\theta + 2\sin 2\theta + \frac{\sin 4\theta}{4}\right\}$$

$$= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$$

$$= \frac{3}{8}\theta + \frac{1}{2}\frac{\tan\theta}{1+\tan^2\theta} + \frac{1}{8}\frac{\tan\theta(1-\tan^2\theta)}{(1+\tan^2\theta)^2}$$

$$= \frac{3}{8}\tan^{-1}(x+1) + \frac{x+1}{2(x^2+2x+2)} + \frac{1}{8}\frac{(x+1)(-x^2-2x)}{(x^2+2x+2)^2}$$

Hence,

$$I = -\frac{1}{4(x^2+2x+2)^2} + \frac{3}{8}\tan^{-1}(x+1) + \frac{x+1}{2(x^2+2x+2)}$$

$$-\frac{(x+1)(x^2+2x)}{8(x^2+2x+2)^2} + C$$

EXAMPLE 23 *Evaluate* : $\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}$

SOLUTION We have,

$$(x^2+2x+3)^2 = x^4+4x^3+10x^2+12x+9$$

$$\therefore\ x^4+4x^3+11x^2+12x+8$$

$$= (x^4+4x^3+10x^2+12x+9)+(x^2-1)$$

$$= (x^2+2x+3)^2+(x^2-1)$$

Hence,

$$I = \int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx$$

$$= \int \frac{(x^2+2x+3)^2+(x^2-1)}{(x^2+2x+3)^2(x+1)}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \int \frac{x-1}{(x^2+2x+3)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{2x-2}{(x^2+2x+3)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{2x+2-4}{(x^2+2x+3)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2}\,dx - 2\int \frac{1}{(x^2+2x+3)^2}\,dx$$

$$= \log|x+1| - \frac{1}{2(x^2+2x+3)} - 2\int \frac{1}{[(x+1)^2+(\sqrt{2})^2]^2}\,dx$$

Putting $x+1 = \sqrt{2}\tan\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$, we get

$$I = \log|x+1| - \frac{1}{2(x^2+2x+3)} - 2\int \frac{\sqrt{2}\sec^2\theta\,d\theta}{(2\tan^2\theta+2)^2}$$

$$= \log|x+1| - \frac{1}{2(x^2+2x+3)} - \frac{1}{\sqrt{2}}\int \cos^2\theta\,d\theta$$

$$= \log|x+1| - \frac{1}{2(x^2+2x+3)} - \frac{1}{2\sqrt{2}}\int (1+\cos 2\theta)\,d\theta$$

$$= \log|x+1| - \frac{1}{2(x^2+2x+3)} - \frac{1}{2\sqrt{2}}(\theta+\sin\theta\cos\theta) + C$$

$$= \log|x+1| - \frac{1}{2(x^2+2x+3)}$$

$$-\frac{1}{2\sqrt{2}}\left[\tan^{-1}\frac{x+1}{\sqrt{2}} + \frac{\sqrt{2}(x+1)}{x^2+2x+3}\right] + C$$

EXAMPLE 24 *Evaluate* : $\int \frac{2x^4+x^3-4x^2+4x+2}{(x^2+4)(x^2-1)^2}\,dx$

SOLUTION We have,

$$\int \frac{2x^4+x^3-4x^2+4x+2}{(x^2+4)(x^2-1)^2}\,dx$$

$$= \int \frac{(2x^4-4x^2+2)+(x^3+4x)}{(x^2+4)(x^2-1)^2}\,dx$$

$$= \int \frac{2(x^2-1)^2+x(x^2+4)}{(x^2+4)(x^2-1)^2}\,dx$$

$$= \int \frac{2}{x^2+4} + \frac{x}{(x^2-1)^2}\,dx$$

$$= 2\int \frac{1}{x^2+2^2}\,dx + \frac{1}{2}\int \frac{2x}{(x^2-1)^2}\,dx$$

$$= 2\int \frac{1}{x^2+2^2}\,dx + \frac{1}{2}\int \frac{1}{(x^2-1)^2}\,d(x^2-1)$$

$$= \tan^{-1}\frac{x}{2} - \frac{1}{2(x^2-1)} + C$$

EXAMPLE 25 *For any natural number m, evaluate*

$$\int (x^{3m}+x^{2m}+x^m)\,(2x^{2m}+3x^m+6)^{1/m}\,dx,\ x>0$$

[IIT 2002]

SOLUTION We have,

$$I = \int (x^{3m}+x^{2m}+x^m)\,(2x^{2m}+3x^m+6)^{1/m}\,dx$$

$$= \int (x^{3m}+x^{2m}+x^m)\,\frac{(2x^{3m}+3x^{2m}+6x^m)^{1/m}}{x}$$

$$= \int (x^{3m-1}+x^{2m-1}+x^{m-1})\,(2x^{3m}+3x^{2m}+6x^m)^{1/m}\,dx$$

$$= \frac{1}{6}\int (2x^{3m}+3x^{2m}+6x^m)^{1/m}\,(6x^{3m-1}+6x^{2m-1}+6x^{m-1})\,dx$$

$$= \frac{1}{6m}\int (2x^{3m}+3x^{2m}+6x^m)^{1/m}\,d(2x^{3m}+3x^{2m}+6x^m)\,dx$$

$$= \frac{1}{6m}\int t^{1/m}\,dt, \text{ where } t = 2x^{3m}+3x^{2m}+6x^m$$

$$= \frac{1}{6m}\,\frac{t^{(1/m)+1}}{\frac{1}{m}+1}$$

$$= \frac{1}{6(m+1)}\,\{2x^{3m}+3x^{2m}+6x^m\}^{(1/m)+1} + C.$$

—❑❑❑—

ALGORITHM TO EVALUATE INTEGRALS OF THE FORM $\int \sin^m x\cos^n x\,dx$, $\int \sin^m x\,dx$ and $\int \cos^n x\,dx$, where $m, n \in N$

STEP I *Find m and n.*

STEP II *If m is odd i.e. power or index of* sin *x is odd, put* cos $x = t$ *and reduce the integrand in terms of t.*

If n is odd i.e. power of cos *x is odd, then put* sin $x = t$. *and reduce the integrand interms of t.*

If m and n both are odd, then either of the above substitutions can be used.

STEP III *Evaluate the integral obtained in step II and replace t by its value.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int \sin^5 x\cos^4 x\,dx$ (ii) $\int x^2\sin^3 x^3\cos^5 x^3\,dx$

SOLUTION (i) Let $I = \int \sin^5 x\cos^4 x\,dx$

Here power of sin x is odd, so we substitute cos $x = t$.

Now,

$$\cos x = t$$

$$\Rightarrow d(\cos x) = dt$$

$$\Rightarrow -\sin x\,dx = dt$$

$$\Rightarrow dx = \frac{-dt}{\sin x}$$

$$\therefore \int \sin^5 x\cos^4 x\,dx$$

$$= \int \sin^5 x\, t^4\left(\frac{-dt}{\sin x}\right)$$

$$= -\int \sin^4 x\, t^5\,dt$$

$$= -\int (1-\cos^2 x)^2 t^4\,dt$$

$$= -\int (1-t^2)^2 t^4\,dt$$

$$= -\int (1-2t^2+t^4)\,t^4\,dt$$

$$= -\int (t^4-2t^6+t^8)\,dt$$

$$= -\left\{\frac{t^5}{5}-\frac{2t^7}{7}+\frac{t^9}{9}\right\}+C$$

$$= -\frac{1}{5}\cos^5 x+\frac{2}{7}\cos^7 x-\frac{1}{9}\cos^9 x+C$$

(ii) Let $I = \int x^2\sin^3 x^3\cos^5 x^3\,dx$.

Here, indices of both sine and cosine are odd. So, we may subsitute either $\sin x^3 = t$ or $\cos x^3 = t$.

Let $\sin x^3 = t$. Then,

$$d(\sin x^3) = dt$$

$$\Rightarrow \cos x^3 . 3x^2\,dx = dt$$

$$\Rightarrow dx = \frac{dt}{3x^2\cos x^3}$$

$$\therefore I = \int x^2 t^3\cos^5 x^3\,\frac{dt}{3x^2\cos x^3}$$

$$= \frac{1}{3}\int t^3\cos^4 x^3\,dt$$

$$= \frac{1}{3}\int t^3(1-\sin^2 x^3)^2\,dt$$

$$= \frac{1}{3}\int t^3(1-t^2)^2\,dt$$

$$= \frac{1}{3}\int (t^3-2t^5+t^7)\,dt$$

$$= \frac{1}{3}\left\{\frac{t^4}{4}-\frac{2t^6}{6}+\frac{t^8}{8}\right\}+C$$

$$= \frac{1}{12}t^4-\frac{1}{9}t^6+\frac{1}{24}t^8+C$$

$$= \frac{1}{12}\sin^4 x^3-\frac{1}{9}\sin^6 x^3+\frac{1}{24}\sin^8 x^3+C$$

EXAMPLE 2 $\int \cos^{2n+1} x\,dx$, $n \in N$.

SOLUTION Here, power of $\cos x$ is odd, so we substitute $\sin x$ as t.

Now,

$t = \sin x$

$\Rightarrow dt = d(\sin x)$

$\Rightarrow dt = \cos x\,dx \Rightarrow dx = \dfrac{dt}{\cos x}$

$\therefore \int \cos^{2n+1} x\,dx$

$$= \int \cos^{2n+1} x\,\frac{dt}{\cos x}$$

$$= \int \cos^{2n} x\,dt$$

$$= \int (1-\sin^2 x)^n\,dt$$

$$= \int (1-t^2)^n\,dt$$

$$= \int \left[{}^nC_0 - {}^nC_1 t^2 + {}^nC_2 t^4 + \ldots + (-1)^n\,{}^nC_n (t^2)^n\right]dt$$

$$= \left[{}^nC_0\,t - {}^nC_1\frac{t^3}{3} + {}^nC_2\frac{t^5}{5} + \ldots + (-1)^n\,{}^nC_n\frac{t^{2n+1}}{2n+1}\right]+C$$

$$= \sin x - {}^nC_1\frac{\sin^3 x}{3} + {}^nC_2\frac{\sin^5 x}{5} + \ldots + (-1)^n\frac{\sin^{2n+1} x}{2n+1}+C$$

—❑❑❑—

ALGORITHM TO EVALUATE INTEGRALS OF THE FORM $\int \sin^m x \cos^n x\,dx$, where $m, n \in Q$ and $m+n$ is a negative even integer.

STEP I Change the integrand in terms of $\tan x$ and $\sec^2 x$ by, dividing numerator and denominator by $\cos^k x$, where $k = -(m+n)$.

STEP II Substitute $\tan x = t$.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int \dfrac{\sin^4 x}{\cos^8 x}\,dx$ (ii) $\int \dfrac{1}{\sqrt{\sin^3 x \cos^5 x}}\,dx$

SOLUTION (i) We have,

$$\int \frac{\sin^4 x}{\cos^8 x}\,dx$$

$$= \int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}}\,dx \qquad \text{[Dividing N}^r\text{ and D}^r\text{ by } \cos^4 x\text{]}$$

$$= \int \tan^4 x \sec^4 x\,dx$$

$$= \int \tan^4 x(1+\tan^2 x)\sec^2 x\,dx$$

$$= \int \tan^4 x(1+\tan^2 x)\sec^2 x\,dx$$

$$= \int \tan^4 x(1+\tan^2 x)\,d(\tan x)$$

$$= \int t^4(1+t^2)\,dt, \text{ where } t = \tan x$$

$$= \frac{t^5}{5}+\frac{t^7}{7}+C$$

$$= \frac{\tan^5 x}{5}+\frac{\tan^7 x}{7}+C$$

(ii) We have,

$$\int \frac{1}{\sqrt{\sin^3 x \cos^5 x}}\,dx$$

$$= \int \frac{1}{\sin^{3/2} x \cos^{5/2} x}\,dx$$

$$= \int \frac{\sec^4 x}{\tan^{3/2} x}\,dx \qquad \text{[Dividing N}^r\text{ and D}^r\text{ by } \cos^4 x\text{]}$$

$$= \int \frac{(1+\tan^2 x)}{\tan^{3/2} x}\sec^2 x\,dx$$

$$= \int \frac{1+\tan^2 x}{\tan^{3/2} x}\,d(\tan x)$$

$$= \int \frac{1+t^2}{t^{3/2}}\,dt, \text{ where } t = \tan x$$

$$= \int (t^{-3/2} + t^{1/2})\, dt$$

$$= \frac{-2}{\sqrt{t}} + \frac{t^{3/2}}{3/2} + C$$

$$= -\frac{2}{\sqrt{\tan x}} + \frac{2}{3}(\tan x)^{3/2} + C$$

EXAMPLE 2 *Evaluate:* $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$

SOLUTION We have,

$$I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$$

$$= \int \frac{1}{\cos^{4/3} x \sin^{8/3} x}\, dx$$

$$= \int \cos^{-4/3} x \sin^{-8/3} x\, dx$$

Since $-\left(\frac{4}{3} + \frac{8}{3}\right) = -4$, which is an even integer. So, we divide both numerator and denominator by $\cos^4 x$.

$$\therefore I = \int \frac{\sec^4 x}{\tan^{8/3} x}\, dx$$

$$= \int \frac{(1 + \tan^2 x)}{\tan^{8/3} x} \sec^2 x\, dx$$

$$= \int \frac{1 + \tan^2 x}{\tan^{8/3} x}\, d(\tan x)$$

$$= \int \frac{1 + t^2}{t^{8/3}}\, dt \text{ where } t = \tan x$$

$$= \int (t^{-8/3} + t^{-2/3})\, dt$$

$$= \frac{-3}{5} t^{-5/3} + 3t^{1/3} + C$$

$$= \frac{-3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C$$

EXAMPLE 3 *Evaluate:* $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\, dx$

SOLUTION We have,

$$\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\, dx = \int \sin^{2/3} x \cos^{-14/3} x\, dx$$

Since,

$\frac{2}{3} + \left(\frac{-14}{3}\right) = -4$. So, we divide and multiply by $\cos^4 x$.

$$\therefore \int \sin^{2/3} x \cos^{-14/3} x\, dx$$

$$= \int \sin^{2/3} x \cos^{-14/3} x \cos^4 x \sec^4 x\, dx$$

$$= \int \frac{\sin^{2/3} x}{\cos^{2/3} x} \sec^4 x\, dx$$

$$= \int \tan^{2/3} x (1 + \tan^2 x)\, d(\tan x)$$

$$= \int (t^{2/3} + t^{8/3})\, dt$$

$$= \frac{3}{5} t^{5/3} + \frac{3}{11} t^{11/3} + C$$

$$= \frac{3}{5} \tan^{5/3} x + \frac{3}{11} \tan^{11/3} x + C$$

EXAMPLE 4 *Evaluate:* $\int \frac{1}{\sqrt[4]{\sin^3 x \cos^5 x}}\, dx$

SOLUTION We have,

$$I = \int \frac{1}{\sqrt[4]{\sin^3 x \cos^5 x}}\, dx$$

$$= \int \sin^{-3/4} x \cos^{-5/4} x\, dx$$

Here, $-\frac{3}{4} + \left(\frac{-5}{4}\right) = -2$. So, we multiply and divide by $\cos^2 x$.

$$\therefore I = \int \sin^{-3/4} x \cos^{3/4} x \sec^2 x\, dx$$

$$= \int \tan^{-3/4} x \sec^2 x\, dx$$

$$= \int \tan^{-3/4} x\, d(\tan x)$$

$$= 4 \tan^{1/4} x + C.$$

EXAMPLE 5 *Evaluate:* $\int \frac{\sqrt{(1 + x^2)^5}}{x^6}\, dx$

SOLUTION Let $I = \int \frac{\sqrt{(1 + x^2)^5}}{x^6}\, dx$

Putting $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$, we get

$$I = \int \frac{\sqrt{(1 + \tan^2\theta)^5}}{\tan^6\theta} \sec^2\theta\, d\theta$$

$$= \int \frac{\sec^7\theta}{\tan^6\theta}\, d\theta$$

$$= \int \frac{1}{\sin^6\theta \cos\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta + \cos^2\theta}{\sin^6\theta \cos\theta}\, d\theta$$

$$= \int \frac{1}{\sin^4\theta \cos\theta} + \frac{\cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta + \cos^2\theta}{\sin^4\theta \cos\theta} + \frac{\cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \frac{1}{\sin^2\theta \cos\theta} + \frac{\cos\theta}{\sin^4\theta} + \frac{\cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta \cos\theta} + \frac{\cos\theta}{\sin^4\theta} + \frac{\cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta} + \frac{\cos\theta}{\sin^4\theta} + + \frac{\cos\theta}{\sin^6\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta + \int \frac{1}{\sin^2\theta}\, d(\sin\theta) + \int \frac{1}{\sin^4\theta}\, d(\sin\theta) + \int \frac{1}{\sin^6\theta}\, d(\sin\theta)$$

$$= \log|\sec\theta + \tan\theta| - \frac{1}{\sin\theta} - \frac{1}{3\sin^3\theta} - \frac{1}{5\sin^5\theta} + C$$

$$= \log\left|x + \sqrt{x^2+1}\right| - \frac{\sqrt{1+x^2}}{x} - \frac{(1+x^2)^{3/2}}{3x^3} - \frac{(1+x^2)^{5/2}}{5x^5} + C$$

EXAMPLE 6 *Evaluate* : $\int \sec^{25/13} x\ \text{cosec}^{27/13} x\, dx$

SOLUTION We have,

$$I = \int \sec^{25/13} x\ \text{cosec}^{27/13} x\, dx$$

$$\Rightarrow I = \int \cos^{-25/13} x \sin^{-27/13} x\, dx$$

Here, $\frac{-25}{13} + \frac{-27}{13} = -4$. So, we multiply and divide by $\cos^4 x$.

$$\therefore I = \int \cos^4 x \cos^{-25/13} x \sin^{-27/13} x \sec^4 x\, dx$$

$$= \int \cos^{27/13} x \sin^{-27/13} x \sec^4 x\, dx$$

$$= \int \frac{\sin^{-27/13} x}{\cos^{-27/13} x} \sec^4 x\, dx$$

$$= \int \tan^{-27/13} x\,(1+\tan^2 x)\sec^2 x\, dx$$

$$= \int t^{-27/13}(1+t^2)\, dt \text{, where } t = \tan x$$

$$= \int (t^{-27/13} + t^{-1/3})\, dt$$

$$= \frac{-13}{14} t^{-14/13} + \frac{3}{2} t^{2/3} + C$$

$$= \frac{-13}{14} (\tan x)^{-14/13} + \frac{3}{2} (\tan x)^{2/3} + C$$

EXAMPLE 7 *Evaluate* : $\int \frac{\cos^4 x}{\sin^3 x\,(\sin^5 x + \cos^5 x)^{3/5}}\, dx$

SOLUTION We have,

$$\int \frac{\cos^4 x}{\sin^3 x\,(\sin^5 x + \cos^5 x)^{3/5}}\, dx$$

$$= \int \frac{\cos^4 x}{\sin^6 x\,(1+\cot^5 x)^{3/5}}\, dx$$

$$= \int \frac{\sec^2 x}{\tan^6 x \left(1 + \frac{1}{\tan^5 x}\right)^{3/5}}\, dx$$

$$= \int \frac{1}{t^6\left(1+\frac{1}{t^5}\right)^{3/5}}\, dt \text{, where } t = \tan x$$

$$= \frac{-1}{5} \int \left(1+\frac{1}{t^5}\right)^{-3/5} \left(\frac{-5}{t^6}\right) dt$$

$$= -\frac{1}{5} \int \left(1+\frac{1}{t^5}\right)^{-3/5} d\left(1+\frac{1}{t^5}\right)$$

$$= -\frac{1}{5} \left\{\frac{\left(1+1/t^5\right)^{2/5}}{2/5}\right\} + C$$

$$= -\frac{1}{2} \left(\frac{t^5+1}{t^5}\right)^{2/5} + C$$

$$= -\frac{1}{2} \left(\frac{\tan^5 x + 1}{\tan^5 x}\right)^{2/5} + C$$

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ALGORITHM TO EVALUATE INTEGRALS OF THE FORM $\int \sin^m x \cos^n x\, dx$, where m and n are even positive integers

Let $z = \cos x + i \sin x$. Then, $\frac{1}{z} = \cos x - i \sin x$,

$z^n = \cos nx + i \sin nx$, and $\frac{1}{z^n} = \cos n x - i \sin n x$.

$$\therefore z + \frac{1}{z} = 2\cos x,\ z - \frac{1}{z} = 2i \sin x,$$

$$z^n + \frac{1}{z^n} = 2\cos n x \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n x$$

$$\Rightarrow \cos x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos n x = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \sin n x = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

We use these results to express $\sin^m x \cos^n x$ as the sum of sines and cosines of multiples of x as discussed below :

We have,

$$\sin^4 x \cos^2 x$$

$$= \left\{\frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}^4 \left\{\frac{1}{2}\left(z + \frac{1}{z}\right)\right\}^2$$

$$= \frac{1}{64}\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$$

$$= \frac{1}{64}\left(z^2-\frac{1}{z^2}\right)^2\left(z-\frac{1}{z}\right)^2$$

$$= \frac{1}{64}\left(z^4-2+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right)$$

$$= \frac{1}{64}\left\{\left(z^6+\frac{1}{z^6}\right)-2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)+4\right\}$$

$$= \frac{1}{64}\{2\cos 6x-4\cos 4x-2\cos 2x+4\}$$

$$= \frac{1}{32}(\cos 6x-2\cos 4x-\cos 2x+2)$$

Following examples will illustrate the use of the above procedure to evaluate integrals.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* : $\int \sin^2 x\cos^4 x\,dx$

SOLUTION Let $z=\cos x+i\sin x$.

Then, $\frac{1}{z}=\cos x-i\sin x$,

$z^n=\cos nx+i\sin nx$

and, $\frac{1}{z^n}=\cos nx-i\sin nx$.

$\therefore\ z+\frac{1}{z}=2\cos x,\ z-\frac{1}{z}=2i\sin x$

$z^n+\frac{1}{z^n}=2\cos nx$ and $z^n-\frac{1}{z^n}=2i\sin nx$.

Now,

$$\sin^2 x\cos^4 x=\left\{\frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}^2\left\{\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}^4$$

$$=-\frac{1}{4}\left(z-\frac{1}{z}\right)^2\times\frac{1}{16}\left(z+\frac{1}{z}\right)^4$$

$$=-\frac{1}{64}\left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$$

$$=-\frac{1}{64}\left(z^4-2+\frac{1}{z^4}\right)\left(z^2+\frac{1}{z^2}+2\right)$$

$$=-\frac{1}{64}\left\{\left(z^6+\frac{1}{z^6}\right)+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4\right\}$$

$$=-\frac{1}{64}\{2\cos 6x+4\cos 4x-2\cos 2x-4\}$$

$$=\frac{1}{32}\{-\cos 6x-2\cos 4x+\cos 2x+4\}$$

$$\therefore\ \int \sin^2 x\cos^4 x\,dx$$

$$=\frac{1}{32}\int\{-\cos 6x-2\cos 4x+\cos 2x+4\}\,dx$$

$$=\frac{1}{32}\left\{-\frac{\sin 6x}{6}-\frac{\sin 4x}{2}+\frac{\sin 2x}{2}+4x\right\}+C$$

EXAMPLE 2 *Evaluate* $\int \sin^6 x\,dx$

SOLUTION Let $z=\cos x+i\sin x$. Then,

$\frac{1}{z}=\cos x-i\sin x$,

$z^n=\cos nx+i\sin n$

and, $\frac{1}{z^n}=\cos nx-i\sin nx$.

Now,

$$\sin^6 x$$

$$=\left\{\frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}^6$$

$$=-\frac{1}{64}\left(z-\frac{1}{z}\right)^6$$

$$=-\frac{1}{64}\left[{}^6C_0\,z^6-{}^6C_1\,z^4+{}^6C_2\,z^2-{}^6C_3+{}^6C_4\frac{1}{z^2}-{}^6C_5\frac{1}{z^4}+{}^6C_6\frac{1}{z^6}\right]$$

$$=-\frac{1}{64}\left\{\left(z^6+\frac{1}{z^6}\right)-6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)-20\right\}$$

$$=-\frac{1}{64}(2\cos 6x-12\cos 4x+30\cos 2x-20)$$

$$=-\frac{1}{32}(\cos 6x-6\cos 4x+15\cos 2x-10)$$

$$\therefore\ \int \sin^6 x\,dx=-\frac{1}{32}\int(\cos 6x-6\cos 4x+15\cos 2x-10)\,dx$$

$$=-\frac{1}{32}\left\{\frac{\sin 6x}{6}-\frac{3}{2}\sin 4x+\frac{15}{2}\sin 2x-10x\right\}+C$$

EXERCISE 10.2

1. $\int \frac{1}{e^x+4}\,dx$
2. $\int \frac{e^x-2}{e^x+2}\,dx$
3. $\int \frac{1}{\cos^2 x\,(1-\tan x)^2}\,dx$
4. $\int \frac{e^{2x}}{(e^x+1)^{1/4}}\,dx$
5. $\int \frac{x^3}{(1+x^2)^2}\,dx$
6. $\int \frac{x^2}{\sqrt{1-x}}\,dx$
7. $\int \frac{x^2}{\sqrt{1+x}}\,dx$
8. $\int \frac{x^5}{\sqrt{1+x^3}}\,dx$

9. $\int \frac{x^5}{\sqrt{1+x^2}}\,dx$

10. $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}\,dx$

11. $\int \frac{1}{x(x^2+1)^3}\,dx$

12. $\int \sec^{4/9} x \operatorname{cosec}^{14/9} x\,dx$

13. $\int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}}\,d\theta$

14. $\int \frac{1}{\sin^3 x \cos^5 x}\,dx$

15. $\int \sin^4 x \cos^2 x\,dx$

16. $\int \sin^6 x \cos^2 x\,dx$

17. $\int \frac{1}{\sin^4 x \cos^2 x}\,dx$

18. $\int \cos^8 x\,dx$

19. $\int \sin^3 x \cos^2 2x\,dx$

20. $\int \sin 3x \cos^3 x\,dx$

21. $\int \frac{1}{\cos^3 x \sqrt{\sin 2x}}\,dx$

22. $\int \sec^5 x \operatorname{cosec}^3 x\,dx$

23. $\int \operatorname{cosec} 2x \sqrt{\tan x}\,dx$

24. $\int \frac{1}{\sqrt[3]{\sin^{11} x \cos x}}\,dx$

25. $\int \frac{\sqrt{\sin^3 2x}}{\sin^5 x}\,dx$

26. $\int \sec^4 x \operatorname{cosec}^2 x\,dx$

27. $\int e^{e^x+x}\,dx$

28. $\int \frac{e^x-1}{e^x+1}\,dx$

29. $\int e^{2x^2+\log_e^x}\,dx$

30. $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx$ [Hint : Put $x+1=t^2$]

31. $\int \frac{\sqrt{1+x^8}}{x^{13}}\,dx$ [Hint : Put $1+\frac{1}{x^8}=t$]

32. $\int \frac{x^7}{(1-x^2)^5}\,dx$ [Hint : Put $1-x^2=t$]

33. $\int \frac{x^7}{(1+x^4)^2}\,dx$ [Hint : Put $1+x^4=t$]

34. $\int \frac{\sqrt{x^2-8}}{x^4}\,dx$

35. $\int \frac{\sqrt{4+x^2}}{x^6}\,dx$ [Hint : Put $x=2\tan\theta$]

36. $\int \frac{x^4}{(1+x^2)^3}\,dx$ [Hint : Put $x=\tan\theta$]

37. $\int \sqrt{e^x-1}\,dx$ [Hint : Put $e^x-1=t^2$]

38. $\int \frac{\log(x+1)-\log x}{x(x+1)}\,dx$

39. $\int \frac{x^3}{(x-1)^{12}}\,dx$

40. $\int x\sqrt[3]{a+x}\,dx$

41. $\int \frac{1}{\cos x \sin^3 x}\,dx$

42. $\int \frac{\sin^4 x}{\cos^2 x}\,dx$

43. $\int \frac{1}{\cos^3 x \sin^3 x}\,dx$

44. $\int \frac{1}{\cos^4 x \sin^4 x}\,dx$

45. $\int \frac{\sqrt{\tan x}}{\sin\cos x}\,dx$

46. $\int \frac{1}{\sin^5 x \cos^5 x}\,dx$

47. $\int \sin^8 x\,dx$

48. $\int \frac{x^{24}}{x^{10}+1}\,dx$

49. $\int \frac{1}{(x^2+2x+10)^3}\,dx$

50. $\int \frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$

51. $\int \frac{(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}}\,dx$

52. $\int \left(\frac{x+1}{x-1}\right)^n dx$

53. $\int \frac{e^{2x}}{\sqrt{e^x+e^\alpha}+\sqrt{e^x+e^\beta}}\,dx$ [Hint : Put $e^x=t, e^x=a, e^\beta=b$]

54. $\int \frac{(x-x^3)^{1/3}}{x^4}\,dx$

55. $\int \sqrt{1+\cos^2 x}\,\sin 2x \cos 2x\,dx$ [Hint : Put $1+\cos^2 x=t^2$]

56. $\int \cos\left\{2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right\}dx$ [Hint : Put $x=\cos 2\theta$]

57. $\int \sqrt{\frac{1-x^{1/n}}{1+x^{1/n}}}\,\frac{1}{x}\,dx$

ANSWERS

1. $-\frac{1}{4}\log(1+4e^{-x})+C$

2. $x+2\log(1+2e^{-x})+C$

3. $\frac{1}{1-\tan x}+C$

4. $\frac{4}{7}(1+e^x)^{7/4}-\frac{4}{3}(1+e^x)^{3/4}+C$

5. $\frac{1}{2}\left[\log(1+x^2)+\frac{1}{1+x^2}\right]+C$

6. $-\frac{2}{15}(3x^2+4x+8)\sqrt{1-x}+C$

7. $\frac{1}{15}(3x^2-4x+8)\sqrt{1+x}+C$

8. $\frac{2}{9}\sqrt{1+x^3}\,(x^3-2)+C$

9. $\frac{1}{15}\sqrt{1+x^2}\,[3(1+x^2)^2-10(1+x^2)+15)+C$

10. $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{3/4}+C$

11. $-\frac{1}{2}\left[\log(1+x^2)-\frac{1}{1+x^2}-\frac{1}{2(1+x^2)^2}-2\log|x|\right]+C$

12. $-\frac{9}{5}(\tan\theta)^{-5/9}+C$

13. $\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{11/7}+C$

14. $-\frac{1}{2}(\tan x)^{-2}+3\log|\tan x|+\frac{3}{2}\tan^2 x+\frac{1}{4}\tan^4 x+C$

15. $\frac{1}{32}\left\{\frac{\sin 6x}{6}-\frac{\sin 4x}{4}-\frac{\sin 2x}{2}+2x\right\}+C$

16. $-\frac{1}{128}\left\{\frac{\sin 8x}{8}-\frac{2}{3}\sin 6x+\sin 4x+2\sin 2x-5x\right\}+C$

17. $-\frac{1}{3}\cot^3 x-2\cot x+\tan x+C$

18. $\frac{1}{128}\left[\frac{\sin 8x}{8}+\frac{4}{3}\sin 6x+7\sin 4x+28\sin 2x+35x\right]+C$

19. $-\cos^3 x+\frac{2}{5}\cos^5 x+\cos x+C$

20. $\frac{3}{2}\sin^2 x-\frac{7}{4}\sin^4 x+\frac{2}{3}\sin^6 x+C$

21. $\sqrt{2}\left(1+\frac{1}{5}\tan^2 x\right)\sqrt{\tan x}+C$

22. $\frac{3}{2}\tan^2 x-\frac{1}{2}\cot^2 x+3\log|\tan x|+\frac{1}{4}\tan^4 x+C$

23. $\sqrt{\tan x}+C$

24. $\frac{-3(1+4\tan^2 x)}{8\tan^2 x\tan^{2/3} x}+C$

25. $\frac{-4\sqrt{2}}{5}\cot^{5/2} x+C$

26. $-\cot^2 x+2\tan x+\frac{\tan^3 x}{3}+C$

27. $e^{e^x}+C$

28. $x+2\log|1+e^{-x}|+C$

29. $\frac{1}{4}e^{2x^2}+C$

30. $x+4\sqrt{x+1}+4\log(\sqrt{x+1}-1)+C$

31. $-\frac{1}{12}\frac{(1+x^8)^{3/2}}{x^{12}}+C$

32. $\frac{x^8}{8(1-x^2)^4}+C$

33. $\frac{1}{4}\left[\log 1+x^4)+\frac{1}{1+x^4}\right]+C$

34. $\frac{(x^2-8)^{3/2}}{24x^3}+C$

35. $\frac{(x^2-6)}{120x^5}+C$

36. $\frac{-x^3}{4(1+x^2)^2}-\frac{3x}{8(1+x^2)}+\frac{3}{8}\tan^{-1}x+C$

37. $2\sqrt{e^x-1}-2\tan^{-1}\sqrt{e^x-1}+C$

38. $-\frac{1}{2}\left\{\log\left(1+\frac{1}{x}\right)\right\}^2+C$

39. $-\frac{1}{8(x-1)^8}-\frac{1}{3(x-1)^9}-\frac{3}{10(x-1)^{10}}-\frac{1}{11(x-1)^{11}}+C$

40. $\frac{3(4x-3a)(a+x)^{4/3}}{28}+C$

41. $\log_e|\tan x|-\frac{1}{2\sin^2 x}+C$

42. $\tan x+\frac{1}{4}\sin 2x-\frac{3}{2}x+C$

43. $\frac{1}{2}(\tan^2 x-\cot^2 x)+2\log|\tan x|+C$

44. $(\tan^2 x-1)\frac{(\tan^4 x+10\tan^2 x+1)}{3\tan^3 x}+C$

45. $2\sqrt{\tan x}+C$

46. $\frac{1}{4}(\tan^4 x-\cot^4 x)+2(\tan^2 x-\cot^2 x)+6\log|\tan x|+C$

47. $\frac{35x}{128}-\frac{1}{4}\sin 2x+\frac{7}{128}\sin 4x+\frac{1}{24}\sin^3 2x+\frac{1}{1024}\sin 8x+C$

48. $\frac{x^{15}}{15}-\frac{x^5}{5}+\frac{1}{5}\tan^{-1}(x^5)+C$

49. $\frac{1}{648}\left[\tan^{-1}\left(\frac{x+1}{3}\right)+\frac{4(x+1)}{x^2+2x+10}+\frac{(x+1)(2-x)(4+x)}{(x^2+2x+10)^3}\right]+C$

50. $\frac{1}{2}\left\{\log(x+\sqrt{x^2+1})\right\}^2+C$

51. $\frac{\left\{x+\sqrt{x^2+1}\right\}^n}{n}+C$

52. $\sum_{r=0}^{n}{}^nC_r\frac{2^r x^{1-r}}{(1-r)}+C$

53. $\frac{1}{(a-b)}\left[\frac{2}{5}(t+a)^{5/2}-\frac{2a}{3}(t+a)^{3/2}-\frac{2}{5}(t+b)^{5/2}+\frac{2b}{3}(t+b)^{3/2}\right]+C$

where $t=e^x, a=e^\alpha$ and $b=e^\beta$.

54. $-\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{3/4}+C$

55. $\frac{2}{5}(1+\cos^2 x)^{3/2}(1+2\sin^2 x)+C$

56. $\left(\frac{1-2x^2}{4}\right)+C$

— ❑❑❑ —

10.5 SOME STANDARD INTEGRALS

In this section, we shall first discuss standard integrals. The following substitutions are very useful to evaluate these integrals.

Expression	Substitution
a^2+x^2	$x=a\tan\theta, a\cot\theta$
a^2-x^2	$x=a\sin\theta, a\cos\theta$
x^2-a^2	$x=a\sec\theta, a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$	$x=\alpha\cos^2\theta+\beta\sin^2\theta$

RESULT 1 $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

or, $\int \frac{1}{a^2+x^2}\,dx = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$

PROOF Let $I = \int \frac{1}{a^2+x^2}\,dx$.

Putting $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$, we get

$$I = \int \frac{1}{a^2+x^2}\,dx$$

$$= \int \frac{1}{a^2+a^2\tan^2\theta}\,a\sec^2\theta\,d\theta$$

$$= \frac{1}{a}\int 1\,.\,d\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Similarly, by putting $x = a\cot\theta$, we get

$$\int \frac{1}{a^2+x^2}\,dx = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$$

RESULT 2 $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

PROOF We have,

$$\int \frac{1}{x^2-a^2}\,dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left\{\log|x-a| - \log|x+a|\right\} + C$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

RESULT 3 $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

PROOF We have,

$$\int \frac{1}{a^2-x^2}\,dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$= \frac{1}{2a}\left\{\log|a+x| - \log|a-x|\right\} + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

RESULT 4 $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right| + C$

PROOF Putting $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$, we get

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}}\,a\sec^2\theta\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \log|\sec\theta+\tan\theta| + C$$

$$= \log\left|\tan\theta+\sqrt{1+\tan^2\theta}\right| + C$$

$$= \log\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right| + C$$

$$= \log\left|x+\sqrt{a^2+x^2}\right| - \log a + C$$

$$= \log\left|x+\sqrt{x^2+a^2}\right| + C_1, \text{ where } C_1 = C - \log a$$

RESULT 5 $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

or, $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}\left(\frac{x}{a}\right)+C$

PROOF Putting $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$, we get

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\,a\cos\theta\,d\theta$$

$$= \int 1\,.\,d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Similarly, by putting $x = a\cos\theta$, we get

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}\left(\frac{x}{a}\right)+C$$

RESULT 6 $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

PROOF Putting $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$, we get

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta-a^2}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \log|\sec\theta+\tan\theta| + C$$

$$= \log\left|\sec\theta+\sqrt{\sec^2\theta-1}\right| + C$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right| + C$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| - \log a + C$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| + C_1, \text{ where } C_1 = C - \log C_1.$$

We shall now illustrate the applications of the above results in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following integrals :*

(i) $\int \frac{x^2}{9+16x^6}\,dx$ (ii) $\int \frac{x^3}{16x^8-25}\,dx$

(iii) $\int \frac{1}{4e^{-x}-9e^{x}}\,dx$ (iv) $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$

(v) $\int \sqrt{\frac{x^4}{a^6+x^6}}\,dx$ (vi) $\int \frac{2^x}{\sqrt{4^x-25}}\,dx$

SOLUTION (i) We have,

$$\int \frac{x^2}{9+16x^6}\,dx$$

$$= \int \frac{x^2}{3^2+(4x^3)^2}\,dx$$

$$= \frac{1}{12}\int \frac{12x^2}{3^2+(4x^3)^2}\,dx$$

$$= \frac{1}{12}\int \frac{1}{3^2+(4x^3)}\,d(4x^3)$$

$$= \frac{1}{12}\int \frac{1}{3^2+t^2}\,dt, \text{ where } t = 4x^3$$

$$= \frac{1}{36}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$= \frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right)+C$$

(ii) We have,

$$\int \frac{x^3}{16x^8-25}\,dx$$

$$= \int \frac{x^3}{(4x^4)^2-5^2}\,dx$$

$$= \frac{1}{16}\int \frac{16x^3}{(4x^4)^2-5^2}\,dx$$

$$= \frac{1}{16}\int \frac{1}{(4x^4)^2-5^2}\,d(4x^4)$$

$$= \frac{1}{16}\int \frac{dt}{t^2-5^2}, \text{ where } t = 4x^4.$$

$$= \frac{1}{16}\times\frac{1}{2\times5}\log\left|\frac{t-5}{t+5}\right|+C$$

$$= \frac{1}{160}\log\left|\frac{4x^4-5}{4x^4+5}\right|+C$$

(iii) We have,

$$\int \frac{1}{4e^{-x}-9e^{x}}\,dx$$

$$= \int \frac{e^x}{4-9e^{2x}}\,dx$$

$$= \int \frac{e^x}{2^2-(3e^x)^2}\,dx$$

$$= \frac{1}{3}\int \frac{3e^x}{2^2-(3e^x)^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{2^2-(3e^x)^2}\,d(3e^x) \qquad \left[\because d(3e^x) = 3e^x\,dx\right]$$

$$= \frac{1}{3}\int \frac{dt}{2^2-t^2}, \text{ where } t = 3e^x$$

$$= \frac{1}{3}\times\frac{1}{2\times2}\log\left|\frac{2+t}{2-t}\right|+e$$

$$= \frac{1}{12}\log\left|\frac{2+3e^x}{2-3e^x}\right|+C$$

(iv) We have,

$$\int \sqrt{\frac{x}{a^3-x^3}}\,dx$$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,dx$$

$$= \frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,dx$$

$$= \frac{2}{3}\int \frac{1}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,d(x^{3/2})$$

$$= \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2-t^2}}, \text{ where } t = x^{3/2}$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right)+C$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$$

(v) $\int \sqrt{\frac{x^4}{a^6+x^6}}\,dx$

$$= \int \frac{x^2}{\sqrt{a^6+x^6}}\,dx$$

$$= \int \frac{x^2}{\sqrt{(a^3)^2+(x^3)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{(a^3)^2+(x^3)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{(a^3)^2+(x^3)^2}}\,d(x^3)$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{(a^3)^2 + t^2}}\,dt, \text{ where } t = x^3$$

$$= \frac{1}{3}\log\left|t + \sqrt{a^6 + t^2}\right| + C$$

$$= \frac{1}{3}\log\left|x^3 + \sqrt{a^6 + x^6}\right| + C$$

(vi) $\int \frac{2^x}{\sqrt{4^x - 25}}\,dx$

$$= \int \frac{2^x}{\sqrt{(2^x)^2 - 5^2}}\,dx$$

$$= \frac{1}{\log_e 2}\int \frac{2^x \log_e 2}{\sqrt{(2^x)^2 - 5^2}}\,dx$$

$$= \frac{1}{\log_e 2}\int \frac{1}{\sqrt{(2^x)^2 - 5^2}}\,d(2^x)$$

$$= \frac{1}{\log_e 2}\int \frac{dt}{\sqrt{t^2 - 5^2}}, \text{ where } t = 2^x$$

$$= \frac{1}{\log_e 2}\log|t + \sqrt{t^2 - 5^2}| + C$$

$$= \frac{1}{\log_e 2}\log\left|2^x + \sqrt{4^x - 25}\right| + C$$

EXAMPLE 2 *Evaluate :* $\int \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

SOLUTION Here, $\sin x + \cos x$ occurs in the differential of $-\cos x + \sin x$. So, we express $9 + 16\sin 2x$ in terms of $-\cos x + \sin x$. We have,

$$(-\cos x + \sin x)^2 = 1 - \sin 2x$$

$$\Rightarrow \sin 2x = 1 - (-\cos x + \sin x)^2$$

$$\Rightarrow 9 + 16\sin 2x = 9 + 16\{1 - (\cos x + \sin x)^2\}$$

$$\Rightarrow 9 + 16\sin 2x = 25 - 16(-\cos x + \sin x)^2$$

$$\therefore \int \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$$

$$= \int \frac{\sin x + \cos x}{25 - 16(-\cos x + \sin x)^2}\,dx$$

$$= \frac{1}{4}\int \frac{4(\sin x + \cos x)}{5^2 - \{4(-\cos x + \sin x)\}^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{5^2 - \{4(-\cos x + \sin x)\}^2}\,d\{4(-\cos x + \sin x)\}$$

$$= \frac{1}{4}\int \frac{1}{5^2 - t^2}\,dt, \text{ where } t = 4(-\cos x + \sin x)$$

$$= \frac{1}{4}\times\frac{1}{2\times 5}\log\left|\frac{5 + t}{5 - t}\right| + C$$

$$= \frac{1}{40}\log\left|\frac{5 + 4(-\cos x + \sin x)}{5 - 4(-\cos x + \sin x)}\right| + C$$

$$= \frac{1}{40}\log\left|\frac{5 - 4\cos x + 4\sin x}{5 + 4\cos x - 4\sin x}\right| + C$$

EXAMPLE 3 *Evaluate :* $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}}\,dx$

SOLUTION Here, $\cos x - \sin x$ occurs in the differential of $\sin x + \cos x$. So, we express $\sin 2x$ in terms of $\sin x + \cos x$.

We have,

$$(\sin x + \cos x)^2 = 1 + \sin 2x$$

$$\Rightarrow \sin 2x = (\sin x + \cos x)^2 - 1$$

$$\Rightarrow 8 - \sin 2x = 8 - (\sin x + \cos x)^2 + 1$$

$$\Rightarrow 8 - \sin 2x = 9 - (\sin x + \cos x)^2$$

$$\therefore \int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}}\,dx$$

$$= \int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2 - (\sin x + \cos x)^2}}\,d(\sin x + \cos x)$$

$$= \int \frac{1}{\sqrt{3^2 - t^2}}\,dt, \text{ where } t = \sin x + \cos x$$

$$= \sin^{-1}\left(\frac{t}{3}\right) + C$$

$$= \sin^{-1}\left\{\frac{\sin x + \cos x}{3}\right\} + C$$

EXAMPLE 4 *Evaluate :* $\int \frac{1}{\sin x + \sec x}\,dx$

SOLUTION We have,

$$\int \frac{1}{\sin x + \sec x}\,dx$$

$$= \int \frac{\cos x}{1 + \sin x\cos x}\,dx$$

$$= \int \frac{2\cos x}{2 + 2\sin x\cos x}\,dx$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x}\,dx$$

$$= \int \frac{\cos x + \sin x}{2 + \sin 2x}\,dx + \int \frac{\cos x - \sin x}{2 + \sin 2x}\,dx$$

$$= \int \frac{1}{2 + \{1 - (\sin x - \cos x)^2\}}\,d(\sin x - \cos x) + \int \frac{1}{2 + \{(\sin x + \cos x)^2 - 1\}}\,d(\sin x + \cos x)$$

$$= \int \frac{1}{(\sqrt{3})^2 - (\sin x - \cos x)^2}\,d(\sin x - \cos x) + \int \frac{1}{1^2 + (\sin x + \cos x)^2}\,d(\sin x + \cos x)$$

$$= \int \frac{1}{(\sqrt{3})^2 - u^2}\,du + \int \frac{1}{1^2 + v^2}\,dv, \text{ where}$$

$u = \sin x - \cos x$ and $v = \sin x + \cos x$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+u}{\sqrt{3}-u} \right| + \tan^{-1}(v) + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\sin-\cos x}{\sqrt{3}-\sin x+\cos x} \right| + \frac{1}{2}\tan^{-1}(\sin x+\cos x) + C$$

EXAMPLE 5 *Evaluate :* $\int \sqrt{\frac{\sin(x-\theta)}{\sin(x+\theta)}}\, dx$

SOLUTION We have,

$$\int \sqrt{\frac{\sin(x-\theta)}{\sin(x+\theta)}}\, dx$$

$$= \int \frac{\sin(x-\theta)}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{\cos^2\theta - \cos^2 x}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= -\cos\theta \int \frac{1}{\sqrt{\cos^2\theta - \cos^2 x}}\, d(\cos x)$$

$$-\sin\theta \int \frac{1}{\sqrt{\sin^2 x - \sin^2\theta}}\, d(\sin x)$$

$$= -\cos\theta \int \frac{1}{\sqrt{\cos^2\theta - u^2}}\, du - \sin\theta \int \frac{dv}{\sqrt{v^2 - \sin^2\theta}}$$

, where $u = \cos x$ and $v = \sin x$

$$= -\cos\theta \sin^{-1}\left(\frac{u}{\cos\theta}\right) - \sin\theta \log \left| v + \sqrt{v^2 - \sin^2\theta} \right| + C$$

$$= -\cos\theta \sin^{-1}\left(\frac{\cos x}{\cos\theta}\right) - \sin\theta \log \left| \sin x + \sqrt{\sin^2 x - \sin^2\theta} \right| + C$$

EXAMPLE 6 *Evaluate :*

(i) $\int \frac{x^2+1}{x^4+1}\, dx$ (ii) $\int \frac{x^2-1}{x^4+x^2+1}\, dx$

(iii) $\int \frac{x^2+4}{x^4+16}\, dx$ (iv) $\int \frac{1}{x^4+1}\, dx$

SOLUTION (i) We have,

$$\int \frac{x^2+1}{x^4+1}\, dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\, dx$$

$$= \int \frac{}{\left(x-\frac{1}{x}\right)^2+2}\, d\left(x-\frac{1}{x}\right)$$

$$= \int \frac{dt}{t^2+(\sqrt{2})^2}, \text{ where } t = x-\frac{1}{x}.$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) + C$$

(ii) $\int \frac{x^2-1}{x^4+x^2+1}\, dx$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}\, dx$$

$$= \int \frac{}{\left(x+\frac{1}{x}\right)^2-1}\, d\left(x+\frac{1}{x}\right)$$

$$= \int \frac{1}{t^2-1}\, dt, \text{ where } t = x+\frac{1}{x}$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{x^2-x+1}{x^2+x+1} \right| + C$$

(iii) $\int \frac{x^2+4}{x^4+16}\, dx$

$$= \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}\, dx$$

$$= \int \frac{}{\left(x-\frac{4}{x}\right)^2+8}\, d\left(x-\frac{4}{x}\right)$$

$$= \int \frac{dt}{t^2+(2\sqrt{2})^2}, \text{ where } t = x-\frac{4}{x}$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C$$

(iv) $\int \frac{1}{x^4+1}\, dx$

$$= \int \frac{1/x^2}{x^2+1/x^2}\, dx$$

$$= \frac{1}{2}\int \frac{2/x^2}{x^2+1/x^2}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)-\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}\,d\left(x-\frac{1}{x}\right) - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}\,d\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dv}{v^2-(\sqrt{2})^2}\,, \text{ where}$$

$$u = x-\frac{1}{x} \text{ and } v = x+\frac{1}{x}.$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$$

EXAMPLE 7 *Evaluate* : $\int \sqrt{\tan\theta}\,d\theta$.

SOLUTION Putting $\tan\theta = x^2$ and $d(\tan\theta) = dx^2$

i.e. $\sec^2\theta\,d\theta = 2x\,dx$, or $d\theta = \frac{2x}{1+x^4}\,dx$, we get

$$\int \sqrt{\tan\theta}\,d\theta$$

$$= \int \sqrt{x^2}\,\frac{2x}{1+x^4}\,dx$$

$$= \int \frac{2x^2}{x^4+1}\,dx$$

$$= \int \frac{2}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)+\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx + \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}\,d\left(x-\frac{1}{x}\right) + \int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}\,d\left(x+\frac{1}{x}\right)$$

$$= \int \frac{du}{u^2+(\sqrt{2})^2} + \int \frac{1}{v^2-(\sqrt{2})^2}\,dv, \text{ where } u = x-\frac{1}{x}, \text{ and } v = x+\frac{1}{x}.$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$$

EXAMPLE 8 *Evaluate* : $\int \frac{1}{\cos^6 x+\sin^6 x}\,dx$

SOLUTION $\int \frac{1}{\cos^6 x+\sin^6 x}\,dx$

$$= \int \frac{1}{(\cos^2 x+\sin^2 x)(\cos^4 x+\sin^4 x-\cos^2 x\sin^2 x)}\,dx$$

$$= \int \frac{1}{(\cos^2 x+\sin^2 x)^2-3\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1}{1-3\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{\sec^4 x}{\sec^4 x-3\tan^2 x}\,dx \qquad [\text{Dividing } N_r \text{ and } D^r \text{ by } \cos^4 x]$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2-3\tan^2 x}\,dx$$

$$= \int \frac{1+\tan^2 x}{1+\tan^4 x-\tan^2 x}\,d(\tan x)$$

$$= \int \frac{t^2+1}{t^4-t^2+1}\,dt, \quad \text{where } t = \tan^2 x$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-1}\,dt$$

$$= \int \frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+1}$$

$$= \int \frac{du}{u^2+1}, \quad \text{where } u = t-\frac{1}{t}$$

$$= \tan^{-1}u + C = \tan^{-1}(\tan x-\cot x) + C$$

EXAMPLE 9 *Evaluate* : $\int (\sqrt{\tan x}+\sqrt{\cot x})\,dx$

SOLUTION We have,

$$I = \int (\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$= \int \left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right)dx$$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}}\, dx$$

Let $\tan x = t^2$. Then, $\sec^2 x\, dx = 2t\, dt$

$$\Rightarrow dx = \frac{2t\, dt}{\sec^2 x} = \frac{2t\, dt}{1+\tan^2 x} \Rightarrow dx = \frac{2t}{1+t^4}\, dt$$

$$\therefore I = \int \frac{t^2+1}{\sqrt{t^2}} \times \frac{2t}{1+t^4}\, dt$$

$$= 2\int \frac{t^2+1}{t^4+1}\, dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\, dt \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } t^2\text{]}$$

$$= 2\int \frac{1}{\left(t-\frac{1}{t}\right)^2+2}\, d\left(t-\frac{1}{t}\right)$$

$$= 2\int \frac{1}{u^2+(\sqrt{2})^2}\, du \text{, where } u = t-\frac{1}{t}$$

$$= \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$= \sqrt{2}\, \tan^{-1}\left\{\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right\}+C \qquad \left[\because u = t-\frac{1}{t} = \sqrt{\tan x}-\sqrt{\cot x}\right]$$

EXAMPLE 10 *Evaluate* : $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}\, dx$

SOLUTION We have,

$$\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}\, dx$$

$$= \int \frac{\cos^2 x+\cos^4 x}{\sin^2 x+\sin^4 x}\cos x\, dx$$

$$= \int \frac{(1-\sin^2 x)+(1-\sin^2 x)^2}{\sin^2 x+\sin^4 x}\cos x\, dx$$

$$= \int \frac{2-3\sin^2 x+\sin^4 x}{\sin^2 x+\sin^4 x}\, d(\sin x)$$

$$= \int \frac{2-3t^2+t^4}{t^2+t^4}\, dt \text{, where } t = \sin x$$

$$= \int \frac{t^4-3t^2+2}{t^4+t^2}\, dt$$

$$= \int \left(1-\frac{4t^2-2}{t^4+t^2}\right) dt$$

$$= \int \left(1-\frac{4}{t^2+1}+\frac{2}{t^2(t^2+1)}\right) dt$$

$$= \int \left(1-\frac{6}{t^2+1}+\frac{2}{t^2}\right) dt$$

$$= t-6\tan^{-1} t-\frac{2}{t}+C$$

$$= \sin x-6\tan^{-1}(\sin x)-2\,\text{cosec}\, x+C$$

EXAMPLE 11 *Evaluate* : $\int \frac{\cos^7 x+\cos^5 x}{\sin^2 x+\sin^4 x}\, dx$

SOLUTION We have,

$$\int \frac{\cos^7 x+\cos^5 x}{\sin^2 x+\sin^4 x}\, dx$$

$$= \int \frac{\cos^6 x+\cos^4 x}{\sin^2 x+\sin^4 x}\cos x\, dx$$

$$= \int \frac{(1-\sin^2 x)^3+(1-\sin^2 x)^2}{\sin^2 x+\sin^4 x}\, d(\sin x)$$

$$= \int \frac{(1-t^2)^3+(1-t^2)^2}{t^2+t^4} \text{, where } t = \sin x$$

$$= \int \frac{(1-t^2)^2(2-t^2)}{t^2+t^4}\, dt$$

$$= \int \left(-t^2+5+\frac{2}{t^2}-\frac{12}{1+t^2}\right) dt$$

$$= -\frac{t^3}{3}+5t-\frac{2}{t}-12\tan^{-1} t+C$$

$$= -\frac{\sin^3 x}{3}+5\sin x-2\,\text{cosec}\, x-12\tan^{-1}(\sin x)+C$$

EXAMPLE 12 *Evaluate* : $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\, dx.$

SOLUTION Putting $x = \sin^2 t$ and $dx = 2\sin t\cos t\, dt$, we get

$$\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\, dx$$

$$= 2\int \frac{1}{1+\sin t}\, dt$$

$$= 2\int \frac{1-\sin t}{\cos^2 t}\, dt$$

$$= 2\int (\sec^2 t-\tan t\sec t)\, dt$$

$$= 2(\tan t-\sec t)+C$$

$$= 2\sqrt{\frac{x}{1-x}}-\frac{2}{\sqrt{1-x}}+C$$

EXAMPLE 13 *Evaluate* : $\int \frac{x^3}{x^{16}+4}\, dx$

SOLUTION Let $I = \int \frac{x^3}{x^{16}+4}\, dx = \int \frac{x^3}{(x^4)^4+4}\, dx$

Putting $x^4 = t$ and $4x^3\, dx = dt$, we get

$$I=\frac{1}{4}\int\frac{1}{t^4+4}\,dt$$

$$=\frac{1}{4}\int\frac{\frac{1}{t^2}}{t^2+\frac{4}{t^2}}\,dt \qquad \text{[Dividing N}^r\text{ and D}^r\text{ by } t^2\text{]}$$

$$=\frac{1}{16}\int\frac{\frac{4}{t^2}}{t^2+\frac{4}{t^2}}\,dt$$

$$=\frac{1}{16}\int\frac{\left(1+\frac{2}{t^2}\right)-\left(1-\frac{2}{t^2}\right)}{t^2+\frac{4}{t^2}}\,dt$$

$$=\frac{1}{16}\int\frac{1+\frac{2}{t^2}}{t^2+\frac{4}{t^2}}\,dt-\frac{1}{16}\int\frac{1-\frac{2}{t^2}}{t^2+\frac{4}{t^2}}\,dt$$

$$=\frac{1}{16}\int\frac{1}{\left(t-\frac{2}{t}\right)^2+4}\,d\left(t-\frac{2}{t}\right)-\frac{1}{16}\int\frac{1}{\left(t+\frac{2}{t}\right)^2-4}\,d\left(t+\frac{2}{t}\right)$$

$$=\frac{1}{16}\int\frac{du}{u^2+2^2}-\frac{1}{16}\int\frac{dv}{v^2-2^2}\text{, where } u=t-\frac{2}{t} \text{ and } v=t+\frac{2}{t}$$

$$=\frac{1}{32}\tan^{-1}\left(\frac{u}{2}\right)-\frac{1}{16}\times\frac{1}{2\times 2}\log\left|\frac{v-2}{v+2}\right|+C$$

$$=\frac{1}{32}\tan^{-1}\left(\frac{t^2-2}{2t}\right)-\frac{1}{64}\log\left|\frac{t^2-2t+2}{t^2+2t+2}\right|+C$$

$$=\frac{1}{32}\tan^{-1}\left(\frac{x^8-2}{2x^4}\right)-\frac{1}{64}\log\left|\frac{x^8-2x^4+2}{x^8+2x^4+2}\right|+C$$

EXAMPLE 14 *Evaluate :* $\int\frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,dx$

SOLUTION Let $I=\int\frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,dx$

$$\Rightarrow I=\int\frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}\,dx \qquad \left[\begin{matrix}\text{Multiplying the}\\ \text{N}^r\text{ and D}^r\text{ by } (x+1)\end{matrix}\right]$$

$$\Rightarrow I=\int\frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}\,dx$$

$$\Rightarrow I=\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}\,dx \text{ [Dividing N}^r\text{ and } D^r\text{ by } x^2\text{]}$$

$$\Rightarrow I=\int\frac{2t\,dt}{(t^2+1)\sqrt{t^2}}\text{, where, } x+\frac{1}{x}+1=t^2$$

$$\Rightarrow I=2\int\frac{1}{t^2+1}\,dt$$

$$=2\tan^{-1}(t)+C$$

$$\Rightarrow I=2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$$

EXAMPLE 15 *Evaluate :* $\int\frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}\,dx$

SOLUTION Let $I=\int\frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}\,dx$. Then,

$$I=\int\frac{\frac{1+x^2}{x^2}}{\left(\frac{1}{x}-x\right)\sqrt{\frac{1+x^2+x^4}{x^2}}}\,dx \quad \text{[Dividing } N^r\text{and } D^r\text{ by } x^2\text{]}$$

$$\Rightarrow I=-\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}+1}}\,dx$$

$$\Rightarrow I=-\int\frac{1}{\left(x-\frac{1}{x}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+3}}\,d\left(x-\frac{1}{x}\right)$$

$$\Rightarrow I=-\int\frac{dt}{t\sqrt{t^2+3}}\text{, where } x-\frac{1}{x}=t$$

$$\Rightarrow I=-\int\frac{u\,du}{(u^2-3)\sqrt{u^2}}\text{, where } t^2+3=u^2 \text{ and } 2t\,dt=2u\,du$$

$$\Rightarrow I=-\int\frac{1}{u^2-3}\,du$$

$$\Rightarrow I=-\frac{1}{2\sqrt{3}}\log\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right|+C$$

$$\Rightarrow I=-\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right|+C$$

$$\Rightarrow I=-\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right|+C$$

EXAMPLE 16 *Evaluate :* $\int\frac{x^4+1}{x^6+1}\,dx$

SOLUTION We have,

$$\int\frac{x^4+1}{x^6+1}\,dx$$

$$=\int\frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}\,dx$$

$$= \int \frac{x^2+1}{x^4-x^2+1}\,dx - 2\int \frac{x^2}{x^6+1}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx - \frac{2}{3}\int \frac{3x^2\,dx}{(x^3)^2+1}\,dx$$

$$= \int \frac{1}{\left(x-\frac{1}{x}\right)^2+1^2}\,d\left(x-\frac{1}{x}\right) - \frac{2}{3}\int \frac{1}{(x^3)^2+1^2}\,d(x^3)$$

$$= \tan^{-1}\left(x-\frac{1}{x}\right) - \frac{2}{3}\tan^{-1}\left(\frac{x^3}{1}\right) + C$$

EXAMPLE 17 *Evaluate :* $\int \frac{x^3+x^2+x-1}{(x^2-1)\sqrt{2-x^2}}\,dx$

SOLUTION We have,

$$\int \frac{x^3+x^2+x-1}{(x^2-1)\sqrt{2-x^2}}\,dx$$

$$= \int \frac{(x^2-1)+x(x^2+1)}{(x^2-1)\sqrt{2-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}\,dx + \int \frac{(x^2+1)x}{(x^2-1)\sqrt{2-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{(\sqrt{2})^2-x^2}}\,dx + \int \frac{(x^2+1)x}{(x^2-1)\sqrt{2-x^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \int \frac{(3-t^2)\,t\,dt}{(1-t^2)\sqrt{t^2}}, \text{ where } 2-x^2=t^2$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \int \frac{3-t^2}{1-t^2}\,dt$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \int \frac{(1-t^2)+2}{1-t^2}\,dt$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \int \left(1+\frac{2}{1-t^2}\right)dt$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - t - 2\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sqrt{2-x^2} - \log\left|\frac{1+\sqrt{2-x^2}}{1-\sqrt{2-x^2}}\right| + C$$

EXAMPLE 18 *Evaluate :*

$$\int \frac{\sin x-\cos x}{(\sin x+\cos x)\sqrt{\sin x\cos x+\sin^2 x\cos^2 x}}\,dx$$

SOLUTION We have,

$$\int \frac{\sin x-\cos x}{(\sin x+\cos x)\sqrt{\sin x\cos x+\sin^2 x\cos^2 x}}\,dx$$

$$= -\int \frac{1}{t\sqrt{\frac{t^2-1}{2}+\left(\frac{t^2-1}{2}\right)^2}}\,dt, \text{ where } t=\sin x+\cos x$$

$$= -\int \frac{1}{t\sqrt{\left(\frac{t^2-1}{2}\right)\left(1+\frac{t^2-1}{2}\right)}}\,dt$$

$$= -\int \frac{2}{t\sqrt{t^4-1}}\,dt$$

$$= -\frac{1}{2}\int \frac{4t^3}{t^4\sqrt{t^4-1}}\,dt$$

$$= -\frac{1}{2}\int \frac{2u\,du}{(u^2+1)\sqrt{u^2}}, \text{ where } t^4-1=u^2$$

$$= -\int \frac{1}{u^2+1}\,du$$

$$= \cot^{-1}u + C$$

$$= \cot^{-1}\sqrt{t^4-1} + C$$

$$= \cot^{-1}\sqrt{(\sin x+\cos x)^4-1} + C$$

$$= \cot^{-1}\left\{\sqrt{\sin^2 2x+\sin 2x}\right\} + C$$

EXAMPLE 19 *Evaluate :* $\int \frac{\sin^3 x/2}{\cos\frac{x}{2}\sqrt{\cos^3 x+\cos^2 x+\cos x}}\,dx$

SOLUTION We have,

$$\int \frac{\sin^3 x/2}{\cos\frac{x}{2}\sqrt{\cos^3 x+\cos^2 x+\cos x}}\,dx$$

$$= \frac{1}{2}\int \frac{(2\sin x/2\cos x/2)(2\sin^2 x/2)}{2\cos^2 x/2\sqrt{\cos^3 x+\cos^2 x+\cos x}}\,dx$$

$$= \frac{1}{2}\int \frac{\sin x(1-\cos x)}{(1+\cos x)\sqrt{\cos^3 x+\cos^2 x+\cos x}}\,dx$$

$$= \frac{-1}{2}\int \frac{1-t}{(1+t)\sqrt{t^3+t^2+t}}\,dt, \text{ where } t=\cos x \text{ and } -\sin x\,dx=dt$$

$$= \frac{1}{2}\int \frac{t^2-1}{(t+1)^2\sqrt{t^3+t^2+t}}\,dt$$

$$= \frac{1}{2}\int \frac{t^2-1}{(t^2+2t+1)\sqrt{t^3+t^2+t}}\,dt$$

$$= \frac{1}{2}\int \frac{1-\frac{1}{t^2}}{\left(t+2+\frac{1}{t}\right)\sqrt{t+\frac{1}{t}+1}}\,dt$$

$$= \frac{1}{2}\int \frac{2u\,du}{(u^2+1)\sqrt{u^2}}, \text{ where } t+\frac{1}{t}+1=u^2, \left(1-\frac{1}{t^2}\right)dt=2u\,du$$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \tan^{-1} u + C$$

$$= \tan^{-1}\sqrt{t+\frac{1}{t}+1} + C$$

$$= \tan^{-1}\sqrt{\cos x + \sec x + 1} + C$$

EXAMPLE 20 *Evaluate :*

$$\int \frac{1}{(1-x^8)}\left\{\cos^{-1}\frac{2x}{1+x^2} + \tan^{-1}\frac{2x}{1-x^2}\right\} dx.$$

SOLUTION We have,

$$\int \frac{1}{(1-x^8)}\left\{\cos^{-1}\frac{2x}{1+x^2} + \tan^{-1}\frac{2x}{1-x^2}\right\} dx$$

$$= \int \frac{1}{(1-x^8)}\left\{\frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2} + \tan^{-1}\frac{2x}{1-x^2}\right\} dx$$

$$= \int \frac{1}{(1-x^8)}\left\{\frac{\pi}{2} - 2\tan^{-1} x + 2\tan^{-1} x\right\} dx$$

$$= \frac{\pi}{2}\int \frac{1}{1-x^8}\,dx$$

$$= \frac{\pi}{4}\int\left(\frac{1}{1-x^4} + \frac{1}{1+x^4}\right) dx$$

$$= \frac{\pi}{4}\int \frac{1}{1-x^4}\,dx + \frac{\pi}{4}\int \frac{1}{1+x^4}\,dx$$

$$= \frac{\pi}{8}\int\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)dx + \frac{\pi}{8}\int\left(\frac{x^2+1}{x^4+1} - \frac{x^2-1}{x^4+1}\right) dx$$

$$= \frac{\pi}{8}\int \frac{1}{1-x^2}\,dx + \frac{\pi}{8}\int \frac{1}{1+x^2}\,dx$$

$$+ \frac{\pi}{8}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{\pi}{8}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{\pi}{8}\int \frac{1}{1-x^2}\,dx + \frac{\pi}{8}\int \frac{1}{1+x^2}\,dx$$

$$+ \frac{\pi}{8}\int \frac{1}{\left(x-\frac{1}{x}\right)^2 + (\sqrt{2})^2} d\left(x-\frac{1}{x}\right) - \frac{\pi}{8}\int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2 - (\sqrt{2})^2}$$

$$= \frac{\pi}{16}\log\left|\frac{1+x}{1-x}\right| + \frac{\pi}{8}\tan^{-1} x$$

$$+ \frac{\pi}{8\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{\pi}{16\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

$$= \frac{\pi}{16}\log\left|\frac{1+x}{1-x}\right| + \frac{\pi}{8}\tan^{-1} x$$

$$+ \frac{\pi}{8\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{\pi}{16\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$$

EXAMPLE 21 (i) $\int \frac{x^8}{x^{12}-1}\,dx$ (ii) $\int \frac{x^2-2}{x\sqrt{x^4+2x^2+4}}\,dx$

SOLUTION (i) Let $I = \int \frac{x^8}{x^{12}-1}\,dx$

$$= \frac{1}{3}\int \frac{(x^3)^2\,3x^2}{(x^3)^4-1}\,dx$$

$$= \frac{1}{3}\int \frac{t^2\,dt}{t^4-1}, \text{ where } t = x^3$$

$$= \frac{1}{3}\int \frac{t^2}{(t^2+1)\,(t^2-1)}\,dt$$

$$= \frac{1}{6}\int\left(\frac{1}{t^2-1} + \frac{1}{t^2+1}\right) dt$$

$$= \frac{1}{6}\left[\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + \tan^{-1} t\right] + C$$

$$= \frac{1}{6}\left[\frac{1}{2}\log\left|\frac{x^3-1}{x^3+1}\right| + \tan^{-1} x^3\right] + C$$

(ii) We have,

$$\int \frac{x^2-2}{x\sqrt{x^4+2x^2+4}}\,dx$$

$$= \int \frac{x^2-2}{x^2\sqrt{x^2+2+\frac{4}{x^2}}}\,dx$$

$$= \int \frac{1-\frac{2}{x^2}}{\sqrt{\left(x+\frac{2}{x}\right)^2 - (\sqrt{2})^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{2}{x}\right)^2 - (\sqrt{2})^2}}\,d\left(x+\frac{2}{x}\right)$$

$$= \log\left|\left(x+\frac{2}{x}\right) + \sqrt{x^2+\frac{4}{x^2}+2}\right| + C$$

EXAMPLE 22 *Evaluate :* $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}\,dx$

SOLUTION We have,

$$\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}\,dx$$

$$= \int \frac{x(x^2-1)}{(x^2+1)(x+1)^2\sqrt{x^3+x^2+x}}\,dx$$

$$= \int \frac{x^3\left(1-\frac{1}{x^2}\right)}{x^3\left(x+\frac{1}{x}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\sqrt{x+\frac{1}{x}+1}}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}\,dx$$

$$I = \int \frac{2t}{(t^2-1)(t^2+1)\sqrt{t^2}}\,dt \text{ , where } x+\frac{1}{x}+1 = t^2$$

$$= \int \frac{2}{(t^2-1)(t^2+1)}\,dt$$

$$= \int \left(\frac{1}{t^2-1}-\frac{1}{t^2+1}\right)dt$$

$$= \int \frac{1}{t^2-1}\,dt - \int \frac{1}{t^2+1}\,dt$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| - \tan^{-1}t + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right| - \tan^{-1}\sqrt{x+\frac{1}{x}+1} + C$$

EXAMPLE 23 *If* $y(x-y)^2 = x$, *then show that*

$$\int \frac{1}{x-3y}\,dx = \frac{1}{2}\log\{(x-y)^2-1\}.$$

SOLUTION We have,

$$y(x-y)^2 = x$$

$\Rightarrow d\{y(x-y)^2\} = dx$ [Taking differentials of both sides]

$\Rightarrow (x-y)^2\,dy + 2y(x-y)(dx-dy) = dx$

$\Rightarrow (x-y)^2\,dy - (x-y)^2\,dx + 2y(x-y)(dx-dy) = [1-(x-y)^2]\,dx$

$\Rightarrow [(x-y)^2 - 2y(x-y)](dy-dx) = [1-(x-y)^2]\,dx$

$\Rightarrow (x-y)(x-3y)(dx-dy) = [(x-y)^2-1]\,dx$

$$\Rightarrow (x-y)\frac{(dx-dy)}{(x-y)^2-1} = \frac{dx}{x-3y}$$

$$\Rightarrow \frac{dx}{x-3y} = \frac{(x-y)\,d(x-y)}{(x-y)^2-1}$$

$$\Rightarrow \int \frac{1}{x-3y}\,dx = \frac{1}{2}\int \frac{2(x-y)\,d(x-y)}{(x-y)^2-1}$$

$$\Rightarrow \int \frac{1}{x-3y}\,dx = \frac{1}{2}\log\{(x-y)^2-1\}$$

EXAMPLE 23 *Evaluate* : $\int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$ [IIT 1987]

SOLUTION We have,

$$I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\,dx$$

$$= \int \sqrt{\cot^2 x - 1}\,dx$$

Putting $\cot x = \sec\theta$ and $-\operatorname{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$, we get

$$I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta\tan\theta}{-\operatorname{cosec}^2 x}\,d\theta$$

$$= -\int \frac{\sec\theta\tan^2\theta}{1+\sec^2\theta}\,d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos\theta+\cos^3\theta}\,d\theta$$

$$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int \frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}$$

$$= -\int \sec\theta\,d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}\,d\theta$$

$$= -\int \sec\theta\,d\theta + 2\int \frac{}{2-\sin^2\theta}\,d(\sin\theta)$$

$$= -\log|\sec\theta+\tan\theta| + 2\times\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + C$$

$$= -\log|\sec\theta+\sqrt{\sec^2\theta-1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\cos^2\theta}}{\sqrt{2}-\sqrt{1-\cos^2\theta}}\right| + C$$

$$= -\log|\cot x+\sqrt{\cot^2 x-1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C$$

EXERCISE 10.3

Evaluate the following integrals :

1. $\int \frac{x^3}{9x^8+16}\,dx$

2. $\int \frac{x}{9-16x^4}\,dx$

3. $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}$

4. $\int \frac{1}{\cos x + \operatorname{cosec} x}\, dx$

5. $\int \frac{1}{\cos^4 x + \cos^2 x \sin^2 + \sin^4 x}\, dx$

6. $\int \frac{\cos x}{1 + \sin^2 x + \sin^4 x}\, dx$

7. $\int \frac{1}{1 - \sin^4 x}\, dx$

8. $\int \frac{x^2 - 1}{x\sqrt{x^4 + 3x^2 + 1}}\, dx$

9. $\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}}\, dx$

10. $\int \frac{1}{1 - x^8}\, dx$

11. $\int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1}\, dx$

12. $\int \frac{1}{\sin^4 x + \cos^4 x}\, dx$

13. $\int \frac{1}{a \sin x + b \sec x}\, dx, a, b > 0$ and $b > a/2$.

14. $\int \frac{2 \sin \theta + \sin 2\theta}{(\cos \theta - 1)\sqrt{\cos \theta + \cos^2 \theta + \cos^3 \theta}}\, d\theta$

[Hint : Put $\cos \theta = t$]

15. Prove that

$$\int \frac{x}{(x^2 + a^2)\sqrt{x^2 + b^2}}\, dx = \begin{cases} \frac{1}{\sqrt{a^2 - b^2}} \tan^{-1}\left\{\sqrt{\frac{x^2 + b^2}{a^2 - b^2}}\right\}, \text{if } a^2 > b^2 \\ \frac{1}{2\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{x^2 + b^2} - \sqrt{b^2 - a^2}}{\sqrt{x^2 + b^2} + \sqrt{b^2 - a^2}}\right|, \text{if } a^2 < b^2 \end{cases}$$

[Hint : Put $x^2 + b^2 = t^2$]

16. $\int \frac{1}{(x - 3)^2\sqrt{x^2 - 6x + 10}}\, dx$ [Hint : Put $x - 3 = \tan \theta$]

17. $\int \sin^{-1} \frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\, dx$

[Hint : Put $2x + 2 = 3 \tan \theta$]

ANSWERS

1. $\frac{1}{48} \tan^{-1}\left(\frac{3x^4}{4}\right) + C$

2. $\frac{1}{48} \log\left|\frac{3 - 4x^2}{3 + 4x^2}\right| + C$

3. $-\log|\sin x + \cos x - \sqrt{\sin 2x}| + C$

4. $-\frac{1}{2\sqrt{3}} \log\left|\frac{\cos x - \sin x - \sqrt{3}}{\cos x - \sin x + \sqrt{3}}\right|$

$-\frac{1}{2\sqrt{3}} \tan^{-1}(\cos x + \sin x) + C$

5. $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3} \tan x}\right) + C$

6. $\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sin^2 x - 1}{\sqrt{3} \sin x}\right) + \frac{1}{4} \log\left|\frac{\sin^2 x - \sin x + 1}{\sin^2 x + \sin x + 1}\right| + C$

7. $\frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$

8. $\log\left|\frac{(x^2 + 1) + \sqrt{x^4 + 3x^2 + 1}}{x}\right| + C$

9. $\frac{1}{\sqrt{2}} \cos^{-1}\left(\frac{\sqrt{2}x}{x^2 + 1}\right) + C$

10. $\frac{1}{2}\left[\tan^{-1} x + \frac{1}{2} \log\left|\frac{1 + x}{1 - x}\right| + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) - \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right|\right] + C$

11. $\tan^{-1}(e^x - e^{-x}) + C$

12. $-\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \cot 2x) + C$

13. $\frac{1}{2\sqrt{2ab + a^2}} \log\left|\frac{\sqrt{a + 2b} + \sqrt{a}(\sin x - \cos x)}{\sqrt{a + 2b} - \sqrt{a}(\sin x - \cos x)}\right| + \frac{1}{\sqrt{2ab - a^2}} \tan^{-1}\left(\frac{(\sin x + \cos x)\sqrt{a}}{\sqrt{2b - a}}\right) + C$

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10.8 APPLICATIONS OF STANDARD INTEGRALS

In this section, we shall discuss applications of standard integrals to evaluate integrals of the form $\int \frac{1}{ax^2 + bx + c}\, dx$ and $\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$.

In order to evaluate these types of integrals, we use the following algorithms.

ALGORITHM

STEP I *Make the coefficient of x^2 unity, if its is not, by multiplying and dividing by it.*

STEP II *Add and subtract the square of the half of the coefficient of x to express ax^2+bx+c in the form*

$$a\left|\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right|$$

STEP III *Use the suitable formula from the following formulas*

$$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C,$$

$$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$$

$$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$$

EVALUATION OF THE INTEGRALS OF THE FORM $\int\frac{1}{\sqrt{ax^2+bx+c}}dx$

ALGORITHM

STEP I *Make the coefficient of x^2 unity if it is not.*

STEP II *Find half of the coefficient of x.*

STEP III *Add and subtract $\left(\frac{1}{2}\text{ coeff. of }x\right)^2$ inside the square root to express the quantity inside the square root in the form* $\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$ or $\frac{4ac-b^2}{4a^2}-\left(x+\frac{b}{2a}\right)^2$.

STEP IV *Use the suitable formula from the following formulas :*

$$\int\frac{1}{\sqrt{a^2+x^2}}dx=\log\left|x+\sqrt{a^2+x^2}\right|+C$$

$$\int\frac{1}{\sqrt{x^2-a^2}}dx=\log\left|x+\sqrt{x^2-a^2}\right|$$

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$$

ILLUSTRATION 1 *Evaluate :* $\int\frac{1}{x\{6(\log x)^2+7\log x+2\}}dx$

SOLUTION Putting $\log x=t$ and $\frac{1}{x}dx=dt$, we get

$$\int\frac{1}{x\{6(\log x)^2+7(\log x)+2\}}dx$$

$$=\int\frac{1}{6t^2+7t+2}dt$$

$$=\frac{1}{6}\int\frac{1}{t^2+\frac{7}{6}t+\frac{1}{3}}dt$$

$$=\frac{1}{6}\int\frac{1}{t^2+\frac{7}{6}t+\frac{49}{144}-\frac{49}{144}+\frac{1}{3}}dt$$

$$=\frac{1}{6}\int\frac{1}{\left(t+\frac{7}{12}\right)^2-\left(\frac{1}{12}\right)^2}dt$$

$$=\frac{1}{6}\times\frac{1}{2\times1/12}\log\left|\frac{t+\frac{7}{12}-\frac{1}{12}}{t+\frac{7}{12}+\frac{1}{12}}\right|+C$$

$$=\log\left|\frac{2t+1}{3t+2}\right|+C$$

$$=\log\left|\frac{2\log x+1}{3\log x+2}\right|+C$$

ILLUSTRATION 2 *Evaluate :* $\int\frac{\sin 2x}{\sqrt{\sin^4x+4\sin^2x-2}}dx$

SOLUTION We have,

$$\int\frac{\sin 2x}{\sqrt{\sin^4x+4\sin^2x-2}}dx$$

$$=\int\frac{1}{\sqrt{(\sin^2x)^2+4\sin^2x-2}}d(\sin^2x),$$

$$=\int\frac{1}{\sqrt{t^2+4t-2}}dt,\text{ where } t=\sin^2x$$

$$=\int\frac{1}{\sqrt{(t+2)^2-(\sqrt{6})^2}}d(t+2)$$

$$=\log\left|t+2+\sqrt{t^2+4t-2}\right|+C$$

$$=\log\left|(\sin^2x+2)+\sqrt{\sin^4x+4\sin^2x-2}\right|+C$$

ILLUSTRATION 3 *Evaluate :* $\int\sqrt{\text{cosec}\,x-1}\,dx$

SOLUTION We have,

$$\int\sqrt{\text{cosec}\,x-1}\,dx$$

$$=\int\sqrt{\frac{1-\sin x}{\sin x}}dx$$

$$=\int\sqrt{\frac{1-\sin^2x}{\sin x(1+\sin x)}}dx$$

$$=\int\frac{\cos x}{\sqrt{\sin^2x+\sin x}}dx$$

$$=\int\frac{1}{\sqrt{\sin^2x+\sin x}}d(\sin x)$$

$= \int \frac{1}{\sqrt{t^2+t}}\,dt$, where $t = \sin x$

$= \int \frac{1}{\sqrt{\left(t+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dt$

$= \log\left|\left(t+\frac{1}{2}\right)+\sqrt{t^2+t}\right| + C$

$= \log\left|\sin x+\frac{1}{2}+\sqrt{\sin^2 x+\sin x}\right| + C$

EVALUATION OF INTEGRALS OF THE FORM $\int \frac{px+q}{ax^2+bx+c}\,dx$

To evaluate this type of integrals, we use the following algorithm.

ALGORITHM

STEP I *Write the numertor* $px+q$ *in the following form :*

$$px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$$

i.e. $px+q = \lambda(2ax+b)+\mu$

STEP II *Obtain the values of* λ and μ *by equating the coefficients of like powers of* x *on both sides.*

STEP III *Replace* $px+q$ *by* $\lambda(2ax+b)+\mu$ *in the given integral to get*

$$\int \frac{px+q}{ax^2+bx+c}\,dx$$

$$= \lambda\int \frac{2ax+b}{ax^2+bx+c}\,dx + \mu\int \frac{1}{ax^2+bx+c}\,dx$$

STEP IV *Integrate RHS in step III and put the values of* λ *and* μ *obtained in step II.*

Following illustration illustrates the above algorithm.

ILLUSTRATION 1 *Evaluate :*

(i) $\int \frac{x}{x^2+x+1}\,dx$ (ii) $\int \frac{4x+1}{x^2+3x+2}\,dx$

(iii) $\int \frac{2x-3}{x^2+3x-18}\,dx$

SOLUTION (i) Let $x = \lambda\cdot\frac{d}{dx}(x^2+x+1)+\mu$.

Then $x = \lambda(2x+1)+\mu$

Comparing the coefficients of like powers of x, we get

$1 = 2\lambda$ and $\lambda+\mu = 0 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\lambda = -\frac{1}{2}$.

So, $\int \frac{x}{x^2+x+1}\,dx$

$= \int \frac{1/2(2x+1)-1/2}{x^2+x+1}\,dx$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\cdot\frac{1}{(\sqrt{3}/2)}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)+C$

$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.$

(ii) Let $4x+1 = \lambda\cdot\frac{d}{dx}(x^2+3x+2)+\mu$.

Then $4x+1 = \lambda(2x+3)+\mu$

Comparing coefficients of like powers of x, we get

$2\lambda = 4$ and $3\lambda+\mu = 1 \Rightarrow \lambda = 2$ and $\mu = -5$

$\therefore \int \frac{4x+1}{x^2+3x+2}\,dx = \int \frac{2(2x+3)-5}{x^2+3x+2}\,dx$

$= 2\int \frac{2x+3}{x^2+3x+2}\,dx - 5\int \frac{1}{x^2+3x+2}\,dx$

$= 2\log|x^2+3x+2| - 5\int \frac{1}{x^2+3x+(9/4)-(9/4)+2}\,dx$

$= 2\log|x^2+3x+2| - 5\int \frac{1}{(x+3/2)^2-(1/2)^2}\,dx$

$= 2\log|x^2+3x+2| - 5\cdot\frac{1}{2(1/2)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+C$

$= 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right|+C.$

(iii) Let $2x-3 = \lambda\cdot\frac{d}{dx}(x^2+3x-18)+\mu$.

Then $2x-3 = \lambda(2x+3)+\mu$

Comparing coefficients of like powers of x, we get

$2\lambda = 2$ and $3\lambda+\mu = -3 \Rightarrow \lambda = 1$ and $\mu = -6$

So, $\int \frac{2x-3}{x^2+3x-18}\,dx$

$= \int \frac{2x+3-6}{x^2+3x-18}\,dx$

$= \int \frac{2x+3}{x^2+3x-18}\,dx - 6\int \frac{1}{x^2+3x-18}\,dx$

$= \log|x^2+3x-18| - 6\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}-18}\,dx$

$$= \log|x^2+3x-18| - 6\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx$$

$$= \log|x^2+3x-18| - 6\cdot\frac{1}{2\left(\frac{9}{2}\right)}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C$$

$$= \log|x^2+3x-18| - \frac{2}{3}\log\left|\frac{x-3}{x+6}\right| + C.$$

EVALUATION OF INTEGRALS OF THE FORM $\int \frac{P(x)}{ax^2+bx+c}dx$, **where** $P(x)$ **is a polynomial of degree greater than or equal to two**

To evaluate this type of integrals we divide the numerator by the denominator and express the integrand as

$$Q(x) + \frac{R(x)}{ax^2+bx+c}$$

where $R(x)$ is a linear function of x.

$$\therefore \int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx.$$

Now to evaluate the second integral on RHS apply the method discussed earlier.

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{x^3+x+1}{x^2-1}dx$ (ii) $\int \frac{x^2+5x+3}{x^2+3x+2}dx.$

SOLUTION (i) We have,

$$\int \frac{x^3+x+1}{x^2-1}dx$$

$$= \int x + \frac{2x+1}{x^2-1}dx$$

$$= \int x\,dx + \int \frac{2x}{x^2-1}dx + \int \frac{1}{x^2-1}dx$$

$$= \frac{x^2}{2} + \log|x^2-1| + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

(ii) We have,

$$\int \frac{x^2+5x+3}{x^2+3x+2}dx$$

$$= \int \left(1 + \frac{2x+1}{x^2+3x+2}\right)dx$$

$$= \int 1\cdot dx + \int \frac{2x+3-2}{x^2+3x+2}dx$$

$$= \int 1\cdot dx + \int \frac{2x+3}{x^2+3x+2}dx - 2\int \frac{1}{x^2+3x+2}dx$$

$$= x + \log|x^2+3x+2| - 2\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$= x + \log|x^2+3x+2| - 2\cdot\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$= x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + C.$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi$ (ii) $\int \frac{1}{2e^{2x}+3e^x+1}dx$

SOLUTION (i) Let

$$I = \int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi-4\sin\phi}d\phi$$

$$= \int \frac{(4\sin\phi-1)\cos\phi}{6-(1-\sin^2\phi)-4\sin\phi}$$

$$= \int \frac{(4\sin\phi-1)\cos\phi}{\sin^2\phi-4\sin\phi+5}d\phi$$

$$I = \int \frac{4t-1}{t^2-4t+5}dt, \text{ where } t = \sin\phi$$

$$\therefore I = \int \frac{2(2t-4)+7}{t^2-4t+5}dt$$

$$= 2\int \frac{2t-4}{t^2-4t+5}dt + 7\int \frac{1}{t^2-4t+5}dt$$

$$= 2\log|t^2-4t+5| + 7\int \frac{1}{(t-2)^2+1^2}dt$$

$$= 2\log|t^2-4t+5| + 7\tan^{-1}(t-2) + C$$

$$= 2\log|\sin^2\phi-4\sin\phi+5| + 7\tan^{-1}(\sin\phi-2) + C$$

(ii) Let $I = \int \frac{1}{2e^{2x}+3e^x+1}dx$

$$= \int \frac{1}{\frac{2}{e^{-2x}}+\frac{3}{e^{-x}}+1}dx$$

$$= \int \frac{e^{-2x}}{2+3e^{-x}+e^{-2x}}dx$$

$$\therefore \quad I = \int \frac{-t\,dt}{2+3t+t^2}$$

$$= -\int \frac{t}{t^2+3t+2}\,dt$$

$$\therefore I = -\int \frac{\frac{1}{2}(2t+3)-\frac{3}{2}}{t^2+3t+2}\,dt$$

$$= -\frac{1}{2}\int \frac{2t+3}{t^2+3t+2}\,dt + \frac{3}{2}\int \frac{1}{t^2+3t+2}\,dt$$

$$= -\frac{1}{2}\int \frac{2t+3}{t^2+3t+2}\,dt + \frac{3}{2}\int \frac{1}{(t+3/2)^2-(1/2)^2}\,dt$$

$$= -\frac{1}{2}\log|t^2+3t+2| + \frac{3}{2}\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$= -\frac{1}{2}\log|e^{-2x}+3e^{-x}+2| + \frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right| + C$$

EXAMPLE 3 $\int \sqrt{\frac{1+x}{x}}\,dx$.

SOLUTION We have,

$$\int \sqrt{\frac{1+x}{x}}\,dx$$

$$= \int \sqrt{\frac{1+x}{x}}\cdot\sqrt{\frac{1+x}{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{x(1+x)}}\,dx$$

$$= \int \frac{1+x}{\sqrt{x^2+x}}\,dx.$$

$$= \int \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x}}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{x^2+x}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+x}}\,d(x^2+x) + \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \sqrt{x^2+x} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + C$$

$$= \sqrt{x^2+x} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + C.$$

EXAMPLE 4 *Evaluate :*

(i) $\int \sqrt{\frac{a-x}{a+x}}\,dx$

(ii) $\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

SOLUTION (i) We have,

$$\int \sqrt{\frac{a-x}{a+x}}\,dx$$

$$= \int \sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}\,dx$$

$$= \int \frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{a}{\sqrt{a^2-x^2}}\,dx - \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\int \frac{1}{\sqrt{a^2-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\int \frac{1}{\sqrt{a^2-x^2}}\,dx + \frac{1}{2}\int \frac{d(a^2-x^2)}{\sqrt{a^2-x^2}}$$

$$= a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + C$$

(ii) We have,

$$I = \int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx.$$

$$I = \int x\sqrt{\frac{a^2-t}{a^2+t}}\,\frac{dt}{2x} \quad \text{where } t = x^2$$

$$= \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t}}\,dt$$

$$= \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t}\times\frac{a^2-t}{a^2-t}}\,dt$$

$$= \frac{1}{2}\int \frac{a^2-t}{\sqrt{a^4-t^2}}\,dt$$

$$= \frac{1}{2}\int \frac{a^2}{\sqrt{a^4-t^2}}\,dt - \frac{1}{2}\int \frac{t\,dt}{\sqrt{a^4-t^2}}$$

$$= \frac{1}{2}a^2\int \frac{1}{\sqrt{(a^2)^2-t^2}}\,dt + \frac{1}{4}\int \frac{d(a^4-t^2)}{\sqrt{a^4-t^2}}$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{2}\sqrt{a^4-t^2} + C$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4-x^4} + C$$

EXERCISE 10.4

1. $\int \frac{1}{x^2-7x+10}\,dx$
2. $\int \frac{1}{x-x^2-2.5}\,dx$
3. $\int \frac{1}{\sqrt{4x-3-x^2}}\,dx$
4. $\int \frac{1}{\sqrt{2-6x-9x^2}}\,dx$
5. $\int \frac{1}{4x^2+4x+5}\,dx$
6. $\int \frac{1}{\sqrt{5-2x+x^2}}\,dx$
7. $\int \frac{1}{\sqrt{12x-9x^2-2}}\,dx$
8. $\int \frac{8x-11}{\sqrt{5+2x-x^2}}\,dx$
9. $\int \frac{x+2}{x^2+2x+2}\,dx$
10. $\int \frac{x-3}{\sqrt{3-2x-x^2}}\,dx$
11. $\int \frac{3x-1}{4x^2-4x+17}\,dx$
12. $\int \frac{3x-1}{\sqrt{x^2+2x+2}}\,dx$
13. $\int \frac{2x+5}{\sqrt{9x^2+6x+2}}\,dx$
14. $\int \frac{3-4x}{2x^2-3x+1}\,dx$
15. $\int \frac{2-5x}{\sqrt{4x^2+9x+1}}\,dx$
16. $\int \frac{x}{\sqrt{3x^2-11x+2}}\,dx$
17. $\int \frac{\sqrt{x}}{\sqrt{2x+3}}\,dx$
18. $\int \sqrt{\frac{a-x}{x-b}}\,dx$
19. $\int \frac{x^2-x+1}{(x^2+1)^{3/2}}\,dx$
20. $\int \frac{1}{\sqrt{1+e^x+e^{2x}}}\,dx$
21. $\int \frac{1+x}{\sqrt{1-x^2}}\,dx$
22. $\int \frac{3x-1}{x^2+9}\,dx$
23. $\int \sqrt{\frac{1-x}{1+x}}\,dx$
24. $\int \frac{x(1-x^2)}{1+x^4}\,dx$
25. $\int \frac{2x-\sqrt{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$
26. $\int \frac{x+(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}\,dx$

ANSWERS

1. $\frac{1}{3}\log_e\left|\frac{x-5}{x-2}\right|+C$
2. $\frac{2}{3}\tan^{-1}\left(\frac{1-2x}{3}\right)+C$
3. $\sin^{-1}(x-2)+C$
4. $\frac{1}{3}\sin^{-1}\left(\frac{3x+1}{\sqrt{3}}\right)+C$
5. $\frac{1}{4}\tan^{-1}\left(\frac{2x+1}{2}\right)+C$
6. $-\log\left|(1+x)+\sqrt{5-2x+x^2}\right|+C$
7. $\frac{1}{3}\sin^{-1}\left(\frac{3x-2}{\sqrt{2}}\right)+C$
8. $-8\sqrt{5+2x-x^2}-3\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$
9. $\frac{1}{2}\log\left|x^2+2x+2\right|+\tan^{-1}(x+1)+C$
10. $-\sqrt{3-2x-x^2}-4\sin^{-1}\left(\frac{x+1}{2}\right)+C$
11. $\frac{3}{8}\left\{\log(4x^2-4x+17)+\frac{1}{6}\tan^{-1}\left(\frac{2x-1}{4}\right)\right\}+C$
12. $3\sqrt{x^2+2x+2}-4\log\left|x+1+\sqrt{x^2+2x+2}\right|+C$
13. $\frac{2}{9}\sqrt{9x^2+6x+2}+\frac{13}{9}\log\left|(3x+1)+\sqrt{9x^2+6x+2}\right|+C$
14. $-\log\left|2x^2-3x+1\right|+C$
15. $\frac{61}{16}\log\left|(8x+9)+4\sqrt{4x^2+9x+1}\right|-\frac{5}{4}\sqrt{4x^2+9x+1}+C$
16. $\frac{1}{3}\sqrt{3x^2-11x+3}+\frac{11}{6\sqrt{3}}\log\left|x-\frac{11}{6}+\sqrt{x^2-\frac{11x}{3}+\frac{2}{3}}\right|+C$
17. $\frac{1}{2}\sqrt{2x^2+3x}-\frac{3}{4\sqrt{2}}\log\left|x+\frac{3}{4}\sqrt{x^2+\frac{3}{2}x}\right|+C$
18. $\sqrt{(a-x)(x-b)}-(a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}+C$
19. $\frac{1}{\sqrt{x^2+1}}+\log\left|x+\sqrt{x^2+1}\right|+C$
20. $\log\left|\frac{1+e^x-\sqrt{1+e^x+e^{2x}}}{1+e^x+\sqrt{1+e^x+e^{2x}}}\right|+C$
21. $\sin^{-1}x-\sqrt{1-x^2}+C$
22. $\frac{3}{2}\log(x^2+9)-\frac{1}{3}\tan^{-1}\frac{x}{3}+C$
23. $\sin^{-1}x+\sqrt{1-x^2}+C$
24. $\frac{1}{2}\tan^{-1}x^2-\frac{1}{4}\log(x^4+1)+C$
25. $-2\sqrt{1-x^2}-\frac{2}{3}(\sin^{-1}x)^{3/2}+C$
26. $-\frac{1}{9}\left(\sqrt{1-9x^2}+(\cos^{-1}3x)^3\right)+C$

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10.9 INTEGRALS OF THE FORM

$$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx,$$

where $P(x)$ is a polynomial of degree more than two

In order to evaluate integrals of the form

$$\int \frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{\sqrt{ax^2+bx+c}}\,dx$$

we follow the following algorithm.

ALGORITHM

STEP I *Assume that*

$$\int \frac{a_0x^n+a_1x^{n-1}+\ldots.+a^n}{\sqrt{ax^2+bx+c}}\,dx$$

$$=(b_0x^{n-1}+b_1x^{n-2}+..+b_{n-1})\sqrt{ax^2+bx+c}+b_n\int\frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

STEP II *Differentiate both sides with respect to x.*

STEP III *Multiply both sides by* $\sqrt{ax^2+bx+c}$.

STEP IV *Equate the coefficients of the similar powers of x on both sides to determine the values of* $b_0, b_1, \ldots., b_n$.

STEP V *Substitute these values in step I and evaluate the integral on RHS.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

SOLUTION Let $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

$$=(b_0x+b_1)\sqrt{x^2+x+1}+b_2\int\frac{1}{\sqrt{x^2+x+1}}\,dx \quad \ldots(i)$$

Differentiating both sides w.r.to x, we get

$$\frac{x^2+2x+3}{\sqrt{x^2+x+1}}=b_0\sqrt{x^2+x+1}+\frac{(b_0x+b_1)(2x+1)}{2\sqrt{x^2+x+1}}+\frac{b_2}{\sqrt{x^2+x+1}}$$

$$x^2+2x+3=b_0(x^2+x+1)+\frac{1}{2}(b_0x+b_1)(2x+1)+b_2$$

Comparing the coefficients of x^2, x and constant term on both sides, we get

$$2b_0=1, b_0+b_1+\frac{1}{2}b_0=2 \text{ and } b_0+\frac{1}{2}b_1+b_2=3$$

$$\Rightarrow b_0=\frac{1}{2}, b_1=\frac{5}{4} \text{ and } b_2=\frac{15}{8}$$

Substituting these values in (i), we get

$$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$$

$$=\left(\frac{x}{2}+\frac{5}{4}\right)\sqrt{x^2+x+1}+\frac{15}{8}\int\frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$=\frac{(2x+5)}{4}\sqrt{x^2+x+1}+\frac{15}{8}\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$=\frac{(2x+5)}{4}\sqrt{x^2+x+1}+\frac{15}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$$

EXAMPLE 2 *Evaluate :* $\int \frac{6x^3+15x^2-7x+6}{\sqrt{2x^2-2x+1}}\,dx$

SOLUTION Let

$$\int \frac{6x^3+15x^2-7x+6}{\sqrt{2x^2-2x+1}}\,dx$$

$$=(b_0x^2+b_1x+b_2)\sqrt{2x^2-2x+1}+b_3\int\frac{1}{\sqrt{2x^2-2x+1}}\,dx \quad \ldots(i)$$

Differentiating both sides w.r.to x, we get

$$\frac{6x^3+15x^2-7x+6}{\sqrt{2x^2-2x+1}}$$

$$=(2b_0x+b_1)\sqrt{2x^2-2x+1}+\frac{(b_0x^2+b_1x+b_2)(2x-1)}{\sqrt{2x^2-2x+1}}+\frac{b_3}{\sqrt{2x^2-2x+1}}$$

$$\Rightarrow 6x^3+15x^2-7x+6$$
$$=(2b_0x+b_1)(2x^2-2x+1)+(b_0x^2+b_1x+b_2)(2x-1)+b_3$$

On comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$6b_0=6, -5b_0+4b_1=15, 2b_0-3b_1+2b_2=-7$$

and, $b_1-b_2+b_3=6$

$$\Rightarrow b_0=1, b_1=5, b_2=3 \text{ and } b_3=4.$$

Substituting these values in (i), we get

$$\int \frac{6x^3+15x^2-7x+6}{\sqrt{2x^2-2x+1}}\,dx$$

$= (x^2+5x+3)\sqrt{2x^2-2x+1} + 4\int \frac{1}{\sqrt{2x^2-2x+1}}\,dx$

$= (x^2+5x+3)\sqrt{2x^2-2x+1} + 2\sqrt{2}\int \frac{1}{\sqrt{x^2-x+\frac{1}{2}}}\,dx$

$= (x^2+5x+3)\sqrt{2x^2-2x+1} + 2\sqrt{2}\int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$

$= (x^2+5x+3)\sqrt{2x^2-2x+1} + 2\sqrt{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+\frac{1}{2}}\right| + C$

EXERCISE 10.5

Evaluate the following integrals :

1. $\int \frac{x^2+x+1}{\sqrt{x^2+2x+3}}\,dx$
2. $\int \frac{x^2-x+1}{\sqrt{2x^2-x+2}}\,dx$
3. $\int \frac{x^2+2x+3}{\sqrt{x^2+1}}$
4. $\int \frac{x^2+1}{\sqrt{x^2+4}}\,dx$
5. $\int \frac{x^3+2x^2+x-7}{\sqrt{x^2+2x-3}}\,dx$
6. $\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}}\,dx$
7. $\int \frac{x^2}{\sqrt{1-2x-x^2}}\,dx$
8. $\int \frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx$
9. $\int \frac{3x^3}{\sqrt{x^2+4x+5}}\,dx$
10. $\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}\,dx$

ANSWERS

1. $\frac{x-1}{2}\sqrt{x^2+2x+3}+C$
2. $\frac{(4x-5)}{16}\sqrt{2x^2-x+2}+\frac{11\sqrt{2}}{64}\log\left|\left(x-\frac{1}{4}\right)+\sqrt{x^2-\frac{x}{2}+1}\right|+C$
3. $\frac{x+4}{2}\sqrt{x^2+1}+\frac{5}{2}\log\left|x+\sqrt{x^2+1}\right|+C$
4. $\frac{x}{2}\sqrt{x^2+4}-\log\left|\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right|+C$
5. $\frac{1}{6}(2x^2+x-9)\sqrt{x^2+2x+3} - 6\log\left|(x+1)+\sqrt{x^2+2x-3}\right|+C$
6. $\frac{1}{24}(8x^2+50x-163)\sqrt{x^2+x+1} + \frac{85}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$
7. $\frac{1}{2}(3-x)\sqrt{1-2x-x^2}+2\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$
8. $x\sqrt{x^2-2x+5}-5\log\left|x-1+\sqrt{x^2-2x+5}\right|+C$
9. $(x^2-5x+20)\sqrt{x^2+4x+5}-15\log\left|x+2+\sqrt{x^2+4x+5}\right|+C$
10. $(x^2+5x+36)\sqrt{x^2-4x-7} + 112\log\left|(x-2)+\sqrt{x^2-4x-7}\right|+C$

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10.10 INTEGRALS OF THE FORM

$$\int \frac{1}{a\sin^2 x+b\cos^2 x}\,dx, \int \frac{1}{a+b\sin^2 x}\,dx,$$
$$\int \frac{1}{a+b\cos^2 x}\,dx, \quad \int \frac{1}{(a\sin x+b\cos x)^2}\,dx$$
$$\int \frac{1}{a+b\sin^2 x+c\cos^2 x}\,dx$$

To evaluate this type of integrals, we use the following algorithm :

ALGORITHM

STEP I *Divide numerator and denominator by* $\cos^2 x$.

STEP II *Replace* $\sec^2 x$, *if any, in denomiator by* $1+\tan^2 x$

STEP III *Put* $\tan x = t$ *so that* $\sec^2 x\,dx = dt$.

STEP IV *After performing the above steps, the integral will reduce to the form* $\int \frac{1}{at^2+bt+c}\,dt$ *which can be integrated by the method discussed in section 10.8.*

Following illustrations will illustrate the above procedure.

ILLUSTRATION 1 *Evaluate :*

(i) $\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}\,dx$

(ii) $\int \frac{1}{1+3\sin^2 x+8\cos^2 x}\,dx$

SOLUTION (i) Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$I = \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx$$

$$= \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2}\, dx$$

$$= \int \frac{dt}{a^2 t^2 + b^2}, \text{ where } t = \tan x$$

$$= \frac{1}{a} \int \frac{1}{(at)^2 + b^2}\, d(at)$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{at}{b}\right) + C$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right) + C.$$

(ii) Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$I = \int \frac{1}{1 + 3\sin^2 x + 8\cos^2 x}\, dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x + 8}\, dx$$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x + 3\tan^2 x + 8}\, dx$$

$$= \int \frac{\sec^2 x\, dx}{4\tan^2 x + 9}$$

$$= \int \frac{dt}{4t^2 + 9}, \text{ where } \tan x = t$$

$$= \frac{1}{2} \int \frac{1}{(2t)^2 + 3^2}\, d(2t)$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2t}{3}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2 \tan x}{3}\right) + C.$$

10.11 INTEGRALS OF THE FORM

$$\int \frac{1}{a \sin x + b \cos x}\, dx, \int \frac{1}{a + b \sin x}\, dx,$$

$$\int \frac{1}{a + b \cos x}\, dx, \qquad \int \frac{1}{a \sin x + b \cos x + c}\, dx$$

To evaluate this type of integrals we follow the following algorithm :

ALGORITHM

STEP I *Put* $\sin x = \dfrac{2 \tan x/2}{1 + \tan^2 x/2}$, $\cos x = \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$.

STEP II *Replace* $1 + \tan^2 \dfrac{x}{2}$ *in the numerator by* $\sec^2 \dfrac{x}{2}$

STEP III *Put* $\tan \dfrac{x}{2} = t$ *so that* $\dfrac{1}{2} \sec^2 \dfrac{x}{2}\, dx = dt$

After performing these three steps the integral reduces to the form $\int \dfrac{1}{at^2 + bt + c}\, dt$ *which can be evaluated by the methods discussed earlier.*

Following illustration will illustrate the procedure.

ILLUSTRATION *Evaluate:*

(i) $\int \dfrac{1}{1 + \sin x + \cos x}\, dx$ (ii) $\int \dfrac{1}{2 + \cos x}\, dx$

(iii) $\int \dfrac{1 + \sin x}{\sin x\,(1 + \cos x)}\, dx$ (iv) $\int \dfrac{1}{1 - 2\sin x}\, dx$

SOLUTION (i) Replacing $\sin x$ by $\dfrac{2 \tan x/2}{1 + \tan^2 x/2}$ and, $\cos x$ by $\dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$$I = \int \frac{1}{1 + \sin x + \cos x}\, dx$$

$$= \int \frac{1}{1 + \dfrac{2 \tan x/2}{1 + \tan^2 x/2} + \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}}\, dx$$

$$= \int \frac{1 + \tan^2 x/2}{1 + \tan^2 x/2 + 2 \tan x/2 + 1 - \tan^2 x/2}\, dx$$

$$= \int \frac{\sec^2 x/2}{2 + 2 \tan x/2}\, dx$$

$$= \int \frac{1}{1 + \tan \dfrac{x}{2}}\, d\left(1 + \tan \frac{x}{2}\right)$$

$$= \log\left\{\tan \frac{x}{2} + 1\right\} + C.$$

(ii) Let $I = \int \dfrac{1}{2 + \cos x}\, dx$.

Putting $\cos x = \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$$I = \int \frac{1}{2 + \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}}\, dx$$

$$= \int \frac{1 + \tan^2 x/2}{2(1 + \tan^2 x/2) + (1 - \tan^2 x/2)}\, dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 3}\, dx$$

$$= 2 \int \frac{1}{\left(\tan \dfrac{x}{2}\right)^2 + (\sqrt{3})^2}\, d\left(\tan \frac{x}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

(iii) Let $I = \int \frac{1+\sin x}{\sin x\,(1+\cos x)}\,dx$.

Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I = \int \frac{1+\frac{2\tan x/2}{1+\tan^2 x/2}}{\frac{2\tan x/2}{1+\tan^2 x/2}\left(1+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}\,dx$$

$$= \int \frac{(1+\tan^2 x/2+2\tan x/2)(1+\tan^2 x/2)}{2\tan x/2\,(1+\tan^2 x/2+1-\tan^2 x/2)}\,dx$$

$$= \int \frac{(1+\tan x/2)^2 \sec^2 x/2}{4\tan x/2}\,dx$$

$$= \frac{1}{2}\int \frac{1+t^2+2t}{t}\,dt, \text{ where } t = \tan\frac{x}{2}$$

$$= \frac{1}{2}\int \left(\frac{1}{t}+t+2\right)dt$$

$$= \frac{1}{2}\left[\log|t| + \frac{t^2}{2} + 2t\right] + C$$

$$= \frac{1}{2}\left[\log|\tan x/2| + \frac{\tan^2 x/2}{2} + 2\tan x/2\right] + C$$

(iv) Let $I = \int \frac{1}{1-2\sin x}\,dx$.

Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, we get

$$I = \int \frac{1}{1-\frac{4\tan x/2}{1+\tan^2 x/2}}\,dx$$

$$= \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2-4\tan x/2}\,dx$$

$$= \int \frac{\sec^2 x/2}{1+\tan^2 x/2-4\tan x/2}\,dx$$

$$= 2\int \frac{1}{\left(\tan\frac{x}{2}-2\right)^2-(\sqrt{3})^2}\,d\left(\tan\frac{x}{2}\right)$$

$$= 2\int \frac{dt}{(t-2)^2-(\sqrt{3})^2}, \text{ where } t = \tan\frac{x}{2}$$

$$= 2\cdot\frac{1}{2\sqrt{3}}\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + C$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\tan x/2-2-\sqrt{3}}{\tan x/2-2+\sqrt{3}}\right| + C$$

10.12 ALTERNATIVE METHOD TO EVALUATE INTEGRALS OF THE FORM $\int \frac{1}{a\sin x + b\cos x}\,dx$

To evaluate this type of integrals we substitute $a = r\cos\theta$, $b = r\sin\theta$ and so $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

$\therefore\ a\sin x + b\cos x = r\cos\theta\sin x + r\sin\theta\cos x = r\sin(x+\theta)$

So, $\int \frac{1}{a\sin x + b\cos x}\,dx$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + C.$$

Following illustration illustrates the above procedure:

ILLUSTRATION *Evaluate:*

(i) $\int \frac{1}{\sqrt{3}\sin x + \cos x}\,dx$ (ii) $\int \frac{1}{\sin x + \sqrt{3}\cos x}\,dx$.

SOLUTION (i) Let $\sqrt{3} = r\sin\theta$ and $1 = r\cos\theta$. Then,

$$r = \sqrt{(\sqrt{3})^2+1^2} = 2 \text{ and, } \tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}.$$

$\therefore \int \frac{1}{\sqrt{3}\sin x + \cos x}\,dx$

$$= \int \frac{1}{r\sin\theta\sin x + r\cos\theta\cos x}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\cos(x-\theta)}\,dx$$

$$= \frac{1}{r}\int \sec(x-\theta)\,dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right| + C$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right| + C.$$

(ii) Let $1 = r\cos\theta$ and $\sqrt{3} = r\sin\theta$. Then,

$r = \sqrt{1^2+(\sqrt{3})^2} = 2$ and, $\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \pi/3$.

$\therefore \int \frac{1}{\sin x + \sqrt{3}\cos x}\,dx$

$$= \frac{1}{r}\int \frac{1}{\sin x\cos\theta + \cos x\sin\theta}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right| + C.$$

10.13 INTEGRALS OF THE FORM $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$

To evaluate this type of integrals we express the numerator as follows:

Numerator = λ (Diff. of denominator) + μ (denominator)

i.e. $(a\sin x + b\cos x) = \lambda\cdot\frac{d}{dx}(c\sin x + d\cos x)$

$$+ \mu\,(c\sin x + d\cos x)$$

where λ and μ are constants to be determined by comparing the coefficients of $\sin x$ and $\cos x$ on both sides.

$$\therefore \int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$$

$$= \int \frac{\lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)}{c\sin x + d\cos x}\,dx$$

$$= \int \mu\,dx + \lambda\int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}\,dx$$

$$= \mu x + \lambda\log|\,c\sin x + d\cos x\,| + K.$$

Following illustrations will illustrate the above method :

ILLUSTRATION 1 *Evaluate :*

(i) $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$ (ii) $\int \frac{1}{1+\tan x}\,dx$

SOLUTION (i) We have,

$$I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx.$$

Let

$$3\sin x + 2\cos x$$

$$= \lambda\cdot\frac{d}{dx}(3\cos x + 2\sin x) + \mu(3\cos x + 2\sin x)$$

$$\Rightarrow 3\sin x + 2\cos x$$

$$= \lambda(-3\sin x + 2\cos x) + \mu(3\cos x + 2\sin x)$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$-3\lambda + 2\mu = 3 \text{ and } 2\lambda + 3\mu = 2$$

$$\Rightarrow \lambda = \frac{-5}{13} \text{ and } \mu = \frac{12}{13}.$$

$$\therefore I = \int \frac{-(5/13)(-3\sin x + 2\cos x) + (12/13)(3\cos x + 2\sin x)}{3\cos x + 2\sin x}\,dx$$

$$= \frac{12}{13}\int 1\cdot dx - \frac{5}{13}\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$$

$$= \frac{12}{13}x - \frac{5}{13}\int \frac{dt}{t}, \quad \text{where } t = 3\cos x + 2\sin x$$

$$= \frac{12}{13}x - \frac{5}{13}\log|\,t\,| + C$$

$$= \frac{12}{13}x - \frac{5}{13}\log|\,3\cos x + 2\sin x\,| + C.$$

(ii) Let $I = \int \frac{1}{1+\tan x}\,dx = \int \frac{\cos x}{\cos x + \sin x}\,dx.$

Let

$$\cos x = \lambda\cdot\frac{d}{dx}(\cos x + \sin x) + \mu(\cos x + \sin x).$$

$$\Rightarrow \cos x = \lambda(-\sin x + \cos x) + \mu(\sin x + \cos x)$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$-\lambda + \mu = 0 \text{ and } \lambda + \mu = 1 \Rightarrow \lambda = \mu = \frac{1}{2}.$$

$$\therefore I = \int \frac{\cos x}{\cos x + \sin x}\,dx$$

$$= \int \frac{1/2(-\sin x + \cos x) + 1/2(\cos x + \sin x)}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{-\sin x + \cos x}{\cos x + \sin x}\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\cos x + \sin x}\,d(\cos x + \sin x) + \frac{1}{2}\int 1\cdot dx,$$

$$= \frac{1}{2}\log|\,\sin x + \cos x\,| + \frac{1}{2}x + C$$

ILLUSTRATION 2 *Evaluate :* $\int \frac{\cos^2 x\sin x}{\sin x - \cos x}\,dx.$

SOLUTION Let $I = \int \frac{\cos^2 x\sin x}{\sin x - \cos x}\,dx$. Then,

$$I = \frac{1}{2}\int \frac{\cos x(2\sin x\cos x)}{\sin x - \cos x}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{\cos x\,[1 - (\sin x - \cos x)^2]}{\sin x - \cos x}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{\cos x}{\sin x - \cos x}\,dx - \frac{1}{2}\int (\sin x - \cos x)\cos x\,dx$$

$$\Rightarrow I = \frac{1}{4}\int \frac{2\cos x}{\sin x - \cos x} - \frac{1}{4}\int (\sin 2x - 1 - \cos 2x)\,dx$$

$$\Rightarrow I = \frac{1}{4}\int \frac{(\sin x + \cos x) - (\sin x - \cos x)}{\sin x - \cos x}\,dx$$

$$-\frac{1}{4}\int (\sin 2x - 1 - \cos 2x)\,dx$$

$$\Rightarrow I = \frac{1}{4}\log|\sin x - \cos x| - \frac{1}{4}x + \frac{1}{8}\cos 2x + \frac{1}{4}x + \frac{1}{8}\sin 2x + C$$

ILLUSTRATION 3 *Evaluate :* $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}\,dx$

SOLUTION We have,

$$I = \int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}\,dx$$

$$= \int \frac{1 + 2\tan x}{(2 - \tan x)^2}\,dx \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } \cos^2 x]$$

$$= \int \frac{(\sec^2 x - \tan^2 x) + 2\tan x}{(2 - \tan x)^2}\,dx$$

$$= \int \frac{\sec^2 x}{(2 - \tan x)^2}\,dx + \int \frac{\tan x\,(2 - \tan x)}{(2 - \tan x)^2}\,dx$$

$$= -\int \frac{-\sec^2 x}{(2 - \tan x)^2}\,dx + \int \frac{\tan x}{(2 - \tan x)}\,dx$$

$$= -\int \frac{1}{(2 - \tan x)^2}\,d(2 - \tan x) + \int \frac{\sin x}{2\cos x - \sin x}\,dx$$

$$= \frac{1}{(2 - \tan x)} + I_1, \text{ where } I_1 = \int \frac{\sin x}{2\cos x - \sin x}\,dx$$

To evaluate I_1, let

$$\sin x = K(2\cos x - \sin x) + L(-2\sin x - \cos x)$$

$$\Rightarrow 1 = -K - 2L \text{ and } 0 = 2K - L$$

$$\Rightarrow K = -\frac{1}{5} \text{ and } L = -\frac{2}{5}$$

$$\therefore I_1 = \int \frac{\sin x}{2\cos x - \sin x}\,dx$$

$$= \int \frac{K(2\cos x - \sin x) + L(-2\sin x - \cos x)}{2\cos x - \sin x}\,dx$$

$$= \int K\,dx + L\int \frac{-2\sin x - \cos x}{2\cos x - \sin x}\,dx$$

$$= Kx + L\log|2\cos x - \sin x| + C$$

$$= -\frac{1}{5}x - \frac{2}{5}\log|2\cos x - \sin x| + C$$

Hence, $I = \frac{1}{2 - \tan x} - \frac{x}{5} - \frac{2}{5}\log|2\cos x - \sin x| + C.$

ILLUSTRATION 4 *Prove that :*

$$\int \frac{\sin x}{\sin 5x}\,dx$$

$$= \frac{1}{5}\sin\left(\frac{2\pi}{5}\right)\log\left\{\frac{\sin(x - 2\pi/5)}{\sin(x + 2\pi/5)}\right\} - \frac{1}{5}\sin\frac{\pi}{5}\log\left\{\frac{\sin(x - \pi/5)}{\sin(x + \pi/5)}\right\} + C$$

SOLUTION We have,

$$(\cos x + i\sin x)^5 = \cos 5x + i\sin 5x$$

$$\Rightarrow \cos 5x + i\sin 5x = \left({}^5C_0\cos^5 x - {}^5C_2\cos^3 x\sin^2 x + {}^5C_4\cos^4 x\sin x\right) + i\left({}^5C_1\cos^4 x\sin x - {}^5C_3\cos^2 x\sin^3 x + {}^5C_5\sin^5 x\right)$$

On equating imaginary parts, we get

$$\sin 5x = 5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x$$

$$\Rightarrow \frac{\sin 5x}{\sin x} = 5\cos^4 x - 10\cos^2 x\sin^2 x + \sin^4 x$$

$$\Rightarrow \frac{\sin 5x}{\sin x} = 5\cos^4 x - 10\cos^2 x(1 - \cos^2 x) + (1 - \cos^2 x)^2$$

$$\Rightarrow \frac{\sin 5x}{\sin x} = 16\cos^4 x - 12\cos^2 x + 1$$

Now,

$$16\cos^4 x - 12\cos^2 x + 1 = 0$$

$$\Rightarrow \cos^2 x = \frac{12 \pm \sqrt{144 - 64}}{32} = \frac{12 \pm 4\sqrt{5}}{32}$$

$$\Rightarrow \cos^2 x = \frac{6 \pm 2\sqrt{5}}{16}$$

$$\Rightarrow \cos^2 x = \left(\frac{\sqrt{5} \pm 1}{4}\right)^2$$

$$\Rightarrow \cos x = \pm\left(\frac{\sqrt{5} + 1}{4}\right)$$

$$\Rightarrow \cos x = \frac{\sqrt{5} + 1}{4}, \frac{\sqrt{5} - 1}{4}, -\left(\frac{\sqrt{5} + 1}{4}\right), -\left(\frac{\sqrt{5} - 1}{4}\right)$$

$$\Rightarrow \cos x = \cos\frac{\pi}{5}, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{3\pi}{5}$$

$$\Rightarrow x = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}.$$

$$\therefore 16\cos^4 x - 12\cos^2 x + 1$$

$$= 16(\cos x - \cos\pi/5)(\cos x - \cos 2\pi/5)(\cos x - \cos 3\pi/5)(\cos x - \cos 4\pi/5)$$

$$= 16(\cos x - \cos\pi/5)(\cos x - \cos 2\pi/5)(\cos x + \cos 2\pi/5)(\cos x + \cos\pi/5)$$

$$= 16(\cos^2 x - \cos^2\pi/5)(\cos^2 x - \cos^2 2\pi/5)$$

$$= 16(\sin^2\pi/5 - \sin^2 x)(\sin^2 2\pi/5 - \sin^2 x)$$

$$\therefore \frac{\sin 5x}{\sin x} = 16(\sin^2\pi/5 - \sin^2 x)(\sin^2 2\pi/5 - \sin^2 x)$$

$$\Rightarrow \frac{\sin x}{\sin 5x}$$

$$= \frac{1}{16}\frac{1}{(\sin^2 \pi/5 - \sin^2 x)(\sin^2 2\pi/5 - \sin^2 x)}$$

$$= \frac{1}{16}\frac{1}{(\sin^2 2\pi/5 - \sin^2 \pi/5)} \times \left\{\frac{1}{\sin^2 \pi/5 - \sin^2 x} - \frac{1}{\sin^2 2\pi/5 - \sin^2 x}\right\}$$

$$= \frac{1}{16} \cdot \frac{1}{\left\{\frac{10+2\sqrt{5}}{16} - \frac{10-2\sqrt{5}}{16}\right\}} \times \left\{\frac{1}{\sin^2 \pi/5 - \sin^2 x} - \frac{1}{\sin^2 2\pi/5 - \sin^2 x}\right\}$$

$$= \frac{1}{4\sqrt{5}}\left\{\frac{1}{\sin(\pi/5 + x)\sin(\pi/5 - x)} - \frac{1}{\sin(2\pi/5 + x)\sin(2\pi/5 - x)}\right\}$$

$$= \frac{1}{4\sqrt{5}}\left[\frac{1}{\sin 2\pi/5} \cdot \frac{\sin\{(\pi/5 + x) + (\pi/5 - x)\}}{\sin(\pi/5 + x)\sin(\pi/5 - x)} - \frac{1}{\sin 4\pi/5} \cdot \frac{\sin\{(2\pi/5 + x) + (2\pi/5 - x)\}}{\sin(2\pi/5 + x)\sin(2\pi/5 - x)}\right]$$

$$= \frac{1}{4\sqrt{5}}\left[\frac{1}{\sin\frac{2\pi}{5}}\left\{\cot\left(\frac{\pi}{5} - x\right) + \cot\left(\frac{\pi}{5} + x\right)\right\} - \frac{1}{\sin\frac{4\pi}{5}}\left\{\cot\left(\frac{2\pi}{5} - x\right) + \cot\left(\frac{2\pi}{5} + x\right)\right\}\right]$$

$$= \frac{1}{4\sqrt{5}\sin 2\pi/5 \sin \pi/5}\left[\sin\frac{\pi}{5}\left\{\cot\left(\frac{\pi}{5} - x\right) + \cot\left(\frac{\pi}{5} + x\right)\right\} - \sin\frac{2\pi}{5}\left\{\cot\left(\frac{2\pi}{5} - x\right) + \cot\left(\frac{2\pi}{5} + x\right)\right\}\right]$$

$$= \frac{1}{5}\left[\sin\frac{\pi}{5}\left\{\cot\left(\frac{\pi}{5} - x\right) + \cot\left(\frac{\pi}{5} + x\right)\right\} - \sin\frac{2\pi}{5}\left\{\cot\left(\frac{2\pi}{5} - x\right) + \cot\left(\frac{2\pi}{5} + x\right)\right\}\right]$$

$$\therefore \int \frac{\sin x}{\sin 5x}\,dx$$

$$= \frac{1}{5}\sin\frac{\pi}{5}\int\left\{\cot\left(\frac{\pi}{5} - x\right) + \cot\left(\frac{\pi}{5} + x\right)\right\}dx - \frac{1}{5}\sin\frac{2\pi}{5}\int\left\{\cot\left(\frac{2\pi}{5} - x\right) + \cot\left(\frac{2\pi}{5} + x\right)\right\}dx$$

$$= \frac{1}{5}\sin\frac{\pi}{5}\left\{-\log\sin\left(\frac{\pi}{5} - x\right) + \log\sin\left(\frac{\pi}{5} + x\right)\right\} - \frac{1}{5}\sin\frac{2\pi}{5}\left\{-\log\sin\left(\frac{2\pi}{5} - x\right) + \log\sin\left(\frac{2\pi}{5} + x\right)\right\} + C$$

$$= \frac{1}{5}\sin\frac{2\pi}{5}\log\left|\frac{\sin\left(\frac{2\pi}{5} - x\right)}{\sin\left(\frac{2\pi}{5} + x\right)}\right| - \frac{1}{5}\sin\frac{\pi}{5}\log\left|\frac{\sin\left(\frac{\pi}{5} - x\right)}{\sin\left(\frac{\pi}{5} + x\right)}\right| + C$$

10.14 INTEGRALS OF THE FORM $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$

To evaluate this type of integrals, we express the numerator as follows:

Numerator = λ (denominator) + μ (Diff. of denominator) + ν

i.e., $(a\sin x + b\cos x + c)$

$$= \lambda(p\sin x + q\cos x + r) + \mu(p\cos x - q\sin x) + \nu$$

where λ, μ, ν are constants to be determined by comparing the coefficients of $\sin x$, $\cos x$ and constant term on both sides.

$$\therefore \int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$$

$$= \int \lambda\,dx + \mu\int \frac{\text{Diff. of denominator}}{\text{Denominator}}\,dx + \nu\int \frac{1}{p\sin x + q\cos x + r}\,dx$$

$$= \lambda x + \mu\log|\text{Denominator}| + \nu\int \frac{1}{p\sin x + q\cos x + r}\,dx.$$

The integral on RHS can be evaluated by the method discussed in section 10.13.

ILLUSTARATION 1 *Evaluate:* $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

SOLUTION Let

$$I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx.$$

Let,

$$3\cos x + 2 = \lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu.$$

Comparing the coefficients of $\sin x$, $\cos x$ and constant term on both sides, we get

$$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2$$

$$\Rightarrow \lambda = \frac{6}{5},\ \mu = \frac{3}{5} \text{ and } \nu = -\frac{8}{5}.$$

$$\therefore I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}\,dx$$

$$= \lambda\int dx + \mu\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx + \nu\int \frac{1}{\sin x + 2\cos x + 3}\,dx$$

$$= \lambda x + \mu\log|\sin x + 2\cos x + 3| + \nu I_1, \text{ where}$$

$$I_1 = \int \frac{1}{\sin x + 2\cos x + 3}\,dx.$$

Putting $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$, $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3}\, dx$$

$$= \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)}\, dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}\, dx$$

$$= \int \frac{2dt}{t^2+2t+5}, \text{ where } t = \tan\frac{x}{2}$$

$$= 2\int \frac{dt}{(t+1)^2+2^2}$$

$$= \tan^{-1}\left(\frac{t+1}{2}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right).$$

Hence,

$$I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C,$$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$.

10.15 INTEGRALS OF THE FORM

$$\int \frac{1}{(a+b\cos x)^2}\, dx, \int \frac{1}{(a+b\sin x)^2}\, dx$$

In order to evaluate this type of integrals, we follow the following algorithm.

ALGORITHM

STEP I *Define* $P = \frac{\sin x}{a+b\cos x}$, *or* $P = \frac{\cos x}{a+b\sin x}$ *according as the integral to be evaluated is of the form*

$$\int \frac{1}{(a+b\cos x)^2}\, dx, \text{ or } \int \frac{1}{(a+b\sin x)^2}\, dx.$$

STEP II *Find* $\frac{dP}{dx}$ *and express it in terms of* $\frac{1}{a+b\cos x}$, *or* $\frac{1}{a+b\sin x}$ *as the case may be.*

STEP III *Integrate both sides of the expression obtained in step II to obtain the value of the required integral.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* : $\int \frac{1}{(5+4\cos x)^2}\, dx$

SOLUTION Let $P = \frac{\sin x}{(5+4\cos x)}$. Then,

$$\frac{dP}{dx} = \frac{(5+4\cos x)\cos x + 4\sin^2 x}{(5+4\cos x)^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{5\cos x + 4}{(5+4\cos x)^2}$$

Let $5+4\cos x = y$. Then, $\cos x = \frac{y-5}{4}$.

$$\therefore \frac{dP}{dx} = \frac{\frac{5(y-5)}{4}+4}{y^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{5y-9}{4y^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{5}{4}\cdot\frac{1}{y} - \frac{9}{4}\cdot\frac{1}{y^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{5}{4}\cdot\frac{1}{5+4\cos x} - \frac{9}{4}\cdot\frac{1}{(5+4\cos x)^2}$$

Integrating both sides w.r.to x, we get

$$P = \frac{5}{4}\int \frac{1}{5+4\cos x}\, dx - \frac{9}{4}\int \frac{1}{(5+4\cos x)^2}\, dx$$

$$\Rightarrow \frac{9}{4}\int \frac{1}{(5+4\cos x)^2}\, dx = \frac{5}{4}\int \frac{1}{5+4\cos x}\, dx - P$$

$$\Rightarrow \int \frac{1}{(5+4\cos x)^2}\, dx = \frac{5}{9}\int \frac{1+\tan^2 x/2}{9+\tan^2 x/2}\, dx - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right)$$

$$\Rightarrow \int \frac{1}{(5+4\cos x)^2}\, dx = \frac{10}{9}\int \frac{1}{9+t^2}\, dt - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right)$$

, where $t = \tan x/2$.

$$\Rightarrow \int \frac{1}{(5+4\cos x)^2}\, dx = \frac{10}{9}\times\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right) + C$$

$$\Rightarrow \int \frac{1}{(5+4\cos x)^2}\, dx = \frac{10}{27}\tan^{-1}\left(\frac{\tan x/2}{3}\right) - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right) + C$$

EXAMPLE 2 *Evaluate* : $\int \frac{1}{(16+9\sin x)^2}\, dx$

SOLUTION Let $P = \frac{\cos x}{16+9\sin x}$. Then,

$$\frac{dP}{dx} = \frac{-(16+9\sin x)\sin x - 9\cos^2 x}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{-16\sin x - 9}{(16+9\sin x)^2}$$

Let $16+9\sin x = y$. Then, $\sin x = \frac{y-16}{9}$.

$$\therefore \frac{dP}{dx} = \frac{-16\left(\frac{y-16}{9}\right)-9}{y^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{-16y+175}{9y^2}$$

$$\Rightarrow \frac{dP}{dx} = -\frac{16}{9y} + \frac{175}{9y^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{-16}{9(16+9\sin x)} + \frac{175}{9(16+9\sin x)^2}$$

Integrating both sides w.r.to x, we get

$$P = \frac{-16}{9}\int \frac{1}{16+9\sin x}\,dx + \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx$$

$$\Rightarrow P = \frac{-16}{9}\int \frac{\sec^2 x/2}{16+16\tan^2 x/2+18\tan x/2}\,dx + \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx$$

$$\Rightarrow \frac{\cos x}{16+9\sin x} = \frac{-32}{9}\int \frac{dt}{16t^2+18t+16}\Big] + \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx \quad \text{where } t = \tan x/2$$

$$\Rightarrow \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx = \frac{\cos x}{16+9\sin x} + \frac{16}{9}\int \frac{1}{8t^2+9t+8}\,dt$$

$$\Rightarrow \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx = \frac{\cos x}{16+9\sin x} + \frac{2}{9}\int \frac{1}{t^2+\frac{9}{8}t+1}\,dt$$

$$\Rightarrow \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx = \frac{\cos x}{16+9\sin x} + \frac{2}{9}\int \frac{1}{\left(t+\frac{9}{16}\right)^2+\left(\frac{\sqrt{175}}{16}\right)^2}\,dt$$

$$\Rightarrow \frac{175}{9}\int \frac{1}{(16+9\sin x)^2}\,dx = \frac{\cos x}{16+9\sin x} + \frac{2}{9}\times\frac{16}{\sqrt{175}}\tan^{-1}\left(\frac{16t+9}{\sqrt{175}}\right)+C$$

$$\Rightarrow \int \frac{1}{(16+9\sin x)^2}\,dx = \frac{9}{175}\frac{\cos x}{(16+9\sin x)^2} + \frac{2}{(175)^{3/2}}\tan^{-1}\left(\frac{16\tan\frac{x}{2}+9}{\sqrt{175}}\right)+C$$

EXAMPLE 3 *Evaluate :* $\int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$.

SOLUTION Let $I = \int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$. Then,

$$I = \frac{b}{a}\int \frac{\frac{a^2}{b}+a\sin x}{(b+a\sin x)^2}\,dx$$

$$\Rightarrow I = \frac{b}{a}\int \frac{\frac{a^2}{b}-b+(b+a\sin x)}{(b+a\sin x)^2}\,dx$$

$$\Rightarrow I = \frac{a^2-b^2}{a}\int \frac{1}{(b+a\sin x)^2}\,dx + \frac{b}{a}\int \frac{1}{(b+a\sin x)}\,dx \quad \ldots(i)$$

Now, let $P = \frac{\cos x}{b+a\sin x}$. Then,

$$\frac{dP}{dx} = \frac{-\sin x(b+a\sin x) - \cos x \cdot a\cos x}{(b+a\sin x)^2}$$

$$\Rightarrow \frac{dP}{dx} = \frac{-b\sin x - a}{(b+a\sin x)^2}$$

$$\Rightarrow \frac{dP}{dx} = -\frac{b}{a}\left\{\frac{a\sin x+\frac{a^2}{b}}{(b+a\sin x)^2}\right\}$$

$$\Rightarrow \frac{dP}{dx} = -\frac{b}{a}\left\{\frac{b+a\sin x+\frac{a^2-b^2}{b}}{(b+a\sin x)^2}\right\}$$

$$\Rightarrow \frac{dP}{dx} = -\frac{b}{a}\frac{1}{b+a\sin x} - \frac{a^2-b^2}{a}\cdot\frac{1}{(b+a\sin x)^2}$$

Integrating both sides w.r.to x, we get

$$P = -\frac{b}{a}\int \frac{1}{b+a\sin x}\,dx - \frac{a^2-b^2}{a}\int \frac{1}{(b+a\sin x)^2}\,dx$$

$$\Rightarrow \frac{a^2-b^2}{a}\int \frac{1}{(b+a\sin x)^2}\,dx = -\frac{b}{a}\int \frac{1}{b+a\sin x}\,dx - P \quad \ldots(ii)$$

From (i) and (ii), we get

$$I = -\frac{b}{a}\int \frac{1}{b+a\sin x}\,dx - P + \frac{b}{a}\int \frac{1}{b+a\sin x}\,dx + C$$

$$\Rightarrow I = -P + C$$

$$= -\frac{\cos x}{b+a\sin x} + C$$

ALITER See Ex 30 on page 10.63.

EXAMPLE 4 *Evaluate :* $\int \frac{1}{(\sin x + a \sec x)^2}\, dx,\ |a| > \frac{1}{2}$

SOLUTION We have,

$$I = \int \frac{1}{(\sin x + a \sec x)^2}\, dx$$

$$\Rightarrow I = \int \frac{\cos^2 x}{(\sin x \cos x + a)^2}\, dx$$

$$\Rightarrow I = \int \frac{\cos^2 x}{(a^2 + 2a \sin x \cos x + \sin^2 x \cos^2 x)}\, dx$$

$$\Rightarrow I = \int \frac{\cos^2 x}{\left(a^2 + a \sin 2x + \frac{1}{4}\sin^2 2x\right)}\, dx$$

$$\Rightarrow I = \int \frac{4\cos^2 x}{(4a^2 + 4a \sin 2x + \sin^2 2x)}\, dx$$

$$\Rightarrow I = 2\int \frac{1 + \cos 2x}{(2a + \sin 2x)^2}\, dx$$

$$\Rightarrow I = \int \frac{2}{(2a + \sin 2x)^2}\, dx + \int \frac{2\cos 2x}{(2a + \sin 2x)^2}\, dx$$

$$\Rightarrow I = \int \frac{2}{(2a + \sin 2x)^2}\, dx + \int \frac{1}{(2a + \sin 2x)^2}\, d(2a + \sin 2x)$$

$$\Rightarrow I = \int \frac{1}{(2a + \sin t)^2}\, dt - \frac{1}{(2a + \sin 2x)} + C, \text{ where } t = 2x.$$

$$\Rightarrow I = I_1 - \frac{1}{(2a + \sin 2x)} + C \qquad \ldots(i)$$

where $I_1 = \int \frac{1}{(2a + \sin t)^2}\, dt$

To evaluate I_1, let us define P given by

$$P = \frac{\cos t}{2a + \sin t}$$

Then,

$$\frac{dP}{dt} = \frac{-(2a + \sin t)\sin t - \cos^2 t}{(2a + \sin t)^2}$$

$$\Rightarrow \frac{dP}{dt} = \frac{-2a \sin t - 1}{(2a + \sin t)^2}$$

$$\Rightarrow \frac{dP}{dX} = \frac{-2a(\sin t + 2a) + (4a^2 - 1)}{(2a + \sin t)^2}$$

$$\Rightarrow \frac{dP}{dt} = \frac{-2a}{(2a + \sin t)} + \frac{(4a^2 - 1)}{(2a + \sin t)^2}$$

Integrating both sides w.r. to, t, we get

$$P = -2a\int \frac{1}{2a + \sin t}\, dt + (4a^2 - 1)\int \frac{1}{(2a + \sin t)^2}\, dt$$

$$\Rightarrow \frac{\cos t}{2a + \sin t} = -2a \int \frac{1 + \tan^2 t/2}{2\tan t/2 + 2a(1 + \tan^2 t/2)}\, dt + (4a^2 - 1)I$$

$$\Rightarrow \frac{\cos t}{2a + \sin t} = -\int \frac{\sec^2 t/2}{1 + \tan^2 \frac{t}{2} + \frac{1}{a}\tan \frac{t}{2}}\, dt + (4a^2 - 1) I_1$$

$$\Rightarrow \frac{\cos t}{2a + \sin t} = -\int \frac{2\, du}{1 + u^2 + \frac{1}{a}u} + (4a^2 - 1) I_1, \text{ where } \tan\left(\frac{t}{2}\right) = u$$

$$\Rightarrow \frac{\cos t}{2a + \sin t} = -2\int \frac{1}{\left(u + \frac{1}{2a}\right)^2 + \left(1 - \frac{1}{4a^2}\right)}\, du + (4a^2 - 1) I_1$$

$$\Rightarrow \frac{\cos t}{2a + \sin t} = -2 \times \frac{2a}{\sqrt{4a^2 - 1}} \tan^{-1}\left(\frac{2au + 1}{\sqrt{4a^2 - 1}}\right) + (4a^2 - 1) I_1$$

$$\Rightarrow I_1 = \frac{1}{(4a^2 - 1)} \frac{\cos t}{2a + \sin t} + \frac{4a}{(4a^2 - 1)^{3/2}} \tan^{-1}\left(\frac{2au + 1}{\sqrt{4a^2 - 1}}\right)$$

$$\Rightarrow I_1 = \frac{1}{(4a^2 - 1)} \frac{\cos t}{2a + \sin t} + \frac{4a}{(4a^2 - 1)^{3/2}} \tan^{-1}\left(\frac{2a \tan t/2 + 1}{\sqrt{4a^2 - 1}}\right)$$

$$\Rightarrow I_1 = \left(\frac{1}{4a^2 - 1}\right) \frac{\cos 2x}{2a + \sin 2x} + \frac{4a}{(4a^2 - 1)^{3/2}} \tan^{-1}\left(\frac{2a \tan x + 1}{\sqrt{4a^2 - 1}}\right)$$

Substituting the value of I_1 in (i), we get

$$\Rightarrow I_1 = \left(\frac{1}{4a^2 - 1}\right) \frac{\cos 2x}{2a + \sin 2x} + \frac{4a}{(4a^2 - 1)^{3/2}} \tan^{-1}\left(\frac{2a \tan x + 1}{\sqrt{4a^2 - 1}}\right) - \frac{1}{2a + \sin 2x} + C$$

EXERCISE 10.6

Evaluate the following integrals :

1. $\int \frac{1}{4\sin^2 x + 4 \sin x \cos x + 5\cos^2 x}\, dx$

2. $\int \frac{1}{1 - \sin^4 x}\, dx$

3. $\int \frac{1}{(2\sin x + 3\cos x)^2}\, dx$

4. $\int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)}\, dx$

5. $\int \frac{1}{5+4\sin x}\,dx$

6. $\int \frac{1}{2+\sin x+\cos x}\,dx$

7. $\int \frac{1}{\sin^2 x+\sin 2x}\,dx$

8. $\int \frac{3\sin x+2}{\sin x+2\cos x+3}\,dx$

9. $\int \frac{3\sin x+2\cos x}{2\sin x+3\cos x}\,dx$

10. $\int \frac{1}{1-2a\cos x+a^2}\,dx, 0<a<1.$

11. $\int \frac{1}{\sin x\,(2+\cos x-2\sin x)}\,dx$

12. $\int \frac{1}{(a+b\cos x)^2}\,dx$

13. $\int \frac{b+a\cos x}{(a+b\cos x)^2}\,dx$

14. $\int \frac{1}{(\sin x+2\sec x)^2}\,dx$

15. $\int \frac{1}{(3+4\sin x)^2}\,dx$

ANSWERS

1. $\frac{1}{4}\tan^{-1}\left\{\frac{2\tan x+1}{2}\right\}+C$

2. $\frac{1}{2}\left\{\tan x+\frac{1}{\sqrt{2}}\tan(\sqrt{2}\tan x)\right\}+C$

3. $-\frac{1}{2(2\tan x+3)}+C$

4. $\frac{1}{5}\log\left\{\frac{\tan x-2}{2\tan x+1}\right\}+C$

5. $\frac{2}{3}\tan^{-1}\left\{\frac{5\tan\frac{x}{2}+4}{3}\right\}+C$

6. $\sqrt{2}\tan^{-1}\left\{\frac{\tan(x/2)+1}{\sqrt{2}}\right\}+C$

7. $\frac{1}{2}\log\left(\frac{\tan x}{\tan x+2}\right)+C$

8. $\frac{6}{5}x+\frac{3}{5}\log|\sin x+2\cos x+3|-\frac{8}{5}\tan^{-1}\left\{\frac{\tan\frac{x}{2}+1}{2}\right\}+C$

9. $\frac{12}{13}x-\frac{5}{13}\log|2\sin x+3\cos x|+C$

10. $\frac{2}{1-a^2}\tan^{-1}\left\{\left(\frac{1+a}{1-a}\right)\tan\frac{x}{2}\right\}+C$

11. $\frac{1}{3}\log\tan\frac{x}{2}+\frac{5}{3}\log\left\{\tan\frac{x}{2}-3\right\}-\log\left\{\tan\frac{x}{2}-1\right\}+C$

12. $\frac{2b}{(b^2-a^2)^{3/2}}\tan^{-1}\left\{\sqrt{\frac{b-a}{b+a}}\tan\frac{x}{2}\right\}+\frac{a}{a^2-b^2}\left(\frac{\sin x}{b+a\cos x}\right)+C$

13. $\frac{\sin x}{b+a\cos x}+C$

14. $\frac{\cos 2x-15}{15(4+\sin 2x)}+\frac{14}{15\sqrt{15}}\sin^{-1}\left(\frac{4\sin 2x+1}{4+\sin 2x}\right)+C$

15. $\frac{12}{49\sqrt{2}}\log\left|\frac{3\tan\frac{x}{2}+4-\sqrt{7}}{3\tan\frac{x}{2}+4+\sqrt{7}}\right|-\frac{4}{7}\left(\frac{\cos x}{3+4\sin x}\right)+C$

HINTS TO SELECTED PROBLEMS

2. We have,

$$\int \frac{1}{1-\sin^4 x}\,dx$$

$$=\int \frac{1}{(1-\sin^2 x)(1+\sin^2 x)}\,dx$$

$$=\frac{1}{2}\int\left(\frac{1}{1-\sin^2 x}+\frac{1}{1+\sin^2 x}\right)dx$$

$$=\frac{1}{2}\int \frac{1}{1-\sin^2 x}\,dx+\frac{1}{2}\int \frac{1}{1+\sin^2 x}\,dx$$

11. We have,

$$\int \frac{1}{\sin x\,(2+\cos x-2\sin x)}\,dx$$

$$=\int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}\left(2+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}-4\sin\frac{x}{2}\cos\frac{x}{2}\right)}\,dx$$

Now, divide N^r and D^r by $\cos^4\frac{x}{2}$ and then put $\tan\frac{x}{2}=t$.

—❑❑❑—

10.16 INTEGRATION BY PARTS

THEOREM *If u and v are two functions of x, then*

$$\int uv\,dx = u\left(\int v\,dx\right) - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx$$

i.e., the integral of the product of two functions

= (First function) × (integral of second function)

– integral of {(Diff. of first function) × (integral of second function)}

PROOF For any two functions $f(x)$ and $g(x)$, we have

$$\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}$$

$$\therefore \int\left\{f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}\right\}dx = f(x)\,g(x)$$

$$\text{or, } \int\left\{f(x)\cdot\frac{d}{dx}\{g(x)\}\right\}dx + \int\left\{g(x)\cdot\frac{d}{dx}\{f(x)\}\right\}dx = f(x)\,g(x)$$

$$\text{or, } \int\left\{f(x)\cdot\frac{d}{dx}\{g(x)\}\right\}dx = f(x)\cdot g(x) - \int\left\{g(x)\cdot\frac{d}{dx}\{f(x)\}\right\}dx$$

Let $f(x) = u$ and $\frac{d}{dx}\{g(x)\} = v$ so that $g(x) = \int v\,dx$.

$$\therefore \int uv\,dx = u\left(\int v\,dx\right) - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}dx$$

NOTE 1 *Proper choice of first and second function*

Integration with the help of the above rule is called the integration by parts. In the above rule there are two terms on RHS and in both the terms the integral of the second function is involved. Therefore in the product of two functions if one of the two functions is not directly integrable (e.g., $\log x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) we take it as the first function and the remaining function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integral both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple function and the function thus obtained under the integral sign is easily integrable than the original function.

NOTE 2 *We can also choose the first function as the function which comes first in the word ILATE, where*

I – stands for the inverse trigonometric function ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.)

L – stands for the logarithmic functions

A – stands for the algebraic functions

T – stands for the trigonometric functions

E – stands for the exponential functions

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int x^2\sin x\,dx$ (ii) $\int x^2 e^x\,dx$

SOLUTION (i) We have,

$$\int \underset{\text{I}}{x^2}\ \underset{\text{II}}{\sin x}\,dx$$

$$= x^2\left(\int \sin x\,dx\right) - \int\left\{\frac{d}{dx}(x^2)\cdot\int \sin x\,dx\right\}dx$$

$$= -x^2\cos x - \int 2x(-\cos x)\,dx$$

$$= -x^2\cos x + 2\int \underset{\text{I}}{x}\ \underset{\text{II}}{\cos x}\,dx$$

$$= -x^2\cos x + 2\left[x\left(\int \cos x\,dx\right) - \int\left\{\frac{d}{dx}(x)\cdot\int \cos x\,dx\right\}dx\right]$$

$$= -x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right]$$

$$= -x^2\cos x + 2[x\sin x + \cos x] + C.$$

(ii) We have,

$$\int \underset{\text{I}}{x^2}\ \underset{\text{II}}{e^x}\,dx = x^2\left(\int e^x\,dx\right) - \int\left\{\frac{d}{dx}(x^2)\cdot\int e^x\,dx\right\}dx$$

$$= x^2e^x - \int 2x\cdot e^x\,dx = x^2e^x - 2\int \underset{\text{I}}{x}\ \underset{\text{II}}{e^x}\,dx$$

$$= x^2e^x - 2\left[x\left(\int e^x\,dx\right) - \int\left\{\frac{d}{dx}(x)\cdot\int e^x\,dx\right\}dx\right]$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] = x^2e^x - 2[xe^x - e^x] + C.$$

EXAMPLE 2 *Evaluate :*

(i) $\int \log x\,dx$ (ii) $\int (\log x)^2\,dx$

SOLUTION (i) We have,

$$\int \underset{\text{I}}{\log x}\cdot\underset{\text{II}}{1}\,dx$$

$$= \log x\cdot\left(\int 1\cdot dx\right) - \int\left\{\frac{d}{dx}(\log x)\cdot\int 1\,dx\right\}dx$$

$$= (\log x)\,x - \int \frac{1}{x}\cdot x\,dx$$

$$= x(\log x) - \int 1\cdot dx$$

$$= x(\log x) - x + C.$$

(ii) $\int \underset{\text{I}}{(\log x)^2}\cdot\underset{\text{II}}{1}\,dx$

$$= (\log x)^2\left(\int 1\cdot dx\right) - \int\left\{\frac{d}{dx}(\log x)^2\cdot\int 1\cdot dx\right\}dx$$

$= (\log x)^2 x - \int 2 \log x \cdot \frac{1}{x} \cdot x \, dx$

$= x (\log x)^2 - 2 \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1} \, dx$

$= x (\log x)^2 - 2 \left[(\log x) \left(\int 1 \cdot dx \right) - \int \left\{ \frac{d}{dx} (\log x) \int 1 \cdot dx \right\} dx \right]$

$= x (\log x)^2 - 2 \left[(\log x) x - \int \frac{1}{x} \cdot x \, dx \right]$

$= x (\log x)^2 - 2 [x \log x - x] + C.$

EXAMPLE 3 *Evaluate :*

(i) $\int \sin^{-1} x \, dx$ (ii) $\int \tan^{-1} x \, dx$ (iii) $\int \sec^{-1} x \, dx.$

SOLUTION (i) Put $\sin^{-1} x = t$ so that $x = \sin t$ and $dx = \cos t \, dt$

$\therefore \int \sin^{-1} x \, dx$

$= \int \underset{\text{I}}{t} \underset{\text{II}}{\cos t} \, dt$

$= t \sin t - \int 1 \cdot (\sin t) \, dt$

$= t \sin t - \int \sin t \, dt$

$= t \sin t + \cos t + C$

$= x \sin^{-1} x + \sqrt{1 - \sin^2 t} + C$

$= x \sin^{-1} x + \sqrt{1 - x^2} + C.$

(ii) Put $\tan^{-1} x = t$ so that $x = \tan t$ and $dx = \sec^2 t \, dt$

$\therefore \int \tan^{-1} x \, dx$

$= \int \underset{\text{I}}{t} \underset{\text{II}}{\sec^2 t} \, dt$

$= t (\tan t) - \int 1 \cdot \tan t \, dt$

$= t \tan t + \log | \cos t | + C$

$= x \tan^{-1} x + \log \left| \frac{1}{\sqrt{1 + x^2}} \right| + C \qquad \left[\because \tan t = x, \; \therefore \cos t = \frac{1}{\sqrt{1 + x^2}} \right]$

$= x \tan^{-1} x - \frac{1}{2} \log | 1 + x^2 | + C.$

(iii) Put $\sec^{-1} x = t$ so that $x = \sec t$ and $dx = \sec t \tan t \, dt$

$\therefore \int \sec^{-1} x \, dx$

$= \int \underset{\text{I}}{t} \underset{\text{II}}{(\sec t \tan t)} \, dt$

$= t (\sec t) - \int 1 \cdot \sec t \, dt$

$= t \sec t - \log | (\sec t + \tan t) | + C$

$= t \sec t - \log | \sec t + \sqrt{\sec^2 t - 1} | + C$

$= x (\sec^{-1} x) - \log | x + \sqrt{x^2 - 1} | + C.$

EXAMPLE 4 *Evaluate :*

(i) $\int x \tan^{-1} x \, dx$ (ii) $\int \frac{\log x}{x^2} dx$

(iii) $\int \frac{x - \sin x}{1 - \cos x} dx$ (iv) $\int \log (1 + x^2) \, dx.$

SOLUTION (i) $\int \underset{\text{II}}{x} \underset{\text{I}}{\tan^{-1} x} \, dx$

$= (\tan^{-1} x) \frac{x^2}{2} - \int \frac{1}{1 + x^2} \cdot \frac{x^2}{2} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{x^2 + 1} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{x^2 + 1} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C.$

(ii) $\int \frac{\log x}{x^2} dx$

$= \int \underset{\text{II}}{\frac{1}{x^2}} \underset{\text{I}}{\log x} \, dx$

$= (\log x) \left(-\frac{1}{x} \right) - \int \frac{1}{x} \left(-\frac{1}{x} \right) dx$

$= -\frac{1}{x} \log x + \int x^{-2} dx = -\frac{1}{x} \log x - \frac{1}{x} + C$

$= -\frac{1}{x} (1 + \log x) + C.$

(iii) $\int \frac{x - \sin x}{1 - \cos x} dx$

$= \int \frac{x}{1 - \cos x} dx - \int \frac{\sin x}{1 - \cos x} dx$

$= \int \frac{x}{2} \operatorname{cosec}^2 \frac{x}{2} dx - \int \frac{2 \sin x/2 \cos x/2}{2 \sin^2 x/2} dx$

$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\operatorname{cosec}^2 \frac{x}{2}} - \int \cot \frac{x}{2} dx$

$= \frac{1}{2} \left\{ x \left(-2 \cot \frac{x}{2} \right) - \int 1 \cdot \left(-2 \cot \frac{x}{2} \right) dx \right\} - \int \cot \frac{x}{2} dx + C$

$= -x \cot \frac{x}{2} + \int \cot \frac{x}{2} dx - \int \cot \frac{x}{2} dx + C$

$= -x \cot \frac{x}{2} + C.$

(iv) $\int \log (1 + x^2) \, dx$

$= \int \underset{\text{I}}{\log (1 + x^2)} \cdot \underset{\text{II}}{1} \, dx$

$= x \log (1 + x^2) - \int \left(\frac{1}{1 + x^2} \cdot 2x \right) x \, dx$

$= x \log(1+x^2) - 2\int \frac{x^2}{x^2+1}\,dx$

$= x \log(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,dx$

$= x \log(x^2+1) - 2\int 1 - \frac{1}{x^2+1}\,dx$

$= x \log(x^2+1) - 2\{x - \tan^{-1} x\} + C$

$= x \log(x^2+1) - 2x + 2\tan^{-1} x + C.$

EXAMPLE 5 *Evaluate :*

(i) $\int \sec^3 x\,dx$ (ii) $\int \sin\sqrt{x}\,dx$

(iii) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$ (iv) $\int (\sin^{-1} x)^2\,dx.$

SOLUTION (i) We have,

$\int \sec^3 x\,dx$

$= \int \underset{\text{I}}{\sec x} \cdot \underset{\text{II}}{\sec^2 x}\,dx$

$= \sec x\left(\int \sec^2 x\,dx\right) - \int \left\{\left(\frac{d}{dx}(\sec x)\right)\left(\int \sec^2 x\,dx\right)\right\} dx$

$= \sec x \tan x - \int \sec x \tan x \tan x\,dx$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$\therefore \int \sec^3 x\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \log|\sec x + \tan x| + C.$

$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \log|\sec x + \tan x| + C$

$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\sec\tan x + \frac{1}{2}\log|\sec x + \tan x| + C.$

(ii) Let $I = \int \sin\sqrt{x}\,dx.$

Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}\,dx = dt$ or $dx = 2\sqrt{x}\,dt$

$\therefore \int \sin\sqrt{x}\,dx$

$= \int \sin t\, 2t\,dt$

$= 2\int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\,dt$

$= 2\left[t(-\cos t) - \int 1 \cdot (-\cos t)\cdot dt\right]$

$= 2\left[-t\cos t + \int \cos t\,dt\right]$

$= 2[-t\cos t + \sin t] + C$

$= -2[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}] + C.$

(iii) Put $x = \sin t$ so that $dx = \cos t\,dt$ and $t = \sin^{-1} x.$

$\therefore \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$

$= \int \frac{t}{(1-\sin^2 t)^{3/2}} \cdot \cos t\,dt$

$= \int \underset{\text{I}}{t} \cdot \underset{\text{II}}{\sec^2 t}\,dt$

$= t \cdot \tan t - \int 1 \cdot \tan t\,dt$

$= t\tan t + \log|\cos t| + C$

$= t \cdot \frac{\sin t}{\sqrt{1-\sin^2 t}} + \log|\sqrt{1-\sin^2 t}| + C$

$= (\sin^{-1} x) \cdot \frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C$

$= \frac{x}{\sqrt{1-x^2}}\sin^{-1} x + \frac{1}{2}\log|1-x^2| + C.$

(iv) Put $\sin^{-1} x = t$, or $x = \sin t$ so that $dx = \cos t\,dt$

$\therefore \int (\sin^{-1} x)^2\,dx$

$= \int \underset{\text{I}}{t^2} \underset{\text{II}}{\cos t}\,dt$

$= t^2(\sin t) - \int 2t \sin t\,dt$

$= t^2 \sin t - 2\int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\,dt$

$= t^2 \sin t - 2\left[t(-\cos t) - \int 1 \cdot (-\cos t)\,dt\right]$

$= t^2 \sin t - 2\left[-t\cos t + \int \cos t\,dt\right]$

$= t^2 \sin t - 2[-t\cos t + \sin t] + C.$

$= t^2 \sin t - 2[-t\sqrt{1-\sin^2 t} + \sin t] + C$

$= x(\sin^{-1} x)^2 - 2[-\sin^{-1} x \cdot \sqrt{1-x^2} + x] + C.$

EXAMPLE 6 *Evaluate :* $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$

SOLUTION We have,

$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$

$= \int \frac{\sin^{-1}\sqrt{x} - \left\{\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right\}}{\frac{\pi}{2}}\,dx \quad \left[\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right]$

$= \frac{2}{\pi}\int \left\{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right\} dx$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int 1\cdot dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x + C \qquad ...(i)$$

Now,

$$\int \sin^{-1}\sqrt{x}\,dx$$

$$= \int_{\text{I}\ \text{II}} \theta \sin 2\theta\,d\theta, \quad \text{where } x = \sin^2\theta$$

$$= -\theta\frac{\cos 2\theta}{2} + \int \frac{1}{2}\cos 2\theta\,d\theta$$

$$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x} \qquad ...(ii)$$

From (i) and (ii), we get

$$I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right] - x + C.$$

$$= \frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}] - x + C.$$

EXAMPLE 7 *Evaluate :*

(i) $\int \cot^{-1}(1-x+x^2)\,dx$

(ii) $\int \tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\sec^2\theta\,d\theta$

SOLUTION We have,

$$\int \cot^{-1}(1-x+x^2)\,dx$$

$$= \int \cot^{-1}\{1-x(1-x)\}\,dx$$

$$= \int \tan^{-1}\left\{\frac{1}{1-x(1-x)}\right\}dx$$

$$= \int \tan^{-1}\left\{\frac{x+(1-x)}{1-x(1-x)}\right\}dx$$

$$= \int \{\tan^{-1}x + \tan^{-1}(1-x)\}\,dx$$

$$= \int \tan^{-1}x\,dx + \int \tan^{-1}(1-x)\,dx$$

$$= I_1 + I_2, \text{ where} \qquad ...(i)$$

$I_1 = \int \tan^{-1}x\,dx$ and $I_2 = \int \tan^{-1}(1-x)\,dx.$

Now,

$$I_1 = \int \tan^{-1}x\,dx$$

$$= \int_{\text{I}\ \text{II}} \tan^{-1}x\ 1\,dx$$

$$= x\tan^{-1}x - \int \frac{x}{1+x}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{1+x^2}d(1+x^2)$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) \qquad ...(ii)$$

and,

$$I_2 = \int \tan^{-1}(1-x)\,dx$$

$$= -\int \tan^{-1}(1-x)\,d(1-x)$$

$$= -\left[(1-x)\tan^{-1}(1-x) - \frac{1}{2}\log\{1+(1-x)^2\}\right] \text{ [Using (ii)]}$$

Substituting the values of I_1 and I_2 in (i), we get

$$\int \cot^{-1}(1-x+x^2)\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

$$-(1-x)\tan^{-1}(1-x) + \frac{1}{2}\log\{1+(1-x)^2\} + C$$

(ii) Let $I = \int \tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right).\sec^2\theta\,d\theta$. Then,

$$I = \int \tan^{-1}\left(\frac{1+\cos 2\theta}{2-\sin 2\theta}\right)\sec^2\theta\,d\theta$$

$$= \int \tan^{-1}\left(\frac{1}{1-\tan\theta+\tan^2\theta}\right)\sec^2\theta\,d\theta$$

$$= \int \tan^{-1}\left(\frac{1}{1-t+t^2}\right)dt, \quad \text{where } t = \tan\theta$$

$$= \int \tan^{-1}\left\{\frac{t+(1-t)}{1-t(1-t)}\right\}dt$$

$$= \int \{\tan^{-1}t + \tan^{-1}(1-t)\}\,dt$$

$$= t\tan^{-1}t - \frac{1}{2}\log(1+t^2)$$

$$-(1-t)\tan^{-1}(1-t) + \frac{1}{2}\log\{1+(1-t)^2\} + C \quad \text{[Using (i)]}$$

$$= \theta\tan\theta + \log\cos\theta - (1-\tan\theta)\tan^{-1}(1-\tan\theta)$$

$$+\frac{1}{2}\log(2-2\tan\theta+\tan^2\theta) + C$$

EXAMPLE 8 *Evaluate :* $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

SOLUTION Putting $x = a\tan^2\theta$ and $dx = 2a\tan\theta\sec^2\theta\,d\theta$, we get

$$\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

$$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\ 2a\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\tan\theta\sec^2\theta}\,d\theta$$

$$= 2a\left[\theta\frac{\tan^2\theta}{2}-\int\frac{\tan^2\theta}{2}\,d\theta\right]+C$$

$$= 2a\left[\frac{\theta}{2}\tan^2\theta-\frac{1}{2}\int(\sec^2\theta-1)\,d\theta\right]+C$$

$$= 2a\left[\frac{\theta}{2}\tan^2\theta-\frac{1}{2}(\tan\theta-\theta)\right]+C$$

$$= a\left[\theta\tan^2\theta-\tan\theta+\theta\right]+C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]+C$$

ALITER Putting $x = a\tan^2\theta$ in $\sin^{-1}\sqrt{\frac{x}{a+x}}$, we get

$$\sin^{-1}\sqrt{\frac{x}{a+x}} = \sin^{-1}(\sin\theta) = \theta = \tan^{-1}\sqrt{\frac{x}{a}}$$

$$\therefore \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

$$= \int \tan^{-1}\sqrt{\frac{x}{a}}\,dx$$

$$= 2a\int \underset{\text{II}}{t}\ \underset{\text{I}}{\tan^{-1}t}\ dt, \text{ where } \frac{x}{a}=t^2$$

$$= 2a\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\int\frac{t^2}{t^2+1}\,dt\right]$$

$$= a\left[t^2\tan^{-1}t-(t-\tan^{-1}t)\right]+C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]+C$$

EXAMPLE 9 *Evalute :* $\int \frac{\sqrt{x^2+1}\,\{\log(x^2+1)-2\log x\}}{x^4}\,dx$

SOLUTION We have,

$$I=\int\sqrt{x^2+1}\,\frac{\{\log(x^2+1)-2\log x\}}{x^4}\,dx$$

$$= \int\frac{\sqrt{x^2+1}\,\{\log(x^2+1)-\log x^2\}}{x^4}\,dx$$

$$= \int\frac{\sqrt{x^2+1}\,\log\left(1+\frac{1}{x^2}\right)}{x^4}\,dx$$

$$= \int\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)\frac{1}{x^3}\,dx$$

$$= -\frac{1}{2}\int\sqrt{t}\log t\,dt\ ,\quad \text{where}\quad t=1+\frac{1}{x^2}$$

$$= -\frac{1}{2}\int \underset{\text{II}}{\sqrt{t}}\ \underset{\text{I}}{\log t}\,dt$$

$$= -\frac{1}{2}\left[\frac{2}{3}(\log t)\,t^{3/2}-\frac{2}{3}\int\frac{1}{t}.t^{3/2}\,dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log t-\frac{4}{9}t^{3/2}\right]+C$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\log\left(1+\frac{1}{x^2}\right)+\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C$$

EXAMPLE 10 *Evaluate :* $\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx$

SOLUTION We have,

$$I=\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx$$

$$= \int x\sec x.\frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$$

$$= \int \underset{\text{I}}{x\sec x}.\underset{\text{II}}{\frac{d(x\sin x+\cos x)}{(x\sin x+\cos x)^2}}$$

$$= x\sec x\times-\frac{1}{x\sin x+\cos x}$$

$$-\int(x\sec x\tan x+\sec x)\times\frac{-1}{x\sin x+\cos x}\,dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x}+\int\sec^2 x\,dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x}+\tan x+C$$

EXAMPLE 11 *Evaluate :* $\int\frac{\cos^2 x+\sin 2x}{(2\cos x-\sin x)^2}\,dx$

SOLUTION We have,

$$I=\int\frac{\cos^2 x+\sin 2x}{(2\cos x-\sin x)^2}\,dx$$

$$= \int\frac{\cos x(2\sin x+\cos x)}{(2\cos x-\sin x)^2}$$

$$= -\int \underset{\text{I}}{\cos x}\ \underset{\text{II}}{\frac{d(2\cos x-\sin x)}{(2\cos x-\sin x)^2}}$$

$$= -\left[\cos x\times\frac{-1}{(2\cos x-\sin x)}-\int(-\sin x)\,\frac{-1}{(2\cos x-\sin x)}\,dx\right]$$

$$= \frac{\cos x}{2\cos x-\sin x}+\int\frac{\sin x}{2\cos x-\sin x}\,dx$$

$$= \frac{\cos x}{2\cos x-\sin x}+I_1\ ,\ \text{where}$$

$$I_1=\int\frac{\sin x}{2\cos x-\sin x}\,dx.$$

To evaluate the integral I_1, let us express $\sin x$ in terms of the denominator $2\cos x - \sin x$ and its derivative. So, let

$\sin x = K(2\cos x - \sin x) + L(-2\sin x - \cos x)$

On equating the coefficient of $\sin x$ and $\cos x$, we get

$1 = -K - 2L$ and $0 = 2K - L$

$\Rightarrow K = \frac{-1}{5}$ and $L = \frac{-2}{5}$

$$\therefore I_1 = \int \frac{K(2\cos x - \sin x) + L(-2\sin x - \cos x)}{2\cos x - \sin x}\, dx$$

$$\Rightarrow I_1 = \int K\, dx + L \int \frac{-2\sin x - \cos x}{2\cos x - \sin x}\, dx$$

$$\Rightarrow I_1 = Kx + L \log |2\cos x - \sin x|$$

$$\Rightarrow I_1 = \frac{-1}{5}x - \frac{2}{5}\log |2\cos x - \sin x|$$

Substituting the value of I_1, we get

$$I = \frac{\cos x}{2\cos x - \sin x} - \frac{1}{5}x - \frac{2}{5}\log |2\cos x - \sin x| + C$$

EXAMPLE 12 *Evaluare :* $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$

SOLUTION We have,

$$I = \int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$$

$$= \int e^{\sin x} x\cos x\, dx - \int e^{\sin x}\tan x \sec x\, dx$$

$$= \int_{\text{I}\ \text{II}} x\, e^{\sin x}\, d(\sin x) - \int_{\text{I}} e^{\sin x}\, d(\sec x)_{\text{II}}$$

$$= \left\{x e^{\sin x} - \int e^{\sin x}\, dx\right\} - \left\{e^{\sin x}\sec x - \int e^{\sin x}\cos x \cdot \sec x\, dx\right\}$$

$$= x e^{\sin x} - \int e^{\sin x}\, dx - e^{\sin x}\cdot \sec x + \int e^{\sin x}\, dx + C$$

$$= x e^{\sin x} - e^{\sin x}\sec x + C$$

$$= e^{\sin x}(x - \sec x) + C$$

EXAMPLE 13 *Evaluate :* $\int \sec^2 x \log(1 + \sin^2 x)\, dx$

SOLUTION We have,

$$I = \int_{\text{II}} \sec^2 x\ \log\ (1 + \sin^2 x)_{\text{I}}\, dx$$

$$= \log(1 + \sin^2 x)\ \tan x - \int \frac{2\sin x \cos x \tan x}{1 + \sin^2 x} dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2\int \frac{\sin^2 x}{1 + \sin^2 x}\, dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2\int \frac{1 + \sin^2 x - 1}{1 + \sin^2 x}\, dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2\int \left(1 - \frac{1}{1 + \sin^2 x}\right)dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2x + 2\int \frac{1}{1 + \sin^2 x}\, dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2x + 2\int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\, dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2x + 2\int \frac{\sec^2 x}{1 + 2\tan^2 x}\, dx$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2x + \sqrt{2}\int \frac{1}{1^2 + (\sqrt{2}\tan x)^2}\, d(\sqrt{2}\tan x)$$

$$= \tan x \cdot \log(1 + \sin^2 x) - 2x + \sqrt{2}\tan^{-1}(\sqrt{2}\tan x) + C.$$

EXAMPLE 14 *Evaluate :* $\int \cos 2x \log(1 + \tan x)\, dx$

SOLUTION We have,

$$I = \int_{\text{II}} \cos 2x\ \log\ (1 + \tan x)_{\text{I}}\, dx$$

$$= \log(1 + \tan x)\cdot \frac{\sin 2x}{2} - \int \frac{1}{1 + \tan x}\sec^2 x \cdot \frac{\sin 2x}{2}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - \int \frac{\tan x}{1 + \tan x}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - \int \left(1 - \frac{1}{1 + \tan x}\right)dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - \int dx + \int \frac{1}{1 + \tan x}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - x + \int \frac{\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - x$$

$$+ \frac{1}{2}\int \frac{(\sin x + \cos x) + (\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - x + \frac{1}{2}\int 1 + \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\sin 2x \log(1 + \tan x) - x + \frac{1}{2}\{x + \log |\sin x + \cos x|\} + C$$

EXAMPLE 15 *Evaluate :* $\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\, dx$

SOLUTION We have,

$$I = \int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\, dx$$

$$= \int_{\text{I}} x^2 \cdot \frac{d(x\tan x + 1)}{(x\tan x + 1)^2}_{\text{II}}\, dx$$

$$= x^2 \times \frac{-1}{x\tan x + 1} - 2\int x \times \frac{-1}{(x\tan x + 1)}\, dx$$

$$= \frac{-x^2}{x\tan x + 1} + 2\int \frac{x\cos x}{x\sin x + \cos x}\, dx$$

$$= \frac{-x^2}{x\tan x+1} + 2\int \frac{d(x\sin x+\cos x)}{x\sin x+\cos x}$$

$$= \frac{-x^2}{x\tan x+1} + 2\log|x\sin x+\cos x| + C.$$

EXAMPLE 16 *Evaluate:* $\int x\sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right\}dx$

SOLUTION Putting $x = 2a\cos\theta$ and $dx = -2a\sin\theta\, d\theta$, we get

$$\int x\sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right\}dx$$

$$= \int 2a\cos\theta\sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{2a-2a\cos\theta}{a}}\right\}(-2a\sin\theta)\,d\theta$$

$$= -4a^2\int \cos\theta\sin\theta\sin^{-1}\left(\sin\frac{\theta}{2}\right)d\theta$$

$$= -a^2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin 2\theta}\ d\theta$$

$$= -a^2\left[-\theta\frac{\cos 2\theta}{2} + \int \frac{\cos 2\theta}{2}\,d\theta\right]$$

$$= -a^2\left[-\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta\right] + C$$

$$= \frac{a^2}{2}\theta\,(2\cos^2\theta - 1) - \frac{a^2}{2}\sin\theta\cos\theta + C$$

$$= \frac{a^2}{2}\left(\frac{x^2}{2a^2}-1\right)\cos^{-1}\left(\frac{x}{2a}\right) - \frac{ax}{4}\sqrt{1-\frac{x^2}{4a^2}} + C$$

EXAMPLE 17 *Evaluate:* $\int \cos 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ **[IIT 1994]**

SOLUTION We have,

$$\int \cos 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

$$= \frac{1}{2}\int \cos 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int \cos 2\theta\log\left(\frac{1+\sin\theta}{1-\sin 2\theta}\right)d\theta$$

$$= \frac{1}{4}\int \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right).2\cos 2\theta\,d\theta$$

$$= \frac{1}{4}\int \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)d(\sin 2\theta)$$

$$= \frac{1}{4}\int \log\left(\frac{1+t}{1-t}\right)dt$$

$$= \frac{1}{4}\int \{\log(1+t) - \log(1-t)\}\,dt$$

$$= \frac{1}{4}\left[\int \{\log(1+t) - \log(1-t)\}\,t - \int\left\{\frac{1}{1+t}+\frac{1}{1-t}\right\}t\,dt\right]$$

$$= \frac{1}{4}\left[t\log\left(\frac{1+t}{1-t}\right) + \int \frac{-2t}{1-t^2}\,dt\right]$$

$$= \frac{1}{4}\left[t\log\left(\frac{1+t}{1-t}\right) + \log(1-t^2)\right] + C$$

$$= \frac{1}{4}\left[\sin 2\theta\log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right) + \log(1-\sin^2 2\theta)\right] + C$$

$$= \frac{1}{4}\sin 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^2 + \frac{1}{4}\log\cos^2 2\theta + C$$

$$= \frac{1}{2}\sin 2\theta\log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| + \frac{1}{2}\log\cos 2\theta + C$$

EXAMPLE 18 *Evaluate:* $\int (\log x)\sin^{-1}x\,dx$

SOLUTION We have,

$$\int \log x\,dx = x\log x - x + C$$

$$\therefore\ I = \int \underset{\text{II}}{\log x}\ \underset{\text{I}}{\sin^{-1}x}\,dx$$

$$= (\sin^{-1}x)(x\log x - x) - \int \frac{(x\log x - x)}{\sqrt{1-x^2}}\,dx$$

$$= (\sin^{-1}x)(x\log x - x) - \int \frac{x}{\sqrt{1-x^2}}\log x\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= (\sin^{-1}x)(x\log x - x) + \frac{1}{2}\int \underset{\text{I}}{\log x}\ \underset{\text{II}}{\left(\frac{-2x}{\sqrt{1-x^2}}\right)}dx - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$= (\sin^{-1}x)(x\log x - x) + \frac{1}{2}\left\{(\log x).2\sqrt{1-x^2} - \int \frac{1}{x}.2\sqrt{1-x^2}\,dx\right\}$$

$$-\frac{1}{2}\times 2\sqrt{1-x^2}$$

$$= (\sin^{-1}x)(x\log x - x) + (\log x)\sqrt{1-x^2} - \int \frac{\sqrt{1-x^2}}{x}dx - \sqrt{1-x^2}$$

$$= (\sin^{-1}x)(x\log x - x) + \sqrt{1-x^2}.\log x - \sqrt{1-x^2} - \int \frac{\sqrt{1-x^2}}{x}\,dx$$

$$= (\sin^{-1}x)(x\log x - x) + \sqrt{1-x^2}.\log x - \sqrt{1-x^2} - I_1 \quad \text{...(i)}$$

where, $I_1 = \int \frac{\sqrt{1-x^2}}{x}\,dx$

Putting $x = \sin\theta$ and $dx = \cos\theta\,d\theta$, we get

$$I_1 = \int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= \int (\operatorname{cosec}\theta - \sin\theta)\,d\theta$$

$$= \log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta$$

$$= \log\left|\frac{1-\cos\theta}{\sin\theta}\right| + \cos\theta$$

$$= \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$I = (\sin^{-1}x)(x\log x - x) + \sqrt{1-x^2}\,.\log x - \sqrt{1-x^2} - \log\left\{\frac{1-\sqrt{1-x^2}}{x}\right\} - \sqrt{1-x^2} + C$$

$$= (\sin^{-1}x)(x\log x - x) + \sqrt{1-x^2}\,.\log x - 2\sqrt{1-x^2} - \log\left\{\frac{1-\sqrt{1-x^2}}{x}\right\} + C$$

EXAMPLE 19 *Evaluate :* $\int \sin 2x \tan^{-1}(\sin x)\,dx$

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$\int \sin 2x \tan^{-1}(\sin x)\,dx$$

$$= 2\int \underset{\text{II}}{t}\ \underset{\text{I}}{\tan^{-1}t}\,dt$$

$$= 2\left[(\tan^{-1}t)\frac{t^2}{2} - \int \frac{1}{1+t^2}\cdot\frac{t^2}{2}\,dt\right]$$

$$= t^2(\tan^{-1}t) - \int \frac{t^2+1-1}{1+t^2}\,dt$$

$$= t^2\tan^{-1}t - t + \tan^{-1}t + C$$

$$= \sin^2 x\,.\tan^{-1}(\sin x) - \sin x + \tan^{-1}(\sin x) + C$$

EXAMPLE 20 *Evaluate :* $\int \tan^{-1}\left\{\sqrt{\sqrt{x}-1}\right\}dx$

SOLUTION Putting $x = \sec^4\theta$ and $dx = 4\sec^4\theta\tan\theta$, we have

$$I = \int \tan^{-1}(\tan\theta)\,.\,4\sec^4\theta\tan\theta\,d\theta$$

$$= 4\int \theta\sec^4\theta\tan\theta\,d\theta$$

$$= 4\int \theta\sec^3\theta\sec\theta\tan\theta\,d\theta$$

$$= 4\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^3\theta}\,d(\sec\theta)$$

$$= 4\left\{\theta\,.\frac{\sec^4\theta}{4} - \frac{1}{4}\int \sec^4\theta\,d\theta\right\}$$

$$= \theta\sec^4\theta - \int \sec^4\theta\,d\theta$$

$$= \theta\sec^4\theta - \int (1+\tan^2\theta)\sec^2\theta\,d\theta$$

$$= \theta\sec^4\theta - \tan\theta - \frac{\tan^3\theta}{3} + C$$

$$= \theta\sec^4\theta - \sqrt{\sec^2\theta - 1} - \frac{1}{3}(\sec^2\theta - 1)^{3/2} + C$$

$$= x\tan^{-1}(\sqrt{\sqrt{x}-1}) - \sqrt{\sqrt{x}-1} - \frac{1}{3}(\sqrt{x}-1)^{3/2} + C$$

EXAMPLE 21 *Evaluate :* $\int \left\{\log(1+\cos x) - x\tan\frac{x}{2}\right\}dx$

SOLUTION We have,

$$\int \left\{\log(1+\cos x) - x\tan\frac{x}{2}\right\}dx$$

$$= \int \underset{\text{II}}{1}\,.\,\underset{\text{I}}{\log(1+\cos x)}\,dx - \int x\tan\frac{x}{2}\,dx$$

$$= x\log(1+\cos x) - \int \frac{x(-\sin x)}{1+\cos x}\,dx - \int x\tan\frac{x}{2}\,dx$$

$$= x\log(1+\cos x) + \int x\tan\frac{x}{2}\,dx - \int x\tan\frac{x}{2}\,dx + C$$

$$= x\log(1+\cos x) + C$$

EXAMPLE 22 *Evaluate :* $\int \frac{x\,e^x}{\sqrt{1+e^x}}\,dx$

SOLUTION We have,

$$I = \int \frac{x\,e^x}{\sqrt{1+e^x}}\,dx$$

$$= \int \underset{\text{I}}{x}\,.\,\underset{\text{II}}{\frac{d(1+e^x)}{\sqrt{1+e^x}}}$$

$$= x\,.\,2\sqrt{1+e^x} - 2\int \sqrt{1+e^x}\,dx$$

$$= 2x\sqrt{1+e^x} - 4\int \frac{t^2}{t^2-1}\,dt, \text{ where } 1+e^x = t^2$$

$$= 2x\sqrt{1+e^x} - 4\int\left(1 + \frac{1}{t^2-1}\right)dt$$

$$= 2x\sqrt{1+e^x} - 4t - 2\log\left|\frac{t-1}{t+1}\right| + C$$

Hence,

$$I = 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$$

EXAMPLE 23 *Evaluate :* $\int \frac{x\log x}{\sqrt{(x^2-1)^3}}\,dx$

SOLUTION We have,

$$I = \int \frac{x\log x}{\sqrt{(x^2-1)^3}}\,dx$$

$$= \int \log x\,.\,\frac{x}{(x^2-1)^{3/2}}\,dx$$

$$= \frac{1}{2}\int \log x\,.\,\frac{2x}{(x^2-1)^{3/2}}\,dx$$

$$= \frac{1}{2}\int \underset{\text{I}}{\log x}\,(x^2-1)^{-3/2}\,d(x^2-1)$$

$$= \frac{1}{2}\left[-2(\log x)(x^2-1)^{-1/2} - \int \frac{1}{x}\times -2(x^2-1)^{-1/2}\,dx\right]$$

$$= -\frac{\log x}{\sqrt{x^2-1}} + \int \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \frac{-\log x}{\sqrt{x^2-1}} + \sec^{-1} x + C$$

EXAMPLE 24 *Evaluate :* $\int \sqrt{x^6+1}\left|\frac{\log(x^6+1) - 6\log x}{x^{10}}\right| dx$

SOLUTION We have,

$$I = \int \sqrt{x^6+1}\left|\frac{\log(x^6+1) - 6\log x}{x^{10}}\right| dx$$

$$= \int \sqrt{x^6+1}\left|\frac{\log(x^6+1) - \log x^6}{x^{10}}\right| dx$$

$$= \int \sqrt{x^6+1}\,\frac{\log\left(1+\frac{1}{x^6}\right)}{x^{10}}\,dx$$

$$= \int \sqrt{1+\frac{1}{x^6}}\,\log\left(1+\frac{1}{x^6}\right).\frac{1}{x^7}\,dx$$

$$= -\frac{1}{6}\int \underset{\text{II}}{\sqrt{t}}\,\underset{\text{I}}{\log t}\,dt, \quad \text{where } 1+\frac{1}{x^6} = t$$

$$= -\frac{1}{6}\left[\frac{2}{3}t^{3/2}\log t - \frac{2}{3}\int \frac{t^{3/2}}{t}\,dt\right]$$

$$= -\frac{1}{6}\left[\frac{2}{3}t^{3/2}\log t - \frac{4}{9}t^{3/2}\right] + C$$

$$= -\frac{1}{9}\left(1+\frac{1}{x^6}\right)^{3/2}\log\left(1+\frac{1}{x^6}\right) + \frac{2}{9}\left(1+\frac{1}{x^6}\right)^{3/2} + C$$

EXAMPLE 25 *If $f(x)$ is a polynomial function of the nth degree prove that*

$$\int e^x f(x)\,dx = e^x\left\{f(x) - f'(x) + f''(x) - f'''(x) + \ldots + (-1)^n f^n(x)\right\}$$

, where $f^n(x) = \frac{d^n f(x)}{dx^n}$.

SOLUTION We have,

$$\int \underset{\text{II}}{e^x}\,\underset{\text{I}}{f(x)}\,dx$$

$$= e^x f(x) - \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{f'(x)}\,dx$$

$$= e^x f(x) - \left\{e^x f'(x) - \int e^x f''(x)\,dx\right\}$$

$$= e^x f(x) - e^x f'(x) + \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{f''(x)}\,dx$$

$$= e^x f(x) - e^x f'(x) + e^x f''(x) - \left\{e^x f'''(x) - \int f^{IV}(x)\,e^x\,dx\right\}$$

$$= e^x f(x) - e^x f'(x) + e^x f''(x) - e^x f'''(x) + \int e^x f^{IV}(x)\,dx$$

$$= e^x\left\{f(x) - f'(x) + f''(x) - f'''(x)\right\} + \int e^x f^{IV}(x)\,dx$$

Continuing in this manner, we get

$$\int e^x f(x)\,dx = e^x\left\{f(x) - f'(x) + f''(x) - f'''(x) + \ldots + (-1)^n f^n(x)\right\}$$

EXAMPLE 26 *Evaluate :* $\int\left(3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right)dx$

SOLUTION We have,

$$\int\left\{3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right\}dx$$

$$= \int \underset{\text{II}}{3x^2}\,\underset{\text{I}}{\tan\frac{1}{x}}\,dx - \int x\sec^2\frac{1}{x}\,dx$$

$$= x^3\tan\left(\frac{1}{x}\right) - \int\left(\sec^2\frac{1}{x}\right)\times -\frac{1}{x^2}\times x^3\,dx - \int x\sec^2\frac{1}{x}\,dx$$

$$= x^3\tan\frac{1}{x} + \int x\sec^2\frac{1}{x}\,dx - \int x\sec^2\frac{1}{x}\,dx + C$$

$$= x^3\tan\left(\frac{1}{x}\right) + C$$

EXAMPLE 27 *Evaluate :*

$$\int e^{x\sin x+\cos x}\cdot\left|\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right| dx$$

SOLUTION We have,

$$\int e^{x\sin x+\cos x}\cdot\left|\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right| dx$$

$$= \int e^{x\sin x+\cos x}.x^2\cos x\,dx + \int e^{x\sin x+\cos x}\,\frac{\cos x - x\sin x}{(x\cos x)^2}\,dx$$

$$= \int \underset{\text{I}}{x}\,\underset{\text{II}}{e^{x\sin x+\cos x}}\,d(x\sin x+\cos x) + \int \underset{I}{e^{x\sin x+\cos x}}\,\underset{\text{II}}{\frac{d(x\cos x)}{(x\cos x)^2}}$$

$$= x\,e^{x\sin x+\cos x} - \int e^{x\sin x+\cos x}\,dx + e^{x\sin x+\cos x}\times\frac{-1}{x\cos x}$$

$$- \int e^{x\sin x+\cos x}.x\cos x\times\frac{-1}{x\cos x}\,dx + C$$

$$= x\,e^{x\sin x+\cos x} - \frac{e^{x\sin x+\cos x}}{x\cos x} + C$$

EXAMPLE 28 *Evaluate :* $\int \frac{1}{(x^2+a^2)^2}\,dx$

SOLUTION We have,

$$\int \frac{1}{(x^2+a^2)}\,dx$$

$$= \int \underset{\text{I}}{\frac{1}{x^2+a^2}}.\underset{\text{II}}{1}\,dx$$

$$= \frac{x}{x^2+a^2} + \int \frac{2x.x}{(x^2+a^2)^2}\,dx$$

$$= \frac{x}{x^2+a^2} + 2\int \frac{x^2}{(x^2+a^2)^2}\,dx$$

$$= \frac{x}{x^2+a^2} + 2\int \frac{x^2+a^2-a^2}{(x^2+a^2)^2}\,dx$$

$$= \frac{x}{x^2+a^2} + 2\int \frac{1}{x^2+a^2}\,dx - 2a^2\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$\therefore\ 2a^2\int \frac{1}{(x^2+a^2)^2}\,dx = \frac{x}{x^2+a^2} + \int \frac{1}{x^2+a^2}\,dx$$

$$= \frac{x}{x^2+a^2} + \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\Rightarrow \int \frac{1}{(x^2+a^2)^2}\,dx = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)$$

Hence, $\int \frac{1}{(x^2+a^2)^2}\,dx = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C$

EXAMPLE 29 *Evaluate :* $\int \frac{1}{(x^2+a^2)^3}\,dx$

SOLUTION Let $I = \int \frac{1}{(x^2+a^2)^3}\,dx$.

We have,

$$\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$= \int \underset{\text{I}}{\frac{1}{(x^2+a^2)^2}} \cdot \underset{\text{II}}{1}\,dx$$

$$= \frac{x}{(x^2+a^2)^2} + 4\int \frac{x^2}{(x^2+a^2)^3}\,dx$$

$$= \frac{x}{(x^2+a^2)^2} + 4\int \frac{x^2+a^2-a^2}{(x^2+a^2)^3}\,dx$$

$$= \frac{x}{(x^2+a^2)^2} + 4\int \frac{1}{(x^2+a^2)^2}\,dx - 4a^2\int \frac{1}{(x^2+a^2)^3}\,dx$$

$$\therefore\ 4a^2\int \frac{1}{(x^2+a^2)^3}\,dx = \frac{x}{(x^2+a^2)^2} + 3\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$\Rightarrow \int \frac{1}{(x^2+a^2)^3}\,dx = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$= \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}\left\{\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{2a}\right\} + C$$

[Using Ex. 28]

$$= \frac{x}{4a^2(x^2+a^2)^2} + \frac{3x}{8a^4(x^2+a^2)} + \frac{3}{8a^5}\tan^{-1}\frac{x}{2a} + C$$

EXAMPLE 30 *Evaluate :* $\int \frac{b+a\sin x}{(a+b\sin x)^2}\,dx$

SOLUTION We have,

$$\int \frac{b+a\sin x}{(a+b\sin x)^2}\,dx$$

$$= \int \frac{b(\cos^2 x+\sin^2 x)+a\sin x}{(a+b\sin x)^2}\,dx$$

$$= \int \frac{b\cos^2 x}{(a+b\sin x)^2}\,dx + \int \frac{\sin x}{a+b\sin x}\,dx$$

$$= \int \cos x\,\frac{b\cos x}{(a+b\sin x)^2}\,dx + \int \frac{\sin x}{a+b\sin x}\,dx$$

$$= \int \cos x\,\frac{d(a+b\sin x)}{(a+b\sin x)^2} + \int \frac{\sin x}{a+b\sin x}\,dx$$

$$= -\frac{\cos x}{a+b\sin x} - \int \frac{\sin x}{(a+b\sin x)}\,dx + \int \frac{\sin x}{(a+b\sin x)}\,dx + C$$

$$= -\frac{\cos x}{a+b\sin x} + C.$$

EXAMPLE 31 *Evaluate :* $\int \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\}dx$ **[IIT 2001]**

SOLUTIONS We have,

$$I = \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\}dx$$

$$\Rightarrow \sin^{-1}\left\{\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right\}dx$$

Substituting $2x+2 = 3\tan\theta$ and $dx = \frac{3}{2}\sec^2\theta\,d\theta$, we get

$$I = \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\times\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\,d\theta$$

$$= \frac{3}{2}\left[\theta\tan\theta - \int \tan\theta\,d\theta\right]$$

$$= \frac{3}{2}\left[\theta\tan\theta - \log|\sec\theta|\right]$$

$$= \frac{3}{2}\left[\left(\frac{2x+2}{3}\right)\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right] + C$$

$$= \frac{3}{2}\left[\left(\frac{2x+2}{3}\right)\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\sqrt{4x^2+8x+13}\right] + C$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$$

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10.17 INTEGRALS OF THE FORM $\int e^x\{f(x)+f'(x)\}\,dx$

THEOREM Prove that : $\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x) + C$.

PROOF We have, $\int e^x\{f(x)+f'(x)\}\,dx$

$$= \int \underset{\text{II}}{e^x}\ \underset{\text{I}}{f(x)}\,dx + \int e^x f'(x)\,dx$$

$= f(x) \cdot e^x - \int f'(x)\, e^x\, dx + \int e^x f'(x)\, dx + C$

$= e^x f(x) + C.$

Thus, to evaluate integrals of the type $\int e^x \{f(x) + f'(x)\}\, dx$, *we first express the integral as the sum of two integrals* $\int e^x f(x)\, dx$ *and* $\int e^x f'(x)\, dx$ *and then integrate the integral involving* $e^x f(x)$ *as integrand by parts taking* e^x *as the second function.*

NOTE The above theorem is also true if we have e^{kx} in place of e^x and $kf(x)$ in place of $f(x)$.

i.e. $\int e^{kx} \{k f(x) + f'(x)\}\, dx = e^{kx} f(x) + C.$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int e^x (\tan x + \log \sec x)\, dx$

(ii) $\int e^x \left\{ \frac{1 + \sin x \cos x}{\cos^2 x} \right\} dx$

(iii) $\int \left\{ \log (\log x) + \frac{1}{(\log x)^2} \right\} dx$

(iv) $\int e^{2x} \left\{ \frac{1 + \sin 2x}{1 + \cos 2x} \right\} dx$

SOLUTION (i) $\int e^x (\tan x + \log \sec x)\, dx$

$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log \sec x}\, dx + \int e^x \tan x\, dx$

$= (\log \sec x)\, e^x - \int \frac{1}{\sec x} \cdot \sec x \tan x\, e^x\, dx + \int e^x \tan x\, dx + C$

$= e^x \log \sec x - \int e^x \tan x\, dx + \int e^x \tan x\, dx + C$

$= e^x \cdot \log \sec x + C$

(ii) $\int e^x \left\{ \frac{1 + \sin x \cos x}{\cos^2 x} \right\} dx$

$= \int e^x \left\{ \frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} \right\} dx$

$= \int e^x (\sec^2 x + \tan x)\, dx$

$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\tan x}\, dx + \int e^x \sec^2 x\, dx$

$= (\tan x)\, e^x - \int \sec^2 x\, e^x\, dx + \int e^x \sec^2 x\, dx + C$

$= e^x \tan x + C$

(iii) Let $I = \int \left\{ \log (\log x) + \frac{1}{(\log x)^2} \right\} dx.$

$\therefore I = \int \left\{ \log t + \frac{1}{t^2} \right\} e^t\, dt$, where $t = \log x$

$= \int \left\{ \log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right\} e^t\, dt$

$= \int \left\{ \log t + \frac{1}{t} \right\} e^t\, dt + \int \left\{ -\frac{1}{t} + \frac{1}{t^2} \right\} e^t\, dt$

$= \int \underset{\text{II}}{e^t} \underset{\text{I}}{\log t}\, dt + \int e^t \cdot \frac{1}{t}\, dt + \int \underset{\text{II}}{e^t} \underset{\text{I}}{(-1/t)}\, dt + \int e^t \frac{1}{t^2}\, dt$

$= (\log t)\, e^t - \int \frac{1}{t} \cdot e^t\, dt + \int e^t \cdot \frac{1}{t}\, dt + \left(\frac{-1}{t} \right) e^t - \int \frac{1}{t^2} \cdot e^t\, dt + \int e^t \frac{1}{t^2}\, dt + C$

$= e^t \cdot \log t - \frac{1}{t} e^t + C$

$= x \log (\log x) - \frac{x}{\log x} + C$

(iv) $\int e^{2x} \left\{ \frac{1 + \sin 2x}{1 + \cos 2x} \right\} dx$

$= \int e^{2x} \left\{ \frac{1 + 2 \sin x \cos x}{2 \cos^2 x \,.} \right\} dx$

$= \int e^{2x} \left\{ \frac{1}{2} \sec^2 x + \tan x \right\} dx$

$= \int \underset{\text{II}}{e^{2x}} \cdot \underset{\text{I}}{\tan x}\, dx + \frac{1}{2} \int e^{2x} \cdot \sec^2 x\, dx$

$= (\tan x) \frac{e^{2x}}{2} - \int \sec^2 x \cdot \frac{e^{2x}}{2}\, dx + \frac{1}{2} \int e^{2x} \sec^2 x\, dx + C$

$= \frac{1}{2} e^{2x} \tan x \cdot - \frac{1}{2} \int e^{2x} \sec^2 x\, dx + \frac{1}{2} \int e^{2x} \sec^2 x\, dx + C$

$= \frac{1}{2} e^{2x} \tan x + C$

EXAMPLE 2 (i) $\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx$ (ii) $\int e^x \frac{x^3 - x + 2}{(x^2+1)^2}\, dx$

SOLUTION (i) We have,

$I = \int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx$

$= \int e^x \frac{1 + x^2 - 2x}{(1+x^2)^2}\, dx$

$$= \int e^x \left\{ \frac{1}{1+x^2} + \frac{-2x}{(x^2+1)^2} \right\} dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\frac{1}{x^2+1}} dx + \int e^x \left\{ \frac{-2x}{(x^2+1)^2} \right\} dx$$

$$= \frac{1}{x^2+1} \cdot e^x - \int \frac{-2x}{(x^2+1)^2} e^x dx + \int e^x \frac{(-2x)}{(x^2+1)^2} dx + C$$

$$= \frac{e^x}{x^2+1} + C$$

(ii) We have,

$$I = \int e^x \frac{x^3 - x + 2}{(x^2+1)^2} dx$$

$$= \int e^x \frac{x^3 + x - 2x + 2}{(x^2+1)^2} dx$$

$$= \int e^x \frac{x(x^2+1) - 2x + 2}{(x^2+1)^2} dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{2-2x}{(x^2+1)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{1-x^2}{(x^2+1)^2} + \frac{x^2-1}{(x^2+1)^2} + \frac{2-2x}{(x^2+1)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{1-x^2}{(x^2+1)^2} + \frac{x^2-2x+1}{(x^2+1)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{1-x^2}{(x^2+1)^2} + \frac{(x^2+1)-2x}{(x^2+1)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{1-x^2}{(x^2+1)^2} + \frac{1}{x^2+1} + \frac{-2x}{(x^2+1)^2} \right) dx$$

$$= \int e^x \left(\frac{x}{x^2+1} + \frac{1-x^2}{(x^2+1)^2} \right) dx + \int e^x \left(\frac{1}{x^2+1} + \frac{-2x}{(x^2+1)^2} \right) dx$$

$$= e^x \cdot \frac{x}{x^2+1} + e^x \cdot \frac{1}{x^2+1} + C \quad \left[\text{Using} : \int e^x (f(x) + f'(x))\, dx = e^x f(x) \right]$$

$$= e^x \left(\frac{x+1}{x^2+1} \right) + C$$

EXAMPLE 3 *Evaluate* : $\int e^x \frac{2-x^2}{(1-x)\sqrt{1-x^2}} dx$

SOLUTION We have,

$$\int e^x \frac{2-x^2}{(1-x)\sqrt{1-x^2}} dx$$

$$= \int e^x \frac{1+(1-x^2)}{(1-x)\sqrt{1-x^2}} dx$$

$$= \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \right) dx$$

$$= \int e^x \left(\frac{1}{(1-x)^{3/2}\sqrt{1+x}} + \sqrt{\frac{1+x}{1-x}} \right) dx$$

$$= \int e^x \left(\sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} + \sqrt{\frac{1+x}{1-x}} \right) dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sqrt{\frac{1-x}{1-x}}} dx + \int e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} dx$$

$$= \sqrt{\frac{1+x}{1-x}} \cdot e^x - \int e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} dx$$

$$+ \int e^x \sqrt{\frac{1-x}{1+x}} \frac{1}{(1-x)^2} dx + C$$

$$= e^x \sqrt{\frac{1+x}{1-x}} + C$$

EXAMPLE 4 *Evaluate* : $\int e^{2ax} \frac{1-\cos 2ax}{1+\sin 2ax} dx$

SOLUTION Let $I = \int e^{2ax} \frac{1-\cos 2ax}{1+\sin 2ax} dx$

$$I = \frac{1}{a} \int e^{2t} \frac{1-\cos 2t}{1+\sin 2t} dt \text{, where } ax = t$$

$$\Rightarrow I = \frac{1}{a} \int e^{2t} \frac{1-\sin(\pi/2+2t)}{1-\cos(\pi/2+2t)} dt$$

$$\Rightarrow I = \frac{1}{a} \int e^{2t} \frac{1-2\sin(\pi/4+t)\cos(\pi/4+t)}{2\sin^2(\pi/4+t)} dt$$

$$\Rightarrow I = \frac{1}{a} \int e^{2t} \left(\frac{1}{2}\operatorname{cosec}^2(\pi/4+t) - \cot(\pi/4+t) \right) dt$$

$$\Rightarrow I = \frac{1}{2a} \int \underset{\text{I}}{e^{2t}} \underset{\text{II}}{\operatorname{cosec}^2(\pi/4+t)}\, dt - \frac{1}{a} \int e^{2t} \cot(\pi/4+t)\, dt$$

$$- \frac{1}{a} \int e^{2t} \cdot \cot(\pi/4+t)\, dt$$

$$\Rightarrow I = -\frac{1}{2a} e^{2t} \cot(\pi/4+t) + \frac{1}{a} \int e^{2t} \cot(\pi/4+t)\, dt$$

$$- \frac{1}{a} \int e^{2t} \cot(\pi/4+t)\, dt + C$$

$$\Rightarrow I = -\frac{1}{2a} e^{2t} \cot(\pi/4+t) + C$$

$$\Rightarrow I = -\frac{1}{2a} e^{2ax} \cot(\pi/4+ax) + C$$

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10.18 INTEGRALS OF THE FORM

$$\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx$$

In order to evaluae this type of integrals, we use integration by parts as illustrated below.

Let $I = \int e^{ax}\sin bx\,dx$. Then,

$$I = \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\sin bx}\,dx$$

$$= -e^{ax}\frac{\cos bx}{b} - \int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\cos bx}\,dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left\{e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right\}$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I.$$

$$\therefore \quad I + I\cdot\frac{a^2}{b^2} = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

$$\Rightarrow I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

Thus,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

Similarly,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int e^{ax}\cos(bx+c)\,dx$ (ii) $\int \sin(\log x)\,dx$

(iii) $\int e^{x}\cos^2 x\,dx$

SOLUTION (i) Let $I = \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\cos(bx+c)}\,dx$

Intergrating by parts, we have

$$I = \frac{e^{ax}}{b}\sin(bx+c) - \int ae^{ax}\frac{\sin(bx+c)}{b}dx$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\sin(bx+c)}\,dx$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\left[-e^{ax}\frac{\cos(bx+c)}{b} - \int -a\,e^{ax}\frac{\cos(bx+c)}{b}dx\right]$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\left[-\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b}\int e^{ax}\cos(bx+c)\,dx\right]$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\left[-\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b}I\right]$$

$$I = \frac{e^{ax}}{b}\sin(bx+c) + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}I$$

$$\Rightarrow I + \frac{a^2}{b^2}I = \frac{e^{ax}}{b^2}\{b\sin(bx+c) + a\cos(bx+c)\}$$

$$\Rightarrow I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{b^2}\{a\cos(bx+c) + b\sin(bx+c)\}$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\}$$

Hence,

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\} + C_1$$

(ii) Let $I = \int \sin(\log x)\,dx$. Then,

$$I = \int \underset{\text{II}}{\sin t} \underset{\text{I}}{e^{t}}\,dt, \text{ where } t = \log x$$

$$\Rightarrow I = -e^{t}\cos t - \int e^{t}(-\cos t)\,dt$$

$$\Rightarrow I = -e^{t}\cos t + \int \underset{\text{I}}{e^{t}} \underset{\text{II}}{\cos t}\,dt$$

$$\Rightarrow I = -e^{t}\cos t + \left\{e^{t}\sin t - \int e^{t}\sin t\,dt\right\}$$

$$\Rightarrow I = -e^{t}\cos t + e^{t}\sin t - I$$

$$\Rightarrow 2I = e^{t}(\sin t - \cos t)$$

$$\Rightarrow I = \frac{e^{t}}{2}(\sin t - \cos t) + C$$

Hence,

$$\int \sin(\log x)\,dx = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$$

(iii) Let $I = \int e^{x}\cos^2 x\,dx$

$$= \int e^{x}\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int e^x (1+\cos 2x)\, dx$$

$$I = \frac{1}{2}\int e^x\, dx + \frac{1}{2}\int e^x \cos 2x\, dx$$

$$= \frac{1}{2}e^x + \frac{1}{2} I_1 + C, \qquad \ldots(i)$$

where $I_1 = \int e^x \cos 2x\, dx$

Now,

$$I_1 = \int \underset{\text{I}}{e^x} \underset{\text{II}}{\cos 2x}\, dx$$

$$= e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin 2x}{2}\, dx \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}e^x \sin 2x - \frac{1}{2}\int \underset{\text{I}}{e^x} \underset{\text{II}}{\sin 2x}\, dx$$

$$= \frac{1}{2}e^x \sin 2x - \frac{1}{2}\left[-e^x \frac{\cos 2x}{2} - \int e^x \left(\frac{-\cos 2x}{2} \right) dx \right]$$

$$= \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\, dx$$

$$\therefore \quad I_1 = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4} I_1$$

$$\Rightarrow I_1 + \frac{1}{4} I_1 = \frac{1}{4}e^x (\cos 2x + 2\sin 2x)$$

$$\Rightarrow \quad \frac{5}{4} I_1 = \frac{1}{4}e^x (\cos 2x + 2\sin 2x)$$

$$\Rightarrow \quad I_1 = \frac{e^x}{5}(\cos 2x + 2\sin 2x) \qquad \ldots(ii)$$

$$\therefore I = \frac{1}{2}e^x + \frac{e^x}{10}(\cos 2x + 2\sin 2x) + C \qquad \text{[Using (i) and (ii)]}$$

EXAMPLE 2 *If* $I_1 = \int e^{ax} \cos bx\, dx$ *and* $I_2 = \int e^{ax} \sin bx\, dx$, *prove that*

(i) $(a^2+b^2)(I_1^2 + I_2^2) = e^{2ax}$ (ii) $\tan^{-1}\left(\frac{I_2}{I_1}\right) + \tan^{-1}\frac{b}{a} = bx.$

SOLUTION We have,

$$I_1 = \int e^{ax} \cos bx\, dx$$

$$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

and,

$$I_2 = \int e^{ax} \sin bx\, dx$$

$$= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

(i) We have,

$$I_1^2 + I_2^2$$

$$= \frac{e^{2ax}}{(a^2+b^2)^2}\left[(a\cos bx + b\sin bx)^2 + (a\sin bx - b\cos bx)^2\right]$$

$$= \frac{e^{2ax}}{(a^2+b^2)^2}(a^2+b^2) = \frac{e^{2ax}}{a^2+b^2}$$

$$\therefore \quad (a^2+b^2)(I_1^2 + I_2^2) = e^{2ax}$$

(ii) We have,

$$\frac{I_2}{I_1} = \frac{a\sin bx - b\cos bx}{a\cos bx + b\sin bx}$$

$$\Rightarrow \frac{I_2}{I_1} = \frac{\tan bx - \frac{b}{a}}{1 + \frac{b}{a}\tan bx} \qquad \text{[Diving } N^r \text{ and } D^r \text{ by } \cos bx\text{]}$$

$$\Rightarrow \frac{I_2}{I_1} = \frac{\tan bx - \tan\left\{\tan^{-1}(b/a)\right\}}{1 + \tan bx \,.\, \tan^{-1}(b/a)}$$

$$\Rightarrow \frac{I_2}{I_1} = \tan\left\{bx - \tan^{-1}\left(\frac{b}{a}\right)\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{I_2}{I_1}\right) = bx - \tan^{-1}\frac{b}{a}$$

$$\Rightarrow \tan^{-1}\left(\frac{I_2}{I_1}\right) + \tan^{-1}\left(\frac{b}{a}\right) = bx.$$

EXAMPLE 3 *Evaluate :* $\int x^2 e^x \sin x\, dx.$

SOLUTION We have,

$$\int \underset{\text{I}}{x^2} \underset{\text{II}}{e^x \sin x}\, dx$$

$$= x^2 \frac{e^x}{2}(\sin x - \cos x) - \int x\, e^x (\sin x - \cos x)\, dx$$

$$= \frac{1}{2}e^x x^2 (\sin x - \cos x) - \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x \sin x}\, dx + \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x \cos x}\, dx$$

$$= \frac{1}{2}e^x x^2 (\sin x - \cos x)$$

$$-\left[x\frac{e^x}{2}(\sin x - \cos x) - \frac{1}{2}\int e^x (\sin x - \cos x)\, dx\right]$$

$$+\left[x\frac{e^x}{2}(\cos x + \sin x) - \frac{1}{2}\int e^x (\cos x + \sin x)\, dx\right]$$

$$= \frac{e^x}{2}x^2(\sin x - \cos x) - \frac{1}{2}e^x \,.\, x(\sin x - \cos x) + \frac{e^x}{2}x(\cos x + \sin x)$$

$$+\frac{1}{2}\int e^x \{(\sin x - \cos x) - (\cos x + \sin x)\}\, dx$$

$$= \frac{e^x}{2}x^2(\sin x - \cos x) - \frac{e^x}{2}x(\sin x - \cos x - \cos x - \sin x)$$

$$-\int e^x \cos x\, dx$$

$$= \frac{e^x}{2}\,.\,x^2(\sin x - \cos x) + e^x \,.\, x\cos x - \frac{e^x}{2}(\cos x + \sin x) + C$$

$$= \frac{e^x}{2}\left[(x^2-1)\sin x - (x-1)^2 \cos x\right] + C$$

—❑❑❑—

10.19 SOME IMPORTANT INTEGRALS

(i) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\log\left|x+\sqrt{a^2+x^2}\right| + C$

(iii) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right| + C$

PROOF (i) Let $I = \int \sqrt{a^2-x^2}\,dx$.

Integrating by parts, we get

$$I = \int \underset{\text{I}}{\sqrt{a^2-x^2}}\ \underset{\text{II}}{1}\ dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{1}{2}(a^2-x^2)^{-1/2}(0-2x)\,x\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right) + C.$$

$$\therefore\ 2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C.$$

(ii) Let $I = \int \sqrt{a^2+x^2}\,dx$.

Integrating by parts, we get

$$I = \int \underset{\text{I}}{\sqrt{a^2+x^2}}\ \underset{\text{II}}{1}\cdot dx$$

$$= \sqrt{a^2+x^2}\,x - \int \frac{1}{2}(a^2+x^2)^{-1/2}(0+2x)\,x\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - I + a^2\log\left|x+\sqrt{a^2+x^2}\right|$$

$$\therefore\ 2I = x\sqrt{a^2+x^2} + a^2\log\left|x+\sqrt{a^2+x^2}\right|$$

$$\Rightarrow I = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\log\left|x+\sqrt{a^2+x^2}\right| + C.$$

(iii) Let $I = \sqrt{x^2-a^2}\,dx$.

$$\Rightarrow \quad I = \int \underset{\text{I}}{\sqrt{x^2-a^2}}\ \underset{\text{II}}{1}\cdot dx$$

Integrating by parts, we get

$$I = \sqrt{x^2-a^2}\,x - \int \frac{1}{2}(x^2-a^2)^{-1/2}(2x)\,x\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - I - a^2\log\left|x+\sqrt{x^2-a^2}\right|$$

$$\therefore\ 2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right|$$

$$\Rightarrow I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right| + C.$$

INTEGRALS OF THE FORM $\int \sqrt{ax^2+bx+c}\,dx$

To evaluate this type of integrals we express ax^2+bx+c as the sum or difference of two squares and apply formulae given above as the case may be.

ILLUSTRATION *Evaluate :*

(i) $\int \sqrt{7x-10-x^2}\,dx$ (ii) $\int \sqrt{4x^2+9}\,dx$

SOLUTION (i) We have,

$$\int \sqrt{-(x^2-7x+10)}\,dx$$

$$= \int \sqrt{-\left(x^2-7x+\frac{49}{4}-\frac{49}{4}+10\right)}\,dx$$

$$= \int \sqrt{-\left\{\left(x-\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2\right\}}\,dx$$

$$= \int \sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left(x-\frac{7}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2} + \frac{1}{2}\cdot\left(\frac{3}{2}\right)^2\sin^{-1}\left(\frac{x-7/2}{3/2}\right) + C$$

$$= \frac{1}{4}(2x-7)\sqrt{7x-10-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right) + C.$$

(ii) We have,

$$\int \sqrt{4x^2+9}\,dx$$

$$= 2\int \sqrt{x^2+\frac{9}{4}}\,dx = 2\int \sqrt{x^2+\left(\frac{3}{2}\right)^2}\,dx$$

$$= 2\left[\frac{1}{2}x\sqrt{x^2+\frac{9}{4}} + \frac{1}{2}\cdot\left(\frac{3}{2}\right)^2 \log\left|x+\sqrt{x^2+\frac{9}{4}}\right|\right] + C$$

$$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\log|2x+\sqrt{4x^2+9}| + C.$$

INTEGRALS OF THE FORM $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

To evaluate integrals of the type $\int (px+q)\sqrt{ax^2+bx+c}\,dx$, we express the linear factor $px+q$ as follows:

$$px+q = \lambda\cdot\frac{d}{dx}(ax^2+bx+c)+\mu$$

where λ and μ are constants to be determined by equating the coefficients of similar terms on both sides. So we have

$$2a\lambda = p \text{ and } b\lambda+\mu = q$$

$$\Rightarrow \quad \lambda = \frac{p}{2a} \text{ and } \mu = q-\frac{bp}{2a}$$

$$\therefore \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \left(\frac{q-bp}{2a}\right)\int \sqrt{ax^2+b+c}\,dx$$

The first integral on RHS is evaluated by putting $ax^2+bx+c=t$ and the second integral is evaluated by the method discussed earlier.

ILLUSTRATIVE EXAMPLES

EXAMPLE *Evaluate :*

(i) $\int x\sqrt{1+x-x^2}\,dx$ (ii) $\int (x+1)\sqrt{1-x-x^2}\,dx.$

SOLUTION Let $x = \lambda\cdot\frac{d}{dx}(1+x-x^2)+\mu.$

$\Rightarrow x = \lambda(1-2x)+\mu$

Comparing the coefficients of like powers of x, we get

$1 = -2\lambda$ and $\lambda+\mu = 0$

$\Rightarrow \lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}.$

$\therefore x = -\frac{1}{2}(1-2x)+\frac{1}{2}$

So, $\int x\sqrt{1+x-x^2}\,dx$

$$= \int\left\{-\frac{1}{2}(1-2x)+\frac{1}{2}\right\}\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int \sqrt{1+x-x^2}\,d(1+x-x^2) + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx,$$

$$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left[\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right)\right] + C$$

$$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left[\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right] + C$$

(ii) Let $x+1 = \lambda\cdot\frac{d}{dx}(1-x-x^2)+\mu$. Then,

$$x+1 = \lambda(-1-2x)+\mu$$

Comparing the coefficients of like powers of x, we get

$-2\lambda = 1$ and $\mu-\lambda = 1$

$\Rightarrow \lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}$

$\therefore x+1 = -\frac{1}{2}(-1-2x)+\frac{1}{2}.$

So, $\int (x+1)\sqrt{1-x-x^2}\,dx$

$$= \int\left\{-\frac{1}{2}(-1-2x)+\frac{1}{2}\right\}\sqrt{1-x-x^2}\,dx$$

$$= -\frac{1}{2}\int (-1-2x)\sqrt{1-x-x^2}\,dx + \frac{1}{2}\int \sqrt{1-x-x^2}\,dx$$

$$= \frac{1}{2}\int \sqrt{1-x-x^2}\,d(1-x-x^2) + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= -\frac{1}{3}(1-x-x^2)^{3/2} + \frac{1}{8}(2x+1)\sqrt{1-x-x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$$

EXAMPLE 2 Evaluate : $\int \sqrt{x+a\sqrt{ax-a^2}}\,dx$, $0<a<2.$

SOLUTION We have,

$$I = \int \sqrt{x+a\sqrt{ax-a^2}}\,dx$$

$$= \int \sqrt{\frac{t^2+a^2}{a}+at}\,\frac{2t}{a}\,dt \quad \text{, where } ax-a^2 = t^2$$

$$= \frac{2}{a^{3/2}}\int t\sqrt{t^2+a^2t+a^2}\,dt$$

$$= \frac{1}{a^{3/2}}\int (2t+a^2-a^2)\sqrt{t^2+a^2t+a^2}\,dt$$

$$= \frac{1}{a^{3/2}}\int (2t+a^2)\sqrt{t^2+a^2t+a^2}\,dt - \sqrt{a}\int \sqrt{t^2+a^2t+a^2}\,dt$$

$$= \frac{1}{a^{3/2}} \frac{(t^2+a^2t+a^2)^{3/2}}{\frac{3}{2}} - \sqrt{a} \int \sqrt{\left(t+\frac{a^2}{2}\right)^2 + a^2 - \frac{a^4}{4}}\, dt$$

$$= \frac{2}{3a^{3/2}} (t^2+a^2t+a^2)^{3/2} - \sqrt{a} \int \sqrt{\left(t+\frac{a^2}{2}\right)^2 + \frac{a^2}{4}(4-a^2)}\, dt$$

$$= \frac{2}{a^{3/2}} (t^2+a^2t+a^2)^{3/2} - \sqrt{a} \int \sqrt{\left(t+\frac{a^2}{2}\right)^2 + \left(\frac{a}{2}\sqrt{4-a^2}\right)^2}\, dt$$

$$= \frac{2}{a^{3/2}} (t^2+at+a^2)^{3/2} - \frac{\sqrt{a}}{2}\left[\left(t+\frac{a^2}{2}\right)\sqrt{t^2+a^2t+a^2}\right.$$
$$\left. + \log\left|\left(t+\frac{a^2}{2}\right) + \sqrt{t^2+a^2t+a^2}\right|\right] + C$$

$$= \frac{2}{a^{3/2}} \left\{ax + a\sqrt{ax-a^2}\right\}^{3/2}$$
$$- \frac{\sqrt{a}}{2}\left[\left\{\sqrt{ax-a^2} + \frac{a^2}{2}\sqrt{ax+a\sqrt{ax-a^2}}\right\}\right.$$
$$\left. + \log\left\{\left(\sqrt{ax-a^2}+\frac{a^2}{2}\right) + \sqrt{ax+a^2\sqrt{ax-a^2}}\right\}\right] + C$$

EXAMPLE 3 *Evaluate :* $\int \frac{x}{x-\sqrt{x^2-1}}\, dx$

SOLUTION We have,

$$\int \frac{x}{x-\sqrt{x^2-1}}\, dx$$
$$= \int x\,(x+\sqrt{x^2-1})\, dx$$
$$= \int x^2\, dx + \int x\sqrt{x^2-1}\, dx$$
$$= \frac{x^3}{3} + \frac{1}{2}\int 2x\,.\sqrt{x^2-1}\, dx$$
$$= \frac{x^3}{3} + \frac{1}{2}\times\frac{2}{3}(x^2-1)^{3/2} + C$$
$$= \frac{x^3}{3} + \frac{1}{3}(x^2-1)^{3/2} + C$$

EXAMPLE 4 *Evaluate :* $\int x\sqrt{\frac{1+x}{1-x}}\, dx$

SOLUTION We have,

$$\int x\sqrt{\frac{1+x}{1-x}}\, dx$$
$$= \int \frac{x(1+x)}{\sqrt{1-x^2}}\, dx$$
$$= \int \frac{x}{\sqrt{1-x^2}}\, dx + \int \frac{x^2}{\sqrt{1-x^2}}\, dx$$
$$= -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\, dx - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\, dx$$
$$= -\sqrt{1-x^2} - \int \sqrt{1-x^2}\, dx + \int \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= -\sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + \sin^{-1}x + C$$
$$= -\sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$$

EXAMPLE 5 *Evaluate :* $\int \frac{x^2}{\sqrt{1-2x-x^2}}\, dx$

SOLUTION We have,

$$\int \frac{x^2}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \frac{-x^2}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \frac{(1-2x-x^2)+(2x-1)}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \sqrt{1-2x-x^2}\, dx - \int \frac{2x-1}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \sqrt{1-2x-x^2}\, dx + \int \frac{-2x-2+3}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \sqrt{1-2x-x^2}\, dx + \int \frac{-2x-2}{\sqrt{1-2x-x^2}}\, dx + 3\int \frac{1}{\sqrt{1-2x-x^2}}\, dx$$
$$= -\int \sqrt{(\sqrt{2})^2-(x+1)^2}\, dx$$
$$+ \int \frac{1}{\sqrt{1-2x-x^2}}\, d(1-2x-x^2) + 3\int \frac{1}{\sqrt{(\sqrt{2})^2-(x+1)^2}}\, dx$$
$$= -\frac{1}{2}\left[(x+1)\sqrt{1-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{2}}\right]$$
$$+ 2\sqrt{1-2x-x^2} + 3\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

EXAMPLE 6 *Evaluate :* $\int x^3\sqrt{\frac{1+x^2}{1-x^2}}\, dx$

SOLUTION We have,

$$I = \int x^3\sqrt{\frac{1+x^2}{1-x^2}}\, dx$$
$$= \int \sqrt{\frac{1+x^2}{1-x^2}}\, .x^2\, .x\, dx$$
$$= \frac{1}{2}\int \sqrt{\frac{1+x^2}{1-x^2}}\, x^2\, .2x\, dx$$
$$= \frac{1}{2}\int \sqrt{\frac{1+x^2}{1-x^2}}\, .x^2\, .d(x^2)$$
$$= \frac{1}{2}\int \sqrt{\frac{1+t}{1-t}}\, t\, dt, \text{ where } t = x^2$$
$$= \frac{1}{2}\int \frac{t+t^2}{\sqrt{1-t^2}}\, dt$$
$$= -\frac{1}{2}\sqrt{1-t^2} - \frac{1}{4}t\sqrt{1-t^2} + \frac{1}{4}\sin^{-1}t + C \qquad \text{[Using Ex 4]}$$
$$= -\frac{1}{2}\sqrt{1-x^4} - \frac{1}{4}x^2\sqrt{1-x^4} + \frac{1}{4}\sin^{-1}x^2 + C$$

EXERCISE 10.7

Evaluate the following integrals :

1. $\int \tan^{-1}\sqrt{x}\,dx$
2. $\int \frac{\sin^{-1}x}{\sqrt{x+1}}\,dx$
3. $\int x\tan^2 x\,dx$
4. $\int x\cos^2 x\,dx$
5. $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx$
6. $\int \frac{\sin^{-1}\sqrt{x}}{\sqrt{1-x}}\,dx$
7. $\int x^2\cos^2 x\,dx$
8. $\int x(\tan^{-1}x)^2\,dx$
9. $\int e^{3x}(\sin 2x-\cos 2x)\,dx$
10. $\int \frac{x^2e^x}{(x+2)^2}\,dx$
11. $\int x^2e^x\cos x\,dx$
12. $\int \frac{\log(\cos x)}{\cos^2 x}\,dx$
13. $\int x\sin x\cos x\,dx$
14. $\int x^3e^{2x}\,dx$
15. $\int \frac{x\cos x}{\sin^3 x}\,dx$
16. $\int e^x\sin^2 x\,dx$
17. $\int x^3e^{x^2}\,dx$
18. $\int e^{-x^2}x^5\,dx$
19. $\int (x^2+3x+5)\cos 2x\,dx$
20. $\int x\log(1+x^3)\,dx$
21. $\int \frac{(\log x-1)}{(\log x)^2}\,dx$
22. $\int xe^{x^2}(x^2+1)\,dx$
23. $\int \frac{xe^x}{(1+x)^2}\,dx$
24. $\int e^{\cos x}\left|\frac{x\sin^3 x+\cos x}{\sin^2 x}\right|dx$
25. $\int e^x\frac{(x^3-x+2)}{(x^2+1)^2}\,dx$
26. $\int x\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right)dx$
27. $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$
28. $\int \frac{\tan^{-1}x}{x^2(1+x^2)}\,dx$
29. $\int \frac{x^2\tan^{-1}x}{1+x^2}\,dx$
30. $\int \sin(x^{1/3})\,dx$
31. $\int \frac{x^2}{(x\cos x-\sin x)^2}\,dx$
32. $\int \sec^2 x\log(1+\sin^2 x)\,dx$
33. $\int a^x\left|\log x+\log a\log\left(\frac{x^x}{e^x}\right)\right|dx$
34. $\int \frac{1}{x-\sqrt{x^2-1}}\,dx$
35. $\int \frac{\sqrt{1+x^{2n}}\{\log(1+x^{2n})-2n\log x\}}{x^{3n+1}}\,dx, n\in N$
36. $\int \frac{b+a\cos x}{(a+b\cos x)^2}\,dx$
37. $\int \left(x+1+\frac{x^2+x+1}{\sqrt{x^2+1}}\right)e^x\,dx$
38. $\int e^{3x}\frac{2+3\sin 2x}{1+\cos 2x}\,dx$

ANSWERS

1. $x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+C$
2. $2\sqrt{x+1}\sin^{-1}x+4\sqrt{1-x}+C$
3. $x\tan x-\frac{x^2}{2}+\log|\cos x|+C$
4. $\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$
5. $\sqrt{1+x^2}\tan^{-1}x-\log(x+\sqrt{x^2+1})+C$
6. $2\left[\sqrt{x}-\sqrt{1-x}\sin^{-1}\sqrt{x}\right]+C$
7. $\frac{x^3}{6}+\frac{x^2}{4}\sin 2x+\frac{1}{4}x\cos 2x-\frac{1}{8}\sin 2x+C$
8. $\frac{x^2+1}{2}(\tan^{-1}x)^2-x\tan^{-1}x+\frac{1}{2}\log(1+x^2)+C$
9. $\frac{e^{3x}}{13}(\sin 2x-5\cos 2x)+C$
10. $e^x\left(\frac{x-2}{x+2}\right)+C$
11. $\frac{1}{2}e^x\{(x^2-1)\cos x+(x-1)^2\sin x\}+C$
12. $\tan x.\log\cos x+\tan x-x+C$
13. $\frac{1}{8}\sin 2x-\frac{1}{4}x\cos 2x+C$
14. $e^{2x}\left\{\frac{x^3}{2}-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right\}+C$
15. $-\frac{1}{2}(x\operatorname{cosec}^2 x+\cot x)+C$
16. $\frac{e^x}{2}\left\{1-\frac{2\sin 2x+\cos 2x}{5}\right\}+C$
17. $\frac{e^{x^2}(x^2-1)}{2}+C$
18. $-\frac{1}{2}e^{-x^2}(x^4+2x^2+2)+C$
19. $\left(\frac{x}{2}+\frac{3}{4}\right)\cos 2x+\left(\frac{2x^2+6x+9}{4}\right)\sin 2x+C$
20. $\frac{1}{2}x^2\log(1+x^3)-\frac{3}{4}x^2+\frac{1}{4}\log(x^2-x+1)-\frac{1}{2}\log(x+1)+\frac{\sqrt{3}}{2}\tan^{-1}\frac{2x-1}{\sqrt{3}}+C$
21. $\frac{x}{\log x}+C$
22. $\frac{1}{2}x^2e^{x^2}+C$

23. $\frac{e^x}{x+1}+C$

24. $e^{\sin x}(x-\sec x)+C$

25. $e^x\frac{x+1}{x^2+1}+C$

26. $\frac{1}{4}(x^2-2a^2)\cos^{-1}\frac{x}{2a}-\frac{x}{8}\sqrt{4a^2-x^2}+C$

27. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\frac{1}{2}\log(1-x^2)+C$

28. $\log\left\{\frac{|x|}{\sqrt{1+x^2}}\right\}-\frac{1}{x}\tan^{-1}x-\frac{1}{2}(\tan^{-1}x)^2+C$

29. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-\frac{1}{2}(\tan^{-1}x)^2+C$

30. $3\left[(2-x^{2/3})\cos x^{1/3}+2x^{1/3}\sin(x^{1/3})\right]+C$

31. $\frac{x\,\text{cosec}\,x}{x\cos x-\sin x}-\cot x+C$

32. $\tan x\log(1+\sin^2 x)-2x+\sqrt{2}\tan^{-1}\left(\sqrt{2}\tan x\right)+C$

33. $a^x x(\log x-1)+C$

34. $\frac{x^2}{2}+\frac{1}{2}\left\{x\sqrt{x^2-1}-\log(x+\sqrt{x^2-1})\right\}+C$

35. $\frac{-2}{9n}\frac{(1+x^{2n})^{3/2}}{x^{3n}}\left\{\frac{3}{2}\log(1+x^{2n})-3n\log x-1\right\}+C$

HINT TO SELECTED PROBLEMS

36. $\int\frac{b+a\cos x}{(a+b\cos x)^2}dx=\int\frac{b(\cos^2x+\sin^2x)+a\cos x}{(a+b\cos x)^2}dx$

$=\int\underset{\text{I}}{\sin x}\underset{\text{II}}{\frac{b\sin x}{(a+b\cos x)^2}}dx+\int\frac{\cos x}{(a+b\cos x)}dx]$

37. Given integral $=\int\left|\frac{(x+1)\sqrt{x^2+1}}{\sqrt{x^2+1}}+\frac{x^2+x+1}{\sqrt{x^2+1}}\right|e^x\,dx$

$=\int(x+1)e^x\,dx+\int\left(\sqrt{x^2+1}+\frac{x}{\sqrt{x^2+1}}\right)e^x\,dx.$

—❑❑❑—

10.20 INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS

(I) PARTIAL FRACTIONS *If $f(x)$ and $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.*

If degree of $f(x)$ < degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of $f(x)\geq$ degree of $g(x)$ then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide $f(x)$ by $g(x)$ so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x)+\frac{\psi(x)}{g(x)}$ where $\phi(x)$ and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of $g(x)$. Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of $g(x)$. Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

The resolution of $\frac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of $g(x)$ as discussed below.

CASE I When denominator is expressible as the product of non-repeating linear factors.

Let $g(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}$$

where $A_1, A_2, \ldots A_n$ are constants and can be determined by equating the numerator on RHS to the numerator on LHS and then substituting $x=a_1, a_2, \ldots, a_n$.

CASE II When the denominator $g(x)$ is expressible as the product of the linear factors such that some of them are repeating.

Let $g(x)=(x-a)^k(x-a_1)(x-a_2)\ldots(x-a_r)$. Then, we assume that

$$\frac{f(x)}{g(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a}+\frac{B_2}{x-a_2}+\ldots+\frac{B_r}{(x-a_r)}$$

i.e., corresponding to non-repeating factors we assume as in Case I and for each repeating factor $(x-a)^k$, we assume partial fractions $\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}$, where $A_1, A_2, \ldots, A_k$ are constants.

Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by comparing coefficients of equal powers of x on both sides.

CASE III When some of the factors of denominator $g(x)$ are quadratic but non-repeating.

Corresponding to each quadratic factor ax^2+bx+c, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type

$$\frac{A(2ax+b)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}.$$

The following examples will illustrate the use of partial fractions to evaluate integrals of ration algebraic functions.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:* $\int \frac{x^2-8x+7}{(x^2-3x-10)^2}\,dx$

SOLUTION We have,

$$\frac{x^2-8x+7}{(x^2-3x-10)^2}=\frac{(x-7)(x-1)}{(x-5)^2(x+2)^2}$$

Let $\frac{(x-7)(x-1)}{(x-5)^2(x+2)^2}=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2}$. Then,

$$(x-7)(x-1)=A(x-5)(x+2)^2+B(x+2)^2$$
$$+C(x+2)(x-5)^2+D(x-5)^2 \quad \ldots(i)$$

Putting $x=5$ and $x=-2$ respectively, we get

$$-8=B(7)^2 \Rightarrow B=\frac{-8}{49}$$

and,

$$27=D(-7)^2 \Rightarrow D=\frac{27}{49}$$

Now, putting $x=0$, in (i) we get

$$7=-20A+4B+50C+25D$$

$$\Rightarrow 7=-20A-\frac{32}{49}+50C+27\times\frac{25}{49}$$

$$\Rightarrow 20A-50C=\frac{400}{49}$$

$$\Rightarrow 2A-5C=\frac{40}{49} \quad \ldots(ii)$$

Putting $x=1$ in (i), we get

$$0=-16A+9B+48C+16D$$

$$\Rightarrow 0=-16A+48C-\frac{72}{49}+\frac{442}{49}$$

$$\Rightarrow 16A-48C=\frac{360}{49}$$

$$\Rightarrow 2A-6C=\frac{45}{49} \quad \ldots(iii)$$

Solving (ii) and (iii), we get

$$A=-\frac{45}{98}, C=\frac{-15}{49}$$

$$\therefore \frac{(x-7)(x-1)}{(x-5)^2(x+2)^2}$$
$$=-\frac{45}{98}\cdot\frac{1}{x-5}-\frac{8}{49}\cdot\frac{1}{(x-5)^2}-\frac{15}{49}\frac{1}{x+2}+\frac{27}{49}\frac{1}{(x+2)^2}$$

Integrating both sides, we get

$$\int \frac{(x-7)(x-1)}{(x-5)^2x+2)^2}\,dx$$
$$=\frac{-45}{98}\int\frac{1}{x-5}\,dx-\frac{8}{49}\int\frac{1}{(x-5)^2}\,dx-\frac{15}{49}\int\frac{1}{x+2}\,dx$$
$$+\frac{27}{49}\int\frac{1}{(x+2)^2}\,dx$$
$$=\frac{-45}{98}\log|x-5|+\frac{8}{49}\frac{1}{x-5}-\frac{15}{49}\log|x+2|-\frac{27}{49}\frac{1}{(x+2)}+C$$

EXAMPLE 2 *Evaluate:*

$$\int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}\,dx$$

SOLUTION We have,

$$\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}$$
$$=x+\frac{3x^4+3x^3-13x^2+4}{x^5-5x^3+4x}$$
$$=x+\frac{3x^4+3x^3-13x^2+4}{x(x^2-4)(x^2-1)}$$
$$=x+\frac{3x^4+3x^3-13x^2+4}{x(x-2)(x+2)(x-1)(x+1)}$$

So, let

$$\frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}=x+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}+\frac{D}{x-2}+\frac{E}{x+2}$$

$$\Rightarrow A=\frac{4}{4}=1, B=\frac{1-2+3-9+4}{1(1-2)(1+2)(1+1)}=\frac{-3}{-6}=\frac{1}{2}$$

$$C=\frac{1-2-3-9+4}{(-1)(-1-2)(-1+2)(-1-1)}=\frac{-9}{-6}=\frac{3}{2},$$

$$D=\frac{64-32+24-36+4}{2(2+2)(2-1)(2+1)}=\frac{24}{24}=1$$

and, $E=\frac{64-32-24-36+4}{(-2)(-4)(-3)(-1)}=\frac{-24}{24}=-1$

$$\therefore \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}$$

$$=x+\frac{1}{x}+\frac{1}{2(x-1)}+\frac{3}{2(x+1)}+\frac{1}{x-2}-\frac{1}{x+2}$$

$$\Rightarrow \int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}\,dx$$

$$=\frac{x^2}{2}+\log x+\frac{1}{2}\log|x-1|+\frac{3}{2}\log|x+1|+\log|x-2|$$

$$-\log|x+2|+C$$

EXAMPLE 3 *Evaluate:* $\int \frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$

SOLUTION We have,

$$\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}=x-2+\frac{4x^3+4x^2+4x+4}{x^4+2x^3+2x^2}$$

So, let

$$\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}=x-2+\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2x+2} \quad ...(i)$$

$$\Rightarrow x^5+2x^3+4x+4=(x-2)x^2(x^2+2x+2)+Ax(x^2+2x+2)$$

$$+B(x^2+2x+2)+(Cx+D)x^2$$

Putting $x=0$, we get

$$4=2B \Rightarrow B=2$$

On equating the coefficient of similar powers of x, we get

$$2=-2+A+C, 0=-4+2A+B+D, 4=2A+2B$$

$\therefore A=0, C=4$ and $D=2$

Substituting the value of A, B, C and D in (i), we get

$$\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}=x-2+\frac{2}{x^2}+\frac{4x+2}{x^2+2x+2}$$

$$\Rightarrow \int \frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$$

$$=\int x-2\,dx+2\int\frac{1}{x^2}dx+\int\frac{4x+2}{x^2+2x+2}\,dx$$

$$=\frac{x^2}{2}-2x-\frac{2}{x}+2\int\frac{2x+2-1}{x^2+2x+2}\,dx$$

$$=\frac{x^2}{2}-2x-\frac{2}{x}+2\int\frac{2x+2}{x^2+2x+2}\,dx-2\int\frac{1}{x^2+2x+2}\,dx$$

$$=\frac{x^2}{2}-2x-\frac{2}{x}+2\log(x^2+2x+2)-2\tan^{-1}(x+1)+C$$

EXAMPLE 4 *Evaluate:* $\int \frac{1}{\sin x(2\cos^2 x-1)}\,dx$

SOLUTION Putting $\cos x=t$, we get

$$I=\int\frac{1}{\sin x(2\cos^2 x-1)}\,dx$$

$$=\int\frac{1}{\sin x(2t^2-1)}\times-\frac{dt}{\sin x}$$

$$=-\int\frac{1}{(1-t^2)(2t^2-1)}\,dt$$

$$\therefore I=-\int\left(\frac{1}{1-t^2}+\frac{2}{2t^2-1}\right)dt$$

$$=-\int\frac{1}{1-t^2}dt-2\int\frac{1}{2t^2-1}\,dt$$

$$=-\int\frac{1}{1^2-t^2}dt-\sqrt{2}\int\frac{1}{(\sqrt{2}t)^2-1^2}d(\sqrt{2}t)$$

$$=-\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|-\frac{\sqrt{2}}{2}\log\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right|+C$$

$$=-\frac{1}{2}\log\left|\frac{1+\cos x}{1-\cos x}\right|-\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C$$

EXAMPLE 5 *Evaluate:* $\int \frac{1}{x^6-a^6+a^2x^2(x^2-a^2)}\,dx$

SOLUTION We have,

$$\int\frac{1}{x^6-a^6+a^2x^2(x^2-a^2)}\,dx$$

$$=\int\frac{1}{(x^2-a^2)(x^4+a^4+2a^2x^2)}$$

$$=\int\frac{1}{(x^2-a^2)(x^2+a^2)^2}\,dx$$

Let $x^2=y$. Then,

$$\frac{1}{(x^2-a^2)(x^2+a^2)^2}=\frac{1}{(y-a^2)(y+a^2)^2}$$

Let $\frac{1}{(y-a^2)(y+a^2)^2}=\frac{A}{y-a^2}+\frac{B}{y+a^2}+\frac{C}{(y+a^2)^2}$

$$\Rightarrow 1=A(y+a^2)^2+B(y+a^2)(y-a^2)+C(y-a^2)$$

Putting $y=a^2$ and $y=-a^2$, respectively we get

$$A = \frac{1}{4a^4}, C = \frac{-1}{2a^2}$$

Now, putting $y = 0$, we get

$1 = Aa^4 - Ba^4 - Ca^2$

$\Rightarrow 1 = \frac{1}{4} - Ba^4 + \frac{1}{2}$

$\Rightarrow B = -\frac{1}{4a^4}$

$$\therefore \frac{1}{(y-a^2)(y+a^2)^2} = \frac{1}{4a^4}\cdot\frac{1}{y-a^2} - \frac{1}{4a^4}\cdot\frac{1}{y+a^2} - \frac{1}{2a^2}\frac{1}{(y+a^2)^2}$$

$$\Rightarrow \frac{1}{(x^2-a^2)(x^2+a^2)^2} = \frac{1}{4a^4}\cdot\frac{1}{x^2-a^2} - \frac{1}{4a^4}\cdot\frac{1}{x^2+a^2} - \frac{1}{2a^2}\frac{1}{(x^2+a^2)^2}$$

$$\Rightarrow \int \frac{1}{(x^2-a^2)(x^2+a^2)^2}\,dx$$

$$= \frac{1}{4a^4}\int \frac{1}{x^2-a^2}\,dx - \frac{1}{4a^4}\int \frac{1}{x^2+a^2}\,dx - \frac{1}{2a^2}\int \frac{1}{(x^2+a^2)^2}\,dx$$

$$= \frac{1}{4a^4}\times\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| - \frac{1}{4a^5}\tan^{-1}\frac{x}{a}$$

$$-\frac{1}{2a^2}\left[\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right] + C$$

[Using Ex 28 on page 10.61]

$$= \frac{1}{8a^5}\log\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^5}\tan^{-1}\frac{x}{a} - \frac{x}{4a^4(x^2+a^2)} + C$$

EXAMPLE 6 *Evaluate:* $\int \frac{x+1}{x(1+x\,e^x)^2}\,dx$ **[IIT 1996]**

SOLUTION We have,

$$I = \int \frac{x+1}{x(1+x\,e^x)^2}\,dx$$

$$= \int \frac{(x+1)\,e^x}{x\,e^x\,.\,(1+x\,e^x)^2}\,dx$$

$$= \int \frac{1}{xe^x(1+x\,e^x)^2}\,d(x\,e^x)$$

$$= \int \frac{1}{t(1+t)^2}\,dt, \text{ where } t = x\,e^x.$$

Let $\frac{1}{t(1+t)^2} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$

$\Rightarrow 1 = A(1+t)^2 + Bt(1+t) + Ct$

Putting $t = 0$ and $t = -1$ respecitvely, we get

$A = 1, C = -1$

Now, putting $t = 1$, we get

$1 = 4A + 2B + C \Rightarrow 1 = 4 + 2B - 1 \Rightarrow B = -1$

$$\therefore \frac{1}{t(1+t)^2} = \frac{1}{t} - \frac{1}{1+t} - \frac{1}{(1+t)^2}$$

$$\therefore I = \int \frac{1}{t(1+t)^2}\,dt$$

$$= \int \left(\frac{1}{t} - \frac{1}{1+t} - \frac{1}{(1+t)^2}\right)dt$$

$$= \log|t| - \log|1+t| + \frac{1}{1+t} + C$$

$$= \log(xe^x) - \log(1+x\,e^x) + \frac{1}{1+x\,e^x} + C$$

EXAMPLE 7 *Evaluate:* $\int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^x}{(e^{2x}+4)(e^{2x}-1)^2}\,dx$

SOLUTION We have,

$$I = \int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^x}{(e^{2x}+4)(e^{2x}-1)^2}\,dx$$

$$= \int \frac{(2e^{4x} + e^{3x} - 4e^{2x} + 4e^x + 2)\,e^x}{(e^{2x}+4)(e^{2x}-1)^2}\,dx$$

$$= \int \left|\frac{2(e^x)^4 + (e^x)^3 - 4(e^x)^2 + 4e^x + 2}{\{(e^x)^2+4\}\{(e^x)^2-1\}^2}\right| d(e^x)$$

$$= \int \frac{2t^4 + t^3 - 4t^2 + 4t + 2}{(t^2+4)(t^2-1)^2}, \text{ where } t = e^x$$

$$= \int \frac{(2t^4 - 4t^2 + 2) + (t^3 + 4t)}{(t^2+4)(t^2-1)^2}\,dt$$

$$= \int \frac{2(t^2-1)^2 + t(t^2+4)}{(t^2+4)(t^2-1)^2}\,dt$$

$$= \int \frac{2(t^2-1)^2 + t(t^2+4)}{(t^2+4)(t^2-1)^2}\,dt$$

$$= \int \frac{2}{t^2+4} + \frac{t}{(t^2-1)^2}\,dt$$

$$= 2\int \frac{1}{t^2+2^2}\,dt + \frac{1}{2}\int \frac{2t}{(t^2-1)^2}\,dt$$

$$= 2\int \frac{1}{t^2+2^2} + \frac{1}{2}\int \frac{1}{(t^2-1)^2}\,d(t^2-1)$$

$$= \tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2(t^2-1)} + C$$

$$= \tan^{-1}\left(\frac{e^x}{2}\right) - \frac{1}{2(e^{2x}-1)} + C$$

EXAMPLE 8 *Evaluate:* $\int \frac{1}{x}\left\{\log_{ex} e\,.\,\log_{e^2 x} e\,.\,\log_{e^3 x} e\right\}dx$

SOLUTION We have,

$$\int \frac{1}{x}\left(\log_{(ex)} e\,.\,\log_{(e^2 x)} e\,.\,\log_{(e^3 x)} e\right)dx$$

$$= \int \frac{1}{x \log_e ex \cdot \log_e e^2x \cdot \log_e e^3 x}\, dx$$

$$= \int \frac{1}{x\left(\log_e e + \log_e x\right)\left(\log_e e^2 + \log_e x\right)\left(\log_e e^3 + \log_e x\right)}\, dx$$

$$= \int \frac{1}{(1+\log_e x)(2+\log_e x)(3+\log_e x)}\, d\left(\log_e x\right)$$

$$= \int \frac{1}{(1+t)(2+t)(3+t)}\, dt, \quad \text{where } t = \log_e x$$

$$= \int \left(\frac{1}{2}\cdot\frac{1}{1+t} - \frac{1}{2+t} + \frac{1}{3+t}\right) dt \qquad \text{[Using partial fractions]}$$

$$= \frac{1}{2}\log|1+t| - \log(2+t) + \log(3+t) + C$$

$$= \frac{1}{2}\log\left|1+\log_e x\right| - \log\left|2+\log_e x\right| + \log\left|3+\log_e x\right| + C$$

$$= \frac{1}{2}\log\{\log_e ex\} - \log\{\log_e e^2x\} + \log\{\log_e e^3x\} + C$$

EXAMPLE 9 *Evaluate:* $\int \frac{3x+1}{(x-1)^3(x+1)}\, dx$

SOLUTION Let

$$\frac{3x+1}{(x-1)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \qquad \ldots(i)$$

Multiplying both sides by $(x+1)$ and then putting $x=-1$, we get

$$A = \frac{-2}{(-2)^3} = \frac{1}{4}$$

Multiplying both sides by $(x-1)^3$ and then putting $x=1$, we get

$$D = \frac{4}{2} = 2$$

From (i), we get

$$3x+1 = A(x-1)^3 + B(x-1)^2(x+1) + C(x-1)(x+1) + D(x+1)$$

Putting $x = 0$, we get

$$1 = -A + B - C + D$$

$$\Rightarrow 1 = -\frac{1}{4} - B - C + 2$$

$$\Rightarrow B - C = \frac{-3}{4}$$

Putting $x = 2$, we get

$$7 = A + 3B + 3C + 3D$$

$$\Rightarrow 7 = \frac{1}{4} + 3B + 3C + 6$$

$$\Rightarrow 3B + 3C = \frac{3}{4}$$

$$\Rightarrow B + C = \frac{1}{4}$$

Solving $B + C = \frac{1}{4}$ and $B - C = \frac{-3}{4}$, we get

$$B = -\frac{1}{4} \text{ and } C = \frac{1}{2}$$

Substituting the values of A, B, C and D in (i), we get

$$\Rightarrow \frac{3x+1}{(x-1)^3(x+1)} = \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{2}{(x-1)^3}$$

$$\Rightarrow \int \frac{3x+1}{(x-1)^3(x+1)}\, dx$$

$$= \frac{1}{4}\int \frac{1}{x+1}\, dx - \frac{1}{4}\int \frac{1}{x-1}\, dx + \frac{1}{2}\int \frac{1}{(x-1)^2}\, dx + 2\int \frac{1}{(x-1)^3}\, dx$$

$$= \frac{1}{4}\log|x+1| - \frac{1}{4}\log|x-1| - \frac{1}{2(x-1)} - \frac{1}{(x-1)^2} + C$$

EXAMPLE 10 *Evaluate:* $\int \frac{1}{4+\tan x+\cot x}\, dx$

SOLUTION We have, $I = \int \frac{\tan x}{\tan^2 x + 4\tan x + 1}\, dx$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{t}{t^2+4t+1} \times \frac{dt}{1+t^2}$$

$$= \int \frac{t}{(t^2+1)(t^2+4t+1)}\, dt$$

Let $\frac{t}{(t^2+1)(t^2+4t+1)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4t+1}$

$$\Rightarrow t = (At+B)(t^2+4t+1) + (Ct+D)(t^2+1)$$

On equating coefficients of like powers of t on both sides, we get

$$0 = B + D$$

$$1 = 4B + A + C$$

$$0 = B + 4A + D$$

and, $$0 = A + C$$

Solving these equations, we get

$$A = 0, B = \frac{1}{4}, C = 0 \text{ and } D = -\frac{1}{4}$$

$$\therefore \frac{t}{(t^2+1)(t^2+4t+1)} = \frac{1}{4(t^2+1)} - \frac{1}{4(t^2+4t+1)}$$

$$\Rightarrow I = \frac{1}{4}\int \frac{1}{t^2+1}\, dt - \frac{1}{4}\int \frac{1}{t^2+4t+1}\, dt$$

$$= \frac{1}{4}\int \frac{1}{t^2+1^2}\, dt - \frac{1}{4}\int \frac{1}{(t+2)^2 - (\sqrt{3})^2}\, dt$$

$$= \frac{1}{4}\tan^{-1}(t) - \frac{1}{8\sqrt{3}}\log\left|\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right| + C$$

$$= \frac{1}{4}x - \frac{1}{8\sqrt{3}}\log\left|\frac{\tan x+2-\sqrt{3}}{\tan x+2+\sqrt{3}}\right| + C$$

EXAMPLE 11 *Evaluate :* $\int \frac{1}{x+\sqrt{x^2-x+1}}\,dx$

SOLUTION Let $I = \int \frac{1}{x+\sqrt{x^2-x+1}}\,dx$

Let $x+\sqrt{x^2-x+1} = t$. Then,

$\sqrt{x^2-x+1} = t-x$

$\Rightarrow x^2-x+1 = (t-x)^2$

$\Rightarrow -x+1 = t^2-2tx$

$\Rightarrow x = \frac{t^2-1}{2t-1}$

$\therefore dx = \frac{(2t-1)\,2t-2(t^2-1)}{(2t-1)^2}\,dt = \frac{2t^2-2t+2}{(2t-1)^2}\,dt$

Substituting these values, we get

$$I = \int \frac{1}{t}\times\frac{2t^2-2t+2}{(2t-1)^2}\,dt = 2\int \frac{t^2-t+1}{t(2t-1)^2}\,dt$$

Let $\frac{t^2-t+1}{t(2t-1)^2} = \frac{A}{t} + \frac{B}{2t-1} + \frac{C}{(2t-1)^2}$ ...(i)

Multiplying both sides by t and putting $t = 0$, we get

$$A = 1$$

Similarly, multiplying both sides of (i) by $(2t-1)^2$ and putting $t = \frac{1}{2}$, we get

$$C = \frac{\frac{1}{4}-\frac{1}{2}+1}{\frac{1}{2}} = \frac{3}{2}$$

From (i), we have

$t^2-t+1 = A(2t-1)^2 + B(2t-1)t + Ct$

On equating coefficient of t^2 on both sides, we get

$1 = 4A+2B \Rightarrow B = -\frac{3}{2}$

Substituting the values of A, B, C in (i), we get

$$\frac{t^2-t+1}{t(2t-1)^2} = \frac{1}{t} - \frac{3}{2(2t-1)} + \frac{3}{2}\frac{1}{(2t-1)^2}$$

$$\therefore I = 2\int \frac{1}{t}\,dt - 3\int \frac{1}{2t-1}\,dt + \frac{3}{2}\int \frac{1}{(2t-1)^2}\,dt$$

$$\Rightarrow I = 2\log t - \frac{3}{2}\log(2t-1) - \frac{3}{4}\frac{1}{(2t-1)} + C$$

$$\Rightarrow I = 2\log\left(x+\sqrt{x^2-x+1}\right) - \frac{3}{4}\log\left\{(2x-1)+\sqrt{x^2-x+1}\right\} - \frac{3}{4\left\{(2x-1)+\sqrt{x^2-x+1}\right\}} + C$$

EXAMPLE 12 *Evaluate* $\int \sqrt[3]{\tan x}\,dx$.

SOLUTION Putting $\tan x = t^{3/2}$ and $\sec^2 x\,dx = \frac{3}{2}t^{1/2}\,dt$ or, $dx = \frac{3t^{1/2}}{2(1+t^3)}\,dt$, we get

$$I = \int \frac{t^{1/2}\times 3t^{1/2}}{2(1+t^3)}\,dt = \frac{3}{2}\int \frac{t}{1+t^3}\,dt = \frac{3}{2}\int \frac{t}{(1+t)(t^2-t+1)}\,dt$$

Now, let

$$\frac{t}{(1+t)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$$

$\Rightarrow t = A(t^2-t+1) + (Bt+C)(t+1)$

On equating the coefficients of like power of t, we get

$A+B = 0, -A+B+C = 1$ and $A+C = 0$

$\Rightarrow A = -\frac{1}{3}, B = \frac{1}{3} = C$

$$\therefore \frac{t}{(1+t)(t^2-t+1)} = \frac{-1}{3(t+1)} + \frac{t+1}{3(t^2-t+1)}$$

$$\Rightarrow I = \frac{3}{2}\int -\frac{1}{3(t+1)} + \frac{t+1}{3(t^2-t+1)}\,dt$$

$$= -\frac{1}{2}\int \frac{1}{t+1}\,dt + \frac{1}{2}\int \frac{t+1}{t^2-t+1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{4}\int \frac{2t-1+3}{t^2-t+1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{4}\int \frac{2t-1}{t^2-t+1}\,dt + \frac{3}{4}\int \frac{1}{t^2-t+1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{4}\int \frac{2t-1}{t^2-t+1}\,dt + \frac{3}{4}\int \frac{1}{t^2-t+1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{4}\log|t^2-t+1| + \frac{3}{4}\int \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{4}\log|t^2-t+1| + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{2}\log\left|\tan^{2/3}x+1\right| + \frac{1}{4}\log\left|\tan^{4/3}x - \tan^{2/3}x + 1\right| + \frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\tan^{2/3}x-1}{\sqrt{3}}\right\} + C$$

EXAMPLE 13 *Evaluate*: $\int \frac{1+x^{2/3}}{1+x}\,dx$

SOLUTION Putting $x = t^3$ and $dx = 3t^2\,dt$, we get

$$I = \int \frac{1+t^2}{1+t^3}\,.3t^2\,dt$$

$$= 3\int \frac{t^4+t^2}{t^3+1}\,dt$$

$$= 3\int \left(t + \frac{t^2-t}{t^3+1}\right)dt$$

$$= 3\int t\,dt + 3\int \frac{t^2-t}{(t+1)(t^2-t+1)}\,dt$$

$$= \frac{3t^2}{2} + 3\int \frac{t^2-t}{(t+1)(t^2-t+1)}\,dt$$

Let $\frac{t^2-t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

$\Rightarrow A(t^2-t+1) + (Bt+C)(t+1) = t^2-t$

On equating the coefficient of like powers of t, we get

$$A+B = 1, -A+C+B = -1, A+C = 0$$

$\Rightarrow A = \frac{2}{3}, B = \frac{1}{3}, C = -\frac{2}{3}$

$$\frac{t^2-t}{(t+1)(t^2-t+1)} = \frac{2}{3(t+1)} + \frac{t-2}{3(t^2-t+1)}$$

Hence,

$$I = \frac{3}{2}t^2 + 2\int \frac{1}{t+1}\,dt + \int \frac{t-2}{t^2-t+1}\,dt$$

$$= \frac{3}{2}t^2 + 2\log(t+1) + \frac{1}{2}\int \frac{2t-1-3}{t^2-t+1}\,dt$$

$$= \frac{3}{2}t^2 + 2\log(t+1) + \frac{1}{2}\int \frac{2t-1}{t^2-t+1}\,dt - \frac{3}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dt$$

$$= \frac{3}{2}t^2 + 2\log(t+1) + \frac{1}{2}\log(t^2-t+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= \frac{3}{2}x^{2/3} + 2\log(x^{1/3}+1) + \frac{1}{2}\log(x^{2/3}-x^{1/3}+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^{1/3}-1}{\sqrt{3}}\right) + C$$

EXAMPLE 14 *Evaluate*: $\int \frac{2x^4-4x^3+3x^2+2x-1}{(x^3-2x^2+2x-1)^2}\,dx$

SOLUTION We have,

$$x^3-2x^2+2x-1 = (x-1)(x^2-x+1)$$

and, $2x^4-4x^3+3x^2+2x-1 = 2(x^2-x+1)^2 - 3(x-1)^2$

$$\therefore I = \int \frac{2x^4-4x^3+3x^2+2x-1}{(x^3-2x^2+2x-1)^2}\,dx$$

$$= \int \frac{2(x^2-x+1)^2 - 3(x-1)^2}{(x-1)^2(x^2-x+1)^2}\,dx$$

$$= 2\int \frac{1}{(x-1)^2}\,dx - 3\int \frac{1}{(x^2-x+1)^2}\,dx$$

$$= \frac{-2}{x-1} - 3\int \frac{1}{\left\{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}^2}\,dx$$

$$= \frac{-2}{x-1} - 3\int \frac{1}{(t^2+a^2)^2}\,dt, \text{ where } t = x-\frac{1}{2} \text{ and } a = \frac{\sqrt{3}}{2}$$

$$= \frac{-2}{x-1} - 3\left\{\frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{t}{2a}\right)\right\} + C$$

[See ex. 28 on page 10.61]

$$= \frac{-2}{x-1} - \frac{2x-1}{x^2-x+1} - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

EXAMPLE 15 *Evaluate*: $\int \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\,d\theta$

SOLUTION We have,

$$I = \int \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\,d\theta$$

$$= \int \frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^6\theta\text{]}$$

$$= \frac{1}{3}\int \frac{d(1+\tan^3\theta)}{(1+\tan^3\theta)^2}$$

$$= -\frac{1}{3(1+\tan^3\theta)} + C$$

EXAMPLE 16 *Evaluate*: $\int \frac{1+x\cos x}{x(1-x^2e^{2\sin x})}\,dx$

SOLUTION We have,

$$I = \int \frac{1+x\cos x}{x\{1-(xe^{\sin x})^2\}}\,dx$$

$$= \int \frac{(1+x\cos x)e^{\sin x}}{xe^{\sin x}\{1-(xe^{\sin x})^2\}}\,dx$$

$$= \int \frac{1}{xe^{\sin x}\{1-(xe^{\sin x})^2\}}\,d(xe^{\sin x})$$

$$= \int \frac{dt}{t(1-t^2)}, \text{ where } t = xe^{\sin x}$$

$$= \int \left(\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right)dt$$

$$= \log|t| - \frac{1}{2}\log|1-t| - \frac{1}{2}\log|1+t| + C$$

$$= \frac{1}{2}\log\left|\frac{t^2}{1-t^2}\right| + C = \frac{1}{2}\log\left|\frac{x^2 e^{2\sin x}}{1-x^2 e^{2\sin x}}\right| + C$$

EXAMPLE 17 *Evaluate :* $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$ [IIT 1999]

SOLUTION Let $I = \int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$, Then,

$$I = \int \frac{(x^3+x)+2(x+1)}{(x^2+1)^2(x+1)}\,dx$$

$$= \int \frac{x}{(x^2+1)(x+1)}\,dx + 2\int \frac{1}{(x^2+1)^2}\,dx$$

$= I_1 + 2I_2$, where

$$I_1 = \int \frac{x}{(x^2+1)(x+1)}\,dx \text{ and } I_2 = \int \frac{1}{(x^2+1)^2}\,dx$$

Now,

$$I_1 = \int \frac{x}{(x^2+1)(x+1)}\,dx$$

$$= \int \frac{1}{2}\left(\frac{x+1}{x^2+1} - \frac{1}{x+1}\right)dx \qquad \text{[Using partial fractions]}$$

$$= \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\log|x+1|$$

and,

$$I_2 = \int \frac{1}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\frac{x}{(x^2+1)} + \frac{1}{2}\tan^{-1}x.$$

[Replacing a by 1 in Ex 29 page 10.63]

Hence,

$$I = I_1 + 2I_2$$

$$= \frac{1}{4}\log(x^2+1) + \frac{x}{x^2+1} + \frac{3}{2}\tan^{-1}x - \frac{1}{2}\log|x+1| + C$$

EXERCISE 10.8

Evaluate the following integrals :

1. $\int \frac{x^5}{1-x^9}\,dx$ [**Hint** : Put $x^3 = t$]
2. $\int \frac{1}{x(1+x+x^2+x^3)}\,dx$
3. $\int \frac{1}{x^3-2x^2-x+2}\,dx$
4. $\int \frac{1}{\tan x + 4\cot x + 4}\,dx$
5. $\int \frac{\tan x}{\tan^2 x + \tan x + 1}\,dx$
6. $\int \frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}\,dx$
7. $\int \frac{x^5-x^4-26x^2-24x-25}{(x^2+4x+5)^2(x^2+4)^2}\,dx$
8. $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}\,dx$
9. $\int \frac{x^9}{(x^4-1)^2}\,dx$
10. $\int \frac{4x^2-8x}{(x-1)^2(x^2+1)^2}\,dx$
11. $\int \frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}\,dx$
12. $\int \frac{(3+x^2)^2 x^3}{(1+x^2)^3}\,dx$
13. $\int \frac{\sec x}{1+\operatorname{cosec} x}\,dx$
14. $\int \frac{4x^4-15x^2+11x+13}{2x^3+3x^2-3x-2}\,dx$
15. $\int \frac{x^4-x^3+5x^2+x+3}{(x+1)(x^2-x+1)^2}\,dx$

ANSWERS

1. $-\frac{1}{9}\log(x^3-1) + \frac{1}{18}\log(x^6+x^3+1) - \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x^3+1}{\sqrt{3}}\right) + C$
2. $\log|x| - \frac{1}{2}\log(x+1) - \frac{1}{2}\tan^{-1}x - \frac{1}{4}\log(1+x^2) + C$
3. $\frac{1}{3}\log|x-2| - \frac{1}{2}\log|x-1| + \frac{1}{6}\log|x+1| + C$

4. $\frac{4}{25}x-\frac{3}{25}\log|\tan x+2|+\frac{2}{5(\tan x+2)}$

$-\frac{3}{25}\log|\cos x|+C$

5. $x-\frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{2\tan x+1}{\sqrt{3}}\right\}+C$

6. $\frac{3-7x-2x^2}{2(x^3-x^2-x+1)}+\log\frac{|x-1|}{(x+1)^2}+C$

7. $\frac{-x}{8(x^2+4)}-\frac{2x+5}{2(x^2+4x+5)}-\frac{1}{16}\tan^{-1}\frac{x}{2}-\tan^{-1}(x+2)+C$

8. $\frac{1}{4}\left|\log\left(\frac{\sqrt{x^2+1}}{|x-1|}\right)+\tan^{-1}x-\frac{7}{(x-1)^2}\right|+C$

9. $\frac{1}{4}\left|\frac{2x^6-3x^2}{x^4-1}+\frac{3}{2}\log\left|\frac{x^2-1}{x^2+1}\right|\right|+C$

10. $\frac{3x^2-x}{(x-1)(x^2+1)}+\log\left|\frac{(x-1)^2}{x^2+1}\right|+\tan^{-1}x+C$

11. $\frac{1}{x^2(x^2+1)}+\log\sqrt{x^2+1}+C$

12. $\frac{1}{2}x^2+\frac{3}{2}\log(1+x^2)+\frac{1}{(1+x^2)^2}+C$

14. $x^2-3x+\frac{13}{9}\log|x-1|-\frac{5}{9}\log(x+2)-\frac{8}{9}\log(2x+1)+C$

15. $\log|x+1|+\frac{14\sqrt{3}}{9}\tan^{-1}\left(\frac{2x-1}{3}\right)+\frac{4x-5}{3(x^2-x+1)}+C$

HINT TO SELECTED PROBLEMS

6. Given integral $=\int\frac{5-3x+6x^2+5x^3-x^4}{(x-1)^3(x+1)^2}\,dx$

$\therefore\ \frac{4x^4-15x^2+11x+13}{2x^3+3x^2-3x-2}$

14. We have, $2x^3+3x^2-3x-2=(x-1)(x+2)(2x+1)$

$=(2x-3)+\frac{6x+7}{2x^3+3x^2-3x-2}=2x-3+\frac{6x+7}{(x-1)(x+2)(2x+1)}$

—❑ ❑ ❑ —

10.21 INTEGRATION SOME IRRATIONAL ALGEBRAIC FUNCTIONS

In the previous sections, we have learnt some methods to integrate some irrational functions in standard forms. In this section, we shall discuss methods of evaluating integrals involving irrational algebraic functions.

INTEGRALS OF THE FORM $\int\frac{\phi(x)}{P(x)\sqrt{Q(x)}}\,dx$, where $P(x)$ and $Q(x)$ are linear expressions in x.

To evaluate this type of integrals, we substitute $Q(x)=t^2$. This substitution will reduce the integral to one of the standard forms.

Following illustration will illustrate the above procedure.

ILLUSTRATION *Evaluate :*

(i) $\int\frac{1}{(x+1)\sqrt{x^2-1}}\,dx$ (ii) $\int\frac{1}{(x-1)\sqrt{x^2+4}}\,dx$

SOLUTION (i) Let $I=\int\frac{1}{(x+1)\sqrt{x^2-1}}\,dx$.

Let $x+1=\frac{1}{t}$ Then, $dx=-\frac{1}{t^2}\,dt$.

$\therefore\ I=\int\frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}\cdot\left(-\frac{1}{t^2}\right)dt$

$=-\int\frac{dt}{\sqrt{1-2t}}$

$=-\int(1-2t)^{-1/2}\,dt$

$=-\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)}+C$

$=\sqrt{1-2t}+C$

$=\sqrt{1-\frac{2}{x+1}}+C$

$=\sqrt{\frac{x-1}{x+1}}+C$

(ii) Let $I=\int\frac{1}{(x-1)\sqrt{x^2+4}}\,dx$.

Let $x-1=\frac{1}{t}$ Then, $dx=-\frac{1}{t^2}\,dt$

$\therefore\ I=\int\frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+4}}$

$=-\int\frac{dt}{\sqrt{5t^2+2t+1}}$

$$\int \frac{dt}{\sqrt{t^2+\frac{2}{5}t+\frac{1}{5}}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{\left(t^2+\frac{2}{5}t+\frac{1}{25}\right)+\frac{1}{5}-\frac{1}{25}}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\left(t+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}}dt$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(t+\frac{1}{5}\right)+\sqrt{\left(t+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}\right|+C$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(t+\frac{1}{5}\right)+\sqrt{\left(t^2+\frac{2}{5}t+\frac{1}{5}\right)}\right|+C$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\frac{1}{(x-1)^2}+\frac{2}{5(x-1)}+\frac{1}{5}}\right|+C$$

$$= -\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}+\sqrt{\frac{x^2+4}{5(x-1)^2}}\right|+C$$

INTEGRALS OF THE FORM $\int \frac{\phi(x)}{P\sqrt{Q}}dx$, where p and q both are pure quadratic expression in x i.e. $p = ax^2 + b$ and $q = cx^2 + d$

To evaluate this type of integrals we put $x = \frac{1}{t}$ and then $c + dt^2 = u^2$ *i.e.*, to evaluate integrals of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$, we put $x = \frac{1}{t}$ to obtain $\int \frac{-tdt}{(a+bt^2)\sqrt{c+dt^2}}$ and then we substitute $c + dt^2 = u^2$.

The following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* : $\int \frac{1}{x^2\sqrt{1+x^2}}dx$

SOLUTION Let $I = \int \frac{1}{x^2\sqrt{1+x^2}}dx$.

Let $x = \frac{1}{t}$ Then $-\frac{1}{x^2}dx = dt \Rightarrow dx = -x^2\,dt$

$\therefore\ I = \int \frac{-dt}{\sqrt{1+\frac{1}{t^2}}}$

$= -\int \frac{t\,dt}{\sqrt{t^2+1}}$

$= -\int \frac{u\,du}{\sqrt{u^2}}$, where $t^2 + 1 = u^2$

$= \int -1 \cdot du$

$= -u + C$

$= -\sqrt{t^2+1} + C$

$= -\sqrt{\frac{1}{x^2}+1} + C$

$= -\frac{\sqrt{1+x^2}}{x} + C.$

EXAMPLE 2 *Evaluate* : $\int \frac{\sqrt{1+x^2}}{1-x^2}dx.$

SOLUTION We have,

$$I = \int \frac{\sqrt{1+x^2}}{1-x^2}dx$$

$$= \int \frac{\sqrt{1+x^2}}{1-x^2} \times \sqrt{\frac{1+x^2}{1+x^2}}\,dx$$

$$= \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2}}dx$$

$$= \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx + \int \frac{x^2}{(1-x^2)\sqrt{1+x^2}}dx$$

$$= \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx - \int \frac{-x^2}{(1-x^2)\sqrt{1+x^2}}dx$$

$$= \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx - \int \frac{(1-x^2)-1}{(1-x^2)\sqrt{1+x^2}}dx$$

$$= \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx - \int \left|\frac{1-x^2}{(1-x^2)\sqrt{1+x^2}} - \frac{1}{(1-x^2)\sqrt{1+x^2}}\right|dx$$

$$= \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx - \int \frac{1}{\sqrt{1+x^2}}dx + \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx$$

$$= 2\int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx - \int \frac{1}{\sqrt{1+x^2}}dx$$

$$= 2I_1 - \log|x + \sqrt{1+x^2}| + C, \text{ where}$$

$$I_1 = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}dx.$$

Let $x=\frac{1}{t}$, Then, $dx=-\frac{1}{t^2}dt$.

$$\therefore I_1=\int\frac{\left(-\frac{1}{t^2}\right)}{\left(1-\frac{1}{t^2}\right)\sqrt{1+\frac{1}{t^2}}}dt$$

$$=-\int\frac{t}{(t^2-1)\sqrt{t^2+1}}dt$$

$$=-\int\frac{u\,du}{(u^2-2)\sqrt{u^2}},\ \text{where } t^2+1=u^2.$$

$$=-\int\frac{du}{u^2-(\sqrt{2})^2}$$

$$=-\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|$$

$$=-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}}\right|$$

$$=-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}\right|$$

$$=-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}\,x}{\sqrt{1+x^2}+\sqrt{2}\,x}\right|$$

$$\therefore I=-\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}\,x}{\sqrt{1+x^2}+\sqrt{2}\,x}\right|-\log\left|x+\sqrt{1+x^2}\right|+C.$$

EXAMPLE 3 *Evaluate:* $\int\frac{2-x^2}{(1+x)\sqrt{1-x^2}}dx$

SOLUTION We have,

$$\int\frac{2-x^2}{(1+x)\sqrt{1-x^2}}dx$$

$$=\int\frac{(1-x^2)+1}{(1+x)\sqrt{1-x^2}}dx$$

$$=\int\frac{1-x^2}{(1+x)\sqrt{1-x^2}}dx+\int\frac{1}{(1+x)\sqrt{1-x^2}}dx$$

$$=\int\frac{1-x}{\sqrt{1-x^2}}dx+\int\frac{1}{(1+x)\sqrt{1-x^2}}dx$$

$$=\int\frac{1}{\sqrt{1-x^2}}dx+\frac{1}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx+\int\frac{1}{(1+x)\sqrt{1-x^2}}dx$$

$$=\sin^{-1}x+\sqrt{1-x^2}+I_1,\ \text{where } I_1=\int\frac{1}{(1+x)\sqrt{1-x^2}}dx$$

Putting $1+x=\frac{1}{t}$, $dx=\frac{-1}{t^2}dt$ and $x=\frac{1-t}{t}$ in I_1, we get

$$I_1=\int\frac{1}{\frac{1}{t}\sqrt{1-\left(\frac{1-t}{t}\right)^2}}\times\frac{-1}{t^2}dt$$

$$=-\int\frac{1}{\sqrt{2t-1}}dt$$

$$=-\sqrt{2t-1}$$

$$=-\sqrt{\frac{2}{1-x}-1}=-\sqrt{\frac{1-x}{1+x}}$$

$$\therefore\int\frac{2-x^2}{(1+x)\sqrt{1-x^2}}dx=\sin^{-1}x+\sqrt{1-x^2}-\sqrt{\frac{1-x}{1+x}}+C$$

EXAMPLE 4 *Evaluate:* $\int\sqrt{\tan^2x+2}\,dx$.

SOLUTION Let $I=\int\sqrt{\tan^2x+2}\,dx$.

Putting $\tan^2x+2=t^2$ and $2\tan x\sec^2x\,dx=2t\,dt$,

or $dx=\frac{2t\,dt}{2\tan x\sec^2x\sqrt{t^2-2}\,(t^2-1)}=\frac{t\,dt}{\sqrt{t^2-2}\,(t^2-1)}$, we get

$$I=\int\sqrt{t^2}\cdot\frac{t\,dt}{\sqrt{t^2-2}\,(t^2-1)}$$

$$=\int\frac{t^2}{(t^2-1)\sqrt{t^2-2}}dt$$

$$=\int\frac{(t^2-1)+1}{(t^2-1)\sqrt{t^2-2}}dt$$

$$=\int\frac{1}{\sqrt{t^2-2}}dt+\int\frac{1}{t^2-1\sqrt{t^2-2}}dt$$

$$=\log\left|t+\sqrt{t^2-2}\right|+I_1\ ,\ \text{where } I_1=\int\frac{1}{(t^2-1)\sqrt{t^2-2}}dt$$

Putting $t=\frac{1}{u}$ and $dt=-\frac{1}{u^2}du$ in I_1, we get

$$I_1=-\int\frac{u\,du}{(1-u^2)\sqrt{1-2u^2}}$$

Now, putting $1-2u^2=v^2$ and $-2u\,du=v\,dv$, we get

$$I_1=\frac{1}{2}\int\frac{v\,dv}{\left(1-\frac{1-v^2}{2}\right)\sqrt{v^2}}$$

$$=\int\frac{dv}{(1+v^2)}$$

$$=\tan^{-1}v$$

$$=\tan^{-1}\sqrt{1-2u^2}=\tan^{-1}\left(\frac{\sqrt{t^2-2}}{t}\right)$$

$$\therefore I=\log\left|t+\sqrt{t^2-2}\right|+\tan^{-1}\left(\frac{\sqrt{t^2-2}}{t}\right)+C$$

$$=\log\left|\sqrt{\tan^2x+2}+\tan x\right|+\tan^{-1}\left(\frac{\tan x}{\sqrt{\tan^2x+2}}\right)+C$$

EXAMPLE 5 *Evaluate :* $\int \frac{1}{(x^2+3x+2)\sqrt{x^2-1}}dx$

SOLUTION We have,

$$I = \int \frac{1}{(x^2+3x+2)\sqrt{x^2-1}}dx$$

$$= \int \frac{1}{(x+1)(x+2)\sqrt{x^2-1}}dx$$

$$= \int \left(\frac{1}{x+1}-\frac{1}{x+2}\right)\frac{1}{\sqrt{x^2-1}}dx$$

$$= \int \frac{1}{(x+1)\sqrt{x^2-1}}dx - \int \frac{1}{(x+2)\sqrt{x^2-1}}dx$$

$= I_1 - I_2$, where

$I_1 = \int \frac{1}{(x+1)\sqrt{x^2-1}}dx$ and, $I_2 = \int \frac{1}{(x+2)\sqrt{x^2-1}}dx$

Putting $x+1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we have

$$I_1 = -\int \frac{1}{\sqrt{1-2t}}dt = \sqrt{1-2t} = \sqrt{\frac{x-1}{x+1}}$$

Putting $x+2 = \frac{1}{u}$ and $dx = \frac{-1}{u^2}du$, we have

$$I_2 = -\int \frac{1}{\sqrt{(1-2u)^2-u^2}}du$$

$$= -\int \frac{1}{\sqrt{1-4u+3u^2}}du$$

$$= -\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{u^2-\frac{4}{3}u+\frac{1}{3}}}du$$

$$= -\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\left(u-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}}du$$

$$= -\frac{1}{\sqrt{3}}\log\left|\left(u-\frac{2}{3}\right)+\sqrt{u^2-\frac{4}{3}u+\frac{1}{3}}\right|$$

$$= -\frac{1}{\sqrt{3}}\log\left|\frac{-2x-1}{3(x+2)}+\sqrt{\frac{x^2-1}{3(x+2)^2}}\right|$$

Hence,

$$I = \sqrt{\frac{x-1}{x+1}} - \frac{1}{\sqrt{3}}\log\left|\frac{-2x-1+\sqrt{3}\sqrt{x^2-1}}{3(x+2)}\right| + C$$

EXAMPLE 6 *Evaluate :* $\int \frac{x^3+x^2+x-1}{(x^2-1)\sqrt{2-x^2}}dx$

SOLUTION We have,

$$I = \int \frac{x^3+x^2+x-1}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{(x^2-1)+(x^3+x)}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}dx + \int \frac{x^3+x}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}dx + \int \frac{x(x^2-1)+2x}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}dx + \int \frac{x}{\sqrt{2-x^2}}dx + \int \frac{2x}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}dx + \int \frac{x}{\sqrt{2-x^2}}dx + \int \frac{(x-1)+(x+1)}{(x^2-1)\sqrt{2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2-x^2}}dx + \int \frac{x}{\sqrt{2-x^2}}dx + \int \frac{1}{(x+1)\sqrt{2-x^2}}dx + \int \frac{1}{(x-1)\sqrt{2-x^2}}dx$$

$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sqrt{2-x^2} + I_1 + I_2 + C$, where

$I_1 = \int \frac{1}{(x+1)\sqrt{2-x^2}}dx$ and $I_2 = \int \frac{1}{(x-1)\sqrt{2-x^2}}dx$

Putting $x+1 = \frac{1}{u}$ in I_1 and $x-1 = \frac{1}{v}$ in I_2, we obtain

$$I_1 = \log\left|\frac{x+1}{2+x+\sqrt{2-x^2}}\right| \text{ and } I_2 = \log\left|\frac{x-1}{2-x+\sqrt{2-x^2}}\right|$$

Hence,

$$I = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sqrt{2-x^2} + \log\left|\frac{x+1}{2+x+\sqrt{2-x^2}}\right| + \log\left|\frac{x-1}{2-x+\sqrt{2-x^2}}\right| + C$$

EXAMPLE 8 *Evaluate :* $\int \frac{x+4}{(x-1)(x+2)^2\sqrt{x^2+x+1}}dx$

SOLUTION Resolving $\frac{x+4}{(x-1)(x+2)^2}$ into partial fractions, we obtain

$$\frac{x+4}{(x-1)(x+2)^2} = \frac{5}{9}\cdot\frac{1}{x-1} - \frac{2}{3(x+2)^2} - \frac{5}{9(x+2)}$$

$$I = \int \frac{x+4}{(x-1)(x+2)^2\sqrt{x^2+x+1}}dx$$

$$= \int \left\{\frac{5}{9}\cdot\frac{1}{x-1} - \frac{5}{9}\cdot\frac{1}{x+2} - \frac{2}{3}\cdot\frac{1}{(x+2)^2}\right\}\frac{1}{\sqrt{x^2+x+1}}dx$$

$$= \frac{5}{9}\int \frac{1}{(x-1)\sqrt{x^2+x+1}}dx - \frac{5}{9}\int \frac{1}{(x+2)\sqrt{x^2+x+1}}dx - \frac{2}{3}\int \frac{1}{(x+2)^2\sqrt{x^2+x+1}}dx$$

$$= \frac{5}{9}I_1 - \frac{5}{9}I_2 - \frac{2}{3}I_3 \quad \text{...(i)}$$

where

$$I_1=\int \frac{1}{(x-1)\sqrt{x^2+x+1}}dx, I_2=\int \frac{1}{(x+2)\sqrt{x^2+x+1}}dx$$

and, $I_3=\int \frac{1}{(x+2)^2\sqrt{x^2+x+1}}dx$

Putting $x-1=\frac{1}{t}$ in I_1, we get

$$I_1=-\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}(x+1)+2\sqrt{x^2+x+1}}{2\sqrt{3}(x-1)}\right|$$

Putting $x+2=\frac{1}{u}$ in I_2 and I_3, we get

$$I_2=-\frac{1}{\sqrt{3}}\log\left|\frac{-\sqrt{3}x+2\sqrt{x^2+x+1}}{2\sqrt{3}(x+2)}\right|$$

$$I_3=-\int\frac{u\,du}{\sqrt{3u^2-3u+1}}$$

$$=-\frac{1}{6}\int\frac{6u-3+3}{\sqrt{3u^2-3u+1}}du$$

$$=-\frac{1}{6}\int\frac{6u-3}{\sqrt{3u^2-3u+1}}du-\frac{1}{2\sqrt{3}}\int\frac{1}{\sqrt{u^2-u+\frac{1}{3}}}du$$

$$=-\frac{1}{3}\sqrt{3u^2-3u+1}-\frac{1}{2\sqrt{3}}\log\left|\left(u-\frac{1}{2}\right)+\sqrt{u^2-u+\frac{1}{3}}\right|$$

$$=-\frac{\sqrt{x^2+x+1}}{3(x+2)}-\frac{1}{2\sqrt{3}}\log\left|\frac{-\sqrt{3}x+2\sqrt{x^2+x+1}}{2\sqrt{3}(x+2)}\right|$$

Hence,

$$I=\frac{-5}{9\sqrt{3}}\log\left|\frac{\sqrt{3}(x+1)+2\sqrt{x^2+x+1}}{2\sqrt{3}(x-1)}\right|+\frac{5}{9\sqrt{3}}\log\left|\frac{-\sqrt{3}x+2\sqrt{x^2+x+1}}{2\sqrt{3}(x+2)}\right|+\frac{2}{9}\frac{\sqrt{x^2+x+1}}{x+2}+\frac{1}{3\sqrt{3}}\log\left|\frac{-\sqrt{3}x+2\sqrt{x^2+x+1}}{2\sqrt{3}(x+2)}\right|+C$$

EXAMPLE 9 *Evaluate :* $\int\frac{\log(1+x^2)}{x^3}dx$

SOLUTION Let $I=\int\frac{\log(1+x^2)}{x^3}dx$

$$\Rightarrow I=\int \underset{\text{I}}{\log(1+x^2)}\cdot\underset{\text{II}}{x^{-3}}\,dx$$

$$=-\frac{1}{2x^2}\log(1+x^2)+\int\frac{x\,dx}{x^2(1+x^2)}$$

$$=-\frac{1}{2x^2}\log(1+x^2)+\frac{1}{2}\int\frac{dt}{t(1+t)},\ \text{where } t=x^2$$

$$=-\frac{1}{2x^2}\log(1+x^2)+\frac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=-\frac{1}{2x^2}\log(1+x^2)+\frac{1}{2}\log\left|\frac{t}{t+1}\right|+C,\ \text{where } t=x^2.$$

EXAMPLE 10 *Evalate :* $\int\frac{1}{1-\sin^4 x}dx$

SOLUTION We have,

$$I=\int\frac{1}{1-\sin^4 x}dx$$

$$=\int\frac{1}{\cos^2 x(1+\sin^2 x)}dx$$

$$=\int\frac{\sec^4 x}{\sec^2 x+\tan^2 x}dx$$

$$=\int\frac{1+\tan^2 x}{1+2\tan^2 x}d(\tan x)$$

$$=\int\frac{1+t^2}{1+2t^2}dt,\ t=\tan x$$

$$=\frac{1}{2}\int 1\,.\,dt+\frac{1}{2}\int\frac{1}{1+2t^2}dt$$

$$=\frac{1}{2}t+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,t)+C$$

$$=\frac{1}{2}\tan x+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+C$$

EXERCISE 10.9

Evaluate the following integrals :

1. $\int\frac{1}{(x-1)\sqrt{x+2}}dx$
2. $\int\frac{1}{(x-1)\sqrt{2x+3}}dx$
3. $\int\frac{x+1}{(x-1)\sqrt{x+2}}dx$
4. $\int\frac{x^2}{(x-1)\sqrt{x+2}}dx$
5. $\int\frac{x}{(x-3)\sqrt{x+1}}dx$
6. $\int\frac{1}{(x^2+1)\sqrt{x}}dx$
7. $\int\frac{x}{(x^2+2x+2)\sqrt{x+1}}dx$
8. $\int\frac{1}{(x-1)\sqrt{x^2+1}}dx$
9. $\int\frac{1}{(x+1)\sqrt{x^2+x+1}}dx$
10. $\int\frac{1}{(x^2-1)\sqrt{x^2+1}}dx$
11. $\int\frac{x}{(x^2+4)\sqrt{x^2+1}}dx$
12. $\int\frac{1}{(1+x^2)\sqrt{1-x^2}}dx$

13. $\int \frac{1}{(2x^2+3)\sqrt{x^2-4}}dx$ 14. $\int \frac{x}{(x^2+4)\sqrt{x^2+9}}dx$

15. $\int \frac{\left\{1-\sqrt{1+x+x^2}\right\}^2}{x^2\sqrt{1+x+x^2}}\,dx$

ANSWERS

1. $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+C$

2. $\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{2x+3}-\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right|+C$

3. $2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+C$

4. $\frac{2}{3}(x+2)^{3/2}-2\sqrt{x+2}+\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+C$

5. $2\sqrt{x+1}+\frac{3}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C$

6. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right|+C$

7. $\frac{1}{\sqrt{2}}\log\left|\frac{(x+2)-\sqrt{2}(x+1)}{(x+2)+\sqrt{2}(x+1)}\right|+C$

8. $-\frac{1}{\sqrt{2}}\log\left|\left(t+\frac{1}{2}\right)+\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{4}}\right|+C$, where $t=\frac{1}{x-1}$.

9. $-\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+C$

10. $-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}\,x+\sqrt{x^2+1}}{\sqrt{2}\,x-\sqrt{x^2+1}}\right|+C$

11. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{x^2+1}{3}}\right)+C$

12. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{1-x^2}{2x^2}}\right)+C$

13. $\frac{1}{2\sqrt{33}}\log\left|\frac{\sqrt{11}\,x+\sqrt{3x^2-12}}{\sqrt{11}\,x-\sqrt{3x^2-12}}\right|+C$

14. $\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{x^2+9}-\sqrt{5}}{\sqrt{x^2+9}+\sqrt{5}}\right|+C$

15. $-2\sqrt{1+x+x^2}-\log\left|x+\frac{1}{2}+\sqrt{1+x+x^2}\right|+\frac{2}{x}+C$

—❑❑❑—

INTEGRALS OF THE FORM $\int f(x,(ax+b)^{m_1/n_1},(ax+b)^{m_2/n_2})\,dx$, *where* f *is a function and* m_1, n_1, m_2, n_2 *are integers*

To evaluate this type of integrals, we follow the following algorithm.

ALGORITHM

STEP I *Obtain* n_1, n_2, n_3 .....

STEP II *Find the l.c.m of* n_1, n_2, n_3 .... *Let it be s.*

STEP III *Substitute* $ax+b=t^s$. *This substitution will reduce the integrand to a rational function of t.*

STEP IV *Evaluate the integral obtained in step III by the methods discussed in the previous sections.*

STEP V *Replace t by* $(ax+b)^{1/s}$.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\int \frac{1}{\sqrt[3]{x+1}+\sqrt{x+1}}dx$.

SOLUTION Let $I=\int \frac{1}{(x+1)^{1/3}+(x+1)^{1/2}}dx$.

Here, l.c.m of 2 and 3 is 6. So, we substitue $x+1=t^6$ and $dx=6t^5dt$.

$\therefore I=\int \frac{1}{t^2+t^3}6t^5\,dt$

$=6\int \frac{t^3}{t+1}dt$

$=6\int \frac{t^3+1-1}{t+1}dt$

$=6\int \frac{t^3+1}{t+1}dt-6\int \frac{1}{t+1}dt$

$=6\int (t^2-t+1)\,dt-6\int \frac{1}{t+1}dt$

$=6\left(\frac{t^3}{3}-\frac{t^2}{2}+t\right)-6\log|t+1|+C$

$=2t^3-3t^2+6t-6\log|t+1|+C$

$$= 2(x+1)^{1/2} - 3(x+1)^{1/3} + 6(x+1)^{1/6} - 6\log\left|(x+1)^{1/6}+1\right| + C$$

EXAMPLE 2 *Evaluate:* $\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}dx$

SOLUTION We have,

$$I = \int \frac{x^{1/2}}{x^{1/2}-x^{1/3}}dx$$

Clearly, l.c.m. of 2 and 3 is 6. So, by

Putting $x = t^6$ and $dx = 6t^5\,dt$, we get

$$I = \int \frac{t^3}{t^3-t^2}\cdot 6t^5\,dt$$

$$= 6\int \frac{t^6}{t-1}dt$$

$$= 6\int \frac{t^6-1+1}{t-1}dt$$

$$= 6\int t^5+t^4+t^2+t+1+\frac{1}{t-1}dt$$

$$= 6\left\{\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^3}{3}+\frac{t^2}{2}+t+\log(t-1)\right\}+C$$

$$= 6\left\{\frac{x}{6}+\frac{x^{5/6}}{5}+\frac{x^{1/2}}{3}+\frac{x^{1/3}}{2}+x^{1/6}+\log(x^{1/6}-1)\right\}+C$$

EXAMPLE 3 *Evaluate:* $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}+2.\sqrt[4]{x}}\,dx$

SOLUTION We have,

$$I = \int \frac{1}{\sqrt{x}+\sqrt[3]{x}+2.\sqrt[4]{x}}\,dx$$

$$= \int \frac{1}{x^{1/2}+x^{1/3}+2x^{1/4}}dx$$

We find that the l.c.m. of 2, 3 and 4 is 12. So, we substitute $x = t^{12}$ and $dx = 12t^{11}\,dt$.

$$\therefore\ I = \int \frac{2t^{11}}{t^6+t^4+2t^3}dt$$

$$= 12\int \frac{t^8}{t^3+t+2}dt$$

$$= 12\int t^5 - t^3 - 2t^2 + t + 4 + \frac{3t^2-6t-8}{t^3+t+2}dt$$

$$= 12\left\{\frac{t^6}{6}-\frac{t^4}{4}-\frac{2}{3}t^3+\frac{t^2}{2}+4t\right\}+12\int \frac{3t^2-6t-8}{t^3+t+2}dt$$

$$= (2t^6-3t^4-8t^3+6t^2+48t)+12I_1 \qquad ...(i)$$

where $I_1 = \int \frac{3t^2-6t+8}{t^3+t+2}dt$

We have, $t^3+t+2 = (t^3+1)+(t+1) = (t+1)(t^2-t+2)$.

So, let

$$\frac{3t^2-6t+8}{t^3+t+2} = \frac{A}{t+1}+\frac{Bt+C}{t^2-t+2}$$

$$\Rightarrow 3t^2-6t+8 = A(t^2-t+2)+(Bt+C)(t+1)$$

Putting $t = -1, 0$ and 1 successively, we get

$$17 = 4A,\ 8 = 2A+C \text{ and } 5 = 2A+2B+2C$$

$$\Rightarrow A = \frac{17}{4},\ C = -\frac{1}{2} \text{ and } B = -\frac{5}{4}$$

$$\therefore \frac{3t^2-6t+8}{t^3+t+2} = \frac{17}{4}\cdot\frac{1}{t+1}-\frac{5t+2}{2(t^2-t+2)}$$

$$\Rightarrow I_1 = \int \frac{3t^2-6t+8}{t^3+t+2}dt$$

$$= \frac{17}{4}\int \frac{1}{t+1}dt - \frac{1}{2}\int \frac{5t+2}{t^2-t+2}dt$$

$$= \frac{17}{4}\log|t+1| - \frac{5}{4}\int \frac{2t+\frac{4}{5}}{t^2-t+2}dt$$

$$= \frac{17}{4}\log|t+1| - \frac{5}{4}\int \frac{2t-1+\frac{9}{5}}{t^2-t+2}dt$$

$$= \frac{17}{4}\log|t+1| - \frac{5}{4}\int \frac{2t-1}{t^2-t+2} - \frac{9}{4}\int \frac{1}{t^2-t+2}dt$$

$$= \frac{17}{4}\log|t+1| - \frac{5}{4}\log(t^2-t+2) - \frac{9}{4}\int \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}dt$$

$$= \frac{17}{4}\log|t+1| - \frac{5}{4}\log(t^2-t+2) - \frac{9}{2\sqrt{7}}\tan^{-1}\left(\frac{2t-1}{\sqrt{7}}\right)+C$$

Substituting the value of I_1 in (i), we get

$$I = 2t^6-3t^4-8t^3+6t^2+48t+51\log|t+1| - 15\log(t^2-t+2)+\frac{54}{\sqrt{7}}\tan^{-1}\left(\frac{2t+1}{\sqrt{7}}\right)+C,\ \text{where } t = x^{1/12}.$$

EXAMPLE 4 *Evaluate:* $\int \left|\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\log(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right|dx$ **[IIT 1992]**

SOLUTION Let

$$\int \left\{\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\log(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right\}dx = I_1+I_2, \text{ where}$$

$$I_1 = \int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx \text{ and } I_2 = \int \frac{\log(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$$

Clearly, LCM of 3 and 4 is 12. So, putting $x = t^{12}$ and $dx = 12t^{11}\,dt$, we get

$$I_1 = \int \frac{12\,t^{11}}{t^4+t^3}\,dt$$

$$= 12\int \frac{t^8}{t+1}\,dt$$

$$= 12\int \left\{t^7 - t^6 + t^5 - t^4 + t^2 + t - 1 + \frac{1}{t+1}\right\} dt$$

$$= 12\left\{\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t + \log|t+1|\right\}$$

$$= 12\left[\frac{x^{2/3}}{8} - \frac{x^{7/12}}{7} + \frac{x^{1/2}}{6} - \frac{x^{5/12}}{5} + \frac{x^{1/4}}{3} + \frac{x^{1/6}}{2} - x^{1/12} + \log|x^{1/12}+1|\right]$$

Now, l.c.m. of 2, 3 and 6 is 6. So, putting $x = u^6$ and $dx = 6u^5\,du$ in I_2, we get

$$I_2 = \int \frac{\log(1+u)}{u^2+u^3}\cdot 6u^5\,du$$

$$= 6\int \frac{u^3}{u+1}\log(1+u)\,dx$$

$$= 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)\log(1+u)\,du$$

$$= 6\int \underset{\text{II}}{(u^2-u+1)}\,\underset{\text{I}}{\log(u+1)}\,du - 6\int \frac{1}{u+1}\log(1+u)\,du$$

$$= 6\left[\log(u+1)\cdot\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right) - \int \frac{1}{u+1}\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right)du\right] - \frac{6}{2}\{\log(1+u)\}^2$$

$$= 6\left[\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right)\log(u+1)\right] - \int \frac{2u^3 - 3u^2 + 6u}{u+1}\,du - 3\{\log(1+u)\}^2$$

$$= (2u^3 - 3u^2 + 6u)\log(u+1) - 3[\log(1+u)]^2 - \int\left(2u^2 - 5u + 11 - \frac{11}{u+1}\right)du$$

$$= (2u^3 - 3u^2 + 6u)\log(u+1) - 3(\log(1+u))^2 - \left\{\frac{2u^3}{3} - \frac{5u^2}{2} + 11u - 11\log(u+1)\right\}$$

$$= (2x^{1/2} - 3x^{1/3} + 6x^{1/6})\log(x^{1/6}+1) - 3[\log(1+x^{1/6}]^2 - \frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3} + 11x^{1/6} - 11\log(x^{1/6}+1)$$

Substituting the values of I_1 and I_2 in (i) we get the value of the given integral.

EXERCISE 10.10

Evaluate the following integrals :

1. $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}}\,dx$
2. $\int \frac{\sqrt{x}}{\sqrt[3]{x^2} - \sqrt[4]{x}}\,dx$
3. $\int \frac{1}{\sqrt[3]{x}\,(\sqrt[3]{x} - 1)}\,dx$
4. $\int \frac{\sqrt[3]{x}}{x(\sqrt{x} + \sqrt[3]{x})}\,dx$
5. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3} + 1}\,dx$
6. $\int \frac{1}{x(\sqrt{x} + \sqrt[5]{x^2})}\,dx$
7. $\int \frac{x}{\sqrt{x+1} + \sqrt[3]{x+1}}\,dx$
8. $\int \frac{x^2 + \sqrt{x+1}}{\sqrt[3]{x+1}}\,dx$

ANSWERS

1. $2\sqrt{x} - 4x^{1/4} + 4\log(1 + x^{1/4}) + C$
2. $\frac{6}{5}\{x^{5/6} + 2x^{5/12} + 2\log|x^{5/12} - 1|\} + C$
3. $3x^{1/3} + 3\log|x^{1/3} - 1| + C$
4. $\log\left\{\frac{x}{(x^{1/6}+1)^6}\right\} + C$
5. $\frac{4}{3}\{x^{3/4} - \log(x^{3/4}+1)\} + C$
6. $\log\frac{x}{(1 + \sqrt[10]{x})^{10}} + \frac{10}{\sqrt[10]{x}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{3}\cdot\frac{1}{\sqrt[10]{x^3}} - \frac{5}{2}\cdot\frac{1}{\sqrt[5]{x^2}} + C$
7. $6\left\{\frac{1}{9}(x+1)^{3/2} - \frac{1}{8}(x+1)^{4/3} + \frac{1}{7}(x+7)^{7/6} - \frac{1}{6}(x+1) + \frac{1}{5}(x+1)^{5/6} - \frac{1}{4}(x+1)^{2/3}\right\} + C$
8. $6\sqrt[3]{(1+x)^2}\left\{\frac{(1+x)^2}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4}\right\} + C$

INEGRALS OF THE FORM $\int x^m(a+bx^n)^p\,dx$, *where m, n and p are rational numbers*

To evaluate this type of integrals, we follow the following steps :

STEP I *Obtain p. If p is an integer, the integral reduces to the integral of a rational function by using the substitution* $x = t^s$, *where s is the l.c.m. of the denominators of the fractions m and n. If p is not an integer, go to step II.*

STEP II *Compute* $\frac{m+1}{n}$. *If* $\frac{m+1}{n}$ *is an integer, says, then make the substitution* $a+bx^n=t^s$., *where* $s=\frac{m+1}{n}$

If $\frac{m+1}{n}$ *is not an integer, then go to step III.*

STEP III *Compute* $\frac{m+1}{n}+p$. *If* $\frac{m+1}{n}+p$ *is an integer, then substitute* $ax^{-n}+b=t^s$, *where s is the denominator of the fraction p.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\int \sqrt{x}\,(1+\sqrt[3]{x})^4\,dx$

SOLUTION Here, $m=\frac{1}{2}$ and $n=\frac{1}{3}$

Clearly, l.c.m. of 2 and 3 is 6. So, we substitute $x=t^6$ and $dx=6t^5\,dt$.

$$\therefore \int \sqrt{x}\,(1+\sqrt[3]{x})^4\,dx$$

$$=\int t^3(1+t^2)^4.6t^5\,dt$$

$$=6\int t^8(1+4t^2+6t^4+4t^6+t^8)\,dt$$

$$=6\int (t^8+4t^{10}+6t^{12}+4t^{14}+t^{16})\,dt$$

$$=6\left\{\frac{t^9}{9}+\frac{4t^{11}}{11}+\frac{6t^{13}}{13}+\frac{4t^{15}}{15}+\frac{t^{17}}{17}\right\}+C$$

$$=\frac{2}{3}x^{3/2}+\frac{24}{11}x^{11/6}+\frac{36}{13}x^{13/6}+\frac{8}{5}x^{5/2}+\frac{6}{17}x^{17/6}+C$$

EXAMPLE 2 *Evaluate :* $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$

SOLUTION We have,

$$\int x^5\sqrt[3]{(1+x^3)^2}\,dx=\int x^5(1+x^3)^{2/3}\,dx$$

Here, $m=5, n=3$ and $p=\frac{2}{3}$.

$$\therefore \frac{m+1}{n}=\frac{6}{3}=2 \text{ is an integer.}$$

So, we substitute $1+x^3=t^2$ and $3x^2\,dx=2t\,dt$ to obtain

$$\int x^5(1+x^3)^{2/3}\,dx$$

$$=\int x^5(t^2)^{2/3}.\frac{2t}{3x^2}\,dt$$

$$=\frac{2}{3}\int x^3t^{7/3}\,dt$$

$$=\frac{2}{3}\int (t^2-1)t^{7/3}\,dt$$

$$=\frac{2}{3}\int (t^{13/3}-t^{7/3})\,dt$$

$$=\frac{2}{3}\left[\frac{3}{16}t^{16/3}-\frac{3}{10}t^{10/3}\right]+C$$

$$=\frac{1}{8}(1+x^3)^{8/3}-\frac{1}{5}(1+x^3)^{5/3}+C$$

EXAMPLE 3 *Evaluate :* $\int x^{13/2}(1+x^{5/2})^{1/2}\,dx$

SOLUTION Here, $m=\frac{13}{2}, n=\frac{5}{2}$ and $p=\frac{1}{2}$

We have, $\frac{m+1}{n}=\frac{\frac{13}{2}+1}{\frac{5}{2}}=3$, an integer.

So, we substitute $1+x^{5/2}=t^3$ and $\frac{5}{2}x^{3/2}\,dx=3t^2\,dt$

$$\therefore \int x^{13/2}(1+x^{5/2})^{1/2}\,dx$$

$$=\int x^{13/2}t^{3/2}.\frac{6t^2}{5x^{3/2}}\,dt$$

$$=\frac{6}{5}\int x^5.t^{7/2}\,dt$$

$$=\frac{6}{5}\int (t^3-1)^2t^{7/2}\,dt$$

$$=\frac{6}{5}\int (t^6-2t^3+1)t^{7/2}\,dt$$

$$=\frac{6}{5}\int (t^{19/2}-2t^{13/2}+t^{7/2})\,dt$$

$$=\frac{6}{5}\left\{\frac{2}{21}t^{21/2}-\frac{4}{15}t^{15/2}+\frac{2}{9}t^{9/2}\right\}+C$$

$$=\frac{4}{35}(1+x^{5/2})^{7/2}-\frac{8}{25}(1+x^{5/2})^{5/2}+\frac{4}{15}(1+x^{5/2})^{3/2}+C$$

EXAMPLE 4 *Evaluate :* $\int \frac{1}{x^{11}\sqrt{1+x^4}}\,dx$.

SOLUTION We have,

$$I=\int \frac{1}{x^{11}\sqrt{1+x^4}}\,dx$$

$$=\int x^{-11}(1+x^4)^{-1/2}\,dx.$$

Here, $m=-11, n=4$ and $p=-\frac{1}{2}$

We have,

$$\frac{m+1}{n}+p=\frac{-11+1}{4}+\left(\frac{-1}{2}\right)$$

$$=\frac{-10}{4}-\frac{1}{2}=-3, \quad \text{which is an integer.}$$

Denominator of p is $s=2$. So, we substitute $ax^{-n}+b=t^s$ i.e. $x^{-4}+1=t^2$ and $-4x^{-5}\,dx=2t\,dt$.

$$\therefore I=\int x^{-11}\,[x^{-4}(x^{-4}+1)]^{-1/2}\,dx$$

$$=\int x^{-9}\,(x^{-4}+1)^{-1/2}\,dx$$

$$=\int x^{-9}\,(t^2)^{-1}\cdot\frac{2t}{-4x^{-5}}\,dt$$

$$=-\frac{1}{2}\int x^{-4}\,t^{-1}\,dt$$

$$=-\frac{1}{2}\int\frac{(t^2-1)}{t}\,dt$$

$$=-\frac{1}{2}\int\left(t-\frac{1}{t}\right)dt$$

$$=-\frac{1}{2}\left[\frac{t^2}{2}-\log t\right]+C$$

$$=\frac{-t^2}{4}+\frac{1}{2}\log t+C$$

$$=\frac{-(1+x^{-4})}{4}+\frac{1}{2}\log\sqrt{1+x^{-4}}+C$$

EXAMPLE 4 *Evaluate:* $\int\sqrt[3]{x}\left\{\sqrt[3]{1+\sqrt[3]{x^4}}\right\}dx$

SOLUTION Let $1+x^{4/3}=t^3$. Then, $\frac{4}{3}x^{1/3}\,dx=3t^2\,dt$

$$\therefore\int\sqrt[3]{x}\left\{\sqrt[3]{1+\sqrt[3]{x^4}}\right\}dx$$

$$=\frac{9}{4}\int t^3\,dt$$

$$=\frac{9t^4}{16}+C=\frac{9}{16}\,(1+x^{4/3})^{4/3}+C$$

EXERCISE 10.11

Evaluate the following integrals :

1. $\int x^{-1}(1+x^{1/3})^{-3}\,dx$
2. $\int\frac{1}{x\sqrt[3]{x^2+1}}dx$
3. $\int\frac{1}{\sqrt[3]{1+x^3}}dx$
4. $\int\frac{1}{\sqrt[4]{1+x^4}}dx$
5. $\int\frac{\sqrt{1-x^4}}{x^5}\,dx$
6. $\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$
7. $\int\frac{\sqrt{1+\sqrt{x}}}{x}\,dx$
8. $\int\frac{1}{x\sqrt[3]{1+x^5}}dx$
9. $\int\frac{\sqrt[3]{1+x^3}}{x^2}\,dx$
10. $\int\sqrt[3]{x(1-x^2)}\,dx$
11. $\int\frac{1}{\sqrt{x}\,(x^{1/4}+1)^{10}}\,dx$
12. $\int x^{1/3}(2+x^{1/2})^2\,dx$
13. $\int x^{-2/3}\,(1+x^{2/3})^{-1}\,dx$
14. $\int x^{-2/3}\,(1+x^{1/3})^{1/2}\,dx$
15. $\int x^{-11}\,(1+x^4)^{1/2}\,dx$

ANSWERS

1. $3\left|\log\left|\frac{x^{1/3}}{1+x^{1/3}}\right|+\frac{2x^{1/3}+3}{2\,(1+x^{1/3})^2}\right|+C$

2. $\frac{1}{2}\log\{(1+x^2)^{1/3}-1\}-\frac{1}{4}\log\left|(1+x^2)^{2/3}+(1+x^2)^{1/3}+1\right|$
$+\frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\,(x^2+1)^{1/3}}{\sqrt{3}}\right\}+C$

3. $\frac{1}{6}\log\frac{u^2+u+1}{(u-1)^2}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2u+1}{\sqrt{3}}+C$, where $u=\frac{(x^3+1)^{1/3}}{x}$

4. $\frac{1}{4}\log\left|\frac{(1+x^4)^{1/4}+x}{(1+x)^{1/4}-x}\right|-\frac{1}{2}\tan^{-1}\left\{\frac{(x^4+1)^{1/4}}{x}\right\}+C$

5. $\frac{1}{4}\log\left|\frac{\sqrt{1-x^4}+1}{x^2}\right|-\frac{1}{4}\,\frac{\sqrt{1-x^4}}{x^4}+C$

6. $\frac{3}{7}(4\sqrt{x}+x^{1/4}-3)\,(1+x^{1/4})^{1/3}+C$

7. $6u+2\log\left|\frac{u-1}{\sqrt{u^2+u+1}}\right|-2\sqrt{3}\tan^{-1}\left\{\frac{2u+1}{\sqrt{3}}\right\}+C$,
$u=(1+\sqrt{x})^{1/3}$

8. $\frac{1}{5}\log\left|\frac{u-1}{\sqrt{u^2+u+1}}\right|+\frac{\sqrt{3}}{5}\tan^{-1}\left\{\frac{1+2u}{\sqrt{3}}\right\}+C, u=(1+x^5)^{1/3}$

9. $-\frac{1}{3}\log\left|\frac{(1+x^3)^{1/3}+x}{\sqrt{(1+x^3)^{2/3}+x\,(1+x^3)^{1/3}+x^2}}\right|-\frac{(1+x^3)^{1/3}}{x}+C$
$+\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{x+2\,(1+x^3)^{1/3}}{x\sqrt{3}}\right\}$

10. $\frac{u}{2\,(u^3+1)}-\frac{1}{6}\log\frac{u+1}{\sqrt{u^2-u+1}}-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2u-1}{\sqrt{3}}+C$,
$u=\left(\frac{1-x^2}{x^2}\right)^{1/3}$

11. $-\frac{1}{2(1+x^{1/4})^8}+\frac{4}{9(1+x^{1/4})^9}+C$

12. $3x^{4/3}+\frac{3}{7}x^{7/3}+\frac{24}{11}x^{11/6}+C$

13. $3\tan^{-1}x^{1/3}+C$ 14. $2(1+x^{1/3})^{3/2}+C$

15. $-\frac{1}{10}\left(1+\frac{1}{x^4}\right)^{5/2}+\frac{1}{3}\left(1+\frac{1}{x^4}\right)^{3/2}-\frac{1}{2}\left(1+\frac{1}{x^4}\right)^{1/2}+C$

—❑❑❑—

10.14 REDUCTION FORMULAE

Sometimes, we come across some functions whose integrals are not immediately reducible to one of the standard forms which we have learnt in the earlier sections. In such cases, we express the given integral in terms of another integral in which the integrand is of the same type but is of lower degree or order. Such a reltion is called a reduction formulae. *Thus, a reduction formula for an integral is a relation which connects the given integral with another integral in which the integrand is of the same type but is of lower degree or order.*

In this section, we shall obtain various reduction formulae for trigonometrical, algebraic and other type of functions.

REDUCTION FORMULAE FOR $\int \sin^n x\,dx$:

Let $I_n = \int \sin^n x\,dx$. Then,

$$I_n = \int \underset{\text{I}}{\sin^{n-1}x}\ \underset{\text{II}}{\sin x\,dx}$$

$$= -\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

$$= -\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x+(n-1)\int (\sin^{n-2}x-\sin^n x)\,dx$$

$$= -\sin^{n-1}x\cos x+(n-1)\,I_{n-2}-(n-1)\,I_n$$

$$\therefore\ n\,I_n = -\sin^{n-1}x\cos x+(n-1)\,I_{n-2}$$

$$\Rightarrow I_n = -\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}I_{n-2}$$

Thus,

$$\int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int \sin^{n-2}x\,dx$$

REDUCTION FORMULA FOR $\int \cos^n x\,dx$

Let $I_n = \int \cos^n x\,dx$. Then,

$$I_n = \int \underset{\text{I}}{\cos^{n-1}x}\ \underset{\text{II}}{\cos x\,dx}$$

$$= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\sin^2 x\,dx$$

$$= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x\sin x+(n-1)\int (\cos^{n-2}x-\cos^n x)\,dx$$

$$= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\,dx-(n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1}x\sin x+(n-1)\,I_{n-2}-(n-1)\,I_n$$

$$\therefore\ I_n(1+n-1) = \cos^{n-1}x\sin x+(n-1)\,I_{n-2}$$

$$\Rightarrow I_n = \frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}I_{n-2}$$

Thus, $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int \cos^{n-2}x\,dx$

REDUCTION FORMULA FOR $\int \tan^n x\,dx$

Let $I_n = \int \tan^n x\,dx$. Then,

$$I_n = \int \tan^{n-2}x\tan^2 x\,dx$$

$$= \int \tan^{n-2}x(\sec^2 x-1)\,dx$$

$$= \int \tan^{n-2}x\sec^2 x\,dx-\int \tan^{n-2}x\,dx$$

$$= \int \tan^{n-2}x\,d(\tan x)-\int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1}-\int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1}-I_{n-2}$$

Hence, $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1}-\int \tan^{n-2}x\,dx$

REDUCTION FORMULA FOR $\int \operatorname{cosec}^n x\,dx$

Let $I_n = \int \operatorname{cosec}^n x\,dx$. Then,

$$I_n = \int \operatorname{cosec}^n x\,dx$$

$$= \int \underset{\text{I}}{\operatorname{cosec}^{n-2}x}\ \underset{\text{II}}{\operatorname{cosec}^2 x\,dx}$$

$$= -\operatorname{cosec}^{n-2}x\cot x-(n-2)\int \operatorname{cosec}^{n-2}x(\operatorname{cosec}^2 x-1)\,dx$$

$$= -\operatorname{cosec}^{n-2}x\cot x-(n-2)\int \operatorname{cosec}^n x\,dx+(n-2)\int \operatorname{cosec}^{n-2}x\,dx$$

$$= -\operatorname{cosec}^{n-2}x\cot x-(n-2)\,I_n+(n-2)\,I_{n-2}$$

$$\therefore\ I_n(1+n-2) = -\operatorname{cosec}^{n-2}x\cot x+(n-2)\,I_{n-2}$$

$$\Rightarrow I_n = -\frac{\text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Hence,

$$\int \text{cosec}^n x\, dx = -\frac{\text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \text{cosec}^{n-2} x\, dx$$

REDUCTION FORMULA FOR $\int \sec^n x\, dx$:

Let $I_n = \int \sec^n x\, dx$. Then,

$$I_n = \int \underset{\text{I}}{\sec^{n-2} x} \underset{\text{II}}{\sec^2 x\, dx}$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Hence, $\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx.$

REDUCTION FORMULA FOR $\int \cot^n x\, dx$

Let $I_n = \int \cot^n x\, dx$. Then,

$$I_n = \int \cot^n x\, dx$$

$$= \int \cot^{n-2} x \cot^2 x\, dx$$

$$= \int \cot^{n-2} x (\text{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x \,\text{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$

$$= -\int \cot^{n-2} x\, d(\cot x) - \int \cot^{n-2} x\, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$$

Hence, $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$

REDUCTION FORMULA FOR $\int \sin^m x \cos^n x\, dx$

We shall obtain reduction formula for $\int \sin^m x \cos^n x\, dx$ by connecting it with any one of the following six integrals :

(i) $\int \sin^{m-2} x \cos^n x\, dx$

(ii) $\int \sin^m x \cos^{n-2} x\, dx$

(iii) $\int \sin^{m+2} x \cos^n x\, dx$

(iv) $\int \sin^m x \cos^{n+2} x\, dx$

(v) $\int \sin^{m-2} x \cos^{n+2} x\, dx$

(vi) $\int \sin^{m+2} x \cos^{n-2} x\, dx$

In order to connect the given integral with any one of the above six integrals, we follow the following algorithm.

ALGORITHM

STEP I *Define* $P = \sin^{\lambda+1} x \cos^{\mu+1} x$, *where* λ *and* μ *are the smallest of the indices of* $\sin x$ *and* $\cos x$ *in the two integrals to be connected.*

STEP II *Find* $\frac{dP}{dx}$

STEP III *Express* $\frac{dP}{dx}$ *in terms of the integrands of the two integrals to be connected.*

STEP IV *Integrate both sides of the expression obtained in step III and obtain the required formula by transposing terms on two sides of the equality sign.*

We illustrate the above algorithm in the following reduction formulae :

(i) Prove that :

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos n+1\, x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x\, dx$$

PROOF The smaller of the indices of $\sin x$ and $\cos x$ in the two integrals to be connected are $m-2$ and n respectively. So, we define

$$P = \sin^{m-1} x \cos^{n+1} x$$

$$P = \sin^{m-1} x \cos^{n+1} x$$

$$\therefore \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^{n+2} x - (n+1) \sin^m x \cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^n x \cos^2 x - (n+1) \sin^m x \cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^n x (1 - \sin^2 x) - (n+1) \sin^m x \cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^n x - (m-1) \sin^m x \cos^n x - (n+1) \sin^m x \cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^n x - (m-1+n+1) \sin^m x \cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m-1) \sin^{m-2} x \cos^n x - (m+n) \sin^m x \cos^n x$$

Integrating both sides w.r. to x, we obtain

$$P = (m-1)\int \sin^{m-2}x\cos^n x\,dx - (m+n)\int \sin^m x\cos^n x\,dx$$

$$\Rightarrow (m+n)\int \sin^m x\cos^n x\,dx = -P + (m-1)\int \sin^{m-2}x\cos^n x\,dx$$

$$\Rightarrow \int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx$$

(ii) Prove that :

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

PROOF We observe that the smaller of the indices of $\sin x$ and $\cos x$ in the two integrals to be connected are m and $n-2$ respectively. So, let

$$P = \sin^{m+1}x\cos^{n-1}x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (n-1)\sin^{m+2}x\cos^{n-2}x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (n-1)\sin^m x\sin^2 x\cos^{n-2}x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (n-1)\sin^m x\cos^{n-2}x(1-\cos^2 x)$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (n-1)\sin^m x\cos^{n-2}x + (n-1)\sin^m x\cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m+n)\sin^m x\cos^n x - (n-1)\sin^m x\cos^{n-2}x$$

Integrating both sides w.r. to x, we get

$$P = (m+n)\int \sin^m x\cos^n x\,dx - (n-1)\int \sin^m x\cos^{n-2}x\,dx$$

$$\Rightarrow (m+n)\int \sin^m x\cos^n x\,dx = P + (n-1)\int \sin^m x\cos^{n-2}x\,dx$$

$$\Rightarrow \int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

(iii) Prove that

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2}x\cos^n x\,dx$$

PROOF Here, the smaller of the indices of $\sin x$ and $\cos x$ in the two integrals to be connected are m and n respectively. So, let

$$P = \sin^{m+1}\cos^{n+1}x$$

$$\frac{dP}{dx} = (m+1)\sin^m x\cos^{n+2}x - (n+1)\sin^{m+2}x\cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x\cdot\cos^2 x - (n+1)\sin^{m+2}x\cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x(1-\sin^2 x) - (n+1)\sin^{m+2}x\cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (m+1)\sin^{m+2}x\cos^n x - (n+1)\sin^{m+2}x\cos^n x$$

$$\Rightarrow \frac{dP}{dx} = (m+1)\sin^m x\cos^n x - (m+n+2)\sin^{m+2}x\cos^n x$$

Integrating both sides w.r. to x, we get

$$P = (m+1)\int \sin^m x\cos^n x\,dx - (m+n+2)\int \sin^{m+2}x\cos^n x\,dx$$

$$\therefore (m+1)\int \sin^m x\cos^n x\,dx = P + (m+n+2)\int \sin^{m+2}x\cos^n x\,dx$$

$$\Rightarrow \int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2}x\cos^n x\,dx.$$

(iv) Prove that :

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x\cos^{n+2}x\,dx$$

Proceed as above.

(v) Prove that :

$$\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x\,dx$$

Proceed as above.

(vi) $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}x\cos^{n-2}x\,dx$$

Proceed as above.

REDUCTION FORMULA FOR $\int \cos^m x\sin nx\,dx$:

Prove that :

$$I_{m,n} = \int \cos^m x\sin nx\,dx = -\frac{\cos^m x\cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1}x\sin(n-1)x\,dx$$

PROOF We have,

$$I_{m,n} = \int \underset{\text{I}}{\cos^m x}\ \underset{\text{II}}{\sin nx}\,dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x\sin x\cos nx\,dx$$

Now,

$$\sin(n-1)x = \sin nx\cos x - \cos nx\sin x$$

$\Rightarrow \cos nx \sin x = \sin nx \cos x - \sin (n-1)x$

$$\therefore I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \{\sin nx \cos x - \sin (n-1)x\}\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \{\cos^m x \sin nx - \cos^{m-1} x \sin (n-1)x\}\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)x\, dx$$

$$\therefore \left(1+\frac{m}{n}\right)\int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)x\, dx$$

$$\Rightarrow \int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \sin (n-1)x\, dx$$

REDUCTION FORMULA FOR $\int \cos^m x \cos nx\, dx$:

Prove that :

$$I_{m,n} = \int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos (n-1)x\, dx$$

PROOF We have,

$$I_{m,n} = \int \cos^m x \cos nx\, dx = \cos^m x \frac{\sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin x \sin nx\, dx$$

Now, $\cos (n-1)x = \cos nx \cos x + \sin nx \sin x$

$\Rightarrow \sin x \sin nx = \cos (n-1)x - \cos nx \cos x.$

Thus,

$$I_{m,n} = \int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \{\cos (n-1)x - \cos nx \cos x\}\, dx$$

$$\Rightarrow \int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos (n-1)x\, dx - \frac{m}{n}\int \cos^m x \cos nx\, dx$$

$$\Rightarrow \left(1+\frac{m}{n}\right)\int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos (n-1)x\, dx$$

$$\Rightarrow \int \cos^m nx \cos nx\, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos (n-1)x\, dx$$

REDUCTION FORMULA FOR $\int \sin^m x \sin nx\, dx$

Prove that :

$$I_{m,n} = \int \sin^m x \sin nx\, dx = \frac{n \sin^m x \cos nx}{m^2-n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2-n^2} + \frac{m(m-1)}{m^2-n^2}\int \sin^{m-2} x \sin nx\, dx$$

PROOF We have,

$$\int \underset{\text{I}}{\sin^m x} \underset{\text{II}}{\sin nx}\, dx = -\frac{\sin^m x \cos nx}{n} + \frac{m}{n}\int \sin^{m-1} x \cos x \cos nx\, dx \quad \ldots(i)$$

Now,

$$\int \underset{\text{I}}{\sin^{m-1} x \cos x} \underset{\text{II}}{\cos nx}\, dx = \sin^{m-1} x \cos x \frac{\sin nx}{n} - \int \{(m-1)\sin^{m-2} x \cos^2 x - \sin^m x\}\frac{\sin nx}{n}\, dx$$

$$= \frac{\sin^{m-1} x \cos x \sin nx}{n} - \frac{1}{n}\int \{(m-1)\sin^{m-2} x (1-\sin^2 x) - \sin^m x\}\sin nx\, dx$$

$$\Rightarrow \int \sin^{m-1} x \cos x \cos nx\, dx = \frac{\sin^{m-1} x \cos x \sin nx}{n} - \frac{m-1}{n}\int \sin^{m-2} x \sin nx\, dx + \frac{m}{n}\int \sin^m x \sin nx\, dx$$

Substituting this value in (i), we obtain

$$\int \sin^m x \sin nx\, dx = -\frac{\sin^m x \cos nx}{n} + \frac{m}{n^2}\sin^{m-1} x \cos x \sin nx - \frac{m(m-1)}{n^2}\int \sin^{m-2} x \sin nx\, dx + \frac{m^2}{n^2}\int \sin^m x \sin nx\, dx$$

$$\Rightarrow \left(1-\frac{m^2}{n^2}\right)\int \sin^m x \sin nx\, dx = \frac{-\sin^m x \cos nx}{n} + \frac{m}{n^2}\sin^{m-1} x \cos x \sin nx - \frac{m(m-1)}{n^2}\int \sin^{m-2} x \cos nx\, dx$$

$$\Rightarrow \int \sin^m x \sin nx\, dx = n\frac{\sin^m x \cos nx}{m^2-n^2} - \frac{m}{m^2-n^2}\sin^{m-1} x \cos x \cos nx$$

$$+\frac{m(m-1)}{m^2-n^2}\int \sin^{m-2}x\cos nx\,dx$$

Similarly, we can prove that

$$\int \sin^m \cos nx\,dx=\frac{n\sin^m x\sin nx}{m^2-n^2}-m\frac{\sin^{m-1}x\cos x\cos nx}{m^2-n^2}+\frac{m(m-1)}{m^2-n^2}\int \sin^{m-2}x\cos n x\,dx$$

SOME OTHER REDUCTION FORMULAE

We shall now discuss some more reduction formulae in the following examples.

EXAMPLE 1 *If* $I_{m,n}=\int x^m(1-x)^n\,dx$, *show that :*

$$(m+n)I_{m,n-1}=x^{m+1}(1-x)^{n-1}+(n-1)I_{m,n-2}$$

SOLUTION We have,

$$I_{m,n}=\int \underset{\text{I}}{x^m}\ \underset{\text{I}}{(1-x)^n}\,dx$$

$$=(1-x)^n\frac{x^{m+1}}{m+1}+\frac{n}{m+1}\int (1-x)^{n-1}x^{m+1}\,dx$$

$$=\frac{x^{m+1}(1-x)^n}{m+1}+\frac{n}{m+1}\int (1-x)^{n-1}x^m\,.\,x\,dx$$

$$=\frac{x^{m+1}(1-x)^n}{m+1}-\frac{n}{m+1}\int (1-x)^{n-1}x^m(1-x-1)\,dx$$

$$I_{m,n}=\frac{x^{m+1}(1-x)^n}{m+1}-\frac{n}{m+1}\int (1-x)^n x^m\,dx+\frac{n}{m+1}\int (1-x)^{n-1}x^m\,dx$$

$$\Rightarrow I_{m,n}=\frac{x^{m+1}(1-x)^n}{m+1}-\frac{n}{m+1}I_{m,n}+\frac{n}{m+1}I_{m,n-1}$$

$$\Rightarrow \left(1+\frac{n}{m+1}\right)I_{m,n}=\frac{x^{m+1}(1-x)^n}{m+1}+\frac{n}{m+1}I_{m,n-1}$$

$$\Rightarrow (m+n+1)I_{m,n}=x^{m+1}(1-x)^n+nI_{m,n-1}$$

Replacing n by $n-1$, we get

$$(m+n)I_{m,n-1}=x^{m+1}(1-x)^{n-1}+(n-1)I_{m,n-2}$$

EXAMPLE 2 *If* $I_n=\int x^n(a-x)^{1/2}\,dx$, *prove that*

$$(2n+3)I_n=2anI_{n-1}-2x^n(a-x)^{3/2}$$

SOLUTION We have,

$$I_n=\int \underset{\text{I}}{x^n}\ \underset{\text{II}}{(a-x)^{1/2}}\,dx$$

$$\Rightarrow I_n=\frac{-2}{3}x^n(a-x)^{3/2}+\frac{2n}{3}\int x^{n-1}(a-x)^{3/2}\,dx$$

$$\Rightarrow I_n=\frac{-2}{3}x^n(a-x)^{3/2}+\frac{2n}{3}\int x^{n-1}(a-x)(a-x)^{1/2}\,dx$$

$$\Rightarrow I_n=-\frac{2}{3}x^n(a-x)^{3/2}+\frac{2an}{3}\int x^{n-1}(a-x)^{1/2}\,dx-\frac{2n}{3}\int x^n(a-x)^{1/2}\,dx$$

$$\Rightarrow I_n=\frac{-2}{3}x^n(a-x)^{3/2}+\frac{2an}{3}I_{n-1}-\frac{2n}{3}I_n$$

$$\Rightarrow \left(1+\frac{2n}{3}\right)I_n=\frac{-2}{3}x^n(a-x)^{3/2}+\frac{2an}{3}I_{n-1}$$

$$\Rightarrow (2n+3)I_n=-2x^n(a-x)^{3/2}+2anI_{n-1}.$$

EXAMPLE 3 *If* $I_n=\int x^n\sqrt{a^2-x^2}\,dx$, *prove that*

$$(n+2)I_n=-x^{n-1}(a^2-x^2)^{3/2}+(n-1)a^2I_{n-2}$$

SOLUTION We have,

$$I_n=\int x^n\sqrt{a^2-x^2}\,dx$$

$$\Rightarrow I_n=\int x^{n-1}\sqrt{a^2-x^2}\,x\,dx$$

$$\Rightarrow I_n=-\frac{1}{2}\int x^{n-1}\sqrt{a^2-x^2}\,(-2x)\,dx$$

$$\Rightarrow I_n=-\frac{1}{2}\int \underset{\text{I}}{x^{n-1}}\ \underset{\text{II}}{\sqrt{a^2-x^2}}\,d(a^2-x^2)$$

$$\Rightarrow I_n=-\frac{1}{2}\left[\frac{2}{3}x^{n-1}(a^2-x^2)^{3/2}-\frac{2}{3}(n-1)\int x^{n-2}(a^2-x^2)^{3/2}\,dx\right]$$

$$\Rightarrow I_n=-\frac{1}{2}\left[\frac{2}{3}x^{n-1}(a^2-x^2)^{3/2}-\frac{2(n-1)}{3}\int x^{n-2}(a^2-x^2)\sqrt{a^2-x^2}\,dx\right]$$

$$\Rightarrow I_n=-\frac{1}{2}\left[\frac{2}{3}x^{n-1}(a^2-x^2)^{3/2}-\frac{2(n-1)a^2}{3}\int x^{n-2}\sqrt{a^2-x^2}\,dx+\frac{2(n-1)}{3}\int x^n\sqrt{a^2-x^2}\,dx\right]$$

$$\Rightarrow I_n=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{a^2(n-1)}{3}I_{n-2}-\frac{(n-1)}{3}I_n$$

$$\Rightarrow I_n\left(1+\frac{n-1}{3}\right)=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{a^2(n-1)}{3}I_{n-2}$$

$$\Rightarrow (n+2)I_n=-x^{n-1}(a^2-x^2)^{3/2}+(n-1)a^2I_{n-2}$$

EXAMPLE 4 *If* $I_n=\int (\sin x+\cos x)^n\,dx$, *then show that*

$$nI_n=(\sin x+\cos x)^{n-1}(\sin x-\cos x)+2(n-1)I_{n-2}.$$

SOLUTION We have,

$$I_n=\int (\sin x+\cos x)^n\,dx$$

$$=\int \underset{\text{I}}{(\sin x+\cos x)^{n-1}}\ \underset{\text{II}}{(\sin x+\cos x)}\,dx$$

$$=(\sin x+\cos x)^{n-1}(-\cos x+\sin x)$$

$$-(n-1)\int (\sin x+\cos x)^{n-2}(\cos x-\sin x)(-\cos x+\sin x)\,dx$$

$$= (\sin x+\cos x)^{n-1}(\sin x-\cos x) + (n-1)\int (\sin x+\cos x)^{n-2}(\sin x-\cos x)^2\,dx$$

$$= (\sin x+\cos x)^{n-1}(\sin x-\cos x) + (n-1)\int (\sin x+\cos x)^{n-2}\left\{2-(\sin x+\cos x)^2\right\}dx$$

$$= (\sin x+\cos x)^{n-1}(\sin x-\cos x) + 2(n-1)\int (\sin x+\cos x)^{n-2}\,dx - (n-1)\int (\sin x+\cos x)^n\,dx$$

$$\therefore\ I_n = (\sin x+\cos x)^{n-1}(\sin x-\cos x) + 2(n-1)\int (\sin x+\cos x)^{n-2}\,dx - (n-1)I_n$$

$$\Rightarrow n I_n = (\sin x+\cos x)^{n-1}(\sin x-\cos x) + 2(n-1)I_{n-2}.$$

EXERCISE 10.12

1. If $I_n = \int \frac{1}{(x^2+a^2)^n}\,dx$, prove that

$$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}I_{n-1}$$

2. If $I_n = \int \sqrt{(a^2+x^2)^n}\,dx$, prove that

$$I_n = \frac{x\sqrt{(a^2+x^2)^n}}{n+1} + \frac{na^2}{n+1}\int (a^2+x^2)^{n/2-1}\,dx$$

3. If $I_n = \int e^{ax}\cos^n x\,dx$, prove that

$$(a^2+n^2)I_n = (a\cos x + n\sin x)e^{ax}\cos^{n-1}x + n(n-1)I_{n-2}$$

4. If $I_{m,n} = \int e^{mx}x^n\,dx$, prove that

$$I_{m,n} = \frac{x^n e^{mx}}{m} - \frac{n}{m}I_{m,n-1}$$

5. If $I_{m,n} = \int \frac{e^{mx}}{x^n}\,dx$, prove that

$$I_{m,n} = \frac{-e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1}I_{m,n-1}$$

6. If $I_n = \int x^n e^{1/x}\,dx$, prove that

$$(n+1)!\,I_n = I_0 + e^{1/x}(x^2 + 2!\,x^3 + 3!\,x^4 + \ldots + n!\,x^{n+1})$$

7. If $I_n = \int e^{ax}\sin^n x\,dx$, prove that

$$I_n = \frac{\sin^{n-1}x\,e^{ax}(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}I_{n-2}$$

8. If $I_n = \int x\,\mathrm{cosec}^n x\,dx$, $n \geq 3$. Prove that

$$I_n = -\frac{1}{n-1}\mathrm{cosec}^{n-2}x\,(x\cot x + 1) + \frac{n-2}{n-1}I_{n-2}$$

CHAPTER

11

DEFINITE INTEGRALS

11.1 THE NEWTON-LEIBNITZ FORMULA OR THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $\phi(x)$ is one of the primitives or antiderivatives of a function $f(x)$ defined on $[a, b]$, then the definite integral of $f(x)$ over $[a, b]$ is given by $\phi(b) - \phi(a)$ and is denoted by $\int_a^b f(x)\,dx$.

Thus, if $\frac{d}{dx}(\phi(x)) = f(x)$, then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

The numbers a and b are called the limits of integration 'a' is called the *lower limit* and 'b' is the *upper limit*. The interval $[a, b]$ is called the interval of integration.

REMARK $\int_a^b f(x)\,dx$ *is read as "the integral of $f(x)$ over $[a, b]$" or "the integral of $f(x)$ from a to b".*

11.2 EVALUATION OF DEFINITE INTEGRALS

If $f(x)$ is a continuous function defined on $[a, b]$, then to evaluate $\int_a^b f(x)\,dx$, we use the following algorithm.

ALGORITHM

STEP I *Find a primitive or antiderivative of $f(x)$ by computing $\int f(x)\,dx$. Let this be $\phi(x)$.*

STEP II *Compute the values of $\phi(x)$ at $a = b$ and at $x = a$.*

STEP III *Calculate $\phi(b) - \phi(a)$. The number so obtained is the required value of $\int_a^b f(x)\,dx$.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$

SOLUTION We have,

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \frac{1}{b^2-a^2}\int_0^\infty \left(\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right)dx$$

$$= \frac{1}{(b^2-a^2)}\left[\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^\infty$$

$$= \frac{1}{(b^2-a^2)}\left[\left\{\frac{1}{a}\tan^{-1}\infty - \frac{1}{b}\tan^{-1}\infty\right\} - \left\{\frac{1}{a}\tan^{-1}0 - \frac{1}{b}\tan^{-1}0\right\}\right]$$

$$= \frac{1}{(b^2-a^2)}\left[\left\{\frac{\pi}{2a} - \frac{\pi}{2b}\right\} - 0\right]$$

$$= \frac{\pi}{2ab(a+b)}$$

EXAMPLE 2 *Prove that :* $\int_0^{\pi/4} \frac{x^2}{(x\sin x + \cos x)^2}\,dx = \frac{4-\pi}{4+\pi}$

SOLUTION We have,

$$\int_0^{\pi/4} \frac{x^2}{(x\sin x + \cos x)^2}\,dx$$

$$= \int_0^{\pi/4} \frac{x\cos x \,.\, x\sec x}{(x\sin x+\cos x)^2}\,dx$$

$$= \int_0^{\pi/4} \underset{\text{I}}{x\sec x}\ \underset{\text{II}}{\frac{d(x\sin x+\cos x)}{(x\sin x+\cos x)^2}}$$

$$= \left[x\sec x\times\frac{-1}{x\sin x+\cos x}\right]_0^{\pi/4} + \int_0^{\pi/4}\frac{(\sec x + x\sec x\tan x)}{x\sin x+\cos x}\,dx$$

$$= \left[\frac{-x\sec x}{x\sin x+\cos x}\right]_0^{\pi/4} + \int_0^{\pi/4}\sec^2 x\,dx$$

$$= \left[\frac{-x\sec x}{x\sin x+\cos x}\right]_0^{\pi/4} + \left[\tan x\right]_0^{\pi/4}$$

$$= \left\{\frac{-\frac{\pi}{4}\sqrt{2}}{\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}} - 0\right\} + (1-0) = \frac{-2\pi}{\pi+4}+1 = \frac{4-\pi}{\pi+4}$$

11.3 EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

When the variable in a definite integral is changed, the substitution in terms of new variable should be effected at three places.

(i) in the integrand,

(ii) in the differential, say, dx

(iii) in the limits

The limits of the new variable, say, t are simply the values of t corresponding to the values of the original variable, say, x, and so they can be easily obtained by putting the values of x in the substitutional relation between x and t. The method is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate*: $\displaystyle\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

SOLUTION Let $I = \displaystyle\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx.$

Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$.

Also, $x = 0 \Rightarrow \theta = \sin^{-1}0 = 0$

and, $x = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$$\therefore\ I = \int_0^{\pi/6}\frac{\theta\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \int_0^{\pi/6}\underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin\theta}\,d\theta$$

$$= \left[-\theta\cos\theta\right]_0^{\pi/6} + \int_0^{\pi/6}\cos\theta\,d\theta$$

$$= \left[-\theta\cos\theta\right]_0^{\pi/6} + \left[\sin\theta\right]_0^{\pi/6}$$

$$= -\frac{\pi}{6}\cos\frac{\pi}{6} + 0 + \sin\frac{\pi}{6} - \sin 0$$

$$= -\frac{\sqrt{3}\pi}{12}+\frac{1}{2}$$

EXAMPLE 2 *Evaluate*: $\displaystyle\int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$

SOLUTION We have,

$$I = \int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$= \int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{1}{\sqrt{2}\sin x\cos x}\,d(-\cos x+\sin x)$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{1}{\sqrt{1-(-\cos x+\sin x)^2}}\,d(-\cos x+\sin x)$$

Let $-\cos x+\sin x = t$. Then,

$x = 0 \Rightarrow t = -1$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore\ I = \sqrt{2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \sqrt{2}\left[\sin^{-1}t\right]_{-1}^{1}$$

$$= \sqrt{2}\left\{\sin^{-1}1 - \sin^{-1}(-1)\right\}$$

$$= \sqrt{2}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}$$

$= \sqrt{2}\,\pi$

EXAMPLE 3 *For any $n > 1$, evaluate the integral*

$$\int_0^\infty \frac{1}{(x+\sqrt{1+x^2})^n}\,dx$$

SOLUTION Let $I = \int_0^\infty \frac{1}{(x+\sqrt{1+x^2})^n}\,dx$.

Let $x+\sqrt{1+x^2} = t$. Then,

$\sqrt{1+x^2} = t - x$

$\Rightarrow \quad 1+x^2 = (t-x)^2$

$\Rightarrow \quad 1 = t^2 - 2tx$

$\Rightarrow \quad x = \frac{t^2-1}{2t} = \frac{1}{2}\left(t - \frac{1}{t}\right)$

$\Rightarrow \quad dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$

Also, $x = 0 \Rightarrow t = 1$

and, $x \to \infty \Rightarrow t \to \infty$

$$\therefore I = \int_1^\infty \frac{1}{t^n} \times \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\int_1^\infty \left(t^{-n} + t^{-n-2}\right)dt$$

$$= \frac{1}{2}\left[\frac{t^{-n+1}}{-n+1} + \frac{t^{-(n+1)}}{-(n+1)}\right]_1^\infty$$

$$= \frac{1}{2}\left[\frac{t^{1-n}}{1-n} - \frac{t^{-(n+1)}}{n+1}\right]_1^\infty$$

$$= \frac{1}{2}\left[0 - \left(\frac{1}{1-n} - \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left\{\frac{-2n}{1-n^2}\right\} = \frac{n}{n^2-1}$$

EXAMPLE 4 *Evaluate :* $\int x^2 \log(1-x^2)\,dx$, *and hence deduce that*

$$\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \ldots\ldots = \frac{8}{9} - \frac{2}{3}\log 2.$$

SOLUTION We have,

$$\int \underset{\text{II}}{x^2} \underset{\text{I}}{\log(1-x^2)}\,dx$$

$$= \log(1-x^2)\frac{x^3}{3} + \frac{2}{3}\int \frac{x^4}{1-x^2}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int \frac{(1-x^4-1)}{1-x^2}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int (1+x^2)\,dx + \frac{2}{3}\int \frac{1}{1-x^2}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\left\{x + \frac{x^3}{3}\right\} + \frac{2}{3}\times\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}x - \frac{2}{9}x^3 + \frac{1}{3}\log\left|\frac{1+x}{1-x}\right| + C$$

$$\therefore \int_0^1 x^2 \log(1-x^2)\,dx$$

$$= \left[\frac{x^3}{3}\log(1-x^2) - \frac{2}{3}x - \frac{2}{9}x^3 + \frac{1}{3}\log\left|\frac{1+x}{1-x}\right|\right]_0^1$$

$$= \left[\frac{x^3}{3}\log(1+x) + \frac{x^3}{3}\log(1-x) - \frac{2}{3}x - \frac{2}{9}x^3 + \frac{1}{3}\log(1+x) - \frac{1}{3}\log(1-x)\right]_0^1$$

$$= \left[\left(\frac{x^3+1}{3}\right)\log(1+x) - \frac{2}{3}x - \frac{2}{9}x^3 + \frac{(x^3-1)}{3}\log(1-x)\right]_0^1$$

$$= \frac{2}{3}\log 2 - \frac{2}{3} - \frac{2}{9} + \frac{1}{3}\lim_{x\to 1}(x^3-1)\log(1-x)$$

$$= \frac{2}{3}\log 2 - \frac{8}{9} + \frac{1}{3}\times 0 \qquad \left[\because \lim_{x\to 1}(x^3-1)\log(1-x) = 0\right]$$

$$= \frac{2}{3}\log 2 - \frac{8}{9} \qquad \ldots(i)$$

Also, for $-1 < x < 1$, we have

$$\log(1-x^2) = -2\left[\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots.\right]$$

$$\therefore \int_0^1 x^2 \log(1-x^2)\,dx$$

$$= -2\int_0^1 x^2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots.\right)dx$$

$$= -2\left[\frac{x^5}{2.5} + \frac{x^7}{4.7} + \frac{x^9}{6.9} + \ldots.\right]_0^1$$

$$= -2\left[\frac{1}{2.5} + \frac{1}{4.7} + \frac{1}{6.9} + \ldots.\right]$$

$$= -\left[\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \ldots.\right] \qquad \ldots(ii)$$

From (i) and (ii), we get

$$-\left[\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9}\ldots.\right] = \frac{2}{3}\log 2 - \frac{8}{9}$$

$$\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\ldots=\frac{8}{9}-\frac{2}{3}\log 2$$

EXAMPLE 5 *Evaluate :* $\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x(1-x)}}\,dx.$

SOLUTION We have,

$$I=\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x-x^2}}\,dx$$

$$=\int_0^1\left(\frac{1}{1+x}-\frac{1}{2+x}\right)\frac{1}{\sqrt{x-x^2}}\,dx$$

$$=\int_0^1 \frac{1}{(1+x)\sqrt{x-x^2}}\,dx-\int_0^1 \frac{1}{(2+x)\sqrt{x-x^2}}\,dx$$

$$=I_1-I_2,$$

where,

$$I_1=\int_0^1 \frac{1}{(1+x)\sqrt{x-x^2}}\,dx \text{ and } I_2=\int_0^1 \frac{1}{(2+x)\sqrt{x-x^2}}\,dx$$

Let $1+x=\frac{1}{t}$. Then, $dx=-\frac{1}{t^2}$ and $x=\frac{1-t}{t}$

Also, $x=0 \Rightarrow t=1$ and $x=1 \Rightarrow t=\frac{1}{2}$

$$\therefore\ I_1=-\int_1^{1/2}\frac{dt}{\sqrt{-2t^2+3t-1}}$$

$$=-\frac{1}{\sqrt{2}}\int_1^{1/2}\frac{1}{\sqrt{\left(\frac{1}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}\,dt$$

$$=-\frac{1}{\sqrt{2}}\left[\sin^{-1}(4t-3)\right]_1^{1/2}$$

$$=-\frac{1}{\sqrt{2}}\{\sin^{-1}(-1)-\sin^{-1}(1)\}$$

$$=\frac{\pi}{\sqrt{2}}$$

Let $2+x=\frac{1}{u}$. Then, $dx=-\frac{1}{u^2}\,du$ and $x=\frac{1-2u}{u}$

Also, $x=0 \Rightarrow u=\frac{1}{2}$ and, $x=1 \Rightarrow u=\frac{1}{3}$

$$\therefore\ I_2=\int_{1/2}^{1/3}\frac{-du}{\sqrt{-6u^2+5u-1}}$$

$$=-\frac{1}{\sqrt{6}}\int_{1/2}^{1/3}\frac{1}{\sqrt{-\left(u^2-\frac{5u}{6}+\frac{1}{6}\right)}}\,du$$

$$=-\frac{1}{\sqrt{6}}\int_{1/2}^{1/3}\frac{1}{\sqrt{\left(\frac{1}{12}\right)^2-\left(u-\frac{5}{12}\right)^2}}\,du$$

$$=-\frac{1}{\sqrt{6}}\left[\sin^{-1}(12u-5)\right]_{1/2}^{1/3}$$

$$=-\frac{1}{\sqrt{6}}\{\sin^{-1}(-1)-\sin^{-1}(1)\}$$

$$=\frac{-1}{\sqrt{6}}\left\{-\frac{\pi}{2}-\frac{\pi}{2}\right\}=\frac{\pi}{\sqrt{6}}$$

Hence, $I=I_1-I_2=\frac{\pi}{\sqrt{2}}-\frac{\pi}{\sqrt{6}}=\frac{\pi}{2\sqrt{3}}(\sqrt{6}-\sqrt{2})$.

EXAMPLE 6 *Prove that :*

$$\int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}\,dx=\frac{\pi}{4ab^2(a+b)}, \text{ where } a, b>0.$$

SOLUTION We have,

$$\frac{d}{dx}(a^2\cos^2 x+b^2\sin^2 x)=2(b^2-a^2)\sin x\cos x.$$

$$\therefore\ I=\int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}\,dx$$

$$=\frac{1}{2(b^2-a^2)}\int_0^{\pi/2} x\cdot\frac{2(b^2-a^2)\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}\,dx$$

$$=\frac{1}{2(b^2-a^2)}\int_0^{\pi/2} \underset{\text{I}}{x}\cdot\frac{d(a^2\cos^2 x+b^2\sin^2 x)}{\underset{\text{II}}{(a^2\cos^2 x+b^2\sin^2 x)^2}}$$

$$=\frac{1}{2(b^2-a^2)}\left[\left\{x\times\frac{-1}{a^2\cos^2 x+b^2\sin^2 x}\right\}_0^{\pi/2}-\int_0^{\pi/2}\frac{-1}{a^2\cos^2 x+b^2\sin^2 x}\,dx\right]$$

$$=\frac{1}{2(b^2-a^2)}\left[\left\{\frac{-x}{a^2\cos^2 x+b^2\sin^2 x}\right\}_0^{\pi/2}+\int_0^{\pi/2}\frac{\sec^2 x}{a^2+b^2\tan^2 x}\,dx\right]$$

$$=\frac{1}{2(b^2-a^2)}\left[\left(-\frac{\pi}{2b^2}-0\right)+\frac{1}{b}\int_0^{\pi/2}\frac{b\sec^2 x}{a^2+(b\tan x)^2}\right]$$

$$=\frac{-\pi}{4b^2(b^2-a^2)}+\frac{1}{2b(b^2-a^2)}\int_0^{\pi/2}\frac{1}{a^2+(b\tan x)^2}\,d(b\tan x)$$

$$= \frac{-\pi}{4b^2(b^2-a^2)} + \frac{1}{2b(b^2-a^2)} \cdot \frac{1}{a}\left[\tan^{-1}\left(\frac{b\tan x}{a}\right)\right]_0^{\pi/2}$$

$$= \frac{-\pi}{4b^2(b^2-a^2)} + \frac{1}{2ba(b^2-a^2)} \{\tan^{-1}\infty - \tan^{-1} 0\}$$

$$= \frac{-\pi}{4b^2(b^2-a^2)} + \frac{1}{2ab(b^2-a^2)}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4ab^2(b^2-a^2)} (-a+b)$$

$$= \frac{\pi}{4ab^2(a+b)}$$

EXAMPLE 7 *Evaluate :* $\int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}\, dx$

SOLUTION Let $I = \int_0^{\pi/2} (\sqrt{\sin x} + \sqrt{\cos x})^{-4}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{1}{\{\sqrt{\sin x} + \sqrt{\cos x}\}^4}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{\cos^2 x \{\sqrt{\tan x} + 1\}^4}\, dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(\sqrt{\tan x} + 1)^4}\, dx$$

Let $\tan x = t$. Then, $\sec^2 x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$

and, $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$$\therefore I = \int_0^{\infty} \frac{1}{(\sqrt{t}+1)^4}\, dt$$

Now, let $t = u^2$. Then, $dt = 2u\, du$

Also $t = 0 \Rightarrow u = 0$ and $t = \infty \Rightarrow u = \infty$.

$$\therefore I = \int_0^{\infty} \frac{2u\, du}{(u+1)^4}$$

$$= 2\int_0^{\infty} \frac{(u+1)-1}{(u+1)^4}\, du$$

$$= 2\int_0^{\infty} \left\{\frac{1}{(u+1)^3} - \frac{1}{(u+1)^4}\right\} du$$

$$= 2\left[-\frac{1}{2(u+1)^2} + \frac{1}{3(u+1)^3}\right]_0^{\infty}$$

$$= 2\left[0 - \left(-\frac{1}{2} + \frac{1}{3}\right)\right] = \frac{1}{3}$$

EXAMPLE 8 *Evaluate :* $\int_0^{\pi/2} \frac{\sec^2 x}{(\sec x + \tan x)^n}\, dx$, where $n > 1$

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sec^2 x}{(\sec x + \tan x)^n}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{1}{[\tan x + \sqrt{1+\tan^2 x}]^n}\, d(\tan x)$$

Let $\tan x = t$. Then, $\sec^2 x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \pi/2 \Rightarrow t = \tan \pi/2 = \infty$.

$$\therefore I = \int_0^{\infty} \frac{1}{[t + \sqrt{1+t^2}]^n}\, dt = \frac{n}{n^2-1}$$ [See Ex. 3]

EXAMPLE 9 *Evaluate :* $\int_0^1 \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\, dx$.

SOLUTION Let $I = \int_0^1 \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\, dx$. Then,

$$I = \int_0^1 \frac{[(1-x^2)+1]}{(1+x)\sqrt{1-x^2}}\, dx$$

$$= \int_0^1 \frac{\sqrt{1-x^2}}{1+x}\, dx + \int_0^1 \frac{1}{(1+x)\sqrt{1-x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx + \int_0^1 \frac{1}{(1+x)\sqrt{1-x^2}}\, dx$$

$= I_1 + I_2$, where

$I_1 = \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ and $I_2 = \int_0^1 \frac{1}{(1+x)\sqrt{1-x^2}}\, dx$

Let $x = \cos\theta$. Then, $dx = -\sin\theta\, d\theta$.

Also, $x = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

and, $x = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$.

$$\therefore I_1 = \int_{\pi/2}^0 \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times -\sin\theta\, d\theta$$

$$= -\int_{\pi/2}^0 \tan\frac{\theta}{2} \sin\theta\, d\theta$$

$$= -\int_{\pi/2}^0 2\sin^2\frac{\theta}{2}\, d\theta$$

$$= -\int_{\pi/2}^{0} (1-\cos\theta)\, d\theta$$

$$= -\left[\theta - \sin\theta\right]_{\pi/2}^{0}$$

$$= -\left[0 - \left(\frac{\pi}{2} - 1\right)\right] = \frac{\pi}{2} - 1$$

Now, let $1 + x = \frac{1}{t}$. Then, $dx = -\frac{1}{t^2}\, dt$ and $x = \frac{1-t}{t}$

Also, $x = 0 \Rightarrow t = 1$ and $x = 1 \Rightarrow t = \frac{1}{2}$

$$\therefore\ I_2 = -\int_1^{1/2} \frac{1}{\sqrt{t^2 - (1-t)^2}}\, dt = -\int_1^{1/2} \frac{1}{\sqrt{2t-1}} = -\left[\sqrt{2t-1}\right]_1^{1/2} = 1$$

Hence, $I = I_1 + I_2 = \frac{\pi}{2} - 1 + 1 = \frac{\pi}{2}$

ALITER Put $x = \sin\theta$ and $dx = \cos\theta\, d\theta$.

EXAMPLE 10 *Evaluate :* $\int_0^{\pi/2} \frac{\sin^3 x}{(1+\cos^2 x)\sqrt{1+\cos^2 x + \cos^4 x}}\, dx$

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sin^3 x}{(1+\cos^2 x)\sqrt{1+\cos^2 x + \cos^4 x}}\, dx$

$$= \int_0^{\pi/2} \frac{(1-\cos^2 x)\sin x}{(1+\cos^2 x)\sqrt{1+\cos^2 x + \cos^4 x}}\, dx$$

Let $\cos x = u$. Then, $-\sin x\, dx = du \Rightarrow \sin x\, dx = -du$.

Also, $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \cos\frac{\pi}{2} = 0$

$$\therefore\ I = -\int_1^0 \frac{(1-u^2)}{(1+u^2)\sqrt{1+u^2+u^4}}\, du$$

$$= \int_1^0 \frac{\left(1 - \frac{1}{u^2}\right) du}{\left(u + \frac{1}{u}\right)\sqrt{\left(u+\frac{1}{u}\right)^2 - 1}}$$

Let $u + \frac{1}{u} = v$. Then, $\left(1 - \frac{1}{u^2}\right) du = dv$

Also, $u \to 0 \Rightarrow v \to \infty$ and $u = 1 \Rightarrow v = 2$

$$\therefore\ I = \int_2^{\infty} \frac{1}{v\sqrt{v^2 - (1)^2}}\, dv$$

$$= \left[\sec^{-1} v\right]_2^{\infty}$$

$$= \sec^{-1}\infty - \sec^{-1} 2$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

EXAMPLE 11 *Prove that :*

$$\int_0^{\pi} \frac{1}{a^2 - 2a\cos x + 1}\, dx = \begin{cases} \frac{\pi}{a^2-1}, & \text{if } a > 1 \\ \frac{\pi}{1-a^2}, & \text{if } 0 < a < 1 \end{cases}$$

SOLUTION We have,

$$I = \int_0^{\pi} \frac{1}{a^2 - 2a\cos x + 1}\, dx$$

$$= \int_0^{\pi} \frac{1 + \tan^2 x/2}{(1+a^2)(1+\tan^2 x/2) - 2a(1-\tan^2 x/2)}\, dx$$

$$= \int_0^{\pi} \frac{\sec^2 x/2}{(1-a)^2 + \tan^2 x/2\,.\,(1+a)^2}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $\sec^2\frac{x}{2}\, dx = 2\, dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan\pi/2 = \infty$.

$$\therefore\ I = 2\int_0^{\infty} \frac{1}{(1-a)^2 + (1+a)^2 t^2}\, dt$$

$$= \frac{2}{(1+a)^2} \int_0^{\infty} \frac{1}{\left(\frac{1-a}{1+a}\right)^2 + t^2}\, dt$$

$$= \frac{2}{(1+a)^2} \times \frac{1}{\left|\frac{1-a}{1+a}\right|} \left[\tan^{-1}\left\{t\left|\frac{1+a}{1-a}\right|\right\}\right]_0^{\infty}$$

$$= \frac{2}{(1+a)\,|1-a|} \{\tan^{-1}\infty - \tan^{-1} 0\}$$

$$= \frac{2}{(1+a)\,|1-a|} \times \left\{\frac{\pi}{2} - 0\right\}$$

$$= \frac{\pi}{(1+a)\,|1-a|} = \begin{cases} \frac{\pi}{1-a^2}, & \text{if } 0 < a < 1 \\ \frac{-\pi}{(1-a^2)} = \frac{\pi}{a^2-1}, & \text{if } a > 1 \end{cases}$$

EXAMPLE 12 *Evaluate :* $\int_0^{\pi/2} \frac{1 + 2\cos x}{(2+\cos x)^2}\, dx$

SOLUTION Let $I = \int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{(\cos^2 x + \sin^2 x) + 2\cos x}{(2+\cos x)^2}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x + \cos x \,.\, (\cos x + 2)}{(2+\cos x)^2}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x}{(2+\cos x)^2}\, dx + \int_0^{\pi/2} \frac{\cos x}{(2+\cos x)}\, dx$$

$$= \int_0^{\pi/2} \sin x \,.\, \frac{\sin x}{(2+\cos x)^2}\, dx + \int_0^{\pi/2} \frac{\cos x}{(2+\cos x)}\, dx$$

$$= -\int_0^{\pi/2} \sin x\, \frac{d(2+\cos x)}{(2+\cos x)^2} + \int_0^{\pi/2} \frac{\cos x}{(2+\cos x)}\, dx$$

$$= -\left[\left\{\frac{-\sin x}{2+\cos x}\right\}_0^{\pi/2} - \int_0^{\pi/2} \cos x \times \frac{-1}{2+\cos x}\, dx\right] + \int_0^{\pi/2} \frac{\cos x}{(2+\cos x)}\, dx$$

$$= \left[\frac{\sin x}{2+\cos x}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos x}{2+\cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{2+\cos x}\, dx$$

$$= \frac{1}{2}.$$

EXAMPLE 13 *Evaluate :* $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta$

SOLUTION Let $I = \int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta$

$$= \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

Let $\sqrt{2}\sin\theta = \sin t$. Then, $\sqrt{2}\cos\theta\, d\theta = \cos t\, dt$

Also, $\theta = 0 \Rightarrow \sin t = \sqrt{2}\sin 0 = 0 \Rightarrow t = 0$

and, $\theta = \frac{\pi}{4} \Rightarrow \sin t = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow t = \frac{\pi}{2}$

$$\therefore I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} (1 - \sin^2 t)^{3/2} \cos t\, dt$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4 t\, dt$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{3 + 4\cos 2t + \cos 4t}{8}\, dt$$

$$\Rightarrow I = \frac{1}{8\sqrt{2}} \int_0^{\pi/2} (3 + 4\cos 2t + \cos 4t)\, dt$$

$$\Rightarrow I = \frac{1}{8\sqrt{2}} \left[3t + 2\sin 2t + \frac{\sin 4t}{4}\right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{3\pi}{16\sqrt{2}}$$

EXAMPLE 14 *Evaluate :* $\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)}\, dx$

SOLUTION We have,

$$I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)}\, dx$$

$$= \int_2^3 \frac{(2x^5 - 2x^3) + (x^4 + 2x^2 + 1)}{(x^2+1)^2 (x^2-1)}\, dx$$

$$= \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2 (x^2-1)}\, dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2}\, dx + \int_2^3 \frac{1}{x^2-1}\, dx$$

$= I_1 + I_2$, where

$$I_1 = \int_2^3 \frac{2x^3}{(x^2+1)^2}\, dx \text{ and } I_2 = \int_2^3 \frac{1}{x^2-1}\, dx$$

Let $x^2 + 1 = u$. Then, $2x\, dx = du$.

Also, $x = 2 \Rightarrow u = 5$ and $x = 3 \Rightarrow u = 10$.

$$\therefore I_1 = \int_5^{10} \frac{u-1}{u^2}\, du$$

$$= \int_5^{10} \left(\frac{1}{u} - \frac{1}{u^2}\right) du$$

$$= \left[\log u + \frac{1}{u}\right]_5^{10}$$

$$= \left(\log 10 + \frac{1}{10}\right) - \left(\log 5 + \frac{1}{5}\right)$$

$$= \log\frac{10}{5} + \frac{1}{10} - \frac{1}{5}$$

$$= \log 2 - \frac{1}{10}$$

$$I_2 = \int_2^3 \frac{1}{x^2-1}\, dx$$

$$= \frac{1}{2}\left[\log\left|\frac{x-1}{x+1}\right|\right]_2^3$$

$$= \frac{1}{2}\left\{\log\frac{2}{4} - \log\frac{1}{3}\right\}$$

$$= \frac{1}{2}\log\frac{3}{2}$$

Hence,

$$I = I_1 + I_2$$

$$= \log 2 - \frac{1}{10} + \frac{1}{2}\log\frac{3}{2}$$

$$= \frac{1}{2}\log 4 + \frac{1}{2}\log\frac{3}{2} - \frac{1}{10}$$

$$= \frac{1}{2}\log\left(4\times\frac{3}{2}\right) - \frac{1}{10}$$

$$= \frac{1}{2}\log 6 - \frac{1}{10}.$$

EXAMPLE 15 *Prove that :*

$$\int_0^{\pi} \frac{\sin^2 x}{a^2 - 2ab\cos x + b^2}\,dx = \begin{cases} \frac{\pi}{2a^2}, & \text{if } a > b > 0 \\ \frac{\pi}{2b^2}, & \text{if } b > a > 0 \end{cases}$$

SOLUTION Let $I = \int_0^{\pi} \frac{\sin^2 x}{a^2 - 2ab\cos x + b^2}\,dx.$

Replacing $\sin x$ by $\frac{2\tan(x/2)}{1+\tan^2(x/2)}$ and $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$,

we get

$$I = \int_0^{\pi} \frac{4\tan^2 x/2}{(a^2+b^2)(1+\tan^2 x/2) - 2ab(1-\tan^2 x/2)} \times \frac{1}{1+\tan^2 x/2}\,dx$$

Let $\tan\frac{x}{2} = t$. Then, $\sec^2\frac{x}{2}\,dx = 2\,dt, \Rightarrow \left(1+\tan^2\frac{x}{2}\right)dx = 2dt.$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan\frac{\pi}{2} = \infty.$

$$\therefore\ I = \int_0^{\infty} \frac{4t^2}{(a^2+b^2)(1+t^2) - 2ab(1-t^2)} \times \frac{1}{1+t^2}\cdot\frac{2dt}{1+t^2}$$

$$= 8\int_0^{\infty} \frac{t^2}{\{(a^2+b^2)(1+t^2) - 2ab(1-t^2)\}(1+t^2)^2}\,dt$$

$$= 8\int_0^{\infty} \frac{t^2}{\{(a-b)^2 + (a+b)^2 t^2\}(1+t^2)^2}\,dt$$

$$= \frac{8}{(a+b)^2}\int_0^{\infty} \frac{t^2}{(t^2+1)^2\left\{t^2 + \left(\frac{a-b}{a+b}\right)^2\right\}}\,dt$$

Let $t^2 = y$. Then,

$$\frac{t^2}{(t^2+1)^2\left\{t^2+\left(\frac{a-b}{a+b}\right)^2\right\}} = \frac{y}{(y+1)^2\left\{y+\left(\frac{a-b}{a+b}\right)^2\right\}}$$

Further, let

$$\frac{y}{(y+1)^2\left\{y+\left(\frac{a-b}{a+b}\right)^2\right\}} = \frac{A}{y+1} + \frac{B}{(y+1)^2} + \frac{C}{y+\left(\frac{a-b}{a+b}\right)^2} \quad ...(i)$$

Multiplying both sides of (i) by $(y+1)^2$ and putting $y = -1$, we get

$$B = \frac{-1}{\left(\frac{a-b}{a+b}\right)^2 - 1} = \frac{(a+b)^2}{4ab}.$$

Multiplying both sides of (i) by $\left\{y+\left(\frac{a-b}{a+b}\right)^2\right\}$ and putting $y = -\left(\frac{a-b}{a+b}\right)^2$, we get

$$C = \frac{-\left(\frac{a-b}{a+b}\right)^2}{\left\{1-\left(\frac{a-b}{a+b}\right)^2\right\}^2} = \frac{-(a+b)^2(a-b)^2}{16a^2b^2}$$

Putting $y = 0$ in (i), we get

$$0 = A + B + C\left(\frac{a+b}{a-b}\right)^2$$

$$\Rightarrow A = -B - C\left(\frac{a+b}{a-b}\right)^2 = -\frac{(a+b)^2}{4ab} + \frac{(a+b)^4}{16a^2b^2}$$

$$\Rightarrow A = \frac{(a-b)^2(a+b)^2}{16a^2b^2}$$

Substituting the value of A, B and C in (i), we get

$$\frac{y}{(y+1)^2\left\{y+\left(\frac{a-b}{a+b}\right)^2\right\}}$$

$$= \frac{(a^2-b^2)^2}{16a^2b^2}\cdot\frac{1}{y+1} + \frac{(a+b)^2}{4ab}\cdot\frac{1}{(y+1)^2} - \frac{(a^2-b^2)}{16a^2b^2}\cdot\frac{1}{\left\{y+\left(\frac{a-b}{a+b}\right)^2\right\}}$$

$$\Rightarrow \frac{t^2}{(t^2+1)^2\left\{t^2+\left(\frac{a-b}{a+b}\right)^2\right\}}$$

$$= \frac{(a^2-b^2)^2}{16a^2b^2}\left\{\frac{1}{t^2+1} - \frac{1}{t^2+\left(\frac{a-b}{a+b}\right)^2} + \frac{4ab}{(a-b)^2}\cdot\frac{1}{(t^2+1)^2}\right\}$$

$$\therefore I=\frac{8}{(a+b)^2}\cdot\frac{(a^2-b^2)^2}{16a^2b^2}\left[\int_0^\infty\frac{1}{t^2+1}\,dt-\int_0^\infty\frac{1}{t^2+\left(\frac{a-b}{a+b}\right)^2}\,dt+\frac{4ab}{(a-b)^2}\int_0^\infty\frac{1}{(t^2+1)^2}\,dt\right]$$

$$=\frac{(a-b)^2}{2a^2b^2}\left[\left[\tan^{-1}t\right]_0^\infty-\left|\frac{a+b}{a-b}\right|\left\{\tan^{-1}\left(\left|\frac{a+b}{a-b}\right|t\right)\right\}_0^\infty+\frac{4ab}{(a-b)^2}I_1\right],\text{ where }I_1=\int_0^\infty\frac{1}{(t^2+1^2}\,dt$$

$$\Rightarrow I=\frac{(a-b)^2}{2a^2b^2}\left[\frac{\pi}{2}-\left|\frac{a+b}{a-b}\right|\frac{\pi}{2}+\frac{4ab}{(a-b)^2}I_1\right]\qquad\text{...(ii)}$$

Now,

$$I_1=\int_0^\infty\frac{1}{(t^2+1)^2}\,dt=\int_0^{\pi/2}\frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta=\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}=\frac{\pi}{4}$$

Substituting the value of I_1 in (ii), we get

$$I=\frac{(a-b)^2}{2a^2b^2}\left[\frac{\pi}{2}-\left|\frac{a+b}{a-b}\right|\frac{\pi}{2}+\frac{4ab}{(a-b)^2}\cdot\frac{\pi}{4}\right]$$

$$\Rightarrow I=\frac{\pi(a-b)^2}{8a^2b^2}\left[2+\frac{4ab}{(a-b)^2}-2\frac{(a+b)}{|a-b|}\right]\qquad\left[\because a,b>0\ \therefore a+b>0\right]$$

$$\Rightarrow I=\frac{\pi(a-b)^2}{8a^2b^2}\left[\frac{2(a^2+b^2)}{(a-b)^2}-\frac{2(a+b)}{|a-b|}\right]$$

$$=\begin{cases}\dfrac{\pi(a-b)^2}{8a^2b^2}\left[\dfrac{2(a^2+b^2)}{(a-b)^2}-\dfrac{2(a+b)}{a-b}\right], & \text{if } a>b>0\\[2ex] \dfrac{\pi(a-b)^2}{8a^2b^2}\left[\dfrac{2(a^2+b^2)}{(a-b)^2}+\dfrac{2(a+b)}{a-b}\right], & \text{if } 0<a<b\end{cases}$$

$$=\begin{cases}\dfrac{\pi}{2a^2}, & \text{if } a>b>0\\[2ex] \dfrac{\pi}{2b^2}, & \text{if } 0<a<b\end{cases}$$

EXAMPLE 16 *Evaluate*: $\int_0^1\log(\sqrt{1-x}+\sqrt{1+x})\,dx$. [IIT 1988]

SOLUTION Let $I=\int_0^1\log(\sqrt{1-x}+\sqrt{1+x})\,dx$.

Let $x=\cos 2\theta$. Then, $dx=-2\sin 2\theta$

Also, $x=0\Rightarrow\cos 2\theta=0\Rightarrow 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$

and, $x=1\Rightarrow\cos 2\theta=1\Rightarrow 2\theta=0\Rightarrow\theta=0$.

$$\therefore I=-2\int_{\pi/4}^0\log(\sqrt{1-\cos 2\theta}+\sqrt{1+\cos 2\theta})\sin 2\theta\,d\theta$$

$$=-2\int_{\pi/4}^0\log\{\sqrt{2}(\sin\theta+\cos\theta)\}\sin 2\theta\,d\theta$$

$$=-2\int_{\pi/4}^0(\log\sqrt{2})\sin 2\theta\,d\theta-2\int_{\pi/4}^0\log(\sin\theta+\cos\theta)\sin 2\theta\,d\theta$$

$$=-2\log\sqrt{2}\int_{\pi/4}^0\sin 2\theta\,d\theta-2\int_{\pi/4}^0\underset{\text{I}}{\log(\sin\theta+\cos\theta)}\,\underset{\text{II}}{\sin 2\theta}\,d\theta$$

$$=-2\log\sqrt{2}\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^0-2\left[-\left\{\log(\sin\theta+\cos\theta)\cdot\frac{\cos 2\theta}{2}\right\}_{\pi/4}^0-\int_{\pi/4}^0\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\times\frac{-\cos 2\theta}{2}\,d\theta\right]$$

$$=\log\sqrt{2}-2\left[0+\frac{1}{2}\int_{\pi/4}^0(\cos\theta-\sin\theta)^2\,d\theta\right]$$

$$=\frac{1}{2}\log 2-\int_{\pi/4}^0(1-\sin 2\theta)\,d\theta$$

$$=\frac{1}{2}\log 2-\left[\theta+\frac{\cos 2\theta}{2}\right]_{\pi/4}^0$$

$$=\frac{1}{2}\log 2-\left(\frac{1}{2}-\frac{\pi}{4}\right)$$

$$=\frac{1}{2}\log 2-\frac{1}{2}+\frac{\pi}{4}$$

EXAMPLE 17 *Evaluate*: $\int_0^{\pi/3}\frac{x}{1+\sec x}\,dx$.

SOLUTION We have,

$$I=\int_0^{\pi/3}\frac{x}{1+\sec x}\,dx$$

$$= \int_0^{\pi/3} \frac{x \cos x}{1+\cos x}\, dx$$

$$= \int_0^{\pi/3} x \left(\frac{2\cos^2 x/2 - 1}{2\cos^2 x/2} \right) dx$$

$$= \int_0^{\pi/3} x\, dx - \frac{1}{2} \int_0^{\pi/3} \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 \frac{x}{2}}\, dx$$

$$= \left[\frac{x^2}{2} \right]_0^{\pi/3} - \frac{1}{2} \left[\left\{ 2x \tan \frac{x}{2} \right\}_0^{\pi/3} - 2 \int_0^{\pi/3} \tan \frac{x}{2}\, dx \right]$$

$$= \frac{\pi^2}{18} - \frac{1}{2} \left[\frac{2\pi}{3} \tan \frac{\pi}{6} - 4 \left[\log \sec \frac{x}{2} \right]_0^{\pi/3} \right]$$

$$= \frac{\pi^2}{18} - \frac{1}{2} \left[\frac{2\pi}{3\sqrt{3}} - 4 \left\{ \log \sec \frac{\pi}{6} - \log 1 \right\} \right]$$

$$= \frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2 \left\{ \log \frac{2}{\sqrt{3}} - 0 \right\}$$

$$= \frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2 \log \frac{2}{\sqrt{3}}$$

EXAMPLE 18 *Evaluate :* $\int_0^1 \frac{x^4 (1-x)^4}{1+x^2}\, dx$

SOLUTION We have,

$$\int_0^1 \frac{x^4 (1-x)^4}{1+x^2}\, dx$$

$$= \int_0^1 \frac{x^4 (1+x^2-2x)^2}{(1+x^2)}\, dx$$

$$= \int_0^1 \frac{x^4 \{(1+x^2)^2 - 4x(1+x^2) + 4x^2\}}{(1+x^2)}\, dx$$

$$= \int_0^1 x^4 (1+x^2)\, dx - 4\int_0^1 x^5\, dx + 4\int_0^1 \frac{x^6}{(1+x^2)}\, dx$$

$$= \int_0^1 (x^6 + x^4)\, dx - 4\int_0^1 x^5\, dx + 4\int_0^1 \frac{(x^6+1)-1}{x^2+1}\, dx$$

$$= \int_0^1 (x^6 + x^4)\, dx - 4\int_0^1 x^5\, dx + 4\int_0^1 \left(x^4 - x^2 + 1 - \frac{1}{x^2+1} \right) dx$$

$$= \left[\frac{x^7}{7} + \frac{x^5}{5} \right]_0^1 - 4 \left[\frac{x^6}{6} \right]_0^1 + 4 \left[\frac{x^5}{5} - \frac{x^3}{3} + x - \tan^{-1} x \right]_0^1$$

$$= \frac{1}{7} + \frac{1}{5} - 4\left(\frac{1}{6} - 0 \right) + 4\left(\frac{1}{5} - \frac{1}{3} + 1 - \tan^{-1} 1 \right)$$

$$= \frac{1}{7} + \frac{1}{5} - \frac{2}{3} + \frac{52}{15} - \pi$$

$$= \frac{22}{7} - \pi$$

EXAMPLE 19 *Show that :* $\int_0^{\pi} \frac{1}{a-\cos x}\, dx = \frac{\pi}{\sqrt{a^2-1}}, a > 1.$

Also, deduce that : $\int_0^{\pi} \frac{1}{(2-\cos x)^2}\, dx = \frac{2\pi}{3\sqrt{3}}$

SOLUTION Let

$$I = \int_0^{\pi} \frac{1}{a-\cos x}\, dx$$

$$= \int_0^{\pi} \frac{1+\tan^2 x/2}{a(1+\tan^2 x/2) - (1-\tan^2 x/2)}\, dx$$

$$I = \int_0^{\pi} \frac{\sec^2 x/2}{(a-1) + (a+1)\tan^2 x/2}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $\sec^2 \frac{x}{2}\, dx = 2\, dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore I = \int_0^{\infty} \frac{2\, dt}{(a-1) + (a+1)\, t^2}$$

$$= \frac{2}{a+1} \int_0^1 \frac{1}{\left\{ \sqrt{\frac{a-1}{a+1}} \right\}^2 + t^2}\, dt$$

$$= \frac{2}{(a+1)} \times \frac{1}{\sqrt{\frac{a-1}{a+1}}} \left[\tan^{-1} \left(\sqrt{\frac{a+1}{a-1}}\, t \right) \right]_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-1}} \{ \tan^{-1} \infty - \tan^{-1} 0 \}$$

$$= \frac{\pi}{\sqrt{a^2-1}}$$

Thus, $\int_0^{\pi} \frac{1}{a-\cos x}\, dx = \frac{\pi}{\sqrt{a^2-1}}$

Differentiating both sides w.r.t. a, we get

$$\int_0^{\pi} \frac{d}{da} \left(\frac{1}{a-\cos x} \right) dx = \pi \frac{d}{da} (a^2-1)^{-1/2}$$

$$\Rightarrow \int_0^{\pi} -\frac{1}{(a-\cos x)^2}\,dx = -\frac{\pi}{2}(a^2-1)^{-3/2}.2a$$

$$\Rightarrow \int_0^{\pi} \frac{1}{(a-\cos x)^2}\,dx = \frac{\pi a}{(a^2-1)^{3/2}}$$

Putting $a = 2$, we get

$$\int_0^{\pi} \frac{1}{(2-\cos x)^2}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

EXAMPLE 20 *Find all values of 'a' ($a \neq 0$) for which the equation :*

$$\int_0^x (t^2-8t+13)\,dt = x\sin\frac{a}{x}$$ *has a solution. Find that solution.*

SOLUTION We have,

$$\int_0^x (t^2-8t+13)\,dt = x\sin\frac{a}{x}$$

$$\Rightarrow \left[\frac{t^3}{3}-4t^2+13t\right]_0^x = x\sin\frac{a}{x}$$

$$\Rightarrow \frac{x^3}{3}-4x^2+13x = x\sin\frac{a}{x}$$

Clearly, this equation is meaningful for $x \neq 0$.

Now,

$$\frac{x^3}{3}-4x^2+13x = x\sin\frac{a}{x}$$

$$\Rightarrow x^2-12x+39 = 3\sin\frac{a}{x} \qquad [\because x \neq 0]$$

$$\Rightarrow (x-6)^2+3 = 3\sin\frac{a}{x}$$

Since $(x-6)^2+3 \geq 3$ for all $x \in R$ and $3\sin a/x \leq 3$ for all $x \neq 0$. Therefore, the given equation is meaningful, if each side is equal to 3.

Hence,

$$(x-6)^2+3 = 3 \text{ and } 3\sin\frac{a}{x} = 3$$

$$\Rightarrow (x-6)^2 = 0 \quad \text{and} \quad \sin\frac{a}{x} = 1$$

$$\Rightarrow x = 6 \quad \text{and} \quad \frac{a}{x} = 2n\pi+\frac{\pi}{2}, n \in Z$$

$$\Rightarrow \frac{a}{6} = (4n+1)\frac{\pi}{2}, n \in Z$$

$$\Rightarrow a = 3(4n+1)\pi, n \in Z.$$

EXAMPLE 21 *Solve the equation :* $\int_{\log_e 2}^{x} \frac{1}{\sqrt{e^x-1}}\,dx = \frac{\pi}{6}$

SOLUTION Let $e^x-1 = t^2$. Then,

$$e^x\,dx = 2t\,dt \Rightarrow dx = \frac{2t}{e^x}\,dt = \frac{2t}{t^2+1}\,dt.$$

Also, $x = \log_e 2 \Rightarrow t^2 = e^{\log 2}-1 = 2-1 = 1$.

$$\therefore \int_{\log_e 2}^{x} \frac{1}{\sqrt{e^x-1}}\,dx = \frac{\pi}{6}$$

$$\Rightarrow \int_1^{\sqrt{e^x-1}} \frac{1}{\sqrt{t^2}}\cdot\frac{2t}{t^2+1}\,dt = \frac{\pi}{6}$$

$$\Rightarrow \int_1^{\sqrt{e^x-1}} \frac{1}{t^2+1}\,dt = \frac{\pi}{12}$$

$$\Rightarrow \left[\tan^{-1}t\right]_1^{\sqrt{e^x-1}} = \frac{\pi}{12}$$

$$\Rightarrow \tan^{-1}\sqrt{e^x-1}-\tan^{-1}1 = \frac{\pi}{12}$$

$$\Rightarrow \tan^{-1}\sqrt{e^x-1}-\frac{\pi}{4} = \frac{\pi}{12}$$

$$\Rightarrow \tan^{-1}\sqrt{e^x-1} = \frac{\pi}{3}$$

$$\Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3}$$

$$\Rightarrow \sqrt{e^x-1} = \sqrt{3}$$

$$\Rightarrow e^x-1 = 3 \Rightarrow e^x = 4 \Rightarrow x = \log_e 4.$$

EXAMPLE 22 *If* $\beta+2\int_0^1 x^2e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx$, *then find the value of* β.

SOLUTION We have,

$$\beta+2\int_0^1 x^2e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta-\int_0^1 xe^{-x^2}(-2x)\,dx = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta-\int_0^1 \underset{\text{I}}{x}\,\underset{\text{II}}{d(e^{-x^2})} = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta-\left[\left(xe^{-x^2}\right)_0^1-\int_0^1 e^{-x^2}\,dx\right] = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta - \left[\frac{1}{e} - \int_0^1 e^{-x^2}\,dx\right] = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta - \frac{1}{e} + \int_0^1 e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx$$

$$\Rightarrow \beta = \frac{1}{e}$$

EXAMPLE 23 *Prove that* : $\int_0^{\pi/2} \sin x \,.\, \log \sin x\,dx = \log_e 2 - 1.$

SOLUTION Let $I = \int_0^{\pi/2} \sin x \log \sin x\,dx$. Then,

$$I = \lim_{\theta \to 0} \int_\theta^{\pi/2} \sin x \log \sin x\,dx$$

$$\Rightarrow I = \lim_{\theta \to 0} I_1, \text{ where } I_1 = \int_\theta^{\pi/2} \sin x \log \sin x\,dx$$

Now,

$$I_1 = \int_\theta^{\pi/2} \underset{\text{II}}{\sin x}\ \underset{\text{I}}{\log \sin x}\,dx$$

$$= \left[(-\log \sin x)\cos x\right]_\theta^{\pi/2} + \int_\theta^{\pi/2} \cot x \cos x\,dx$$

$$= (\log \sin \theta)\cos\theta + \int_\theta^{\pi/2} \frac{\cos^2 x}{\sin x}\,dx$$

$$= (\log \sin \theta)\cos\theta + \int_\theta^{\pi/2} \frac{1 - \sin^2 x}{\sin x}\,dx$$

$$= (\log \sin \theta)\cos\theta + \int_0^{\pi/2} (\operatorname{cosec} x - \sin x)\,dx$$

$$= (\log \sin \theta)\cos\theta + \left[\log(\operatorname{cosec} x - \cot x) + \cos x\right]_\theta^{\pi/2}$$

$$= (\log \sin \theta)\cos\theta - [\log(\operatorname{cosec}\theta - \cot\theta) + \cos\theta]$$

$$= (\log \sin \theta)\cos\theta - \log\frac{1 - \cos\theta}{\sin\theta} - \cos\theta$$

$$= (\log \sin \theta)\cos\theta - \log\frac{\sin\theta}{1 + \cos\theta} - \cos\theta$$

$$= (\log \sin \theta)(\cos\theta - 1) + \log(1 + \cos\theta) - \cos\theta$$

$$\therefore I = \lim_{\theta \to 0} I_1$$

$$= \lim_{\theta \to 0} (\log \sin \theta)(\cos\theta - 1) + \log(1 + \cos\theta) - \cos\theta$$

$$= \lim_{\theta \to 0} \frac{\log \sin \theta}{(\cos\theta - 1)^{-1}} + \log_e 2 - 1$$

$$= \lim_{\theta \to 0} \frac{\cot\theta}{\sin\theta\,(\cos\theta - 1)^{-2}} + \log_e 2 - 1$$

$$= \lim_{\theta \to 0} \frac{\cos\theta\,(\cos\theta - 1)^2}{(1 - \cos^2\theta)} + \log_e 2 - 1$$

$$= \lim_{\theta \to 0} \frac{-\cos\theta\,.\,(\cos\theta - 1)}{(\cos\theta + 1)} + \log_e 2 - 1$$

$$= \log_e 2 - 1.$$

EXAMPLE 24 *For* $x > 0$, *let* $f(x) = \int_1^x \frac{\log_e t}{1+t}\,dt$. *Find the function* $f(x) + f\left(\frac{1}{x}\right)$ *and show that* $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

[IIT 2000]

SOLUTION We have,

$$f(x) = \int_1^x \frac{\log_e t}{1+t}\,dt \qquad \ldots(i)$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log_e t}{1+t}\,dt$$

Let $t = \frac{1}{u}$. Then, $dt = -\frac{1}{u^2}\,du$.

Also, $t = 1 \Rightarrow u = 1$ and $t = \frac{1}{x} \Rightarrow u = x$

$$\therefore f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e (1/u)}{1 + \frac{1}{u}} \times \frac{-1}{u^2}\,du$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e u}{(1+u)\,u}\,du$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{(1+t)\,t}\,dt \qquad \ldots(ii)$$

From (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right)$$

$$= \int_1^x \frac{\log_e t}{1+t} + \frac{\log_e t}{(1+t)\,t}\,dt$$

$$= \int_1^x \frac{\log_e t}{1+t} \cdot \left(\frac{1+t}{t}\right) dt$$

$$= \int_1^x \frac{\log_e t}{t}\,dt$$

$$= \int_1^x \log_e t\, d(\log_e t)$$

$$= \left[\frac{(\log_e t)^2}{2}\right]_1^x = \frac{(\log_e x)^2}{2}$$

$$\therefore f(e)+f\left(\frac{1}{e}\right) = \frac{(\log_e e)^2}{2} = \frac{1}{2}$$

EXAMPLE 25 *For $x>0$, let $f(x)=\int_1^x \frac{\log_e t}{1+t+t^2}\,dt$. Show that $f(x)=f\left(\frac{1}{x}\right)$.*

SOLUTION We have,

$$f(x)=\int_1^x \frac{\log_e t}{1+t+t^2}\,dt \quad \text{for all } x>0.$$

$$\therefore f\left(\frac{1}{x}\right)=\int_1^{1/x} \frac{\log_e t}{1+t+t^2}\,dt$$

Now, let $t = \frac{1}{u}$. Then, $dt = -\frac{1}{u^2}\,du$.

Also, $t = 1 \Rightarrow u = 1$ and $t = \frac{1}{x} \Rightarrow u = x$.

$$\therefore f\left(\frac{1}{x}\right)=\int_1^x \frac{\log_e (1/u)}{1+\frac{1}{u}+\frac{1}{u^2}}\times -\frac{1}{u^2}\,du$$

$$= \int_1^x \frac{\log u}{u^2+u+1}\,du$$

$$= f(x).$$

EXAMPLE 26 *If $I_n=\int_0^{\pi/4} \tan^n x\,dx$, prove that*

(i) $I_n+I_{n-2} = \frac{1}{n-1}$

(ii) $I_{n-1}+I_{n+1} = \frac{1}{n}$

(iii) $\frac{1}{n+1}<2I_n<\frac{1}{n-1}$ for all $n = 2, 3, 4, \ldots\ldots$

[IIT 1998]

SOLUTION (i) We have,

$$I_n = \int_0^{\pi/4} \tan^n x\,dx.$$

$$\Rightarrow \quad I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$\therefore I_n + I_{n-2} = \int_0^{\pi/4} \tan^n x\,dx + \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \int_0^{\pi/4} (\tan^2 x+1)\tan^{n-2} x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx$$

Let $\tan x = t$. Then, $\sec^2 x\,dx = dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi/4 \Rightarrow t = \tan\frac{\pi}{4} = 1$.

$$\therefore I_n + I_{n-2} = \int_0^1 t^{n-2}\,dt = \left[\frac{t^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$$

(ii) We have,

$$I_n + I_{n-2} = \frac{1}{n-1} \text{ for all } n>1.$$

Replacing n by $(n+1)$, we get

$$I_{n+1} + I_{n-1} = \frac{1}{n}$$

(iii) We know that

$$0 < \tan x < 1 \qquad \text{for all } x \in (0, \pi/4)$$

$$\Rightarrow \quad \tan^{n+2} x < \tan^n x < \tan^{n-2} x \quad \text{for all } x \in [0, \pi/4]$$

$$\Rightarrow \int_0^{\pi/4} \tan^{n+2} dx < \int_0^{\pi/4} \tan^n x\,dx < \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$\Rightarrow \quad I_{n+2} < I_n < I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n+2} < 2I_n < I_n + I_{n-2}$$

$$\Rightarrow \quad \frac{1}{n+1} < 2I_n < \frac{1}{n-1} \qquad \text{[Using (i)]}$$

EXAMPLE 27 *Prove that :*

$$\int_0^{\pi} \frac{\sin 2nx}{\sin x}\,dx = 0 \text{ for all } n \in N.$$

SOLUTION Let $I_n = \int_0^{\pi} \frac{\sin 2nx}{\sin x}\,dx$. Then,

$$I_{n+1} = \int_0^{\pi} \frac{\sin(2n+2)x}{\sin x}\,dx$$

$$\therefore\ I_{n+1} - I_n = \int_0^{\pi} \frac{\sin(2n+2)x}{\sin x}\,dx - \int_0^{\pi} \frac{\sin 2nx}{\sin x}\,dx$$

$$= \int_0^{\pi} \frac{\sin(2n+2)x - \sin 2nx}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \frac{\sin x \cos(2n+1)x}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \cos(2n+1)x\,dx$$

$$= \frac{2}{2n+1}\Big[\sin(2n+1)x\Big]_0^{\pi}$$

$$= \frac{2}{2n+1}[\sin(2n+1)\pi - \sin 0] = 0$$

$\therefore\ I_{n+1} = I_n$ for all $n \in N$

$\Rightarrow I_n = I_{n-1} = I_{n-2} = \ldots = I_1$

But, $I_1 = \int_0^{\pi} \frac{\sin 2x}{\sin x}\,dx = 2\int_0^{\pi} \cos x\,dx = 2\Big[\sin x\Big]_0^{\pi} = 2 \times 0 = 0$

Hence, $I_n = 0$ for all $n \in N$.

EXAMPLE 28 *Prove that :*

$$\int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}\,dx = \pi \text{ for all } n \in N.$$

SOLUTION Let $I_n = \int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}\,dx$. Then,

$$I_{n+1} = \int_0^{\pi} \frac{\sin(2n+3)x}{\sin x}\,dx$$

$$\therefore\ I_{n+1} - I_n = \int_0^{\pi} \frac{\sin(2n+3)x - \sin(2n+1)x}{\sin x}\,dx$$

$$= \int_0^{\pi} \frac{2\sin x \cos(2n+2)x}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \cos(2n+2)x\,dx$$

$$= \frac{2}{2n+2}\Big[\sin(2n+2)x\Big]_0^{\pi}$$

$$= \frac{1}{n+1}[\sin(2n+2)\pi - \sin 0] = 0 \qquad \text{for all } n \in N.$$

$\Rightarrow I_{n+1} = I_n$ for all $n \in N$

$\Rightarrow I_n = I_{n-1} = I_{n-2} = \ldots = I_1$

But, $I_1 = \int_0^{\pi} \frac{\sin 3x}{\sin x}\,dx$

$$= \int_0^{\pi} (3 - 4\sin^2 x)\,dx$$

$$= \int_0^{\pi} [3 - 2(1 - \cos 2x)]\,dx$$

$$= \int_0^{\pi} (1 + 2\cos 2x)\,dx$$

$$= \Big[x + \sin 2x\Big]_0^{\pi}$$

$$= \pi + \sin 2\pi - \sin 0 = \pi$$

Hence, $I_n = \pi$ of all $n \in N$.

EXAMPLE 29 *If n is an odd positive integer, prove that*

$$I_n = \int_0^{\pi/2} \frac{\sin nx}{\sin x}\,dx = \frac{\pi}{2}.$$

SOLUTION Let $n = 2m + 1$, where $m = 0, 1, 2, \ldots$

$$\therefore\ I_n = \int_0^{\pi/2} \frac{\sin nx}{\sin x}\,dx$$

$$\Rightarrow I_{2m+1} = \int_0^{\pi/2} \frac{\sin(2m+1)x}{\sin x}\,dx \qquad \ldots(i)$$

$$\Rightarrow I_{2m+3} = \int_0^{\pi/2} \frac{\sin(2m+3)x}{\sin x}\,dx \qquad \ldots(ii) \qquad \left[\begin{array}{l}\text{Replacing } m \text{ by} \\ (m+1) \text{ in } (i)\end{array}\right]$$

From (i) and (ii), we get

$$I_{2m+3} - I_{2m+1}$$

$$= \int_0^{\pi/2} \frac{\sin(2m+3)x}{\sin x}\,dx - \int_0^{\pi/2} \frac{\sin(2m+1)x}{\sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2m+3)x - \sin(2m+1)x}{\sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{2 \sin x \cos (2m+2)x}{\sin x}\, dx$$

$$= 2 \int_0^{\pi/2} \cos (2m+2)x\, dx$$

$$= \frac{2}{2m+2}\left[\sin (2m+2)x\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}\left[\sin (m+1)\pi - \sin 0\right]$$

$= 0$ for all $m = 0, 1, 2, \ldots$

$\therefore\ I_{2m+3} = I_{2m+1}$ for all $m = 0, 1, 2, \ldots$.

$\Rightarrow I_{2m+3} = I_{2m+1} = I_{2m-1} = \ldots\ldots = I_1$

$\Rightarrow I_{2m+1} = I_{2m-1} = \ldots\ldots = \frac{\pi}{2}$ $\left[\because I_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x}\, dx = \frac{\pi}{2}\right]$

Hence, $I_n = \int_0^{\pi/2} \frac{\sin nx}{\sin x}\, dx = \frac{\pi}{2}$.

EXAMPLE 30 *Prove that :*

$$\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, d\theta = n\pi \text{ for all } n \in N.$$

SOLUTION Let $I_n = \int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, d\theta$, $n \in N$. Then,

$$I_{n+1} = \int_0^{\pi} \frac{\sin^2 (n+1)\theta}{\sin^2 \theta}\, d\theta$$

$$\therefore\ I_{n+1} - I_n = \int_0^{\pi} \frac{\sin^2 (n+1)\theta}{\sin^2 \theta}\, d\theta - \int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, d\theta$$

$$= \int_0^{\pi} \frac{\sin^2 (n+1)\theta - \sin^2 n\theta}{\sin^2 \theta}\, d\theta$$

$$= \int_0^{\pi} \frac{\sin (2n+1)\theta \sin \theta}{\sin^2 \theta}\, d\theta$$

$$= \int_0^{\pi} \frac{\sin (2n+1)\theta}{\sin \theta}\, d\theta$$

$= \pi$, for all $n \in N$ [See example 28]

$\Rightarrow I_1, I_2, I_3, \ldots$ form an AP with common difference π.

$\Rightarrow I_n = I_1 + (n-1)\pi$ [Using : $a_n = a_1 + (n-1)d$]

$\Rightarrow I_n = \pi + (n-1)\pi$ $\left[\because I_1 = \int_0^{\pi} \frac{\sin^2 \theta}{\sin^2 \theta}\, d\theta = \pi\right]$

$\Rightarrow I_n = n\pi$

Hence, $\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, d\theta = n\pi$ for all $n \in N$.

EXAMPLE 31 *Prove that :*

$$\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$$

SOLUTION Let $I_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx$. Then,

$$I_{n+1} = \int_0^{\pi/2} \frac{\sin^2 (n+1)x}{\sin x}\, dx$$

$$\therefore\ I_{n+1} - I_n = \int_0^{\pi/2} \frac{\sin^2 (n+1)x}{\sin x}\, dx - \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 (n+1)x - \sin^2 x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin (2n+1)x \sin x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \sin (2n+1)x\, dx$$

$$= -\frac{1}{2n+1}\left[\cos (2n+1)x\right]_0^{\pi/2}$$

$$= -\frac{1}{2n+1}[0-1] = \frac{1}{2n+1}$$

$\Rightarrow$ $I_n - I_{n-1} = \frac{1}{2n-1}$ [Replacing n by $(n-1)$]

$$I_{n-1} - I_{n-2} = \frac{1}{2n-3}$$

$$I_{n-2} - I_{n-3} = \frac{1}{2n-5}$$

$$I_{n-3} - I_{n-4} = \frac{1}{2n-7}$$

.. .. ..
.. .. ..
.. .. ..

$$I_3 - I_2 = \frac{1}{5}$$

$$I_2 - I_1 = \frac{1}{3}$$

Adding all these, we get

$$I_n - I_1 = \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-3}+\frac{1}{2n-1}$$

$$\Rightarrow I_n = I_1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-3}+\frac{1}{2n-1}$$

$$\Rightarrow I_n = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-1} \qquad \left[\because I_1 = \int_0^{\pi/2} \sin x\,dx = 1\right]$$

EXAMPLE 32 *If* $I_n = \int_0^1 x^n \tan^{-1} x\,dx$, *prove that*

$$(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$

SOLUTION We have,

$$I_n = \int_0^1 \underset{\text{II}}{x^n} \underset{\text{I}}{\tan^{-1} x}\,dx$$

$$\Rightarrow I_n = \left[\frac{x^{n+1}}{n+1}\tan^{-1}x\right]_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx$$

$$\Rightarrow (n+1)I_n = \left[x^{n+1}\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$$

$$\Rightarrow (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx \qquad \ldots(i)$$

$$\Rightarrow (n-1)I_{n-2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n-1}}{1+x^2}dx \qquad \ldots(ii) \qquad [\text{Replacing } n \text{ by } (n-2) \text{ in (i)}]$$

Adding (i) and (ii), we get

$$(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}+x^{n-1}}{1+x^2}dx$$

$$\Rightarrow (n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \int_0^1 \frac{x^{n-1}(x^2+1)}{1+x^2}dx$$

$$\Rightarrow (n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \int_0^1 x^{n-1}\,dx$$

$$\Rightarrow (n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$

EXAMPLE 33 *If n is a positive integer, prove that*

$$\int_0^a e^{-x}x^n\,dx = n!\left[1 - e^{-a}\left\{1 + a + \frac{a^2}{2!} + \ldots + \frac{a^n}{n!}\right\}\right]$$

Also, deduce the value of $\int_0^\infty e^{-x}x^n\,dx$.

SOLUTION Let

$$I_n = \int_0^a e^{-x}x^n\,dx. \text{ Then,}$$

$$I_n = \int_0^a \underset{\text{II}}{e^{-x}} \underset{\text{I}}{x^n}\,dx$$

$$\Rightarrow I_n = \left[-x^n e^{-x}\right]_0^a + n\int_0^a x^{n-1}e^{-x}\,dx$$

$$\Rightarrow I_n = -a^n e^{-a} + n\,.\,I_{n-1} \qquad \ldots(i)$$

Replacing n by $(n-1), (n-2), \ldots\ldots 3, 2, 1$, we get

$$I_{n-1} = -a^{n-1}e^{-a} + (n-1)I_{n-2}$$

$$I_{n-2} = -a^{n-2}e^{-a} + (n-2)I_{n-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3 = -a^3 e^{-a} + 3I_2$$

$$I_2 = -a^2 e^{-a} + 2I_1$$

$$I_1 = -a\,e^{-a} + I_0$$

$$= -ae^{-a} - e^{-a} + 1 \qquad \left[\because I_0 = \int_0^a e^{-x}dx = -\left[e^{-x}\right]_0^a = 1 - e^{-a}\right]$$

$$\therefore I_n = -a^n e^{-a} + n\left\{-a^{n-1}e^{-a} + (n-1)I_{n-2}\right\}$$

$$= -a^n e^{-a} - n\,a^{n-1}e^{-a} + n(n-1)\left\{-a^{n-2}e^{-a} + (n-2)I_{n-3}\right\}$$

$$= -a^n e^{-a} - n\,a^{n-1}e^{-a} - n(n-1)a^{n-2}e^{-a} + n(n-1)(n-2)I_{n-3}$$

$$= -a^n e^{-a} - n\,.\,a^{n-1}e^{-a} - n(n-1)a^{n-2}e^{-a} + n(n-1)(n-2)\left\{-a^{n-3}e^{-a} + (n-3)I_{n-4}\right\}$$

$$= -a^n e^{-a} - n\,a^{n-1}e^{-a} - n(n-1)a^{n-2}e^{-a} - n(n-1)(n-2)a^{n-3}e^{-a} + n(n-1)(n-2)(n-3)I_{n-4}$$

Continuing in this manner, we get

$$I_n = -a^n e^{-a} - n\,a^{n-1}e^{-a} - n(n-1)a^{n-2}e^{-a} - n(n-1)(n-2)a^{n-3}e^{-a} + \ldots. - n(n-1)(n-2)(n-3)\ldots. 2\,ae^{-a} + n(n-1)(n-2)(n-3)\ldots 2\,.\,1\,I_0$$

$$= -e^{-a}\left[a^n + n\,a^{n-1} + n(n-1)a^{n-2} + n(n-1)(n-2)a^{n-3} + \ldots. + n!\,a\right] + n!\,(-e^{-a}+1) \qquad \left[\because I_0 = \int_0^a e^{-x}dx = 1 - e^{-a}\right]$$

$$= -e^{-a}\left[a^n + n\,a^{n-1} + n(n-1)\,a^{n-2} + n(n-1)(n-2)a^{n-3} + \ldots + n!\,a + n!\right] + n!$$

EXAMPLE 34 *Determine a positive integer $n \le 5$, such that*

$$\int_0^1 e^x (x-1)^n\,dx = 16 - 6e.$$

[IIT 1996]

SOLUTION Let $I_n = \int_0^1 e^x\,(x-1)^n\,dx$.

Let $x - 1 = t$. Then, $dx = dt$

Also, $x = 0 \Rightarrow t = -1$ and $x = 1 \Rightarrow t = 0$

$$\therefore I_n = \int_{-1}^{0} e^{t+1}\,t^n\,dt$$

$$= e\int_{-1}^{0} \underset{\text{II}}{e^t}\ \underset{\text{I}}{t^n}\,dt$$

$$= e\left[\left\{t^n e^t\right\}_{-1}^{0} - n\int_{-1}^{0} t^{n-1}e^t\,dt\right]$$

$$= e\left[-(-1)^n e^{-1} - n\int_{-1}^{0} t^{n-1}e^t\,dt\right]$$

$$= (-1)^{n+1} - n\,.\,e\int_{-1}^{0} t^{n-1}e^{-t}\,dt$$

$$= (-1)^{n+1} - n\,I_{n-1}$$

Putting $n = 1, 2, 3$, we get

$$I_1 = 1 - I_0 = 1 - \int_0^1 e^x\,dx = 1 - (e-1) = 2 - e$$

$$I_2 = -1 - 2I_1 = -1 - 2(2-e) = -5 + 2e$$

$$I_3 = 1 - 3I_2 = 1 - 3(-5+2e) = 16 - 6e$$

Hence, $n = 3$.

EXAMPLE 35 *If* $s_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx$ *and* $v_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$

where $n \in N$, *show that*

(i) $s_{n+1} - s_n = 0$

(ii) $v_{n+1} - v_n = s_{n+1}$

SOLUTION (i) We have,

$$s_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx \text{ and } s_{n+1} = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx$$

$$\therefore s_{n+1} - s_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx$$

$$= 2\int_0^{\pi/2} \frac{\sin x \cos 2nx}{\sin x}\,dx$$

$$= 2\int_0^{\pi/2} \cos 2nx\,dx$$

$$= \frac{2}{2n}\left[\sin 2nx\right]_0^{\pi/2}$$

$$= \frac{1}{n}[\sin n\pi - \sin 0] = 0$$

$$\Rightarrow s_{n+1} - s_n = 0$$

(ii) We have,

$$v_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx \text{ and } v_{n+1} = \int_0^{\pi/2} \frac{\sin^2(n+1)x}{\sin^2 x}\,dx$$

$$\therefore v_{n+1} - v_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x \sin x}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx = s_{n+1}$$

EXAMPLE 36 *Prove that for any positive integer k,*

$$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x]$$

Hence, prove that $\int_0^{\pi/2} \sin 2kx \cot x\,dx = \frac{\pi}{2}$.

[IIT 1990]

SOLUTION We have,

$$2\sin x[\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x]$$

$$= 2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + \ldots + 2\sin x\cos(2k-1)x$$

$$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \ldots + \{\sin 2kx - \sin(2k-2)x\}$$

$$= \sin 2kx$$

$$\therefore 2[\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x] = \frac{\sin 2kx}{\sin x} \quad \ldots(i)$$

Now,

$\sin 2kx \cot x$

$$= \frac{\sin 2kx}{\sin x} \cdot \cos x$$

$$= 2\cos x\,[\cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x]$$

[Using (i)]

$$= 2\cos^2 x + 2\cos 3x \cos x + 2\cos 5x \cos x + \ldots + 2\cos(2k-1)x \cos x$$

$$= (1+\cos 2x) + (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) + \ldots + (\cos 2kx + \cos(2k-2)x)$$

$$= 1 + 2\,\{\cos 2x + \cos 4x + \cos 6x + \cos(2k-2)x\} + \cos 2kx$$

$$\therefore \int_0^{\pi/2} \sin 2kx \cot x\, dx$$

$$= \int_0^{\pi/2} 1 \cdot dx + 2\int_0^{\pi/2} [\cos 2x + \ldots + \cos(2k-2)x]\, dx + \int_0^{\pi/2} \cos 2kx\, dx$$

$$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \frac{1}{2k}\left[\sin 2kx\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}.$$

EXAMPLE 37 *If* $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx$, *where n is a non-negative integer, then show that*

$$I_{n+2} + I_n = 2I_{n+1}$$

Hence, show that $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\, dx = \frac{n\pi}{2}$

SOLUTION We have,

$$I_{n+2} - I_{n+1} = \int_0^{\pi} \frac{1-\cos(n+2)x}{1-\cos x}\, dx - \int_0^{\pi} \frac{1-\cos(n+1)x}{1-\cos x}\, dx$$

$$= \int_0^{\pi} \left\{\frac{1-\cos(n+2)x}{1-\cos x} - \frac{1-\cos(n+1)x}{1-\cos x}\right\} dx$$

$$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}\, dx$$

$$= \int_0^{\pi} \frac{2\sin(n+3/2)x \sin\frac{x}{2}}{2\sin^2\frac{x}{2}}\, dx$$

$$\Rightarrow I_{n+2} - I_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}\, dx \quad \ldots(i)$$

Replacing n by $(n-1)$ in (i), we get

$$\therefore I_{n+1} - I_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\, dx \quad \ldots(ii)$$

From (i) and (ii), we get

$$(I_{n+2} - I_{n-1}) - (I_{n+1} - I_n)$$

$$= \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

$$= \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\, dx$$

$$= 2\int_0^{\pi} \frac{\sin\frac{x}{2}\cos(n+1)x}{\sin\frac{x}{2}}\, dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\, dx$$

$$= \frac{2}{n+1}\left[\sin(n+1)x\right]_0^{\pi}$$

$$= \frac{2}{n+1}[\sin(n+1)\pi - \sin 0]$$

$$= \frac{2}{n+1}(0-0) = 0$$

Hence, $I_{n+2} + I_n = 2I_{n+1}$.

Now,

$I_{n+2} + I_n = 2I_{n+1}$

$\Rightarrow I_{n+2} - I_{n+1} = I_{n+1} - I_n$ for all $n = 0, 1, 2, \ldots$

$\Rightarrow I_{n+2} - I_{n+1} = I_{n+1} - I_n = I_n - I_{n-1} = I_{n-1} - I_{n-2} \ldots = I_1 - I_0$

$\Rightarrow I_{n+2} - I_{n+1} = \pi - 0 \quad \left[\because I_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x}\, dx = \pi \text{ and } I_0 = 0\right]$

$\Rightarrow I_1, I_2, \ldots, I_n, \ldots$ form an AP with common difference π

$\therefore I_n = I_1 + (n-1)\pi = \pi + (n-1)\pi = n\pi$

Hence, $\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx = n\pi$ ...(iii)

Now,

$$\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{1-\cos 2nx}{1-\cos 2x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{1-\cos n\theta}{1-\cos\theta} \qquad [\text{Putting } 2x=\theta \text{ and } dx=\tfrac{1}{2}d\theta]$$

$$= \frac{1}{2}(n\pi) \qquad [\text{Using (iii)}]$$

$$= \frac{n\pi}{2}$$

EXAMPLE 38 *Evaluate* $\int_0^1 (tx+1-x)^n\,dx$, *where* n *is a positive integer and* t *is a parameter independent of* x. *Hence, prove that*

$$\int_0^1 x^k (1-x)^{n-k}\,dx = \frac{1}{{}^nC_k\,.\,(n+1)}$$

SOLUTION We have,

$$I = \int_0^1 (tx+1-x)^n\,dx$$

$$= \int_0^1 [(t-1)x+1]^n\,dx$$

$$= \left[\frac{[(t-1)x+1]^{n+1}}{(n+1)(t-1)}\right]_0^1$$

$$= \frac{1}{n+1}\left\{\frac{t^{n+1}-1}{t-1}\right\}$$

$$= \frac{1}{n+1}\left[t^n + t^{n-1} + t^{n-2} + \ldots + t + 1\right] \qquad \ldots(i)$$

Again,

$$I = \int_0^1 (tx+1-x)^n\,dx$$

$$= \int_0^1 [(1-x)+tx]^n\,dx$$

$$= \int_0^1 \left\{{}^nC_0(1-x)^n + {}^nC_1(1-x)^{n-1}(tx) + {}^nC_2(1-x)^{n-2}(tx)^2 + \ldots + {}^nC_r(1-x)^{n-r}(tx)^r + \ldots + {}^nC_n(tx)^n\right\}dx$$

$$= \int_0^1 \left\{\sum_{r=0}^{n} {}^nC_r\,(1-x)^{n-r}(tx)^r\right\}dx$$

$$= \sum_{r=0}^{n} {}^nC_r \left\{\int_0^1 (1-x)^{n-r}x^r\,dx\right\}t^r \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\sum_{r=0}^{n} {}^nC_r \left\{\int_0^1 (1-x)^{n-r}x^r\,dx\right\}t^r = \frac{1}{n+1}\left\{1+t+t^2+\ldots+t^n\right\}$$

On equating the coefficients of t^k on both sides, we get

$${}^nC_k \left\{\int_0^1 (1-x)^{n-k}x^k\,dx\right\} = \frac{1}{n+1}$$

$$\Rightarrow \int_0^1 (1-x)^{n-k}x^k\,dx = \frac{1}{(n+1)\,.\,{}^nC_k}$$

EXAMPLE 39 *If* $m, n \in N$, *then prove that*

$$I_{m,n} = \int_0^1 x^m\,(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$$

SOLUTION We have,

$$I_{m,n} = \int_0^1 \underset{\text{II}}{x^m}\,\underset{\text{I}}{(1-x)^n}\,dx$$

$$= \left[(1-x)^n\,\frac{x^{m+1}}{m+1}\right]_0^1 + \frac{n}{m+1}\int_0^1 (1-x)^{n-1}x^{m+1}\,dx$$

$$= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}\,dx$$

$$= \frac{n}{m+1}\,I_{m+1,n-1}.$$

$$\Rightarrow I_{m,n} = \frac{n}{m+1}\,I_{m+1,n-1}$$

$$= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\,I_{m+2,n-2} \qquad \left[\because I_{m+1,n-1} = \frac{n-1}{m+2}\,I_{m+2,n-2}\right]$$

$$= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdot\frac{n-2}{m+3}\cdot\ldots\cdot\frac{2}{m+(n-1)}\cdot\frac{1}{m+n}\,I_{m+n,0}$$

$$= \frac{n\,.\,(n-1)(n-2)(n-3)\ldots 2\,.\,1}{(m+1)(m+2)\ldots(m+n)}\int_0^1 x^{m+n}\,dx$$

$$= \frac{m!\,n!}{(m+n)!}\left[\frac{x^{m+n+1}}{m+n+1}\right]_0^1 = \frac{m!\,n!}{(m+n+1)!}$$

EXAMPLE 40 *If* $I_n = \int_0^{\pi/4} \tan^n x\,dx$, *show that* $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}, \frac{1}{I_5+I_7}, \ldots$ *form an A.P. Find the common difference of this progression.*

SOLUTION We have,

$$I_n = \int_0^{\pi/4} \tan^n x\, dx$$

$$\Rightarrow I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x\, dx$$

$$\therefore I_n + I_{n+2}$$

$$= \int_0^{\pi} \tan^n x\, dx + \int_0^{\pi/4} \tan^{n+2} x\, dx$$

$$= \int_0^{\pi/4} \tan^n x \,.\, (1 + \tan^2 x)\, dx$$

$$= \int_0^{\pi/4} \tan^n x \sec^2 x\, dx$$

$$= \int_0^1 t^n\, dt, \text{ where } t = \tan x.$$

$$= \left[\frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}, \quad n = 2, 3, 4, 5, ...$$

$$\Rightarrow \frac{1}{I_n + I_{n+2}} = n+1, n = 2, 3, 4, 5, ...$$

$$\Rightarrow \frac{1}{I_2 + I_4} = 3, \frac{1}{I_3 + I_5} = 4, \frac{1}{I_4 + I_6} = 5, \frac{1}{I_5 + I_7} = 6, ...$$

Clearly, 3, 4, 5, 6, ... is an AP with common difference 1.

Hence,

$\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$, ... is an AP with common difference 1.

EXAMPLE 41 *Evaluate* $\int_0^{\infty} e^{-x} \sin^n x\, dx$, *if n is an even integer.*

SOLUTION Let $I_n = \int_0^{\infty} \underset{\text{II}}{e^{-x}} \underset{\text{I}}{\sin^n x}\, dx$. Then,

$$I_n = \left[-\sin^n x\, e^{-x}\right]_0^{\infty} + n \int_0^{\infty} \sin^{n-1} x \cos x\, e^{-x}\, dx$$

$$\Rightarrow I_n = n \int_0^{+\infty} \underset{\text{I}}{\sin^{n-1} x \cos x}\ \underset{\text{II}}{e^{-x}}\, dx$$

$$\Rightarrow I_n = n \left[\left[-\sin^{n-1} x \cos x\, e^{-x}\right]_0^{\infty} + \int_0^{\infty} \left\{(n-1)\sin^{n-2} x \cos^2 x - \sin^n x\right\} e^{-x}\, dx\right]$$

$$\Rightarrow I_n = n \int_0^{\infty} \left\{(n-1)\sin^{n-2} x\,(1 - \sin^2 x) - \sin^n x\right\} e^{-x}\, dx$$

$$\Rightarrow I_n = n \int_0^{\infty} \left\{(n-1)\sin^{n-2} x - n \sin^n x\right\} e^{-x}\, dx$$

$$\Rightarrow I_n = n(n-1) \int_0^{\infty} e^{-x} \sin^{n-2} x\, dx - n^2 \int_0^{\infty} e^{-x} \sin^n x\, dx$$

$$\Rightarrow I_n = n(n-1) I_{n-2} - n^2 I_n$$

$$\Rightarrow (n^2 + 1) I_n = n(n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$$

Replacing n by $n-2, n-4, n-6, ..., 2$, we get

$$I_{n-2} = \frac{(n-2)(n-3)}{(n-2)^2+1} I_{n-4}$$

$$I_{n-4} = \frac{(n-4)(n-5)}{(n-4)^2+1} I_{n-6}$$

$$\vdots \qquad\qquad \vdots$$

$$I_2 = \frac{2(2-1)}{2^2+1} I_0$$

Multiplying these, we get

$$I_n = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)\ldots 2(2-1)}{(n^2+1)\{(n^2-2)^2+1\} \ldots\ldots\ldots\ldots (2^2+1)} I_0$$

$$= \frac{n!}{\prod_{r=1}^{n/2} \{1 + (2r)^2\}} \qquad \left[\because I_0 = \int_0^{\infty} e^{-x}\, dx = 1\right]$$

EXAMPLE 42 *Given a real valued function f (x)which is monotonic and differentiable, prove that for any real numbers a and b ;*

$$\int_a^b \left\{f^2(x) - f^2(a)\right\} dx = \int_{f(a)}^{f(b)} 2x \left\{b - f^{-1}(x)\right\} dx$$

SOLUTION Since $f(x)$ is differentiable, so it is continuous. Also, $f(x)$ is monotonic. Therefore, $f(x)$ is invertible and $f^{-1}(x)$ exists.

Let $f^{-1}(x) = t$. Then, $x = f(t)$ and $dx = f'(t)\, dt$.

Also, $x = f(a) \Rightarrow f^{-1}\{f(a)\} = t \Rightarrow t = a$,

and, $x = f(b) \Rightarrow f^{-1}\{f(b)\} = t \Rightarrow t = b$.

$$\therefore \int_{f(a)}^{f(b)} 2x \left\{b - f^{-1}(x)\right\} dx$$

$$= \int_a^b 2f(t)(b-t)f'(t)\,dt$$

$$= \int_a^b (b-t)\,d\,|f(t)|^2 \quad (\text{I}, \text{II})$$

$$= \left[(b-t)|f(t)|^2\right]_a^b + \int_a^b |f(t)|^2\,dt$$

$$= -(b-a)|f(a)|^2 + \int_a^b |f(t)|^2\,dt$$

$$= \int_a^b \left[|f(t)|^2 - |f(a)|^2\right]dt$$

$$= \int_a^b |f^2(x) - f^2(a)|\,dx$$

EXAMPLE 43 *Find the function $f(x)$ satisfying the equation*

$$f(x) = x + \int_0^{\log 2} f(y)\,e^x\,(e^y + e^{-y})\,dy$$

SOLUTION We have,

$$f(x) = x + \int_0^{\log 2} f(y)\,e^x\,(e^y + e^{-y})\,dy$$

$$\Rightarrow f(x) = x + \left\{\int_0^{\log 2} f(y)\,e^y\,dy + \int_0^{\log 2} f(y)\,e^{-y}\,dy\right\}e^x$$

$\Rightarrow f(x) = x + (a+b)\,e^x$, where

$$a = \int_0^{\log 2} f(y)\,e^y\,dy \quad \text{and,} \quad b = \int_0^{\log 2} f(y)\,e^{-y}\,dy.$$

Now,

$$a = \int_0^{\log 2} f(y)\,e^y\,dy$$

$$= \int_0^{\log 2} \{y + (a+b)\,e^y\}\,e^y\,dy \qquad \left[\because f(x) = x + (a+b)\,e^x \Rightarrow f(y) = y + (a+b)\,e^y\right]$$

$$= \int_0^{\log 2} y\,e^y\,dy + (a+b)\int_0^{\log 2} e^{2y}\,dy \quad (\text{I}, \text{II})$$

$$= \left[y\,e^y - e^y\right]_0^{\log 2} + \left(\frac{a+b}{2}\right)\left[e^{2y}\right]_0^{\log 2}$$

$$= (2\log 2 - 2 + 1) + \left(\frac{a+b}{2}\right)(4-1)$$

$$= 2\log 2 - 1 + \frac{3}{2}(a+b)$$

$\therefore\ 2a = 4\log 2 - 2 + 3(a+b)$

$\Rightarrow a + 3b = 2 - 4\log 2$ ...(i)

and,

$$b = \int_0^{\log 2} f(y)\,e^{-y}\,dy$$

$$= \int_0^{\log 2} \left[y + (a+b)\,e^y\right]e^{-y}\,dy$$

$$= \int_0^{\log 2} y\,e^{-y}\,dy + \int_0^{\log 2} (a+b)\,dy \quad (\text{I}, \text{II})$$

$$= \left[-y\,e^{-y} - e^{-y}\right]_0^{\log 2} + (a+b)(\log 2 - 0)$$

$$= \left(-\frac{1}{2}\log 2 - \frac{1}{2} + 1\right) + (a+b)\log 2$$

$$= \left(-\frac{1}{2}\log 2 + \frac{1}{2}\right) + (a+b)\log 2$$

$\therefore\ 2b = -\log 2 + 1 + (2a+2b)\log 2$

$\Rightarrow 2a\log 2 + 2b(\log 2 - 1) = (\log 2 - 1)$ ...(ii)

Solving, (i) and (ii), we get

$$a + b = \frac{1 - 3\log 2}{1 + 2\log 2}.$$

Hence, $f(x) = x + \left(\frac{1 - 3\log 2}{1 + 2\log 2}\right)e^x$.

EXAMPLE 44 *Find $f(x)$, if $f'(x) = f(x) + \int_0^1 f(x)\,dx$, given that $f(0) = 1$.*

SOLUTION We have,

$$f'(x) = f(x) + \int_0^1 f(x)\,dx \qquad \text{...(i)}$$

$\Rightarrow f''(x) = f'(x) + 0$ [On differentiating both sides]

$$\Rightarrow \frac{f''(x)}{f'(x)} = 1$$

$\Rightarrow \log|f'(x)| = x + \log C$ [On integrating both sides]

$\Rightarrow f'(x) = C\,e^x$

$\Rightarrow f(x) = C\,e^x + C_1$

Now, $f(0) = 1$

$\Rightarrow C\,e^0 + C_1 = 1$

$\Rightarrow C + C_1 = 1$

$\therefore f(x) = Ce^x + (1 - C)$ ...(ii)

Substituting the values of $f(x)$ and $f'(x)$ in (i), we get

$$Ce^x = Ce^x + (1-C) + \int_0^1 \{Ce^x + (1-C)\}\,dx$$

$$\Rightarrow Ce^x = Ce^x + (1-C) + \left[Ce^x + (1-C)x\right]_0^1$$

$$\Rightarrow Ce^x = Ce^x + (1-C) + Ce + (1-C) - C$$

$$\Rightarrow \quad 0 = 1 - C + Ce + 1 - 2C$$

$$\Rightarrow \quad 0 = 2 + C(e-3)$$

$$\Rightarrow \quad C = \frac{2}{3-e}$$

Putting the value of C in (ii), we get

$$f(x) = \frac{2}{3-e}\cdot e^x + \left(\frac{1-e}{3-e}\right)$$

EXAMPLE 45 *If* $I = \int_0^{\pi} \frac{\cos x}{(x+2)^2}\,dx$ *and* $I_1 = \int_0^{\pi/2} \frac{\cos x \sin x}{x+1}\,dx$, *then evaluate* I_1 *in terms of* I.

SOLUTION Let $x = 2t$. Then, $dx = 2\,dt$.

Also, $x = 0 \Rightarrow t = 0$ and $x = \pi \Rightarrow 2t = \pi \Rightarrow t = \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} \frac{\cos 2t}{(2t+2)^2}\,2\,dt$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \frac{\cos 2t}{(t+1)^2}\,dt$$

$$\Rightarrow 2I = \int_0^{\pi/2} \underset{\text{I}}{\cos 2t} \cdot \underset{\text{II}}{\frac{1}{(t+1)^2}}\,dt$$

$$\Rightarrow 2I = \left[-\frac{\cos 2t}{t+1}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{-2\sin 2t}{t+1}\,dt$$

$$\Rightarrow 2I = \frac{2}{2+\pi} + 1 - 4\int_0^{\pi/2} \frac{\sin t \cos t}{t+1}\,dt$$

$$\Rightarrow 2I = \frac{2}{\pi+2} + 1 - 4I_1$$

$$\Rightarrow 2I = \frac{\pi+4}{\pi+2} - 4I_1$$

$$\Rightarrow I_1 = \frac{I}{2} + \frac{1}{4}\left(\frac{\pi+4}{\pi+2}\right)$$

IMPORTANT NOTE Consider the integral

$$I = \int_0^{\pi} \frac{1}{1+2\sin^2 x}\,dx$$

$$= \int_0^{\pi} \frac{1}{\cos^2 x + 3\sin^2 x}\,dx$$

$$= \int_0^{\pi} \frac{\sec^2 x}{1+3\tan^2 x}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^{\pi} 1 + (\sqrt{3}\tan x)^2\,d(\sqrt{3}\tan x)$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi}$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}(0) - \tan^{-1} 0\right] = 0$$

The above solution is not correct, because the anti-derivative or, primitive of $\frac{1}{1+2\sin^2 x}$ is not continuous at $x = \frac{\pi}{2} \in (0, \pi)$.

Similarly, $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$ cannot be written as

$$\left[\tan^{-1}\left(\frac{1}{x}\right)\right]_{-1}^{1} = \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{2},$$

because $\tan^{-1}\left(\frac{1}{x}\right)$ is not continuous at $x = 0 \in (-1, 1)$.

In fact, we have

$$\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$$

$$= \int_{-1}^{1} \frac{-1}{1+x^2}\,dx$$

$$= -\int_{-1}^{1} \frac{1}{1+x^2}\,dx = -\left[\tan^{-1} x\right]_{-1}^{1} = -\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = -\frac{\pi}{2}.$$

EXAMPLE 46 *If* $u_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$, *show that* $u_1, u_2, u_3 \ldots$ *form a G.P. Also, find* u_n.

SOLUTION We have,

$$u_n = \int_0^{\pi/2} \underset{\text{I}}{\cos^n x} \underset{\text{II}}{\cos nx}\, dx$$

$$\Rightarrow u_n = \left[\frac{\cos^n x \sin nx}{n}\right]_0^{\pi/2} + n\int_0^{\pi/2} \cos^{n-1} x \sin x \frac{\sin nx}{n}\, dx$$

$$\Rightarrow u_n = \int_0^{\pi/2} \cos^{n-1} x \sin x \sin nx\, dx$$

$$\Rightarrow u_n = \int_0^{\pi/2} \cos^{n-1} x \{\cos(n-1)x - \cos x \cos nx\}\, dx$$

$$[\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x]$$

$$\Rightarrow u_n = \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x\, dx - \int_0^{\pi/2} \cos^n x \cos nx\, dx$$

$$\Rightarrow u_n = u_{n-1} - u_n$$

$$\Rightarrow 2u_n = u_{n-1}$$

$$\Rightarrow \frac{u_n}{u_{n-1}} = \frac{1}{2} \text{ for all } n = 2, 3, \ldots.$$

$\Rightarrow u_1, u_2, u_3, \ldots$ form a G.P. with common ratio $\frac{1}{2}$.

Now,

$$u_n = u_1 \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow u_n = \frac{\pi}{4}\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow u_n = \frac{\pi}{2^{n+1}} \qquad \left[\because u_1 = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}\right]$$

EXAMPLE 47 *Prove the identity*

$$\int_0^x \left\{\int_0^u f(t)\, dt\right\} du = \int_0^x (x-u) f(u)\, du$$

SOLUTION We have,

$$\int_0^x \underset{\text{I}}{\left\{\int_0^u f(t)\, dt\right\}} \cdot \underset{\text{II}}{1}\, du = \left[\left\{\int_0^u f(t)\, dt\right\} u\right]_0^x - \int_0^x f(u) \cdot u\, du$$

$$= \int_0^x x f(t)\, dt - \int_0^x u f(u)\, du$$

$$= \int_0^x x f(u)\, du - \int_0^x u f(u)\, du$$

$$= \int_0^x (x-u) f(u)\, du$$

EXAMPLE 48 *If* $f(x) = x + \int_0^1 (xy^2 + x^2 y) f(y)\, dy$, *find* $f(x)$.

SOLUTION We have,

$$f(x) = x + x\left\{\int_0^1 y^2 f(y)\, dy\right\} + \left\{\int_0^1 y f(y)\, dy\right\} x^2 \qquad \ldots(i)$$

$$\Rightarrow f(x) = \left\{1 + \int_0^1 y^2 f(y)\, dy\right\} x + \left\{\int_0^1 y f(y)\, dy\right\} x^2$$

$\Rightarrow f(x)$ is of the form $ax + bx^2$.

Let $f(y) = ay + by^2$.

Now, $\int_0^1 y^2 f(y)\, dy = \int_0^1 (ay + by^2) y^2\, dy$

$$= \int_0^1 (ay^3 + by^4)\, dy$$

$$= \left[\frac{ay^4}{4} + \frac{by^5}{5}\right]_0^1 = \frac{a}{4} + \frac{b}{5}.$$

and, $\int_0^1 y f(y)\, dy = \int_0^1 y(ay + by^2)\, dy$

$$= \int_0^1 (ay^2 + by^3)\, dy$$

$$= \left[\frac{ay^3}{3} + \frac{by^4}{4}\right]_0^1 = \frac{a}{3} + \frac{b}{4}.$$

Putting these values in (i), we get

$$f(x) = x + x\left(\frac{a}{4} + \frac{b}{5}\right) + x^2\left(\frac{a}{3} + \frac{b}{4}\right)$$

But, $f(x) = ax + bx^2$

Comparing the two expressions for $f(x)$, we get

$$1 + \frac{a}{4} + \frac{b}{5} = a \text{ and } \frac{a}{3} + \frac{b}{4} = b$$

Solving these two equations, we get

$$a = \frac{180}{119} \text{ and } b = \frac{80}{119}$$

Substituting these values in $f(x) = ax + bx^2$, we get

$$f(x) = \frac{180}{119}x + \frac{80}{119}x^2.$$

EXAMPLE 49 *If* $f(x) = e^x + \int_0^1 (e^x + te^{-x}) f(t)\,dt$, *find* $f(x)$.

SOLUTION We have,

$$f(x) = e^x + \int_0^1 \{e^x + t\,e^{-x}\} f(t)\,dt$$

$$\Rightarrow f(x) = e^x + e^x \int_0^1 f(t)\,dt + e^{-x}\int_0^1 t f(t)\,dt$$

$$\Rightarrow f(x) = \left\{1 + \int_0^1 f(t)\,dt\right\} e^x + \left\{\int_0^1 t f(t)\,dt\right\} e^{-x}$$

$$\Rightarrow f(x) = A\,e^x + B\,e^{-x},$$

where, $A = 1 + \int_0^1 f(t)\,dt$

and, $B = \int_0^1 t f(t)\,dt$

$$\therefore A = 1 + \int_0^1 (A\,e^t + B\,e^{-t})\,dt$$

$$= 1 + \left[A\,e^t - B\,e^{-t}\right]_0^1$$

$$= 1 + \left(Ae - \frac{B}{e}\right) - (A - B)$$

$$\Rightarrow A = 1 + (e-1)A + \left(1 - \frac{1}{e}\right)B$$

$$\Rightarrow (2-e)A + B(e^{-1} - 1) = 1 \qquad \ldots(i)$$

and,

$$B = \int_0^1 t f(t)\,dt$$

$$= \int_0^1 t\,(A\,e^t + B\,e^{-t})\,dt$$

$$= A\left[te^t - e^t\right]_0^1 + B\left[-t\,e^{-t} - e^{-t}\right]_0^1$$

$$= A + B(1 - 2e^{-1})$$

$$\Rightarrow A - 2Be^{-1} = 0 \qquad \ldots(ii)$$

Solving (i) and (ii), we get

$$A = \frac{2}{5-3e},\ B = \frac{e}{5-3e}$$

Hence, $$f(x) = \left(\frac{2}{5-3e}\right)e^x + \left(\frac{e}{5-3e}\right)e^{-x}$$

EXAMPLE 50 *A function* $f: R \to R$ *satisfies the equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in R$ *and is continuous throughout the domain. If* $I_1 + I_2 + I_3 + I_4 + I_5 = 450$, *when* $I_n = n\int_0^n f(x)\,dx$. *Find* $f(x)$.

SOLUTION We have,

$f(x+y) = f(x) + f(x)$ for all $x, y \in R$.

$\Rightarrow$ $f(0) = f(0) + f(0)$ [Putting $x = y = 0$]

$\Rightarrow$ $f(0) = 0$

Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} \qquad [\because f(0) = 0]$$

$\Rightarrow f'(x) = f'(0)$

$\Rightarrow f(x) = xf'(0) + C$ [Integrating both sides w.r. to x]

Putting $x = 0$ on both sides, we get

$f(0) = C \Rightarrow C = 0$ $[\because f(0) = 0]$

$\therefore f(x) = xf'(0)$ ...(i)

Now,

$$I_n = n\int_0^n f(x)\,dx$$

$$\Rightarrow I_n = n\int_0^n xf'(0)\,dx \qquad \text{[Using (i)]}$$

$$\Rightarrow I_n = nf'(0)\left[\frac{x^2}{2}\right]_0^n$$

$$\Rightarrow I_n = \frac{n^3}{2}f'(0)$$

$\therefore I_1 + I_2 + I_3 + I_4 + I_5 = 450$

$$\Rightarrow \frac{f'(0)}{2}\{1^3 + 2^3 + \ldots + 5^3\} = 450$$

$$\Rightarrow \left\{\frac{5(5+1)}{2}\right\}^2 \frac{f'(0)}{2} = 450$$

$$\Rightarrow \quad f'(0) = 4$$

Therefore, from (i), we get $f(x) = 4x$.

EXAMPLE 51 *Let $f: R \to R$ be defined by $f(x) = \int_1^3 \frac{1}{|t-x|+1}$.*

Then, find $\int_1^3 f(x)\,dx$.

SOLUTION For any $x \in [1, 3]$, we get

$$f(x) = \int_1^3 \frac{1}{|t-x|+1}\,dt$$

$$= \int_1^x \frac{1}{-(t-x)+1}\,dt + \int_x^3 \frac{1}{(t-x)+1}\,dt$$

$$= \int_1^x \frac{1}{x-t+1}\,dt + \int_x^3 \frac{1}{t-x+1}\,dt$$

$$= -[\log(x-t+1)]_1^x + [\log(t-x+1)]_x^3$$

$$= -(\log 1 - \log x) + \log(4-x) - \log 1$$

$$= \log(4-x) + \log x$$

$$= \log(4x-x^2)$$

$$\therefore \int_1^3 f(x)\,dx$$

$$= \int_1^3 \log(4x-x^2)\,dx$$

$$= [x\log(4x-x^2)]_1^3 - \int_1^3 \frac{(4x-2x^2)}{4x-x^2}\,dx$$

$$= 3\log 3 - \log 3 - \int_1^3 \frac{2x-4}{x-4}\,dx$$

$$= 2\log 3 - \int_1^3 \frac{2(x-4)+4}{x-4}\,dx$$

$$= 2\log 3 - [2x + 4\log(x-4|]_1^3$$

$$= 2\log 3\,[4 + 4(\log 1 - \log 3)]$$

$$= 6\log 3 - 4.$$

EXAMPLE 52 *Let $f(x) = \frac{\sin x}{x}$ for all $x \in (0, \pi]$, prove that*

$$\frac{\pi}{2}\int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)\,dx$$

SOLUTION Let $I = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{x}\,dx$ ...(i)

Also,

$$I = \int_0^{\pi} f(\pi-x)\,dx$$

$$= \int_0^{\pi} \frac{\sin(\pi-x)}{\pi-x}\,dx$$

$$= \int_0^{\pi} \frac{\sin x}{\pi-x}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \left\{\frac{\sin x}{x} + \frac{\sin x}{\pi-x}\right\}dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{x(\pi-x)}\,dx \qquad \text{...(iii)}$$

Now,

$$\frac{\pi}{2}\int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x}{x} \times \frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x}{x} \times \frac{\cos x}{\frac{\pi}{2}-x}\,dx$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin 2x}{x\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \frac{\pi}{8}\int_0^{\pi} \frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}\,dt, \text{ where } t = 2x$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin t}{t(\pi-t)}\,dt \qquad \text{...(iv)}$$

From (iii) and (iv), we get

$$\frac{\pi}{2}\int_0^{\pi} f(x) f\left(\frac{\pi}{2}-x\right) dx = \int_0^{\pi} f(x)\, dx$$

EXAMPLE 53 *Evaluate* $\int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}\,dx$ [IIT 1993]

SOLUTION Let

$$I = \int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}\,dx$$

$$= \int_2^3 \frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}\,dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2}\,dx + \int_2^3 \frac{1}{x^2-1}\,dx$$

$$= \int_2^3 \frac{x^2}{(x^2+1)^2}\cdot 2x\,dx + \int_2^3 \frac{1}{x^2-1}\,dx$$

$$= \int_5^{10} \frac{t-1}{t^2}\,dt + \int_2^3 \frac{1}{x^2-1}\,dx \text{ where, } \frac{1}{t} = x^2+1$$

$$= \left[\log t + \frac{1}{t}\right]_5^{10} + \frac{1}{2}\left[\log\left(\frac{x-1}{x+1}\right)\right]_2^3$$

$$= \left(\log 10 + \frac{1}{10}\right) - \left(\log 5 + \frac{1}{5}\right) + \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$= \log 2 - \frac{1}{10} + \frac{1}{2}\log\left(\frac{3}{2}\right)$$

$$= \frac{1}{2}\log 6 - \frac{1}{10}.$$

EXERCISE 11.1

Evaluate the following integrals :

1. $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

2. $\int_0^{\pi/4} \sqrt{2+\tan^2 x}\,dx$

3. $\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,dx, b>a$

4. $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx$

5. $\int_a^b \sqrt{(x-a)(b-x)}\,dx$

6. $\int_{\sqrt{\frac{3a^2+b^2}{4}}}^{\frac{\sqrt{a^2+b^2}}{2}} \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx$

7. $\int_0^{\pi/4} \frac{1}{\cos^4 x - \cos^2 x\sin^2 x + \sin^4 x}\,dx$

8. $\int_0^1 \frac{2x^2+3x+3}{(x+1)(x^2+2x+2)}\,dx$

9. $\int_0^{\pi/4} \sqrt{\tan x}\,dx$

10. $\int_0^1 \frac{1}{(1+x^2)\sqrt{2+x^2}}\,dx$

11. $\int_0^{\pi/4} \cos 2x \cdot \log(1+\tan x)\,dx$

12. $\int_0^{\pi} \cos 2x \log\sin x\,dx$

13. $\int_0^{\pi/4} e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$

14. $\int_{\pi/3}^{\pi/6} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$

15. $\int_{\pi/4}^{\pi/3} \frac{2\sec^6\theta+\tan\theta+\sec^5\theta\tan\theta-2\sec^4\theta\tan\theta+2\sec^3\theta\tan\theta+\sec\theta\tan\theta}{(\sec^2\theta+1)^2\tan^2\theta}\,d\theta$

16. Prove that

$$\int_0^{\pi} \cos mx\cos nx\,dx = \begin{cases} \frac{2n}{n^2-m^2}, & \text{if } (n-m) \text{ is an odd integer} \\ 0, & \text{if } (n-m) \text{ is an even integer} \end{cases}$$

17. If $0 \le \alpha \le 1$ and the positive value of the square root is taken, then prove that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-2\alpha x+x^2}}\,dx = 2$$

What is the value of the integral if $\alpha > 1$?

18. Prove that $\int_0^{\pi} \frac{1}{3+2\sin x+\cos x}\,dx = \frac{\pi}{4}$

19. Evaluate $\int_0^{\pi/4} \tan^6 x\,dx$

20. Evaluate : $\int_1^{e} \frac{(1+\log x)^2}{1+\log x^{x+1}+(\log x^{\sqrt{x}})^2}\,dx$

21. Prove that : $\int_{\sqrt{3}}^{2\sqrt{3}} x^2 (x^2-3)^{3/2}\,dx = \frac{16038}{35}$

22. Evaluate : $\int_{3/4}^{9/25} \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\,dx$

23. Evaluate : $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx$

24. Evaluate : $\int_{-2}^{2} \frac{3x^5+4x^3+2x^2+x-33}{x^2+4}\,dx$

25. $\int_0^{1} \frac{1}{(1+x)(2+x)\sqrt{x-x^2}}\,dx$

26. $\int_0^{\infty} e^{-x} x^n\,dx$

27. If $I_n = \int_0^{\pi/2} x\cos^n x\,dx$, show that

$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$ and hence find I_4.

28. If $I_{m,n} = \int_0^{1} (1-x^m)^n\,dx$, $n \in N$, show that

$(1+mn)\, I_{m,n} = mn\, I_{m,n-1}$

29. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n x\,dx$, $n > 2$, show that

$I_n = \frac{1}{n-1} - I_{n-2}$

30. If $I_n = \int_0^{\pi/2} x^n \sin x\,dx$, prove that

$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

31. If $I_n = \int_0^{2a} x^m \sqrt{2ax-x^2}\,dx$, prove that

$I_m = \frac{(2m+1)\,a}{m+2} I_{m-1}$

Hence, evaluate I_3.

32. If $h(a) = h(b)$, evaluate

$\int_a^{b} \{fogoh(x)\}^{-1} f'\{g(h(x))\}\, g'\{h(x)\}\, h'(x)\,dx$

33. If $I_n = \int_0^{\pi/2} \cos^n x\,dx$, $n \in N$, show that $I_n = \frac{n-1}{n} I_{n-2}$.

34. If $I_n = \int_0^{1} e^x (x-1)^n\,dx$, show that $I_n + n I_{n-1} = (-1)^{n+1}$

35. If $f(\theta) = \frac{d}{d\theta}\int_0^{\theta} \frac{1}{1-\cos\theta\cos x}\,d\theta$, show that

$\frac{df}{d\theta}\sin\theta + 2f\cos\theta = \frac{\pi}{2}$

36. If $I_n = \int_0^{1} x^n \tan^{-1} x\,dx$, prove that

$(n+1) I_n + (n-1) I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$

37. (i) If $\int_1^{\infty} \frac{x^3+3}{x^6(x^2+1)}\,dx = \frac{a+b\pi}{c}$, find a, b, c.

(ii) If $\int_0^{1} \frac{1-x^2}{1+x^2+x^4}\,dx = a\log_e(b)$, find a, b.

38. Show that $\int_0^{a} e^{ax-x^2}\,dx = e^{a^2/4}\int_0^{a} e^{-x^2/4}\,dx$

ANSWERS

1. $\frac{1}{20}\log_e 3$ 2. $\frac{\pi}{6} + \log_e\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)$ 3. π 4. $\frac{\pi}{2}(b-a)$

5. $\frac{\pi}{8}(b-a)^2$ 6. $\frac{\pi}{12}$

7. $\frac{\pi}{2}$ 8. $\log 4 - \cot^{-1} 3$

9. $\frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \log(\sqrt{2}-1)$ 10. $\frac{\pi}{6}$

11. $\frac{3}{4}\log 2 - \frac{\pi}{8}$ 12. $-\frac{\pi}{2}$

13. $1 + \left(\frac{\pi}{4} - \sqrt{2}\right) e^{1/\sqrt{2}}$ 14. $2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

15. $\frac{1}{2}\log_e\left\{\frac{5}{9}(3+2\sqrt{3})\right\} - \frac{2}{15}$

17. $\frac{2}{\alpha}$ 19. $\frac{13}{15} - \frac{\pi}{4}$

20. $\log(1+e)$ 22. $3 - 2\sqrt{3}$

23. $\frac{\pi}{2} - 1$ 24. $8 - \frac{41\pi}{4}$

26. $n!$ 27. $\frac{3\pi^2}{64} - \frac{1}{4}$

31. $\frac{7\pi}{8}a^5$ 32. 0

37. (i) $a = 40 - 6\log 2, b = -9, c = 12$

(ii) $a = \frac{1}{2}, b = 3$

HINTS TO SELECTED PROBLEMS

3. Put $x = (a\cos^2 t + b\sin^2 t)$

6. Put $x^2 = a^2\cos^2 t + b^2\sin^2 t$

15. Put $\sec\theta = x$. 22. Put $x = \sin^2\theta$

27. $I_n = \int_0^{\pi/2} \underset{\text{I}}{\cos^{n-1} x}\ \underset{\text{II}}{(x\cos x)}\, dx$

28. Integrate by parts taking 1 as the second function.

31. $I_m = \int_0^{2a} \underset{\text{I}}{x^{m+1/2}}\ \underset{\text{II}}{\sqrt{2a-x}}\, dx$

37. (i) Put $x = \tan\theta$.

—❑❑❑—

11.4 GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ (See Fig. 11.1) i.e.

$$\int_a^b f(x)\,(dx) = \text{Area } LAMB.$$

In general, $\int_a^b f(x)\,dx$ represents an algebraic sum of the areas of the figures bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$. The areas above x-axis are taken with plus sign and the areas below x-axis are taken with a minus sign. (See Fig. 11.2) i.e.

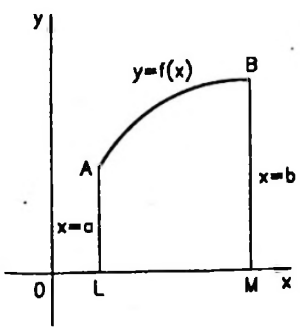


Fig. 11.1

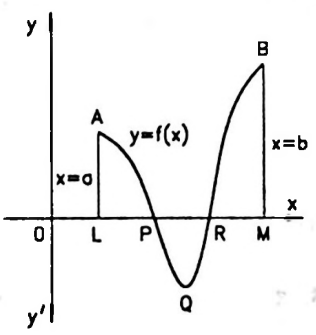


Fig. 11.2

$$\int_a^b f(x)\,dx = \text{Area } LAP - \text{Area } PQR + \text{Area } RBM.$$

11.5 THE DEFINITE INTEGRAL

In this section, we shall consider integration as the limit of the sum of certain number of terms when the number of terms tends to infinity and each term tends to zero. As a matter of fact the summation aspect of definite integral is more fundamental and it was invented far before the differentiation was known.

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n equal parts each of width h by inserting $(n-1)$ points $a + h$, $a + 2h$, $a + 3h$, ..., $a + (n-1)h$ between a and b as shown in Fig. 11.3. Then,

$$nh = b - a \Rightarrow h = \frac{b-a}{n}.$$

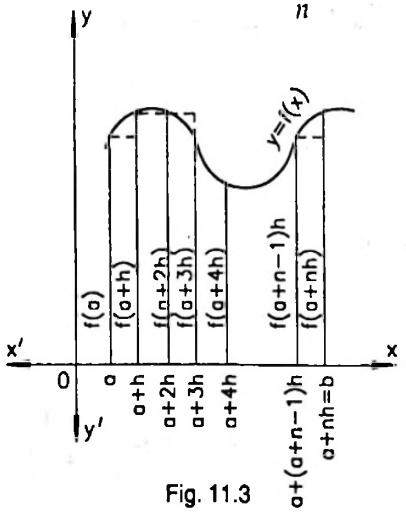


Fig. 11.3

Let S_n denote the sum of the areas of n rectangles shown in Fig. 11.3. Then,

$S_n = h f(a) + h f(a+h) + h f(a+2h) + \ldots + h f(a+(n-1)h)$

$\Rightarrow S_n = h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)]$

Clearly, S_n denotes the area which is close to the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$. It is evident that if n increases, the number of rectangles will increase and the width of rectangles will decrease. Consequently S_n gives closer approximation to the area enclosed by the curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$.

Thus, $\lim_{n\to\infty} S_n$ gives the area of the region bounded by the four curves $y = f(x), y = 0$ (x-axis), $x = a$ and $x = b$. It can be proved that this limit exists for all continuous functions defined on closed integral $[a, b]$ and is defined as the definite integral of $f(x)$ over $[a, b]$.

$$\therefore \int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$$

or,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)],$$

where $h = \dfrac{b-a}{n}$

But, $n \to \infty \Leftrightarrow h \to 0$

$$\therefore \int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)] \quad \ldots(i)$$

The process of evaluating a definite integral by using the above definition is called *integration from first principles* or *integration by ab-initio method* or *integration as the limit of a sum.*

REMARK 1 *In finding S_n in the above section, we have taken the left end points of the subintervals. We can also take the right end-points of the subintervals throughout to obtain:*

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)],$$

$$h = \frac{b-a}{n}.$$

It can be proved that this formula and the formula (i) give the same limit.

The following results will be helpful in evaluating definite integrals as limit of sums.

(i) $1 + 2 + 3 + \ldots + (n-1) = \dfrac{n(n-1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \left\{\dfrac{n(n-1)}{2}\right\}^2$

(iv) $a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\dfrac{r^n - 1}{r-1}\right),\ r \neq 1$

(v) $\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin(a+(n-1)h)$

$$= \frac{\sin\left\{a + \left(\dfrac{n-1}{2}\right)h\right\} \sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}$$

(vi) $\cos a + \cos(a+h) + \cos(a+2h) + \ldots + \cos(a+(n-1)h)$

$$= \frac{\cos\left\{a + \left(\dfrac{n-1}{2}\right)h\right\} \sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}.$$

REMARK 2 The value of S_n in the above discussion is independent of the choice of sub-divisions of the interval $[a, b]$ and it is also independent of the choice of points in the sub-divisons. Thus, if we divide $[a, b]$ into sub-intervals.

$[a, ar], [ar, ar^2], [ar^2, ar^3], \ldots, [ar^{n-1}, ar^n = b]$, where $r > 1$, then we obtain

$$\int_a^b f(x)\,dx = \lim_{r\to 1} (r-1)\,[a f(a) + ar f(ar) + ar^2 f(ar^2) + \ldots + ar^{n-1} f(ar^{n-1})]$$

where $ar^n = b$ or, $n = \dfrac{\log(b/a)}{\log r}$

Clearly, $r \to 1 \Leftrightarrow n \to \infty$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* $\int_a^b x^2\,dx$ *as limit of sums.*

SOLUTION (iii) We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h],$$

where $h = \dfrac{b-a}{n}$.

Here, $f(x) = x^2$.

$$\therefore \int_a^b x^2\,dx$$

$$= \lim_{h\to 0} h\,[a^2 + (a+h)^2 + (a+2h)^2 + (a+3h)^2 + \ldots + [a+(n-1)h]^2]$$

$$= \lim_{h\to 0} h\,[na^2 + 2ah\,(1 + 2 + 3 + \ldots + (n-1) + h^2\,(1^2 + 2^2 + \ldots + (n-1)^2)]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah\cdot\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[(nh)a^2 + a(nh)(nh-h) + \frac{1}{6}(nh)(nh-h)(2nh-h)\right]$$

$$= \lim_{h\to 0}\Big[(b-a)a^2 + a(b-a)(b-a-h)$$

$$+\frac{1}{6}(b-a)(b-a-h)\{2(b-a)-h\}\Big] \quad \left[\because h = \frac{b-a}{n} \Rightarrow nh = b-a\right]$$

$$= \left[(b-a)a^2 + a(b-a)^2 + \frac{2}{6}(b-a)^3\right]$$

$$= \frac{(b-a)}{3}[3a^2 + 3a(b-a) + (b-a)^2]$$

$$= \frac{(b-a)}{3}(3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2)$$

$$= \frac{1}{3}(b-a)(a^2 + ab + b^2)$$

$$= \frac{1}{3}(b^3 - a^3).$$

EXAMPLE 2 *Evaluate* $\int_a^b \sin x\,dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\,dx$$

$$= \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)],$$

where $h = \frac{b-a}{n}$.

Here, $f(x) = \sin x$.

$$\therefore \int_a^b \sin x\,dx$$

$$= \lim_{h\to 0} h\,[\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin(a+(n-1)h)]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left\{a + (n-1)\frac{h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left\{a + \frac{nh}{2} - \frac{h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left\{a + \frac{b-a}{2} - \frac{h}{2}\right\}\sin\left\{\frac{b-a}{2}\right\}}{\sin\frac{h}{2}}\right] \quad [\because nh = b-a]$$

$$= \lim_{h\to 0}\left[\frac{\frac{h}{2}}{\sin\frac{h}{2}} \times 2\sin\left\{\frac{a+b}{2} - \frac{h}{2}\right\}\sin\left\{\frac{b-a}{2}\right\}\right]$$

$$= \lim_{h\to 0}\left\{\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right\}\cdot\lim_{h\to 0} 2\sin\left\{\frac{a+b}{2} - \frac{h}{2}\right\}\sin\left\{\frac{b-a}{2}\right\}$$

$$= 2\sin\left\{\frac{a+b}{2}\right\}\sin\left\{\frac{b-a}{2}\right\}$$

$$= \cos a - \cos b$$

EXAMPLE 3 *Evaluate* $\int_a^b e^x\,dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\,dx$$

$$= \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)],$$

where $h = \frac{b-a}{n}$

Here, $f(x) = e^x$.

$$\therefore \int_a^b e^x\,dx$$

$$= \lim_{h\to 0} h\left[e^a + e^{a+h} + e^{a+2h} + \ldots + e^{a+(n-1)h}\right]$$

$$= \lim_{h\to 0} h\,e^a\left[1 + e^h + e^{2h} + e^{3h} + \ldots + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\,e^a\left[\frac{(e^h)^n - 1}{e^h - 1}\right] \quad \left[\because a + ar + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\,e^a\left(\frac{e^{nh} - 1}{e^h - 1}\right)$$

$$= \lim_{h\to 0} e^a\left\{\frac{(e^{b-a} - 1)}{\left(\frac{e^h - 1}{h}\right)}\right\} = e^a(e^{b-a} - 1) = e^b - e^a.$$

EXAMPLE 4 *Evaluate* $\int_a^b \frac{1}{x}\,dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\,dx = \lim_{r\to 1}(r-1)\,[a f(a) + ar f(ar) + \ldots + ar^{n-1} f(ar^{n-1})],$$

where $ar^n = b$ or, $n = \dfrac{\log(b/a)}{\log r}$

Here, $f(x) = \dfrac{1}{x}$

$$\therefore \int_a^b f(x)\,dx$$

$$= \lim_{r\to 1}(r-1)\left[a\cdot\frac{1}{a} + ar\cdot\frac{1}{ar} + .. + ar^{n-1}\cdot\frac{1}{ar^{n-1}}\right]$$

$$= \lim_{r\to 1}(r-1)\,n$$

$$= \lim_{r\to 1}\frac{(r-1)\log_e(b/a)}{\log_e r}$$

$$= \log\left(\frac{b}{a}\right)\lim_{r\to 1}\frac{(r-1)}{\log\{1+(r-1)\}}$$

$$= \log_e\left(\frac{b}{a}\right)$$

EXAMPLE 5 *Evaluate* $\int_a^b \dfrac{1}{\sqrt{x}}\,dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\,dx$$

$$= \lim_{r\to 1}(r-1)\left[a f(a) + ar f(ar) + ar^2 f(ar^2) + \ldots + ar^n f(ar^n)\right],$$

where $ar^n = b$ or, $n = \dfrac{\log(b/a)}{\log r}$

Here, $f(x) = \dfrac{1}{\sqrt{x}}$.

$$\therefore \int_a^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{r\to 1}(r-1)\left[a\cdot\frac{1}{\sqrt{a}} + ar\cdot\frac{1}{\sqrt{ar}} + ar^2\cdot\frac{1}{\sqrt{ar^2}} + \ldots + ar^{n-1}\cdot\frac{1}{\sqrt{ar^{n-1}}}\right]$$

$$= \lim_{r\to 1}(r-1)\left[\sqrt{a} + \sqrt{ar} + \sqrt{ar^2} + \ldots + \sqrt{ar^{n-1}}\right]$$

$$= \lim_{r\to 1}(r-1)\sqrt{a}\left\{\frac{(\sqrt{r})^n - 1}{\sqrt{r}-1}\right\}$$

$$= \lim_{r\to 1}(\sqrt{r}+1)\sqrt{a}\left\{(\sqrt{r})^n - 1\right\}$$

$$= \lim_{r\to 1}(\sqrt{a}r + \sqrt{a})\,(r^{n/2} - 1)$$

$$= \lim_{r\to 1}(\sqrt{ar} + \sqrt{a})\left\{\left(\frac{b}{a}\right)^{1/2} - 1\right\}$$

$$= 2\sqrt{a}\left\{\frac{\sqrt{b}}{\sqrt{a}} - 1\right\}$$

$$= 2\,(\sqrt{b} - \sqrt{a})$$

—❑❑❑—

11.6 SUMMATION OF SERIES USING INTEGRATION AS THE LIMIT OF A SUM

Using the definition of the definite integral as the limit of a sum, we can evaluate the sum of a certain infinite series by comparing it with some definite integral.

If $f(x)$ is a continuous function defined on $[a, b]$, then we have seen that

$$\int_a^b f(x)\,dx$$

$$= \lim_{n\to\infty} h\,[f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h) + f(a+nh)],$$

where $nh = b - a$.

$$\text{or, } \int_a^b f(x) = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh).$$

If we take $a = 0$ and $b = 1$, then $h = \dfrac{1}{n}$.

$$\therefore \int_0^1 f(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

It follows from this that an infinite series can be expressed as a definite integral, if $1/n$ is common from its every term and all its terms are the functions of $\dfrac{r}{n}$ $(r = 0, 1, 2, \ldots, n)$ i.e. each of its terms is the form $\dfrac{1}{n} f\left(\dfrac{r}{n}\right)$.

By using the following algorithm, we can always express an infinite series as a definite integral.

ALGORITHM

STEP I *Obtain the given series.*

STEP II *Express the series in the form* $\lim_{n\to\infty}\left[\dfrac{1}{n}\Sigma f\left(\dfrac{r}{n}\right)\right]$

STEP III *Replace* Σ *by* $\int$ $\dfrac{r}{n}$ *by* x *and* $\dfrac{1}{n}$ *by* dx.

STEP IV *Obtain lower and upper limits by computing* $\lim_{n\to\infty}\left(\dfrac{r}{n}\right)$ *for the least and greatest values of* r, *respectively.*

STEP V *Evaluate the integral obtained in previous step.*

The value so obtained is the required sum of the given series.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the sum of the following series as* $n \to \infty$.

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{n}{n^2+n^2}.$$

SOLUTION Let

$$S=\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{n}{n^2+r^2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}\right]$$

For the corresponding definite integral,

$$\text{Lower limit}=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0 \quad [\because\ r=1 \text{ for the first term}]$$

$$\text{Upper limit}=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\frac{n}{n}=1 \quad [\because\ r=n \text{ for the last term}]$$

$$\therefore\ S=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}\right]$$

$$=\int_0^1\frac{1}{1+x^2}\,dx=[\tan^{-1}x]_0^1=\frac{\pi}{4}$$

EXAMPLE 2 *Show that the sum of the series*

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}$$

as $n\to\infty$ is log 3.

SOLUTION Let $S=\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+2n}\right]$

$$=\lim_{n\to\infty}\left[\sum_{r=0}^{2n}\frac{1}{n+r}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{n}{n+r}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)}\right]$$

Now,

$$\text{Lower limit}=\lim_{n\to\infty}\frac{r}{n}=\lim_{n\to\infty}\frac{0}{n}=0 \quad [\because\ r=0 \text{ for the first term}]$$

$$\text{Upper limit}=\lim_{n\to 0}\frac{r}{n}=\lim_{n\to\infty}\frac{2n}{n}=2 \quad [\because\ r=2n \text{ for the last term}]$$

$$\therefore\ S=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)}\right]$$

$$=\int_0^1\frac{1}{1+x}\,dx=[\log(1+x)]_0^2$$

$$=\log 3-\log 1=\log 3.$$

EXAMPLE 3 *Find the sum of the series*

$$\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\ldots.+\frac{r^2}{r^3+n^3}+\ldots+\frac{1}{2n}$$

as $n\to\infty$.

SOLUTION Let

$$S=\lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\ldots.+\frac{r^2}{r^3+n^3}+\ldots.+\frac{n^2}{n^3+n^3}\right]$$

$$=\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\left(\frac{(r/n)^2}{1+(r/n)^3}\right)\frac{1}{n}\right]$$

$$=\int_0^1\frac{x^2}{x^3+1}\,dx \qquad \left[\begin{array}{l}\text{Lower limit}=\lim_{n\to\infty}\frac{0}{n}=0\\ \text{Upper limit}=\lim_{n\to\infty}\frac{n}{n}=1\end{array}\right]$$

$$=\frac{1}{3}\int_0^3\frac{d(x^3+1)}{x^3+1}$$

$$=\frac{1}{3}\left[\log(x^3+1)\right]_0^1$$

$$=\frac{1}{3}\log 2$$

EXAMPLE 4 *Find the sum of the series*

$$\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n+2^2)^{3/2}}+\frac{n^2}{(n^2+3^2)^{3/2}}+\ldots+\frac{n^2}{[n^2+(n-1)^2]^{3/2}}$$

as $n\to\infty$.

SOLUTION Let

$$S = \lim_{n\to\infty} \frac{n^2}{(n+2)^{2/2}} + \frac{n^2}{(n^2+3^2)^{3/2}} + \ldots + \frac{n^2}{\{n^2+(n-1)^2\}^{3/2}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{n^2}{(n^2+r^2)^{3/2}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{\left(1+\frac{r^2}{n^2}\right)^{3/2}} \cdot \frac{1}{n}$$

$$= \int_0^1 \frac{1}{(1+x^2)^{3/2}}\, dx$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^{3/2}}\, d\theta \qquad \text{[On putting } x = \tan\theta\text{]}$$

$$= \int_0^{\pi/4} \cos\theta\, d\theta = \left[\sin\theta\right]_0^{\pi/4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

EXAMPLE 5 *Find the sum of the following series as* $n \to \infty$.

$$\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2}\,(2n+2)} + \ldots + \frac{n}{2n\sqrt{n}\,.\,3n}$$

SOLUTION Let

$$S = \lim_{n\to\infty}\left[\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2}\,(2n+2)} + \ldots + \frac{n}{2n\sqrt{n}\,.\,3n}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)\sqrt{r}\,(2n+r)}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left\{\frac{1}{\left(1+\frac{r}{n}\right)\sqrt{2\frac{r}{n}+\frac{r^2}{n^2}}}\right\} \cdot \frac{1}{n}$$

$$= \int_0^1 \frac{1}{(1+x)\sqrt{2x+x^2}}\, dx$$

$$= \int_0^1 \frac{1}{(x+1)\sqrt{(x+1)^2-1^2}}\, dx$$

$$= \left[\sec^{-1}(x+1)\right]_0^1$$

$$= \sec^{-1}2 - \sec^{-1}1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

EXAMPLE 6 *Evaluate :*

$$\lim_{n\to\infty} \frac{[(n+1)(n+2)(n+3)\ldots(n+n)]^{1/n}}{n}$$

SOLUTION Let

$$S = \lim_{n\to\infty} \frac{[(n+1)(n+2)(n+3)\ldots(n+n)]^{1/n}}{n}$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)..\left(\frac{n+n}{n}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$\therefore \log S$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \log\left(1+\frac{3}{n}\right) + \ldots + \log\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left\{\log\left(1+\frac{r}{n}\right)\right\} \cdot \frac{1}{n}$$

$$= \int_0^1 \log(1+x)\, dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x}\, dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1 \left(1-\frac{1}{1+x}\right) dx$$

$$= \log 2 - \left[x - \log(1+x)\right]_0^1$$

$$= \log 2 - [1 - \log 2] = 2\log 2 - 1$$

$$= \log 4 - \log e$$

$$= \log\left(\frac{4}{e}\right)$$

$\therefore\ S = \frac{4}{e}$.

EXAMPLE 7 *Evaluate :* $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$.

SOLUTION Let

$$S = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$$

$$= \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\left\{\frac{1\,.\,2\,.\,3\,.....\,(n-1)\,n}{n^n}\right\}^{1/n}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}....\frac{n-1}{n}\cdot\frac{n}{n}\right\}^{1/n}$$

$\therefore \log S$

$$= \lim_{n\to\infty}\frac{1}{n}\left\{\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+\log\left(\frac{3}{n}\right)+\log\left(\frac{n-1}{n}\right)\right\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

$$= \lim_{n\to\infty}\left\{\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)\right\}\frac{1}{n}$$

$$= \int_0^1 \log x\,dx$$

$$= \left[x\log x - x\right]_0^1$$

$$= -1 - \lim_{x\to 0} x\log x - 0$$

$$= -1 - \lim_{x\to 0}\frac{\log x}{1/x} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= -1 - \lim_{x\to 0}\frac{1/x}{-1/x^2} \qquad \text{[Using L' Hospital's Rule]}$$

$$= -1 + \lim_{x\to 0} x$$

$$= -1$$

$$\therefore\ S = e^{-1} = \frac{1}{e}$$

EXAMPLE 8 *Evaluate :*

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)..\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

SOLUTION Let

$$S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)..\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

$$\Rightarrow \log S = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+...+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\log S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$\Rightarrow \log S = \lim_{n\to\infty}\left\{\sum_{r=1}^{n}\log\left(1+\left(\frac{r}{n}\right)^2\right)^{1/n}\right\}\frac{1}{n}$$

$$\Rightarrow \log S = \int_0^1 \log(1+x^2)\,dx$$

$$\Rightarrow \log S = \int_0^1 1\,.\,\log(1+x^2)\,dx$$

$$\Rightarrow \log S = \left[x\log(1+x^2)\right]_0^1 - \int_0^1 \frac{2x^2}{1+x^2}\,dx$$

$$\Rightarrow \log S = (\log 2 - 0) - 2\int_0^1 \frac{(1+x^2)-1}{1+x^2}\,dx$$

$$\Rightarrow \log S = \log 2 - 2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$\Rightarrow \log S = \log 2 - 2\left[x-\tan^{-1}x\right]_0^1$$

$$\Rightarrow \log S = \log 2 - 2\left[(1-\tan^{-1}1)-0\right]$$

$$\Rightarrow \log S = \log 2 - 2\left(1-\frac{\pi}{4}\right)$$

$$\Rightarrow \log S = \log 2 - 2 + \frac{\pi}{2} = \log 2 + \log e^{(\pi/2-2)} = \log\{2e^{\pi/2-2}\}$$

$$\Rightarrow S = 2e^{(\pi/2-2)}$$

EXAMPLE 9 *Evaluate :*

$$\lim_{n\to\infty}\left[\left\{1+\left(\frac{1}{n}\right)^4\right\}\left\{1+\left(\frac{2}{n}\right)^4\right\}^{1/2}\left\{1+\left(\frac{3}{n}\right)^4\right\}^{1/3}\left\{1+\left(\frac{4}{n}\right)^4\right\}^{1/4}...\,2^{1/n}\right]$$

SOLUTION Let

$$\lim_{n\to\infty}\left[\left\{1+\left(\frac{1}{n}\right)^4\right\}\left\{1+\left(\frac{2}{n}\right)^4\right\}^{1/2}\left\{1+\left(\frac{3}{n}\right)^4\right\}^{1/3}.....\,2^{1/n}\right]$$

$$= \lim_{n\to\infty}\left[\left\{1+\left(\frac{1}{n}\right)^4\right\}\left\{1+\left(\frac{2}{n}\right)^4\right\}^{1/2}\left\{1+\left(\frac{3}{n}\right)^4\right\}^{1/3}\left\{1+\left(\frac{n}{n}\right)^4\right\}^{1/n}\right]$$

$$\therefore \log S = \lim_{n\to\infty}\left[\log\left\{1+\left(\frac{1}{n}\right)^4\right\}+\frac{1}{2}\left\{1+\log\left(\frac{2}{n}\right)^4\right\}\right.$$

$$\left.+\frac{1}{3}\left\{1+\log\left(\frac{3}{n}\right)^4\right\}+...+\frac{1}{n}\left\{1+\log\left(\frac{n}{n}\right)^4\right\}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r} \log\left\{1+\left(\frac{r}{n}\right)^4\right\}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \log\left\{1+\left(\frac{r}{n}\right)^n\right\} \frac{1}{\left(\frac{r}{n}\right)} \cdot \frac{1}{n}$$

$$= \int_0^1 \frac{\log(1+x^4)}{x}\, dx$$

$$= \frac{1}{4}\int_0^1 \frac{\log(1+x^4)}{x^4} \cdot 4x^3\, dx$$

$$= \frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\, dt, \text{ where } x^4 = t$$

$$= \frac{1}{4}\int_0^1 \frac{1}{t}\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ...\right) dt$$

$$= \frac{1}{4}\int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + ...\right) dt$$

$$= \frac{1}{4}\left[t - \frac{t^2}{2^2} + \frac{t^3}{3^2} - \frac{t^4}{4^2} + ...\right]_0^1$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right]$$

$$= \frac{1}{4}\left(\frac{\pi^2}{12}\right)$$

$$= \frac{\pi^2}{48} \qquad \left[\because 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + .. = \frac{\pi^2}{12}\right]$$

$\therefore S = e^{\pi^2/48}$.

EXAMPLE 10 *Evaluate :*

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2} \cdots \left(1+\frac{n^2}{n^2}\right)^{2n/n^2}\right]$$

SOLUTION Let

$$S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2} \cdots \left(1+\frac{n^2}{n^2}\right)^{2n/n^2}\right]$$

$$\Rightarrow \log S$$

$$= \lim_{n\to\infty}\left[\frac{2}{n^2}\log\left(1+\frac{1}{n^2}\right) + \frac{4}{n^2}\log\left(1+\frac{2^2}{n^2}\right) + ... + \frac{2n}{n^2}\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} 2\left(\frac{r}{n}\right)\log\left(1+\frac{r^2}{n^2}\right)\cdot\frac{1}{n}$$

$$= \int_0^1 2x \log(1+x^2)\, dx$$

$$= \int_0^1 \log(1+t)\, dt, \text{ where } t = x^2$$

$$= \left[t\log(1+t)\right]_0^1 - \int_0^1 \frac{t}{1+t}\, dt$$

$$= (\log 2 - \log 1) - \int_0^1 \frac{(t+1)-1}{t+1}\, dt$$

$$= \log 2 - \left[t - \log(t+1)\right]_0^1$$

$$= \log 2 - [1 - \log 2]$$

$$= 2\log 2 - 1 = \log 4 - \log e$$

$$= \log\left(\frac{4}{e}\right)$$

$\therefore S = \frac{4}{e}$.

EXAMPLE 11 *Evaluate :*

$$\lim_{n\to\infty} n\left[\frac{1}{3n^2+8n+4} + \frac{1}{3n^2+16n+16} + ... \text{ to } n \text{ terms}\right]$$

SOLUTION Let

$$\lim_{n\to\infty} n\left[\frac{1}{3n^2+8n+4} + \frac{1}{3n^2+16n+16} + ... \text{ to } n \text{ terms}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{1}{4(n+1)^2-n^2} + \frac{1}{4(n+2)^2-n^2} + ... + \frac{1}{4(n+n)^2-n^2}\right]$$

$$= \lim_{n\to\infty} n\left[\sum_{r=1}^{n} \frac{1}{4(n+r)^2-n^2}\right]$$

$$= \lim_{n\to\infty} n\left[\sum_{r=1}^{n} \frac{1}{4\left(1+\frac{r}{n}\right)^2-1}\right]\frac{1}{n^2}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{4\left(1+\frac{r}{n}\right)^2-1}\cdot\frac{1}{n}$$

$$= \int_0^1 \frac{1}{4(x+1)^2 - 1}\,dx$$

$$= \frac{1}{4}\int_0^1 \frac{1}{(x+1)^2 - \left(\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{4} \times \frac{1}{2\left(\frac{1}{2}\right)}\left[\log\left(\frac{x+1-\frac{1}{2}}{x+1+\frac{1}{2}}\right)\right]_0^1$$

$$= \frac{1}{4}\left[\log\left(\frac{3/2}{5/2}\right) - \log\left(\frac{1/2}{3/2}\right)\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{3}{5}\right) - \log\left(\frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{9}{5}\right)$$

EXAMPLE 12 *Evaluate :*

$$\lim_{n\to\infty}\frac{1}{n^2}\left[\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + 3\sin^3\frac{3\pi}{4n} + \ldots + n\sin^3\frac{n\pi}{4n}\right]$$

SOLUTION Let

$$S = \lim_{n\to\infty}\frac{1}{n^2}\left[\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + 3\sin^3\frac{3\pi}{4n} + \ldots + n\sin^3\frac{n\pi}{4n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\sum_{r=1}^{n} r\cdot\sin^3\frac{r\pi}{4n}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left\{\left(\frac{r}{n}\right)\sin^3\frac{r\pi}{4n}\right\}\frac{1}{n}$$

$$= \int_0^1 x\sin^3\frac{\pi x}{4}\,dx$$

$$= \frac{3}{4}\int_0^1 x\sin\frac{\pi x}{4}\,dx - \frac{1}{4}\int_0^1 x\sin\frac{3\pi x}{4}\,dx$$

$$= \frac{3}{4}\left[-\frac{4}{\pi}x\cos\frac{\pi x}{4} + \frac{16}{\pi^2}\sin\frac{\pi x}{4}\right]_0^1$$

$$-\frac{1}{4}\left[-\frac{4}{3\pi}x\cos\frac{3\pi x}{4} + \frac{16}{9\pi^2}\sin\frac{3\pi x}{4}\right]_0^1$$

$$= \frac{3}{4}\left[\frac{-2\sqrt{2}}{\pi} + \frac{8\sqrt{2}}{\pi^2}\right] - \frac{1}{4}\left[\frac{2\sqrt{2}}{3\pi} + \frac{16}{9\pi^2}\times\frac{1}{\sqrt{2}}\right]$$

$$= \frac{\sqrt{2}}{9\pi^2}(52 - 15\pi)$$

EXERCISE 11.2

1. Evaluate the following integrals using the definition of the definite integral as the limit of a sum :

(i) $\int_a^b e^{-x}\,dx$ (ii) $\int_a^b \frac{1}{x}\,dx$ (iii) $\int_a^b \sin x\,dx$

(iv) $\int_a^b x^{-1/3}\,dx$ (v) $\int_a^b \cos x\,dx$ (vi) $\int_1^2 \frac{1}{x^2}\,dx$

2. Evaluate the following

(i) $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}\right]$

(ii) $\lim_{n\to\infty}\left[\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \ldots. + \frac{n}{(n+n)^2}\right]$

(iii) $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots. + \frac{1}{8n}\right]$

(iv) $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots. + \frac{n}{n^2+n^2}\right]$

(v) $\lim_{n\to\infty}\left[\frac{1^m + 2^m + 3^m + \ldots. + n^m}{n^{m+1}}\right]$

(vi) $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots. + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

(vii) $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \ldots.. + \frac{1}{3n}\right]$

(viii) $\lim_{n\to\infty}\left[\frac{(n-m)^{1/3}}{n} + \frac{(2^2n-m)^{1/3}}{2n} + \ldots + \frac{(n^3-m)^{1/3}}{n^2}\right]$

(ix) $\lim_{n\to\infty}\left[\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \ldots + \frac{1}{nb}\right]$

(x) $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{2}\,(3\sqrt{2}+4\sqrt{n})^2} + \ldots + \frac{1}{49n}\right]$

(xi) $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \frac{3}{n^2}\sec^2\frac{9}{n^2} + \ldots + \frac{1}{n}\sec^2 1\right]$

(xii) $\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\tan\frac{3\pi}{2n}\ldots.\tan\frac{(n-1)\pi}{2n}\right]^{1/n}$

(xiii) Find $\lim_{n\to\infty} S_n$, if

$$S_n = \left[\frac{1}{2n} + \frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \ldots + \frac{1}{\sqrt{3n^2+2n-1}}\right]$$

(xiv) $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\ldots+\frac{1}{n}\right]$

(xv) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{an}\right]$, where $a\in N$

(xvi) $\lim_{n\to\infty}\frac{1}{n^{p+1}}\left[1^p+2^p+3^p+\ldots.+n^p\right], p>-1$

(xvii) $\lim_{n\to\infty}\left[\frac{(n^3+1^3)(n^3+2^3)\ldots(n^3+n^3)}{n^{3n}}\right]^{1/n}$

(xviii) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+\ldots+\frac{1}{n}\right]$

3. If $np = 1$ find

$$\lim_{n\to\infty}\prod_{r=1}^{n}[1+(rp)^2]^{1/r}$$

4. Show that

$$\lim_{n\to\infty}\left[\phi(a)\,\phi\left(a+\frac{t}{n}\right)\phi\left(a+\frac{2t}{n}\right)\ldots.\,\phi\left(a+\frac{nt}{n}\right)\right]^{1/n}=e^{\lambda}, \text{ where}$$

$$\lambda=\frac{1}{t}\int_a^{a+t}\log\phi(x)\,dx$$

ANSWERS

1. (i) $e^{-a}-e^{-b}$ (ii) $\log_e\left(\frac{b}{a}\right)$ (iii) $\cos a-\cos b$ (iv) $\frac{3}{2}(b^{2/3}-a^{2/3})$ (v) $\sin b-\sin a$ (vi) $1/2$

2. (i) $\log_e 2$ (ii) $1/2$ (iii) $3/8$ (iv) $\frac{\pi}{4}$ (v) $\frac{1}{m+1}$ (vi) $\frac{\pi}{2}$ (vii) $\log 3$ (viii) $\frac{3}{2}$ (ix) $\log\frac{b}{a}$ (x) $\frac{1}{14}$ (xi) $\frac{1}{2}\tan 1$ (xii) 1 (xiii) $\frac{\pi}{6}$ (xiv) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (xv) $\log_e a$ (xvi) $\frac{1}{p+1}$ (xviii) $4e^{\pi/\sqrt{3}}-3$ (xviii) $\frac{\pi}{2}$

—❑❑❑—

11.4 PROPERTIES OF DEFINITE INTEGRALS

In this section, we will study some fundamental properties of definite integrals which are very useful in evaluating integrals.

PROPERTY I $\int_a^b f(x)\,dx=\int_a^b f(t)\,dt$

i.e., integration is independent of the change of variable.

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\frac{d}{dx}\{\phi(x)\}=f(x)\Rightarrow\frac{d}{dt}\{\phi(t)\}=f(t).$$

Hence, $\int_a^b f(x)\,dx=[\phi(x)]_a^b=\phi(b)-\phi(a)$ ...(i)

and, $\int_a^b f(t)\,dt=[\phi(t)]_a^b=\phi(b)-\phi(a)$ ...(ii)

From (i) and (ii), we have

$$\int_a^b f(x)\,dx=\int_b^a f(t)\,dt$$

PROPERTY II $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

i.e., if the limits of a definite integral are interchanged then its value changes by minus sign only.

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx=\phi(b)-\phi(a)$$

and,

$$-\int_b^a f(x)\,dx=-[(\phi(a)-\phi(b)]=\phi(b)-\phi(a)$$

$$\therefore\quad\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$$

PROPERTY III $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$, where $a<c<b$.

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx=\phi(b)-\phi(a) \quad ...(i)$$

and,

$$\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

$$=[\phi(c)-\phi(a)]+[\phi(b)-\phi(c)]$$

$= \phi(b) - \phi(a)$ ...(ii)

From (i) and (ii), we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

GENERALIZATION *The above property can be generalized into the following form*

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx,$$

where $a < c_1 < c_2 < c_3 \ldots < c_{n-1} < c_n < b$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 Evaluate :

(i) $\int_0^{\pi} |\cos x|\,dx$ (ii) $\int_0^2 |x^2 + 2x - 3|\,dx$

SOLUTION (i) We have,

$$\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx$$

$$= \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}$$

$$= (1-0) - (0-1) = 2$$

(ii) We have,

$$|x^2 + 2x - 3| = \begin{cases} x^2 + 2x - 3 & \text{for} \quad x < -3 \text{ or } x > 1 \\ -(x^2 + 2x - 3) & \text{for} \quad -3 < x < 1 \end{cases}$$

$$\therefore \int_0^2 |x^2 + 2x - 3|\,dx$$

$$= \int_0^1 |x^2 + 2x - 3|\,dx + \int_1^2 |x^2 + 2x - 3|\,dx$$

$$= -\int_0^1 (x^2 + 2x - 3)\,dx + \int_1^2 (x^2 + 2x - 3)\,dx$$

$$= -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2 = 4.$$

EXAMPLE 2 *Evalutate :*

(i) $\int_0^3 [x]\,dx$ (ii) $\int_0^{1.5} [x^2]\,dx$ [IIT 1988] (iii) $\int_0^2 [x^2]\,dx$

SOLUTION (i) We have,

$$\int_0^3 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx$$

$$= \int_0^1 0\,dx + \int_1^2 1.\,dx + \int_2^3 2\,dx$$

$$= (2-1) + 2(3-2) = 3.$$

(ii) We have,

$$\int_0^{1.5} [x^2]\,dx = \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{1.5} [x^2]\,dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1.\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$$

$$= 0 + (\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) = 2 - \sqrt{2}$$

(iii) We have,

$$\int_0^2 [x^2]\,dx = \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\,dx + \int_{\sqrt{3}}^2 [x^2]\,dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1.\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx$$

$$= 0 + 1.(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= 5 - \sqrt{3} - \sqrt{2}$$

EXAMPLE 3 *Evaluate :* $\int_{-1}^1 (x - [x])\,dx$

SOLUTION We have,

$$\int_{-1}^1 (x - [x])\,dx = \int_{-1}^1 x\,dx - \int_{-1}^1 [x]\,dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^1 - \left[\int_{-1}^0 [x]\,dx + \int_0^1 [x]\,dx\right]$$

$$= 0 - \left[\int_{-1}^0 -1\,dx + \int_0^1 0\,dx\right] = 1$$

EXAMPLE 4 *Evaluate* : $\int_0^2 \{x\}\,dx$, *where* $\{x\}$ *denotes the fractional part of* x.

SOLUTION We have,

$$\int_0^2 \{x\}\,dx = \int_0^1 \{x\}\,dx + \int_1^2 \{x\}\,dx$$

$$= \int_0^1 x\,dx + \int_0^2 (x-1)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{(x-1)^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \frac{1}{2} - 0 = 1.$$

EXAMPLE 5 *Evaluate* : $\int_0^4 \{\sqrt{x}\}$, *where* $\{x\}$ *donotes the fractional part of* x.

SOLUTION We have,

$$\int_0^4 \{\sqrt{x}\}\,dx = \int_0^1 \sqrt{x}\,dx + \int_1^4 (\sqrt{x}-1)\,dx$$

$$= \frac{2}{3} + \frac{2}{3}(8-1) - 3$$

$$= \frac{7}{3}$$

ALITER $\int_0^4 \{\sqrt{x}\}\,dx = \int_0^4 \sqrt{x}\,dx - \int_0^4 [\sqrt{x}]\,dx$

EXAMPLE 6 *Evaluate* : $\int_0^a [x^n]\,dx$

SOLUTION Let k be a natural number such that $k \le a^n < k+1$. Then,

$$\int_0^a [x^n]\,dx$$

$$= \int_0^1 [x^n]\,dx + \int_1^{2^{1/n}} [x^n]\,dx + \int_{2^{1/n}}^{3^{1/n}} [x^n]\,dx + \ldots + \int_{(k-1)^{1/n}}^{k^{1/n}} [x^n]\,dx + \int_{k^{1/n}}^{(a^n)^{1/n}} [x^n]\,dx$$

$$= \sum_{r=1}^{k} \int_{(r-1)^{1/n}}^{r^{1/n}} [x^n]\,dx + \int_{k^{1/n}}^{(a^n)^{1/n}} [x^n]\,dx$$

$$= \sum_{r=1}^{k} \int_{(r-1)^{1/n}}^{r^{1/n}} (r-1)\,dx + \int_{k^{1/n}}^{(a^n)^{1/n}} k\,dx$$

$$= \sum_{r=1}^{k} (r-1)\left[r^{1/n} - (r-1)^{1/n}\right] + k\,(a - k^{1/n})$$

EXAMPLE 7 *Prove that* : $\int_0^x [x]\,dx = x\,[x] - \frac{1}{2}[x]\{[x]+1\}$

SOLUTION Let k be an integer such that $k \le x < k+1$. Then,

$$\int_0^x [x]\,dx = \int_0^k [x]\,dx + \int_k^x [x]\,dx$$

$$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \ldots + \int_{k-1}^k [x]\,dx + \int_k^x [x]\,dx$$

$$= \sum_{r=1}^{k} \int_{r-1}^{r} [x]\,dx + \int_k^x [x]\,dx$$

$$= \sum_{r=1}^{k} \int_{r-1}^{r} (r-1)\,dx + \int_k^x k\,dx$$

$$= \sum_{r=1}^{k} (r-1) + k\,(x-k)$$

$$= (1 + 2 + \ldots + k - 1) + k\,(x-k)$$

$$= \frac{k\,(k-1)}{2} + kx - k^2$$

$$= kx - \frac{k\,(k+1)}{2}$$

$$= [x]\,x - \frac{[x]\{[x]+1\}}{2}$$

EXAMPLE 8 *Show that* $\int_0^x \left(\{x\} - \frac{1}{2}\right)dx = \frac{1}{2}\{x\}^2 - \frac{1}{2}\{x\}$, *where* $\{x\}$ *denotes the fractional part of* x.

SOLUTION We have, $\{x\} = x - [x]$.

Let $k \le x < k+1$, where $k \in N$. Then,

$$\int_0^x \left(\{x\} - \frac{1}{2}\right)dx$$

$$= \int_0^x \left(x - [x] - \frac{1}{2}\right)dx$$

$$= \int_0^x \Big(x-[x]\Big)dx - \int_0^x \frac{1}{2}\,dx$$

$$= \int_0^k \Big(x-[x]\Big)dx + \int_k^x (x-[x])\,dx - \frac{x}{2}$$

$$= \frac{k}{2} + \int_k^x (x-k)\,dx - \frac{x}{2}$$

$$= \frac{k}{2} + \left(\frac{x^2}{2} - kx\right) - \left(\frac{k^2}{2} - k^2\right) - \frac{x}{2}$$

$$= \frac{x^2}{2} - kx + \frac{k(k+1)}{2} - \frac{x}{2}$$

$$= \frac{1}{2}(x^2 - 2kx + k^2) + \frac{1}{2}(k-x)$$

$$= \frac{1}{2}(x-k)^2 - \frac{1}{2}(x-k)$$

$$= \frac{1}{2}\Big(x-[x]\Big)^2 - \frac{1}{2}\Big(x-[x]\Big)$$

$$= \frac{1}{2}[x]^2 - \frac{1}{2}[x]$$

EXAMPLE 9 *Evaluate* : $\int_0^{2\pi} [\sin x + \cos x]\,dx$

SOLUTION We have,

$$\sin x + \cos x = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$$

$$\therefore\ [\sin x + \cos x] = \left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right]$$

The graph of $y = \left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right]$ is obtained from the graph of $y = [\sqrt{2}\sin x]$ by translating it by $\frac{\pi}{4}$ units in the direction of OX'. The graph so obtained is shown in Fig. 11.4

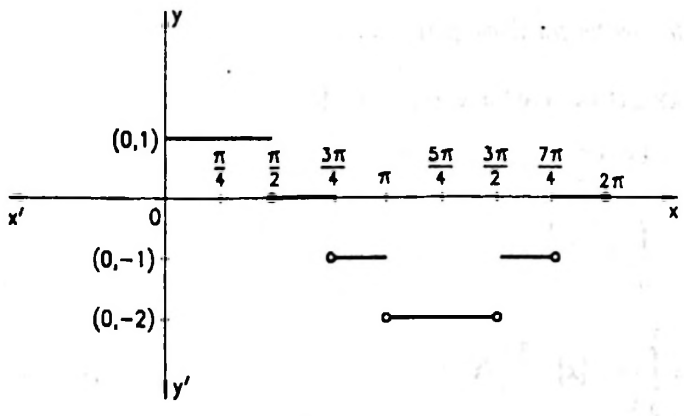


Fig. 11.4

It is evident from the graph of $y = [\sqrt{2}\sin(x + \pi/4)]$ that

$$f(x) = [\sin x + \cos x] = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le 3\pi/4 \\ -1, & 3\pi/4 < x \le \pi \\ -2, & \pi < x < 3\pi/2 \\ -1, & 3\pi/2 \le x < 7\pi/4 \\ 0, & 7\pi/4 \le x < 2\pi \end{cases}$$

$$\therefore \int_0^{2\pi} [\sin x + \cos x]\,dx$$

$$= \int_0^{\pi/2} 1.\,dx + \int_{\pi/2}^{3\pi/4} 0\,dx + \int_{3\pi/4}^{\pi} (-1)\,dx + \int_{\pi}^{3\pi/2} (-2)\,dx + \int_{3\pi/2}^{7\pi/4} (-1)\,dx + \int_{7\pi/4}^{2\pi} 0\,dx$$

$$= \frac{\pi}{2} + 0 - \frac{\pi}{4} - 2\times\frac{\pi}{2} + (-1)\frac{\pi}{4} + 0 = -\pi.$$

EXAMPLE 10 *Evaluate* : $\int_0^2 [x^2 - x + 1]\,dx.$

SOLUTION The graph of $f(x) = x^2 - x + 1$ is shown in Fig. 11.5. It is evident from the graph that

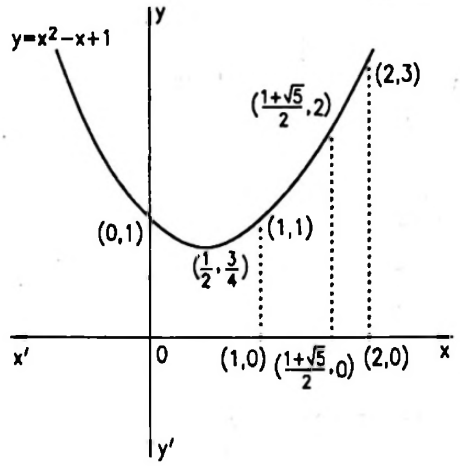


Fig. 11.5

$$\frac{3}{4} \le x^2 - x + 1 < 1 \quad \text{for } 0 < x < 1$$

$$1 \le x^2 - x + 1 < 2 \quad \text{for } 1 \le x < \frac{1+\sqrt{5}}{2}$$

and,

$$2 \le x^2 - x + 1 < 3 \quad \text{for } \frac{1+\sqrt{5}}{2} \le x < 2$$

$$\therefore\ [x^2 - x + 1] = \begin{cases} 0, & \text{for } 0 < x < 1 \\ 1, & \text{for } 1 \le x < \dfrac{1+\sqrt{5}}{2} \\ 2, & \text{for } \dfrac{1+\sqrt{5}}{2} \le x < 2 \end{cases}$$

Hence,

$$\int_0^2 [x^2 - x + 1]\, dx$$

$$= \int_0^1 0\, dx + \int_1^{\frac{1+\sqrt{5}}{2}} 1\, dx + \int_{\frac{1+\sqrt{5}}{2}}^2 2\, dx$$

$$= \left\{\frac{1+\sqrt{5}}{2} - 1\right\} + 2\left\{2 - \frac{1+\sqrt{5}}{2}\right\}$$

$$= 3 - \frac{1+\sqrt{5}}{2} = \frac{5-\sqrt{5}}{2}$$

EXAMPLE 11 *Evaluate* : $\int_0^{n^2} [\sqrt{x}]\, dx$, *where* $n \in N$.

SOLUTION We have,

$$\int_0^{n^2} [\sqrt{x}]\, dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)^2}^{r^2} [\sqrt{x}]\, dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)^2}^{r^2} (r-1)\, dx$$

$$= \sum_{r=1}^{n} (r-1) \int_{(r-1)^2}^{r^2} dx$$

$$= \sum_{r=1}^{n} (r-1)\left\{r^2 - (r-1)^2\right\}$$

$$= \sum_{r=1}^{n} (r-1)(2r-1)$$

$$= \sum_{r=1}^{n} (2r^2 - 3r + 1)$$

$$= 2\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)}{3} - \frac{3n(n+1)}{2} + n$$

$$= \frac{n}{6}[2(n+1)(2n+1) - 9(n+1) + 6]$$

$$= \frac{n}{6}\left[4n^2 + 6n + 2 - 9n - 9 + 6\right]$$

$$= \frac{n}{6}\left[4n^2 - 3n - 1\right]$$

$$= \frac{n(n-1)(4n+1)}{6}$$

EXAMPLE 12 *Evaluate* : $\int_0^{1000} [\tan^{-1} x]\, dx$

SOLUTION We know that

$$\frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \text{ for all } x \in R$$

$$\Rightarrow \quad 0 \le \tan^{-1} x < 1 \qquad \text{for } 0 \le x < \tan 1$$

and, $\quad 1 \le \tan^{-1} x < \frac{\pi}{2} \qquad \text{for } \tan 1 \le x < \infty$

$$\Rightarrow \quad [\tan^{-1} x] = \begin{cases} 0 & \text{for } 0 \le x < \tan 1 \\ 1 & \text{for } \tan 1 \le x < 1000 \end{cases}$$

$$\therefore \int_0^{1000} [\tan^{-1} x]\, dx$$

$$= \int_0^{\tan 1} 0\, dx + \int_{\tan 1}^{1000} 1\, dx$$

$$= (1000 - \tan 1)$$

EXAMPLE 13 *Evaluate* : $\int_0^{\pi} \left(x \sin [x^2 - \pi]\right) dx$

SOLUTION Clearly, $y = x^2 - \pi$ is an increasing function on $[0, \pi]$ and, $-\pi \le x^2 - \pi \le \pi^2 - \pi$

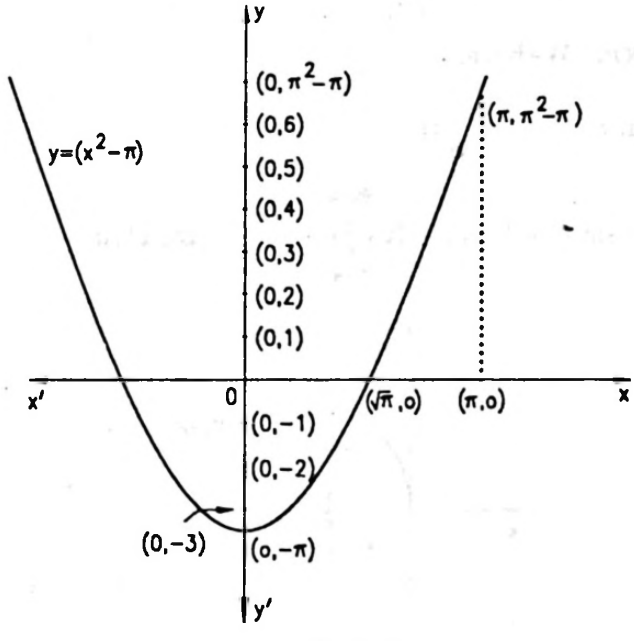


Fig. 11.6

$$\therefore\ [x^2-\pi] = \begin{cases} -4, \text{if } 0 \le x \le \sqrt{\pi-3} \\ -3, \text{if } \sqrt{\pi-3} < x \le \sqrt{\pi-2} \\ -2, \text{if } \sqrt{\pi-2} < x < \sqrt{\pi-1} \\ -1, \text{if } \sqrt{\pi-1} < x < \sqrt{\pi} \\ 0, \text{if } \sqrt{\pi} \le x < \sqrt{\pi+1} \\ 1, \text{if } \sqrt{\pi+1} \le x < \sqrt{\pi+2} \\ 2, \text{if } \sqrt{\pi+2} \le x < \sqrt{\pi+3} \\ 3, \text{if } \sqrt{\pi+3} \le x < \sqrt{\pi+4} \\ 4, \text{if } \sqrt{\pi+4} \le x < \sqrt{\pi+5} \\ 5, \text{if } \sqrt{\pi+5} \le x < \sqrt{\pi+6} \\ 6, \text{if } \sqrt{\pi+6} \le x < \pi \end{cases}$$

Hence,

$$\int_0^{\pi} \left(x \sin [x^2-\pi]\right) dx$$

$$= \int_0^{\sqrt{\pi-3}} x \sin(-4)\, dx + \sum_{r=-3}^{5} \int_{\sqrt{\pi+r}}^{\sqrt{\pi+r+1}} x \sin r\, dx + \int_{\sqrt{\pi+6}}^{\pi} x \sin 6\, dx$$

$$= -\frac{1}{2}(\sin 4)(\pi-3-0) + \frac{1}{2}\sum_{r=-3}^{5} (\sin r)(\pi+r+1-\pi-r) + \frac{1}{2}(\sin 6)(\pi^2-\pi-6)$$

$$= -\frac{1}{2}(\pi-3)\sin 4 + \frac{1}{2}\left(\sum_{r=-3}^{5} \sin r\right) + \frac{1}{2}(\pi^2-\pi-6)\sin 6$$

$$= \frac{1}{2}[-(\pi-3)\sin 4 - \sin 3 - \sin 2 - \sin 1 + \sin 1 + \sin 2 + \sin 3 + \sin 4 + \sin 5 + (\pi^2-\pi-6)\sin 6]$$

$$= \frac{1}{2}[-\pi \sin 4 + 4\sin 4 + \sin 5 - 6\sin 6 + (\pi^2-\pi)\sin 6]$$

EXAMPLE 14 *Evaluate :* $\int_0^{\pi} \Big|\, |\sin x| - |\cos x| \,\Big| dx$

SOLUTION We have,

$$\int_0^{\pi} \Big|\, |\sin x| - |\cos x| \,\Big| dx$$

$$= \int_0^{\pi/4} -(|\sin x| - |\cos x|)\, dx + \int_{\pi/4}^{3\pi/4} (|\sin x| - |\cos x|)\, dx + \int_{3\pi/4}^{\pi} -(|\sin x| - |\cos x|)\, dx$$

$$= -\int_0^{\pi/4} (\sin x - \cos x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx + \int_{\pi/2}^{3\pi/4} (\sin x + \cos x)\, dx + \int_{3\pi/4}^{\pi} -(\sin x + \cos x)\, dx$$

$$= -\Big[-\cos x - \sin x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} + \Big[-\cos x + \sin x\Big]_{\pi/2}^{3\pi/4} - \Big[-\cos x + \sin x\Big]_{3\pi/4}^{\pi}$$

$$= -[-\sqrt{2}+1] + [-1+\sqrt{2}] + [\sqrt{2}-1] - [1-\sqrt{2}]$$

$$= 4\sqrt{2} - 4$$

$$= 4(\sqrt{2}-1)$$

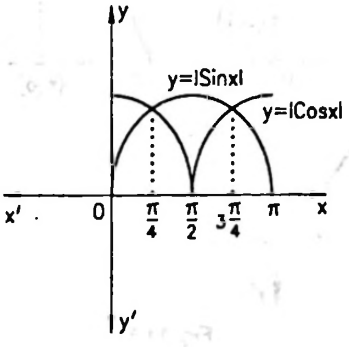


Fig. 11.7

EXAMPLE 15 *Evaluate :* $\int_{-1}^{1} \Big[x\,[1+\sin\ \pi x]\Big] dx$

SOLUTION We have,

$$\int_{-1}^{1} \Big[x\,[1+\sin \pi x]\Big] dx$$

$$= \int_{-1}^{0} \Big[x\,[1+\sin \pi x]\Big] dx + \int_{0}^{1} \Big[x\,[1+\sin \pi x]\Big] dx$$

$$= \int_{-1}^{0} [x \times 0]\, dx + \int_{0}^{1} [x \times 1]\ dx$$

$$= \int_{-1}^{0} 0\, dx + \int_{0}^{1} 0\, dx = 0$$

EXAMPLE 16 *Prove that :* $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \frac{7\pi^2}{72}$

SOLUTION We have,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x, & \text{for } -1 \le x \le 1 \\ \pi - 2\tan^{-1} x, & \text{for } x > 1 \\ -\pi - 2\tan^{-1} x, & \text{for } x < -1 \end{cases}$$

$$\therefore \int_0^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$= \int_0^{1} \frac{1}{1+x^2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx + \int_1^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$= 2\int_0^{1} \frac{\tan^{-1} x}{1+x^2}\, dx + \int_1^{\sqrt{3}} \frac{1}{1+x^2}(\pi - 2\tan^{-1} x)\, dx$$

$$= 2\int_0^1 \tan^{-1}x\, d(\tan^{-1}x) + \pi\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx - 2\int_0^{\sqrt{3}} \tan^{-1}x\, d(\tan^{-1}x)$$

$$= 2\left[\frac{(\tan^{-1}x)^2}{2}\right]_0^1 + \pi\left[\tan^{-1}x\right]_1^{\sqrt{3}} - 2\left[\frac{(\tan^{-1}x)^2}{2}\right]_1^{\sqrt{3}}$$

$$= \left[(\tan^{-1}1)^2 - (\tan^{-1}0)^2\right] + \pi\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right] - \left[(\tan^{-1}\sqrt{3})^2 - (\tan^{-1}1)^2\right]$$

$$= \frac{\pi^2}{16} + \pi\left(\frac{\pi}{3} - \frac{\pi}{4}\right) - \left(\frac{\pi^2}{9} - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^2}{16} + \frac{\pi^2}{12} - \frac{\pi^2}{9} + \frac{\pi^2}{16} = \frac{7\pi^2}{72}$$

EXAMPLE 17 *Evaluate* $\int_0^1 (\{2x\} - 1)(\{3x\} - 1)\,dx$, *where* $\{\cdot\}$ *denotes the fractional part.*

SOLUTION Let

$$I = \int_0^1 (\{2x\} - 1)(\{3x\} - 1)\,dx$$

$$= \int_0^1 (2x - [2x] - 1)(3x - [3x] - 1)\,dx$$

$$\Rightarrow I = \int_0^1 (2x - 1 - [2x])(3x - 1 - [3x])\,dx$$

$$\Rightarrow I = \int_0^{1/3} (2x - 1 - [2x])(3x - 1 - [3x])\,dx + \int_{1/3}^{1/2} (2x - 1 - [2x])\ (3x - 1 - [3x])\,dx + \int_{1/2}^{2/3} (2x - 1 - [2x])(3x - 1 - [3x])\,dx + \int_{2/3}^{1} ((2x - 1) - [2x])\ (3x - 1 - [3x])\,dx$$

$$\Rightarrow I = \int_0^{1/3} (2x - 1)(3x - 1)\,dx + \int_{1/3}^{1/2} (2x - 1)(3x - 1 - 1)\,dx + \int_{1/2}^{2/3} (2x - 1 - 1)(3x - 1 - 1)\,dx + \int_{2/3}^{1} (2x - 1 - 1)(3x - 1 - 2)\,dx$$

$$\Rightarrow I = \int_0^{1/3} (6x^2 - 5x + 1)\,dx + \int_{1/3}^{1/2} (6x^2 - 7x + 2)\,dx + 2\int_{1/2}^{2/3} (3x^2 - 5x + 2)\,dx + 6\int_{2/3}^{1} (x - 1)^2\,dx$$

$$\Rightarrow I = \left[2x^3 - \frac{5}{2}x^2 + x\right]_0^{1/3} + \left[2x^3 - \frac{7}{2}x^2 + 2x\right]_{1/3}^{1/2} + 2\left[x^3 - \frac{5}{2}x^2 + 2x\right]_{1/2}^{2/3} + 6\left[\frac{(x-1)^3}{3}\right]_{2/3}^{1}$$

$$\Rightarrow I = \frac{7}{54} + \frac{5}{216} + \frac{1}{27} + \frac{2}{27} = \frac{28 + 5 + 8 + 16}{216} = \frac{57}{216}$$

EXAMPLE 18 *If* $\int_0^x [x]\,dx = \int_0^{[x]} x\,dx$, *then prove that either* x *is purely fractional or* x *is such that* $\{x\} = \frac{1}{2}$.

Here, $[\cdot]$ *denotes greatest integer function and* $\{\cdot\}$ *denotes fractional part.*

SOLUTION We have,

$$\int_0^x [x]\,dx = x[x] - \frac{1}{2}[x]([x] + 1) \qquad \text{[See Ex. 7]}$$

and, $\int_0^{[x]} x\,dx = \frac{[x]^2}{2}$

$$\therefore \int_0^x [x]\,dx = \int_0^{[x]} x\,dx$$

$$\Rightarrow x[x] - \frac{1}{2}[x]\left([x] + 1\right) = \frac{1}{2}[x]^2$$

$$\Rightarrow 2x[x] - [x]^2 - [x] = [x]^2$$

$$\Rightarrow 2[x]([x] + \{x\}) - 2[x]^2 - [x] = 0$$

$$\Rightarrow 2[x]\{x\} - [x] = 0$$

$$\Rightarrow [x](2\{x\} - 1) = 0$$

$$\Rightarrow [x] = 0 \text{ or, } \{x\} = \frac{1}{2}$$

$$\Rightarrow x = \{x\} \text{ or, } \{x\} = \frac{1}{2}$$

$$\Rightarrow x \text{ is purely fractional or, } \{x\} = \frac{1}{2}.$$

EXAMPLE 19 *Evaluate :*

$\int_0^n f(x)\,dx$, if $f(x) = \sqrt{x-k+1}$; $k-1 \le x < k$, $k = 1, 2, \ldots n$.

SOLUTION We have,

$$\int_0^n f(x)\,dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)}^{r} f(x)\,dx$$

$$= \sum_{r=1}^{n} \int_{r-1}^{r} \sqrt{x-r+1}\,dx$$

$$= \sum_{r=1}^{n} \int_{r-1}^{r} \sqrt{x-(r-1)}\,dx$$

$$= \sum_{r=1}^{n} \int_{0}^{1} \sqrt{t}\,dt, \text{ where } t = x-r+1$$

$$= n\int_0^1 \sqrt{t}\,dt$$

$$= \frac{2n}{3}\left[t^{3/2}\right]_0^1$$

$$= \frac{2n}{3}$$

EXAMPLE 20 *Evaluate* : $\int_0^x [t+1]^3\,dt$, *where* $[\cdot]$ *denotes the greatest integer function and* $x \in R^+$.

SOLUTION Let $I = \int_0^x [t+1]^3\,dt$.

Also, let $n \le x < n+1$, $n = 0, 1, 2, \ldots$.

Then, $[x] = n$

$$\therefore\ I = \int_0^1 [t+1]^3\,dt + \int_1^2 [t+1]^3\,dt + \ldots + \int_{n-1}^{n} [t+1]^3\,dt + \int_n^x [t+1]^3\,dt$$

$$= \sum_{r=1}^{n} \int_{r-1}^{r} [t+1]^3\,dt + \int_n^x [t+1]^3\,dt$$

$$= \sum_{r=1}^{n} \int_{r-1}^{r} r^3\,dt + \int_n^x (n+1)^3\,dt$$

$$= \sum_{r=1}^{n} r^3\{r-(r-1)\} + (n+1)^3(x-n)$$

$$= \sum_{r=1}^{n} r^3 + (n+1)^3(x-n)$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3(x-n)$$

$$= \left\{\frac{[x]([x]+1)}{2}\right\}^2 + ([x]+1)^3(x-[x])$$

EXAMPLE 21 *Evaluate* $\int_0^{\infty} [ne^{-x}]\,dx$, *where* $n \in N, n \ge 1$ *and* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION Let $f(x) = n\,e^{-x}$. Then, the graph of $f(x)$ is as shown in Fig. 11.8.

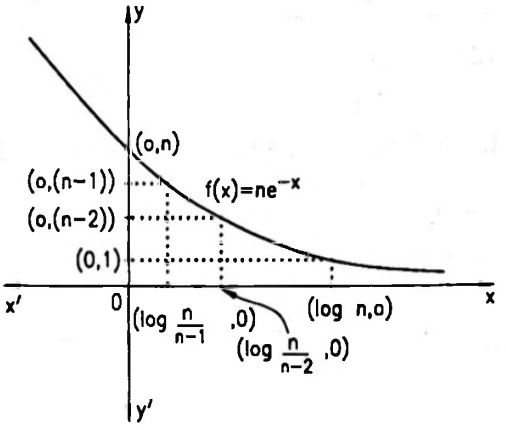


Fig. 11.8

It is evident from the graph of $f(x) = ne^{-x}$, that

$$0 < ne^{-x} \le n \text{ for all } x \ge 0$$

Clearly,

$$(n-r-1) \le ne^{-x} < n-r, \text{ if } \log\left(\frac{n}{n-r}\right) < x < \log\left(\frac{n}{n-r-1}\right)$$

for $r = 0, 1, 2, \ldots, (n-2)$

and, $\quad 0 < n\,e^{-x} < 1 \quad$ for $x > \log n$.

$$\therefore\ [n\,e^{-x}] = \begin{cases} (n-r-1), \text{ if } \log\left(\dfrac{n}{n-r}\right) < x < \log\left(\dfrac{n}{n-r-1}\right) \\ \qquad\qquad r = 0, 1, 2, \ldots, (n-2) \\ 0 \quad , \text{ if } x > \log n \end{cases}$$

Hence,

$$I = \int_0^{\infty} [n\,e^{-x}]\,dx$$

$$= \sum_{r=0}^{n-2} \int_{\log(n/(n-r))}^{\log(n/(n-r-1))} [n\,e^{-x}]\,dx + \int_{\log n}^{\infty} [ne^{-x}]\,dx$$

$$= \sum_{r=0}^{n-2} \int_{\log(n/n-r)}^{\log(n/n-r-1)} (n-r-1)\,dx + \int_{\log x}^{\infty} 0\,dx$$

$$= \sum_{r=0}^{n-2} (n-r-1)\left\{\log\left(\frac{n}{n-r-1}\right) - \log\left(\frac{n}{n-r}\right)\right\}$$

$$= \sum_{r=0}^{n-2} (n-r-1)\log\left(\frac{n-r}{n-r-1}\right)$$

$$= \sum_{r=0}^{n-2} (n-r-1)\{\log(n-r) - \log(n-r-1)\}$$

$$= \sum_{r=0}^{n-2} [\{(n-r)\log(n-r) - (n-r-1)\log(n-r-1)\} - \log(n-r)]$$

$$= \sum_{r=0}^{n-2} \{(n-r)\log(n-r) - (n-r-1)\log(n-r-1)\}$$

$$- \sum_{r=0}^{n-2} \log(n-r)$$

$$= \{n\log n - (n-1)\log(n-1)\}$$
$$+ \{(n-1)\log(n-1) - (n-2)\log(n-2)\}$$
$$+ \{(n-2)\log(n-2) - (n-3)\log(n-3)\} + \ldots.$$
$$+ \{2\log 2 - 1\log 1\} - [\log n + \log(n-1) + \ldots. + \log 2]$$

$$= n\log n - \log\{n(n-1)(n-2)\ldots.2\}$$

$$= n\log n - \log n!$$

$$= \log\left(\frac{n^n}{n!}\right)$$

EXAMPLE 22 *Evaluate :* $\int_1^2 (x^{[x^2]} + [x^2]^x)\,dx$, *where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION We have,

$$\int_1^2 (x^{[x^2]} + [x^2]^x)\,dx$$

$$= \int_1^{\sqrt{2}} (x^{[x^2]} + [x^2]^x)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^{[x^2]} + [x^2]^x)\,dx + \int_{\sqrt{3}}^{2} (x^{[x^2]} + [x^2]^x)\,dx$$

$$= \int_1^{\sqrt{2}} (x+1)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^x)\,dx + \int_{\sqrt{3}}^{2} (x^3 + 3^x)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^{\sqrt{2}} + \left[\frac{x^3}{3} + \frac{2^x}{\log 2}\right]_{\sqrt{2}}^{\sqrt{3}} + \left[\frac{x^4}{4} + \frac{3^x}{\log 3}\right]_{\sqrt{3}}^{2}$$

$$= \frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$$

EXAMPLE 23 *For any differentiable function* $f(x)$, *prove that*

$$\int_0^n f'(x)\left\{[x] - x + \frac{1}{2}\right\}dx = \int_0^n f(x)\,dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r).$$

SOLUTION We have,

$$\int_0^n f'(x)\left\{[x] - x + \frac{1}{2}\right\}dx$$

$$= \int_0^n f'(x)\,[x]\,dx - \int_0^n \underset{\text{I}}{x}\,\underset{\text{II}}{f'(x)}\,dx + \frac{1}{2}\int_0^n f'(x)\,dx$$

$$= \sum_{r=1}^{n} \int_{r-1}^{1} f'(x)\,[x]\,dx - \left[\{xf(x)\}_0^n - \int_0^n f(x)\,dx\right] + \frac{1}{2}\Big[f(x)\Big]_0^n$$

$$= \sum_{r=1}^{n} \int_{r-1}^{r} f'(x)\,(r-1)\,dx - \left[nf(n) - \int_0^n f(x)\,dx\right] + \frac{1}{2}[f(n) - f(0)].$$

$$= \sum_{r=1}^{n} (r-1)\int_{r-1}^{r} f'(x)\,dx - \left[nf(n) - \int_0^n f(x)\,dx\right] + \frac{1}{2}[f(n) - f(0)]$$

$$= \sum_{r=1}^{n} (r-1)\{f(r) - f(r-1)\} - \left[nf(n) - \int_0^n f(x)\,dx\right] + \frac{1}{2}[f(n) - f(0)]$$

$$= \sum_{r=1}^{n} [rf(r) - (r-1)f(r-1)] - \sum_{r=1}^{n} f(r) - \left\{nf(n) - \int_0^n f(x)\,dx\right\}$$
$$+ \frac{1}{2}[f(n) - f(0)]$$

$$= [1.f(1) - 0f(0)] + [2f(2) - 1f(1)] + [3f(3) - 2f(2)]$$
$$+ \ldots + [nf(n) - (n-1)f(n-1)]$$
$$- \sum_{r=1}^{n} f(r) - [nf(n) - \int_0^n f(x)\,dx] + \frac{1}{2}[f(n) - f(0)]$$

$$= nf(n) - \sum_{r=1}^{n} f(r) - nf(n) + \int_0^n f(x)\,dx + \frac{1}{2}f(n) - \frac{1}{2}f(0)$$

$$= \int_0^n f(x)\,dx + \frac{1}{2}f(n) - \sum_{r=1}^{n} f(r) - \frac{1}{2}f(0)$$

$$= \int_0^n f(x)\,dx + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r) + \frac{1}{2}f(0)$$

$$= \int_0^n f(x)\,dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r)$$

EXAMPLE 24 *Prove that :*

$$\int_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\,dt = 1$$

SOLUTION Let $I_1 = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt$ and $I_2 = \int_{1/e}^{\cot x} \frac{1}{t(1+t_2)}\,dt$

Putting $t = \frac{1}{u}$ in I_1, we get

$$I_1 = \int_e^{\cot x} \frac{\frac{1}{u}}{1+\frac{1}{u^2}} \times -\frac{1}{u^2}\,du$$

$$= -\int_e^{\cot x} \frac{1}{u(u^2+1)}\,du$$

$$= \int_{\cot x}^{e} \frac{1}{u(u^2+1)}\,du$$

$$= \int_{\cot x}^{e} \frac{1}{t(t^2+1)}\,dt$$

$$\therefore\ I_1 + I_2 = \int_{\cot x}^{e} \frac{1}{t(t^2+1)}\,dt + \int_{1/e}^{\cot x} \frac{1}{t(t^2+1)}\,dt$$

$$= \int_{1/e}^{e} \frac{1}{t(t^2+1)}\,dt$$

$$= \int_{1/e}^{e} \left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt$$

$$= \left[\log t - \frac{1}{2}\log(t^2+1)\right]_{1/e}^{e}$$

$$= \left[\log e - \frac{1}{2}\log(e^2+1)\right] - \left[\log\left(\frac{1}{e}\right) - \frac{1}{2}\log\left(\frac{1}{e^2}+1\right)\right]$$

$$= 1 - \frac{1}{2}\log(e^2+1) + 1 + \frac{1}{2}\log\left(\frac{e^2+1}{e^2}\right)$$

$$= 2 + \frac{1}{2}\log\left(\frac{e^2+1}{e^2} \times \frac{1}{e^2+1}\right)$$

$$= 2 + \frac{1}{2}\log e^{-2}$$

$$= 2 - 1 = 1$$

EXAMPLE 25 *Given* $\int_0^1 \frac{\sin t}{1+t}\,dt = \alpha$, *find the value of* $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}\,dt$ *in terms of* α.

SOLUTION Let $I = \int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}\,dt = \frac{1}{2}\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{1+(2\pi-t/2)}\,dt$

Let $2\pi - \frac{t}{2} = u$. Then, $dt = -2\,du$.

Also, $t = 4\pi - 2 \Rightarrow u = 1$ and $t = 4\pi \Rightarrow u = 0$

$$\therefore\ I = \frac{1}{2}\int_1^0 \frac{\sin(2\pi-u)}{1+u}(-2\,du) = -\int_0^1 \frac{\sin u}{u+1}\,du = -\alpha$$

EXAMPLE 26 $\int_0^{n\pi} \sin\left[\frac{2x}{\pi}\right]dx$, *where [ ] denotes the greatest integer function.*

SOLUTION We have,

$$\int_0^{n\pi} \sin\left[\frac{2x}{\pi}\right]dx$$

$$= \int_0^{\pi/2} \sin\left[\frac{2x}{\pi}\right]dx + \int_{\pi/2}^{\pi} \sin\left[\frac{2x}{\pi}\right]dx + \ldots + \int_{(2n-1)\pi/2}^{n\pi} \sin\left[\frac{2x}{\pi}\right]dx$$

$$= \int_0^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} \sin 1\,dx + \int_{\pi}^{3\pi/2} \sin 2\,dx + \ldots + \int_{n\pi-\pi/2}^{n\pi} \sin(2n-1)\,dx$$

$$= \frac{\pi}{2}[\sin 1 + \sin 2 + \ldots + \sin(2n-1)]$$

$$= \frac{\pi}{2}\left[\frac{\sin n \sin\frac{(2n-1)}{2}}{\sin\frac{1}{2}}\right]$$

—❑❑❑—

PROPERTY IV $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

PROOF Let $a - x = t$. Then, $dx = -dt$.

Also, when $x = 0$, $t = a$ and when, $x = a$, $t = 0$.

$$\therefore\ \int_0^a f(a-x)\,dx = \int_a^0 f(t)\,(-dt)$$

$$= -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt \qquad \text{[By Prop. (II)]}$$

$$= \int_0^a f(x)\,dx \qquad \text{[By Prop. (I)]}$$

NOTE *This property is useful to evaluate a definite integral without first finding the corresponding indefinite integral which may be difficult or sometimes even impossible to find.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate* : $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$2I = \int_0^{\pi/2} 1\cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}.$$

EXAMPLE 2 *Evaluate* :

(i) $\int_0^{\pi/2} \log\tan x\,dx$ (ii) $\int_0^{\pi/4} \log(1+\tan x)\,dx.$

[IIT 1997]

SOLUTION (i) Let $I = \int_0^{\pi/2} \log\tan x\,dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log\cot x\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log\tan x + \log\cot x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\tan x\cdot\cot x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log 1\cdot dx = \int_0^{\pi/2} 0\cdot dx = 0$$

$$\Rightarrow I = 0.$$

(ii) Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ...(i)

Then,

$$I = \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-x\right)\right\}dx$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{\tan\pi/4-\tan x}{1+\tan\pi/4\tan x}\right)dx$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$\Rightarrow I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\,dx$$

$$\Rightarrow I = \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$\Rightarrow \quad = \int_0^{\pi/2} \log \sin x\, dx - (\log 2) \int_0^{\pi/2} 1 \cdot dx - \int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8} \log 2.$$

EXAMPLE 3 *Evaluate :*

$$(i) \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx \qquad (ii) \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$$

SOLUTION (i) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x}\, dx \qquad ...(ii)$$

Adding (i) and (ii), we obtain

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \sin x \cos x}\right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x}\, dx = 0$$

$$\Rightarrow I = 0.$$

(ii) Let $I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$

Then,

$$I = \int_0^{\pi/2} \{2 \log \sin x - \log (2 \sin x \cos x)\}\, dx$$

$$= \int_0^{\pi/2} \{2 \log \sin x - \log 2 - \log \sin x - \log \cos x\}\, dx$$

$$= \int_0^{\pi/2} \log \sin x\, dx - \int_0^{\pi/2} \log 2\, dx - \int_0^{\pi/2} \log \cos x\, dx$$

$$I = (\log 2)\, [x]_0^{\pi/4} - I$$

$$= \int_0^{\pi/2} \log \sin x\, dx - \frac{\pi}{2} (\log 2) - \int_0^{\pi/2} \log \sin x\, dx$$

$$= -(\log 2) \left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{2} \log 2.$$

EXAMPLE 4 *Evaluate :* $\int_0^1 \cot^{-1} (1 + x^2 - x)\, dx$ [IIT 1998]

SOLUTION We have,

$$\cot^{-1} (1 + x^2 - x) = \cot^{-1} \{1 - x (1 - x)\}$$

$$= \tan^{-1}\left(\frac{1}{1 - x(1-x)}\right)$$

$$= \tan^{-1}\left(\frac{x + (1-x)}{1 - x(1-x)}\right)$$

$$= \tan^{-1} x + \tan^{-1} (1 - x)$$

$$\therefore \quad I = \int_0^1 \cot^{-1} (1 + x^2 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} (1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} \{1 - (1 - x)\}\, dx \quad \text{[Using Prop IV]}$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2 \int_0^1 \tan^{-1} x\, dx$$

$$= 2 \left[\left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1 + x^2}\, dx\right]$$

$$= 2 \left[\left[x \tan^{-1} x\right]_0^1 - \frac{1}{2} \left[\log_e (1 + x^2)\right]_0^1\right]$$

$$= 2 \left[(\tan^{-1} 1 - 0) - \frac{1}{2} \{\log_e 2 - \log_e 1\}\right]$$

$$= 2 \times \frac{\pi}{4} - \log_e 2$$

$$= \frac{\pi}{2} - \log_e 2$$

EXAMPLE 5 *Show that :*

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log_e 2.$$

SOLUTION Let $x = \tan\theta$. Then, $dx = \sec^2\theta\,d\theta$.

Also, $x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

and, $x = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$.

$$\therefore I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log_e 2 \qquad \text{[See Ex 2 (ii)]}$$

EXAMPLE 6 *If f and g are continuous on $[0, a]$ and satisfy $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, show that*

$$\int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

SOLUTION We have,

$$\int_0^a f(x)\,g(x)\,dx$$

$$= \int_0^a f(a-x)\,g(a-x)\,dx$$

$$= \int_0^a f(x)\,(2-g(x))\,dx \qquad [\because g(a-x) = 2-g(x)]$$

$$= 2\int_0^a f(x)\,dx - \int_0^a f(x)\,g(x)\,dx$$

$$\therefore 2\int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\Rightarrow \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

EXAMPLE 7 *Evaluate :* $\displaystyle\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx.$

[IIT 1985]

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$. Then,

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\sin^4(\pi/2-x)+\cos^4(\pi/2-x)}\,dx$$

$$= \int_0^{\pi/2} \frac{(\pi/2-x)\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx - \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx - I$$

$$\therefore 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x\sec^2 x}{1+\tan^4 x}\,dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^4 x\text{]}$$

$$\Rightarrow 2I = \frac{\pi}{4}\int_0^{\pi/2} \frac{2\tan x\sec^2 x\,dx}{1+(\tan^2 x)^2}$$

$$\Rightarrow 2I = \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{1+(\tan^2 x)^2}\,d(\tan^2 x)$$

$$\therefore 2I = \frac{\pi}{4}\int_0^{\infty} \frac{1}{1+t^2}\,dt, \text{ where } t = \tan^2 x.$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow I = \frac{\pi^2}{16}$$

EXAMPLE 8 *Prove that :* $\displaystyle\int_0^{\pi} \frac{x}{1-\cos\alpha\sin x}\,dx = \frac{\pi(\pi-\alpha)}{\sin\alpha}$

[IIT 1986]

SOLUTION Let $I = \displaystyle\int_0^{\pi} \frac{x}{1-\cos\alpha\sin x}\,dx$. Then,

$$I = \int_0^{\pi} \frac{(\pi-x)}{1-\cos\alpha\sin(\pi-x)}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{1-\cos\alpha\sin x}\,dx - \int_0^{\pi} \frac{x}{1-\cos\alpha\sin x}\,dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\sin x}\,dx - I$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\sin x}\,dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1+\tan^2 x/2}{(1+\tan^2 x/2) - 2\cos\alpha\tan x/2}\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $\sec^2\frac{x}{2}\, dx = 2\, dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan\frac{\pi}{2} = \infty$.

$$\therefore I = \frac{\pi}{2} \int_0^{\infty} \frac{2\, dt}{t^2 - 2t\cos\alpha + 1}$$

$$\Rightarrow I = \pi \int_0^{\infty} \frac{1}{(t - \cos\alpha)^2 + (1 - \cos^2\alpha)}\, dt$$

$$\Rightarrow I = \pi \int_0^{\infty} \frac{1}{\sin^2\alpha + (t - \cos\alpha)^2}\, dt$$

$$\Rightarrow I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t - \cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\infty - \tan^{-1}(-\cot\alpha)\right]$$

$$\Rightarrow I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \tan^{-1}(\cot\alpha)\right]$$

$$\Rightarrow I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \alpha\right)\right\}\right]$$

$$\Rightarrow I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right]$$

$$\Rightarrow I = \frac{\pi(\pi - \alpha)}{\sin\alpha}$$

EXAMPLE 9 *Show that :*

$$\int_0^{\pi/2} \sin\theta \log\sin\theta\, d\theta = \log(2/e).$$

SOLUTION Let $I = \int_0^{\pi/2} \sin\theta \log\sin\theta\, d\theta$.

Let $t = \cos\theta$. Then, $dt = -\sin\theta\, d\theta$.

Also, $\theta = 0 \Rightarrow t = \cos 0 = 1$ *and*, $\theta = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{2} = 0$.

$$\therefore I = -\int_1^0 \log\sqrt{1 - t^2}\, dt$$

$$= \int_0^1 \log(1 - t^2)^{1/2}\, dt$$

$$= \frac{1}{2}\int_0^1 \log(1 - t^2)\, dt$$

$$= \frac{1}{2}\int_0^1 [\log(1 + t) + \log(1 - t)]\, dt$$

$$= \frac{1}{2}\int_0^1 \log(1 + t)\, dt + \frac{1}{2}\int_0^1 \log(1 - t)\, dt$$

$$= \frac{1}{2}\int_0^1 \log(1 + t)\, dt + \frac{1}{2}\int_0^1 \log[1 - (1 - t)]\, dt$$

$$= \frac{1}{2}\int_0^1 \log(1 + t)\, dt + \frac{1}{2}\int_0^1 \log t\, dt$$

$$= \frac{1}{2}\left[\left\{t\log(1 + t)\right\}_0^1 - \int_0^1 \frac{t}{1 + t}\, dt\right] + \frac{1}{2}\left[\left\{t\log t\right\}_0^1 - \int_0^1 1\, dt\right]$$

$$= \frac{1}{2}\left[\log 2 - \int_0^1 \frac{t}{1 + t}\, dt\right] + \frac{1}{2}\left[0 - \left[t\right]_0^1\right]$$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\int_0^1 \frac{t + 1 - 1}{t + 1}\, dt - \frac{1}{2}$$

$$= \frac{1}{2}\log 2 - \frac{1}{2} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1 + t}\right) dt$$

$$= \frac{1}{2}\log 2 - \frac{1}{2} - \frac{1}{2}\left[t - \log(1 + t)\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{1}{2} - \frac{1}{2}[(1 - \log 2) - 0]$$

$$= \log 2 - 1 = \log 2 - \log e = \log\left(\frac{2}{e}\right)$$

EXAMPLE 10 *Prove that :* $\int_0^{\pi} \frac{x^2\cos x}{(1 + \sin x)^2}\, dx = \pi(2 - \pi)$.

SOLUTION Let $I = \int_0^{\pi} \frac{x^2\cos x}{(1 + \sin x)^2}\, dx$. Then,

$$I = \int_0^{\pi} \underset{\text{I}}{x^2} \underset{\text{II}}{\frac{d(1 + \sin x)}{(1 + \sin x)^2}}$$

$$= \left[\frac{-x^2}{1 + \sin x}\right]_0^{\pi} - \int_0^{\pi} 2x \times \frac{-1}{1 + \sin x}\, dx$$

$$= \left[\frac{-x^2}{1+\sin x}\right]_0^{\pi} + 2\int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$= -\pi^2 + 2I_1, \text{ where } I_1 = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

Now,

$$I_1 = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$\Rightarrow I_1 = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx$$

$$\Rightarrow I_1 = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx$$

$$\Rightarrow I_1 = \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx - \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$\Rightarrow I_1 = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx - I_1$$

$$\Rightarrow 2I_1 = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

$$\Rightarrow I_1 = \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$\Rightarrow I_1 = \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$\Rightarrow I_1 = \frac{\pi}{2}\int_0^{\pi} (\sec^2 x - \tan x \sec x)\,dx$$

$$\Rightarrow I_1 = \frac{\pi}{2}\left[\tan x - \sec x\right]_0^{\pi}$$

$$\Rightarrow I_1 = \frac{\pi}{2}[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)]$$

$$\Rightarrow I_1 = \frac{\pi}{2}[1+1] = \pi$$

$$\therefore I = -\pi^2 + 2\pi = \pi(2-\pi).$$

EXAMPLE 11 *Evaluate*: $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

SOLUTION Let

$$I = \int_1^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx \qquad \ldots(i)$$

Then,

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{1+\cos(\pi/2-x)+\sin(\pi/2-x)}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{1+\sin x+\cos x}\,dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos x+\sin x}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1+\sin x+\cos x-1}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(1 - \frac{1}{1+\sin x+\cos x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} 1\,.\,dx - \int_0^{\pi/2} \frac{1}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow 2I = \left[x\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}\,dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \int_0^{1} \frac{2\,dt}{2+2t}, \text{ where } t = \tan\frac{x}{2}$$

$$\Rightarrow 2I = \frac{\pi}{2} - [\log(1+t)]_0^1$$

$$\Rightarrow 2I = \frac{\pi}{2} - \log 2$$

$$\Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

EXAMPLE 12 *Evaluate*: $\displaystyle\int_0^{\pi} \frac{x^3\cos^4 x\sin x}{(\pi^2 - 3\pi x + 3x^2)}\,dx$

SOLUTION Let

$$I = \int_0^{\pi} \frac{x^3\cos^4 x\sin x}{(\pi^2 - 3\pi x + 3x^2)}\,dx \qquad \ldots(i)$$

Then,

$$I = \int_0^{\pi} \frac{(\pi-x)^3\cos^4(\pi-x)\sin(\pi-x)}{\pi^2 - 3\pi(\pi-x) + 3(\pi-x)^2}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)^3 \cos^4 x \sin x}{(\pi^2 - 3\pi x + 3x^2)} dx \quad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{(\pi^2 - 3\pi x + 3x^2) \cos^4 x \sin x}{(\pi^2 - 3\pi x + x^2)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \cos^4 x \sin x \, dx$$

$$\Rightarrow 2I = -\pi \int_1^{-1} t^4 \, dt, \text{ where } t = \cos x$$

$$\Rightarrow 2I = -\left[\frac{t^5}{5}\right]_1^{-1} \pi = -\left(-\frac{1}{5} - \frac{1}{5}\right) = \frac{2\pi}{5}$$

$$\Rightarrow I = \frac{\pi}{5}.$$

EXAMPLE 13 *Assuming that* $\int_0^{\pi} \log \sin x \, dx = -\pi \log 2$,

show that $\int_0^{\pi} x^3 \log \sin x \, dx = -\frac{3\pi}{2} \int_0^{\pi} x^2 \log (\sqrt{2} \sin x) \, dx.$

SOLUTION Let

$$I = \int_0^{\pi} x^3 \log \sin x \, dx \quad \ldots(i)$$

Then,

$$I = \int_0^{\pi} (\pi - x)^3 \log \sin (\pi - x) \, dx$$

$$\Rightarrow I = \int_0^{\pi} (\pi - x)^3 \log \sin x \, dx \quad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \{(\pi - x)^3 + x^3\} \log \sin x \, dx$$

$$\Rightarrow 2I = \int_0^{\pi} (\pi^3 + 3\pi x^2 - 3\pi^2 x) \log \sin x \, dx$$

$$\Rightarrow 2I = \pi^3 \int_0^{\pi} \log \sin x \, dx + 3\pi \int_0^{\pi} x^2 \log \sin x \, dx - 3\pi^2 \int_0^{\pi} x \log \sin x \, dx$$

$$\Rightarrow 2I = -\pi^4 \log 2 - 3\pi^2 \int_0^{\pi} x \log \sin x \, dx + 3\pi \int_0^{\pi} x^2 \log \sin x \, dx$$

$$\Rightarrow 2I = -\pi^4 \log 2 - 3\pi^2 I_1 + 3\pi I_2 \quad \ldots(i)$$

where,

$$I_1 = \int_0^{\pi} x \log \sin x \, dx \text{ and } I_2 = \int_0^{\pi} x^2 \log \sin x \, dx$$

Now,

$$I_1 = \int_0^{\pi} x \log \sin x \, dx$$

$$\Rightarrow I_1 = \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx$$

$$\Rightarrow I_1 = \int_0^{\pi} (\pi - x) \log \sin x \, dx$$

$$\Rightarrow I_1 = \int_0^{\pi} \pi \log \sin x \, dx - \int_0^{\pi} x \log \sin x \, dx$$

$$\Rightarrow I_1 = \pi \int_0^{\pi} \log \sin x \, dx - I_1$$

$$\Rightarrow 2I_1 = \pi \int_0^{\pi} \log \sin x \, dx$$

$$\Rightarrow 2I_1 = \pi (-\pi \log 2)$$

$$\Rightarrow I_1 = -\frac{\pi^2}{2} \log 2$$

Putting the value of I_1 in (i), we get

$$2I = -\pi^4 \log 2 - 3\pi^2 \times \frac{-\pi^2}{2} \log 2 + 3\pi I_2$$

$$\Rightarrow 2I = \frac{\pi^4}{2} \log 2 + 3\pi I_2$$

$$\Rightarrow I = \frac{\pi^4}{4} \log 2 + \frac{3\pi}{2} I_2$$

$$\Rightarrow \int_0^{\pi} x^3 \log \sin x \, dx$$

$$= \frac{\pi^4}{4} \log 2 + \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sin x \, dx$$

$$= \frac{3\pi}{2} \times \frac{\pi^3}{6} \log 2 + \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sin x \, dx$$

$$= \frac{3\pi}{2} \times \frac{\pi^3}{3} \log \sqrt{2} + \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sin x \, dx$$

$$= \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sqrt{2} \, dx + \frac{3\pi}{2} \int_0^{\pi} x^2 \log \sin x \, dx$$

$$= \frac{3\pi}{2} \int_0^{\pi} x^2 (\log \sqrt{2} + \log \sin x) \, dx$$

$$= \frac{3\pi}{2} \int_0^{\pi} x^2 \log (\sqrt{2} \sin x) \, dx$$

EXAMPLE 14 *Evaluate :* $\int_0^{\pi/4} \frac{x^2 (\sin 2x - \cos 2x)}{(1 + \sin 2x) \cos^2 x} \, dx$

SOLUTION Let

$$I = \int_0^{\pi/4} \frac{x^2 (\sin 2x - \cos 2x)}{(1 + \sin 2x) \cos^2 x} \, dx$$

Then,

$$I = 2 \int_0^{\pi/4} \frac{x^2 [(1 + \sin 2x) - (1 + \cos 2x)]}{(1 + \sin 2x)(1 + \cos 2x)} \, dx$$

$$\Rightarrow I = 2 \int_0^{\pi/4} x^2 \left\{ \frac{1}{1 + \cos 2x} - \frac{1}{1 + \sin 2x} \right\} dx$$

$$\Rightarrow I = 2 \int_0^{\pi/4} \frac{x^2}{1 + \cos 2x} \, dx - 2 \int_0^{\pi/4} \frac{x^2}{1 + \sin 2x} \, dx$$

$$\Rightarrow I = \int_0^{\pi/4} x^2 \sec^2 x \, dx - 2 \int_0^{\pi/4} \frac{(\pi/4 - x)^2}{1 + \sin 2(\pi/4 - x)} \, dx$$

$$\Rightarrow I = \int_0^{\pi/4} x^2 \sec^2 x \, dx - 2 \int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x\right)^2}{1 + \cos 2x} \, dx$$

$$\Rightarrow I = \int_0^{\pi/4} x^2 \sec^2 x \, dx - \int_0^{\pi/4} \left(\frac{\pi}{4} - x\right)^2 \sec^2 x \, dx$$

$$\Rightarrow I = \int_0^{\pi/4} x^2 \sec^2 x \, dx - \int_0^{\pi/4} \left(\frac{\pi^2}{16} - \frac{\pi}{2} x + x^2\right) \sec^2 x \, dx$$

$$\Rightarrow I = \frac{-\pi^2}{16} \int_0^{\pi/4} \sec^2 x \, dx + \frac{\pi}{2} \int_0^{\pi/4} \underset{\text{I}}{x} \; \underset{\text{II}}{\sec^2 x} \, dx$$

$$\Rightarrow I = \frac{-\pi^2}{16} \left[\tan x \right]_0^{\pi/4} + \frac{\pi}{2} \left[x \tan x + \log \cos x \right]_0^{\pi/4} dx$$

$$\Rightarrow I = -\frac{\pi^2}{16} + \frac{\pi}{2} \left[\frac{\pi}{4} + \log \frac{1}{\sqrt{2}} \right] = \frac{\pi^2}{16} - \frac{\pi}{4} \log 2$$

EXAMPLE 15 *Evaluate :* $\int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} \, dx$

[IIT 1991]

SOLUTION Let

$$I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} \, dx \qquad \ldots(i)$$

Then,

$$I = \int_0^{\pi} \frac{(\pi - x) \sin 2(\pi - x) \sin\left(\frac{\pi}{2} \cos (\pi - x)\right)}{2(\pi - x) - \pi} \, dx$$

$$I = \int_0^{\pi} \frac{(\pi - x) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{\pi - 2x} \, dx$$

$$= \int_0^{\pi} \frac{(x - \pi) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} \, dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(2x - \pi) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{(2x - \pi)} \, dx$$

$$\Rightarrow 2I = \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2} \cos x\right) dx$$

$$\Rightarrow 2I = 2 \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2} \cos x\right) dx$$

$$\Rightarrow I = \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2} \cos x\right) dx$$

Let $\frac{\pi}{2} \cos x = t$. Then, $-\frac{\pi}{2} \sin x \, dx = dt \Rightarrow \sin x \, dx = -\frac{2}{\pi} dt$

Also, $x = 0 \Rightarrow t = \frac{\pi}{2}$ and $x = \pi \Rightarrow t = -\frac{\pi}{2}$

$$\therefore I = -\frac{2}{\pi} \int_{\pi/2}^{-\pi/2} \frac{2t}{\pi} \sin t \, dt$$

$$\Rightarrow I = \frac{2}{\pi} \times \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} t \sin t \, dt$$

$$\Rightarrow I = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} \underset{\text{I}}{t}\ \underset{\text{II}}{\sin t}\, dt$$

$$\Rightarrow I = \frac{4}{\pi^2}\left[-t\cos t + \sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2}\times 2 = \frac{8}{\pi^2}$$

EXAMPLE 16 *Evaluate* : $\int_0^2 \frac{1}{(17+8x-4x^2)\,(e^{6(1-x)}+1)}\,dx$

SOLUTION Let

$$I = \int_0^2 \frac{1}{(17+8x-4x^2)\,(e^{6(1-x)}+1)}\,dx \qquad \ldots(i)$$

Then,

$$I = \int_0^2 \frac{1}{\{17+8(2-x)-4(2-x)^2\}\,\{e^{6\{1-(2-x)\}}+1\}}\,dx$$

$$\Rightarrow I = \int_0^2 \frac{1}{\{17+16-8x-16+16x-4x^2\}\,\{e^{-6(1-x)}+1\}}\,dx$$

$$\Rightarrow I = \int_0^2 \frac{e^{6(1-x)}}{(17+8x-4x^2)\,(e^{6(1-x)}+1)}\,dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^2 \frac{1}{17+8x-4x^2}\,dx$$

$$= \frac{-1}{4}\int_0^2 \frac{1}{x^2-2x-\frac{17}{4}}\,dx$$

$$= -\frac{1}{4}\int_0^2 \frac{1}{(x-1)^2-\frac{21}{4}}\,dx$$

$$= \frac{1}{4}\int_0^2 \frac{1}{\left(\frac{\sqrt{21}}{2}\right)^2-(x-1)^2}$$

$$= \frac{1}{4}\times\frac{1}{\sqrt{21}}\left[\log\left|\frac{\frac{\sqrt{21}}{2}+(x-1)}{\frac{\sqrt{21}}{2}-(x-1)}\right|\right]_0^2$$

$$= \frac{1}{4\sqrt{21}}\left[\log\left|\frac{\sqrt{21}-2+2x}{\sqrt{21}+2-2x}\right|\right]_0^2$$

$$= \frac{1}{4\sqrt{21}}\left[\log\left(\frac{\sqrt{21}+2}{\sqrt{21}-2}\right)-\log\left(\frac{\sqrt{21}-2}{\sqrt{21}+2}\right)\right]$$

$$= \frac{1}{4\sqrt{21}}\log\left(\frac{\sqrt{21}+2}{\sqrt{21}-2}\right)^2$$

$$= \frac{1}{2\sqrt{21}}\log\left(\frac{\sqrt{21}+2}{\sqrt{21}-2}\right)$$

$$\therefore I = \frac{1}{4\sqrt{21}}\log\left(\frac{\sqrt{21}+2}{\sqrt{21}-2}\right)$$

EXAMPLE 17 *If n is an odd natural number, prove that*

$$I = \int_0^\pi \frac{x\sin nx}{\sin x}\,dx = \frac{\pi^2}{2}.$$

SOLUTION Let $n = 2m+1$, where $n = 0, 1, 2, \ldots$ Then,

$$I = \int_0^\pi \frac{x\sin(2m+1)x}{\sin x}\,dx \qquad \ldots(i)$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x)\sin(2m+1)(\pi-x)}{\sin(\pi-x)}\,dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x)\sin\{(2m+1)\pi-(2m+1)x\}}{\sin x}\,dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x)\sin(2m+1)x}{\sin x}\,dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \pi\int_0^\pi \frac{\sin(2m+1)x}{\sin x}\,dx$$

$\Rightarrow 2I = \pi\times\pi$ [See Ex. 28 on page 11.12]

$$\Rightarrow I = \frac{\pi^2}{2}$$

EXAMPLE 18 *Evaluate* : $\int_0^\pi \frac{x^2\sin x}{(2x-\pi)\,(1+\cos^2 x)}\,dx$

SOLUTION Let

$$I = \int_0^\pi \frac{x^2\sin x}{(2x-\pi)\,(1+\cos^2 x)}\,dx \qquad \ldots(i)$$

Then,

$$I = \int_0^\pi \frac{(\pi-x)^2\sin(\pi-x)}{\{2(\pi-x)-\pi\}\,\{1+\cos^2(\pi-x)\}}\,dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x)^2\sin x}{(\pi-2x)\,(1+\cos^2 x)}\,dx$$

$$\Rightarrow I = -\int_0^\pi \frac{(\pi-x)^2\sin x}{(2x-\pi)\,(1+\cos^2 x)}\,dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^\pi \frac{x^2\sin x-(\pi-x)^2\sin x}{(2x-\pi)\,(\pi+\cos^2 x)}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{(-\pi^2 + 2\pi x)\sin x}{(2x-\pi)(1+\cos^2 x)}\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow 2I = -\pi \int_0^{\pi} \frac{1}{1+\cos^2 x}\,d(\cos x)$$

$$\Rightarrow 2I = -\pi \left[\tan^{-1}(\cos x)\right]_0^{\pi}$$

$$\Rightarrow 2I = -\pi \left[\tan^{-1}(-1) - \tan^{-1} 1\right]$$

$$\Rightarrow 2I = -\pi \left[\frac{-\pi}{4} - \frac{\pi}{4}\right]$$

$$\Rightarrow I = \frac{\pi^2}{4}.$$

EXAMPLE 19 *If f, g, h be continuous functions on $[0, a]$ such that $f(a-x) = f(x), g(a-x) = -g(x)$ and $3h(x) - 4h(a-x) = 5$, then prove that*

$$\int_0^a f(x)\,g(x)\,h(x)\,dx = 0.$$

SOLUTION Let $I = \int_0^a f(x)\,g(x)\,h(x)\,dx$

Then,

$$I = \int_0^a f(a-x)\,g(a-x)\,h(a-x)\,dx$$

$$\Rightarrow I = \int_0^a f(x)\{-g(x)\}\left\{\frac{3h(x)-5}{4}\right\}dx \quad \left[\because \begin{array}{l} f(a-x) = f(x), \\ g(a-x) = -g(x) \\ h(a-x) = \dfrac{3h(x)-5}{4} \end{array}\right]$$

$$\Rightarrow I = -\frac{3}{4}\int_0^a f(x)\,g(x)\,h(x)\,dx + \frac{5}{4}\int_0^a f(x)\,g(x)\,dx$$

$$\Rightarrow I = -\frac{3}{4}I + \frac{5}{4}I_1, \text{ where } I_1 = \int_0^a f(x)\,g(x)\,dx$$

$$\Rightarrow \frac{7}{4}I = \frac{5}{4}I_1$$

$$\Rightarrow I = \frac{5}{7}I_1 \qquad \ldots(i)$$

Now,

$$I_1 = \int_0^a f(x)\,g(x)\,dx$$

$$\Rightarrow I_1 = \int_0^a f(a-x)\,g(a-x)\,dx$$

$$\Rightarrow I_1 = -\int_0^a f(x)\,g(x)\,dx \qquad \left[\because \begin{array}{l} f(a-x) = f(x) \\ g(a-x) = -g(x) \end{array}\right]$$

$$\Rightarrow I_1 = -I_1$$

$$\Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0$$

Putting $I_1 = 0$ in (i), we get : $I = 0$.

EXAMPLE 20 *Evaluate :* $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$

[IIT 1999]

SOLUTION Let

$$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx \quad \ldots(i)$$

Then,

$$I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\,dx$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} 1\,dx = \pi$$

$$\Rightarrow I = \frac{\pi}{2}$$

The above property of the definite integrals is in fact a particular case the following property.

—❑❑❑—

PROPERTY V $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

PROOF Let $x = a + b - t$. *Then,* $dx = -dt$.

Also, when $x = a$, $t = b$ and, $x = b \Rightarrow t = a$.

$$\int_a^b f(x)\,dx$$

$$= \int_b^a f(a+b-t)\,(-dt)$$

$$= -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \qquad \text{[By Prop. II]}$$

$$= \int_a^b f(a+b-x)\,dx \qquad \text{[By Prop. I]}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$ (ii) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx.$

SOLUTION (i) Let

$$I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx \qquad ...(i)$$

Then,

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}\,dx$$

$$\Rightarrow \quad I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx = \int_1^2 1\cdot dx = [x]_1^2 = 2-1 = 1$$

$$\Rightarrow \quad I = 1/2.$$

(ii) Let

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \qquad ...(i)$$

Then,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} 1\cdot dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \pi/12.$$

EXAMPLE 2 *Prove that :* $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}.$

SOLUTION Let

$$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx \qquad ...(i)$$

Then,

$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f\{a+b-(a+b-x)\}}\,dx$$

$$\Rightarrow I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$\Rightarrow 2I = \int_a^b 1\,dx = (b-a)$$

$$\Rightarrow \quad I = \left(\frac{b-a}{2}\right).$$

REMARK *The result proved in the above example may be used as a standard result.*

—❑❑❑—

PROPERTY VI

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

PROOF From property III, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad ...(i)$$

Now, $\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-t)\,(-dt)$, where $t = -x$

$$= -\int_a^0 f(-t)\,dt$$

$$= \int_0^a f(-t)\,dt \qquad \text{[By Prop. II]}$$

$$= \int_0^a f(-x)\,dx \qquad \text{[By Prop. I]}$$

$$= \begin{cases} \int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0 & \\ -\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an odd function} \end{cases} \qquad ...(ii)$$

Thus, when $f(x)$ is an even function

$$\text{(i) and (ii)} \Rightarrow \int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

and, when $f(x)$ is an odd function,

$$\text{(i) and (ii)} \Rightarrow \int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

REMARK The graph of an even function is symmetric about y-axis that is the curve on left side of y-axis is exactly identical to curve on its right side. So,

$$\int_0^a f(x)\,dx = \int_{-a}^{0} f(x)\,dx \qquad \text{[see Fig. 11.9].}$$

In case of an odd function the curve is symmetric in opposite quadrants, so

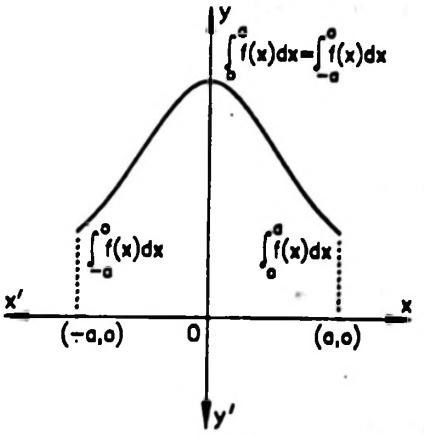


Fig. 11.9

$$\int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx \qquad \text{[see Fig. 11.10]}$$

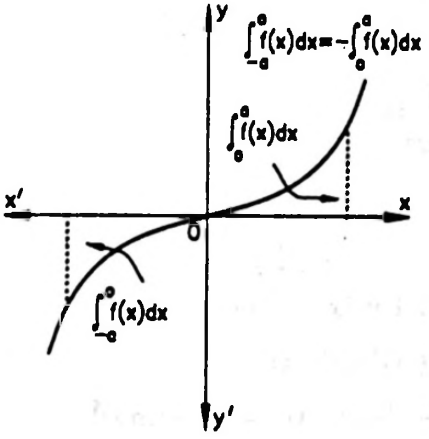


Fig. 11.10

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$ (ii) $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

SOLUTION (i) Let $f(x) = \sin^7 x$. Then,

$f(-x) = \sin^7(-x) = [\sin(-x)]^7 = [-\sin x]^7 = -\sin^7 x = -f(x)$.

So, $f(x)$ is an odd function.

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0$$

(ii) Let $f(x) = \sin^2 x$. Then,

$f(-x) = \sin^2(-x) = [\sin(-x)]^2 = [-\sin x]^2 = \sin^2 x = f(x)$.

Therefore, $f(x)$ is an even function.

So, $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

$= 2\int_0^{\pi/2} \sin^2 x\,dx$ [Using Prop. VII]

$= 2I$, where $I = \int_0^{\pi/2} \sin^2 x\,dx$.

Now,

$I = \int_0^{\pi/2} \sin^2 x\,dx$ ...(i)

$\Rightarrow I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right)dx$ [Using Prop. IV]

$\Rightarrow I = \int_0^{\pi/2} \cos^2 x\,dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\sin^2 x + \cos^2 x)\,dx = \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

Hence, $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{2}$.

EXAMPLE 2 *Evaluate* : $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

SOLUTION (i) Let $f(x) = x^3 \sin^4 x$. Then,

$f(-x) = (-x)^3 \sin^4(-x)$

$= -x^3 (\sin(-x))^4 = -x^3(-\sin x)^4$

$= -x^3 \sin^4 x = -f(x)$.

So, $f(x)$ is an odd function.

Hence,

$$\int_{-\pi/4}^{\pi/4} f(x)\,dx = 0 \text{ i.e., } \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx = 0.$$

EXAMPLE 3 *Evaluate* : $\int_{-\pi/4}^{\frac{\pi}{4}} \frac{x + \pi/4}{2 - \cos 2x}\,dx$

SOLUTION Let $I = \int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x}\,dx$. Then,

$$I = \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x}\,dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x}\,dx$$

$$\Rightarrow I = 0 + \frac{\pi}{2}\int_0^{\pi/4} \frac{1}{2 - \cos 2x}\,dx \left[\because \frac{x}{2 - \cos 2x} \text{ is an odd function & } \frac{1}{2 - \cos 2x} \text{ is an even function}\right]$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/4} \frac{1 + \tan^2 x}{1 + 3\tan^2 x}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{1^2 + (\sqrt{3}\tan x)^2}\,dx$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}}\int_0^{\pi/4} \frac{1}{1^2 + (\sqrt{3}\tan x)^2}\,d(\sqrt{3}\tan x)$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi/4}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}}\{\tan^{-1}\sqrt{3} - \tan^{-1} 0\} = \frac{\pi}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$$

EXAMPLE 4 *Evaluate :*

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2x}{1 + x^2}\right)dx$$

[IIT 1995]

SOLUTION Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1 - x^4} \cos^{-1}\left(\frac{2x}{1 + x^2}\right)dx$. Then,

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{(1 - x^4)}\left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{2x}{1 + x^2}\right)\right\}dx$$

$$\Rightarrow I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1 - x^4}\left(\frac{\pi}{2} - 2\tan^{-1} x\right)dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1 - x^4}\,dx - 2\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1 - x^4}\tan^{-1} x\,dx$$

Since $\frac{x^4}{1 - x^4}$ and $\frac{x^4 \tan^{-1} x}{1 - x^4}$ are even and odd functions respectively.

$$\therefore I = \frac{\pi}{2} \times 2\int_0^{1/\sqrt{3}} \frac{x^4}{1 - x^4}\,dx - 2 \times 0$$

$$\Rightarrow I = -\pi\int_0^{1/\sqrt{3}} \frac{(1 - x^4) - 1}{1 - x^4}\,dx$$

$$\Rightarrow I = -\pi \int_0^{1/\sqrt{3}} \left(1 - \frac{1}{1-x^4}\right) dx$$

$$\Rightarrow I = -\pi \int_0^{1/\sqrt{3}} 1 \, . \, dx + \pi \int_0^{1/\sqrt{3}} \frac{1}{1-x^4} dx$$

$$\Rightarrow I = -\pi \left(\frac{1}{\sqrt{3}} - 0\right) + \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right) dx$$

$$\Rightarrow I = -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + \tan^{-1} x\right]_0^{1/\sqrt{3}}$$

$$\Rightarrow I = -\frac{\pi}{3} + \frac{\pi}{2}\left[\frac{1}{2}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow I = \frac{-\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\frac{1}{2}\log\left(\frac{(\sqrt{3}+1)^2}{2}\right) + \frac{\pi}{6}\right]$$

$$\Rightarrow I = \frac{-\pi}{\sqrt{3}} + \frac{\pi}{4}\log(2+\sqrt{3}) + \frac{\pi^2}{12}$$

EXAMPLE 5 *Evaluate*

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}\left\{\sin^{-1}(1-2x^2) + \cos^{-1}(2x\sqrt{1-x^2})\right\} dx$$

SOLUTION Let

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}\left\{\sin^{-1}(1-2x^2) + \cos^{-1}(2x\sqrt{1-x^2})\right\} dx.$$

Then,

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}\left[\left\{\frac{\pi}{2} - \cos^{-1}(1-2x^2)\right\} + \left\{\frac{\pi}{2} - \sin^{-1}(2x\sqrt{1-x^2}\right\}\right] dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}\left[\frac{\pi}{2} - 2\sin^{-1}x + \frac{\pi}{2} - 2\sin^{-1}x\right] dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}(\pi - 4\sin^{-1}x)\, dx$$

$$= \pi \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4} dx - 4\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x^8}{1-x^4}\sin^{-1}x\, dx$$

$$= 2\pi \int_0^{1/\sqrt{2}} \frac{x^8}{1-x^4} dx - 4 \times 0 \qquad \left[\because \frac{x^8}{1-x^4} \text{ is an even function and } \frac{x^8}{1-x^4}\sin^{-1}x \text{ is an odd function}\right]$$

$$= 2\pi \int_0^{1/\sqrt{2}} \left\{-x^4 - 1 + \frac{1}{2(1-x^2)} + \frac{1}{2(1+x^2)}\right\} dx$$

$$= 2\pi\left[-\frac{x^5}{5} - x + \frac{1}{4}\log\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\tan^{-1}x\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left[-\frac{1}{20\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{4}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= \pi\left[\frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{21}{10\sqrt{2}}\right].$$

EXAMPLE 6 *Prove that :*

$$\int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi} dx = \frac{8}{\pi^2}$$

[IIT 1991]

SOLUTION Let $I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi} dx$

Let $2x - \pi = t$. Then, $2\,dx = dt$.

Also, $x = 0 \Rightarrow t = -\pi$ and $x = \pi \Rightarrow t = \pi$

$$\therefore I = \int_{-\pi}^{\pi} \frac{\left(\frac{\pi+t}{2}\right)\sin(\pi+t)\sin\left\{\frac{\pi}{2}\cos\left(\frac{\pi+t}{2}\right)\right\}}{t}\frac{dt}{2}$$

$$\Rightarrow I = \frac{1}{4}\int_{-\pi}^{\pi} \frac{(\pi+t)\sin t \sin\left\{\frac{\pi}{2}\left(\sin\frac{t}{2}\right)\right\}}{t} dt$$

$$\Rightarrow I = \frac{1}{4}\int_{-\pi}^{\pi} \pi \frac{\sin\,.\sin\left\{\frac{\pi}{2}\sin\frac{t}{2}\right\}}{t} dt + \frac{1}{4}\int_{-\pi}^{\pi} \sin t \sin\left\{\frac{\pi}{2}\sin\frac{t}{2}\right\} dt$$

$$\Rightarrow I = 0 + \frac{2}{4}\int_0^{\pi} \sin t \sin\left\{\frac{\pi}{2}\sin\frac{t}{2}\right\} dt \qquad \text{[Using Prop VI]}$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi} \sin t \sin\left\{\frac{\pi}{2}\sin\frac{t}{2}\right\} dt$$

$$\Rightarrow I = \frac{2}{2}\int_0^{\pi} \sin\frac{t}{2}\cos\frac{t}{2}\sin\left\{\frac{\pi}{2}\sin\frac{t}{2}\right\} dt$$

Let $\frac{\pi}{2}\sin\frac{t}{2} = u$. Then, $\frac{\pi}{4}\cos\frac{t}{2}\,dt = du$

Also, $t = 0 \Rightarrow u = 0$ and $t = \pi \Rightarrow u = \frac{\pi}{2}$

$$\therefore I = \frac{4}{\pi}\int_0^{\pi} \frac{2u}{\pi}\sin u\, du$$

$$= \frac{8}{\pi^2}\int_0^{\pi/2} \underset{\text{I}}{u}\ \underset{\text{II}}{\sin u}\, du$$

$$= \frac{8}{\pi^2}\left[\left[-u\cos u\right]_0^{\pi/2} + \left[\sin u\right]_0^{\pi/2}\right] = \frac{8}{\pi^2}$$

EXAMPLE 7 Prove that

$$\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\,dx = \frac{8}{\pi}$$

SOLUTION Let

$$I = \int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\,dx$$

Let $2x-\pi = t$. Then, $2\,dx = dt$.

Also, $x = 0 \Rightarrow t = -\pi$ and, $x = \pi \Rightarrow t = \pi$.

$$I = \int_{-\pi}^{\pi} \frac{\left(\frac{\pi+t}{2}\right)^2 \sin(\pi+t)\sin\left\{\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{t}{2}\right)\right\}}{t}\,\frac{1}{2}\,dt$$

$$= \frac{1}{8}\int_{-\pi}^{\pi} \frac{(\pi^2+2\pi t+t^2)\sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)}{t}\,dt$$

$$= \frac{1}{8}\left[\pi^2\int_{-\pi}^{\pi} \frac{\sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)}{t}\,dt + 2\pi\int_{-\pi}^{\pi} \sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)dt + \int_{-\pi}^{\pi} t\sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)dt\right]$$

$$= \frac{1}{8}\left[\pi^2\times 0 + 4\pi\int_0^{\pi} \sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)dt + 0\right] \text{[Using prop VI]}$$

$$= \frac{\pi}{2}\int_0^{\pi} \sin t \sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)dt$$

$$= \frac{\pi}{2}\times\frac{4}{\pi^2} \quad \text{[See Ex 6]}$$

$$= \frac{8}{\pi}$$

EXAMPLE 8 *Evaluate:* $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

[IIT 1991]

SOLUTION Let

$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$. Then,

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

$I = I_1 + I_2$, where

$$I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx \text{ and } I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

Since $f(x) = \frac{2x}{1+\cos^2 x}$ is an odd function and $\frac{2x\sin x}{1+\cos^2 x}$ is an even function.

$$I_1 = 0 \text{ and } I_2 = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx.$$

Now,

$$I_2 = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \quad \text{...(i)}$$

$$\Rightarrow I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$$

$$\Rightarrow I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I_2 = 4\pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\sin x\,dx$$

$$2I_2 = -4\pi\int_1^{-1} \frac{1}{1+t^2}\,dt, \text{ where } t = \cos x$$

$$\Rightarrow 2I_2 = -4\pi\left[\tan^{-1} t\right]_1^{-1}$$

$$\Rightarrow 2I_2 = -4\pi(-1\pi/4 - \pi/4) = 2\pi^2$$

$$\Rightarrow I_2 = \pi^2$$

Hence, $I = 0+\pi^2 = \pi^2$.

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PROPERTY VII $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x)+f(-x)\}\,dt$

PROOF We have,

$$I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\Rightarrow I = I_1 + \int_{0}^{a} f(x)\,dx, \text{ where } I_1 = \int_{-a}^{0} f(x)\,dx$$

Let $x = -t$. Then, $dx = -dt$

Also, $x = -a \Rightarrow -t = -a \Rightarrow t = a$

and, $x = 0 \Rightarrow -t = 0 \Rightarrow t = 0$.

$$\therefore I_1 = \int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(-t)\,dt$$

$$= \int_{0}^{a} f(-t)\,dt$$

$$= \int_{0}^{a} f(-x)\,dx$$

Hence,

$$I = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x)+f(-x)\}\,dx$$

Following examples will illustrate the applications of this property.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:* $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)}{e^x+1}\,dx$

SOLUTION Let

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right)+\tan^{-1}\frac{2x}{1-x^2}}{e^x+1}\,dx.$$

Then,

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\frac{\pi}{2}-\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\tan^{-1}\frac{2x}{1-x^2}}{e^x+1}\,dx$$

$$\Rightarrow I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\frac{\pi}{2}-2\tan^{-1}x+2\tan^{-1}x}{e^x+1}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{1}{e^x+1}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_{0}^{1/\sqrt{3}} \left(\frac{1}{e^x+1}+\frac{1}{e^{-x}+1}\right)dx$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x)+f(-x)\}\,dx\right]$$

$$\Rightarrow I = \frac{\pi}{2}\int_{0}^{1/\sqrt{3}} \left|\frac{1}{e^x+1}+\frac{e^x}{e^x+1}\right|dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_{0}^{1/\sqrt{3}} \frac{1+e^x}{e^x+1}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_{0}^{1/\sqrt{3}} 1\,.\,dx = \frac{\pi}{2\sqrt{3}}$$

EXAMPLE 2 *Evaluate:* $\int_{-(\pi/4)^{1/3}}^{(\pi/4)^{1/3}} \frac{x^2}{\{1+\sin^2 x^3\}\{1+e^{x^7}\}}\,dx$

SOLUTION Let

$$\int_{-(\pi/4)^{1/3}}^{(\pi/4)^{1/3}} \frac{x^2}{\{1+\sin^2 x^3\}\{1+e^{x^7}\}}\,dx$$

Then, by using property VII, we have

$$I = \int_{0}^{(\pi/4)^{1/3}} \left\{\frac{x^2}{\{1+\sin^2 x^3\}\{1+e^{x^7}\}}+\frac{(-x)^2}{\{1+\sin^2(-x)^3\}\{1+e^{(-x)^7}\}}\right\}dx$$

$$\Rightarrow I = \int_{0}^{(\pi/4)^{1/3}} \frac{x^2}{\{1+\sin^2 x^3\}\{1+e^{x^7}\}}+\frac{x^2}{\{1+\sin^2 x^3\}\{1+e^{-x^7}\}}\,dx$$

$$\Rightarrow I = \int_{0}^{(\pi/4)^{1/3}} \frac{x^2}{(1+\sin^2 x^3)(1+e^{x^7})}+\frac{x^2 e^{x^7}}{(1+\sin^2 x^3)(e^{x^7}+1)}\,dx$$

$$\Rightarrow I = \int_{0}^{(\pi/4)^{1/3}} \frac{x^2(1+e^{x^7})}{(1+\sin^2 x^3)(1+e^{x^7})}\,dx$$

$$\Rightarrow I = \int_0^{(\pi/4)^{1/3}} \frac{x^2}{(1+\sin^2 x^3)}\,dx$$

$$\Rightarrow I = \frac{1}{3}\int_0^{(\pi/4)} \frac{1}{1+\sin^2 t}\,dt, \text{ where } t = x^3.$$

$$\Rightarrow I = \frac{1}{3}\int_0^{\pi/4} \frac{\sec^2 t}{\sec^2 t + \tan^2 t}\,dt \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } \cos^2 t]$$

$$\Rightarrow I = \frac{1}{3}\int_0^{\pi/4} \frac{\sec^2 t}{1^2 + (\sqrt{2}\tan t)^2}\,dt$$

$$\Rightarrow I = \frac{1}{3\sqrt{2}}\int_0^{\pi/4} \frac{1}{1^2 + (\sqrt{2}\tan t)^2}\,d(\sqrt{2}\tan t)$$

$$\Rightarrow I = \frac{1}{3\sqrt{2}}\left[\tan^{-1}(\sqrt{2}\tan t)\right]_0^{\pi/4}$$

$$\Rightarrow I = \frac{1}{3\sqrt{2}}\{\tan^{-1}\sqrt{2} - \tan^{-1} 0\} = \frac{1}{3\sqrt{2}}\tan^{-1}\sqrt{2}.$$

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PROPERTY VIII $\displaystyle\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

PROOF Using Property III, we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \ldots(i)$$

Consider the integral $\displaystyle\int_a^{2a} f(x)\,dx$. Let $x = 2a - t$. Then, $dx = -dt$. Also, $x = a \Rightarrow t = a$ and, $x = 2a \Rightarrow t = 0$.

$$\therefore \int_0^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt)$$

$$= -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt \qquad [\text{By Prop. II}]$$

$$= \int_0^a f(2a-x)\,dx \qquad [\text{By Prop. I}]$$

$$= \begin{cases} \int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ -\int_0^a f(x)\,dx, & \text{if } f(2a-x) = -f(x) \end{cases} \qquad \ldots(ii)$$

Thus, when $f(2a-x) = f(x)$, then

(i) and (ii) $\Rightarrow \displaystyle\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

and,

when $f(2a-x) = -f(x)$, then

(i) and (ii) $\Rightarrow \displaystyle\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$

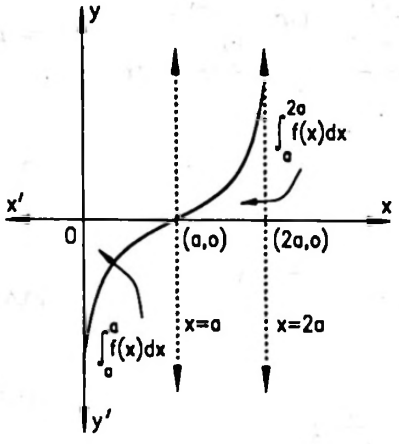


Fig. 11.11

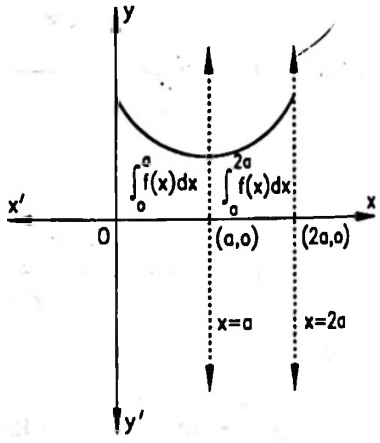


Fig. 11.12

REMARK *If $f(2a-x) = f(x)$, then the graph of $f(x)$ is symmetrical about $x = a$ so, $\displaystyle\int_0^a f(x)\,dx = \int_a^{2a} f(x)\,dx$.*

In case $f(2a-x) = -f(x)$, then the graph of $f(x)$ is as shown in Fig. 11.11.

So, $\displaystyle\int_0^a f(x)\,dx = -\int_a^{2a} f(x)\,dx$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:* $\displaystyle\int_0^{2\pi} \cos^5 x\,dx$.

SOLUTION Let $f(x) = \cos^5 x$. Then,

$$f(2\pi - x) = [\cos(2\pi - x)]^5 = \cos^5 x$$

$$\therefore \int_0^{2\pi} \cos^5 x\,dx$$

$$= 2\int_0^{\pi} \cos^5 x\,dx \qquad \text{[Using Prop. VII]}$$

Now,

$$f(\pi - x) = [\cos(\pi - x)]^5 = -\cos^5 x = -f(x)$$

$$\therefore \int_0^{\pi} \cos^5 x\,dx = 0. \qquad \text{[Using Prop. VIII]}$$

$$\text{Hence, } \int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx = 2\times 0 = 0.$$

EXAMPLE 2 *Prove that*

$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2.$$

SOLUTION Let $I = \int_0^{\pi/2} \log\sin x\,dx.$ ...(i)

Then,

$$I = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log\cos x\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx$$

$$= \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin x\cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \frac{\pi}{2}(\log 2)$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \frac{\pi}{2}\log 2. \qquad \text{...(iii)}$$

Let $I_1 = \int_0^{\pi/2} \log\sin 2x\,dx.$

Putting $2x = t$, we get

$$I_1 = \int_0^{\pi} \log\sin t\,\frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{\pi} \log\sin t\,dt$$

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log\sin t\,dt$$

$$= \int_0^{\pi/2} \log\sin x\,dx = I \qquad \text{[Using Prop. I]}$$

So, from (iii), we get

$$2I = I - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\frac{\pi}{2}\log 2$$

$$\text{Hence, } \int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2$$

EXAMPLE 3 *Evaluate* : $\int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx,\ a, b > 0.$

SOLUTION Let

$$I = \int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx \qquad \text{...(i)}$$

Then,

$$I = \int_0^{\pi} \frac{(\pi - x)}{[a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)]^2}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx$$

$$\Rightarrow 2I = 2\pi\int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx \qquad \text{[Using Prop. VIII]}$$

$$\Rightarrow I = \pi\int_0^{\pi/2} \frac{\sec^4 x}{(a^2 + b^2\tan^2 x)^2} \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^4 x\text{]}$$

$$\Rightarrow I = \pi\int_0^{\pi/2} \frac{(1+\tan^2 x)\sec^2 x}{(a^2 + b^2\tan^2 x)^2}\,dx$$

$$\Rightarrow I = \pi\int_0^{\infty} \frac{1+t^2}{(a^2+b^2t^2)^2}\,dt, \text{ where } t = \tan x$$

$$\Rightarrow I = \frac{\pi}{b^2}\int_0^{\infty} \frac{b^2+b^2t^2}{(a^2+b^2t^2)^2}\,dt$$

$$= \frac{\pi}{b^2}\int_0^{\pi} \frac{(b^2-a^2)+(a^2+b^2t^2)}{(a^2+b^2t^2)^2}\,dt$$

$$\Rightarrow I = \frac{\pi}{b^2}(b^2-a^2)\int_0^{\infty} \frac{1}{(a^2+b^2t^2)}\,dt + \frac{\pi}{b^2}\int_0^{\infty} \frac{1}{(a^2+b^2t^2)}\,dt$$

$$\Rightarrow I = \frac{\pi(b^2-a^2)}{b^2} I_1 + \frac{\pi}{ab^3}\left[\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty},$$

where $I_1 = \int_0^{\infty} \frac{1}{(a^2+b^2t^2)^2}\,dt$

$$\Rightarrow I = \frac{\pi(b^2-a^2)}{b^2} I_1 + \frac{\pi}{ab^3}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$\Rightarrow I = \pi\left(\frac{b^2-a^2}{b^2}\right) I_1 + \frac{\pi}{ab^3}\left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow I = \pi\left(\frac{b^2-a^2}{b^2}\right) I_1 + \frac{\pi^2}{2ab^3} \qquad \text{...(iii)}$$

To evaluate I_1, let us consider the integral

$$\frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \int \frac{1}{(a^2+b^2t^2)}\,dt$$

$$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \int \underset{\text{I}}{\frac{1}{(a^2+b^2t^2)}}\ \underset{\text{II}}{1}\,dt$$

$$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{t}{a^2+b^2t^2} - \int \frac{-2b^2t^2}{(a^2+b^2t^2)^2}\,dt$$

$$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{t}{a^2+b^2t^2} + 2\int \frac{(a^2+b^2t^2)-a^2}{(a^2+b^2t^2)^2}\,dt$$

$$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{t}{a^2+b^2t^2} + 2\int \frac{1}{a^2+b^2t^2}\,dt - 2a^2\int \frac{1}{(a^2+b^2t^2)^2}\,dt$$

$$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{t}{a^2+b^2t^2} + \frac{2}{ab}\tan^{-1}\left(\frac{bt}{a}\right) - 2a^2\int \frac{1}{(a^2+b^2t^2)^2}\,dt$$

$$\Rightarrow 2a^2\int \frac{1}{(a^2+b^2t^2)^2}\,dt = \frac{t}{a^2+b^2t^2} + \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right)$$

$$\Rightarrow 2a^2\int_0^{\infty} \frac{1}{(a^2+b^2t^2)^2}\,dt = \left[\frac{t}{a^2+b^2t^2} + \frac{1}{ab}\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$

$$\Rightarrow 2a^2 I_1 = \left[0 + \frac{1}{ab}\cdot\frac{\pi}{2} - 0\right] = \frac{\pi}{2ab}$$

$$\Rightarrow I_1 = \frac{\pi}{4a^3 b}$$

Substituting the value of I_1 in (iii), we get

$$I = \pi\left(\frac{b^2-a^2}{b^2}\right)\frac{\pi}{4a^3 b} + \frac{\pi^2}{2ab^3} = \frac{\pi^2}{2a^3b^3}(a^2+b^2).$$

EXAMPLE 4 *Show that :*

$$\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

[IIT 1990]

SOLUTION Let

$$I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx \qquad \text{...(i)}$$

Then,

$$I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\}\sin\left(\frac{\pi}{2} - x\right)dx \qquad \text{[Using Prop. IV]}$$

$$= \int_0^{\pi/2} f\{\sin(\pi - 2x)\}\cos x\,dx$$

$$= \int_0^{\pi/2} f\{\sin 2x\}\cos x\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)\,.\,(\sin x + \cos x)\,dx$$

$$= 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\,dx \qquad \text{[Using Prop. VIII]}$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin 2\left(\frac{\pi}{4}-x\right)\right\}\sin\left(\frac{\pi}{4}-x+\frac{\pi}{4}\right)dx \qquad \text{[Using Prop.IV]}$$

$$= 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin\left(\frac{\pi}{2}-2x\right)\right\}\sin\left(\frac{\pi}{2}-x\right)dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

$$\therefore\ I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

Hence,

$$\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx.$$

EXAMPLE 5 *Evaluate :* $\displaystyle\int_0^{\pi}\frac{a^2\sin^2 x + b^2\cos x}{a^4\sin^2 x + b^4\cos^2 x}\,dx$

SOLUTION Let

$$I = \int_0^{\pi}\frac{a^2\sin^2 x + b^2\cos x}{a^4\sin^2 x + b^4\cos^2 x}\,dx$$

Then,

$$I = \int_0^{\pi}\frac{a^2\sin^2 x}{a^4\sin^2 x + b^4\cos^2 x}\,dx + \int_0^{\pi}\frac{b^2\cos x}{a^4\sin^2 x + b^4\cos^2 x}\,dx$$

$$\Rightarrow I = 2a^2\int_0^{\pi/2}\frac{\sin^2 x}{a^4\sin^2 x + b^4\cos^2 x}\,dx + 0 \qquad \text{[Using Prop. VIII]}$$

$$\Rightarrow I = 2a^2\int_0^{\pi/2}\frac{\tan^2 x}{a^4\tan^2 x + b^4}\,dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^2 x\text{]}$$

$$\Rightarrow I = 2a^2\int_0^{\pi/2}\frac{\tan^2 x\sec^2 x}{(a^4\tan^2 x + b^4)(1+\tan^2 x)}\,dx$$

$$\Rightarrow I = 2a^2\int_0^{\infty}\frac{t^2}{(a^4t^2 + b^4)(1+t^2)}\,dt\ \text{, where } t = \tan x$$

$$\Rightarrow I = \frac{2a^2}{b^4-a^4}\int_0^{\infty}\left(\frac{b^4}{a^4t^2+b^4} - \frac{1}{1+t^2}\right)dt$$

$$\Rightarrow I = \frac{2a^2}{b^4-a^4}\left[\frac{b^4}{a^2b^2}\tan^{-1}\left(\frac{a^2t}{b^2}\right) - \tan^{-1}t\right]_0^{\infty}$$

$$\Rightarrow I = \frac{2a^2}{b^4-a^4}\left(\frac{b^2}{a^2}\frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$\Rightarrow I = \frac{2a^2}{b^4-a^4}\times\frac{\pi}{2}\left(\frac{b^2-a^2}{a^2}\right) = \frac{\pi}{b^2+a^2}$$

EXAMPLE 6 *Prove that :*

$$\int_0^{2\pi}\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx = \pi^2$$

[IIT 1996]

SOLUTION Let

$$I = \int_0^{2\pi}\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \text{...(i)}$$

Then,

$$I = \int_0^{2\pi}\frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x) + \cos^{2n}(2\pi-x)}\,dx$$

$$\Rightarrow I = \int_0^{2\pi}\frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{2\pi}\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x} + \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow 2I = \int_0^{2\pi}\frac{2\pi\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow I = \pi\int_0^{2\pi}\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow I = 2\pi\int_0^{\pi}\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \text{[Using Prop. VIII]}$$

$$\Rightarrow I = 4\pi\int_0^{\pi/2}\frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \text{...(iii)} \qquad \text{[Using Prop. VIII]}$$

$$\Rightarrow I = 4\pi\int_0^{\pi/2}\frac{\sin^{2n}(\pi/2-x)}{\sin^{2n}(\pi/2-x) + \cos^{2n}(\pi/2-x)}\,dx$$

$$\Rightarrow I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\, dx \qquad \ldots(iv)$$

Adding (iii) and (iv), we get

$$2I = 4\pi \int_0^{\pi/2} 1 \,.\, dx = 4\pi \times \frac{\pi}{2}$$

$$\Rightarrow I = \pi^2.$$

EXAMPLE 7 *If* $I(u) = \int_0^{\pi} \log_e (1 - 2u\cos x + u^2)\, dx$, *then prove that*

(i) $I(u) = I(-u)$ (ii) $I(u) + I(-u) = I(u^2)$

SOLUTION (i) We have,

$$I(u) = \int_0^{\pi} \log_e (1 - 2u\cos x + u^2)\, dx$$

$$\Rightarrow I(u) = \int_0^{\pi} \log_e (1 - 2u\cos(\pi - x) + u^2)\, dx \quad \text{[Using prop. IV]}$$

$$\Rightarrow I(u) = \int_0^{\pi} \log (1 + 2u\cos x + u^2)\, dx$$

$$\Rightarrow I(u) = I(-u)$$

(ii) We have,

$$I(u) + I(-u)$$

$$= \int_0^{\pi} \left[\log (1 - 2u\cos x + u^2) + \log (1 + 2u\cos x + u^2)\right] du$$

$$= \int_0^{\pi} \log \left\{(1 + u^2)^2 - 4u^2\cos^2 x\right\} dx$$

$$= \int_0^{\pi} \log \left\{1 + u^4 - 2u^2 (2\cos^2 x - 1)\right\} dx$$

$$= \int_0^{\pi} \log \left(1 + u^4 - 2u^2 \cos 2x\right) dx$$

$$= \frac{1}{2} \int_0^{2\pi} \log (1 + u^4 - 2u^2 \cos t)\, dt, \text{ where } 2x = t$$

$$= \frac{1}{2} \times 2 \int_0^{\pi} \log (1 + u^4 - 2u^2 \cos t)\, dt \quad \text{[Using prop. VIII]}$$

$$= \int_0^{\pi} \log (1 + u^4 - 2u^2 \cos t)\, dt = I(u^2).$$

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PROPERTY IX $\int_0^{2a} f(x)\, dx = \int_0^{a} \{f(x) + f(2a - x)\}\, dx$

PROOF We have,

$$I = \int_0^{2a} f(x)\, dx$$

$$= \int_a^{a} f(x)\, dx + \int_a^{2a} f(x)\, dx$$

$$= \int_0^{a} f(x)\, dx + I_1, \text{ where } I_1 = \int_a^{2a} f(x)\, dx$$

Let $2a - x = t$. Then, $-dx = dt \Rightarrow dx = -dt$

Also, $x = a \Rightarrow t = a$ and $x = 2a \Rightarrow t = 0$

$$\therefore I_1 = -\int_a^0 f(2a - t)\, dt$$

$$= \int_0^{a} f(2a - t)\, dt$$

$$= \int_0^{a} f(2a - x)\, dx$$

$$\therefore I = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$$

$$\Rightarrow \int_0^{2a} f(x)\, dx = \int_0^{a} \{f(x) + f(2a - x)\}\, dx.$$

PROPERTY X $\int_a^b f(x)\, dx = (b - a) \int_a^1 f\{(b - a)x + a\}\, dx$

PROOF Let $I = \int_a^b f(x)\, dx$.

Further, let $\frac{x - a}{b - a} = t$. Then, $dx = (b - a)\, dt$

Also $x = a \Rightarrow t = 0$ and $x = b \Rightarrow t = 1$.

$$\therefore I = \int_0^1 f\{(b - a)t + a\} \,.\, (b - a)\, dt$$

$$= (b-a)\int_0^1 f\{(b-a)\,t+a\}\,dt$$

$$= (b-a)\int_0^1 f\{(b-a)\,x+a\}\,dx$$

Hence, $\int_a^b f(x)\,dx = (b-a)\int_0^1 f\{(b-a)\,x+a\}\,dx.$

ILLUSTRATION *Evaluate :*

$$\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx.$$

SOLUTION Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2}\,dx +$ and $I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$

Using Property X, we have

$$I_1 = \int_{-4}^{-5} e^{(x+5)^2}\,dx$$

$$= (-5+4)\int_0^1 e^{\{(-5+4)x-4+5\}^2}\,dx$$

$$= -\int_0^1 e^{(-x+1)^2}\,dx \quad = -\int_0^1 e^{(x-1)^2}\,dx$$

and,

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right)\int_0^1 e^{9\{(2/3-1/3)x+1/3-2/3\}^2}dx$$

$$= \frac{1}{3}\int_0^1 e^{9(x/3-1/3)^2}\,dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2}\,dx$$

$$\therefore\ I = I_1 + 3I_2 = -\int_0^1 e^{(x-1)^2}\,dx + \int_0^1 e^{(x-1)^2}\,dx = 0$$

PROPERTY XI *If $f(x)$ is a periodic function with period T, then*

(i) $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in Z$

(ii) $\int_{,a}^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in Z, a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,\ m, n \in Z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx,\ n \in Z, a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx,\ n \in Z, a, b \in R.$

PROOF (i) We have,

$$\int_0^{nT} f(x)\,dx$$

$$= \sum_{r=1}^{n}\int_{(r-1)T}^{rT} f(x)\,dx$$

$$= \sum_{r=1}^{n}\int_0^T f\{(r-1)\,T+y\}\,dy,\ \text{where } x=(r-1)\,T+y$$

$$= \sum_{r=1}^{n}\int_0^T f(y)\,dy \qquad \left[\because f(x) \text{ is periodic with period } T \Rightarrow \therefore f(y+(r-1)\,T) = f(y)\right]$$

$$= n\int_0^T f(y)\,dy.$$

Similarly, we can prove other parts.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that :* $\int_0^{400\pi} \sqrt{1-\cos 2x}\,dx = 800\sqrt{2}$

SOLUTION We have, $\int_0^{400\pi} \sqrt{1-\cos 2x}$

$$= \int_0^{400\pi} \sqrt{2}\,|\sin x|\,dx$$

$$= \sqrt{2}\times 400\int_0^{\pi} |\sin x|\,dx \quad [\because |\sin x| \text{ is periodic with period } \pi]$$

$$= 400\sqrt{2}\int_0^{\pi} \sin x\,dx = 400\sqrt{2}\,[-\cos x]_0^{\pi} = 800\sqrt{2}.$$

EXAMPLE 2 *Prove that* : $\int_0^{10} (x - [x])\,dx = 5$

SOLUTION Since $x - [x]$ is a periodic function with period one unit. Therefore,

$$\int_0^{10} (x - [x])\,dx$$

$$= 10\int_0^{1} (x - [x])\,dx$$

$$= 10\left[\int_0^1 x\,dx - \int_0^1 [x]\,dx\right]$$

$$= 10\left[\left[\frac{x^2}{2}\right]_0^1 - 0\right] = \frac{10}{2} = 5$$

EXAMPLE 3 *Prove that* : $\int_0^{100} e^{x-[x]}\,dx = 100\,(e-1).$

SOLUTION Since $x - [x]$ is a periodic function with period one unit, therefore so is $e^{x-[x]}$, and hence

$$\int_0^{100} e^{x-[x]}\,dx$$

$$= 100\int_0^1 e^{x-[x]}\,dx$$

$$= 100\int_0^1 e^{x-0}\,dx$$

$$= 100\int_0^1 e^x\,dx = 100\,(e-1).$$

EXAMPLE 4 *Evaluate* : $\int_0^{32\pi/3} \sqrt{1+\cos 2x}\,dx$

SOLUTION We have,

$$\int_0^{32\pi/3} \sqrt{1+\cos 2x}\,dx$$

$$= \sqrt{2}\int_0^{10\pi} |\cos x|\,dx + \sqrt{2}\int_{10\pi}^{32\pi/3} |\cos x|\,dx$$

$$= 10\sqrt{2}\int_0^{\pi} |\cos x|\,dx + \sqrt{2}\int_0^{2\pi/3} |\cos x|\,dx$$

$$= 10\sqrt{2}\left[\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx\right] + \sqrt{2}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{2\pi/3} -\cos x\,dx\right]$$

$$= 10\sqrt{2}\,[1+1] + \sqrt{2}\left[1 - \frac{\sqrt{3}}{2} + 1\right]$$

$$= 20\sqrt{2} + \sqrt{2}\left(2 - \frac{\sqrt{3}}{2}\right) = 22\sqrt{2} - \sqrt{\frac{3}{2}}$$

EXAMPLE 5 *Show that* :

$$\int_0^{n\pi+V} |\sin x|\,dx = (2n+1) - \cos V$$

[IIT 1994]

where n is a positive integer and $0 \le V < \pi$.

SOLUTION We have,

$$\int_0^{n\pi+V} |\sin x|\,dx$$

$$= \int_0^{V} |\sin x|\,dx + \int_{V}^{n\pi+V} |\sin x|\,dx$$

$$= \int_0^{V} \sin x\,dx + n\int_0^{\pi} |\sin x|\,dx \qquad \left[\because |\sin x| \text{ is periodic with period } \pi\right]$$

$$= \int_0^{V} \sin x\,dx + n\int_0^{\pi} \sin x\,dx$$

$$= \left[-\cos x\right]_0^{V} + n\left[-\cos x\right]_0^{\pi}$$

$$= (-\cos V + 1) + n\,(1+1)$$

$$= (2n+1) - \cos V.$$

EXAMPLE 6 *Evaluate* : $\int_0^{n\pi} \left|\frac{\sin 2x}{2+\sin 4x}\right| dx,\ n \in N.$

SOLUTION Since $\sin 2x$ and $2 + \sin 4x$ are periodic functions with periods π and $\frac{\pi}{2}$ respectively. Therefore, $\frac{\sin 2x}{2+\sin 4x}$ is periodic with period π.

$$\therefore \int_0^{n\pi} \left|\frac{\sin 2x}{2+\sin 4x}\right| dx$$

$$= n\int_0^{\pi}\left|\frac{\sin 2x}{2+\sin 4x}\right| dx$$

$$= \frac{n}{2}\int_0^{2\pi}\left|\frac{\sin\theta}{2+\sin 2\theta}\right| d\theta \text{, where } \theta = 2x$$

$$= \frac{n}{2}\int_0^{2\pi}\frac{|\sin\theta|}{2+\sin 2\theta}\,d\theta, \qquad \left[\because\ 2+\sin 2\theta > 0 \text{ for all } \theta\right]$$

$$= \frac{n}{2}\left[\int_0^{\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta - \int_{\pi}^{2\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta\right]$$

$$= \frac{n}{2}\left[\int_0^{\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta - \int_0^{\pi}\frac{\sin(\pi+t)}{2+\sin 2(\pi+t)}\,dt\right]$$

[Putting $\theta - \pi = t$ in the second integral]

$$= \frac{n}{2}\left[\int_0^{\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta + \int_0^{\pi}\frac{\sin t}{2+\sin 2t}\,dt\right]$$

$$= \frac{n}{2}\times 2\int_0^{\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta$$

$$= n\int_0^{\pi}\frac{\sin\theta}{2+\sin 2\theta}\,d\theta$$

$$= \frac{n}{2}\int_0^{\pi}\frac{2\sin\theta}{2+\sin 2\theta}\,d\theta$$

$$= \frac{n}{2}\int_0^{\pi}\frac{(\sin\theta+\cos\theta)+(\sin\theta-\cos\theta)}{2+\sin 2\theta}\,d\theta$$

$$= \frac{n}{2}\int_0^{\pi}\frac{\sin\theta+\cos\theta}{2+\sin 2\theta}\,d\theta + \frac{n}{2}\int_0^{\pi}\frac{\sin\theta-\cos\theta}{2+\sin 2\theta}\,d\theta$$

$$= \frac{n}{2}\int_0^{\pi}\frac{d(\sin\theta-\cos\theta)}{(\sqrt{3})^2-(\sin\theta-\cos\theta)^2} - \frac{n}{2}\int_0^{\pi}\frac{d(\sin\theta+\cos\theta)}{1^2+(\sin\theta+\cos\theta)^2}$$

$$= \frac{n}{2}\times\frac{1}{2\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+\sin\theta-\cos\theta}{\sqrt{3}-\sin\theta+\cos\theta}\right)\right]_0^{\pi} - \frac{n}{2}\left[\tan^{-1}(\sin\theta+\cos\theta)\right]_0^{\pi}$$

$$= \frac{n}{4\sqrt{3}}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 - \frac{n}{2}\left\{\tan^{-1}(-1)-\tan^{-1}1\right\}$$

$$= \frac{n}{2\sqrt{3}}\log(2+\sqrt{3}) + \frac{n\pi}{4}$$

EXAMPLE 7 *Prove that* $\int_0^{2n\pi}[\sin x+\cos x]\,dx$, *where* $n \in N$ *and* $[\cdot]$ *denote, the greatest integral function.*

SOLUTION Since $f(x) = \sin x + \cos x$ is a periodic function with period 2π. Therefore,

$$\int_0^{2n\pi}[\sin x+\cos x]\,dx$$

$$= n\int_0^{2\pi}[\sin x+\cos x]\,dx$$

$$= n(-\pi) \qquad \text{[See Ex 9 on page 11.35]}$$

$$= -n\pi$$

EXAMPLE 8 *Evaluate :* $\int_{-\pi/4}^{n\pi-\pi/4}|\sin x+\cos x|\,dx,\ n \in N$

SOLUTION We know that, if $f(x)$ is a peridic function with period T, then

$$\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx, \text{ where } n \in N \text{ and } a \in R.$$

Since $|\sin x+\cos x|$ is a periodic function with period π. Therefore,

$$\int_{-\pi/4}^{n\pi-\pi/4}|\sin x+\cos x|\,dx$$

$$= n\int_0^{\pi}|\sin x+\cos x|\,dx$$

$$= \sqrt{2}\,n\int_0^{\pi}\left|\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right| dx$$

$$= \sqrt{2}\,n\int_0^{\pi}\left|\sin\left(x+\frac{\pi}{4}\right)\right| dx$$

$$= \sqrt{2}\,n\left[\int_0^{3\pi/4}|\sin(x+\pi/4)|\,dx + \int_{3\pi/4}^{\pi}|\sin(x+\pi/4)|\,dx\right]$$

$$= \sqrt{2}\,n\left[\int_0^{3\pi/4}\sin(x+\pi/4)\,dx - \int_{3\pi/4}^{\pi}\sin(x+\pi/4)\,dx\right]$$

$$= \sqrt{2}\,n\left[-\left[\cos\left(x+\frac{\pi}{4}\right)\right]_0^{3\pi/4} + \left[\cos\left(x+\frac{\pi}{4}\right)\right]_{3\pi/4}^{\pi}\right]$$

$$= \sqrt{2}\,n\left[-\cos\pi+\cos\frac{\pi}{4}+\cos\frac{5\pi}{4}-\cos\pi\right]$$

$$= 2\sqrt{2}n.$$

ALITER Let $x+\frac{\pi}{4} = t$. Then, $dx = dt$

$$\therefore \int_{-\pi/4}^{n\pi-\pi/4} |\sin x+\cos x|\,dx$$

$$= \int_0^{n\pi} \left|\sin\left(t-\frac{\pi}{4}\right)+\cos\left(t-\frac{\pi}{4}\right)\right| dt$$

$$= \int_0^{n\pi} \sqrt{2}\,|\sin t|\,dt = \sqrt{2}\int_0^{n\pi} |\sin t|\,dt$$

$$= \sqrt{2}\,n\int_0^{\pi} |\sin t|\,dt = \sqrt{2}\,n\left(\int_0^{\pi} \sin t\,dt\right) = 2\sqrt{2}\,n.$$

EXAMPLE 9 *Prove that :* $\int_{\pi}^{5\pi/4} \frac{\sin 2x}{\cos^4 x+\sin^4 x}\,dx = \frac{\pi}{4}$.

SOLUTION Let $f(x) = \frac{\sin 2x}{\cos^4 x+\sin^4 x}$.

Clearly, $f(x+\pi) = f(x)$ for all x. So, $f(x)$ is a periodic function with period π.

$$\therefore \int_{\pi}^{5\pi/4} f(x)\,dx = \int_{\pi}^{\pi+\pi/4} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx$$

$$= \int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x+\sin^4 x}\,dx = 2\int_0^{\pi/4} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$$

$$= 2\int_0^{\pi/4} \frac{\tan x\sec^2 x}{1+\tan^4 x}\,dx = \int_0^{\pi/4} \frac{d(\tan^2 x)}{1^2+(\tan^2 x)^2}$$

$$= \left[\tan^{-1}(\tan^2 x)\right]_0^{\pi/4} = \tan^{-1}(1)-\tan^{-1}(0) = \frac{\pi}{4}-0 = \frac{\pi}{4}$$

EXAMPLE 10 *If f is an even function, then prove that*

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx \qquad \text{[IIT 2003]}$$

SOLUTION We have,

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx$$

$$= \int_0^{\pi/4} \left\{f(\cos 2x)\cos x + f\Big(\cos 2(\pi/2-x)\Big)\cos(\pi/2-x)\right\} dx$$

[Using Prop.IX]

$$= \int_0^{\pi/4} \left|f(\cos 2x)\cos x + f(-\cos 2x)\sin x\right| dx$$

$$= \int_0^{\pi/4} f(\cos 2x)(\cos x+\sin x)\,dx \qquad \left[\because f(-\cos 2x) = f(\cos 2x) \text{ as } f \text{ is even}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos\left(\frac{\pi}{4}-x\right)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f\Big(\cos 2(\pi/4-x)\Big)\cos\Big(\pi/4-(\pi/4-x)\Big)dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$

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PROPERTY XII *Leibnitz's rule for the differentiation under the integral sign :*

If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$, and $f(x, t)$ is continuous

(i) $\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right]$

$$= \int_{\phi(x)}^{\psi} \frac{\partial}{\partial x} f(x,t)\,dt + \frac{d\psi(x)}{dx} f(x,\psi(x)) - \frac{d}{dx}(\phi(x) f(x,\phi(x)))$$

If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$, and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

(ii) $\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right)$

$$= \frac{d}{dx}\{\psi(x)\} f(\psi(x)) - \frac{d}{dx}\{\phi(x)\} f(\phi(x))$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* $y = \int_0^x f(t)\sin\{k(x-t)\}\,dt$, *then prove that*

$$\frac{d^2y}{dx^2} + k^2 y = k f(x).$$

SOLUTION We have,

$$y = \int_0^x f(t)\sin\{k(x-t)\}\,dt$$

Differentiating both sides w.r.t. x, we obtain

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}\{f(t)\sin k(x-t)\}\,dt + \frac{d(x)}{dx}\{f(x)\sin k(x-x)\}$$

$$- \frac{d}{dx}(0)\{f(0)\sin k(x-0)\}$$

$$\Rightarrow \frac{dy}{dx} = k\int_0^x f(t)\cos k(x-t)\,dt + f(x)\times 0 - 0$$

$$\Rightarrow \frac{dy}{dx} = k\int_0^x f(t)\cos k(x-t)\,dt$$

Differentiating both sides w.r. to x, we get

$$\frac{d^2y}{dx^2} = k\int_0^x \frac{\partial}{\partial x}\{f(t)\cos k(x-t)\}\,dt + \frac{d}{dx}(x)\{f(x)\cos k(x-x)\}$$

$$- \frac{d}{dx}(0)\{f(0)\cos k(x-0)\}\Big]$$

$$\Rightarrow \frac{d^2y}{dx^2} = k\left[-k\int_0^x f(t)\sin k(x-t)\,dt + f(x) - 0\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -k^2 y + kf(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + k^2 y = kf(x).$$

EXAMPLE 2 *Find* $\dfrac{d}{dx}\left(\displaystyle\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right)$.

SOLUTION We have,

$$\frac{d}{dx}\left\{\int_{x^2}^{x^3} \frac{1}{\log t}\,dt\right\}$$

$$= \frac{d}{dx}(x^3)\cdot\left(\frac{1}{\log x^3}\right) - \frac{d}{dx}(x^2)\cdot\left(\frac{1}{\log x^2}\right)$$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{1}{\log x}(x^2 - x)$$

EXAMPLE 3 *Prove that :*

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt = \frac{\pi}{4} \text{ for all } x.$$

SOLUTION Let $\phi(x) = \displaystyle\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$. Then,

$$\frac{d\phi}{dx} = \int_0^{\sin^2 x} 0\,dt + \frac{d}{dx}(\sin^2 x)\cdot(\sin^{-1}\sqrt{\sin^2 x}) - 0$$

$$+ \int_0^{\cos^2 x} 0\,dt + \frac{d}{dx}(\cos^2 x)\cdot(\cos^{-1}\sqrt{\cos^2 x}) - 0 = 0, \text{ for all } x.$$

$\therefore$ $\phi(x)$ = constant for all x

Let $\phi(x) = k$, for all x ...(i)

$$\Rightarrow \phi\left(\frac{\pi}{4}\right) = k$$

$$\Rightarrow \int_0^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\,dt = k$$

$$\Rightarrow \int_0^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt = k$$

$$\Rightarrow \int_0^{1/2} \frac{\pi}{2}\,dt = k$$

$$\Rightarrow k = \frac{\pi}{4}$$

Putting $k = \pi/4$ in (i), we get

$$\phi(x) = \pi/4 \text{ for all } x.$$

EXAMPLE 4 *If* $\displaystyle\int_x^{2x} \sqrt{5-3\sin^2 t}\,dt + \int_0^y \sin t\,dt = 0$, *find* $\dfrac{dy}{dx}$.

SOLUTION We have,

$$\int_x^{2x} \sqrt{(5-3\sin^2 t}\,dt + \int_0^y \sin t\,dt = 0$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left\{\int_x^{2x} \sqrt{5-3\sin^2 t}\,dt\right\} + \frac{d}{dx}\left\{\int_0^y \sin t\,dt\right\} = 0$$

$$\Rightarrow \frac{d(2x)}{dx}\cdot\sqrt{5-3\sin^2 2x} - \frac{d}{dx}(x)\cdot\sqrt{5-3\sin^2 x}$$

$$+ \frac{dy}{dx}\cdot\sin y - 0 = 0$$

$$\Rightarrow 2(5-3\sin^2 2x) - \sqrt{5-3\sin^2 x} + \sin y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{5-3\sin^2 x} - 2\sqrt{5-3\sin^2 2x}}{\sin y}$$

EXAMPLE 5 Find $\dfrac{d}{dx}\left\{\displaystyle\int_{\sqrt{x}}^{x^2+x} \frac{1}{2+\sqrt{t}}\,dt\right\}$

SOLUTION We have,

$$\frac{d}{dx}\left\{\int_{\sqrt{x}}^{x^2+x}\frac{1}{2+\sqrt{t}}\,dt\right\}$$

$$=\frac{d}{dx}(x^2+x)\left\{\frac{1}{2+\sqrt{x^2+x}}\right\}-\frac{d}{dx}(\sqrt{x})\left\{\frac{1}{2+\sqrt{\sqrt{x}}}\right\}$$

$$=\frac{2x+1}{2+\sqrt{x^2+x}}-\frac{1}{2\sqrt{x}\,(2+\sqrt{\sqrt{x}})}$$

$$=\frac{2x+1}{2+\sqrt{x^2+x}}-\frac{1}{4\sqrt{x}+2\sqrt{x}\sqrt{x}}$$

EXAMPLE 6 *Variables x and y are related by the equation*

$$x=\int_0^y\frac{1}{\sqrt{1+9u^2}}\,du$$

Show that $\frac{d^2y}{dx^2}$ is proportional to y.

SOLUTION We have,

$$x=\int_0^y\frac{1}{\sqrt{1+9u^2}}\,du.$$

$$\therefore\quad\frac{dx}{dy}=\frac{dy}{dy}\left\{\frac{1}{\sqrt{1+9y^2}}\right\}$$

$$\Rightarrow\quad\frac{dx}{dy}=\frac{1}{\sqrt{1+9y^2}}$$

$$\Rightarrow\quad\frac{dy}{dx}=\sqrt{1+9y^2}$$

$$\Rightarrow\quad\frac{d^2y}{dx^2}=\frac{9y}{\sqrt{1+9y^2}}\frac{dy}{dx}$$

$$\Rightarrow\quad\frac{d^2y}{dx^2}=\frac{9y}{\sqrt{1+9y^2}}\times\sqrt{1+9y^2}=9y$$

Hence, $\frac{d^2y}{dx^2}\propto y$.

EXAMPLE 7 *Find the points of maxima/minima of the function $f(x)$ given by*

$$f(x)=\int_0^{x^2}\frac{t^2-5t+4}{2+e^t}\,dt.$$

SOLUTION We have,

$$f(x)=\int_0^{x^2}\frac{t^2-5t+4}{2+e^t}\,dt$$

$$\Rightarrow f'(x)=\frac{d}{dx}(x^2)\left\{\frac{x^4-5x^2+4}{2+e^{x^2}}\right\}-0$$

$$\Rightarrow f'(x)=\frac{2x(x^4-5x^2+4)}{2+e^{x^2}}$$

$$\Rightarrow f'(x)=\frac{2x(x-1)(x+1)(x-2)(x+2)}{2+e^{x^2}}$$

Now, $f'(x)=0\Rightarrow x=0,1,-1,2,-2$.

The changes in signs of $f'(x)$ for different values of x are shown in Fig. 11.13

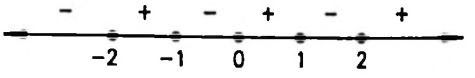


Fig. 11.13

Clearly, $f'(x)$ changes its sign from positive to negative in the neighbourhoods of the points $x=-1,1$. So, $x=1$ and '$x=-1$ are points of local maximum.

Also, $f'(x)$ changes its sign from negative to positive in the neighbourhood of $x=-2,0$ and 2. So, $x=-2,0,2$ are the points of local minimum.

EXAMPLE 8 *Find the equation of the tangent to the curve $y=f(x)$ at $x=1$, where*

$$f(x)=\int_{x^2}^{x^3}\frac{1}{\sqrt{1+t^5}}\,dt$$

SOLUTION We have,

$$y=f(x)=\int_{x^2}^{x^3}\frac{1}{\sqrt{1+t^5}}\,dt$$

$$\therefore\quad\frac{dy}{dx}=\frac{d}{dx}(x^3)\cdot\frac{1}{\sqrt{1+(x^3)^5}}-\frac{d}{dx}(x^2)\cdot\frac{1}{\sqrt{1+(x^2)^5}}$$

$$\Rightarrow\quad\frac{dy}{dx}=\frac{3x^2}{\sqrt{1+x^{15}}}-\frac{2x}{\sqrt{1+x^{10}}}$$

$$\Rightarrow\quad\left(\frac{dy}{dx}\right)_{x=1}=\frac{3}{\sqrt{2}}-\frac{2}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

Clearly, $y=0$ for $x=1$.

So, the equatin of the tangent at $(1,0)$ is

$$y-0=\frac{1}{\sqrt{2}}(x-1)\text{ or, }x-\sqrt{2}\,y-1=0$$

EXAMPLE 9 *If $\phi(x)=\cos x-\int_0^x(x-t)\,\phi(t)\,dt$, then prove that*

$$\phi''(x)+\phi(x)=-\cos x.$$

SOLUTION We have,

$$\phi(x) = \cos x - \int_0^x (x-t)\,\phi(t)\,dt.$$

By using Lebniz's rule, we get

$$\phi'(x) = -\sin x - \left[\int_0^x \phi(t)\,dt + 1\,.\,(x-x)\,\phi(x) - 0\right]$$

$$= -\sin x - \int_0^x \phi(t)\,dt$$

Differentiating both sides w.r.to x, we get

$$\phi''(x) = -\cos x - \phi(x)$$

$$\Rightarrow \phi''(x) + \phi(x) = -\cos x.$$

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PROPERTY XIII *Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha)\,dx$, then*

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx.$

SOLUTION Let $I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx.$

Then,

$$\frac{dI(\alpha)}{d\alpha} = \int_0^1 \frac{x^\alpha \,.\, \log x}{\log x}\,dx$$

$$\Rightarrow \frac{dI(\alpha)}{d\alpha} = \int_0^1 x^\alpha\,dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

$$\therefore I(\alpha) = \log(\alpha+1) + C \qquad \ldots(i)$$

If $\alpha = 0$, then $I(\alpha) = \int_0^1 \frac{x^0 - 1}{\log x}\,dx = 0.$

Putting $\alpha = 0, I = 0$ in (i), we get

$$\therefore 0 = \log(0+1) + C \Rightarrow C = 0.$$

Hence, $I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx = \log(\alpha+1)$

EXAMPLE 2 *Prove that :*

$$\int_0^{\pi/2} \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

SOLUTION Let $I = \int_0^{\pi/2} \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\,dx.$ Then,

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{a^2\tan^2 x + b^2}\,dx$$

$$\Rightarrow I = \int_0^\infty \frac{dt}{a^2t^2 + b^2}, \text{ where } t = \tan x$$

$$\Rightarrow I = \frac{1}{ab}\left[\tan^{-1}\left(\frac{at}{b}\right)\right]_0^\infty = \frac{1}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2ab}.$$

Thus, $\int_0^{\pi/2} \frac{1}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}$ ...(i)

Differentiating both sides with respect to a, we get

$$\int_0^{\pi/2} \frac{-2a\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{-\pi}{2a^2b}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3b} \qquad \ldots(ii)$$

Similarly, by differentiating (i) with respect to b, we get

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab^3} \qquad \ldots(iii)$$

Adding (ii) and (iii), we obtain

$$\int_0^{\pi/2} \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3} = \frac{\pi}{4a^3b^3}(a^2+b^2)$$

EXAMPLE 3 *Evaluate :* $\int_0^1 \frac{x^a - x^b}{\log x}\,dx.$

SOLUTION Let $I(a) = \int_0^1 \frac{x^a - x^b}{\log x}\,dx$ ...(i)

Differentiating w.r.to a, we get

$$\frac{dI}{da} = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - x^b}{\log x}\right)dx$$

$$= \int_0^1 \frac{x^a \log x - 0}{\log x}\,dx$$

$$= \int_1^0 x^a\,dx$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_1^0 = \frac{1}{a+1}$$

On integrating both sides w.r. to a, we get

$\therefore\ I = \log(a+1) + C$ ...(ii)

From (i), we have

$$I(b) = \int_0^1 \frac{x^b - x^b}{\log x}\,dx = 0$$

From (ii), we have

$I(b) = \log(b+1) + C$

$\Rightarrow\quad 0 = \log(b+1) + C$

$\Rightarrow\quad C = -\log(b+1)$

Substituting the vlaue of C in (ii), we get

$$I = \log(a+1) - \log(b+1) = \log\left(\frac{a+1}{b+1}\right)$$

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PROPERTY XIV *If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)\,dx$ is independent of* a.

PROOF Let $\phi(a) = \int_a^{a+T} f(x)\,dx$. Then,

$$\frac{d\phi(a)}{da} = \frac{d}{da}(a+T).f(a+T) - \frac{d(a)}{da}f(a)$$

$$= f(a+T) - f(a)$$

$$= 0 \quad [\because f \text{ is periodic with period } T \therefore f(a+T) = f(a)]$$

Thus, $\frac{d(\phi(a))}{d} = 0 \Rightarrow \phi(a)$ is independent of a.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that the value of $\int_a^{a+\pi/2} (\sin^4 x + \cos^4 x)\,dx$ is independent of a.*

SOLUTION Since $(\sin^4 x + \cos^4 x)$ is a periodic function with period $\pi/2$. So, $\int_a^{a+\pi/2} (\sin^4 x + \cos^4 x)\,dx$ is independent of a.

EXAMPLE 2 *If $f(x)$ is a function satisfying $f(x+a) + f(x) = 0$ for all $x \in R$ and constant a such that $\int_b^{c+b} f(x)\,dx$ is independent of b, then find the least positive value of c.*

SOLUTION We have,

$f(x+a) + f(x) = 0$ for all $x \in R$ ...(i)

$\Rightarrow f(x+a+a) + f(x+a) = 0$ [Replacing x by $x+a$]

$\Rightarrow f(x+2a) + f(x+a) = 0$ ...(ii)

Subtracting (i) from (ii), we get

$f(x+2a) - f(x) = 0$ for all $x \in R$

$\Rightarrow f(x+2a) = f(x)$ for all $x \in R$

So, $f(x)$ is a periodic function with period $2a$.

It is given that $\int_b^{c+b} f(x)\,dx$ is independent of b.

Therefore, the minimum value of c is equal to the period of $f(x)$ i.e. $2a$.

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PROPERTY XV *If $f(x) \geq 0$ on the interval $[a, b]$, then* $\int_a^b f(x)\,dx \geq 0$

PROPERTY XVI *If $f(x) \leq g(x)$ on $[a, b]$, then* $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

PROPERTY XVII *If m and M are the smallest and greatest values of a function $f(x)$ defined on an interval $[a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

PROOF We have,

$$m \leq f(x) \leq M \text{ for all } x \in [a, b]$$

$$\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

PROPERTY XIII *If $f(x)$ is defined on $[a, b]$, then*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

PROPERTY XIX If $f^2(x)$ and $g^2(x)$ are integrable on [a, b], then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leq \left\{\int_a^b f^2(x)\,dx\right\}^{1/2} \left\{\int_a^b g^2(x)\,dx\right\}^{1/2}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that* : $\frac{\pi}{4} < \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \leq 1$.

SOLUTION Since $1 + x^4 < (1 + x^2)^2$. Therefore,

$$\sqrt{1+x^4} < 1 + x^2$$

$$\Rightarrow \frac{1}{\sqrt{1+x^4}} > \frac{1}{1+x^2}$$

Also, $\frac{1}{\sqrt{1+x^4}} \leq 1$.

Thus, $\frac{1}{1+x^2} < \frac{1}{\sqrt{1+x^4}} \leq 1$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx < \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \leq \int_0^1 1\,dx$$

$$\Rightarrow \left[\tan^{-1}x\right]_0^1 < \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \leq 1$$

$$\Rightarrow \frac{\pi}{4} < \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \leq 1.$$

EXAMPLE 2 *Prove that* : $1 \leq \int_0^1 e^{x^2}\,dx \leq e$.

SOLUTION For $0 \leq x \leq 1$, we have

$$e^0 \leq e^{x^2} \leq e^1$$

$$\therefore e^0(1-0) \leq \int_0^1 e^{x^2}\,dx \leq e(1-0)$$

$$\Rightarrow 1 \leq \int_0^1 e^{x^2}\,dx \leq e.$$

EXAMPLE 3 *Prove that :*

$$\frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,dx < \frac{\pi}{4\sqrt{2}}$$

SOLUTION For any $x \in [0, 1]$, we have

$$\Rightarrow 4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$$

$$\Rightarrow \sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2}$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}} < \frac{1}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{4-2x^2}}$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,dx < \int_0^1 \frac{1}{\sqrt{4-2x^2}}\,dx$$

$$\Rightarrow \left[\sin^{-1}\frac{x}{2}\right]_0^1 < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,dx < \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$$

$$\Rightarrow \sin^{-1}\frac{1}{2} - 0 < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,dx < \frac{1}{\sqrt{2}}\left\{\sin^{-1}\frac{1}{\sqrt{2}} - 0\right\}$$

$$\Rightarrow \frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,dx < \frac{\pi}{4\sqrt{2}}$$

EXAMPLE 4 *Prove that :*

$$\log 2 < \int_0^1 \frac{1}{\sqrt{1+x^6}}\,dx < \frac{\pi}{2}$$

SOLUTION For all $x \in [0, 1]$, we have

$$1-x^2 < 1+x^6 < (1+x)^2$$

$$\Rightarrow \quad \sqrt{1-x^2} < \sqrt{1+x^6} < \sqrt{(1+x)^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{(1+x)^2}} < \frac{1}{\sqrt{1+x^6}} < \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \int_0^1 \frac{1}{1+x}\,dx < \int_0^1 \frac{1}{\sqrt{1+x^6}}\,dx < \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad \left[\log(1+x)\right]_0^1 < \int_0^1 \frac{1}{\sqrt{1+x^6}}\,dx < \left[\sin^{-1}x\right]_0^1$$

$$\Rightarrow \log 2 < \int_0^1 \frac{1}{\sqrt{1+x^6}}\,dx < \sin^{-1}1$$

$$\Rightarrow \log 2 < \int_0^1 \frac{1}{\sqrt{1+x^6}}\,dx < \frac{\pi}{2}$$

EXAMPLE 5 *Prove that :*

$$1 < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2}.$$

SOLUTION Let $f(x) = \dfrac{\sin x}{x}$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$

$$\therefore \quad f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{g(x)}{x^2}, \quad \text{where } g(x) = x\cos x - \sin x.$$

Now, $g'(x) = -x\sin x < 0$ for all $x \in (0, \pi/2)$

$\Rightarrow$ $g(x)$ is decreasing on $(0, \pi/2)$

$\Rightarrow$ $g(x) < g(0)$ for all $x \in (0, \pi/2)$

$\Rightarrow$ $g(x) < 0$ for all $x \in (0, \pi/2)$

$\Rightarrow$ $f'(x) < 0$ for all $x \in (0, \pi/2)$

$\Rightarrow$ $f(x)$ is decreasing on $(0, \pi/2)$

$\Rightarrow \lim\limits_{x \to \pi/2^-} f(x) < f(x) < \lim\limits_{x \to 0^+} f(x)$ for all $x \in (0, \pi/2)$

$$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \text{for all } x \in (0, \pi/2)$$

$$\Rightarrow \int_0^{\pi/2} \frac{2}{\pi}\,dx < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \int_0^{\pi/2} 1\,.\,dx \quad \text{for all } x \in (0, \pi/2)$$

$$\Rightarrow 1 < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2} \quad \text{for all } x \in (0, \pi/2)$$

AN IMPORTANT NOTE

Consider the integral $I = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$. We observe that

$$f(x) = \frac{1}{1+x^2} \text{ for all } x \in (-1, 1).$$

$$\therefore \quad I = \int_{-1}^{1} f(x)\,dx > 0$$

However, if we make the substitution $x = \dfrac{1}{t}$, we get

$$I = -\int_{-1}^{1} \frac{1}{1+t^2}\,dt \Rightarrow I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

This is obviously wrong due to the reason that the substitution $x = \dfrac{1}{t}$ is not valid, because the function $x = \dfrac{1}{t}$ is discontinuous at $x = 0 \in (-1, 1)$.

Similarly, if we substitute $\tan\dfrac{x}{2} = t$ in the integeral $I = \int_0^{2\pi} \frac{1}{5-2\cos x}\,dx$, we get $I = \int_0^0 \frac{2dt}{3+7t^2} = 0$. But $\frac{1}{5-2\cos x} > 0$ for all $x \in (0, 2\pi)$. Therefore, $I > 0$. We are obtaining the wrong result by using the substitution $\tan\dfrac{x}{2}$, because $\tan\dfrac{x}{2}$ is not continuous at $x = \pi \in (0, 2\pi)$.

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11.5 INTEGRAL FUNCTION AND ITS PROPERTIES

Let $f(x)$ be a continuous function defined on $[a, b]$, then a function $\phi(x)$ defined by

$$\phi(x) = \int_a^x f(t)\,dt \quad \text{for all } x \in [a, b]$$

is called the integral function of the function $f(x)$.

The integral function has the following properties :

PROPERTY I *The integral function of an integrable function is always continuous.*

PROPERTY II *If $\phi(x)$ is the integral function of a continuous function $f(x)$ defined on $[a, b]$, then $\phi(x)$ is differentiable on (a, b) and $\phi'(x) = f(x)$ for all $x \in (a, b)$.*

NOTE *Determining the primitive of a function or indefinite integration is the process inverse to differentiation whereas determining the integral function of a given function is the limit of sum of an infinite number of small terms. A function may have a primitive but not an integral function and vice-versa.*

PROPERTY III *The integral function of an odd function is an even function*

If $f(x)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

PROOF We have,

$$\phi(-x) = \int_a^{-x} f(t)\,dt$$

$$\Rightarrow \phi(-x) = \int_a^{-a} f(t)\,dt + \int_{-a}^{-x} f(t)\,dt$$

$$\Rightarrow \phi(-x) = -\int_a^{-a} f(t)\,dt - \int_a^x f(-y)\,dy, \text{ where } t = -y$$

$$\Rightarrow \phi(-x) = 0 - \int_a^x f(-y)\,dy \qquad \left[\because f(t) \text{is an odd function} \;\therefore \int_{-a}^{a} f(t)\,dt = 0\right]$$

$$\Rightarrow \phi(-x) = \int_a^x f(y)\,dy \quad [\because f \text{ is an odd function } \therefore f(-y) = -f(y)]$$

$$\Rightarrow \phi(-x) = \int_a^x f(t)\,dt$$

$$\Rightarrow \phi(-x) = \phi(x)$$

Hence, $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

PROPERTY IV *If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.*

PROOF We have,

$$\phi(-x) = \int_0^{-x} f(t)\,dt$$

$$= -\int_0^x f(-y)\,dy \text{ , where } t = -y$$

$$= -\int_0^x f(y)\,dy \quad [\because f \text{ is an evern function } \therefore f(-y) = f(y)]$$

$$= -\int_0^x f(t)\,dt \quad [\because \text{ Integ. is indep. of change of variable}]$$

$$= -\phi(x).$$

Hence, $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

REMARK *If $f(t)$ is an even function, then for non-zero a, $\int_a^x f(t)\,dt$ is not necessarily an odd function. It will be an odd function if $\int_0^a f(t)\,dt = 0$, because*

$$\phi(x) = \int_a^x f(t)\,dt \text{ is an odd function}$$

$$\Rightarrow \phi(-x) = -\phi(x)$$

$$\Rightarrow \int_a^{-x} f(t)\,dt = -\int_a^x f(t)\,dt$$

$$\Rightarrow \int_a^0 f(t)\,dt + \int_0^{-x} f(t)\,dt = -\int_a^0 f(t)\,dt - \int_0^x f(t)\,dt$$

$$\Rightarrow \int_a^0 f(t)\,dt - \int_0^x f(-y)\,dy = -\int_a^0 f(t)\,dt - \int_0^x f(t)\,dt$$

[Putting $t = -y$ in the second integral on LHS]

$$\Rightarrow 2\int_a^0 f(t)\,dt - \int_0^x f(y)\,dy = -\int_0^x f(t)\,dt \qquad [\because f(-y) = f(y)]$$

$$\Rightarrow 2\int_a^0 f(t)\,dt - \int_0^x f(t)\,dt = -\int_0^x f(t)\,dt \qquad \text{[Using Prop I]}$$

$$\Rightarrow 2\int_0^a f(t)\,dt = 0$$

$$\Rightarrow \int_0^a f(t)\,dt = 0.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $f(x) = |x+1|$ for all $x \in [-2, -1]$, then find its integral function.*

SOLUTION Let $\phi(x)$ be the integral function of function $f(x)$ defined on $[-2, 1]$. Then,

$$\phi(x) = \int_{-2}^x f(x)\,dx$$

$$= \int_{-2}^{x} |t+1|\, dt$$

$$= \begin{cases} \int_{-2}^{x} -(t+1)\, dt & , \text{if } -2 \le x < -1 \\ \int_{-2}^{-1} |t+1|\, dt + \int_{-1}^{x} |t+1|\, dt & , \text{if } -1 \le x \le 1 \end{cases}$$

$$= \begin{cases} \int_{-2}^{x} -(t+1)\, dt & , \text{if } -2 \le x \le -1 \\ \int_{-2}^{-1} -(t+1)\, dt + \int_{-1}^{x} (t+1)\, dt & , \text{if } -1 \le x \le 1 \end{cases}$$

$$= \begin{cases} -\left[\left(\frac{t^2}{2}+t\right)\right]_{-1}^{x} & , \text{if } -2 \le x < -1 \\ -\left[\frac{t^2}{2}+t\right]_{-2}^{-1} + \left[\frac{t_2}{2}+t\right]_{-1}^{x} & , \text{if } -1 \le x \le 1 \end{cases}$$

$$= \begin{cases} -\frac{x^2}{2} - x & , \text{if } -2 \le x < -1 \\ \frac{x^2}{2} + x + 1 & , \text{if } -1 \le x \le 1 \end{cases}$$

Since $f(x) = |x+1|$ is continuous on $[-2,-1]$, therefore $\phi(x)$ is also continuous on $[-2,-1]$. Note that $\phi(x)$ is not differentiable at $x = 1$.

EXAMPLE 2 *If* $\phi(x) = -\int_0^x \log\cos y\, dy$, *prove that*

$$\phi(x) = 2\phi\left(\frac{\pi}{4}+\frac{x}{2}\right) - 2\phi\left(\frac{\pi}{4}-\frac{x}{2}\right) - x\log 2$$

Hence or otherwise show that $\phi\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\log 2$.

SOLUTION We have,

$$\phi(x) = -\int_0^x \log\cos y\, dy$$

$$\therefore \quad \phi\left(\frac{\pi}{4}+\frac{x}{2}\right) = -\int_0^{\pi/4+x/2} \log\cos y\, dy$$

and, $\phi\left(\frac{\pi}{4}+\frac{x}{2}\right) = -\int_0^{\pi/4-x/2} \log\cos y\, dy$

Let $I = \phi\left(\frac{\pi}{4}+\frac{x}{2}\right) - \phi\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

Then,

$$I = -\int_0^{\pi/4+x/2} \log\cos y\, dy + \int_0^{\pi/4-x/2} \log\cos y\, dy$$

$$\Rightarrow I = \int_{\pi/4+x/2}^{0} \log\cos y\, dy + \int_0^{\pi/4-x/2} \log\cos y\, dy$$

$$\Rightarrow I = \int_{\pi/4+x/2}^{\pi/4-x/2} \log\cos y\, dy \qquad \ldots(i)$$

$$\Rightarrow I = \int_{\pi/4+x/2}^{\pi/4-x/2} \log\cos\left(\frac{\pi}{4}-\frac{x}{2}+\frac{\pi}{4}+\frac{x}{2}-y\right) dy \quad \text{[Using Prop. XIV]}$$

$$\Rightarrow I = \int_{\pi/4+x/2}^{\pi/4-x/2} \log\cos\left(\frac{\pi}{2}-y\right) dy$$

$$\Rightarrow I = \int_{\pi/4+x/2}^{\pi/4-x/2} \log\sin y\, dy \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/4+x/2}^{\pi/4-x/2} (\log\cos y + \log\sin y)\, dy$$

$$= \int_{\pi/4+x/2}^{\pi/4-x/2} (\log\sin 2y - \log 2)\, dy$$

$$= \int_{\pi/4+x/2}^{\pi/4-x/2} \log\sin 2y\, dy - \int_{\pi/4+x/2}^{\pi/4-x/2} \log 2\, dy$$

$$= \int_{\pi/4+x/2}^{\pi/4-x/2} \log\sin 2y\, dy + x\log 2$$

$$= \frac{1}{2}\int_{x}^{-x} \log\sin\left(\frac{\pi}{2}+t\right) dt = x\log 2, \text{ where } y = \frac{\pi}{4}+\frac{t}{2}$$

$$= \frac{1}{2}\int_{x}^{-x} \log\cos t\, dt + x\log 2$$

$$= -\frac{1}{2}\int_{-x}^{x} \log\cos t\, dt + x\log 2$$

$$= -\int_0^x \log\cos t\, dt + x\log 2 \qquad \text{[Using Prop. XIV]}$$

$$= \phi(x) + x\log 2.$$

$$\therefore \quad 2\left\{\phi\left(\frac{\pi}{4}+\frac{x}{2}\right) - \phi\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \phi(x) + x\log 2$$

$$\Rightarrow \phi(x) = 2\phi\left(\frac{\pi}{4}+\frac{x}{2}\right) - 2\phi\left(\frac{\pi}{4}-\frac{x}{2}\right) - x\log 2.$$

EXAMPLE 3 *Find a function $f: R \to R$ satisfying the following conditions:*

(i) $f(x)$ is continuous on $[0, \infty)$

(ii) $f(x) > 0$ for all $x \in (0, \infty)$

(iii) $\frac{1}{2}\int_0^x f^2(t)\,dt = \frac{1}{x}\left\{\int_0^x f(t)\,dt\right\}^2$

SOLUTION Let $\phi(x)$ be the integral function of function $f(x)$ on $[0, \infty)$. Then,

$$\phi(x) = \int_0^x f(t)\,dt$$

Cleraly, $\phi'(x) = f(x)$ for all $x \in [0, \infty)$.

Now,

$$\frac{1}{2}\int_0^x f^2(t)\,dt = \frac{1}{x}\left\{\int_0^x f(t)\,dt\right\}^2$$

$$\Rightarrow \frac{1}{2}\int_0^x |\phi'(t)|^2\,dt = \frac{1}{x}|\phi(x)|^2$$

$$\Rightarrow \frac{1}{2}|\phi'(x)|^2 = -\frac{1}{x^2}|\phi(x)|^2 + \frac{2}{x}\phi(x)\,\phi'(x)$$

[Differentiating both sides w.r.t. x]

$$\Rightarrow \frac{1}{2}|\phi'(x)|^2 = \frac{2x\,\phi(x)\,\phi'(x) - |\phi(x)|^2}{x^2}$$

$$\Rightarrow x^2|\phi'(x)|^2 = 4x\,\phi(x)\,\phi'(x) - 2|\phi(x)|^2$$

$$\Rightarrow \left\{\frac{x\phi'(x)}{\phi(x)}\right\}^2 - 4\left\{\frac{x\phi'(x)}{\phi(x)}\right\} + 2 = 0$$

[Dividing both sides by $\{\phi(x)^2\}$]

$$\Rightarrow \frac{x\phi'(x)}{\phi(x)} = \frac{4 \pm \sqrt{16-8}}{2}$$

$$\Rightarrow x\frac{\phi'(x)}{\phi(x)} = 2 \pm \sqrt{2}$$

$$\Rightarrow \frac{\phi'(x)}{\phi(x)} = \frac{(2 \pm \sqrt{2})}{x}$$

$$\Rightarrow \int \frac{\phi'(x)}{\phi(x)}\,dx = (2 \pm \sqrt{2})\int \frac{1}{x}\,dx$$

$$\Rightarrow \log\phi(x) = (2 \pm \sqrt{2})\log x + \log C$$

$$\Rightarrow \phi(x) = C x^{2 \pm \sqrt{2}}$$

$$\Rightarrow \phi'(x) = (2 \pm \sqrt{2})\,C\,x^{1 \pm \sqrt{2}}$$

$$\Rightarrow f(x) = (2 \pm \sqrt{2})\,C\,x^{1 \pm \sqrt{2}}$$

$$\Rightarrow f(x) = (2+\sqrt{2})\,C\,x^{1+\sqrt{2}} \qquad \left[\begin{array}{l} f(x) \text{ is continuous on } (0, \infty) \\ \therefore f(x) \neq (2-\sqrt{2})\,C\,x^{1-\sqrt{2}} \end{array}\right]$$

EXAMPLE 4 *If $f(x)$ is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$.*

SOLUTION Let $\phi(x) = \int_0^x f(t)\,dt$. Then, $\phi(x)$ is the integeral function of $f(x)$ defined on $(-\infty, \infty)$ such that $\phi'(x) = f(x)$.

Also, $\phi(x)$ is continuous as $f(x)$ is continiuous

In order to prove that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$, it is sufficient to show that the function $m^2x^2 + \int_0^x f(t)\,dt$ attains value 2 for some x.

Consider the function

$$g(x) = m^2x^2 + \int_0^x f(t)\,dt$$

$$\Rightarrow g(x) = m^2x^2 + \phi(x).$$

Since m^2x^2 is everywhere continuous and $\phi(x)$ is also continous. Therefore, $g(x)$ is continuous for all $x \in (-\infty, \infty)$.

It is given that $\phi(x) \to \infty$ as $|x| \to \infty$ and m^2x^2 also tends to ∞ as $|x| \to \infty$. Therefore,

$$g(x) \to \infty \text{ as } |x| \to \infty$$

Also, $g(0) = 0 + \phi(0) = 0$

Thus, $g(x)$ attains all values between 0 and ∞. Hence, $g(x)$ attains value 2 for some $x \in R$ for all real values of m i.e.

$$g(x) = 2 \text{ for some } x \in R$$

Hence, the line $y = mx$ intersects the curve

$$y^2 + \int_0^x f(t)\,dt = 2 \quad \text{for all } m \in R.$$

EXAMPLE 5 *Given an even function f defined and integrable everywhere and periodic with period 2, let*

$$g(x) = \int_0^x f(t)\,dt \text{ and } g(1) = \alpha.$$

(i) *Prove that g is an odd function such that*

$$g(x+2)-g(x) = g(2) \text{ for all } x.$$

(ii) *Compute g (2) and g (5) in terms of α.*

(iii) *For what value of α, will g (x) be periodic with period 2 ?*

SOLUTION (i) We have,

$$g(x) = \int_0^x f(t)\,dt$$

$$\therefore\ g(-x) = \int_0^{-x} f(t)\,dt$$

$$= -\int_0^x f(-y)\,dy, \text{ where } t = -y.$$

$$= -\int_0^x f(y)\,dy \qquad [\because f \text{ is an even function}]$$

$$= -g(x)$$

Hence, $g(x)$ is an odd function.

Now,

$$g(x+2)-g(2)$$

$$= \int_0^{x+2} f(t)\,dt - \int_0^2 f(t)\,dt$$

$$= \int_0^{x+2} f(t)\,dt + \int_2^0 f(t)\,dt$$

$$= \int_2^{x+2} f(t)\,dt$$

$$= \int_0^x f(y+2)\,dy, \text{ where } t = y+2$$

$$= \int_0^x f(y)\,dy \qquad [\because f \text{ is periodic with period 2}]$$

$$= g(x)$$

$$\therefore\ g(x+2)-g(x) = g(2) \text{ for all } x.$$

(ii) We have, $g(1) = \alpha$

Now,

$$g(x+2)-g(x) = g(2) \text{ for all } x \qquad \ldots(i)$$

$$\Rightarrow\ g(x+2)+g(-x) = g(2) \text{ for all } x \quad [\because g \text{ is an odd function}]$$

$$\Rightarrow\ g(1)+g(1) = g(2) \qquad [\text{Putting } x = -1]$$

$$\Rightarrow\ g(2) = 2g(1) = 2\alpha \qquad [\because g(1) = \alpha]$$

Putting $x = 1$ and $x = 3$ successively in (i), we get

$$g(3)-g(1) = g(2)$$

and, $g(5)-g(3) = g(2)$

$$\Rightarrow\ g(5)-g(1) = 2g(2) \qquad [\text{By adding these two}]$$

$$\Rightarrow\ g(5)-\alpha = 2\times 2\alpha \qquad [\because g(1)=\alpha \text{ and } g(2)=2\alpha]$$

$$\Rightarrow\ g(5) = 5\alpha$$

(iii) If $g(x)$ is periodic function with period 2, then

$$g(x+2) = g(x) \text{ for all } x.$$

But, $g(x+2)-g(x) = g(2)$ for all x

$$\Rightarrow\ g(x+2)-g(x) = 2\alpha \quad \text{for all } x \qquad [\because g(2)=2\alpha]$$

$$\Rightarrow\ 2\alpha = 0 \qquad [\because g(x+2)=g(x)]$$

$$\Rightarrow\ \alpha = 0$$

Hence, $g(x)$ is periodic with period 2, if $\alpha = 0$.

EXAMPLE 6 *Let f be an integrable, odd periodic function with period 2 and let g (x) be its integral function. Prove that :*

(i) *g is an even function with period 2.*

(ii) $g(2n) = 0$ *for all* $n \in Z$

SOLUTION (i) We have,

$$g(x) = \int_0^x f(t)\,dt.$$

$$\therefore\ g(-x) = \int_0^{-x} f(t)\,dt$$

$$= -\int_0^x f(-y)\,dy, \text{ where } t = -y$$

$$= \int_0^x f(y)\,dy \qquad [\because f \text{ is an odd function}]$$

$$= g(x)$$

Thus, g is an even function.

Now,

$$g(x+2) = \int_0^{x+2} f(t)\,dt$$

$$= \int_0^x f(t)\,dt + \int_x^{x+2} f(t)\,dt$$

$$= \int_0^x f(t)\,dt + \int_0^2 f(t)\,dt \quad [\because f \text{ is periodic with period } 2x]$$

$$= g(x)+g(2)$$

Putting $x = 0$, we get

$$g(2) = g(0)+g(2) \Rightarrow g(0) = 0$$

Now, putting $x = -2$, we get

$g(0) = g(-2) + g(2)$

$\Rightarrow \quad 0 = g(2) + g(2)$

$\Rightarrow g(2) = 0$

Putting $g(2) = 0$ in $g(x+2) = g(x) + g(2)$, we get

$g(x+2) = g(x)$ for all x

Hence, g is a periodic function with period 2.

(ii) We have,

$g(x+2) = g(x)$ for all x

and, $g(2) = 0$

$\therefore \quad g(2+2) = g(2) \Rightarrow g(4) = 0$

$g(4+2) = g(4) = 0 \Rightarrow g(6) = 0$

Continuing in this manner, we get

$g(2n) = 0$ for all $n \in Z$.

$$\therefore \quad g(2n) = \int_0^{2n} f(t)\, dt$$

$$= n \int_0^2 f(t)\, dx \qquad [\because f \text{ is periodic with period 2}]$$

$$= n\, g(2).$$

EXAMPLE 7 *Find the greatest value of*

$$F(x) = \int_{5\pi/3}^{x} (6\cos t - 2\sin t)\, dt$$

in the interval $[5\pi/3, 7\pi/4]$.

SOLUTION We have,

$$F(x) = \int_{5\pi/3}^{x} (6\cos t - 2\sin t)\, dt$$

$\Rightarrow F'(x) = 6\cos x - 2\sin x$

For all $x \in [5\pi/3, 7\pi/4]$, we have

$\cos x > 0$ and $\sin x < 0$

$\Rightarrow F'(x) = 6\cos x - 2\sin x > 0$

$\Rightarrow F(x)$ is an increasing function on $[5\pi/3, 7\pi/4]$

$\Rightarrow F(x)$ attains its greatest vaue at $x = 7\pi/4$.

Hence,

Greatest value $= F(7\pi/4)$

$$= \int_{5\pi/3}^{7\pi/4} (6\cos t - 2\sin t)\, dt$$

$$= \left[6\sin t + 2\cos t\right]_{5\pi/3}^{7\pi/4}$$

$$= 3\sqrt{3} - 2\sqrt{2} - 1.$$

EXAMPLE 8 *Let $f(x)$ be a continuous function on $R - \{0\}$ such that $\int_0^a f(x)\, dx$ exists for all $a > 0$. If $g(x) = \int_x^a \frac{f(t)}{t}\, dt$, prove that*

$$\int_0^a f(x)\, dx = \int_0^a g(x)\, dx.$$

SOLUTION We have,

$$g(x) = \int_x^a \frac{f(t)}{t}\, dt = -\int_a^x \frac{f(t)}{t}\, dt.$$

$$\therefore \quad g'(x) = -\frac{f(x)}{x}$$

$\Rightarrow f(x) = -x\, g'(x)$

$$\Rightarrow \int_0^a f(x)\, dx = -\int_0^a x\, g'(x)\, dx$$

$$= -\left[x\, g(x)\right]_0^a + \int_0^a g(x)\, dx$$

$$= -a\, g(a) + \int_0^a g(x)\, dx$$

$$= \int_0^a g(x)\, dx \qquad \left[\because g(a) = \int_a^a \frac{f(t)}{t}\, dt = 0\right]$$

EXAMPLE 9 *A function is defined as $f(x) = x - x^2$, $0 \le x \le 1$ and $f(x+1) = f(x)$ for all $x \in R$.*

(i) *Find an expression for the function $g(x)$ defined as :*

$$g(x) = \int_0^x f(x)\, dx$$

(ii) *Express $g(x)$ as the sum of two functions $g_e(x)$ and $g_0(x)$ so that $g_e(x)$ is an even function and $g_0(x)$ is an odd function.*

SOLUTION We have, $f(x) = x - x^2$; $0 \le x \le 1$ and $f(x+1) = f(x)$ for all $x \in R$. So, $f(x)$ is a periodic function with period 1.

(i) Let $x \in R$. Then, there exists an integer n such that

$$n \le x < n+1.$$

Now,

$$g(x) = \int_0^x f(x)\, dx$$

$$= \int_0^n f(x)\, dx + \int_n^x f(x)\, dx$$

$$= n\int_0^1 f(x)\,dx + \int_0^{x-n} f(x)\,dx \quad [\because f(x) \text{is periodic with period 1}]$$

$$= n\int_0^1 (x-x^2)\,dx + \int_0^{x-n} (x-x^2)\,dx$$

$$= n\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{x-n}$$

$$= \frac{n}{6} + \frac{(x-n)^2}{2} - \frac{(x-n)^3}{3}, \text{ where } n \le x < n+1.$$

(ii) We have,

$$g(x) = \frac{1}{2}[g(x)+g(-x)] + \frac{1}{2}[g(x)-g(-x)]$$

$$= g_e(x) + g_o(x)$$

where, $g_e(x) = \frac{1}{2}[g(x)+g(-x)]$

and $g_0(x) = \frac{1}{2}[g(x)-g(-x)]$ are respectively even and odd functions.

—❑❑❑—

11.6 MEAN VALUE OF A FUNCTION OVER AN INTERVAL

Let $f(x)$ be a continuous function defined on $[a, b]$. Then, there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

The number $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is called the mean value of the function $f(x)$ over $[a, b]$.

For example, the mean value of $f(x) = \sin x$ over $[0, \pi]$ is

$$\frac{1}{(\pi-0)}\int_0^\pi \sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^\pi = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the mean value of* $\frac{1}{4+\cos x}$ *over the interval* $[0, 2\pi]$.

SOLUTION Let $f(x) = \frac{1}{4+\cos x}$. Then,

Mean value of $f(x)$ over $[0, 2\pi]$

$$= \frac{1}{(2\pi-0)}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{4+\cos x}\,dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} \frac{1}{4+\cos x}\,dx \qquad \left[\because \frac{1}{4+\cos x} = \frac{1}{4+\cos(2\pi-x)}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{4+\cos x}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\sec^2 x/2}{5+3\tan^2 x/2}\,dx$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{2\,dt}{5+3t^2}, \text{ where } t = \tan x/2$$

$$= \frac{2}{\pi}\times\frac{1}{\sqrt{5}\,\sqrt{3}}\left[\tan^{-1}\left(\frac{\sqrt{3}t}{\sqrt{5}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{15}\,\pi}\times\frac{\pi}{2} = \frac{1}{\sqrt{15}}$$

EXAMPLE 2 *Find the mean value of* $\sin^2 x$ *over the interval* $[0, \pi]$.

SOLUTION Let $f(x) = \sin^2 x$.

Mean value of $\sin^2 x$ over $[0, \pi]$

$$= \frac{1}{\pi-0}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (1-\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}.$$

EXAMPLE 3 *Find the mean value of* $f(x) = x\,[2x]\ Sgn\ (x-2)$ *on* $[1, 3]$, *where* $[x]$ *denotes the greatest integer function.*

SOLUTION We have,

Mean value of $f(x)$ over $[1, 3]$

$$= \frac{1}{(3-1)} \int_1^3 f(x)\ dx$$

$$= \frac{1}{2} \int_1^3 f(x)\ dx$$

$$= \frac{1}{2}\left[\int_1^{1.5} x\,[2x]\ Sgn\ (x-2)\ dx + \int_{1.5}^{2} x\,[2x]\ sgn\ (x-2)\ dx + \int_2^{2.5} x\,[2x]\ Sgn\ (x-2)\ dx + \int_{2.5}^{3} x\,[2x]\ \ Sgn\ (x-2)\ dx\right]$$

$$= \frac{1}{2}\left[\int_1^{1.5} -2x\ dx + \int_{1.5}^{2} -3x\ dx + \int_2^{2.5} 4x\ dx + \int_{2.5}^{3} 5x\ dx\right]$$

$$= \frac{15}{4}.$$

—❑❑❑—

11.7 IMPROPER INTEGRALS

If $f(x)$ is continuous on $[a, \infty]$, then $\int_a^{\infty} f(x)\ dx$ is called an improper integeral and is defined as

$$\int_a^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_a^b f(x)\ dx \qquad ...(i)$$

If there exists a finite limit on the right hand side of (i), then we say that the improper integral is convergent; otherwise it is divergent.

Similarly, we define $\int_{-\infty}^{b} f(x)\ dx = \lim_{a \to -\infty} \int_a^b f(x)\ dx$

and, $\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{a} f(x)\ dx + \int_a^{\infty} f(x)\ dx$

REMARK *Geometrically, for* $f(x) > 0$ *the improper integral* $\int_a^{\infty} f(x)\ dx$ *gives area of the figure bounded by the curve* $y = f(x)$, *the x-axis and the straight line* $x = a$.

11.8 GAMMA FUNCTION

If n *is a positive rational number, then the improper integral* $\int_0^{\infty} e^{-x} x^{n-1}\ dx$ *is defined as Gamma Function and is denoted by* Γn

i.e. $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\ dx$, where $x \in Q^+$.

ILLUSTRATION 1 We have,

$$\Gamma 1 = \int_0^{\infty} e^{-x} x^0\ dx$$

$$= \lim_{b \to \infty} \int_0^b e^{-x}\ dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b$$

$$= \lim_{b \to \infty} (-e^{-b} + e^0) = 0 + 1 = 1$$

ILLUSTRATION 2 We have,

$$\Gamma 2 = \int_0^{\infty} e^{-x} x^{2-1}\ dx$$

$$= \lim_{b \to \infty} \int_0^b e^{-x}\ x\ dx$$

$$= \lim_{b \to \infty} \left[-x e^{-x} - e^{-x}\right]_0^b$$

$$= \lim_{b \to \infty} \left[(-b e^{-b} - e^{-b}) - (0 - 1)\right]$$

$$= \lim_{b \to \infty} \left(\frac{-b}{e^b} - \frac{1}{e^b}\right) + 1$$

$$= \lim_{b \to \infty} \frac{-b+1}{e^b} + 1$$

$$= \lim_{b \to \infty} -\frac{1}{e^b} + 1 \qquad \text{[Using L' Hospital's Rule]}$$

$$= 1.$$

PROPERTIES OF GAMMA FUNCTION Gamma function has the following properties:

(i) $\Gamma 1 = 1, \Gamma 0 = \infty$ and $\Gamma n + 1 = n\Gamma n$.

For example,

$$\Gamma 5 = 4\Gamma 4 = 4 \times 3\Gamma 3 = 4 \times 3 \times 2\Gamma 2$$

$$= 4 \times 3 \times 2 \times 1\Gamma 1 = 4 \times 3 \times 2 \times 1.$$

(ii) If $n \in N$, then $\Gamma n + 1 = n!$

(iii) $\Gamma_{1/2} = \sqrt{\pi}$

(iv) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \dfrac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$.

ILLUSTRATION *Evaluate* $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$

SOLUTION Using Gamma Function, we have

$$\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\,\Gamma\frac{4+6+2}{2}} = \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\,\Gamma 6}$$

$$= \frac{\left(\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}\right)\left(\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}\right)}{2\times 5!} = \frac{3\pi}{512}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the value of* $\Gamma 2$

SOLUTION We have,

$$\Gamma 2 = \int_0^{\infty} e^{-x} x^{2-1}\, dx$$

$$= \lim_{b\to\infty} \int_0^b e^{-x} x\, dx$$

$$= \lim_{b\to\infty} \left[-x e^{-x} - e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty} \left[(-b e^{-b} - e^{-b}) - (0-1)\right]$$

$$= \lim_{b\to\infty} \left(\frac{-b}{e^b} - \frac{1}{e^b}\right) + 1$$

$$= \lim_{b\to\infty} \frac{-b+1}{e^b} + 1$$

$$= \lim_{b\to\infty} -\frac{1}{e^b} + 1 \qquad \text{[Using L' Hospital's Rule]}$$

$$= 1.$$

EXAMPLE 2 *Evaluate :* $\int_0^1 x^5 (1-x^2)^4\, dx.$

SOLUTION Putting $x = \sin\theta$ and $dx = \cos\theta\, d\theta$, we get

$$\int_0^1 x^5 (1-x^2)^4\, dx$$

$$= \int_0^{\pi/2} \sin^5\theta\, (1-\sin^2\theta)^4 \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \cos^9\theta\, d\theta$$

$$= \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{9+1}{2}}{2\,\Gamma\frac{5+9+2}{2}}$$

$$= \frac{\Gamma 3\,\Gamma 5}{2\,\Gamma 8} = \frac{2!\times 4!}{2\times 7!} = \frac{4!}{7!} = \frac{1}{210}.$$

EXAMPLE 3 *If* $m, n \in N$, *evaluate* $\int_a^b (x-a)^m (b-x)^n\, dx.$

SOLUTION Putting $x = a\cos^2\theta + b\sin^2\theta$, we have

$$dx = 2(b-a)\sin\theta\cos\theta$$

Also, $x = a \Rightarrow (a-b)\sin^2\theta = 0 \Rightarrow \theta = 0$

$$x = b \Rightarrow (b-a)\cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

$$\therefore \int_a^b (x-a)^m (b-x)^n\, dx$$

$$= \int_0^{\pi/2} (b-a)^{m+n} \sin^{2m}\theta \cos^{2n}\theta \,.\, 2(b-a)\sin\theta\cos\theta\, d\theta$$

$$= 2(b-a)^{m+n+1} \int_0^{\pi/2} \sin^{2m+1}\theta \cos^{2n+1}\theta\, d\theta$$

$$= 2(b-a)^{m+n+1} \frac{\Gamma\frac{2m+1+1}{2}\,\Gamma\frac{2n+1+1}{2}}{2\,\Gamma\frac{2m+1+2n+1+2}{2}}$$

$$= (b-a)^{m+n+1} \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}}$$

$$= (b-a)^{m+n+1} \frac{m!\, n!}{(m+n+1)!}$$

EXAMPLE 4 *Prove that* : $\int_0^{\pi} \sin^{2n} x \cos^2 x \, dx$

$$= \left(\frac{2}{2n+2}\right)\left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\cdots\frac{1}{2}\cdot\frac{\pi}{2}.$$

SOLUTION We have,

$$I = \int_0^{\pi} \sin^{2n} x \cos^2 x \, dx$$

$$\Rightarrow I = 2\int_0^{\pi/2} \sin^{2n} x \cos^2 x \, dx$$

$$= 2 \cdot \frac{\Gamma\frac{2n+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{2n+2+2}{2}}$$

$$= \frac{\Gamma\frac{2n+1}{2}\,\Gamma\frac{3}{2}}{\Gamma_{n+2}}$$

$$= \frac{\left(\frac{2n-1}{2}\times\frac{2n-3}{2}\times\frac{2n-5}{2}\times\ldots\times\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}\right)\times\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{(n+1)!}$$

$$= \frac{1\,.\,3\,.\,5\ldots(2n-1)\,\pi}{2^{n+1}\,(n+1)!}$$

$$= \frac{(2n-1)\,(2n-3)\,(2n-5)\ldots3\,.\,1}{2^n\,[(n+1)\,n\,(n-1)\ldots.3..2.1]}\,\frac{\pi}{2}$$

$$= \left(\frac{1}{n+1}\right)\cdot\left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\left(\frac{2n-5}{2n-4}\right)\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \left(\frac{2}{2n+2}\right)\left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\left(\frac{2n-5}{2n-4}\right)\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

EXERCISE 11.3

Evaluate the following integrals :

(i) $\int_0^{2\pi} [2\sin x]$ (ii) $\int_0^{2\pi} [\sqrt{2}\sin x]\,dx$

(iii) $\int_0^{\pi} [2\cos x]\,dx$ (iv) $\left|\int_0^{2\pi} [2\sin x]\,dx\right|$

(v) $\int_1^4 \log [x]\,dx$ (vi) $\int_a^b \frac{|x|}{x}, a<b$

(vii) $\int_{-1}^{1} \left[x\,[1+\sin\pi x]+1\right]dx$

(viii) $\int_{-1}^{3} \left(|x|+|x-1|\right)dx$ (ix) $\int_0^{n^2} [\sqrt{x}]\,dx$

(x) $\int_0^{\sqrt{3}} [x^3-1]\,dx$ (xi) $\int_{-1000}^{0} [\tan^{-1}x]\,dx$

2. Prove that : $\int_{-1}^{\sqrt{3}} \frac{1}{1+x^2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx = \frac{25\pi^2}{144}$

3. Evaluate : $\int_1^3 \frac{1}{x^2+[x]^2+1-2x\,[x]}\,dx$, where $[\cdot]$ denotes the greatest integer function.

4. Evaluate : $\int_{-n}^{n} \min\{x-[x], -x-[-x]\}\,dx$; where $[\cdot]$ denotes the greatest integer function.

5. Evaluate : $\int_{-3/2}^{2} f(x)\,dx$, where $f(x)$ is given by

$$f(x) = \max\left(|t-1|-|t|+|t+1|\right), \frac{-3}{2} \le t \le x.$$

6. Prove that : $\int_0^{\pi} \cos 2x \log\sin x\,dx = -\frac{\pi}{2}$

7. Prove that : $\int_0^1 \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = \frac{\pi}{4\sqrt{2}}$

8. Evaluate : $\int_0^1 \frac{1}{(5+2x-2x^2)\,(1+e^{2-4x})\,dx}$

9. Prove that : $\int_0^{2\pi} \frac{\sin^2\theta}{a-b\cos\theta} = \frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$, if $a>b>0$.

10. Prove that : $\int_0^{\infty} \frac{1}{1+x^4}\,dx = \int_0^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}$

11. Prove that :

$$\int_0^{2\pi} \frac{1}{a+b\cos x+c\sin x}\,dx = \frac{2\pi}{\sqrt{a^2-b^2-c^2}},$$

where $a > \sqrt{b^2+c^2} > 0$

12. Prove that : $\int_0^{\pi} \frac{1}{a^2+b^2-2ab\cos x}\,dx = \frac{\pi}{a^2-b^2}$, if $a>b>0$.

13. Evaluate : $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}\,dx$

14. Evaluate : $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$

15. Prove that : $\int_0^{\pi} \frac{x \sin x}{2-\sin^2 x}\,dx = \frac{\pi^2}{4}$

16. Prove that : $\int_{-\pi}^{\pi} \frac{2x\,(1+\sin x)}{\cos^2 x}\,dx = \pi^2$

17. Prove that : $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)\,dx$

18. If $\int_0^{y} e^{-t^2}\,dt + \int_0^{x^2} \sin^2 t\,dt = 0$, then prove that $\frac{dy}{dx} = -2\sin^2 1$ at $x = 1$.

19. Prove that : $\int_{-a}^{a} \tan x f(\cos x)\,dx = 0$

20. Prove that : $\int_0^{1/2} \frac{1}{(1-2x^2)\sqrt{1-x^2}}\,dx = \frac{1}{2}\log(\sqrt{3}+2)$

21. Prove that : $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\,dx = \frac{\pi^2}{2ab}$

22. Prove that : $\int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2}\,dx = \pi(2-\pi)$

23. Prove that : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi^2}{16}$

24. Prove that : $\int_0^{\pi} \log(1+\cos x)\,dx = -\pi \log 2$

25. Prove that : $\int_0^{\infty} \frac{x \log x}{(1+x^2)^2}\,dx = 0$

26. Prove that : $\int_0^{\pi} \frac{x}{1+\sin\alpha \sin x}\,dx = \frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)$

27. Prove that : $\int_0^{1} \frac{\tan^{-1} x}{x}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}\,dx$

28. Prove that : $\int_{3n\pi}^{3n\pi + 3\pi/n} \frac{4x}{\left[(a^2+b^2)+(a^2-b^2)\cos\frac{2nx}{3}\right]^2}\,dx$

$= \frac{9(2n^2+1)\pi}{n^2}\left(\frac{a^2+b^2}{a^3 b^3}\right)$, where $a, b > 0$.

29. Evaluate : $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx$

30. Evaluate : $\int_{-\pi/2}^{\pi/2} \left[\log\left(\frac{px^2+qx+r}{px^2-qx+r}\right)(a+b)\,|\sin x|\right]dx$

31. Evaluate : $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

32. Evaluate : $\int_0^{\pi} x \log \sin x\,dx$

33. Evaluate : $\int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx$

34. Evaluate : $\int_0^{\pi/2} \frac{\sin 8x \log\cot x}{\cos 2x}\,dx$

35. Evaluate : $\int_0^{\pi/2} \sin 2x \log\tan x\,dx$

36. Evaluate : $\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$

37. Prove that : $\int_1^{\infty} \frac{x^2-2}{x^3\sqrt{x^2-1}}\,dx = 0$

38. Prove that

(i) $\int_{-2}^{2} \frac{3x^5+4x^3+2x^2+x+20}{x^2+4}\,dx = 3\pi+8$

(ii) $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx = \frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$

(iii) $\int_0^{\infty} f(x^n+x^{-n})\,\frac{\log x}{x}\,dx = 0$

(iv) $\int_0^{\infty} f(x^n+x^{-n})\,\frac{\log x}{1+x^2}\,dx = 0$

38. Evaluate : $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{-1/2} dx$

40. Evaluate : $\int_{-1}^{3/2} |x \sin \pi x|\,dx$

41. Evaluate the following integrals :

(i) $\int_0^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\,dx$

(ii) $\int_0^{1} x \log(1+x^2)\,dx$

(iii) $\int_0^{\pi/2} \frac{1}{1+\frac{1}{6}\sin^2 x}\,dx$

42. Evaluate : $\int_0^5 [\sqrt{x}]\, dx$, where $[\cdot]$ denotes the greatest integer function.

43. Evaluate : $\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)\, x\, dx, n \in Z$

44. Prove that : $\int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} \log\left(\frac{1+x}{1-x}\right) dx = \frac{5\pi}{4}$

45. Prove that : $\int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt = \frac{3\pi}{16}$ for all $x \in R$.

46. Prove that : $\int_a^b ([x] + [-x])\, dx = a - b$, where $[x]$ is the grestest integer less than or equal to x.

47. Prove that : $\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x-c)\, dx$

48. If $f(x+T) = f(x)$ for all x, prove that

$$\int_T^{nT} f(x)\, dx = (n-1)\int_0^T f(x)\, dx$$

49. Evaluate the following :

(i) $\int_{-100\pi}^{100\pi} (|\sin x| + |\cos x|)\, dx$ (ii) $\int_0^{50\pi} \sin^3 x\, dx$

(iii) $\int_{\pi}^{100\pi} |\cos x|\, dx$

50. Prove that :

(i) $\left|\int_2^5 \frac{\sin x}{(1+x)^2} dx\right| < \frac{1}{6}$ (ii) $\left|\int_{10}^{19} \frac{\cos x}{1+x^2} dx\right| < \frac{1}{10^7}$

(iii) $\frac{\pi}{2} < \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 x}} dx < \frac{\pi}{2\sqrt{1-k^2}}, 0 < k < 1.$

51. If $y = \int_0^{x^2} \log(1+x)\, dx$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

52. If $y = \int_4^{4x^2} t^4 e^{4t}\, dt$, find $\frac{d^2y}{dx^2}$.

53. Evaluate : $\int_0^{\pi/4} \frac{d}{dx}\left\{\int_1^x \sec^4\theta\, d\theta\right\} dx$

54. If $\int_0^1 \frac{\sin x}{1+x} dx = \alpha$, find the value of $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t} dt$ in terms of α.

55. Find $x \in R$ such that $\int_0^x [x]\, dx = \int_0^{[x]} x\, dx$, where $[\cdot]$ denotes the greatest integer function.

56. If $f(x)$ and $g(x)$ are continuous functions defined on the interval $[0, 4]$ satisfying :

$$f(x) = f(4-x), g(x) + g(4-x) = 3 \text{ and}$$

$\int_0^4 f(x)\, dx = 2$, then find the value of $\int_0^4 f(x)\, g(x)\, dx$.

ANSWERS

1. (i) $-\pi$ (ii) $-\pi$ (iii) $-\frac{\pi}{2}$

(iv) 0 (v) $\log 6$ (vi) $|b| - |a|$

(vii) 2 (viii) 9 (ix) $\frac{n(n-1)(4n+1)}{6}$

(x) $4\sqrt{3} - 1 - 3\sqrt{2} - 3\sqrt{3} - 3\sqrt{4} - 3\sqrt{5}$ (xi) $3\tan 1 - 2000$

3. $\frac{\pi}{2}$ 4. $\frac{n}{2}$ 5. $\frac{41}{8}$

8. $\frac{1}{\sqrt{11}} \log\left(\frac{\sqrt{11}+1}{\sqrt{10}}\right)$ 13. $\pi(\sqrt{2}-1)$ 14. $\frac{1}{2}$

29. $\frac{\pi}{2} - 1$ 30. 0

31. $\frac{\pi}{4} - \frac{1}{2}\log 2$ 32. $-\pi^2 \log 2$

33. $\frac{\pi^2}{2\sqrt{2}}$ 34. 0 35. 0

36. 2 39. $4\log(4/3)$

40. $\frac{3\pi+1}{\pi^2}$ 41. (i) $\frac{16\pi}{3} - 2\sqrt{3}$

(ii) $\log 2 - \frac{1}{2}$ (iii). $\frac{\pi}{2}\sqrt{\frac{6}{7}}$

42. 5 43. 0

49. (i) 800 (ii) 0 (iii) 180

51. $\frac{dy}{dx} = 2x\log(1+x^2)$

$$\frac{d^2y}{dx^2} = \frac{2}{1+x^2}[2x^2 + (1+x^2)\log(1+x^2)]$$

52. $2048\, x^9 e^{16x^2}$, $2048\, x^8 e^{16x^2}(9 + 32x^2)$ 54. $-\alpha$

55. $0 \le x < 1, x = n + \frac{1}{2}, n \in Z - \{0\}$ 56. 3

HINT TO SELECTED PROBLEMS

1. (x) $\int_0^{\sqrt{3}} [x^3-1]\,dx = \sum_{r=1}^{5} \int_{(r-1)^{1/3}}^{r^{1/3}} [x^3-1]\,dx + \int_{5^{1/3}}^{\sqrt{3}} [x^3-1]\,dx$

$$= \sum_{r=1}^{5} (r-2)\{r^{1/3}-(r-1)^{1/3}\} + 4(\sqrt{3}-5^{1/3})$$

5. $f(x) = \begin{cases} -x, & -3/2 \le x < -1 \\ x+2, & -1 \le x < 0 \\ -x+2, & 0 \le x < 1 \\ x, & 1 \le x < 2 \end{cases}$

7. $I = \int_0^1 \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = \int_0^1 \frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx$

$$= -\int_{\infty}^{2} \frac{dt}{t\sqrt{t^2-2}}, \quad \text{where } t = x+\frac{1}{x}$$

8. Use: $\int_0^a f(x)\,dx = \int_a^a f(a-x)\,dx$ and then add the two integrals]

9. Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a-b\cos\theta}\,d\theta$. Then, $I = 2\int_0^{\pi} \frac{\sin^2\theta}{a-b\cos\theta}\,d\theta$

$$\Rightarrow I = 2\int_0^{\pi} \frac{\sin^2(\pi-\theta)}{a-b\cos(\pi-\theta)}\,d\theta = 2\int_0^{\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$$

$$\therefore 2I = 2\int_0^{\pi} \frac{\sin^2\theta}{a^2-b^2\cos^2\theta}\,d\theta = 4\int_0^{\pi/2} \frac{\sin^2\theta}{a^2-b^2\cos^2\theta}\,d\theta.$$

Now divide N^r and D^r by $\cos^4\theta$ and put $\tan\theta = t$]

10. Let $I = \int_0^{\infty} \frac{1}{1+x^4}\,dx$. Putting $x = \frac{1}{t}$, we get

$$I = \int_0^{\infty} \frac{x^2}{1+x^4}\,dx$$

$$\therefore 2I = \int_0^{\infty} \frac{1+x^2}{1+x^4}\,dx = \int_0^{\infty} \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)\right]_0^{\infty}$$

13. Use: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and Add

22. $I = \int_0^{\pi} x^2 \frac{d(1+\sin x)}{(1+\sin x)^2}$

41. Put $x = \sec^4\theta$

➥❑❑❑➥

CHAPTER

12

AREAS OF BOUNDED REGIONS

12.1 INTRODUCTION

In the previous chapter, we have learnt about definite integrals. The computation of areas of figures bounded by plane curves also reduces to the evaluation of a definite integral. In this chapter, we shall discuss problems on finding the areas of bounded plane regions. This requires the knowledge of graphs and tracing of curves which we have learnt in chapter 2 on graphs of functions.

12.2 AREA OF BOUNDED REGIONS

THEOREM *Let $f(x)$ be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by*

$$\int_a^b f(x)\,dx \text{ or, } \int_a^b y\,dx .$$

PROOF Let AD be the curve $y = f(x)$ between the ordinates BA ($x = a$) and CD ($x = b$). Then the required area is the area $ABCD$.

Let $P(x, y)$ be any point on the curve and $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point on it. Draw ordinates PL and QM. Then $PL = y$, $QM = y + \Delta y$ and $LM = \Delta x$.

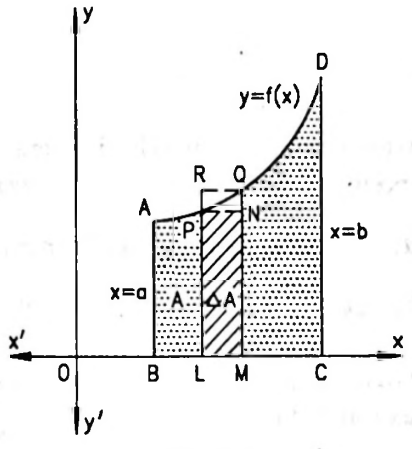


Fig. 12.1

Let A denote the area $BLPA$, and let $A + \Delta A$ be the area $BMQA$. Then, ΔA = area $LMQP$. Also, area $LMNP = y\,\Delta x$, and area $LMQR = (y + \Delta y)\,\Delta x$.

Now,

Area of rectangle $LMNP \le$ Area $LMQP$

$\le$ Area of rectangle $LMQR$

$$\Rightarrow \quad y\,\Delta x \le \Delta A \le (y + \Delta y)\,\Delta x$$

$$\Rightarrow \quad y \le \frac{\Delta A}{\Delta x} \le y + \Delta y$$

$$\Rightarrow \lim_{\Delta x \to 0} y \le \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \le \lim_{\Delta y \to 0} y + \Delta y$$

$$\Rightarrow \quad y \le \frac{dA}{dx} \le y$$

$$\Rightarrow \quad \frac{dA}{dx} = y$$

$$\Rightarrow \int_a^b \frac{dA}{dx}\,dx = \int_a^b y\,dx \qquad \left[\begin{array}{l}\text{Integrating between}\\ \text{the limits } a \text{ and } b\end{array}\right]$$

$$\Rightarrow \left[A\right]_{x=a}^{x=b} = \int_a^b y\,dx$$

$$\Rightarrow (\text{Area } A \text{ when } x = b) - (\text{Area } A \text{ when } x = a) = \int_a^b y\,dx$$

$$\Rightarrow \text{Area } ABCD - 0 = \int_a^b y\,dx$$

$\left[\begin{array}{l}\text{When } x = a,\ PL \text{ coincides with } AB\\ \text{So, area } ABLP = 0 \text{ when } x = a\end{array}\right]$

$$\Rightarrow \text{Area } ABCD = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

REMARK 1 *If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by*

$$\left| \int_a^b y\,dx \right|.$$

REMARK 2 *The area bounded by the curve $x = f(y)$, the y axis and the abscissae $y = c$ and $y = d$ is given by*

$$\int_c^d f(y)\,dy \text{ or, } \int_c^d x\,dy$$

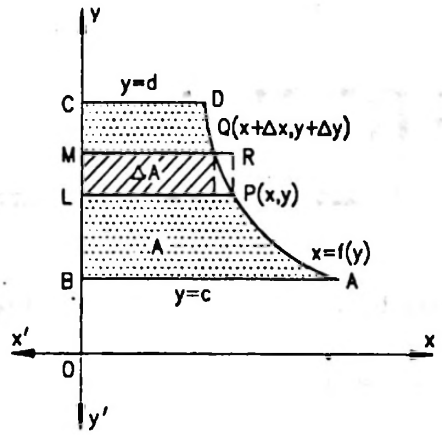


Fig. 12.2

The following algorithm is very helpful to find the area of bounded regions.

ALGORITHM

STEP I *Make a rough sketch showing the area to be found.*

STEP II *Slice the area into horizontal or vertical strips as the case may be.*

STEP III *Consider a representative strip and the corresponding approximating rectangle.*

STEP IV *Find the area of the approximating rectangle. If the representative strip is parallel to y-axis, then its width is taken as Δx and if it is parallel to x-axis, then its width is taken as Δy. In Fig. 12.3, RLMQ is the approximating rectangle of area $y\Delta x$ and in Fig. '12.3, the area of the approximating rectangle RLM is $x\,\Delta y$.*

STEP V *Find the limits within which the approximating rectangle can move. In Fig. 12.3, the approximating rectangle of area*

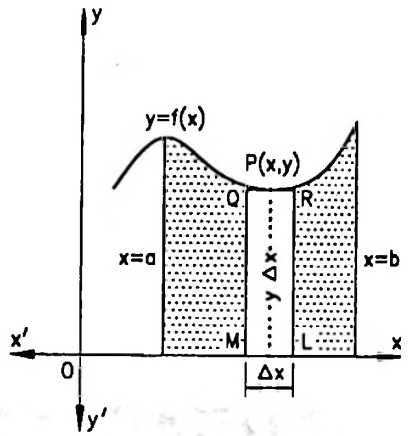


Fig. 12.3

$y\,\Delta x$ can move between $x = a$ and $x = b$, therefore the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is given by $\int_a^b y\,dx$. In Fig. 12.4, the approximating rectangle of area $x\,\Delta y$ can move between $y = c$ and $y = d$, therefore area bounded by $x = \phi(y)$, $x = 0, y = c$ and $y = d$ is given by $\int_c^d x\,dy$.

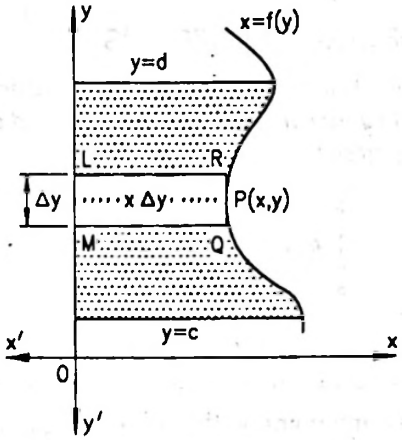


Fig. 12.4

The procedure is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the area of the region included between the parabola $y = \frac{3x^2}{4}$ and the line $3x - 2y + 12 = 0$.*

SOLUTION The given equations are

$$y = \frac{3x^2}{4} \quad ...(i)$$

and, $3x - 2y + 12 = 0$ ...(ii)

Equation (i) represents a parabola having vertex at the origin, axis along the positive direction of y-axis and opens upwards. A free hand sketch of the parabola $y = \frac{3x^2}{4}$ is shown in Fig. 12.5.

Equation $3x - 2y + 12 = 0$ represents a straight line. Putting $y = 0$ and $x = 0$ respectively in (ii), we obtain $x = -4$ and $y = 6$ respectively. So, the straight line given by (ii) meets x-axis at $(-4, 0)$ and y-axis at $(0, 6)$.

A rough sketch of the curves represented by (i) and (ii) is shown in Fig. 12.5.

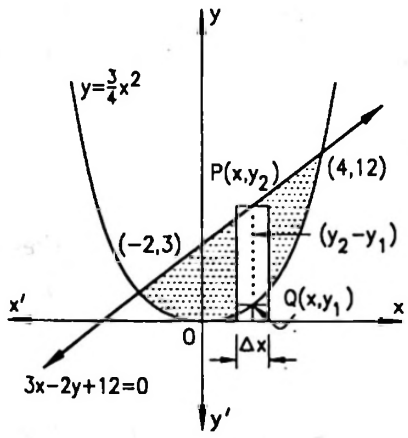


Fig. 12.5

In order to find the points of intersection of the given parabola and the line, we solve (i) and (ii) simultaneously solving.

These two equations simultaneously, we find that the two curves intersect at the points (– 2, 3) and (4, 12). Here, we slice the shaded area into vertical strips. We find that each vertical strip runs from the parabola to the line. So, the approximating rectangle shown in Fig. 12.5 has width Δx, length $(y_2 - y_1)$ and the area $(y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = -2$ to $x = 4$. So,

Required area

$$= \int_{-2}^{4} (y_2 - y_1)\,dx$$

$$= \int_{-2}^{4} \left(\frac{3x+12}{2} - \frac{3}{4}x^2\right) dx \qquad \left[\because (x, y_1) \text{ and } (x, y_2) \text{ lie on (i) and (ii) respectively } \therefore\ y_1 = \frac{3x+12}{2} \text{ and } y_2 = \frac{3x^2}{4}\right]$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

= 27 sq. units.

EXAMPLE 2 *Find the area of the region bounded by the curves* $y = x^2 + 2$, $y = x$, $x = 0$ *and* $x = 3$.

SOLUTION The equations of the given curves are

$$y = x^2 + 2 \quad \text{...(i)}$$
$$y = x \quad \text{...(ii)},$$
$$x = 0 \quad \text{...(iii)}$$

and, $$x = 3 \quad \text{...(iv)}$$

Now, $y = x^2 + 2 \Rightarrow (x-0)^2 = (y-2)$ ...(i)

So, equation (i) represents a parabola with vertex at (0, 2) and axis along positive direction of y-axis. This parabola opens upwards as shown in Fig. 12.6. Clearly, $y = x$ is a line passing through the origin and makes 45° angle with x-axis, $x = 0$ is y-axis and $x = 3$ is a line parallel to y-axis at a distance of 3 units from it. A rough sketch of these curves is shown in Fig. 12.6 and the region bounded by these curves is shaded.

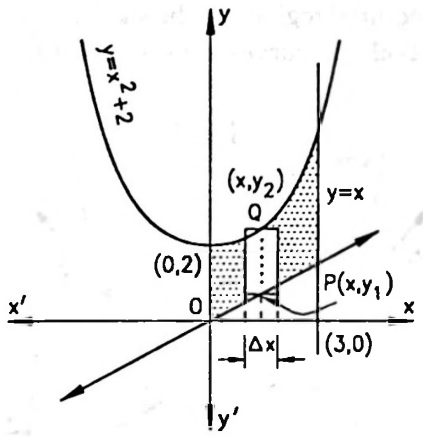


Fig. 12.6

Here, we slice this region into vertical strips. We observe that each vertical strip runs from the line $y = x$ to the parabola $y = x^2 + 2$. So, the approximating rectangle shown in Fig. 12.6 has, width $= \Delta x$, length $= (y_2 - y_1)$ and area $= (y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = 0$ to $x = 3$. So,

Required area

$$= \int_{0}^{3} (y_2 - y_1)\,dx$$

$$= \int_{0}^{3} [(x^2 + 2) - x]\,dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on respectively (ii) and (i)}\ \therefore\ y_1 = x \text{ and } y_2 = x^2 + 2\right]$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

$$= 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

EXAMPLE 3 *Find the area of the region* $\{(x, y) : x^2 \le y \le |x|\}$.

SOLUTION Let

$$R = \{(x, y) : x^2 \le y \le |x|\}$$

$$= \{(x, y) : x^2 \le y\} \cap \{(x, y) : y \le |x|\}$$

$$= \{(x, y) : x^2 \le y\} \cap [\{(x, y) : y \le x, x \ge 0\} \cup \{(x, y) : y \le -x, x < 0\}]$$

$$= R_1 \cap (R_2 \cup R_3),$$

where

$$R_1 = \{(x, y) : x^2 \le y\},$$

$$R_2 = \{(x, y) : y \le x, x \ge 0\}$$

and, $R_3 = \{(x, y) : y \le -x, x < 0\}$.

REGION R_1: Clearly, $x^2 = y$ represents a parabola with vertex at (0, 0), positive direction of y-axis as its axis and it opens upwards. The interior of the parabola is the region R_1.

REGION R_2: Clearly, $y = x$, $x \ge 0$ is a line passing through the origin and making an angle of 45° with the positive direction of x-axis. So, R_2 is the region lying below the line $y = x$.

REGION R_3: Clearly, $y = x$, $x < 0$ is a line passing through the origin and making an angle of 135° with the positive direction of x-axis. So, R_3 is the region lying below the line $y = -x$.

Hence, the required region R is the shaded region shown in Fig. 12.7. Since both the curves are symmetrical about y-axis. So,

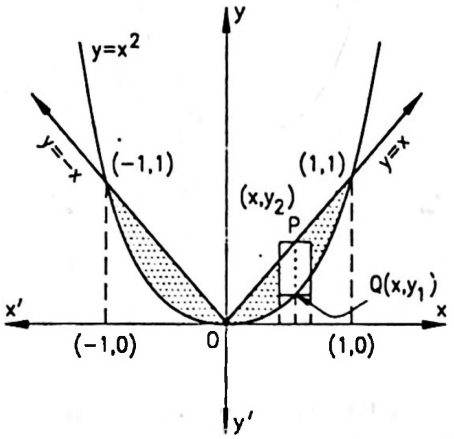


Fig. 12.7

Required area = 2 (Shaded area in first quadrant)

$$= 2\int_0^1 (x - x^2)\,dx = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq. units}$$

EXAMPLE 4 *Find the area of the region*

$$\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\}.$$

SOLUTION Let

$R = \{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\}$

$= \{(x, y) : 0 \le y \le x^2 + 1\} \cap \{(x, y) : 0 \le y \le x + 1\} \cap \{(x, y) : 0 \le x \le 2\}$
$= R_1 \cap R_2 \cap R_3$,

where

$$R_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$

$$R_2 = \{(x, y) : 0 \le y \le x + 1\}$$

and, $R_3 = \{(x, y) : 0 \le x \le 2\}$.

REGION R_1: We have $y = x^2 + 1$ or $(x - 0)^2 = (y - 1)$. Clearly, this equation represents a parabola with vertex at (0, 1), axis along the positive direction of y-axis and it opens upwards. Since, $0 \le y \le x^2 + 1 \Leftrightarrow y \ge 0$ and $y \le x^2 + 1$. So, R_1 is the region lying above x-axis and outside the parabola $y = x^2 + 1$.

REGION R_2: Clearly, $y = x + 1$ represents a straight line cutting x-axis at (−1, 0) and y-axis at (0, 1). Since $0 \le y \le x + 1 \Leftrightarrow y \ge 0$ and $y \le x + 1$. Therefore, R_2 is the region lying above x-axis and below the line $y = x + 1$.

REGION R_3: Clearly, R_3 is the region lying between the lines $x = 0$ i.e. y-axis and $x = 2$.

Hence, the required region R is the shaded region as shown in Fig. 12.8.

Solving $y = x^2 + 1$ and $y = x + 1$, obtain that the two curves intersect at the points (0, 1) and (1, 2).

Now, we slice the shaded region into vertical strips. We observe that vertical strips change their character at the point B. Draw a line BC parallel to the y-axis which divides the area $OABDEO$ into two portions $OABCO$ and $BDECB$. For the area $OABCO$, the approximating rectangle has, length = y_1, width = Δx and area = $y_1\,\Delta x$. As it can move from $x = 0$ to $x = 1$.

So, area $OABCO = \int_0^1 y_1\,dx$

$$= \int_0^1 (x^2 + 1)\,dx \quad \left[\because P(x, y_1) \text{ lies on } y = x^2 + 1,\ \therefore\ y_1 = x^2 + 1\right]$$

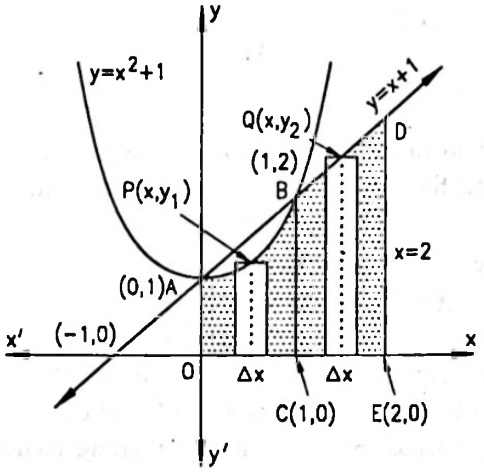


Fig. 12.8

For the area $BDECB$, the approximating rectangle shown in Fig. 12.8 has, length = y_2, width = Δx and area = $y_2\,\Delta x$. Since it can move from $x = 1$ to $x = 2$.

So, area $BDECB = \int_1^2 y_2\,dx$

$$= \int_1^2 (x + 1)\,dx \quad \left[\because Q(x, y_2) \text{ lies on } y = x + 1,\ \therefore\ y_2 = x + 1\right]$$

Hence,

Required area

= Area $OABCO$ + Area $BDECB$

$$= \int_0^1 (x^2 + 1)\,dx + \int_1^2 (x + 1)\,dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2$$

$$= \frac{23}{6} \text{ sq. units}$$

EXAMPLE 5 *Find the area bounded by the curve* $y^2 = 4a^2(x - 1)$ *and the lines* $x = 1$ *and* $y = 4a$.

SOLUTION We have,

$$y^2 = 4a^2(x-1) \text{ or, } (y-0)^2 = 4a^2(x-1).$$

Clearly, this equation represents a parabola with vertex at (1, 0) as shown in Fig. 12.9. The area enclosed by $y^2 = 4a^2(x-1)$, $x = 1$ and $y = 4a$ is the area of shaded portion in Fig. 12.9.

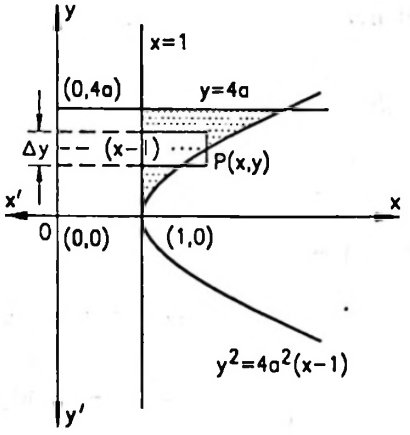


Fig. 12.9

When we slice the area of the shaded portion in horizontal strips, we observe that each strip has its one end on the line $x = 1$ and the other end on the parabola $y^2 = 4a^2(x-1)$. So, the approximating rectangle shown in Fig. 12.9 has, length $= x - 1$, width $= \Delta y$ and area $= (x-1)\,\Delta y$. Since, the approximating rectangle can move from $y = 0$ to $y = 4a$.

$\therefore$ Required area

$$= \int_0^{4a} (x-1)\,dy$$

$$= \int_0^{4a} \frac{y^2}{4a^2}\,dy \qquad \left[\because P(x,y) \text{ lies on } y^2 = 4a^2(x-1) \;\therefore\; x - 1 = y^2/4a^2\right]$$

$$= \frac{1}{4a^2}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{1}{4a^2}\left(\frac{64a^3}{3}\right) = \frac{16a}{3} \text{ sq. units}$$

EXAMPLE 6 *Find the area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.*

SOLUTION The curve $y = 2x - x^2$ representes a parabola opening downward and crossing x-axis at (0, 0) and (2, 0). Clearly, $y = -x$ represents a line passing through the origin and making 135° with x-axis. A rough sketch of the two curves is shown in Fig. 12.10. The region whose area is to be found is shaded in Fig. 12.10. Here, we slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 12.10, we have,

Width $= \Delta x$

Length $= (y_1 - y_2)$

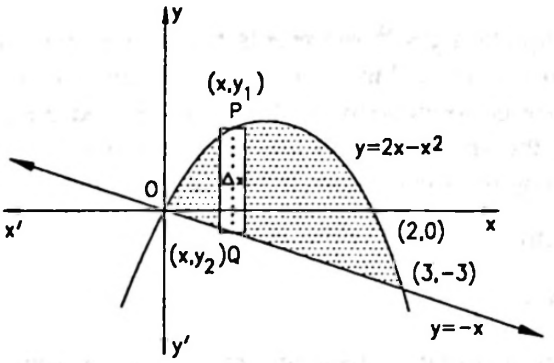


Fig. 12.10

$\therefore$ Area $= (y_1 - y_2)\,\Delta x$

The approximating rectangle can move horizontal between $x = 0$ and $x = 3$.

$\therefore$ Required area

$$= \int_0^3 (y_1 - y_2)\,dx$$

$$= \int_0^3 \{2x - x^2 - (-x)\}\,dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2x - x^2 \text{ and } y = -x \text{ respectively } \therefore\; y_1 = 2x - x^2 \text{ and } y_2 = -x\right]$$

$$= \int_0^3 (3x - x^2)\,dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{9}{2} \text{ sq. units}$$

EXAMPLE 7 *Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.*

SOLUTION The equation $y = 2x - x^2$ represents a parabola opening downwards having vertex at (1, 1) and crossing x-axis at (0, 0) and (2, 0).

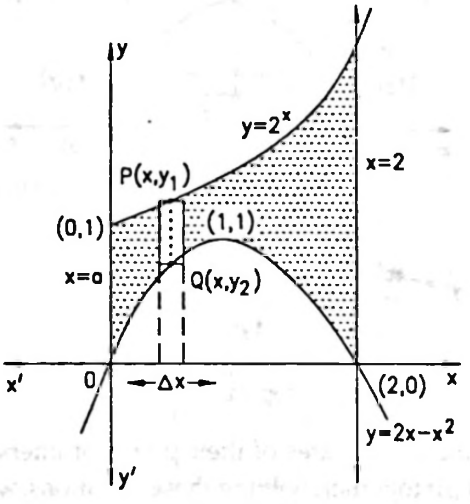


Fig. 12.11

The equation $y = 2^x$ represents the exponential curve as shown in Fig. 12.11. Lines $x = 0$ and $x = 2$ are shown in Fig. 12.11. The area bounded by these curves is shaded in Fig. 12.11. We slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 12.11, we have,

Length $= (y_1 - y_2)$, width $= \Delta x$

$\therefore$ Area $= (y_1 - y_2)\,\Delta x$

The approximating rectangle can move horizontally between $x = 0$ and $x = 2$. So,

Required area

$$= \int_0^2 (y_1 - y_2)\,dx$$

$$= \int_0^2 (2^x - 2x + x^2)\,dx \qquad \left[\begin{array}{l}\because\ P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on} \\ y = 2^x \text{ and } y = 2x - x^2 \text{ respectively.} \\ \therefore\ y_1 = 2^x \text{ and } y_2 = 2x - x^2\end{array}\right]$$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2} = \frac{3}{\log 2} - \frac{4}{3}$$

EXAMPLE 8 *Find the area bounded by the curvess* $y^2 = 4a\,(x + a)$ *and* $y^2 = 4b\,(b - x)$, *where* $a, b > 0$.

SOLUTION The two curves are :

$$y^2 = 4a\,(x + a) \qquad \ldots(i)$$

and, $$y^2 = -4a\,(x - b) \qquad \ldots(ii)$$

Clearly, these two curves represent two parabolas having their vertices at $(-a, 0)$ and $(b, 0)$ respectively as shown in Fig. 12.12.

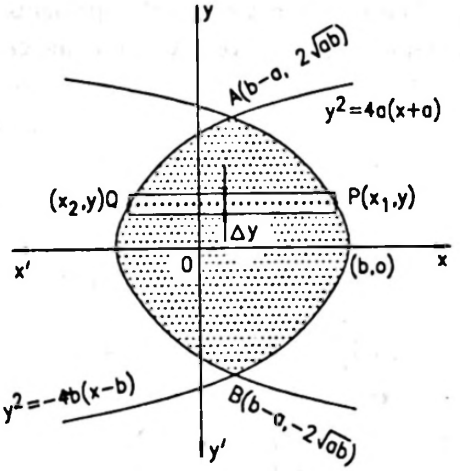


Fig. 12.12

To find the coordinates of their points of intersection, we solve (i) and (ii) together. Solving these equations, we find that the two curves intersect at $A\,(b - a, 2\sqrt{ab})$ and $B\,(b - a, -2\sqrt{ab})$. We have to find the area of the shaded region in Fig. 12.12. Let us slice this region into horizontal strips. For the approximating rectangle shown in Fig. 12.12, we have

Length $= PQ = (x_1 - x_2)$, Width $= \Delta y$ and Area $= (x_2 - x_1)\,\Delta y$

The approximating rectangle can move vertically between B and A.

$\therefore$ Required area

$$= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} (x_1 - x_2)\,dy$$

$$= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left\{\left(b - \frac{y^2}{4b}\right) - \left(\frac{y^2}{4a} - a\right)\right\} dy$$

$$\left[\begin{array}{r}\because\ P(x_1, y) \text{ and } Q(x_2, y) \text{ lie on} \\ \text{(ii) and (i) respectively} \\ \therefore\ y^2 = -4b\,(x_1 - b) \text{ and } y^2 = 4a\,(x_2 + a)\end{array}\right]$$

$$= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left\{a + b - \left(\frac{a+b}{4ab}\right)y^2\right\} dy$$

$$= 2\int_0^{2\sqrt{ab}} \left\{a + b - \left(\frac{a+b}{4ab}\right)y^2\right\} dy$$

$$= 2\left[(a+b)\,y - \left(\frac{a+b}{4ab}\right)\frac{y^3}{3}\right]_0^{2\sqrt{ab}}$$

$$= 2\left[(a+b)\,2\sqrt{ab} - \left(\frac{a+b}{4ab}\right)\frac{8ab\sqrt{ab}}{3}\right]$$

$$= 2\left[(a+b)\,2\sqrt{ab} - \frac{2}{3}(a+b)\sqrt{ab}\right]$$

$$= \frac{8}{3}(a+b)\sqrt{ab} \text{ sq. units.}$$

EXAMPLE 9 *Sketch the curves and identify the region bounded by the curves* $x = \frac{1}{2}$, $x = 2$, $y = \log x$ *and* $y = 2^x$. *Find the area of this region.*

[IIT 1991]

SOLUTION Since the inverse of a logarithmic function is an exponential function and vice-versa and these two curves are on the opposite sides of the line $y = x$. Thus, $y = 2^x$ and $y = \log x$ do not intersect. Their graphs are shown in Fig. 12.13. The shaded region in Fig. 12.13 shows the area bounded by the given curves. Let us slice this region into vertical strips as shown in Fig. 12.13. For the approximating rectangle shown in Fig. 12.13, we have

Length $= (y_1 - y_2)$, Width $= \Delta x$

Area $= (y_1 - y_2)\,dx$

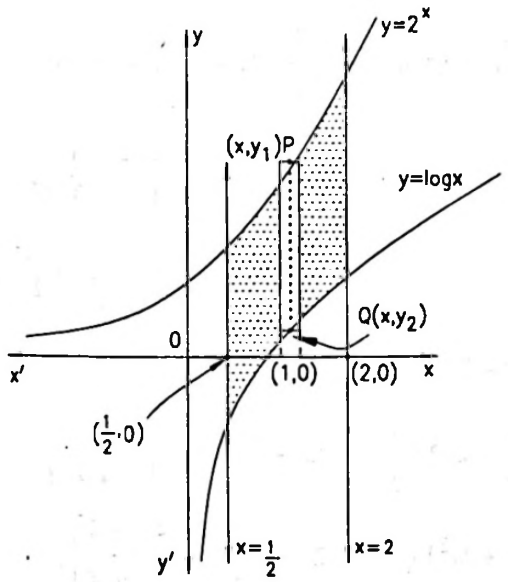


Fig. 12.13

As the approximating rectangle can move horizontally between $x = \frac{1}{2}$ and $x = 2$.

$\therefore$ Required area

$$= \int_{1/2}^{2} (y_1 - y_2)\, dx$$

$$= \int_{1/2}^{2} (2^x - \log x)\, dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2^x \text{ and } y = \log x \text{ respectively } \therefore y_1 = 2^x \text{ and } y_2 = \log x\right]$$

$$= \left[\frac{2^x}{\log 2} - x \log x + x\right]_{1/2}^{2}$$

$$= \left\{\frac{4}{\log 2} - 2 \log 2 + 2\right\} - \left\{\frac{\sqrt{2}}{\log 2} + \frac{1}{2} \log 2 + \frac{1}{2}\right\}$$

$$= \frac{(4 - \sqrt{2})}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \text{ sq. units}$$

EXAMPLE 10 *Let $[x]$ denote the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4[\sqrt{x}]y$ and $y^2 = 4[\sqrt{y}]x$ that lie within the square $\{(x, y) \mid 1 \le x \le 4, 1 \le y \le 4\}$. Find the area of the part of the square that is enclosed by the two curves and the line $x + y = 3$.*

SOLUTION We have,

$1 \le x \le 4$ and $1 \le y \le 4$

$\Rightarrow 1 \le \sqrt{x} \le 2$ and $1 \le \sqrt{y} \le 2$

$\Rightarrow [\sqrt{x}] = 1$ and $[\sqrt{y}] = 1$ for all (x, y) lying within the square.

Thus, the curves $x^2 = 4[\sqrt{x}]y$ and $y^2 = 4[\sqrt{y}]x$ are the parts of the two parabolas $x^2 = 4y$ and $y^2 = 4x$ respectively lying within the given square as shown in Fig. 12.14. We have to find the area of the shaded region. Let us slice this region into vertical strips. We observe that the vertical strips change their character at point E. So, we draw a line EG parallel to y-axis which divides the area into two regions EGF and ECG.

In the region ECF, the approximating rectangle has length $= (y_2 - y_1)$ and width $= \Delta x$ and it can move from $x = 1$ to $x = 2$.

$\therefore$ A_1

= Area of region EGF

$$= \int_{1}^{2} (y_2 - y_1)\, dx$$

$$= \int_{1}^{2} [\sqrt{4x} - (3 - x)]\, dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y^2 = 4x \text{ and } x + y = 3 \text{ respectively}\right]$$

$\therefore$ A_2 = Area of region ECF

$$= \int_{2}^{4} (y_4 - y_3)\, dx$$

$$= \int_{2}^{4} \left(\sqrt{4x} - \frac{x^2}{4}\right) dx \qquad \left[\because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on } x^2 = 4y \text{ and } y^2 = 4x \text{ respectively } \therefore y_3 = \frac{x^2}{4} \text{ and } y_4 = \sqrt{4}x\right]$$

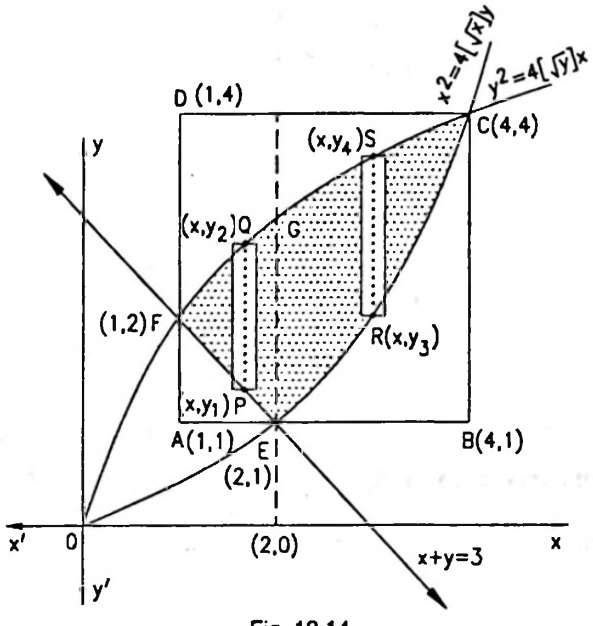


Fig. 12.14

Hence,

required area = $A_1 + A_2$

$$= \int_{1}^{2} (2\sqrt{x} - 3 + x)\, dx + \int_{2}^{4} \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{4}{3} x^{3/2} - 3x + \frac{x^2}{2}\right]_{1}^{2} + \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12}\right]_{2}^{4}$$

$$= \left(\frac{8\sqrt{2}}{3} - 6 + 2\right) - \left(\frac{4}{3} - 3 + \frac{1}{2}\right) + \left(\frac{32}{3} - \frac{64}{12}\right) - \left(\frac{8\sqrt{2}}{3} - \frac{8}{12}\right)$$

$$= \frac{8\sqrt{2}}{3} - 4 + \frac{7}{6} + \frac{16}{3} - \frac{8\sqrt{2}}{3} + \frac{2}{3}$$

$$= \frac{19}{6} \text{ sq. units.}$$

EXAMPLE 11 *The parabola $y^2 = ax$ cuts $x^2 - y^2 = 2a^2$ at P and Q. Tangent at P to the hyperbola cuts the parabola again in R. Find the area of the curvilinear triangle PQR.*

SOLUTION Solving $y^2 = ax$ and $x^2 - y^2 = 2a^2$ together, we obtain the coordinates of P and Q as $(2a, a\sqrt{2})$ and $(2a, -a\sqrt{2})$ respectively. The equation of the tangent to $x^2 - y^2 = 2a^2$ at $P(2a, a\sqrt{2})$ is

$$2ax - a\sqrt{2}y = 2a^2$$

or,
$$2x - \sqrt{2}y = 2a$$

Solving this equation with $y^2 = ax$, we obtain the coordinates of R as $(a/2, -a/\sqrt{2})$

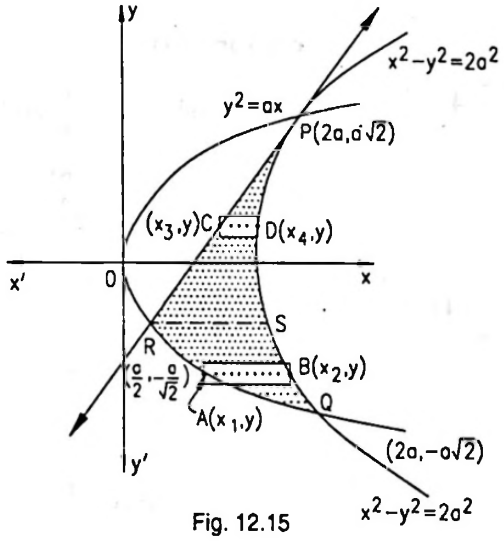


Fig. 12.15

Now,

Area of curve linear ΔPQR

= Area of curvilinear ΔQRS + Area of curvilinear trianglePRS.

$$= \int_{-a\sqrt{2}}^{-a/\sqrt{2}} (x_2 - x_1)\, dy + \int_{-a/\sqrt{2}}^{a\sqrt{2}} (x_4 - x_3)\, dy$$

$$= \int_{-a\sqrt{2}}^{-a/\sqrt{2}} \left(\sqrt{2a^2 + y^2} - \frac{y^2}{a}\right) dy + \int_{-a/\sqrt{2}}^{a\sqrt{2}} \left\{\sqrt{2a^2 + y^2} - \left(a + \frac{1}{\sqrt{2}}y\right)\right\} dy$$

$$= \int_{-a\sqrt{2}}^{-a/\sqrt{2}} \sqrt{2a^2 + y^2}\, dy + \int_{-a/\sqrt{2}}^{a\sqrt{2}} \sqrt{2a^2 + y^2}\, dy \quad - \int_{-a\sqrt{2}}^{-a/\sqrt{2}} \frac{y^2}{a}\, dy - \int_{-a/\sqrt{2}}^{a\sqrt{2}} \left(a + \frac{y}{\sqrt{2}}\right) dy$$

$$= \int_{-a\sqrt{2}}^{a\sqrt{2}} \sqrt{2a^2 + y^2}\, dy - \frac{1}{a} \int_{-a\sqrt{2}}^{a/\sqrt{2}} y^2\, dy \quad - \int_{-a/\sqrt{2}}^{a\sqrt{2}} \left(a + \frac{y}{\sqrt{2}}\right) dy$$

$$= 2\int_0^{a\sqrt{2}} \sqrt{2a^2 + y^2}\, dy - \frac{1}{a} \int_{-a\sqrt{2}}^{-a/\sqrt{2}} y^2\, dy \quad - \int_{-a/\sqrt{2}}^{a\sqrt{2}} \left(a + \frac{y}{\sqrt{2}}\right) dy$$

$$= \left[y\sqrt{2a^2 + y^2} + 2a^2 \log\{y + \sqrt{2a^2 + y^2}\}\right]_0^{a\sqrt{2}} - \frac{1}{3a}\left[y^3\right]_{-a\sqrt{2}}^{a/\sqrt{2}}$$
$$- \left[ay + \frac{y^2}{2\sqrt{2}}\right]_{-a/\sqrt{2}}^{a\sqrt{2}}$$

$$= [2\sqrt{2}\,a^2 + 2a^2 \log(a\sqrt{2} + 2a) - 2a^2 \log(\sqrt{2}a)]$$
$$- \frac{1}{3a}\left[\frac{-a^3}{2\sqrt{2}} + 2\sqrt{2}a^3\right] - \left[\left(\sqrt{2}a^2 + \frac{a^2}{\sqrt{2}}\right) - \left(\frac{-a^2}{\sqrt{2}} + \frac{a^2}{4\sqrt{2}}\right)\right]$$

$$= \left[2\sqrt{2a^2} + 2a^2 \log(1 + \sqrt{2}) - \frac{7a^2}{6\sqrt{2}} - \frac{3a^2}{\sqrt{2}} - \frac{3a^2}{4\sqrt{2}}\right]$$

$$= \left[2a^2 \log(1 + \sqrt{2}) - \frac{11\sqrt{2}}{24}a^2\right] \text{sq. units}$$

EXAMPLE 12 *Find all values of the parameter a $(a \geq 1)$ for which the area of the figure bounded by the pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]\,x^2$, $y = \frac{1}{2}[a]\,x^2$ is the greatest. Here $[\cdot]$ denotes the greatest integer function.*

SOLUTION The curves $y = [a]\,x^2$ and $y = \frac{1}{2}[a]\,x^2$ represent parabolas which are symmetric about y-axis. The equation $y^2 - 3y + 2 = 0$ gives a pair of straight lines $y = 1$, $y = 2$ which are parallel to x-axis. The shaded region in Fig. 12.16 determines the area bounded by the two parabolas and two lines. Let us slice this region into horizontal strips. For the approximating rectangle shown in Fig. 12.16, we have

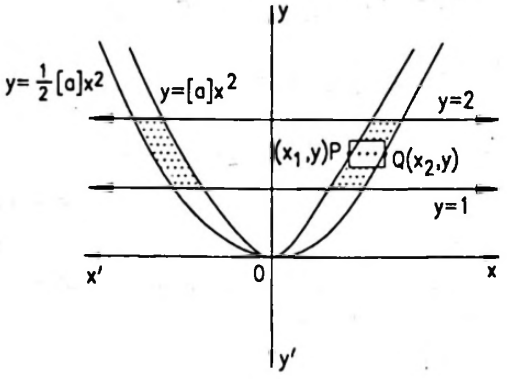


Fig. 12.16

Length $= x_2 - x_1$, Width $= \Delta y$, Area $= (x_2 - x_1)\, dy$

As it can move vertically between $y = 1$ and $y = 2$. So,

Required area

$$= 2\int_1^2 (x_2 - x_1)\, dy$$

$$= 2\int_1^2 \left(\sqrt{\frac{2y}{[a]}} - \sqrt{\frac{y}{[a]}}\right) dy \quad \left[\begin{array}{l}\because\ P(x_1, y) \text{ and } Q(x_2, y) \text{ lie on} \\ y = [a]\,x^2 \text{ and } y = \frac{1}{2}[a]\,x^2 \text{ respectively.} \\ \therefore\ [a]\,x_1^2 = y \text{ and } \frac{1}{2}[a]\,x_2^2 = y\end{array}\right]$$

$$= 2\int_1^2 \left(\frac{\sqrt{2}-1}{\sqrt{[a]}}\right)\sqrt{y}\,dy$$

$$= \frac{2(\sqrt{2}-1)}{\sqrt{[a]}}\int_1^2 \sqrt{y}\,dy$$

$$= \frac{2(\sqrt{2}-1)}{\sqrt{[a]}} \times \frac{2}{3}\left[y^{3/2}\right]_1^2$$

$$= \frac{4(\sqrt{2}-1)}{3\sqrt{[a]}}(2^{3/2}-1)$$

$$= \frac{4}{3}\frac{(5-\sqrt{2}-2\sqrt{2})}{\sqrt{[a]}}$$

Clearly, A will be greatest, if $[a]$ is least. It is given that $a \geq 1$. Therefore, the least value of $[a]$ is 1.

Thus, A is greatest when

$$[a] = 1 \Rightarrow a \in [1, 2).$$

EXAMPLE 13 *Find the area enclosed by the x-axis and the curve which passes through the point* (0, 2) *and satisfies the differential equation*

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0.$$

SOLUTION We have,

$$x\left(\frac{dy}{dx}\right)^2 - 2y\left(\frac{dy}{dx}\right) - x = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{2y \pm \sqrt{4y^2+4x^2}}{2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \pm \sqrt{x^2+y^2}}{x}$$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = v \pm \sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \pm\sqrt{1+v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \pm\frac{dx}{x}$$

$$\Rightarrow \log\left(v + \sqrt{1+v^2}\right) = \pm\log x + \log C$$

$$\Rightarrow v + \sqrt{1+v^2} = Cx \quad \text{or,} \quad v + \sqrt{1+v^2} = \frac{C}{x}$$

$$\Rightarrow y + \sqrt{x^2+y^2} = Cx^2 \quad \text{or,} \quad y + \sqrt{x^2+y^2} = C$$

$$\Rightarrow (Cx^2 - y)^2 = x^2 + y^2 \quad \text{or,} \quad (C-y)^2 = x^2 + y^2$$

$$\Rightarrow C^2x^4 - 2Cx^2y = x^2 \quad \text{or,} \quad C^2 - 2cy = x^2$$

$$\Rightarrow C^2x^2 - 2Cy = 1 \quad \text{or,} \quad x^2 + 2Cy - C^2 = 0$$

$$\Rightarrow x^2 - \frac{2}{C}y - \frac{1}{C^2} = 0 \quad \text{or,} \quad x^2 + 2Cy - C^2 = 0$$

Since the curve passes through (0, 2). Therefore,

$$x^2 - \frac{2}{C}y - \frac{1}{C^2} = 0 \Rightarrow -\frac{4}{C} - \frac{1}{C^2} = 0 \Rightarrow C = -\frac{1}{4}$$

and, $\quad x^2 + 2Cy - C^2 = 0 \Rightarrow 4C - C^2 = 0 \Rightarrow C = 4$

Substituting these values of C in the respective equations, we obtain the equation of the required curve as

$$x^2 + 8y - 16 = 0 \Rightarrow x^2 = -8(y-2)$$

This represents a parabola opening downwards and cutting x-axis at $(\pm 4, 0)$. The shaded region in Fig. 12.17 represents the required area. We slice this region into vertical strips. For the approximating rectangle shown in Fig. 12.17, we have

Length $= y$, width $= \Delta x$, Area $= y\,\Delta x$.

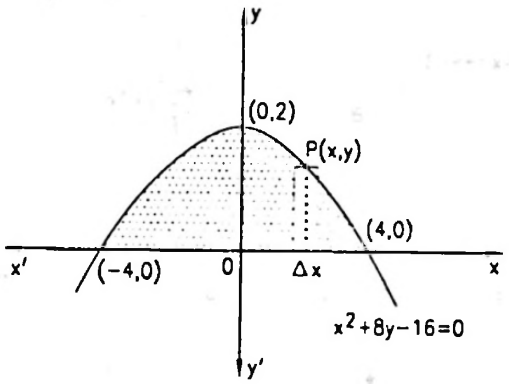


Fig. 12.17

As the approximating rectangle can move between $x = -4$ and $x = 4$. So,

$$\text{Required area} = \int_{-4}^{4} y\,dx$$

$$= \int_{-4}^{4} \frac{16-x^2}{8}\,dx$$

$$= 2\int_0^4 \left(2 - \frac{x^2}{8}\right)dx$$

$$= 2\left[2x - \frac{x^3}{24}\right]_0^4$$

$$= \frac{32}{3} \text{ sq. units.}$$

EXAMPLE 14 *Find the area of the closed figure bounded by* $x = -1$, $y = 0$, $y = x^2 + x + 1$ *and the tangent to the curve* $y = x^2 + x + 1$ *at* $(1, 3)$.

SOLUTION We have,

$$y = x^2 + x + 1$$

$$\Rightarrow \frac{dy}{dx} = 2x + 1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,3)} = 3.$$

The equation of the tangent to $y = x^2 + x + 1$ at $(1, 3)$ is

$$y - 3 = 3(x - 1) \Rightarrow y = 3x.$$

The equation $y = x^2 + x + 1$ represents a parabola opening upward and having vertex at $\left(-\frac{1}{2}, \frac{3}{4}\right)$. Its graph is shown in Fig. 12.18. The area enclosed by $x = -1$, $y = 0$, $y = x^2 + x + 1$ and $y = 3x$ is shaded in Fig. 12.18. We slice the shaded region into vertical strips.

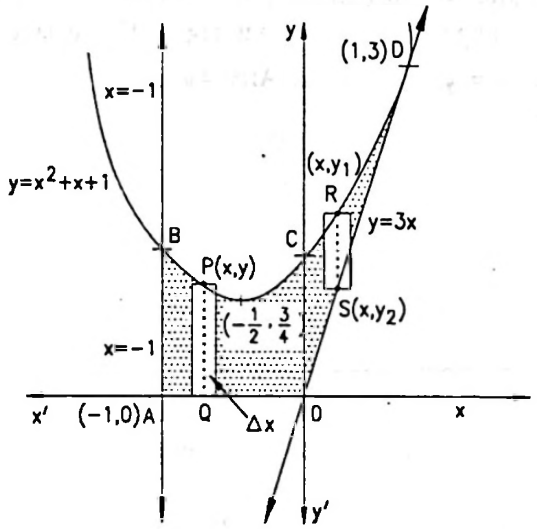


Fig. 12.18

Required area = Area of region $OABC$ + Area of region OCD

$$= \int_{-1}^{0} y\,dx + \int_{0}^{1} (y_1 - y_2)\,dx$$

$$= \int_{-1}^{0} (x^2 + x + 1)\,dx + \int_{0}^{1} (x^2 + x + 1 - 3x)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} + \left[\frac{x^3}{3} - x^2 + x\right]_{0}^{1}$$

$$= -\left(-\frac{1}{3} + \frac{1}{2} - 1\right) + \left(\frac{1}{3} - 1 + 1\right)$$

$$= \frac{7}{6} \text{ sq. units.}$$

EXAMPLE 15 *Find the area of the figure bounded by the parabola* $(y - 2)^2 = (x - 1)$, *the tangent to it at the point with ordinate 3, and the x-axis.*

SOLUTION Putting $y = 3$ in $(y - 2)^2 = x - 1$, we obtain $x = 2$.

The equation of the parabola is $(y - 2)^2 = (x - 1)$

Differentiating w.r.t x, we get

$$2(y - 2)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y - 2)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{1}{2}.$$

The equation of the tangent at $(2, 3)$ to the parabola $(y - 2)^2 = (x - 1)$ is

$$y - 3 = \frac{1}{2}(x - 2)$$

or, $x - 2y + 4 = 0$

The parabola $(y - 2)^2 = (x - 1)$ cuts x-axis at $Q(5, 0)$. The tangent $P(2, 3)$ cuts x-axis

The shaded region in Fig. 12.19 is the region enclosed by the given curves. We slice this region into horizontal strips. The approximating rectangle shown in Fig. 12.19 has length

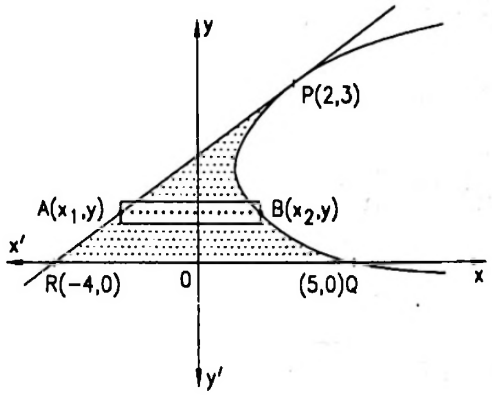


Fig. 12.19

$= x_2 - x_1$, width $= \Delta y$ and it can move between $y = 0$ and $y = 3$.

$\therefore$ Required area

$$= \int_{0}^{3} (x_2 - x_1)\,dy$$

$$= \int_{0}^{3} |(y^2 - 4y + 5) - (2y - 4)|\,dy$$

$$= \int_{0}^{3} (y^2 - 6y + 9)\,dy$$

$$= \left[\frac{y^3}{3} - 3y^2 + 9y\right]_0^3 = 9 - 27 + 27 = 9 \text{ sq. units}$$

EXAMPLE 16 *Find the smaller of the region bounded by the curves* $(x-1)^2 + 4(y-1)^2 = 4$ *and* $4(y-1)^2 = 3(x-1)$.

SOLUTION The equation $(x-1)^2 + 4(y-1)^2 = 4$ represents an ellipse with centre (1, 1) and the equation $4(y-1)^2 = 3(x-1)$ respresents a parabola with vertex (1, 1). If we shift the origin at (1, 1), then the equation of the two curves become

$$X^2 + 4Y^2 = 4 \quad \text{...(i)}$$

and,
$$4Y^2 = 3X \quad \text{...(ii)}$$

Since the area of the region bounded by the given curves does not change by the shifting of origin. So,

Required area = Area enclosed by (i) and (ii) in the new system.

Solving (i) and (ii), we obtain the coordinates of their points of intersection as marked in Fig. 12.20. The region enclosed by the curves (i) and (ii) is shaded in Fig. 12.20. For the approximating rectangle *LM* shown in Fig. 12.20, we have

Length = $(x_2 - x_1)$, width = dy.

As it can move vertically from $B(1, -\sqrt{3}/2)$ to $A(1, \sqrt{3}/2)$. Therefore,

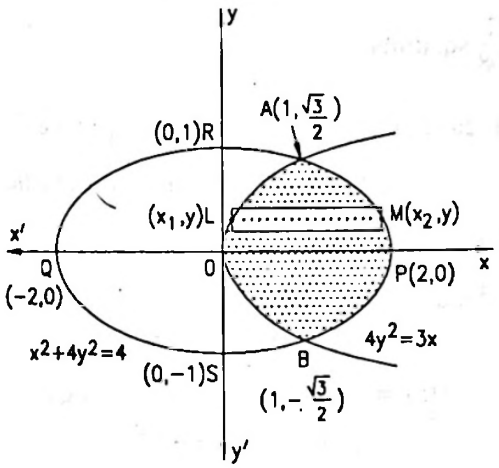


Fig. 12.20

Required area

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (x_2 - x_1)\, dy$$

$$= 2\int_0^{\sqrt{3}/2} (x_2 - x_1)\, dy \qquad \left[\because \text{Both the curves are symmetric about X-axis}\right]$$

$$= 2\int_0^{\sqrt{3}/2} \left\{\sqrt{4 - 4y^2} - \frac{4y^2}{3}\right\} dy$$

$$= 4\int_0^{\sqrt{3}/2} \sqrt{1 - y^2}\, dy - \frac{8}{3}\int_0^{\sqrt{3}/2} y^2\, dy$$

$$= 2\left[y\sqrt{1 - y^2} + \sin^{-1} y\right]_0^{\sqrt{3}/2} - \frac{8}{9}\left[y^3\right]_0^{\sqrt{3}/2}$$

$$= 2\left[\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right] - \frac{8}{9} \times \frac{3\sqrt{3}}{8}$$

$$= \left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}}\right) \text{sq. units}$$

EXAMPLE 17 *Find the area of the region bounded by*

$$|y| = 1 - x^2.$$

SOLUTION We have,

$$(1 - x^2) = |y|$$

$$= \begin{cases} y, \text{ if } y \geq 0 \\ -y, \text{ if } y < 0 \end{cases}$$

Thus, the two curves are :

$$y = 1 - x^2, \text{ for } y \geq 0$$

and,
$$y = x^2 - 1, \text{ for } y < 0.$$

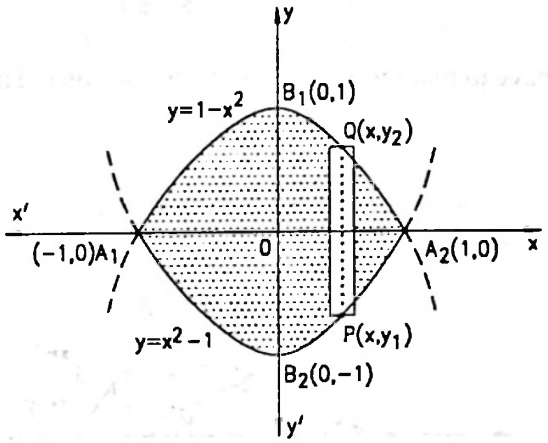


Fig. 12.21

The graphs of these two curves are shown in Fig. 12.21.

We have to find the area of the shaded region in Fig. 12.21. We slice this region into vertical strips. The approximating rectangle shown in Fig. 12.21 has

Length = $y_2 - y_1 = (1 - x^2) - (x^2 - 1) = 2(1 - x^2)$, width = Δy

It can move horizontally between $A_1(-1, 0)$ and $A_2(1, 0)$

$\therefore$ Required area

$$= \int_{-1}^{1} (y_2 - y_1)\, dx$$

$$= 2\int_{-1}^{1} (1 - x^2)\, dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 2\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{8}{3} \text{ sq. units}$$

EXAMPLE 18 *Let* $f(x) = \text{maximum } \{x^2, (1-x)^2, 2x(1-x)\}$. *Determine the area of the region bounded by the curve* $y = f(x)$, *x-axis,* $x = 0$ *and* $x = 1$.

[IIT 1997]

SOLUTION Let us first draw the graphs of the following curves :

$$y = x^2 \quad ...(i)$$
$$y = (1-x)^2 \quad ...(ii)$$
$$y = 2x(1-x) \quad ...(iii)$$

in the interval [0, 1].

All these curves are parabolas. Their points of intersection are marked in Fig. 12.22.

It is evident from the graph that

$$f(x) = \begin{cases} (1-x)^2, & 0 \le x \le \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \le x \le \frac{2}{3} \\ x^2, & \frac{2}{3} \le x \le 1 \end{cases}$$

We have to find the area of the shaded region in Fig. 12.22.

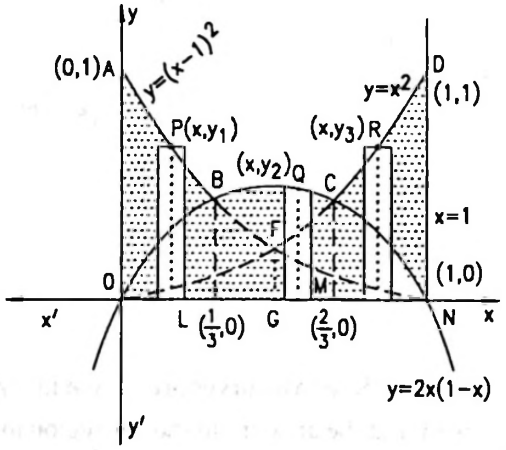


Fig. 12.22

Required Area

= (Area $OABLO$) + (Area $LBCM$) + (Area ($CMND$)

$$= \int_0^{1/3} y_1\,dx + \int_{1/3}^{2/3} y_2\,dx + \int_{2/3}^{1} y_3\,dx$$

$$= \int_0^{1/3} (x-1)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2\,dx$$

$$= \left[\frac{(x-1)^3}{3}\right]_0^{1/3} + \left[x^2 - \frac{2x^3}{3}\right]_{1/3}^{2/3} + \left[\frac{x^3}{3}\right]_{2/3}^{1}$$

$$= \frac{1}{3}\left[\frac{-8}{27} + 1\right] + \left[\left(\frac{4}{9} - \frac{16}{81}\right) - \left(\frac{1}{9} - \frac{2}{81}\right)\right] + \frac{1}{3}\left[1 - \frac{8}{27}\right]$$

$$= \frac{38}{81} + \frac{20}{81} - \frac{7}{81} = \frac{51}{81} = \frac{17}{27} \text{ sq. units}$$

EXAMPLE 19 *Let*

$$f(x) = \begin{cases} \min\{(x-1)^2, 2x(1-x)\}, & \text{if } 0 \le x < 1/2 \\ \min\{2x(1-x), x^2\}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Draw a rough sketch of the curve $y = f(x)$ *and find the area bounded by the curve and x-axis.*

SOLUTION Proceeding as in Ex. 18, the graph of $f(x)$ will be the curve $OBFGNO$ and $f(x)$ will be given by

$$f(x) = \begin{cases} 2x(1-x), & 0 \le x \le \frac{1}{3} \\ (x-1)^2, & \frac{1}{3} \le x \le \frac{1}{2} \\ x^2, & \frac{1}{2} \le x \le \frac{2}{3} \\ 2x(1-x), & \frac{2}{3} \le x \le 1 \end{cases}$$

Required area = Area of the region $OBLO$ + Area of the region $BFGL$ + Area of the region $FCMG$ + Area of the region $CMND$

$$= \int_0^{1/3} 2x(1-x)\,dx + \int_{1/3}^{1/2} (x-1)^2\,dx + \int_{1/2}^{2/3} x^2\,dx + \int_{2/3}^{1} 2x(1-x)\,dx$$

$$= \frac{31}{108} \text{ sq. units}$$

EXAMPLE 20 *Let* $f(x) = \text{minimum } \left\{e^x, \frac{3}{2}, 1 + e^{-x}\right\}, 0 \le x \le 1$. *Find the area bounded by* $y = f(x)$, *x-axis, y-axis and the line* $x = 1$.

SOLUTION It is evident from the graph of $y = e^x, y = \frac{3}{2}$ and $y = 1 + e^{-x}$ that

$$f(x) = \begin{cases} e^x, & 0 \le x \le \log(3/2) \\ 3/2, & \log 3/2 \le x \le \log 2 \\ 1 + e^{-x}, & \log 2 \le x \le 1 \end{cases}$$

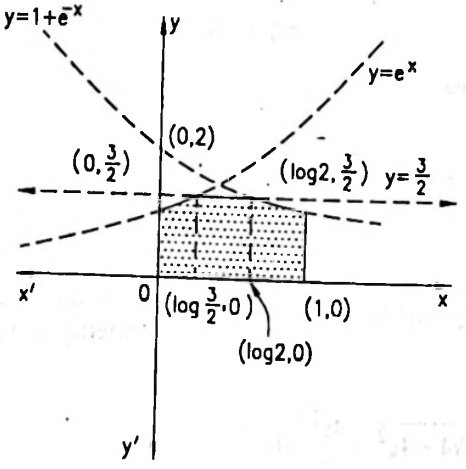


Fig. 12.23

$\therefore$ Required area

$$= \int_0^{\log 3/2} e^x\,dx + \int_{\log(3/2)}^{\log 2} \frac{3}{2}\,dx + \int_{\log 2}^{1} (1+e^{-x})\,dx$$

$$= \left[e^x\right]_0^{\log(3/2)} + \frac{3}{2}\left[x\right]_{\log 3/2}^{\log 2} + \left[1-e^{-x}\right]_{\log 2}^{1}$$

$$= \left\{\frac{3}{2}-1\right\} + \frac{3}{2}\left\{\log 2 - \log\frac{3}{2}\right\} + \left\{1-\frac{1}{e}\right\} - \left\{\log 2 - \frac{1}{2}\right\}$$

$$= \left\{2 - \frac{1}{e} + \frac{3}{2}\log\frac{4}{3} - \log 2\right\} \text{ sq. units}$$

EXAMPLE 21 *Compute the area of the region bounded by the curves* $y = ex\log x$ *and* $y = \dfrac{\log x}{ex}$.

[IIT 1990]

SOLUTION The graphs of the curves $y = ex\log x$ and $y = \dfrac{\log x}{ex}$ are shown in Fig. 12.24.

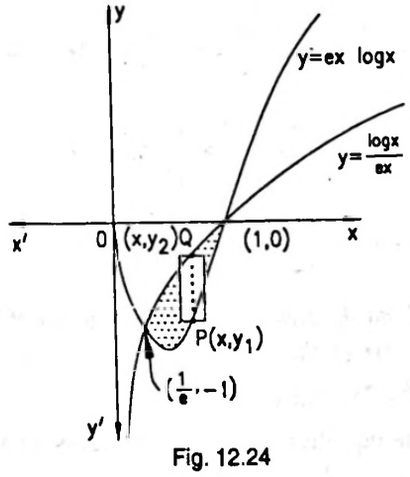


Fig. 12.24

The abscissa of the point of intersection of the curves $y = ex\log x$ and $y = \dfrac{\log x}{ex}$ are given by

$$ex\log x = \frac{\log x}{ex}$$

$$\Rightarrow e^2x^2\log x = \log x$$

$$\Rightarrow (e^2x^2-1)\log x = 0$$

$$\Rightarrow x = \frac{1}{e} \text{ and } x = 1$$

Required area

$$= \int_{1/e}^{1} (y_1 - y_2)\,dx$$

$$= \int_{1/e}^{1} \left\{ex\log x - \frac{\log x}{ex}\right\} dx$$

$$= e\int_{1/e}^{1} x\log x\,dx - \frac{1}{e}\int_{1/e}^{1} \frac{\log x}{x}\,dx$$

$$= e\left[(\log x)\frac{x^2}{2} - \frac{x^2}{4}\right]_{1/e}^{1} - \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1}$$

$$= e\left[\left(0-\frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right)\right] - \frac{1}{e}\left[0-\frac{1}{2}\right]$$

$$= e\left(-\frac{1}{4} + \frac{3}{4e^2}\right) + \frac{1}{2e}$$

$$= \frac{5}{4e} - \frac{e}{4}.$$

EXAMPLE 22 *Let* $f(x)$ *be defined by*

$$f(x) = \max\ (4\sin x, 4 - 2\sin x),\ 0 \le x \le 2\pi.$$

Draw a sketch of $y = f(x)$ *and compute the area bounded by the curve* $y = f(x)$, *the y-axis, the x-axis and the ordinate at* $x = 2\pi$.

SOLUTION The graph of $f(x)$ is shown by the continuous curve shown in Fig. 12.25. At the points of intersection of $y = 4\sin x$ and $y = 4 - 2\sin x$, we have

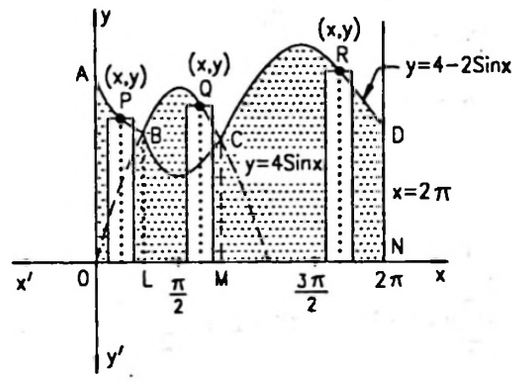


Fig. 12.25

$$4\sin x = 4 - 2\sin x \Rightarrow \sin x = \frac{2}{3}$$

Thus, the coordinates of B and C are $\left(\sin^{-1}\frac{2}{3}, \frac{8}{3}\right)$ and $\left(\pi - \sin^{-1}\frac{2}{3}, \frac{8}{3}\right)$ respectively.

Now, rquired area

= Area of region $OABL$ + Area of region $LBCM$ + Area of region $MCDN$

$$= \int_0^{\sin^{-1}2/3} (4-2\sin x)\,dx + \int_{\sin^{-1}2/3}^{\pi-\sin^{-1}2/3} 4\sin x\,dx + \int_{\pi-\sin^{-1}2/3}^{2\pi} (4-2\sin x)\,dx$$

$$= \left[4x + 2\cos x\right]_0^{\sin^{-1}2/3} - 4\left[\cos x\right]_{\sin^{-1}2/3}^{\pi-\sin^{-1}2/3} + \left[4x + 2\cos x\right]_{\pi-\sin^{-1}2/3}^{2\pi}$$

$$= \left[4\sin^{-1}\frac{2}{3} + 2\frac{\sqrt{5}}{3} - 2\right] - 4\left[-\frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3}\right]$$

$$+ \left[(8\pi + 2) - (4\pi - 4\sin^{-1}\frac{2}{3}) - 2\left(1 - \frac{\sqrt{5}}{3}\right)\right]$$

$$= \left(4\pi + 8\sin^{-1}\frac{2}{3} + 4\sqrt{5}\right) \text{ sq. units}$$

EXAMPLE 23 *Let $f(x)$ be a continuous function defined by*

$$f(x) = \begin{cases} 4\sin\frac{x}{2}, x \le \frac{\pi}{2} \\ A(x-\pi)^2 + 2, x > \frac{\pi}{2} \end{cases}$$

Find the area bounded by the curve $y = f(x)$, the x-axis and the line $\pi y = 4\sqrt{2}(x - \pi)$.

SOLUTION Since $f(x)$ is everywhere continuous. Therefore,

$$\lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \to \pi/2^-} 4\sin\frac{x}{2} = \lim_{x \to \pi/2^+} \{A(x-\pi)^2\} + 2 = 4\sin\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{4}{\sqrt{2}} = \frac{A\pi^2}{4} + 2 = \frac{4}{\sqrt{2}}$$

$$\Rightarrow \quad A = \frac{8}{\pi^2}(\sqrt{2} - 1)$$

Thus,

$$f(x) = \begin{cases} 4\sin(x/2) & , x \le \frac{\pi}{2} \\ \frac{8}{\pi^2}(\sqrt{2} - 1)(x-\pi)^2 + 2, & x > \frac{\pi}{2} \end{cases}$$

For $x \le \frac{\pi}{2}$, we have $y = 4\sin\frac{x}{2}$ and for $x > \frac{\pi}{2}$, $y = \frac{8}{\pi^2}(\sqrt{2} - 1)(x - \pi)^2 + 2$, which represents a parabola having vertex at $(\pi, 2)$ and opens upward. The equation $\pi y = 4\sqrt{2}(x - \pi)$ represents a straight line passing through $(\pi, 0)$. The shaded region shown in Fig. 12.26 repreents the desired region.

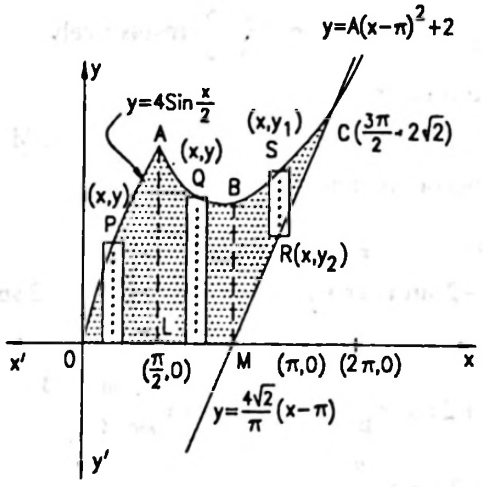


Fig. 12.26

Now,

Required area

= Area of region $OALO$ + Area of region $LABML$ + Area of region $BMCB$

$$= \int_0^{\pi/2} 4\sin\frac{x}{2}\,dx + \int_{\pi/2}^{\pi} \{A(x-\pi)^2 + 2\}\,dx$$

$$+ \int_{\pi}^{3\pi/2} \left\{\{A(x-\pi)^2 + 2\} - \frac{4\sqrt{2}}{\pi}(x-\pi)\right\} dx$$

$$= -8\left[\cos\frac{x}{2}\right]_0^{\pi/2} + \left[\frac{A}{3}(x-\pi)^3 + 2x\right]_{\pi/2}^{\pi}$$

$$+ \left[\frac{A(x-\pi)^3}{3} + 2x - \frac{2\sqrt{2}}{\pi}(x-\pi)^2\right]_{\pi}^{3\pi/2}$$

$$= -8\left(\frac{1}{\sqrt{2}} - 1\right) + 2\pi + \frac{A\pi^3}{24} - \pi + \frac{A\pi^3}{24} + 3\pi - \frac{\pi}{\sqrt{2}} - 2\pi$$

$$= -4\sqrt{2} + 8 + 2\pi - \frac{\pi}{\sqrt{2}} + \frac{\pi^3}{12} \times \frac{8}{\pi^2}(\sqrt{2} - 1)$$

$$= 4\sqrt{2}(\sqrt{2} - 1) + \frac{\pi}{3\sqrt{2}}(4\sqrt{2} + 1) \text{ sq. units.}$$

EXAMPLE 24 *Find the area lying on the same side of the axis of x, as the positive part of the axis of y and which is contained by $y^2 = 4ax$, $x^2 + y^2 = 2ax$ and $x = y + 2a$.*

SOLUTION The equation $y^2 = 4ax$ represents a parabola and $x^2 + y^2 = 2ax$ or $(x-a)^2 + (y-0)^2 = a^2$ represents a circle having its centre at $(a, 0)$. We have to find the area of the shaded region in Fig. 12.27. The line $x = y + 2a$ and the parabola $y^2 = 4ax$ intersect at $B\left(a(\sqrt{3}+1)^2, 2a(\sqrt{3}+1)\right)$.

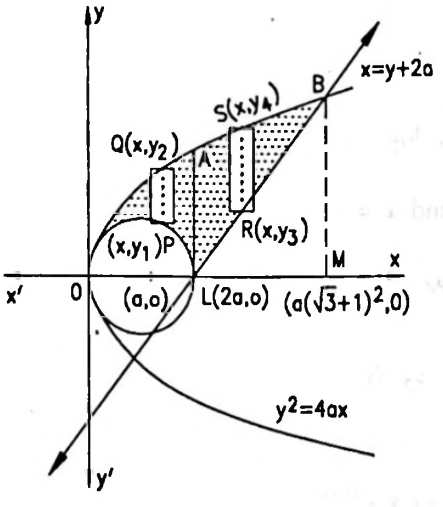


Fig. 12.27

Now, Required area

= Area of region $OLAO$ + Area of region $ALBA$

$$= \int_0^{2a} (y_2 - y_1)\,dx + \int_{2a}^{a(\sqrt{3}+1)^2} (y_4 - y_3)\,dx$$

$$= \int_0^{2a} \left[\sqrt{4ax} - \sqrt{a^2 - (x-a)^2}\right] dx + \int_{2a}^{(\sqrt{3}+1)^2 a} [\sqrt{4ax} - (x-2a)]\,dx$$

$$= \int_0^{2a} \sqrt{4ax}\,dx + \int_{2a}^{(\sqrt{3}+1)^2 a} \sqrt{4ax}\,dx - \int_0^{2a} \sqrt{a^2-(x-a)^2}\,dx - \int_{2a}^{(\sqrt{3}+1)^2 a} (x-2a)\,dx$$

$$= \int_0^{(\sqrt{3}+1)^2 a} \sqrt{4ax}\,dx - \int_0^{2a} \sqrt{a^2-(x-a)^2}\,dx - \int_{2a}^{(\sqrt{3}+1)^2 a} (x-2a)\,dx$$

$$= \frac{4}{3}\sqrt{a}\left[x^{3/2}\right]_0^{(\sqrt{3}+1)^2 a} - \left[\frac{1}{2}(x-a)\sqrt{a^2-(x-a)^2} + \frac{1}{2}a^2 \sin^{-1}\frac{(x-a)}{a}\right]_0^{2a} - \left[\frac{x^2}{2} - 2ax\right]_{2a}^{(\sqrt{3}+1)^2 a}$$

$$= \frac{4}{3}(\sqrt{3}+1)^3 a^2 - \left[\frac{1}{2}a^2 \sin^{-1}(1) - \frac{1}{2}a^2 \sin^{-1}(-1)\right]$$

$$- \left[-2a^2(\sqrt{3}+1)^2 + \frac{(\sqrt{3}+1)^4 a^2}{2} + 2a^2\right]$$

$$= \frac{4}{3}(\sqrt{3}+1)^3 a^2 - \frac{\pi a^2}{2} + 2a^2(\sqrt{3}+1)^2 - \frac{(\sqrt{3}+1)^4}{2}a^2 - 2a^2$$

$$= \frac{4}{3}(10+6\sqrt{3})a^2 - \frac{\pi a^2}{2} + a^2(\sqrt{3}+1)^2\left\{2 - \frac{(\sqrt{3}+1)^2}{2}\right\} - 2a^2$$

$$= \frac{4}{3}(10+6\sqrt{3})a^2 - \frac{\pi a^2}{2} - a^2(\sqrt{3}+1)^2\sqrt{3} - 2a^2$$

$$= \frac{4}{3}(10+6\sqrt{3})a^2 - \frac{\pi a^2}{2} - \sqrt{3}a^2(4+2\sqrt{3}) - 2a^2$$

$$= \frac{16}{3}a^2 + 4\sqrt{3}a^2 - \frac{\pi a^2}{2}$$

$$= (32 + 24\sqrt{3} - 3\pi)\frac{a^2}{6} \text{ sq. units}$$

EXAMPLE 25 *Let $f(x)$ be a non-negative differentiable function satisfying the differnetial equation $\frac{dy}{dx} = x$ for $x \le \sqrt{2}$ and the differential equation $y\frac{dy}{dx} = x$ for $x > \sqrt{2}$. Find the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = 0$ and $x = 2$.*

SOLUTION Since $f(x)$ is differentiable at $x = \sqrt{2}$. Therefore,

(LHD at $x = \sqrt{2}$) = (RHD at $x = \sqrt{2}$)

$$\Rightarrow \lim_{x \to \sqrt{2}^-} \frac{dy}{dx} = \lim_{x \to \sqrt{2}^+} \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \lim_{x \to \sqrt{2}^-} x = \lim_{x \to \sqrt{2}^+} \frac{x}{f(x)} \qquad \left[\frac{dy}{dx} = \begin{cases} x, & x \le \sqrt{2} \\ \frac{x}{y}, & x > \sqrt{2} \end{cases}\right]$$

$$\Rightarrow \sqrt{2} = \frac{\sqrt{2}}{f(\sqrt{2})} \qquad [\because f(x) \text{ is continuous at } x = \sqrt{2}]$$

$$\Rightarrow f(\sqrt{2}) = 1$$

Now, $\frac{dy}{dx} = x$ for $x \le \sqrt{2}$

$$\Rightarrow dy = x\,dx$$

$$\Rightarrow y = \frac{x^2}{2} + C_1$$

But, $y = 1$ when $x = \sqrt{2}$. Therefore,

$$1 = \frac{2}{2} + C_1 \Rightarrow C_1 = 0$$

$$\therefore y = \frac{x^2}{2} \text{ for } x \le \sqrt{2}.$$

Also, $y\frac{dy}{dx} = x$ for $x > \sqrt{2}$

$$\Rightarrow y\,dy = x\,dx \quad \text{for } x > \sqrt{2}$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_2$$

But, $y = 1$ when $x = \sqrt{2}$. Therefore,

$$\frac{1}{2} = 1 + C_2 \Rightarrow C_2 = -\frac{1}{2}$$

$$\therefore \frac{y^2}{2} = \frac{x^2}{2} - \frac{1}{2} \quad \text{for } x > \sqrt{2}$$

$$\Rightarrow y^2 = x^2 - 1 \quad \text{for } x > \sqrt{2}$$

Thus, we have,

$$f(x) = \begin{cases} \frac{x^2}{2}, & x \le \sqrt{2} \\ \sqrt{x^2-1}, & x > \sqrt{2} \end{cases}$$

The graph of the curve $y = f(x)$ is shown in Fig. 12.28. We have to find the area of the shaded region.

Now,

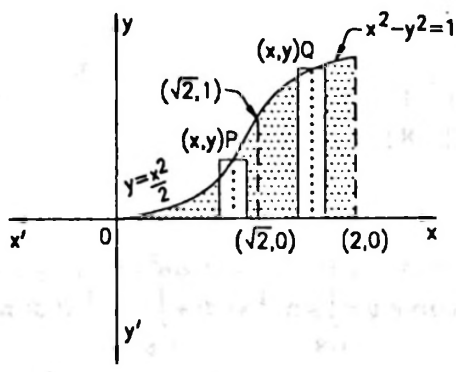


Fig. 12.28

Required area

$$= \int_0^{\sqrt{2}} \frac{x^2}{2}\,dx + \int_{\sqrt{2}}^{2} \sqrt{x^2-1}\,dx$$

$$= \frac{1}{6}\left[x^3\right]_0^{\sqrt{2}} + \left[\frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)\right]_{\sqrt{2}}^{2}$$

$$= \frac{2\sqrt{2}}{6} + \sqrt{3} + \frac{1}{2}\log(2+\sqrt{3}) - \frac{1}{\sqrt{2}} - \frac{1}{2}\log(1+\sqrt{2})$$

$$= \left[\sqrt{3} - \frac{1}{3\sqrt{2}} + \frac{1}{2}\log\frac{2+\sqrt{3}}{1+\sqrt{2}}\right] \text{sq. units}$$

EXAMPLE 26 *Find the area bounded by the curve*

$$y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt,$$

where $0 \le x \le \frac{\pi}{2}$, *y-axis and the curve satisfying the differential equation* $y(x+y^3)\,dx = x(y^3-x)\,dy$ *passing through* $(4,-2)$.

SOLUTION We have,

$$y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt,\ 0 \le x \le \frac{\pi}{2}$$

By using Leibnitz is rule, we have

$$\frac{dy}{dx} = 2\sin x\cos x\sin^{-1}(\sin x) - 2\sin x\cos x\cos^{-1}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow y = C \text{ (constant) for all } x \in [0, \pi/2]$$

But, for $x = \frac{\pi}{4}$, we have

$$y = \int_{1/8}^{1/2} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{1/2} \cos^{-1}\sqrt{t}\,dt$$

$$= \int_{1/8}^{1/\sqrt{2}} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt$$

$$= \int_{1/8}^{1/2} \frac{\pi}{2}\,dt$$

$$= \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3\pi}{16}$$

$$\therefore \quad C = \frac{3\pi}{16}$$

Thus, the curve $y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$ reduces to $y = 3\pi/16$.

The second curve satisfies the differential equation

$$y(x+y^3)\,dx = x(y^3-x)\,dy$$

$$\Rightarrow \quad x(y - dx + x\,dy) = y^3(x\,dy - y\,dx)$$

$$\Rightarrow \quad x\,d(xy) = x^2y^3\,d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} = \left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

On integrating, we get

$$-\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C_1$$

It passes through $(4,-2)$. Therefore,

$$\frac{1}{8} = \frac{1}{8} + C_1 \Rightarrow C_1 = 0$$

$$\therefore \quad -\frac{1}{xy} = \frac{y^2}{2x^2}$$

$$\Rightarrow \quad y^3 = -2x$$

$$\Rightarrow \quad y = (-2x)^{1/3}$$

Thus, the two curves are $y = \frac{3\pi}{16}$ and $y = (-2x)^{1/3}$. Their graphs are shown in Fig. 12.29. We have to find the area of the shaded region. For the approximating rectangle shown in Fig. 12.29, we have, length $= 0 - x$, width $= \Delta y$. As it can move

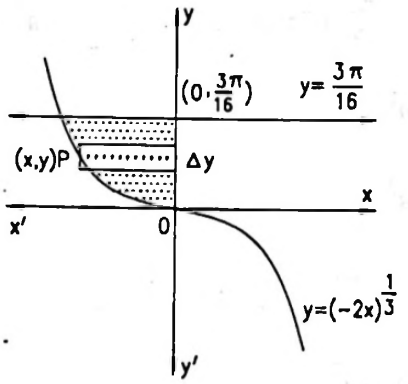


Fig. 12.29

between $y = 0$ and $y = \frac{3\pi}{16}$. Therefore,

Required area

$$= \int_0^{3\pi/16} -x\,dy$$

$$= \int_0^{3\pi/16} \frac{y^3}{2}\,dy$$

$$= \frac{1}{8}\left[y^4\right]_0^{3\pi/16}$$

$$= \frac{1}{8}\left(\frac{3\pi}{16}\right)^4 \text{ sq. units.}$$

EXAMPLE 27 *A tangent is drawn to the curve $x^2+2x-4ky+3=0$ at a point whose abscissa is 3. The tangent is perpendicular to the line $x-2y+3=0$. Find the area bounded by the curve, this tangent, the x-axis and the ordinate $x=-1$.*

SOLUTION We have,

$$x^2+2x-4ky+3=0$$

Differentiating w.r.t. x, we get

$$2x+2-4k\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x+1}{2k}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=3}=\frac{4}{2k}=\frac{2}{k}$$

So, the slope of the tangent to the curve

$$x^2+2x-4ky+3=0 \text{ at } x=3 \text{ is } \frac{2}{k}.$$

It is given that the tangent at $x=3$ is perpendicular to the line $x-2y+3=0$. Therefore,

$$\frac{2}{k}\times\frac{1}{2}=-1 \Rightarrow k=-1$$

Putting $k=-1$ in $x^2+2x-4ky+3=0$, we obtain

$$x^2+2x+4y+3=0$$

$$\Rightarrow \quad (x+1)^2=-4\left(y+\frac{1}{2}\right),$$

which is a parabola with its vertices at $(-1,-1/2)$ and opens downward.

Putting $x=3$ in the equation of the parabola $x^2+2x+4y+3=0$ we get $y=-9/2$

The equation of the tangent at $(3,-9/2)$ is

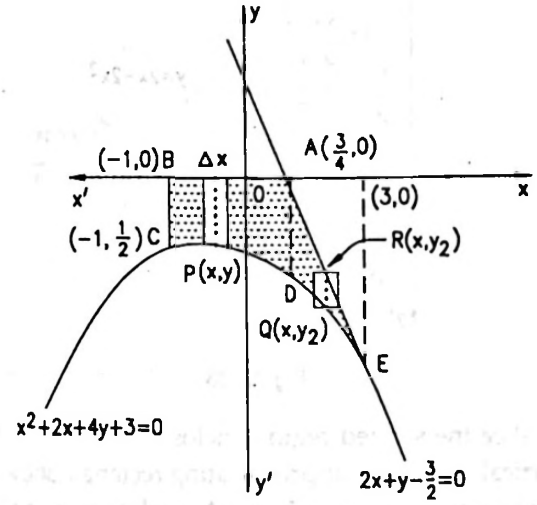


Fig. 12.30

$$y+\frac{9}{2}=-2(x-3) \text{ or, } 2x+y-\frac{3}{2}=0$$

In Fig. 12.30, we have to find the area of the shaded region.

Required area

= Area of the region $ABCD$ + Area of the region AED.

$$=\left|\int_{-1}^{3/4} y\,dx\right|+\left|\int_{3/4}^{3}(y_1-y_2)\,dx\right|$$

$$=\left|\int_{-1}^{3/4}\left(\frac{-x^2-2x-3}{4}\right)dx\right|+\left|\int_{3/4}^{3}\left\{\left(\frac{-x^2-2x-3}{4}\right)-\left(\frac{3}{2}-2x\right)\right\}dx\right|$$

$$=\left|\left[-\frac{1}{4}\left(\frac{x^3}{3}+x^2+3x\right)\right]_{-1}^{3/4}\right|$$

$$+\left|\left[-\frac{1}{4}\left(\frac{x^3}{3}+x^2+3x\right)-\left(\frac{3x}{2}-x^2\right)\right]_{3/4}^{3}\right|$$

$$=\frac{109}{48} \text{ sq. units}$$

EXAMPLE 28 *If the line $y=mx$ bisects the area enclosed by the lines $x=0$, $y=0$, $x=\frac{3}{2}$ and the curve $y=1+4x-x^2$. Find the value of m.*

SOLUTION We have,

$$y=1+4x-x^2$$

$$\Rightarrow \quad x^2-4x=-y+1$$

$$\Rightarrow \quad (x-2)^2=-(y-5)$$

This equation represents a parabola having vertex at (2, 5) and opens downward. The area enclosed by this parabola and the lines $x=0, y=0, x=3/2$ is shaded in Fig. 12.31.

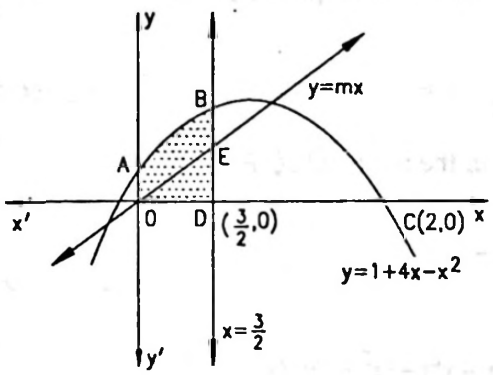


Fig. 12.31

Since $y=mx$ bisects the area of the shaded region. Therefore,

Area of the region $ODBAO$ = 2 Area of the region $ODEO$

$$\Rightarrow \int_0^{3/2} (1+4x-x^2)\,dx = 2\int_0^{3/2} mx\,dx$$

$$\Rightarrow \left[x+2x^2-\frac{x^3}{3}\right]_0^{3/2} = 2m\left[\frac{x^2}{2}\right]_0^{3/2}$$

$$\Rightarrow \frac{3}{2}+\frac{9}{2}-\frac{9}{8} = m\times\frac{9}{4}$$

$$\Rightarrow \frac{39}{8} = \frac{9m}{4}$$

$$\Rightarrow m = \frac{13}{6}$$

EXAMPLE 29 *In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?*

[IIT 1994]

SOLUTION The parabola $y = 4x - x^2$ opens downward and crosses x-axis at (0, 0) and (4, 0). The parabola $y = x^2 - x$ opens upward and crosses x-axis at (0, 0) and (1, 0). The graphs of these two parabolas are shown in Fig. 12.32. The two parabolas intersect at O (0, 0) and P (5/2 , 15/4).

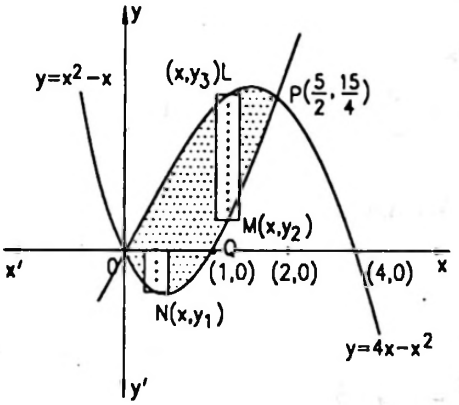


Fig. 12.32

Let A_1 = Area of the region $OQPLO$

and, A_2 = Area of the region $OQNO$.

Then,

$A_1 + A_2$

= Area of the region $ONQ\,PLO$

$$= \int_0^{5/2} (y_3 - y_2)\,dx$$

$$= \int_0^{5/2} |(4x-x^2)-(x^2-x)|\,dx$$

$$= \int_0^{5/2} (5x-2x^2)\,dx$$

$$= \left[\frac{5}{2}x^2-\frac{2}{3}x^3\right]_0^{5/2}$$

$$= \frac{5}{2}\times\frac{25}{4}-\frac{2}{3}\times\frac{125}{8}$$

$$= \frac{125}{24} \text{ sq. units}$$

and,

A_2 = Area of the region $OQNO$

$$= \left|\int_0^1 y_1\,dx\right|$$

$$= \left|\int_0^1 (x^2-x)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^1\right|$$

$$= \left|\frac{1}{3}-\frac{1}{2}\right| = \frac{1}{6}.$$

$$\therefore A_1 = (A_1+A_2) - A_2 = \frac{125}{24}-\frac{1}{6} = \frac{121}{24}$$

Hence, required ratio = $A_1 : A_2 = \frac{121}{24} : \frac{1}{6} = 121 : 4$

EXAMPLE 30 *Indicate the region bounded by the curves $y = x \log x$ and $y = 2x - 2x^2$ and obtain the area enclosed by them.*

SOLUTION The graph of $y = x \log x$ is similar to the graph of $y = \log x$ and $y = 2x - 2x^2$ represents a parabola opening downward. It crosses x-axis at (0, 0) and (1, 0). Thus, $y = x \log x$ and $y = 2x - 2x^2$ intersect at (1, 0). The area enclosed by these two curves is shaded in Fig. 12.33.

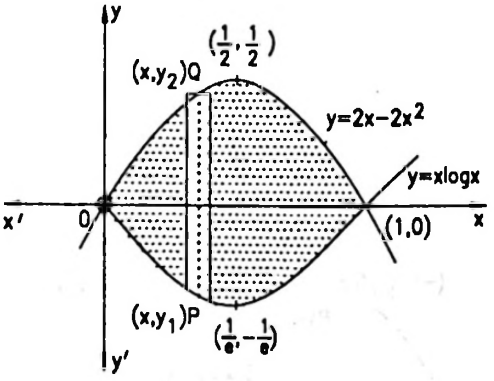


Fig. 12.33

We slice the shaded region enclosed by these two curves into vertical strips. The approximating rectangle shown in Fig. 12.33 has length = $y_2 - y_1$, width = Δx and it can move between $x = 0$ and $x = 1$. So,

Required area

$$= \int_0^1 (y_2 - y_1)\,dx$$

$$= \int_0^1 (2x - 2x^2 - x\log x)\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3 - \frac{x^2}{2}\log x + \frac{x^2}{4}\right]_0^1$$

$$= \left\{1 - \frac{2}{3} - \frac{1}{2}\log 1 + \frac{1}{4}\right\} - \frac{1}{2}\lim_{x\to 0} x^2 \log x$$

$$= \frac{7}{12} - \lim_{x\to 0}\frac{\log x}{1/x^2}$$

$$= \frac{7}{12} - \lim_{x\to 0}\frac{1/x}{-2/x^3} \qquad \text{[Using L' Hospital's Rule]}$$

$$= \frac{7}{12} + \lim_{x\to 0}\frac{x^2}{2}$$

$$= \frac{7}{12} \text{ sq. units}$$

EXAMPLE 31 *Find the smaller area enclosed by $y = f(x)$, where $f(x)$ is a polynomial of least degree satisfying $\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e$ and the circle $x^2 + y^2 = 2$ above the x-axis.*

SOLUTION It is given that

$$\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e \text{ exists finitely.}$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^3} = 0 \text{ and } e^{\lim_{x\to 0}\frac{f(x)}{x^4}} = e$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^3} = 0 \text{ and } \lim_{x\to 0}\frac{f(x)}{x^4} = 1.$$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$. Then,

$$\lim_{x\to 0}\frac{f(x)}{x^3} = 0 \Rightarrow a_0 = a_1 = a_2 = 0 = a_3$$

Now,

$$\lim_{x\to 0}\frac{f(x)}{x^4} = 1$$

$$\Rightarrow \lim_{x\to 0} a_4 + a_5 x + a_6 x^2 + \ldots = 1$$

$$\Rightarrow a_4 = 1$$

Since $f(x)$ is a polynomial of least degree satisfying the above conditions. Therefore,

$$a_5 = a_6 = \ldots = 0.$$

Thus, $f(x) = x^4$.

We have to find the area of the smaller region enclosed by $y = x^4$, $x^2 + y^2 = 2$ and the x-axis. The two curves intersect at $(1, 1)$ and $(-1, 1)$. Since both the curves are symmetrical about y-axis. So,

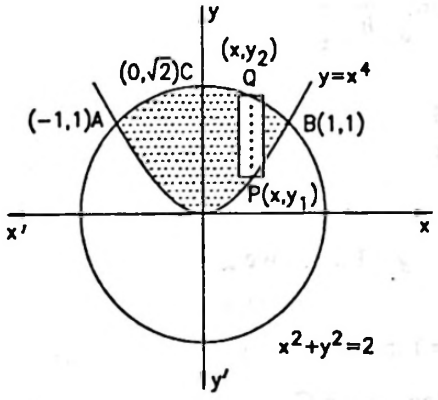


Fig. 12.34

Required area

= 2 (Area of the region $OCBO$)

$$= 2\int_0^1 (y_2 - y_1)\,dx$$

$$= 2\int_0^1 (\sqrt{2 - x^2} - x^4)\,dx$$

$$= 2\left[\frac{1}{2}x\sqrt{2 - x^2} + \frac{1}{2}(\sqrt{2})^2 \sin^{-1}\frac{x}{\sqrt{2}} - \frac{x^5}{5}\right]_0^1$$

$$= 2\left[\frac{1}{2} + \sin^{-1}\frac{1}{\sqrt{2}} - \frac{1}{5}\right]$$

$$= 2\left[\frac{\pi}{4} + \frac{3}{10}\right]$$

$$= \left(\frac{\pi}{2} + \frac{3}{5}\right) \text{sq. units.}$$

EXAMPLE 32 *Let $f(x)$ be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all $x, y \in R$, $y \neq 0$ and $\lim_{x\to 0}\frac{f(1+x)}{x} = 3$. Find the area bounded by the curve $y = f(x)$, the y-axis and the line $y = 3$.*

SOLUTION We have,

$$f\left(\frac{x}{y}\right) = f(x) - f(y) \text{ for all } x, y \in R, y \neq 0 \qquad \ldots(i)$$

and, $$\lim_{x\to 0}\frac{f(1+x)}{x} = 3 \qquad \ldots(ii)$$

Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} \qquad \left[\because f(x)-f(y) = f\left(\frac{x}{y}\right)\right]$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}\times x}$$

$$= \frac{3}{x} \qquad \left[\because \lim_{x\to 0}\frac{f(1+x)}{x} = 3\right]$$

$\Rightarrow f(x) = 3\log x + \log C$ ...(iii) [On integrating]

Putting $x = y = 1$ in (i), we get :

$$f(1) = 0$$

Putting $x = 1$ in (iii), we get

$$f(1) = 3\log 1 + \log C$$

$\Rightarrow \log C = 0$

Putting $\log C = 0$ in (iii), we obtain

$f(x) = 3\log x$ for all $x \in R - \{0\}$.

In Fig. 12.35, we have to find the area of the shaded region. Let us slice this region into horizontal strips. For the approximating rectangle shown in Fig. 12.35, we have

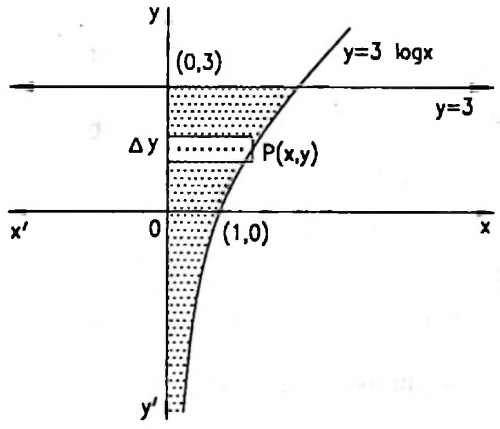


Fig. 12.35

Length = x, Width = Δy and it can move from $y = -\infty$ to $y = 3$. So,

$$\text{Required area} = \int_{-\infty}^{3} x\,dy$$

$$= \int_{-\infty}^{3} e^{y/3}\,dy \qquad [\because y = 3\log x \Rightarrow x = e^{y/3}]$$

$$= \left[3e^{y/3}\right]_{-\infty}^{3}$$

$$= 3e - 0 = 3e \text{ sq. units}$$

EXAMPLE 33 *Find all the possible values of $b > 0$, so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum.*

[IIT 1997]

SOLUTION The given parabolas are

$$y = x - bx^2 \qquad ...(i)$$

and,

$$y = \frac{x^2}{b} \qquad ...(ii)$$

The abscissae of their points of intersection are given by

$$\frac{x^2}{b} = x - bx^2$$

$$\Rightarrow x^2(1+b^2) - bx = 0$$

$$\Rightarrow x = 0, x = \frac{b}{1+b^2}$$

Thus, the two parabolas intersect at $O(0,0)$ and $A\left(\frac{b}{1+b^2}, \frac{b}{(1+b^2)^2}\right)$

The region enclosed by the two parabolas is shaded in Fig. 12.36. To find the area of this aregion, let us slice it into vertical strips. The approximating rectangle shown in Fig. 12.36 has length = $y_2 - y_1$, width = Δx and it can move between $x = 0$ and $x = \frac{b}{1+b^2}$. So, the area A enclosed by the two parabolas is given by

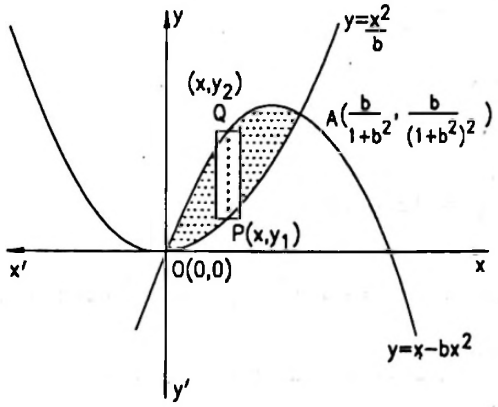


Fig. 12.36

$$A = \int_0^{\frac{b}{1+b^2}} (y_2 - y_1)\,dx$$

$$= \int_0^{\frac{b}{1+b^2}} \left(x - bx^2 - \frac{x^2}{b}\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{bx^3}{3} - \frac{x^3}{3b}\right]_0^{\frac{b}{1+b^2}}$$

$$= \frac{1}{2}\left(\frac{b}{1+b^2}\right)^2 - \frac{1}{3b}(1+b^2)\left(\frac{b}{1+b^2}\right)^3$$

$$= \frac{1}{2}\left(\frac{b}{1+b^2}\right)^2 - \frac{1}{3}\left(\frac{b}{1+b^2}\right)^2 = \frac{1}{6}\frac{b^2}{(1+b^2)^2}$$

Now,

$$A = \frac{1}{6}\frac{b^2}{(1+b^2)^2}$$

$$\Rightarrow \frac{dA}{db} = \frac{1}{6}\left[\frac{(1+b^2)^2 \times 2b - 2b^2(1+b^2)(2b)}{(1+b^2)^4}\right]$$

$$\Rightarrow \frac{dA}{db} = \frac{b}{3(1+b^2)^3}[1+b^2-2b^2]$$

$$\Rightarrow \frac{dA}{db} = \frac{b(1-b)(1+b)}{3(1+b^2)^3}$$

For maximum value of A, we must have

$$\frac{dA}{db} = 0$$

$$\Rightarrow b(1-b)(1+b) = 0$$

$$\Rightarrow 1-b = 0$$

$$\Rightarrow b = 1 \qquad [\because b > 0]$$

Since $b > 0$. Therefore, $1 + b > 0$. Thus, $\frac{dA}{db}$ changes its sign from positive to negative in the neighbourhood of $b = 1$. Therefore, A is maximum when $b = 1$.

EXAMPLE 34 *Let P be a point on the parabola $y^2 = 4x$ with the ordinate y satisfying $1 < y \le 2$. Then normal to the parabola at P intersects the x-axis in N and a line parallel to y-axis through P intersects the x-axis in M. Find the maximum difference in the areas of triangles PMN and PMS, where S is the focus of the parabola.*

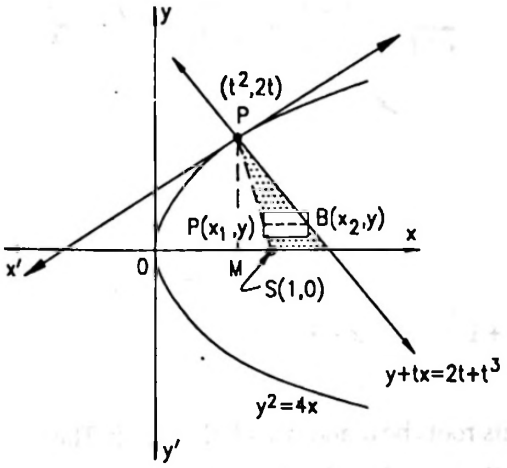


Fig. 12.37

SOLUTION Let $P(t^2, 2t)$ be a point on the parabola such that $1 < 2t \le 2$ i.e. $\frac{1}{2} < t \le 1$.

We have,

$$y^2 = 4x$$

$$\Rightarrow 2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{2}{2t} = \frac{1}{t}$$

The equation of the normal at P to the parabola $y^2 = 4x$ is

$$y - 2t = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - t^2)$$

$$\Rightarrow y - 2t = -t(x - t^2)$$

$$\Rightarrow y + tx = 2t + t^3 \qquad \ldots(i)$$

This meets X-axis at $N(2 + t^2, 0)$

Let

A = Area of ΔPMN – Area of ΔPMS

= Area of the region PSN

$$= \int_0^{2t} (x_2 - x_1)\,dy$$

$$= \int_0^{2t} \left\{\left(\frac{2t + t^3 - y}{t}\right) - \left(\frac{(t^2-1)y}{2t} + t^2\right)\right\} dy \qquad \left[\because \text{Eqn. of SP is } y = \frac{2t}{t^2-1}(x - t^2)\right]$$

$$= \int_0^{2t} \left\{2 + t^2 - \frac{y}{t} - \left(\frac{t^2-1}{2t}\right)y - t^2\right\} dy$$

$$= \int_0^{2t} \left(2 - \frac{t^2+1}{2t}y\right) dy$$

$$= \left[2y - \left(\frac{t^2+1}{4t}\right)y^2\right]_0^{2t}$$

$$= 4t - (t^2+1)t$$

$$= 3t - t^3$$

Now, $\frac{dA}{dt} = 1 - 3t^2$ for all t

For maximum value of A, we must have

$$\frac{dA}{dt} = 0 \Rightarrow 3 - 3t^2 = 0 \Rightarrow t = \pm 1$$

We have, $\frac{d^2A}{dt^2} = -6t$

Clearly, $\frac{d^2A}{dt^2} = -6 < 0$ for $t = 1$.

Thus, A is maximum when $t = 1$. The maximum value of A is given by

$$A = 3 - 1 = 2 \text{ sq. units.}$$

EXAMPLE 35 *The two curves $y^2 = 4a(x+2)$, $a > 0$ and $x^2 + y^2 = 4$ intersect each other on points A and B. Find all possible values of 'a' so that the area of the region bounded by the parabola and the chord AB is maximum.*

SOLUTION The equations of the two curves are :

$$y^2 = 4a(x+2) \quad ...(i)$$

and,
$$x^2 + y^2 = 4 \quad ...(ii)$$

Clearly, $y^2 = 4a(x+2)$ represents a parabola with vertex $(-2, 0)$ and opening rightward. The equation $x^2 + y^2 = 4$ represents a circle of radius 2 having centre at the origin.

The x-coordinate of the point of intersection of (i) and (ii) is given by

$$x^2 + 4a(x+2) = 4$$
$$\Rightarrow x^2 + 4ax + 8a - 4 = 0$$
$$\Rightarrow x = -2, x = 2 - 4a.$$

Thus, the two curves intersect at

$$(-2, 0) \text{ and } (2 - 4a, \pm\sqrt{3 + 8a - 16a^2})$$

Let A denote the area enclosed by the parabola (i) and chord AB shown in Fig. 12.38. Then,

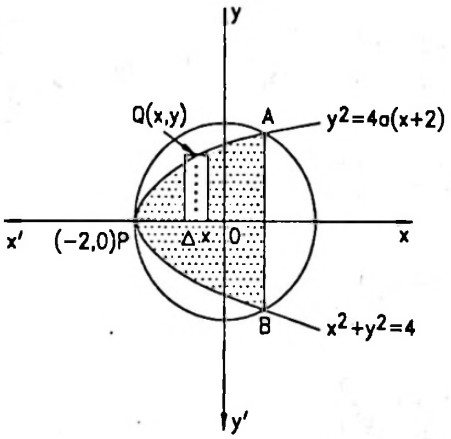


Fig. 12.38

$$A = 2\int_{-2}^{2-4a} y\,dx$$

$$= 2\int_{-2}^{2-4a} \sqrt{4a(x+2)}\,dx$$

$$= 4\sqrt{a}\int_{-2}^{2-4a} \sqrt{x+2}\,dx$$

$$= \frac{8\sqrt{a}}{3}\left[(x+2)^{3/2}\right]_{-2}^{2-4a}$$

$$A = \frac{8\sqrt{a}}{3}(4-4a)^{3/2}$$

$$\Rightarrow A^2 = \frac{64a}{9}(4-4a)^3$$

Clearly, A will be maximum or minimum according as A^2 is maximum or minimum.

Let $f(a) = A^2 = \frac{64}{9}a(4-4a)^3$. Then,

$$\frac{df(a)}{da} = \frac{64}{9}[(4-4a)^3 + 3a(4-4a)^2(-4)]$$

$$\Rightarrow \frac{df(a)}{da} = \frac{64}{9}(4-4a)^2(4-16a)$$

For maximum or minimum, we must have

$$\frac{df(a)}{da} = 0 \Rightarrow a = 1, \frac{1}{4}.$$

Clearly, $\frac{df(a)}{da}$ changes its sign from positive to negative in the neighbourhood of $a = \frac{1}{4}$.

Thus, $f(a) = A^2$ is maximum when $a = \frac{1}{4}$

EXAMPLE 36 *Find the least value of the area bounded by the line $y = mx + 1$ and the parabola $y = x^2 + 2x - 3$ when m is a parameter.*

SOLUTION The parabola $y = x^2 + 2x - 3$ opens upward and cuts x-axis at $(-3, 0)$ and $(1, 0)$. The area enclosed by the line $y = mx + 1$ and the parabola is shaded in Fig. 12.39. The abscissae of the points of intersection of the parabola $y = x^2 + 2x - 3$ and the line $y = mx + 1$ are given by

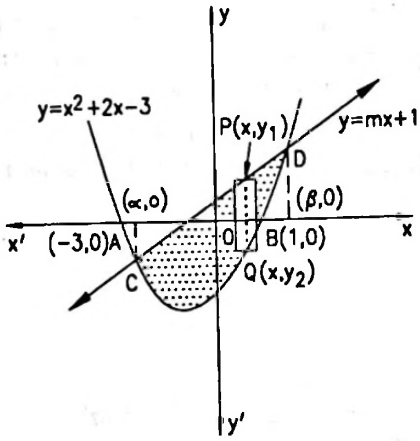


Fig. 12.39

$$mx + 1 = x^2 + 2x - 3$$
$$\Rightarrow x^2 + x(2-m) - 4 = 0 \quad ...(i)$$

Let its roots be α and β such that $\alpha < \beta$. Then,

$$\alpha + \beta = m - 2 \text{ and } \alpha\beta = -4 \quad ...(ii)$$

Let A denote the area of the shaded region. Then,

$$A = \int_{\alpha}^{\beta} (y_1 - y_2)\, dx$$

$$= \int_{\alpha}^{\beta} |(mx+1) - (x^2 + 2x - 3)|\, dx$$

$$= \left[\frac{m}{2}x^2 + x - \frac{x^3}{3} - x^2 + 3x\right]_{\alpha}^{\beta}$$

$$= \left[4x + \left(\frac{m}{2} - 1\right)x^2 - \frac{x^3}{3}\right]_{\alpha}^{\beta}$$

$$= 4(\beta - \alpha) + \left(\frac{m}{2} - 1\right)(\beta^2 - \alpha^2) - \frac{1}{3}(\beta^3 - \alpha^3)$$

$$= (\beta - \alpha)\left[4 + \left(\frac{m-2}{2}\right)(\beta + \alpha) - \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2)\right]$$

$$= \sqrt{(\beta+\alpha)^2 - 4\alpha\beta}\left[4 + \left(\frac{m-2}{2}\right)(\beta+\alpha) - \frac{1}{3}\{(\beta+\alpha)^2 - \alpha\beta\}\right]$$

$$= \sqrt{(m-2)^2 + 16}\left[4 + \frac{(m-2)^2}{2} - \frac{1}{3}\{(m-2)^2 + 4\}\right] \quad \text{[Using (iii)]}$$

$$= \sqrt{m^2 - 4m + 20}\left[\frac{8}{3} + \frac{1}{6}(m-2)^2\right]$$

$$= \frac{1}{6}(m^2 - 4m + 20)^{3/2}$$

$$= \frac{1}{6}[(m-2)^2 + 4^2]^{3/2}$$

Clearly, A will be least when $m = 2$ and the least value of A is given by

$$A = \frac{1}{6}(4^2)^{3/2} = \frac{32}{3} \text{ sq. units}$$

EXAMPLE 37 *Let $f(x)$ be a differentiable function satisfying $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$ for all $x, y \in R$ and $f(1) = 1$. Find the area bounded by the curves $[f(x)]^{1/3} + [f(y)]^{2/3} = 0$ and $[f(x)]^{2/3} + [f(y)]^{2/3} = 12$.*

SOLUTION We have,

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2) \text{ for all } x, y \in R$$

$$\Rightarrow \frac{f(x+y)}{x+y} - \frac{f(x-y)}{x-y} = 4xy \text{ for all } x, y \in R, y \neq \pm x$$

$$\Rightarrow \frac{f(x+h)}{x+h} - \frac{f(x-h)}{x-h} = 4xh$$

$$\Rightarrow \frac{f(x+h) - f(x) + f(x)}{x+h} - \frac{f(x-h) - f(x) + f(x)}{x-h} = 4hx$$

$$\Rightarrow \frac{f(x+h) - f(x)}{x+h} - \frac{f(x-h) - f(x)}{x-h} = 4hx - \frac{f(x)}{x+h} + \frac{f(x)}{x-h}$$

$$\Rightarrow \frac{f(x-h) - f(x)}{x+h} - \frac{f(x-h) - f(x)}{x-h} = 4hx + \frac{2hf(x)}{x^2 - h^2}$$

$$\Rightarrow \left\{\frac{f(x+h) - f(x)}{h}\right\}\left(\frac{1}{x+h}\right) + \left\{\frac{f(x-h) - f(x)}{-h}\right\}\left(\frac{1}{x-h}\right) = 4x + \frac{2f(x)}{x^2 - h^2}$$

Taking limit of both sides as $h \to 0$, we get

$$\frac{f'(x)}{x} + \frac{f'(x)}{x} = 4x + \frac{2f(x)}{x^2}$$

$$\Rightarrow \frac{2f'(x)}{x} = 4x + \frac{2f(x)}{x^2}$$

$$\Rightarrow f'(x) = 2x^2 + \frac{f(x)}{x}$$

$$\Rightarrow \frac{df(x)}{dx} - \frac{f(x)}{x} = 2x^2$$

This is a linear differential equation with I.F. $= e^{\int -\frac{1}{x}dx} = \frac{1}{x}$.

Multiplying both sides of the above equation by $\frac{1}{x}$ and integrating, we get

$$\frac{f(x)}{x} = x^2 + C$$

$$\Rightarrow f(x) = x^3 + Cx$$

$$\Rightarrow f(1) = 1 + C$$

$$\Rightarrow C = 0 \qquad [\because f(1) = 1]$$

$$\therefore f(x) = x^3.$$

Thus, the two curves are :

$$x + y^2 = 0 \qquad \text{...(i)}$$

and,

$$x^2 + y^2 = 12 \qquad \text{...(ii)}$$

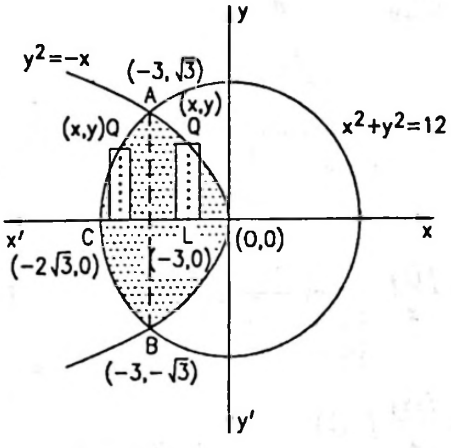


Fig. 12.40

The area bounded by these curves is shaded in Fig. 12.40.

Required area

$$= 2\left[\int_{-2\sqrt{3}}^{-3} \sqrt{12 - x^2}\, dx + \int_{-3}^{0} \sqrt{-x}\, dx\right]$$

$$= 2\left[\left\{\frac{x}{2}\sqrt{12-x^2}+6\sin^{-1}\frac{x}{2\sqrt{3}}\right\}_{-2\sqrt{3}}^{-3}+\left\{\left(\frac{-2}{3}\right)(-x)^{3/2}\right\}_{-3}^{0}\right]$$

$$= 2\left[\left(\frac{-3\sqrt{3}}{2}-6\times\frac{\pi}{3}\right)-\left(0+6\times\frac{-\pi}{2}\right)+\left(\frac{-2}{3}\right)(0-3^{3/2})\right]$$

$$= 2\left[\frac{-3\sqrt{3}}{2}-2\pi+3\pi+2\sqrt{3}\right]$$

$$= 2\left[\pi+\frac{\sqrt{3}}{2}\right]$$

$$= (2\pi+\sqrt{3}) \text{ sq. units.}$$

EXAMPLE 38 *Let f be a differentiable function satisfying the condition :*

$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}, \text{ where } y\neq 0, f(y)\neq 0 \text{ for all } x, y\in R \text{ and } f'(1)=2,$$

then find out the area enclosed by $y=f(x)$, $x^2+y^2=2$ and x-axis.

SOLUTION We have,

$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)} \text{ for all } x, y\in R \text{ such that } y\neq 0 \text{ and } f(y)\neq 0.$$

$$\therefore\ f(1)=\frac{f(1)}{f(1)}$$

$$\Rightarrow\ |f(1)|^2-f(1)=0$$

$$\Rightarrow\ f(1)=0 \text{ or, } f(1)=1$$

$$\Rightarrow\ f(1)=1 \qquad [\because f(1)\neq 0]$$

Now,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\left[\frac{f(x+h)}{f(x)}-1\right]}{h}$$

$$=f(x)\cdot\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)-1}{h} \qquad \left[\because \frac{f(x)}{f(y)}=f\left(\frac{x}{y}\right)\right]$$

$$=\frac{f(x)}{x}\cdot\left\{\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\left(1+\frac{h}{x}\right)-1}\right\} \qquad [\because f(1)=1]$$

$$=\frac{f(x)}{x}f'(1)$$

$$\Rightarrow\ \frac{d(f(x))}{dx}=\frac{2f(x)}{x} \qquad [\because f'(1)=2]$$

$$\Rightarrow\ \frac{df(x)}{f(x)}=\frac{2}{x}dx$$

Integrating both sides, we get

$$\log f(x)=2\log x+\log C$$

$$\Rightarrow\ f(x)=Cx^2 \qquad \text{...(i)}$$

But, $f(1)=1$ Therefore, $C=1$.

Putting $C=1$ in (i), we obtain

$$f(x)=x^2.$$

Thus, we have to find the area enclosed by the curves $y=x^2$, $x^2+y^2=2$ and x-axis. Graphs of these curves and region enclosed are shown in Fig. 12.41. Clearly, $y=x^2$ and $x^2+y^2=2$ intersect at (1, 1) and (– 1, 1). The shaded region in Fig. 12.41 is sliced into horizontal strips. The approximating rectangle shown in Fig. 12.41 is of length $=(x_2-x_1)$, width $=\Delta y$ and it can move vertically between $y=0$ and $y=1$. So,

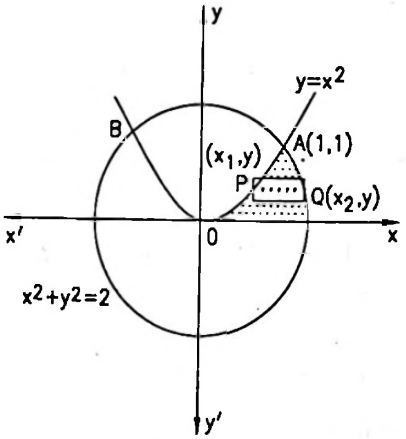


Fig. 12.41

Required area

$$=\int_0^1 (x_2-x_1)\,dy$$

$$=\int_0^1 (\sqrt{2-y^2}-\sqrt{y})\,dy$$

$$=\left[\frac{1}{2}y\sqrt{2-y^2}+\frac{2}{2}\sin^{-1}\frac{y}{\sqrt{2}}-\frac{2}{3}y^{3/2}\right]_0^1$$

$$=\frac{1}{2}+\sin^{-1}\frac{1}{\sqrt{2}}-\frac{2}{3}=\left(\frac{\pi}{4}-\frac{1}{6}\right) \text{ sq. units}$$

EXAMPLE 39 *A square is constructed by joining the points $A(-1, 1)$, $B(-1, -1)$, $C(1, -1)$ and $D(1, 1)$. Considering A, B, C and D as centres, four circles each of 2 unit radius are described on the square. Find the area common to these circles.*

SOLUTION We have to find the area of the shaded region in Fig. 12.42. From the symmetry,

Required area = 4 (Area of the region $ORQO$) ...(i)

Clearly, R is the point of intersection of the circle having centre at $(-1, -1)$ and radius 2 i.e. $(x+1)^2+(y+1)^2=2^2$ with x-axis. Therefore, abscissa of R of given by

$$(x+1)^2+(0+1)^2=2^2 \ \Rightarrow\ x=\sqrt{3}-1.$$

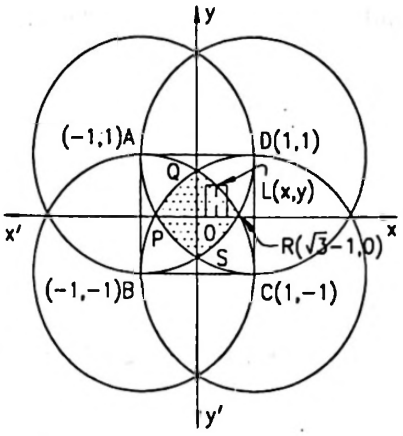


Fig. 12.42

$\therefore$ Area of the region $ORQO$

$$= \int_0^{\sqrt{3}-1} y\,dx$$

$$= \int_0^{\sqrt{3}-1} \{-1+\sqrt{2^2-(x+1)^2}\}\,dx \qquad \left[\begin{array}{l}\because\ L(x,y) \text{ lies on} \\ (x+1)^2+(y+1)^2=2^2 \\ \therefore\ y=-1+\sqrt{2^2-(x+1)^2}\end{array}\right]$$

$$= \left[-x+\frac{1}{2}(x+1)\sqrt{2^2-(x+1)^2}+\frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$$

$$= \left\{-(\sqrt{3}+1)+\frac{\sqrt{3}}{2}+\frac{4}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right\}-\left\{\frac{1}{2}\sqrt{3}+\frac{4}{2}\sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= -\sqrt{3}+1+\frac{\sqrt{3}}{2}+2\times\frac{\pi}{3}-\frac{\sqrt{3}}{2}-2\times\frac{\pi}{6}$$

$$= \left(\frac{\pi}{3}-\sqrt{3}\right)$$

Hence, required area $= 4\left(\frac{\pi}{3}-\sqrt{3}\right)$ sq. units.

EXAMPLE 40 *Find the curve passing through the origin O, such that the ratio of the area of the triangle MTP, formed by the tangent at some point P of the curve, ordinate MP of this point and x-axis, to the area of the curvilinear triangle OMP is equal to a constant number such that* $|k| > \frac{1}{2}$.

SOLUTION Let $y = f(x)$ be the equation of the rquired curve. The equation of the tangent at any point $P(x, y)$ to this curve is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts X-axis at point $T\left(x - y\frac{dx}{dy}, 0\right)$.

Now,

A_1 = Area of curve linear triangle OMP

$$= \left|\int_0^x y\,dx\right|$$

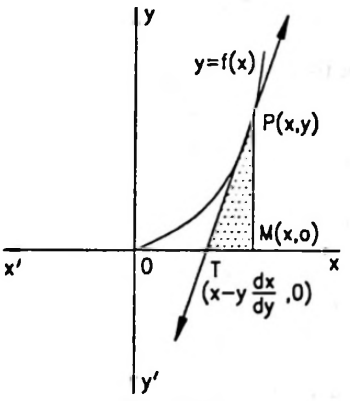


Fig. 12.43

A_2 = Area of ΔMTP

$$= \left|\frac{1}{2}(TM \times MP)\right|$$

$$= \left|\frac{1}{2}\left(y\frac{dx}{dy}\times y\right)\right|$$

$$= \left|\frac{1}{2}y^2\frac{dx}{dy}\right|$$

It is given that

$$\frac{A_2}{A_1} = k$$

$$\Rightarrow \frac{\left|\frac{1}{2}y^2\frac{dx}{dy}\right|}{\left|\int_0^x y\,dx\right|} = k$$

$$\Rightarrow \frac{1}{2}y^2 = k\left|\left(\int_0^x y\,dx\right)\frac{dy}{dx}\right|$$

$$\Rightarrow \frac{1}{2}y^2 = \pm k\left(\int_0^x y\,dx\right)\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{y}{\left(\int_0^x y\,dx\right)} = \pm 2k\ \frac{\frac{dy}{dx}}{y}$$

Integrating both sides, w.r. to x we get

$$\log\left(\int_0^x y\,dx\right) = \pm 2k\log y + \log C$$

$$\Rightarrow \int_0^x y\,dx = C\,y^{\pm 2k}$$

Differentiating both sides w.r.t. x, we get

$$y = (\pm 2k\,C)\,y^{\pm 2k-1}\frac{dy}{dx}$$

$\Rightarrow \quad dx = (\pm 2k\,C)\, y^{\pm 2k-2}\, dy$

Integrating both sides, we get

$$x = \pm 2k\,C\,\frac{y^{\pm 2k-1}}{(\pm 2k-1)} + C_1$$

It passes through the origin (0, 0). Therefore, $C_1 = 0$

Hence, the equations of the required curves are given by

$$x = \frac{\pm 2k\,C}{(\pm 2k-1)}\, y^{\pm 2k-1}$$

EXAMPLE 41 *Find a continuous function $f(x)$ satisfying*

$$x^4 - 4x^2 \le f(x) \le 2x^2 - x^3 \text{ for all } x \in [0, 2]$$

such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the y-axis and the line $x = t$ $(0 \le t \le 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y-axis and the line $x = t$ $(0 \le t \le 2)$.

SOLUTION Let A_1 be the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, y-axis and $x = t$.

Since $f(x) \ge x^4 - 2x^2$ for all $x \in [0, 2]$.

$$\therefore \quad A_1 = \int_0^t |f(x) - (x^4 - 4x^2)|\, dx \qquad \ldots(i)$$

Let A_2 be the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y-axis and $x = t$. Then,

$$A_2 = \int_0^t |(2x^2 - x^3) - f(x)|\, dx \qquad \left[\because f(x) \le (2x^2 - x^3) \text{ for all } x \in [0, 2]\right]$$

It is given that

$$A_1 = k A_2$$

$$\Rightarrow \int_0^t |f(x) - (x^4 - 4x^2)|\, dx = k\left[\int_0^t |(2x^2 - x^3) - f(x)|\, dx\right]$$

Differentiating both sides w.r.t. t, we get

$$f(t) - (t^4 - 4t^2) = k\,[(2t^2 - t^3) - f(t)]$$

$$\Rightarrow \quad (1 + k) f(t) = k\,(2t^2 - t^3) + (t^4 - 4t^2)$$

$$\Rightarrow f(t) = \frac{k}{k+1}(2t^2 - t^3) + \frac{t^4 - 4t^2}{k+1} \text{ for all } t \in [0, 2]$$

Hence,

$$f(x) = \frac{k}{k+1}(2x^2 - x^3) + \frac{(x^4 - 4x^2)}{k+1}$$

$$= \frac{1}{k+1}\{x^4 - kx^3 + 2(k-2)x^2\}.$$

EXAMPLE 42 *Find $f(x)$ satisfying $f(x) > x$ for all $x > 1$ if it is given that the area bounded by the curve $y = f(x)$, $y = x$ and the lines $x = 1$, $x = t$ is $(t + \sqrt{1 + t^2}) - \sqrt{2} - 1$ sq. units for all $t > 1$.*

SOLUTION It is given that $f(x) > x$ for all $x > 1$. So,

Area bounded by $y = f(x)$, $y = x$ and the lines $x = 1$, $x = t$ is given by

$$\int_1^t |f(x) - x|\, dx$$

But this area is given equal to $(t + \sqrt{1 + t^2} - \sqrt{2} - 1)$ sq.units. Therefore,

$$\int_1^t |f(x) - x|\, dx = t + \sqrt{1 + t^2} - \sqrt{2} - 1 \text{ for all } t > 1$$

Differentiating both sides w.r.t. t, we get

$$f(t) - t = 1 + \frac{t}{\sqrt{1 + t^2}} \qquad \text{for all } t > 1$$

$$\Rightarrow \quad f(t) = t + 1 + \frac{t}{\sqrt{1 + t^2}} \qquad \text{for all } t > 1.$$

Hence, $f(x) = x + 1 + \dfrac{x}{\sqrt{1 + x^2}}$ for all $x > 1$.

EXAMPLE 43 *Let C_1, C_2 and C_3 be the graphs of the function $y = x^2$, $y = 2x$, $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P parallel to the axes meet C_2 and C_3 at Q and R respectively as shown in Fig. 12.44. If for every position P on C_1 the area of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.*

[IIT 1998]

SOLUTION Let $P(t, t^2)$ be any point on curve C_1.

Since PQ is parallel to x-axis. Therefore, y-coordinate of Q is t^2. As Q lies on $y = 2x$. So, its x-coordinate is $t^2/2$.

Thus, the coordinates of Q are $\left(t^2/2, t^2\right)$.

It is evident from the figure that the x-coordinate of R is t and so its coordinates are $(t, f(t))$.

To find the area of the region OPQ, we slice it into horizontal strips. The approximating rectangle shown in Fig. 12.44 has length $= x_2 - x_1$, width $= \Delta y$ and it runs from $y = 0$ to $y = t^2$.

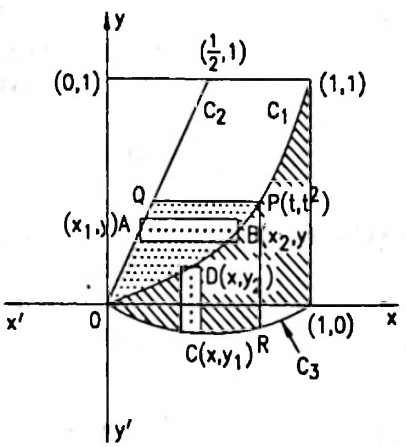


Fig. 12.44

$\therefore$ Area of the region OPQ

$$= \int_0^{t^2} (x_2 - x_1)\, dy$$

$$= \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right) dy \qquad \left[\because A(x_1, y) \text{ and } B(x_2, y) \text{ lie on } y = 2x \text{ and } y = x^2 \text{ respectively. } \therefore y = 2x_1 \text{ and } y = x_2^2 \Rightarrow x_1 = y/2 \text{ and } x_2 = \sqrt{y}\right]$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^2}{4}\right]_0^{t^2}$$

$$= \frac{2}{3} t^3 - \frac{t^4}{4}$$

The region ORP is sliced into vertical strips. For the approximating rectangle shown in Fig. 12.44, Length $= y_2 - y_1$, width $= \Delta x$ and it can move from $x = 0$ to $x = t$.

$\therefore$ Area of the region ORP

$$= \int_0^t (y_2 - y_1)\, dx$$

$$= \int_0^t (x^2 - f(x))\, dx \qquad \left[\because C(x, y_1) \text{ and } D(x, y_2) \text{ lie on } y = f(x) \text{ and } y = x^2 \text{ respectively. } \therefore y_1 = f(x) \text{ and } y_2 = x^2\right]$$

$$= \int_0^t x^2\, dx - \int_0^t f(x)\, dx$$

$$= \frac{t^3}{3} - \int_0^t f(x)\, dx$$

It is given that for all positions of point P

Area of the region OPQ = Area of the region ORP

$$\Rightarrow \frac{2}{3} t^3 - \frac{t^4}{4} = \frac{t^3}{3} - \int_0^t f(x)\, dx \qquad \text{for all } t \in [0, 1]$$

$$\Rightarrow \int_0^t f(x)\, dx = -\frac{t^3}{3} + \frac{t^4}{4} \qquad \text{for all } t \in [0, 1]$$

Differentiating both sides w.r.t. t, we get

$f(t) = -t^2 + t^3$ for all $t \in [0, 1]$

Hence, $f(x) = x^3 - x^2$ for all $x \in [0, 1]$.

EXAMPLE 44 *Consider a square with vertices at (1, 1), (−1, 1), (−1, −1) and (1, −1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.*

[IIT 1995]

SOLUTION Recall that a parabola is the locus of a point which is equidistant from a fixed point and a fixed line. The fixed point is the focus and the fixed line is the directrix. The points which are nearer to the fixed point (focus) than the fixed line (directrix) lie inside the parabola.

Curve C_1 represents a parabola having focus at (0, 0) and directrix AD. So, its equation is

$$(y - 0)^2 = -2\left(x - \frac{1}{2}\right) \qquad \ldots(i)$$

Similarly, equations of parabolas C_2, C_3 and C_4 are respectively

$$(y - 0)^2 = 2\left(x + \frac{1}{2}\right) \qquad \ldots(ii)$$

$$(x - 0)^2 = -2\left(y - \frac{1}{2}\right) \qquad \ldots(iii)$$

and, $$(x - 0)^2 = 2\left(y + \frac{1}{2}\right) \qquad \ldots(iv)$$

The region S is the shaded region enclosed by these four parabolas.

Parabolas C_1 and C_3 intersect at point E whose coordinates are obtained by solving the equations (i) and (iii) simultaneously

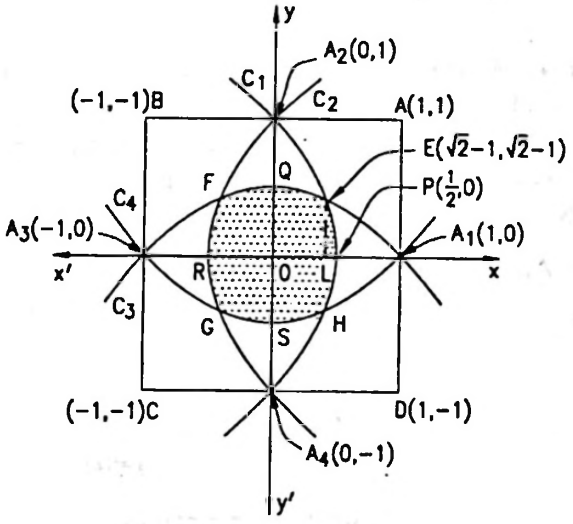


Fig. 12.45

We have,

$$y^2 + 2x - 1 = 0$$

and, $$x^2 + 2y - 1 = 0$$

Subtracting these two, we get

$$y^2 - x^2 + 2(x - y) = 0$$

$$\Rightarrow (y - x)(y + x - 2) = 0$$

$$\Rightarrow y = x \text{ or } y = 2 - x$$

Putting $y = x$ in $y^2 + 2x - 1 = 0$, we get

$$y^2 + 2y - 1 = 0 \Rightarrow y = \sqrt{2} - 1$$

Thus, the coordinate of E are $(\sqrt{2} - 1, \sqrt{2} - 1)$.

From the symmetry, we observe that

Area of region S

$= 4$ (Area of region $OPEQO$)

$= 4$ (Area of region $OLEQO$ + Area of region $LPEL$)

$$= \left[\int_0^{\sqrt{2}-1} \frac{1-x^2}{2}\,dx + \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\,dx \right]$$

$$= 4\left[\left\{ \frac{1}{2}\left(x - \frac{x^3}{3} \right) \right\}_0^{\sqrt{2}-1} - \frac{1}{3}\left\{ (1-2x)^{3/2} \right\}_{\sqrt{2}-1}^{1/2} \right]$$

$$= 4\left[\frac{1}{2}(\sqrt{2}-1)\left\{ 1 - \frac{(\sqrt{2}-1)^2}{3} \right\} - \frac{1}{3}\left\{ 0 - (3-2\sqrt{2})^{3/2} \right\} \right]$$

$$= 4\left[(\sqrt{2}-1)\frac{\sqrt{2}}{3} + \frac{1}{3}(\sqrt{2}-1)^3 \right]$$

$$= \frac{4}{3}(\sqrt{2}-1)[3-\sqrt{2}]$$

$$= \frac{4}{3}(4\sqrt{2}-5) \text{ sq. units}$$

EXAMPLE 45 *A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in the ratio $p : q$, find the equation of the curve.*

SOLUTION Let

A_1 = Area of the curve linear triangle $OLPO$

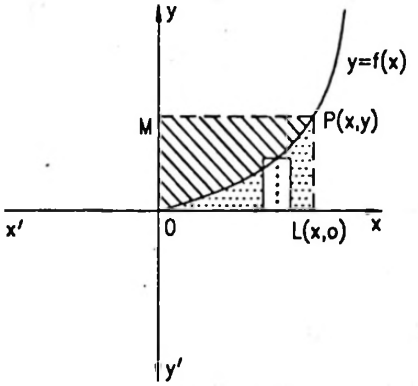


Fig. 12.46

$$= \int_0^x y\,dx$$

A_2 = Area of curve linear triangle $OMPO$

= Area of rectangle $OLPM$ – Area of curve linear triangle $OLPO$

$$= xy - \int_0^x y\,dx$$

It is given that

$$\frac{A_2}{A_1} = \frac{p}{q}$$

$$\Rightarrow \frac{xy - \int_0^x y\,dx}{\int_0^x y\,dx} = \frac{p}{q}$$

$$\Rightarrow \frac{xy}{\int_0^x y\,dx} = 1 + \frac{p}{q}$$

$$\Rightarrow xy = \left(\frac{p+q}{q} \right) \int_0^x y\,dx$$

Differentiating both sides w.r.t. x, we get

$$y + x\frac{dy}{dx} = \left(\frac{p+q}{q} \right) y$$

$$\Rightarrow x\frac{dy}{dx} = \frac{p}{q}y$$

$$\Rightarrow \frac{dy}{y} = \frac{p}{q}\frac{dx}{x}$$

$$\Rightarrow \log y = \frac{p}{q}\log x + \log C$$

$$\Rightarrow y = C x^{p/q}$$

This is the equation of the required curve.

EXAMPLE 46 *Let $f(x)$ be a continuous function given by*

$$f(x) = \begin{cases} 2x & , |x| \le 1 \\ x^2 + ax + b & , |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

[IIT 1999]

SOLUTION We have,

$$f(x) = \begin{cases} 2x & , -1 \le x \le 1 \\ x^2 + ax + b & , x < -1 \text{ or } x > 1 \end{cases}$$

Since $f(x)$ is everywhere continuous so it is continuous at $x = \pm 1$.

$$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \text{ and } \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\Rightarrow 1 - a + b = -2 \quad \text{and} \quad 2 = 1 + a + b$$

$$\Rightarrow a - b = 3 \quad \text{and} \quad a + b = 1$$

$\Rightarrow a = 2, b = -1$

$\therefore f(x) = \begin{cases} 2x & , -1 \le x \le 1 \\ x^2 + 2x - 1, & x < -1 \text{ or } x > 1 \end{cases}$

The graph of the curve $y = f(x)$ is shown in Fig. 12.47 and the region on the left of the line $8x + 1 = 0$ enclosed by the curves $x = -2y^2$ and $y = f(x)$ in the third quadrant is shaded in Fig. 12.47.

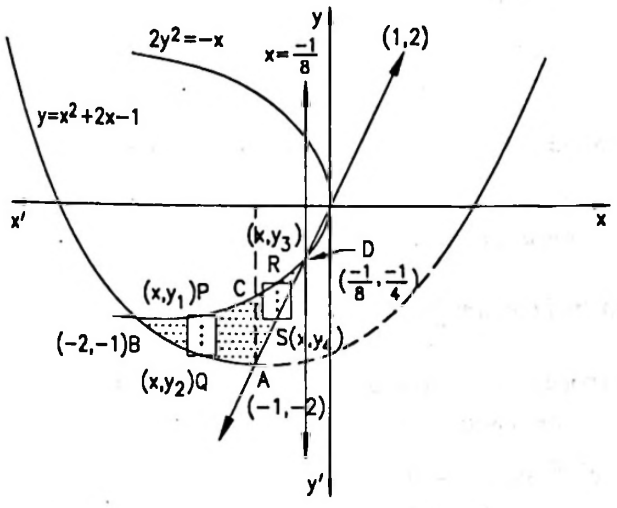


Fig. 12.47

Required area

$$= \int_{-2}^{-1} (y_1 - y_2)\,dx + \int_{-1}^{-1/8} (y_3 - y_4)\,dx$$

$$= \int_{-2}^{-1} \left\{\sqrt{\frac{-x}{2}} - x^2 - 2x + 1\right\} dx + \int_{-1}^{-1/8} \left\{\sqrt{\frac{-x}{2}} - 2x\right\} dx$$

$$= \left[-\frac{1}{\sqrt{2}} \times \frac{2}{3}(-x)^{3/2} - \frac{x^3}{3} - x^2 + x\right]_{-2}^{-1} + \left[\frac{-2}{3\sqrt{2}}(-x)^{3/2} - x^2\right]_{-1}^{-1/8}$$

$$= \left[\left\{-\frac{2}{3\sqrt{2}} + \frac{1}{3} - 1 - 1\right\} - \left\{-\frac{2}{3\sqrt{2}} \times 2\sqrt{2} + \frac{8}{3} - 4 - 2\right\}\right]$$

$$+ \left[\left\{\frac{-2}{3\sqrt{2}} \times \frac{1}{16\sqrt{2}} - \frac{1}{64}\right\} - \left\{\frac{-2}{3\sqrt{2}} - 1\right\}\right]$$

$$= \left[\left(\frac{-2}{3\sqrt{2}} - \frac{5}{3}\right) - \left(-\frac{4}{3} + \frac{8}{3} - 6\right)\right] + \left[\left(-\frac{1}{48} - \frac{1}{64}\right) - \left(-\frac{2}{3\sqrt{2}} - 1\right)\right]$$

$$= -\frac{2}{3\sqrt{2}} - \frac{5}{3} + \frac{14}{3} - \frac{1}{48} - \frac{1}{64} + \frac{2}{3\sqrt{2}} + 1$$

$$= 4 - \frac{7}{192} = \frac{761}{192} \text{ sq. units.}$$

EXAMPLE 47 *Find the area of the region bounded by the curves* $y = \frac{4 - x^2}{4 + x^2}$, $25y^2 = 9x$ *and* $y = \frac{3}{5}|x| - \frac{6}{5}$ *which contains (1, 0) point in its interior.*

SOLUTION We have,

$$y = \frac{4 - x^2}{4 + x^2}$$

We observe that all powers of x in the above equation are even, so it is symmetric about y-axis. The curve intersects y axis at (0, 1). Also,

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{4 - x^2}{4 + x^2} = -1.$$

Therefore, as $x \to \infty$, $y \to -1$ i.e. $y = -1$ is an asympotote to the curve.

Now,

$$y = \frac{4 - x^2}{4 + x^2} \Rightarrow x = \pm 2\sqrt{\frac{1 - y}{y + 1}}$$

This means that x is imaginary for $y > 1$ or $y < -1$ i.e. the curve lies between $y = -1$ and $y = 1$. At (0, 1) the tangent to the curve $y = \frac{4 - x^2}{4 + x^2}$ is $y = 1$. Thus, the graph of $y = \frac{4 - x^2}{4 + x^2}$ is as shown in Fig. 12.48.

The equation $25y^2 = 9x$ represents a parabola with vertex (0, 0) and axis OX. $y = \frac{3}{5}|x| - \frac{6}{5}$ represents a pair of lines passing through $(0, -6/5)$. The shaded region in Fig. 12.48 represents the area enclosed by the given curves.

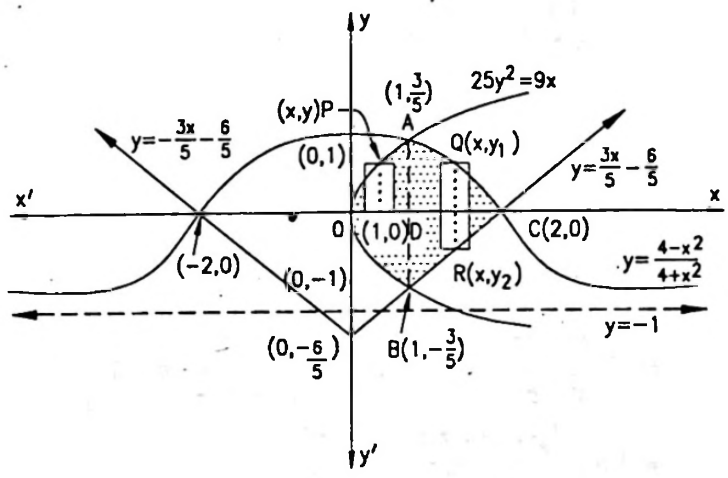


Fig. 12.48

Now,

Required area

= Area of the region $OABO$ + Area of the region $ADBCA$

= 2 (Area of the region $OADO$) + Area of the region $ADBCA$

$$= 2\int_0^1 y\,dx + \int_1^2 (y_1 - y_2)\,dx$$

$$= 2\int_0^1 \frac{3}{5}x + \int_1^2 \left|\frac{4 - x^2}{4 + x^2} - \frac{3x - 6}{5}\right| dx$$

$$= \frac{6}{5}\left[\frac{x^2}{2}\right]_0^1 + \int_1^2 \left\{-1 + \frac{8}{4+x^2} - \frac{3}{5}(x-2)\right\} dx$$

$$= \frac{3}{5} + \left[-x + 4\tan^{-1}\frac{x}{2} - \frac{3}{5}\times\frac{(x-2)^2}{2}\right]_1^2$$

$$= \frac{3}{5} + \left[(-2 + 4\tan^{-1}1) - \left\{-1 + 4\tan^{-1}\frac{1}{2} - \frac{3}{10}\right\}\right]$$

$$= \frac{3}{5} + \left[-2 + \pi + \frac{13}{10} - 4\tan^{-1}\frac{1}{2}\right]$$

$$= \left\{\pi - 4\tan^{-1}\frac{1}{2} - \frac{1}{10}\right\} \text{ sq. units.}$$

EXAMPLE 48 *Let $b \neq 0$ and for $j = 0, 1, 2, ..., n$ let S_j be the area of the region bounded by y-axis and the curve $x e^{ay} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, ..., S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$.*

[IIT 2001]

SOLUTION We have,

$$S_j = \left|\int_{j\pi/b}^{(j+1)\pi/b} x\, dy\right|$$

$$= \left|\int_{j\pi/b}^{(j+1)\pi/b} e^{-ay}\sin by\, dy\right|$$

$$= \left|\left[\frac{e^{-ay}}{a^2+b^2}(-a\sin by - b\cos by)\right]_{j\pi/b}^{(j+1)\frac{\pi}{b}}\right|$$

$$= \left|\frac{e^{-a(j+1)\pi/b}}{a^2+b^2}\times -b(-1)^{j+1} - \frac{e^{-aj\pi/b}}{a^2+b^2}\times -b(-1)^j\right|$$

$$= |b|\left|\frac{e^{-a(j+1)\pi/b}}{a^2+b^2} + \frac{e^{-aj\pi/b}}{a^2+b^2}\right|$$

$$= |b|\frac{e^{-aj\pi/b}}{a^2+b^2}\left\{e^{-a\pi/b}+1\right\}, j = 0, 1\,2, ..., n$$

Now,

$$\frac{S_{j+1}}{S_j} = |b|\frac{e^{-a(j+1)\pi/b}}{a^2+b^2}\left\{e^{-a\pi/b}+1\right\} + |b|\frac{e^{-aj\pi/b}}{a^2+b^2}(e^{-a\pi/b}+1)$$

$$= e^{-a\pi/b} \qquad \text{for all } j = 0, 1, 2, ..., n$$

Hence, $S_0, S_1, S_2, ..., S_n$ are in G.P. with common ration $e^{-a\pi/b}$.

For $a = -1$ and $b = \pi$, we have

$$S_j = \frac{e^j\pi(e+1)}{\pi^2+1}, j = 0, 1, 2, .., n$$

$$\therefore \sum_{j=0}^{n} S_j = \sum_{j=0}^{n}\frac{e^j\pi(e+1)}{\pi^2+1}$$

$$= \frac{(e+1)\pi}{\pi^2+1}(1 + e + e^2 + ... + e^n)$$

$$= \frac{(e+1)\pi}{\pi^2+1}\left\{\frac{e^{n+1}-1}{e-1}\right\}$$

EXAMPLE 49 *If $\alpha, \beta > 0$ prove that the areas $S_0, S_1, S_2, ...$, bounded by the x-axis and half-waves of the curve $y = e^{-\alpha x}\sin\beta x$; $x \ge 0$ form a G.P. with common ratio $e^{-\pi\alpha/\beta}$.*

Also, find the sum $\sum_{n=0}^{\infty} S_n$.

SOLUTION The curve $y = e^{-\alpha x}\sin\beta x$ cuts x-axis at the points whose abscissae are given by

$$e^{-\alpha x}\sin\beta x = 0$$

$$\Rightarrow \quad \sin\beta x = 0$$

$$\Rightarrow \quad \beta x = n\pi, n = 0, 1, 2, ... \qquad [\because x > 0]$$

$$\Rightarrow \quad x_n = \frac{n\pi}{\beta}, n = 0, 1, 2, ...$$

The graph of $y = e^{-\alpha x}\sin\beta x$ is shown in Fig. 12.49. We observe that

$$e^{-\alpha x}\sin\beta x \ge 0 \text{ for all } x \in \left[\frac{2n\pi}{\beta}, \frac{(2n+1)\pi}{\beta}\right]$$

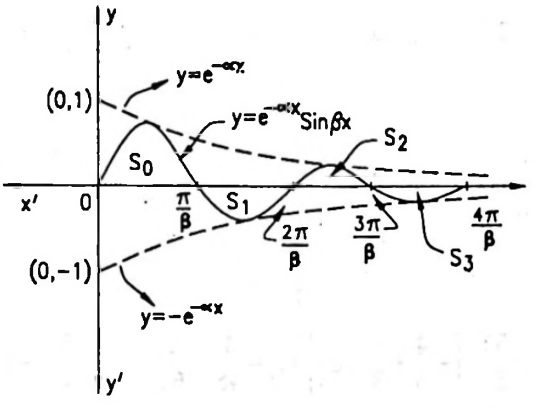


Fig. 12.49

and,

$$e^{-\alpha x}\sin\beta x \le 0 \text{ for all } x \in \left[(2n+1)\frac{\pi}{\beta}, (2n+2)\frac{\pi}{\beta}\right], n = 0, 1, 2, ...$$

Now,

$$S_n = \left|\int_{n\pi/\beta}^{(n+1\pi/\beta} y\, dx\right|$$

$$= \left| \int_{n\pi/\beta}^{(n+1\pi/\beta} e^{-\alpha x} \sin \beta x \, dx \right|$$

$$= \left| \left[\frac{e^{-\alpha x}}{\alpha^2+\beta^2} (-\alpha \sin \beta x - \beta \cos \beta x) \right]_{n\pi/\beta}^{(n+1)\pi/\beta} \right|$$

$$= \left| \frac{e^{-\alpha(n+1)\pi/\beta}}{\alpha^2+\beta^2} \{0 - \beta(-1)^{n+1}\} - \frac{e^{-\alpha n\pi/\beta}}{\alpha^2+\beta^2} \{0 - \beta(-1)^n\} \right|$$

$$= \left| e^{-\alpha(n+1)\pi/\beta} \cdot \frac{\beta}{\alpha^2+\beta^2} + e^{-\alpha n\pi/\beta} \cdot \frac{\beta}{\alpha^2+\beta^2} \right|$$

$$= \frac{|\beta|}{(\alpha^2+\beta^2)} e^{-\alpha n\pi/\beta} (e^{-\alpha\pi/\beta} + 1)$$

Now,

$$\frac{S_{n+1}}{S_n}$$

$$= \frac{|\beta| e^{-\alpha(n+1)\pi/\beta}}{\alpha^2+\beta^2} (e^{-\alpha\pi/\beta} + 1) + |\beta| e^{-\alpha n\pi/\beta} (e^{-\alpha\pi/\beta} + 1)$$

$= e^{-\pi\alpha/\beta}$, which is a constant.

Hence, $S_0, S_1, S_2, \ldots$ form a G.P. with common ratio $e^{-\pi\alpha/\beta}$.

Now,

$$S_0 = \frac{|\beta|}{\alpha^2+\beta^2} \left(e^{-\alpha\pi/\beta} + 1\right)$$

$$\therefore \sum_{n=0}^{\infty} S_n = S_0 + S_1 + S_2 + \ldots$$

$$= \frac{S_0}{1 - e^{-\pi\alpha/\beta}} = \frac{|\beta|}{(\alpha^2+\beta^2)} \left| \frac{1+e^{-\pi\alpha/\beta}}{1-e^{-\pi\alpha/\beta}} \right|$$

EXAMPLE 50 *Sketch and find the area bounded by the curve* $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ *and* $x^2 + y^2 = a^2, a > 0$. *If the lines* $|x| + |y| = a$ *divide the area in two parts find their ratio in first quadrant only.*

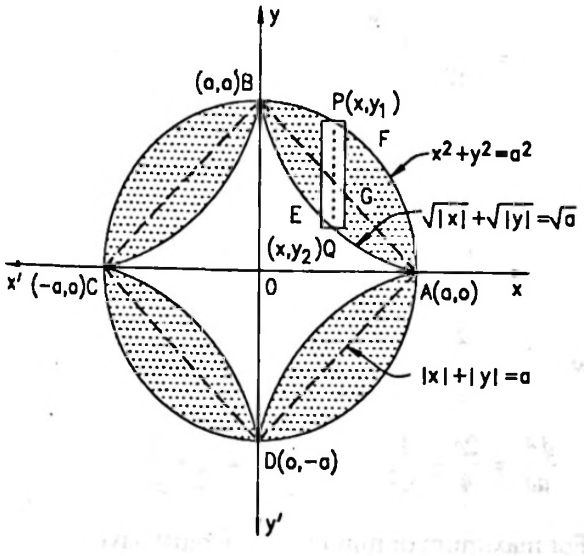


Fig. 12.50

SOLUTION The shaded region in Fig. 12.50 represents the region enclosed by $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$.

From the symmetry, it is evident that

Required area

= 4 [Area of the region bounded by the two curves in first quadrant only]

$$= 4 \int_0^a (y_1 - y_2) \, dx$$

$$= 4 \int_0^a \{\sqrt{a^2 - x^2} - (\sqrt{a} - \sqrt{x})^2\} \, dx$$

$$= 4 \int_0^a (\sqrt{a^2 - x^2} - a - x + 2\sqrt{a}\sqrt{x}) \, dx$$

$$= 4 \left[\frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} - ax - \frac{x^2}{2} + \frac{4\sqrt{a}}{3} x^{3/2} \right]_0^a$$

$$= 4 \left[\left\{ \frac{1}{2} a^2 \times \frac{\pi}{2} - a^2 - \frac{a^2}{2} + \frac{4}{3} a^2 \right\} - 0 \right]$$

$$= \left\{ \pi a^2 - \frac{2}{3} a^2 \right\} \text{ sq. units.}$$

Now,

A_1 = Area included between the line and circle in first quadrant

$$= \frac{\pi a^2}{4} - \frac{1}{2} a^2 = \frac{(\pi - 2) a^2}{4}$$

A_2 = Area included between the line and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$

$$= \left\{ \pi a^2 - \frac{2}{3} a^2 \right\} - A_1$$

$$= \left\{ \pi a^2 - \frac{2}{3} a^2 \right\} - \left(\frac{\pi - 2}{4} \right) a^2$$

$$= \frac{3\pi}{4} a^2 - \frac{1}{6} a^2$$

$$= (9\pi - 2) \frac{a^2}{12}$$

$$\therefore \text{ Required ratio } = \frac{A_2}{A_1} = \frac{1}{3} \left(\frac{9\pi - 2}{\pi - 2} \right)$$

EXAMPLE 51 *Sketch the region bounded by the curves* $|x+y| \le 1, |x-y| \le 1$ *and* $3x^2 + 3y^2 = 1$. *Also, find its area.*

SOLUTION We have,

$$|x+y| \le 1 \Leftrightarrow -1 \le x+y \le 1$$

and, $|x-y| \le 1 \Leftrightarrow -1 \le x-y \le 1$

Clearly, $|x+y| \le 1$ represents that part the xy-plane which lies between the lines $x+y=1$ and $x+y=-1$.

Also, $|x-y| \le 1$ represents the region lying between the lines $x-y=1$ and $x-y=-1$.

The equation $3x^2+3y^2=1$ represents a circle having centre at the origin and radius $\frac{1}{\sqrt{3}}$.

The required region is shaded in Fig. 12.51.

Clearly,

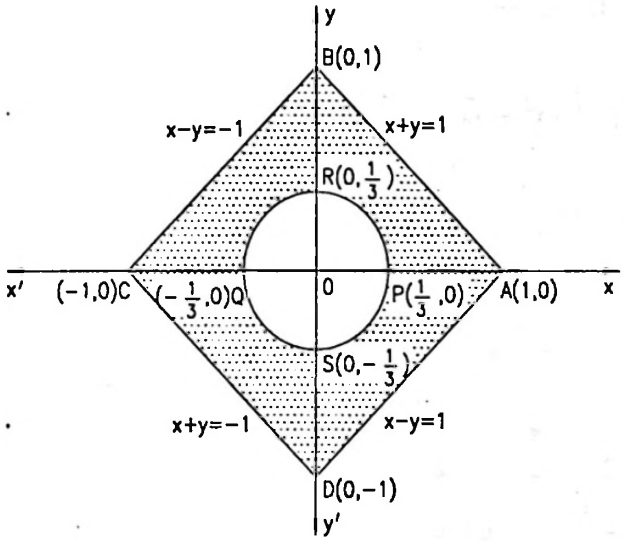


Fig. 12.51

Required area

= 4 [Area enclosed by $x+y=1$ and $x^2+y^2=1$ in the first-quadrant]

= 4 (Area of ΔOAB − Area of quadrant $OPRO$)

$$= 4\left\{\frac{1}{2}\times 1\times 1-\frac{\pi}{4}\left(\frac{1}{\sqrt{3}}\right)^2\right\}$$

$$= 4\left(\frac{1}{2}-\frac{\pi}{12}\right)$$

$$= \left(2-\frac{\pi}{3}\right) \text{ sq. units}$$

EXAMPLE 52 *Find a (>0) for which the area of the curvilinear trapezoid bounded by the curve $y=\frac{1}{6}x+x^{-2}$ and the lines whose combined equation is $y(x^2-3ax+2a^2)=0$, takes the least value. Also plot the curvilinear trapezoid.*

SOLUTION We have,

$$y = \frac{1}{6}x + \frac{1}{x^2}$$

We observe that $y \to \infty$ as $x \to 0$ and $y \to 0$ as $x \to \pm\infty$.

Also, $\frac{dy}{dx} = \frac{1}{6}-\frac{2}{x^3}$ and, $\frac{d^2y}{dx^2}=\frac{6}{x^4}$

$\frac{dy}{dx}=0 \Rightarrow x=(12)^{1/3}$.

Clearly, $\frac{d^2y}{dx^2}>0$ for $x=12^{1/3}$.

So, $x=12^{1/3}$ is a point of local minimum.

The curve $y=\frac{x}{6}+\frac{1}{x^2}$ crosses x-axis at $((-6)^{1/3},0)$.

The rough sketch of $y=\frac{x}{6}+\frac{1}{x^2}$ is as shown in Fig. 12.52. The equation $y(x^2-3ax+2a^2)=0$ gives $y=0, x=a, x=2a$

The shaded region in Fig. 12.52 represents the region enclosed by $y=\frac{1}{6}x+x^{-2}$ and $y(x^2-3ax+2a^2)=0$

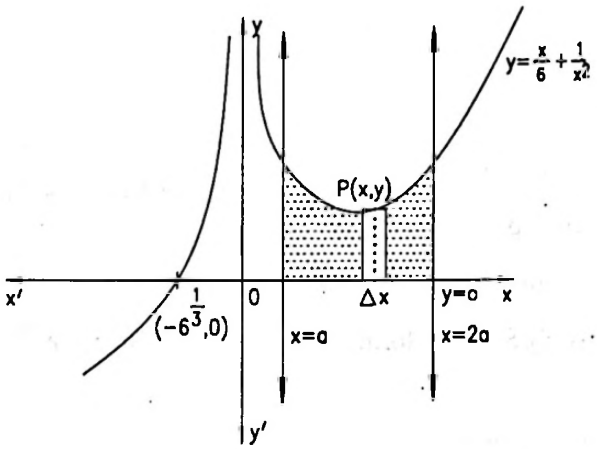


Fig. 12.52

Let A be the area enclosed. Then,

$$A = \int_a^{2a} y\,dx$$

$$= \int_a^{2a}\left(\frac{x}{6}+\frac{1}{x^2}\right)dx$$

$$= \left[\frac{x^2}{12}-\frac{1}{x}\right]_a^{2a}$$

$$= \frac{a^2}{3}-\frac{1}{2a}-\frac{a^2}{12}+\frac{1}{a}$$

$$= \frac{a^2}{4}+\frac{1}{2a}$$

Now, $A = \frac{a^2}{4}+\frac{1}{2a}$

$$\Rightarrow \quad \frac{dA}{da} = \frac{2a}{4}-\frac{1}{2a^2} \quad \text{and} \quad \frac{d^2A}{da^2}=\frac{1}{2}+\frac{1}{a^3}$$

For maximum or minimum, we must have

$$\frac{dA}{da} = 0 \Rightarrow \frac{2a}{4} - \frac{1}{2a^2} = 0$$

$$\Rightarrow 4a^3 - 4 = 0 \Rightarrow a = 1.$$

Clearly, $\frac{d^2A}{da^2} = \frac{1}{2} + 1 > 0$ for $a = 1$.

Thus, A is minimum when $a = 1$.

EXAMPLE 53 *Find the area of the curvilinear triangle formed by the figure* $xy^2 = a^2(x-a)$, *line* $y = \frac{a}{2}$ *and the circle* $x^2 + y^2 = a^2$ *as sides.*

SOLUTION For the curve $xy^2 = a^2(x-a)$, we observe the following points :

(i) The curve is symmetric about x-axis as all powers of x are even

(ii) It cuts x-axis at $(a, 0)$.

(iii) Values of y are imaginary for $0 < x < a$.

(iv) We have, $y^2 - a^2 = -\frac{a^3}{x}$. Therefore, $y^2 - a^2 \to 0$ as $x \to \pm\infty$ i.e. $y \to \pm a$ as $x \to \pm\infty$.

Thus, the graph of $xy^2 = a^2(x-a)$ is as shown in Fig. 12.53. The equation $x^2 + y^2 = a^2$ represents a circle with centre at $(0, 0)$ and radius a.

The shaded region in Fig. 12.53 is the region enclosed by the curvelinear triangle.

Required area

$$= \int_0^{a/2} (x_2 - x_1)\, dy$$

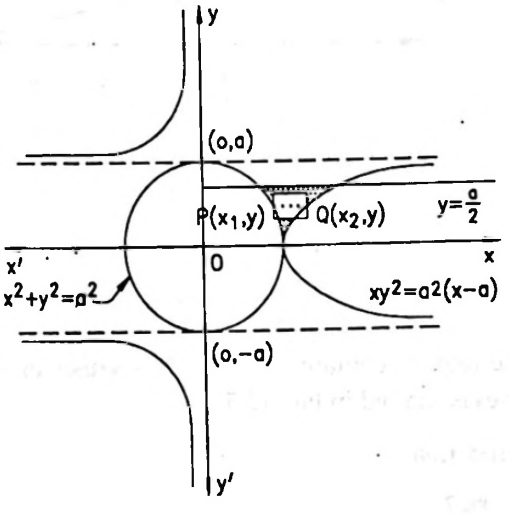


Fig. 12.53

$$= \int_0^{a/2} \left| \frac{a^3}{a^2 - y^2} - \sqrt{a^2 - y^2} \right| dy$$

$$= \frac{a^3}{2a}\left[\log\frac{a+y}{a-y}\right]_0^{a/2} - \left[\frac{1}{2}y\sqrt{a^2-y^2} + \frac{1}{2}a^2\sin^{-1}\frac{y}{a}\right]_0^{a/2}$$

$$= \frac{a^2}{2}\log 3 - \frac{a}{4}\times\frac{\sqrt{3}a}{2} - \frac{1}{2}\frac{a^2\pi}{6}$$

$$= \left[\frac{a^2}{2}\left\{\log 3 - \frac{\sqrt{3}}{4}\right\} - \frac{\pi a^2}{12}\right] \text{sq. units}$$

EXAMPLE 54 *Draw the rough sketch and find the area enclosed by the curve* $\{y - e^{-\sqrt{1-x^2}}\}^2 = 1 - x^2$.

SOLUTION We observe that in the given curve

(i) all powers of x are even so it is symmetric about y-axis.

(ii) values of y-axist for $x \in [-1, 1]$

(iii) intersects y-axis at $\left(0, \frac{1}{e} + 1\right)$ and $\left(0, \frac{1}{e} - 1\right)$

(iv) For $x = \pm 1$, $y = 1$. So, the curve passes through (1, 1) and (– 1, 1).

The rough sketch of the curve is shown in Fig. 12.54.

We have,

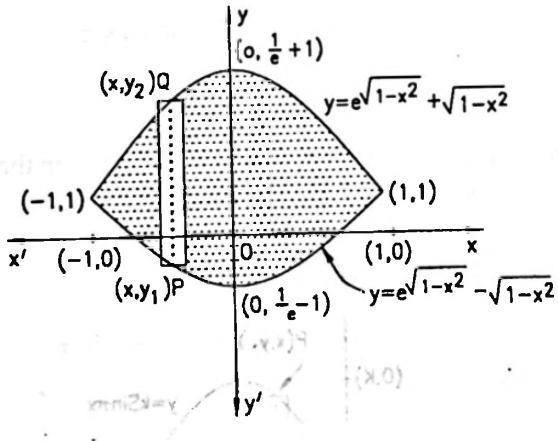


Fig. 12.54

$$\{y - e^{\sqrt{1-x^2}}\}^2 = 1 - x^2$$

$$\Rightarrow y - e^{\sqrt{1-x^2}} = \pm\sqrt{1-x^2}$$

$$\Rightarrow y = e^{\sqrt{1-x^2}} \pm \sqrt{1-x^2}$$

The shaded region in Fig. 12.54 represents the region enclosed by the curve. The two branches of the curve are given by $y = e^{\sqrt{1-x^2}} + \sqrt{1-x^2}$ and $y = e^{\sqrt{1-x^2}} - \sqrt{1-x^2}$.

The shaded region in Fig. 12.54 is sliced into vertical strips. The approximating rectangle shown in Fig. 12.54 has length $= y_2 - y_1$, with $= \Delta x$ and it can move between $x = -1$ and $x = 1$. So,

Required area

$$= \int_{-1}^{1} (y_2 - y_1)\, dx$$

$$= \int_{-1}^{1} \left\{(e^{\sqrt{1-x^2}} + \sqrt{1-x^2}) - (e^{\sqrt{1-x^2}} - \sqrt{1-x^2})\right\} dx$$

$$\left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y_1 = e^{\sqrt{1-x^2}} - \sqrt{1-x^2} \text{ and } y_2 = e^{\sqrt{1-x^2}} + \sqrt{1-x^2} \text{ respectively.}\right]$$

$$= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$= 4\int_{0}^{1} \sqrt{1-x^2}\, dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$= \pi$ sq. units.

EXAMPLE 54 *The area between the two curves defined by* $y = |x - (x)|$ *and* $y = k \sin \pi x$, $k \geq 1$ *in the interval* $0 \leq x \leq 1$ *is found to be equal to* 2 sq. units. *Plot the common area and find the constant k. Here ( ) denotes the integer nearest to x e.g.* $(1.3) = 1$, $(1.5) = 2$, $(1.6) = 2$ *and so on.*

SOLUTION We have,

$$y = |x - (x)| = \begin{cases} |x|, & 0 < x < \frac{1}{2} \\ |x-1|, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

The graphs of $y = k \sin \pi x$ and $y = |x - (x)|$ in the interval $[0, 1]$ are shown in Fig. 12.55. The common region is the shaded region in the figure.

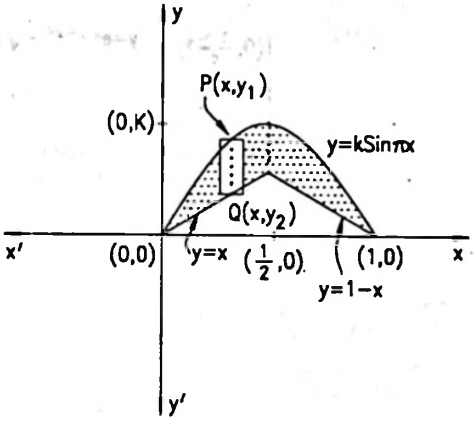


Fig. 12.55

Now,

Area of the shaded region

$$= 2\int_{0}^{1/2} (y_1 - y_2)\, dx$$

$$= 2\int_{0}^{1/2} (k \sin \pi x - x)\, dx$$

$$= 2\left[\frac{-k}{\pi}\cos \pi x - \frac{x^2}{2}\right]_0^{1/2}$$

$$= 2\left[-\frac{1}{8} + \frac{k}{\pi}\right]$$

But the area of the shaded region is given to be equal to 2.

$$\therefore\ 2\left(-\frac{1}{8} + \frac{k}{\pi}\right) = 2 \Rightarrow k = \frac{9\pi}{8}.$$

EXAMPLE 55 *Find the area of the region containing the points satisfying*

$$|y| + \frac{1}{2} \leq e^{-|x|}, \max(|x|, |y|) \leq 2.$$

SOLUTION We observe that

$$\max(|x|, |y|) \leq 2$$

represents the region enclosed by the square whose sides are $x = 2$, $x = -2$, $y = 2$ and $y = -2$.

In order to draw the graph of $|y| + \frac{1}{2} \leq e^{-|x|}$, let us consider the curve given by

$$y + \frac{1}{2} = e^{-x}$$

or $$y = e^{-x} - \frac{1}{2}$$

The graph of this curve is shown in Fig. 12.56.

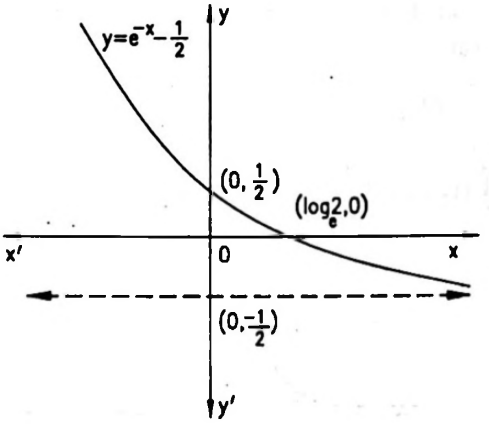


Fig. 12.56

By using transformations of graphs (see chapter 2), the graph of $|y| = e^{-|x|} - \frac{1}{2}$ is as shown in Fig. 12.57.

The region containing the points satisfying the two inequalities is shaded in Fig. 12.57.

Required area

$$= 4\int_{0}^{\log 2} y_1\, dx$$

$$= 4\int_{0}^{\log 2} \left(e^{-x} - \frac{1}{2}\right) dx$$

$$= 4\left[-e^{-x} - \frac{x}{2}\right]_0^{\log 2}$$

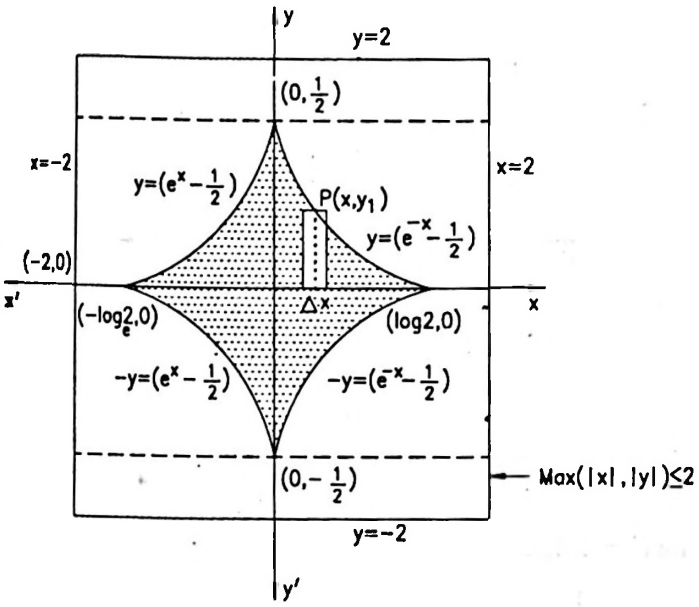


Fig. 12.57

$$= 4\left\{-\frac{1}{2}-\frac{1}{2}\log 2+1\right\}$$

$$= 2(1-\log 2) = 2\log(e/2) \text{ sq. units}$$

EXAMPLE 56 *Find the total area bounded by the curves* $y = \cos x - \cos^2 x$ *and* $y = x^2\left(x^2 - \frac{\pi^2}{4}\right)$.

SOLUTION The shaded region in Fig. 12.58 shows the region bounded by the two curves. Since both the curves are symmetric about y-axis.

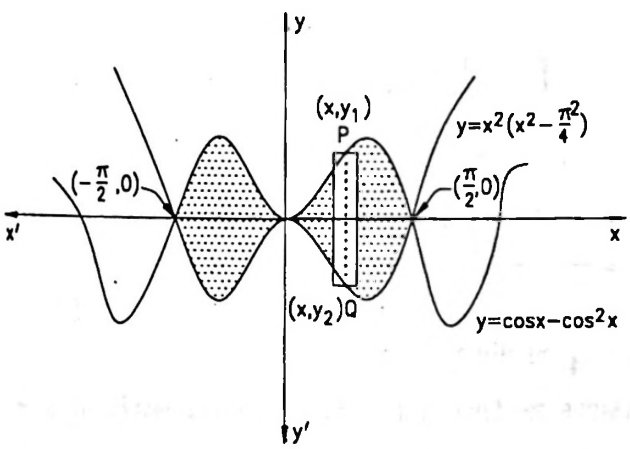


Fig. 12.58

Therefore,

Required area

$$= 2\int_0^{\pi/2} (y_1 - y_2)\, dx$$

$$= 2\int_0^{\pi/2} \left[\cos x - \cos^2 x - x^2\left(x^2 - \frac{\pi^2}{4}\right)\right] dx$$

$$= 2\left[\sin x - \frac{x}{2} - \frac{\sin 2x}{4} - \frac{x^5}{5} + \frac{\pi^2 x^3}{12}\right]_0^{\pi/2}$$

$$= \left\{2 - \frac{\pi}{2} + \frac{\pi^5}{120}\right\} \text{ sq. units}$$

EXAMPLE 57 *The functions* $y = 1 + \cos x$ *and* $y = 1 + \cos(x - \alpha)$, *where* $0 < \alpha < \pi/2$ *are given on the interval* $[0, \pi]$. *At what value of* α *is the area of the figure bounded by the curves* $y = 1 + \cos x$, $y = 1 + \cos(x - \alpha)$, $x = 0$, *is equal to the area of the figure bounded by the curves* $y = 1 + \cos(x - \alpha)$, $y = 1$ *and* $x = \pi$.

SOLUTION Clearly, $A_1 A_2 A_3 A_1$ represents the region bounded by $y = 1 + \cos x$, $y = 1 + \cos(x - \alpha)$and $x = 0$ and $B_1 B_2 B_3 B_1$ represents the region enclosed by the curves $y = 1 + \cos(x - \alpha)$, $y = 1$ and $x = \pi$.

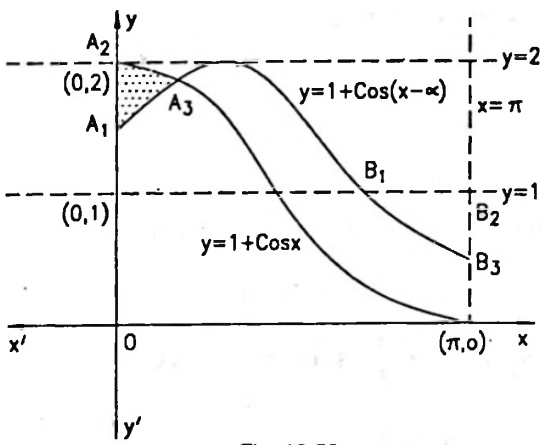


Fig. 12.59

Curves $y = 1 + \cos x$ and $y = 1 + \cos(x - \alpha)$ intersect at the point whose abscissa is given by

$$1 + \cos x = 1 + \cos(x - \alpha)$$

$$\Rightarrow \quad \cos x = \cos(x - \alpha)$$

$$\Rightarrow \quad x = \alpha - x$$

$$\Rightarrow \quad x = \frac{\alpha}{2}$$

$$\therefore \quad S_1 = \text{Area of the region } A_1 A_2 A_3 A_1$$

$$= \int_0^{\alpha/2} |(1 + \cos x) - (1 + \cos(x - \alpha))|\, dx$$

$$= \int_0^{\alpha/2} |(\cos x - \cos(x - \alpha)|\, dx$$

$$= \left[\sin x - \sin(x - \alpha)\right]_0^{\alpha/2}$$

$$= \sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right) - \sin\alpha$$

$$= 2\sin\frac{\alpha}{2} - \sin\alpha$$

The x-coordinate of the point of intersection of $y = 1$ and $y = 1 + \cos(x - \alpha)$ is given by

$$1 = 1 + \cos(x - \alpha)$$

$$\Rightarrow \cos(x - \alpha) = 0$$

$$\Rightarrow x - \alpha = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2} + \alpha$$

$\therefore$ S_2 = Area of the region $B_1 B_2 B_3 B_1$

$$= \int_{\pi/2+\alpha}^{\pi} \{1 - (1 + \cos(x - \alpha))\}\, dx$$

$$= \int_{\pi/2+\alpha}^{\pi} -\cos(x - \alpha)\, dx$$

$$= \Big[-\sin(x - \alpha)\Big]_{\pi/2+\alpha}^{\pi}$$

$$= -\sin(\pi - \alpha) + \sin\frac{\pi}{2}$$

$$= 1 - \sin\alpha$$

We have to find the values of α for which $S_1 = S_2$.

Now,

$$S_1 = S_2 \Rightarrow 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha$$

$$\Rightarrow 2\sin\frac{\alpha}{2} = 1$$

$$\Rightarrow \sin\frac{\alpha}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{\alpha}{2} = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{3}$$

EXAMPLE 58 *Draw the rough sketch of the curve $y^2 = (1 - x^2)^3$ and compute the area enclosed by it.*

SOLUTION The equation of the given curve is $y^2 = (1 - x^2)^3$. We observe that the powers of x and y both are even. So, it is symmetric about both the axes. The curve meets the coordinate axes at (1, 0), (− 1, 0), (0, 1) and (0, − 1). The values of y are imaginary for $|x| > 1$.

We have,

$$y^2 = (1 - x^2)^3$$

$$\Rightarrow y = (1 - x^2)^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = -3x\sqrt{1 - x^2}$$

Clearly, $\frac{dy}{dx} = 0$ at $x = \pm 1$

Thus, x-axis is the tangent to the curve at (1, 0) and (− 1, 0).

Also, $\frac{dy}{dx} = 0$ at (0, 1) and (0, − 1). So, tangents at these points are parallel to x-axis. Keeping all these points in mind, we obtain the rough sketch of the curve $y^2 = (1 - x^2)^3$ as shown in Fig. 12.60. The shaded region is the region enclosed by the curve.

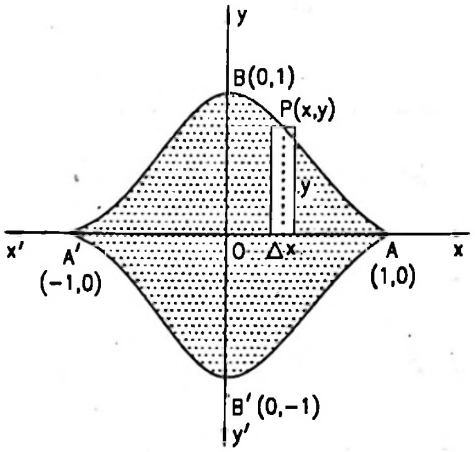


Fig. 12.60

Required Area

$$= 4\int_0^1 y\, dx$$

$$= 4\int_0^1 (1 - x^2)^{3/2}\, dx$$

$$= 4\int_0^{\pi/2} (1 - \sin^2\theta)^{3/2}\cos\theta\, d\theta \qquad [\text{Putting } x = \sin\theta]$$

$$= 4\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4\left(\frac{\Gamma_{5/2}\Gamma_{1/2}}{2\Gamma_3}\right)$$

$$= \frac{4 \times \frac{3}{2} \times \frac{1}{2}\Gamma_{1/2}\Gamma_{1/2}}{2 \times 2!}$$

$$= \frac{3\pi}{4} \text{ sq. units.}$$

EXAMPLE 59 *Find the area of the region enclosed by the curve*

$$5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$$

SOLUTION Comparing the given equation with the general equation of second degree i.e. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = 5, h = 3, b = 2, g = \frac{7}{2}, f = 3 \text{ and } c = 6$$

We have, $abc + 2fgh - af^2 - bg^2 - ch^2 = 60 - 45 - \frac{49}{2} - 54 \neq 0$

and, $h^2 - ab = 9 - 10 < 0$.

So, the given equation represents an ellipse.

Rewriting the given equation as a quadratic in y, we obtain

$$2y^2 + 6(1+x)y + 5x^2 + 7x + 6 = 0$$

$$\Rightarrow y = \frac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$$

Clearly, the values of y are real for $x \in [1, 3]$

When $x = 1$, we get $y = -3$

and, $x = 3 \Rightarrow y = -6$.

The rough sketch of the ellipse is as shown in Fig. 12.61.

Let $f(x) = \dfrac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}$

and, $g(x) = \dfrac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2}$

The region enclosed by the curves $y = f(x)$ and $y = g(x)$ is the region bounded by the given ellipse.

The shaded region in Fig. 12.61 is sliced into vertical strips. For the approximating rectangle shown in Fig. 12.61, we have length $= y_1 - y_2$, width $= \Delta x$ and it can move between $x = 1$ and $x = 3$. So,

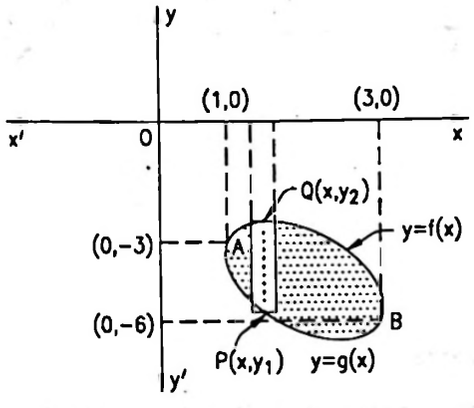


Fig. 12.61

Required area

$$= \left| \int_1^3 (y_1 - y_2)\, dx \right|$$

$$= \left| \int_1^3 |g(x) - f(x)|\, dx \right| \quad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = g(x) \text{ and } y = f(x) \text{ respectively} \right]$$

$$= \left| -\int_1^3 \sqrt{(3-x)(x-1)}\, dx \right|$$

$$= \left| -\int_1^3 \sqrt{-x^2 + 4x - 3}\, \right| dx$$

$$= \left| -\int_1^3 \sqrt{-(x^2 - 4x + 4 - 4 + 3)}\, dx \right|$$

$$= \left| -\int_1^3 \sqrt{1^2 - (x-2)^2}\, dx \right|$$

$$= \left| -\left[\frac{1}{2}(x-2)\sqrt{-x^2 + 4x - 3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1}\right) \right]_1^3 \right|$$

$$= \left| -\left[\left\{ \frac{1}{2}\sin^{-1} 1 \right\} - \left\{ \frac{1}{2}\sin^{-1}(-1) \right\} \right] \right|$$

$$= \left| -\frac{\pi}{2} \right| = \frac{\pi}{2} \text{ sq. units}$$

EXAMPLE 60 *Let $O(0, 0)$, $A(2, 0)$ and $B(1, 1/\sqrt{3})$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min\{d(P, OB), d(P, AB)\}$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.*

[IIT 1997]

SOLUTION We have,

$$OB = AB$$

So, the triangle OAB is isosceles.

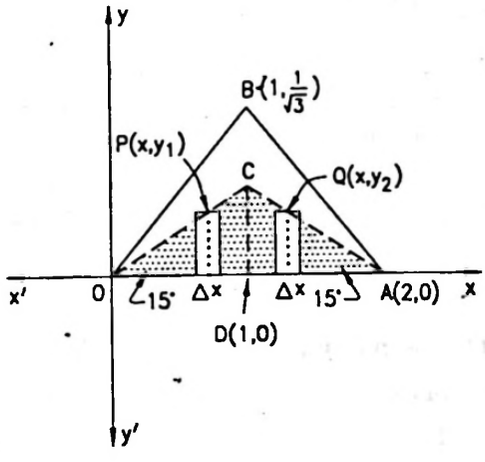


Fig. 12.62

Also,

Slope of $OB = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \angle AOB = \angle OAB = 30°$

Now,

$d(P, OA) \le \min\{d(P, OB), d(P, AB)\}$

$\Rightarrow d(P, OA) \le d(P, OB)$ and $d(P, OA) \le d(P, AB)$

$\Rightarrow$ P lies below the bisector of $\angle AOB$ and also that of $\angle OAB$.

$\Rightarrow$ P lies in the region OAC, where C is the point of intersection of bisectors of $\angle AOB$ and $\angle OAB$.

Equation of OC is

$$y = (\tan 15°)\, x$$

or,
$$y = \left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right) x \qquad \ldots(i)$$

Equation of AC is

$$y - 0 = (\tan 165°)(x - 2)$$

or, $$y = -\tan 15°\,(x - 2)$$

or, $$y = -\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)(x-2) \quad ...(ii)$$

Now,

required area

$$= \int_0^1 y_1\,dx + \int_1^2 y_2\,dx$$

$$= \int_0^1 \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)x\,dx + \int_1^2 -\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)(x-2)\,dx \quad \left[\because P \text{ and } Q \text{ lie on (i) and (ii) respectively}\right]$$

$$= \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left[\frac{x^2}{2}\right]_0^1 - \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left[\frac{(x-2)^2}{2}\right]_1^2$$

$$= \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\times\frac{1}{2} - \frac{\sqrt{3}-1}{\sqrt{3}+1}\left[0-\frac{1}{2}\right]$$

$$= \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \text{sq. units.}$$

EXAMPLE 61 *A curve $y = f(x)$ passes through the point $P(1, 1)$, the normal to the curve at P is $a(y-1)+(x-1)=0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also, obtain the area bounded by the y-axis, the curve and the normal to the curve at P.*

[IIT 1996]

SOLUTION We have,

Slope of the normal at $P(x, y)$

$$= \text{Slope of the line } a(y-1)+(x-1)=0$$

$$= -\frac{1}{a}$$

$\therefore$ Slope the tangent at $P = a$.

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = a \quad ...(i)$$

It is given that the slope of the tangent at any point on the curve $y = f(x)$ is proportional to the ordinate of the point. Therefore,

$$\frac{dy}{dx} \propto y$$

$$\Rightarrow \frac{dy}{dx} = \lambda y \quad ...(ii)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{P(1,1)} = \lambda$$

$$\Rightarrow a = \lambda \quad \text{[Using (i)]}$$

Putting $\lambda = a$ in (ii), we get

$$\frac{dy}{dx} = ay$$

$$\Rightarrow \frac{dy}{y} = a\,dx$$

$$\Rightarrow \log y = ax + \log C$$

$$\Rightarrow y = Ce^{ax}$$

This passes through $P(1, 1)$. Therefore, $C = e^{-a}$.

Thus, the equation of the curve is

$$y = e^{-a}e^{ax} \text{ or, } y = e^{a(x-1)} \quad ...(iii)$$

The graph of $y = e^{a(x-1)}$ is shown in Fig. 12.63. It cuts y-axis at $Q(0, e^{-a})$. The normal to (iii) at $P(1, 1)$ meets y-axis at $R\left(0, 1+\frac{1}{a}\right)$.

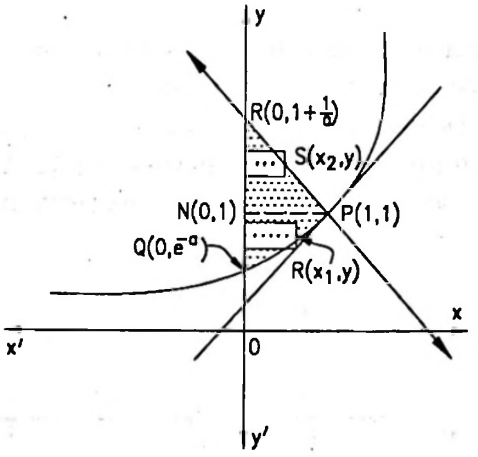


Fig. 12.63

Let PN be perpendicular from P on y-axis.

Now,

Required area

= Area of the region PQN + Area of the region PNR

$$= \int_{e^{-a}}^{1} x_1\,dy + \int_1^{1+1/a} x_2\,dy$$

$$= \frac{1}{a}\int_{e^{-a}}^{1} (\log y + a)\,dy + \int_1^{1+1/a} \{(1+a) - ay\}\,dy$$

$$= \frac{1}{a}\left[y(\log y - 1) + ay\right]_{e^{-a}}^{1} + \left[(1+a)y - \frac{a}{2}y^2\right]_a^{1+1/a}$$

$$= \frac{1}{a}\left[(-1+a) - e^{-a}(-a-1) - ae^{-a}\right]$$

$$+\left[(1+a)\left(1+\frac{1}{a}-1\right) - \frac{a}{2}\left\{\left(1+\frac{1}{a}\right)^2 - 1^2\right\}\right]$$

$$= \frac{1}{a}\left[-1+a+ae^{-a}+e^{-a}-ae^{-a}\right] + \left[\frac{1+a}{a} - \frac{a}{2}\left(2+\frac{1}{a}\right)\frac{1}{a}\right]$$

$$= \frac{1}{a}(-1+a+e^{-a}) + \left(\frac{1}{a}+1-1-\frac{1}{2a}\right)$$

$$= \frac{-1}{a} + 1 + \frac{e^{-a}}{a} + \frac{1}{2a}$$

$$= \left(1 + \frac{e^{-a}}{a} - \frac{1}{2a}\right) \text{ sq. units}$$

EXAMPLE 62 *Find the area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of the two minima of the curve.*

SOLUTION The given curve is symmetric about y-axis as its equation contains all even powers of x. It cuts x-axis at $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ and $(0, 0)$. By equating the lowest degree term to zero, we obtain the equation of the tangent at the origin as $y = 0$ i.e. x-axis.

Now, $y = 2x^4 - x^2$

$$\Rightarrow \frac{dy}{dx} = 8x^3 - 2x \text{ and } \frac{d^2y}{dx^2} = 24x^2 - 2$$

For points of local maximum or minimum, we must have

$$\frac{dy}{dx} = 0 \Rightarrow x = 0, \pm\frac{1}{2}$$

Clearly, $\left(\frac{d^2y}{dx^2}\right)_{x = \pm 1/2} = 6 - 2 > 0$

So, $x = \pm\frac{1}{2}$ are the points of local minima of the curve $y = 2x^4 - x^2$.

The graph of $y = 2x^4 - x^2$ is shown in Fig. 12.64. The shaded region is the region enclosed by this curve and the ordinates at the points of minima of the curve.

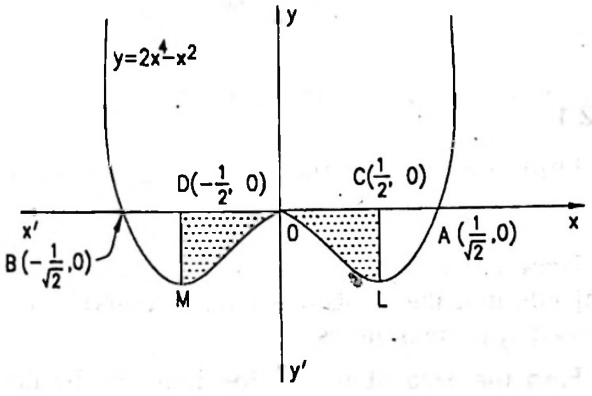


Fig. 12.64

As the curve is symmetrical about y-axis. So,

Required area

$$= 2\left|\int_0^{1/2} y\,dx\right|$$

$$= 2\left|\int_0^{1/2} (2x^4 - x^2)\,dx\right|$$

$$= 2\left|\left[\frac{2}{5}x^5 - \frac{x^3}{3}\right]_0^{1/2}\right|$$

$$= 2\left|\left(\frac{2}{5}\times\frac{1}{32} - \frac{1}{3}\times\frac{1}{8}\right)\right|$$

$$= 2\left|\frac{1}{80} - \frac{1}{24}\right| = \frac{7}{120} \text{ sq. units.}$$

EXAMPLE 63 *Find the area of the region bounded by the curves $y = \frac{x^2}{4a}$ and $y = \frac{8a^3}{x^2 + 4a^2}$.*

SOLUTION The equation $y = \frac{8a^3}{x^2 + 4a^2}$ represents a curve which is symmetrical about y-axis and it cuts y-axis at $(2a, 0)$. We observe that $y \to 0$ as $x \to \infty$. So, x-axis is an asymptote to the curve. The equation $y = \frac{x^2}{4a}$ represents a parabola. At the point of intersection of these two curves, we have

$$\frac{x^2}{4a} = \frac{8a^3}{x^2 + 4a^2}$$

$$\Rightarrow x^4 + 4a^2x^2 - 32a^4 = 0$$

$$\Rightarrow (x^2 - 4a^2)(x^2 + 8a^2) = 0$$

$$\Rightarrow x^2 - 4a^2 = 0$$

$$\Rightarrow x = \pm 2a.$$

Thus, the two curves intersect at $(2a, 0)$ and $(-2a, 0)$. The region bounded by the two curves is shaded in Fig. 12.65. Since both the curves are symmetrical about y-axis.

So, required area

$= 2$ (Area of the region $OACNO$)

$$= 2\left[\int_0^{2a} (y_1 - y_2)\,dx\right]$$

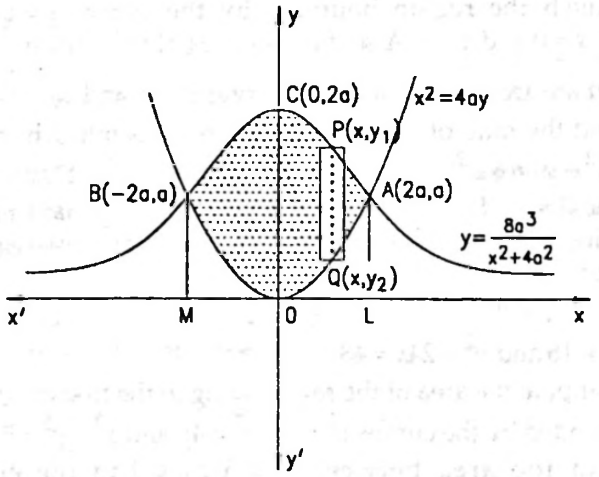


Fig. 12.65

$$= 2\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2}-\frac{x^2}{4a}\right)dx$$

$$= 16a^3\int_0^{2a}\frac{1}{x^2+4a^2}\,dx-\frac{1}{2a}\int_0^{2a}x^2\,dx$$

$$= 16a^3\times\frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a}-\frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= 8a^2\,(\tan^{-1}1-\tan^{-1}0)-\frac{1}{6a}(8a^3-0)$$

$$= 8a^2\times\frac{\pi}{4}-\frac{4}{3}a^2$$

$$= \left(2\pi a^2-\frac{4}{3}a^2\right)$$

$$= \left(2\pi-\frac{4}{3}\right)a^2 \text{ sq. units.}$$

EXAMPLE 64 *Find the area of the region bounded by the curves*

$$y = x^2,\ y = |\,2-x^2\,| \text{ and } y = 2,$$

which lies to the right of the line $x = 1$.

[IIT 2002]

SOLUTION The region bounded by the given curves on the right side of $x = 1$ is shaded in Fig. 12.66. Let us slice the shaded region into vertical strips.

Required area

= Area of the region $APQA$ + Area of the region $AQRA$

$$= \int_1^{\sqrt{2}}(y_2-y_1)\,dx+\int_{\sqrt{2}}^{2}(2-y)\,dx$$

$$= \int_1^{\sqrt{2}}\{x^2-(2-x^2)\}\,dx+\int_{\sqrt{2}}^{2}(4-x^2)\,dx$$

$$= \int_1^{\sqrt{2}}(2x^2-2)\,dx+\int_{\sqrt{2}}^{2}(4-x^2)\,dx$$

$$= \left[\frac{2}{3}x^3-2x\right]_1^{\sqrt{2}}+\left[4x-\frac{x^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= \left(\frac{4\sqrt{2}}{3}-2\sqrt{2}\right)-\left(\frac{2}{3}-2\right)+\left(8-\frac{8}{3}\right)-\left(4\sqrt{2}-\frac{2\sqrt{2}}{3}\right)$$

$$= \left(-4\sqrt{2}+\frac{20}{3}\right)$$

$$= \left(\frac{20-12\sqrt{2}}{3}\right)\text{sq. units}$$

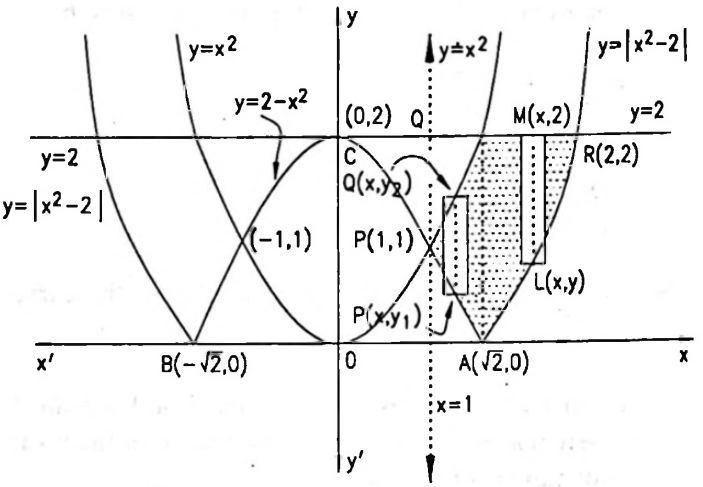


Fig. 12.66

EXERCISE 12.1

1. Find the area bounded by the curves $y = e^x$, $y = |\,x-1\,|$ and $x = 2$.
2. Sketch the region bounded by the curves $y = x^2+2$, $y = -x$, $x = 0$ and $x = 1$. Also, find the area of this region.
3. Find the area bounded by the curves $y^2 = x$ and $2y^2 = 3-x$.
4. Find the ratio of the areas of two regions into which the circle $x^2+y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$.
5. The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Sketch one of these regions and find its area.
6. Compute the area of the region bounded by the parabolas $y^2+8x = 16$ and $y^2-24x = 48$.
7. Compute the area of the region lying in the first quadrant and bounded by the curves $y^2 = 4x$, $x^2 = 4y$ and $x^2+y^2 = 5$.
8. Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.
9. Find the area of the closed figure bounded by $y = 2\cos^2 x\,(1+\sin^2 x)$ on the interval $[0, 2\pi]$ and the x-axis.
10. Draw a sketch of the curves $y = \sin x$ and $y = \cos 3x$ in $[0, \pi]$ indicating the points where they intersect. Find the area enclosed by the two curves.
11. Find the area of the region bounded by the curve $C: y = \tan x$, the tangent drawn to C at $x = \pi/4$ and the x-axis.
12. Find the area bounded by the curves $x = y^2$ and $x = 3-2y^2$.
13. Let $f(x)$ be a non-negative differentiable function satisfying the differential equation $\frac{dy}{dx} = x$ for $x \le \sqrt{2}$ and the differential equation $y\frac{dy}{dx} = x$ for $x > \sqrt{2}$. Find the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = 0$ and $x = 2$.

14. Find the area bounded by the curve $y = x^2 - 2x + 5$, the tangent to it at the point (2, 5) and the axes of coordinates.

15. Show that the area cut off a parabola by any double ordinate is two-thirds of the corresponding rectangle contained by that double ordinate and its distance from the vertex.

16. Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

17. Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

18. Find the area of the region bounded by the curves $x^2 = y$, $y = x + 2$ and x-axis.

19. Find the area of the region enclosed by the parabolas $y = 5x^2$ and $y = 2x^2 + 9$.

20. Find the area of the region bounded by $y = x - 1$ and $(y - 1^2 = 4(x + 1)$.

21. Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $x + y = 0$.

22. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y - 1 = 0$.

23. Find the area of the region lying in the first quadrant and included between the curves $x^2 + y^2 = 3a^2$, $x^2 = 2ay$ and $y^2 = 2ax$, $a > 0$.

24. Find the area of the region

$$R = \{(x, y) \mid x^2 + y^2 \le 6y, y^2 \le 8x, x + y \le 6\}$$

25. Find the area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $2x \le 5$.

26. Find the area enclosed by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the straight line $y = 3x - 15$.

27. Find the area enclosed by the curves $3x^2 + 5y = 32$ and $y = |x - 2|$.

28. Find the area enclosed by the curves $x^2 + y^2 = 4$, $x^2 + \sqrt{2}y = 0$ and $y = x$.

29. Find the area enclosed by the parabolas $x = -2y^2$ and $x = 1 - 3y^2$.

30. Find the area lying above x-axis and enclosed by the curve $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$.

31. Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$.

32. Find the area bounded by the cures $x^2 + y^2 = 4$, $x^2 + \sqrt{2}y = 0$ and $x = y$.

33. Find the area enclosed between the curves $y = \log_e(x + e)$ and $x = \log_e(1/y)$ and the x-axis.

34. Find the area of the closed figure bounded by $y = 2\cos^2 x(1 + \sin^2 x)$ on the interval $[0, 2\pi]$ and the x-axis.

35. For what value of m, is the area bounded by $y = x^2 - 3$ and $y = mx + 2$ is least ? Calculate this area.

36. A normal to the curve $x^2 + ax - y + 2 = 0$ at the point whose abscissa 1, is parallel to the line, $y = x$. Find the area in the first quadrant bounded by the curve, this normal and the axis of x.

37. Let $S(\lambda)$ be the area included between the parabola $y = x^2 + 2x - 3$ and the line $y = \lambda x + 1$. Find $S(-1)$ and find the least value of $S(\lambda)$.

38. Show that the area common to the ellipses $a^2x^2 + b^2y^2 = 1$, $b^2x^2 + a^2y^2 = 1$, where $0 < a < b$ is $\frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$.

39. Find the area bounded by $y = \frac{3}{|x|}$ and $y = 2 - |2 - x|$.

40. Find the area of the figure bounded by the parabola $y = -x^2 - 2x + 3$, the line tangnet to it at the point $M(2, -5)$ and the y-axis.

41. A curve is given by $x = 1 - 3t^2$ and $y = t - 3t^3$, shade the region bounded by this curve and $y = \sqrt{2}x^2$. Also, find the area of the bounded region.

42. A function $f(x)$ is defined by $y = f(x)$, where $x = 2t - |t|$ and $y = t^2 + t|t|$. Draw the graph of $f(x)$ in the interval $[-1, 1]$ and find the area bounded by $y = f(x)$, $x = 0$ and $y = [x + 1]$, where $[\cdot]$ denotes the greatest integer function.

43. Find the ratio of the areas in which the curve $y = \left[\frac{x^3}{100} + \frac{x}{35}\right]$ divides the circle $x^2 + y^2 - 4x + 2y + 1 = 0$. Here, $[\cdot]$ denotes the greatest integer function.

44. Find the area bounded by the curves in each of the following

(i) $y = \sqrt{25 - x^2}$, $4y = |4 - x^2|$ and $x = 0$

(ii) $4y = |4 - x^2|$, $y = 7 - |x|$

(iii) $y = |x - 1|$, $y = \sqrt{5 - x^2}$

(iv) $y = \log_e x$ and $y = (\log_e x)^2$

45. Find the area of the region bounded by the two branches of the curve $(y - x)^2 = x^3$ and the line $x = 1$.

46. Find the area enclosed by $x^2 + y^2 = 4$, $y = x^2 + x + 1$, $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$ and x-axis. Here $[\cdot]$ denotes the greatest integer function.

47. Let $f(x) = \max\{\sin x, \cos, x\ 1/2\}$. Find the area of the region bounded by the curves $y = f(x)$, x-axis, y-axis and $x = 2\pi$.

48. The curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area enclosed by the curve, the axis of x and the line $x = 4$ is 8 square units. Determine a, b.

49. Sketch the region bounded by the curves $y = x^2$ and $y = \frac{2}{1 + x^2}$ and find its area.

50. Find the area of the region lying inside the circle $x^2 + (y - 1)^2 = 1$ and outside the epllipse $C^2x^2 + y^2 = C^2$, where $C = (\sqrt{2} - 1)$.

51. Find the area of the region bounded by the x-axis and

$$y = \begin{cases} \tan x, -\pi/3 \le x \le \pi/3 \\ \cot x, \pi/6 \le x \le 3\pi/2 \end{cases}$$

52. Find the area of the curvelinear triangle formed by $y = \tan x, y = \frac{2}{3}\cos x$ and $x = 0$.

53. Find the area bounded by the curves $x = |y^2 - 1|$ and $y = x - 5$.

54. Find the area of the region bounded by the curves $y = \log_e x$ and $y = \sin^4 \pi x$ and y-axis.

ANSWERS

1. $(e^2 - 2)$ sq. units
2. $\frac{17}{6}$ sq. units
3. 4 sq. units
4. $(4\pi + \sqrt{3}) : (8\pi - \sqrt{3})$
5. $2\sqrt{2}$ sq. units
6. $32\sqrt{\frac{2}{3}}$ sq. units
7. $\frac{2}{3} + \frac{5}{2}\sin^{-1}\frac{3}{5}$ sq. units
8. $2\sqrt{2}$ sq.units
9. $5\pi/2$ sq. units.
10. Points : $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{3\pi}{4}$

    Area $= \frac{2}{3\sqrt{2}}\left[4\sqrt{1 - \frac{1}{\sqrt{2}}} + 2\sqrt{1 + \frac{1}{\sqrt{2}}} - 1\right]$ sq. units
11. $\frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$ sq. units
12. 4 sq. units
13. $\left[\sqrt{3} - \frac{1}{3\sqrt{2}} - \frac{1}{2}\log\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)\right]$ sq. units
14. 8/3 sq. units
16. $\frac{16}{3}ab$ sq. units
17. 15 : 49
18. 5/6 sq. units
19. $12\sqrt{3}$ sq. units
20. 64/3 sq. units
21. 9/2 sq. units
22. 16/3 sq. units
23. $a^2\left[\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right]$ sq.units
24. $\frac{1}{12}(27\pi - 2)$ sq. units
25. $\left(\frac{\pi}{6} + \frac{1}{8} - \frac{\sqrt{3}}{8}\right)$ sq.units.
26. $\frac{73}{6}$ sq. units
27. $\frac{33}{2}$ sq. units
28. $\frac{3\pi + 11}{3}$ sq. units
29. 4/3 sq. units
30. $\left(4 + 25\sin^{-1}\frac{4}{5}\right)$ sq. units
31. $\frac{11}{4}$ sq. units
32. $\left(\pi + \frac{1}{3}\right)$ sq. units
33. 2 sq. units
34. $5\pi/2$ sq. units.
35. $m = 0, \frac{20\sqrt{5}}{3}$ sq. units
36. $\frac{7}{6}$ sq.units
37. $\frac{125}{6}, \frac{32}{3}$ sq. units
39. $\left(2 - \frac{3}{2}\log_e 3\right)$ sq. units
40. 8/3 sq. units
43. $\frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$
44. (i) $\left[2 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)\right]$ sq. units (ii) 32 sq. units

    (iii) $\frac{5\pi - 2}{4}$ sq. units (iv) $(3 - e)$ sq. units
45. 4/5 sq. units
46. $\left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right)$ sq. units
47. $\left(\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}\right)$ sq. units
48. $a = 3, b = -1$
49. $\left(\pi - \frac{2}{3}\right)$ sq. units.
50. $\left[\pi\left(1 - \frac{1}{2\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\right]$ sq. units
51. $\log\left(\frac{3}{2}\right)$ sq. units
52. $\left[\frac{1}{3} + \log\left(\frac{\sqrt{3}}{2}\right)\right]$ sq. units
53. $\frac{109}{6}$ sq. units
54. $\frac{11}{8}$ sq. unists

HINTS TO SELECTED PROBLEMS

5. Required area $= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$

13. $f(x)$ is differentiable at $x = \sqrt{2}$

$\Rightarrow$ (LHD at $x = \sqrt{2}$) = (RHD at $x = \sqrt{2}$)

$\Rightarrow \lim_{x \Rightarrow \sqrt{2}^-} \frac{dy}{dx} = \lim_{x \to \sqrt{2}^+} \frac{dy}{dx}$

$\Rightarrow \lim_{x \to \sqrt{2}} x = \lim_{x \to \sqrt{2}} \frac{x}{f(x)}$

$\Rightarrow \sqrt{2} = \frac{\sqrt{2}}{f(\sqrt{2})}$

$\Rightarrow f(\sqrt{2}) = 1.$

31. Required area $= \left|\int_0^1 y\, dx\right| + \int_1^2 y\, dx + \left|\int_2^3 y\, dx\right|$

33. To draw the graph of $y = \log_e (x + e)$, translate $y = \log x$ along OX' through e-units. And $x = \log_e (1/y)$

$\Rightarrow y = e^{-x}$.

43. We have, $y = \left[\frac{x^3}{100} + \frac{x}{35}\right] = 0$ for $x \in [0, 4]$. So, required ratio is same as the ratio of the regions in which x-axis divides the circle.

46. for $-2 \le x \le 2$, we have $1 < \sin^2 \frac{x}{4} + \cos \frac{x}{4} < 2$

$\therefore\ y = \left[\sin^2 \frac{x}{4} + \cos \frac{x}{4}\right] = 1$

47. Required area

$$= \int_0^{\pi/4} \cos x\, dx + \int_{\pi/4}^{5\pi/6} \sin dx + \int_{5\pi/6}^{5\pi/3} \frac{1}{2}\, dx + \int_{5\pi/3}^{2\pi} \cos x\, dx$$

—❑❑❑—

CHAPTER

13

DIFFERENTIAL EQUATIONS

13.1 SOME DEFINITIONS

DIFFERENTIAL EQUATION *An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.*

For instance,

(i) $\frac{dy}{dx} = 2xy$ (ii) $\frac{d^2y}{dx^2} = 4x$

(iii) $\frac{dy}{dx} = \sin x + \cos x$ (iv) $\frac{dy}{dx} + 2xy = x^3$

(v) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$ (vi) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

(vii) $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(viii) $(x^2 + y^2)\,dx - 2xy\,dy = 0$ (ix) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(1 + \frac{dy}{dx}\right)^3 = 0$

are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION *The order of a differential equation is the order of the highest order derivative appearing in the equation.*

For example, in the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$, the order of highest order derivative is 2. So, it is a differential equation of order 2. The equation $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$ is of the order 3, because the order of highest order derivative in it is 3.

NOTE *The order of a differential equation is a positive integer.*

DEGREE OF A DIFFERENTIAL EQUATION *The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.*

In other words, the degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

ILLUSTRATION 1 Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0.$$

In this equation the power of highest order derivative is 1. So, it is a differential equation of degree 1.

ILLUSTRATION 2 Consider the differential equation

$$x\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0.$$

In this equation, the order of the highest order derivative is 3 and its power is 2. So, it is a differential equation of order 3 and degree 2.

ILLUSTRATION 3 The differential equation

$y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ when expressed as a polynomial in derivatives becomes

$$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - 1) = 0$$

In this equation, the power of highest order derivative is 2. So, its degree is 2.

ILLUSTRATION 4 Consider the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\left(\frac{d^2y}{dx^2}\right).$$

The order of highest order differential coefficient is 2. So, its order is 2. To find its degree we express the differential equation as a polynomial in derivatives. When expressed as a polynomial in derivatives it becomes

$$k^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

Clearly, the power of the highest order differential coefficient is 2. So, its degree is 2.

ILLUSTRATION 5 The differential equation $(x^2+y^2)\,dx - 2xy\,dy = 0$ may be written as

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

So, it is a differential equation of order 1 and degree 1.

ILLUSTRATION 6 Consider the differential equation

$$y = px + \sqrt{a^2p^2+b^2}, \text{ where } p = \frac{dy}{dx}.$$

The order of the highest order derivative is 1. So, its order is 1. To determine its degree we express it as a polynomial in differential coefficients as follows:

$$y = px + \sqrt{a^2p^2+b^2}$$

$$\Rightarrow (y-px)^2 = a^2p^2+b^2$$

$$\Rightarrow p^2(x^2-a^2) - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow (x^2-a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 - b^2 = 0.$$

Clearly, the power of highest order differential coefficient is 2. So, its degree is 2.

LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation is a linear differential equation if it is expressible in the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non-linear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation if

(i) *its degree is more than one.*

(ii) *any of the differential coefficient has exponent more than one*

(iii) *exponent of the dependent variable is more than one.*

(iv) *products containing dependent variable and its differential coefficients are present.*

ILLUSTRATION 7 The differential equation

$$\left(\frac{d^3y}{dx^3}\right)^3 - 6\left(\frac{d^2y}{dx^2}\right)^2 - 4y = 0,$$

is a non-linear differential equation, because its degree is 3, more than one.

ILLUSTRATION 8 The differential equation

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 9y = x,$$

is a non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

ILLUSTRATION 9 The differential equation $(x^2+y^2)\,dx - 2xy\,dy = 0$ is a non-linear differential equation, because the exponent of dependent variable y is 2 and it involves the product of y and $\frac{dy}{dx}$.

ILLUSTRATION 10 Consider the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x.$$

This is a linear differential equation of order 2 and degree 1.

13.2 SOLUTION OF A DIFFERENTIAL EQUATION

SOLUTION *The solution of a differential equation is a relation between the variables involved which satisfies the differential equation. Such a relation and the derivatives obtained therefrom when substituted in the differential equation, makes left hand, and right hand sides identically equal.*

For example, $y = e^x$ is a solution of the differential equation

$$\frac{dy}{dx} = y.$$

Consider the differential equation $\frac{d^2y}{dx^2} + y = 0.$ ...(i)

Also, $y = A\cos x + B\sin x$ ...(ii)

where A and B are arbitrary constants.

Differentiating (ii), w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin x + B\cos x.$$

Differentiating this w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0.$$

This shows that $y = A\cos x + B\sin x$ satisfies the differential equation (i) and hence it is a solution of (i).

It can be easily verified that $y = 3\cos x + 2\sin x$, $y = A\cos x$, $y = B\sin x$ etc., are also solutions of (i). We find that the solution $y = 3\cos x + 2\sin x$ does not contain any arbitrary constant whereas solutions $y = A\cos x$, $y = B\sin x$ contain only one arbitrary constant. The solution $y = A\cos x + B\sin x$ contains

two arbitrary constants, so it is known as the general solution of (i) whereas all other solutions are particular solutions.

GENERAL SOLUTION *The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.*

For example, $y = A\cos x + B\sin x$ is the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

But, $y = A\cos x$ is not the general solution as it contains one arbitrary constant.

PARTICULAR SOLUTION *Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.*

For example, $y = 3\cos x + 2\sin x$ is a particular solution of the differential equation (i).

ILLUSTRATION 1 *Show that* $y = a\cos(\log x) + b\sin(\log x)$ *is a solution of the differential equation* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

SOLUTION We have,

$$y = a\cos(\log x) + b\sin(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating both sides w.r.t. x, we obtain

$$x\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx} = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$$

which is same as the given differential equation.

Hence, $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the given differential equation.

ILLUSTRATION 2 *Show that* $xy = ae^x + be^{-x} + x^2$ *is a solution of the differential equation*

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$$

SOLUTION We are given that

$$xy = ae^x + be^{-x} + x^2 \qquad \ldots (i)$$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2 \qquad \ldots (ii)$$

Now, $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$

$$= [ae^x + be^{-x} + 2] - [ae^x + be^{-x} + x^2] + x^2 - 2$$

$$= 0 \qquad \text{[Using (i) and (ii)]}$$

Thus, $xy = ae^x + be^{-x} + x^2$ is a solution of the given differential equation.

ILLUSTRATION 3 *Show that* $y = [x + \sqrt{x^2+1}]^m$ *satisfies the differential equation* $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$

SOLUTION We have,

$$y = [x + \sqrt{x^2+1}]^m$$

$$\Rightarrow \quad \frac{dy}{dx} = m(x + \sqrt{x^2+1})^{m-1}\frac{d}{dx}(x + \sqrt{x^2+1})$$

$$\Rightarrow \quad \frac{dy}{dx} = m(x + \sqrt{x^2+1})^{m-1}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = m\frac{(x + \sqrt{x^2+1})^m}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{my}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad (x^2+1)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

Differentiating both sides w.r.t. x, we get

$$(x^2+1)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$$

$$\Rightarrow \quad (x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$$

13.3 FORMATION OF DIFFERENTIAL EQUATIONS

Consider a family of exponential curves $y = Ae^x$, where A is an arbitrary constant. For different values of A, we get different members of the family. Differentiating the relation $y = Ae^x$ w.r.t. x, we get

$$\frac{dy}{dx} = Ae^x.$$

Eliminating the arbitrary constant between $y = Ae^x$ and $\frac{dy}{dx} = Ae^x$, we get $\frac{dy}{dx} = y$. This is the differential equation of the family of curves represented by $y = Ae^x$.

Thus, by eliminating one arbitrary constant, a differential equation of first order is obtained.

Now consider the family of curves given by

$$y = A\cos 2x + B\sin 2x \qquad ...(i)$$

where A and B are arbitrary constants.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x \qquad ...(iii)$$

Eliminating A and B from equations (i), (ii) and (iii), we get

$$\frac{d^2y}{dx^2} = -4y \Rightarrow \frac{d^2y}{dx^2} + 4y = 0.$$

Here we note that by eliminating two arbitrary constants, a differential equation of second order is obtained.

Similarly one can see that by eliminating three arbitrary constants a differential equation of third order is obtained.

Thus, from the examples cited above it can be concluded that if an equation involves n arbitrary constants, a differential equation of nth order can be obtained by eliminating these n arbitrary constants.

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

In order to formulate a differential equation, we may use the following algorithm.

ALGORITHM

STEP I *Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constants.*

STEP II *Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.*

STEP III *Differentiate the relation in step I n times with respect to x.*

STEP IV *Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step III and an equation in Step I.*

The equation so obtained is the desired differential equation.

The following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the differential equation of all circles touching the*
(i) *x-axis at the origin* (ii) *y-axis at the origin*

SOLUTION (i) The equation of the family of circles touching x-axis at the origin is

$$(x-0)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ay = 0 \qquad ...(i)$$

where a is an arbitrary constant.

This equation contains only one arbitrary constant, we differentiate it once w.r.t. x, so that

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow a = \frac{x + y\,(dy/dx)}{dy/dx} \qquad ...(ii)$$

Putting the value of a from (ii) in (i), we get

$$x^2 + y^2 = 2y\left\{\frac{x + y\,(dy/dx)}{dy/dx}\right\}$$

$$\Rightarrow (x^2 - y^2)\frac{dy}{dx} = 2xy$$

This is the required differential equation.

(ii) The equation of the family of circles touching y-axis at the origin is

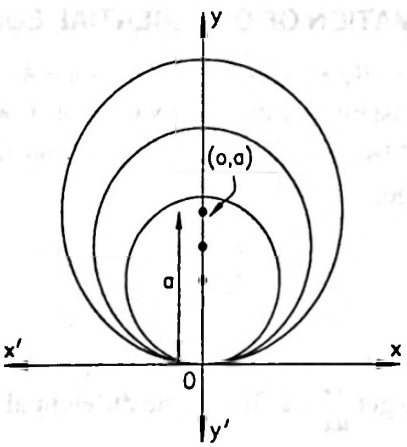


Fig. 13.1

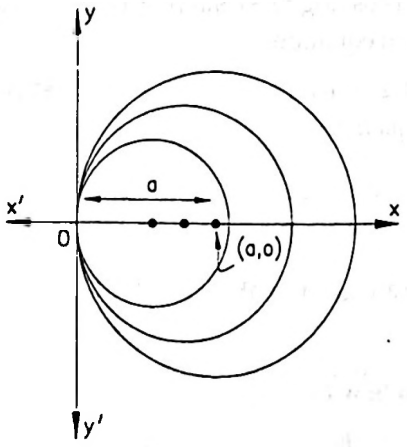


Fig. 13.2

$$(x-a)^2+(y-0)^2 = a^2$$

$$\Rightarrow x^2+y^2-2ax = 0 \qquad ...(i)$$

where a is an arbitrary constant.

This equation contains only one arbitrary constant, we differentiate it only once w.r.t. x, so that

$$2x+2y\frac{dy}{dx}-2a = 0$$

$$\Rightarrow a = x+y\frac{dy}{dx} \qquad ...(ii)$$

Putting the value of a from (ii) in (i), we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow y^2-x^2 = 2xy\frac{dy}{dx}.$$

This is the required differential equation.

EXAMPLE 2 *Obtain the differential equation of all circles of radius r.*

SOLUTION The equation of the family of circles of radius r is

$$(x-a)^2+(y-b)^2 = r^2 \qquad ...(i)$$

where a and b are arbitrary constants.

Since equation (i) contains two arbitrary constants, we differentiate it two times w.r.t. x and the differential equation will be of second order.

Differentiating (i) w.r.t. x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow (x-a)+(y-b)\frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \qquad ...(iii)$$

From (iii), we have

$$y-b = -\frac{1+(dy/dx)^2}{d^2y/dx^2} \qquad ...(iv)$$

Putting the value of $(y-b)$ in (ii), we obtain

$$x-a = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{d^2y/dx^2} \qquad ...(v)$$

Substituting the values of $(x-a)$ and $(y-b)$ in (i), we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2}{(d^2y/dx^2)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{(d^2y/dx^2)^2} = r^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2.$$

This is the required differential equation.

EXAMPLE 3 *Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.*

SOLUTION The equation of circles in the first quadrant which touch the coordinate axes is

$$(x-a)^2+(y-a)^2 = a^2 \qquad ...(i)$$

where a is an arbitrary constant. This equation contains one arbitrary constant, so we shall differentiate it once only and we shall get a differential equation of first order.

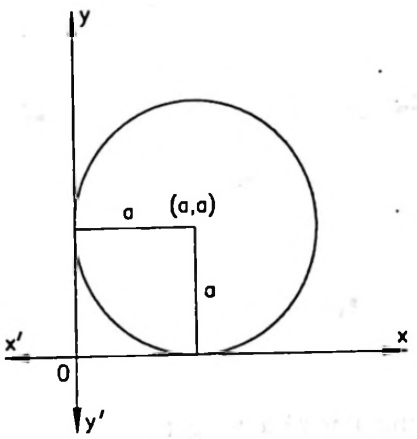


Fig. 13.3

Differentiating (i) w.r.t. x, we get

$$2(x-a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow x-a+(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow a = \frac{x+y\frac{dy}{dx}}{1+\frac{dy}{dx}}$$

$$\Rightarrow a = \frac{x+py}{1+p}, \text{ where } p = \frac{dy}{dx}.$$

Substituting the value of a in (i), we get

$$\left(x-\frac{x+py}{1+p}\right)^2+\left(y-\frac{x+py}{1+p}\right)^2 = \left(\frac{x+py}{1+p}\right)^2$$

$$\Rightarrow (xp-py)^2+(y-x)^2 = (x+py)^2$$

$$\Rightarrow (x-y)^2p^2+(x-y)^2 = (x+py)^2$$

$$\Rightarrow (x-y)^2(p^2+1) = (x+py)^2$$

$$\Rightarrow (x-y)^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(x+y\frac{dy}{dx}\right)^2.$$

This is the required differential equation.

EXAMPLE 4 *Show that the differential equation that represents all parabolas having their axis of symmetry coincident with the axis of x is* $yy_2 + y_1^2 = 0$.

SOLUTION The equation that represents a family of parabolas having their axis of symmetry coincident with the axis of x is

$$y^2 = 4a(x - h) \qquad ...(i)$$

This equation contains two arbitrary constants, so we shall differentiate it twice to obtain a second order differential equation.

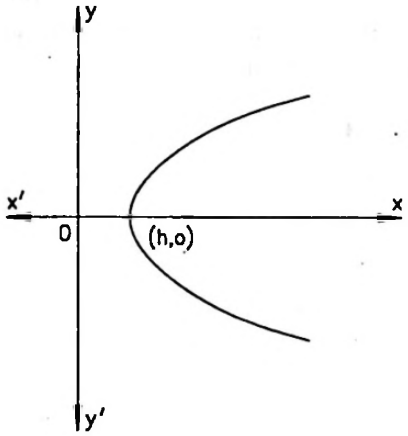


Fig. 13.4

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow y\frac{dy}{dx} = 2a \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad yy_2 + y_1^2 = 0,$$

which is the required differential equation.

EXAMPLE 5 *Find the differential equation of all non-horizontal lines in a plane.*

SOLUTION The equation of the family of all non-horizontal line in a plane is given by

$$ax + by = 1 \qquad ...(i)$$

where a, b are arbitrary constants such that $a \neq 0$.

Differentiating (i) w.r.t y, we get

$$a\frac{dx}{dy} + b = 0$$

Differentiating this w.r.t. y, we get

$$a\frac{d^2x}{dy^2} = 0$$

$$\Rightarrow \frac{d^2x}{dy^2} = 0 \qquad [\because a \neq 0]$$

Hence, the differential equation of all non-horizontal lines in a plane is $\frac{d^2x}{dy^2} = 0$.

EXAMPLE 6 *Find the differential equation of all non-vertical lines in a plane.*

SOLUTION The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

Now,

$$ax + by = 1$$

$$\Rightarrow \quad a + b\frac{dy}{dx} = 0 \qquad \text{[Differentiating w.r.t. x]}$$

$$\Rightarrow \quad b\frac{d^2y}{dx} = 0 \qquad \text{[Differentiating w.r.t. x]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 0 \qquad [\because b \neq 0]$$

Hence, the differential equation is $\frac{d^2y}{dx^2} = 0$

EXAMPLE 7 *Find the differential equation of all conics whose centre lies at the origin.*

SOLUTION The equation

$$ax^2 + 2hxy + by^2 = 1 \qquad ...(i)$$

represents the family of all conics whose centre lies at the origin for different values of a, h, b.

Differentiating (i) w.r.t. x, we get

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \quad ax + h\left(x\frac{dy}{dx} + y\right) + by\frac{dy}{dx} = 0$$

$$\Rightarrow \quad ax^2 + h\left(x^2\frac{dy}{dx} + yx\right) + byx\frac{dy}{dx} = 0 \qquad ...(ii)$$

[Multiplying both sides by x]

Subtracting (ii) from (i), we get

$$h\left(xy - x^2\frac{dy}{dx}\right) + b\left(y^2 - xy\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \quad (hx + by)\,y - x\frac{dy}{dx}(hx + by) = 1$$

$$\Rightarrow \quad (hx + by)\left(y - x\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \quad hx + by = \frac{1}{y - x\frac{dy}{dx}} \qquad ...(iii)$$

Differentiating w.r.t. x, we get

$$h+b\frac{dy}{dx}=\frac{-\left(\frac{dy}{dx}-x\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)}{\left(y-x\frac{dy}{dx}\right)^2}$$

$$\Rightarrow\ h+b\frac{dy}{dx}=\frac{x\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^2}$$

$$\Rightarrow\ hx+bx\frac{dy}{dx}=\frac{x^2\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^2} \qquad ...(iv)$$

Subtracting (iv) from (iii), we get

$$by-bx\frac{dy}{dx}=\frac{1}{y-x\frac{dy}{dx}}-\frac{x^2\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^2}$$

$$\Rightarrow\ b\left(y-x\frac{dy}{dx}\right)=\frac{y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^2}$$

$$\Rightarrow\ b=\frac{y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^3}$$

Differentiating w.r.t. x, we get

$$0=\frac{\frac{dy}{dx}-\frac{dy}{dx}-3x\frac{d^2y}{dx^2}-x^2\frac{d^3y}{dx^3}}{\left(y-x\frac{dy}{dx}\right)^3}+\frac{3\left(y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}\right)x\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^4}$$

$$\Rightarrow\ 0=\frac{-\left(3x\frac{d^2y}{dx^2}+x^2\frac{d^3y}{dx^3}\right)}{\left(y-x\frac{dy}{dx}\right)^3}+\frac{3x\left(y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}\right)\frac{d^2y}{dx^2}}{\left(y-x\frac{dy}{dx}\right)^4}$$

$$\Rightarrow\ \left(3x\frac{d^2y}{dx^2}+x^2\frac{d^3y}{dx^3}\right)\left(y-x\frac{dy}{dx}\right)=3x\frac{d^2y}{dx^2}\left(y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}\right)$$

which is the required differential equation.

EXAMPLE 8 *Find the differential equation of all conics whose axes coincide with the axes of coordinates.*

SOLUTION The equation

$$ax^2+by^2=1, a, b\neq 0 \qquad ...(i)$$

represents the family of all conics whose axes coincide with the axes of coordinates for different values of a and b.

Differentiating (i) w.r.t. x, we get

$$2ax+2by\frac{dy}{dx}=0$$

$$\Rightarrow\ ax+by\frac{dy}{dx}=0 \qquad ...(i)$$

$$\Rightarrow\ a+b\left(\frac{dy}{dx}\right)^2+by\frac{d^2y}{dx^2}=0 \qquad ...(iii)$$

Multiplying (iii) by x and subtracting from (i), we get

$$b\left[y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}\right]=0$$

$$\Rightarrow\ y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}=0$$

$$\Rightarrow\ y\frac{dy}{dx}=x\left\{\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right\}$$

which is the required differential equation.

EXAMPLE 9 *Find the differential equation of all straight lines which are at a constant distance p from the origin.*

SOLUTION The equation

$$x\cos\alpha+y\sin\alpha=p \qquad ...(i)$$

represents a family of straight lines which are at a constant distance p from the origin for different values of α.

Differentiating (i) w.r.t. x, we get

$$\cos\alpha+\frac{dy}{dx}\sin\alpha=0$$

$$\Rightarrow\ \tan\alpha=-\frac{1}{\frac{dy}{dx}}$$

$$\Rightarrow\ \sin\alpha=\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\ \text{and,}\ \cos\alpha=\frac{-dy/dx}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Substituting the values of $\sin\alpha$ and $\cos\alpha$ in (i), we get

$$\frac{-x\frac{dy}{dx}+y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=p$$

$$\Rightarrow\ \left(y-x\frac{dy}{dx}\right)^2=p^2\left(1+\left(\frac{dy}{dx}\right)^2\right),$$

which is the required differential equation.

EXAMPLE 10 *Find the differential equation corresponding to* $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$, *where* C_1, C_2, C_3 *are arbitrary constants and* m_1, m_2, m_3 *are roots of the equation* $m^3 - 7m + 6 = 0$.

SOLUTION Since m_1, m_2, m_3 are the roots of the equation

$$m^3 - 7m + 6 = 0$$

$$\therefore \quad m_1 m_2 + m_2 m_3 + m_3 m_1 = -7 \qquad ...(i)$$

$$m_1 + m_2 + m_3 = 0 \qquad ...(ii)$$

and, $m_1 m_2 m_3 = -6$ ...(iii)

Now,

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} \qquad ...(iv)$$

$$\Rightarrow \frac{dy}{dx} = C_1 m_1 e^{m_1 x} + C_2 m_2 e^{m_2 x} + C_3 m_3 e^{m_3 x}$$

$$\Rightarrow \frac{dy}{dx} = m_1 (y - C_2 e^{m_2 x} - C_3 e^{m_3 x}) + C_2 m_2 e^{m_2 x} + C_3 m_3 e^{m_3 x}$$

[Using (iv)]

$$\Rightarrow \frac{dy}{dx} = m_1 y + C_2 (m_2 - m_1) e^{m_2 x} + C_3 (m_3 - m_1) e^{m_3 x} \qquad ...(v)$$

$$\Rightarrow \frac{d^2y}{dx^2} = m_1 \frac{dy}{dx} + C_2 m_2 (m_2 - m_1) e^{m_2 x} + C_3 m_3 (m_3 - m_1) e^{m_3 x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m_1 \frac{dy}{dx} + m_2 \left\{ \frac{dy}{dx} - m_1 y - C_3 (m_3 - m_1) e^{m_3 x} \right\}$$

$$+ C_3 m_3 (m_3 - m_1) e^{m_3 x} \qquad \text{[Using (v)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (m_1 + m_2) \frac{dy}{dx} - m_1 m_2 y$$

$$+ C_3 (m_3 - m_1)(m_3 - m_2) e^{m_3 x} \qquad ...(vi)$$

$$\Rightarrow \frac{d^3y}{dx^3} = (m_1 + m_2) \frac{d^2y}{dx^2} - m_1 m_2 \frac{dy}{dx}$$

$$+ C_3 (m_3 - m_1)(m_3 - m_2) m_3 e^{m_3 x}$$

$$\Rightarrow \frac{d^3y}{dx^3} = (m_1 + m_2) \frac{d^2y}{dx^2} - m_1 m_2 \frac{dy}{dx}$$

$$+ m_3 \left| \frac{d^2y}{dx^2} - (m_1 + m_2) \frac{dy}{dx} + m_1 m_2 y \right| \qquad \text{[Using (vi)]}$$

$$\Rightarrow \frac{d^3y}{dx^3} = (m_1 + m_2 + m_3) \frac{d^2y}{dx^2} - (m_1 m_2 + m_2 m_3 + m_3 m_1) \frac{dy}{dx}$$

$$+ m_1 m_2 m_3 y$$

$$\Rightarrow \frac{d^3y}{dx^3} = 0 \frac{d^2y}{dx^2} + 7 \frac{dy}{dx} - 6y \qquad \text{[Using (i), (ii) and (iii)]}$$

$$\Rightarrow \frac{d^3y}{dx^3} - 7 \frac{dy}{dx} + 6y = 0$$

EXAMPLE 11 *Find the differential equation of all parabolas whose axis of symmetry is parallel to x-axis.*

SOLUTION The general equation of all parabolas whose axis of symmetry is parallel to x-axis is

$$x = ay^2 + by + c \qquad ...(i)$$

where a, b, c are arbitrary constants.

Differentiating (i) w.r.t. y, we get

$$\frac{dx}{dy} = 2ay + b$$

$$\Rightarrow \frac{d^2x}{dy^2} = 2a$$

$$\Rightarrow \frac{d^3x}{dy^3} = 0, \text{ which is the required differential equation.}$$

EXAMPLE 12 *Find the differential equation of all parabolas whose axis of symmetry is along x-axis.*

SOLUTION The equation

$$x = ay^2 + b \qquad ...(i)$$

represents a family of parabolas having their axis of symmetry along x-axis, where a, b are arbitrary constants.

Differentiating (i) w.r.t. y, we get

$$\frac{dx}{dy} = 2ay + b$$

$$\Rightarrow \frac{d^2x}{dy^2} = 2a$$

$$\Rightarrow \frac{d^3x}{dy^3} = 0, \text{ which is the required differential equation.}$$

EXAMPLE 13 *Find the differential equation of the parabolas having their vertices at the origin and foci on the x-axis.*

SOLUTION The equation

$$y^2 = 4ax, a \neq 0 \qquad ...(i)$$

represents a family of parabolas having their vertices at the origin and foci on x-axis.

Differentiating (i) w.r.t. x, we get

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \quad y \frac{dy}{dx} = 2a$$

$$\Rightarrow \quad a = \frac{1}{2} y \frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$y^2 = 2yx \frac{dy}{dx}$$

$\Rightarrow \quad y = 2x\dfrac{dy}{dx}$,

which is the required differential equation.

EXAMPLE 14 *Find the differential equation of hyperbolas with the coordinate axes as transverse and conjugate axes.*

SOLUTION The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(i)$$

represents a family of hyperbolas whose transverse and conjugate axes are along the coordinate axes for differents values of a and b.

Differentiating (i) w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} - \frac{y}{b^2}\frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{a^2} - \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 - \frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \qquad ...(iii)$$

Multiplying (iii) by x and subtracting from (ii), we get

$$-\frac{y}{b^2}\frac{dy}{dx} + \frac{x}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{xy}{b^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2},$$

which is the required differential equation.

EXAMPLE 15 *Form the differential equation that represents all the parabolas each of which has a latusrectum $4a$ and whose axes are parallel to the x-axis.*

SOLUTION The equation

$$(y-k)^2 = 4a(x-h) \qquad ...(i)$$

represents a family of parabolas of latusrectum $4a$ and whose axes are parallel to the x-axis.

Differentiating (i) w.r.t. x, we get

$$2(y-k)\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad (y-k)\frac{dy}{dx} = 2a \qquad ...(ii)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 + (y-k)\frac{d^2y}{dx^2} = 0 \qquad ...(iii)$$

From (ii) and (iii), we obtain

$$\left(\frac{dy}{dx}\right)^2 + \frac{2a}{\dfrac{dy}{dx}}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 + 2a\frac{d^2y}{dx^2} = 0,$$ which is the required differential equation.

EXAMPLE 16 *Show that the one parameter family of confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ satisfy the differential equation*

$$\left(x + y\frac{dy}{dx}\right)\left(x\frac{dy}{dx} - y\right) = (a^2 - b^2)\frac{dy}{dx}.$$

SOLUTION We have,

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(b^2+\lambda) + y(a^2+\lambda)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad xb^2 + a^2y\frac{dy}{dx} = -\lambda\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \lambda = -\frac{xb^2 + a^2y\dfrac{dy}{dx}}{x + y\dfrac{dy}{dx}}$$

$$\Rightarrow \quad a^2+\lambda = a^2 - \frac{xb^2 + a^2y\dfrac{dy}{dx}}{x + y\dfrac{dy}{dx}} \text{ and, } b^2+\lambda = b^2 - \frac{xb^2 + a^2y\dfrac{dy}{dx}}{x + y\dfrac{dy}{dx}}$$

$$\Rightarrow \quad a^2+\lambda = \frac{(a^2-b^2)x}{x + y\dfrac{dy}{dx}} \text{ and, } b^2+\lambda = \frac{(b^2-a^2)y\dfrac{dy}{dx}}{x + y\dfrac{dy}{dx}}$$

Substituting the values of $a^2+\lambda$ and $b^2+\lambda$ in (i), we obtain

$$\frac{x^2\left(x + y\dfrac{dy}{dx}\right)}{(a^2-b^2)x} + \frac{y^2\left(x + y\dfrac{dy}{dx}\right)}{(b^2-a^2)y\dfrac{dy}{dx}} = 1$$

$$\Rightarrow \quad \left(x + y\frac{dy}{dx}\right)\left(x - \frac{y}{\dfrac{dy}{dx}}\right) = (a^2-b^2)$$

$$\Rightarrow \quad \left(x + y\frac{dy}{dx}\right)\left(x\frac{dy}{dx} - y\right) = (a^2-b^2)\frac{dy}{dx},$$

which is the required differential equation.

EXAMPLE 17 *Find the differential equation of circles passing through the points of intersection of unit circle with centre at the origin and the line bisecting the first quadrant.*

SOLUTION The equation of the unit circle with centre at the origin is $x^2 + y^2 = 1$ and the line bisecting the first quadrant is $y = x$.

So, equation of the family of circles passing through the intersection of these two is given by

$$x^2 + y^2 - 1 + \lambda (y - x) = 0 \qquad ...(i)$$

where λ is a parameter.

Differentiating (i) w.r. t. x, we get

$$2x + 2y\frac{dy}{dx} + \lambda\left(\frac{dy}{dx} - 1\right) = 0$$

$$\Rightarrow \qquad \lambda = \frac{-2\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx} - 1}$$

Putting the value of λ in (i), we obtain the required differential equation as

$$\left(\frac{dy}{dx} - 1\right)(x^2 + y^2 - 1) - 2\left(x + y\frac{dy}{dx}\right) = 0$$

EXAMPLE 18 *Prove that the differential equation of all the conics touching the y-axis at the origin and having their centres on the x-axis is*

$$x^2 y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

SOLUTION The equation of all conics whose axes are parallel to the coordinate axes and centres at the origin is

$$Ax^2 + By^2 = 1 \qquad ...(i)$$

Let the centre be $(a, 0)$. Then, equation (i) becomes

$$A(x - a)^2 + B y^2 = 1 \qquad ...(ii)$$

If it touches y-axis at the origin, then

$$Aa^2 + By^2 = 1 \qquad [\text{Putting } x = 0 \text{ in (ii)}]$$

must have equal roots i.e. $Aa^2 + By^2 = 1$ must have equal roots.

$$\therefore \quad Aa^2 = 1 \Rightarrow A = \frac{1}{a^2}.$$

Putting $A = \frac{1}{a^2}$ in (ii), we get

$$\frac{x^2}{a^2} - \frac{2x}{a} + By^2 = 0 \qquad ...(iii)$$

as the equation of the family of conics satisfying the given properties.

Differentiating (iii) w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2}{a} + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} - \frac{1}{a} + By\frac{dy}{dx} = 0 \qquad ...(iv)$$

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{a^2} + B\left(\frac{dy}{dx}\right)^2 + By\frac{d^2y}{dx^2} = 0 \qquad ...(v)$$

Eliminating $\frac{1}{a^2}, \frac{1}{a}$ and B from (iii), (iv) and (v), we get

$$\begin{vmatrix} x^2 & -2x & y^2 \\ x & -1 & y\frac{dy}{dx} \\ 1 & 0 & \frac{dy}{dx} + y\frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(y^2 - 2xy\frac{dy}{dx}\right) + (2x - x^2)\left(\frac{dy}{dx} + y\frac{d^2y}{dx^2}\right) = 0$$

$$\Rightarrow \quad x^2 y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0,$$

which is the required differential equation.

EXAMPLE 19 *Obtain the differential equation satisfied by the locus of the foot of the perpendicular drawn from the centre of the family of ellipse to their tangents.*

SOLUTION The equation of any tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2 m^2 + b^2} \qquad ...(i)$$

Let $P(h, k)$ be the foot of the perpendicular drawn from the origin $O(0, 0)$ to (i).

Since (h, k) lies on (i). Therefore,

$$k = mh \pm \sqrt{a^2 m^2 + b^2} \qquad ...(ii)$$

Slope of $OP = \frac{k}{h}$.

Since OP is perpendicular to (i). Therefore,

$$\frac{k}{h} \times m = -1 \Rightarrow m = \frac{-h}{k}$$

Putting the value of m in (ii), we get

$$k = -\frac{h^2}{k} \pm \sqrt{\frac{a^2h^2}{k^2} + b^2}$$

$$\Rightarrow \quad (k^2 + h^2)^2 = a^2 h^2 + b^2 k^2$$

Thus, the locus of (h, k) is

$$(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2 \qquad ...(iii)$$

We have to find the differential equation satisfied by (iii).

Differentiating (iii) w.r.t. x, we get

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2a^2 x + 2b^2 y\frac{dy}{dx}$$

$$\Rightarrow \quad 2(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = a^2 x + b^2 y\frac{dy}{dx} \qquad ...(iv)$$

Differentiating (iv) w.r.to x, we get

$$4\left\{x+y\frac{dy}{dx}\right\}^2+2(x^2+y^2)\left\{1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}$$

$$=a^2+b^2\left\{\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right\} \quad ...(v)$$

Multiplying (iv) by x and subtracting from (iii), we get

$$(x^2+y^2)\left\{x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)\right\}=b^2\left\{y^2-xy\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad (x^2+y^2)\left(y^2-x^2-2xy\frac{dy}{dx}\right)=b^2y\left(y-x\frac{dy}{dx}\right) \quad ...(vi)$$

Multiplying (v) by x and subtracting from (iv), we get

$$2(x^2+y^2)\left\{x+y\frac{dy}{dx}-x-xy\frac{d^2y}{dx^2}-x\left(\frac{dy}{dx}\right)^2\right\}-4\left(x+y\frac{dy}{dx}\right)^2$$

$$=b^2\left\{y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}\right\}$$

$$\Rightarrow 2(x^2+y^2)\left\{y\frac{dy}{dx}-xy\frac{d^2y}{dx^2}-x\left(\frac{dy}{dx}\right)^2\right\}-4\left(x+y\frac{dy}{dx}\right)^2$$

$$=b^2\left\{y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}\right\} \quad ...(vii)$$

Eliminating b^2 from (vi) and (vii), we obtain the required differential equation as

$$\frac{(x^2+y^2)\left(y^2-x^2-2xy\frac{dy}{dx}\right)}{y\left(y-x\frac{dy}{dx}\right)}$$

$$=\frac{2(x^2+y^2)\left\{y\frac{dy}{dx}-xy\frac{d^2y}{dx^2}-x\left(\frac{dy}{dx}\right)^2\right\}-4\left(x+y\frac{dy}{dx}\right)^2}{\left\{y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}\right\}}$$

EXERCISE 13.1

1. Find the differential equations of the family of curves in each of the following :

   (i) $y = Ae^{3x} + Be^{5x}$

   (ii) $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda} = 1$, λ is parameter.

   (iii) $xy = ae^x + be^{-x} + x^2$

   (iv) $y = a\sin(mx+b)$

2. Find the differential equation associated with the primitive $x^2+y^2+2ax+2by+c=0$, where a, b, c are arbitrary constants.
3. Find the differential equation of all lines in a plane which are equally inclined to the coordinate axes.
4. Find the differential equation of all parabolas whose axis of symmetry is parallel to y axis.
5. Find the differential equation of all circles in a plane.
6. Assume that a rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.
7. A population grows at the rate of 5% per year. If $x = x(t)$ denotes the number of individuals in the population after t years. Write the differential equation involving the rate of change of x.
8. Obtain the differential equation governing the motion of a particle of mass m that is projected vertically upwards in a medium in which the resistance is k times the square of the particle velocity.
9. Find a differential equation whose primitive is $c^2+2cy+a^2-x^2=0$, where c is an arbitrary constants and a is a definite constant.
10. Obtain the differential equation of the curve $y = a_1e^{nx} + a_2\cos nx + a_3\sin nx$, where n is a given constant and a_1, a_2, a_3 are arbitrary constants.

ANSWERS

1. (i) $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$

   (ii) $\frac{dy}{dx}=\frac{xy}{x^2-a^2}$

   (iii) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0$

   (iv) $y\frac{d^3y}{dx^3}=\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$

2. $3\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}=0$

3. $\frac{dy}{dx}=\pm1$

4. $\frac{d^3y}{dx^3}=0$

5. $\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$

6. $\frac{dr}{dt} = -k$, r is the radius.

7. $\frac{dx}{dt} = \frac{5x}{100}$

8. $m\frac{dk}{dt} = -mg - kv^2$.

9. $x^2 + 2xy\frac{dy}{dx} + (a^2 - x^2)\left(\frac{dy}{dx}\right)^2 = 0$

10. $y_3 + y_2(n^2 - n) - n^3 y_1 = 0$

—❑❑❑—

13.4 FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS AND THEIR GEOMETRICAL INTERPRETATIONS

A first order and first degree differential equation involves the independent variable x (say), dependent variable y (say) and the differential coefficeint of y w.r t. x i.e. $\frac{dy}{dx}$. So, it can be put in any one of the following forms :

$$\frac{dy}{dx} = f(x, y) \text{ or } f\left(x, y, \frac{dy}{dx}\right) = 0 \text{ or } f(x, y)\,dx + g(x, y)\,dy = 0,$$

where $f(x, y)$ and $g(x, y)$ are functions of x and y.

GEOMETRICAL INTERPRETATION The general form of a first order and first degree differential equation is

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \text{...(i)}$$

We know that the tangent of the direction of a curve in Cartesian rectangular coordinates at any point is given by $\frac{dy}{dx}$, so the equation in (i) can be known as an equation which establishes the relationship between the coordinates of a point and the slope of the tangent i.e., dy/dx to the integral curve at that point. Solving the differential equation given by (i) means finding those curves for which the direction of tangent at each point coincides with the direction of the field. All the curves represented by the general solution when taken together will give the locus of the differential equation. Since there is one arbitrary constant in the general solution of the equation of first order, the locus of the equation can be said to be made up of single infinity of curves.

13.5 METHODS OF SOLVING FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATIONS

A first order and first degree differential equation can be expressed in any one of the following forms :

$$f(x, y)\,dx + g(x, y)\,dy = 0$$

or, $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$,

or, $\frac{dy}{dx} = \phi(x, y)$,

where $f(x, y)$, $g(x, y)$ and $\phi(x, y)$ are functions of x, y.

It is not always possible to solve this type of equations. The solution of this type of differential equation is possible only when it falls under the category of some standard forms. In this section, we shall be discussing some of the standard forms and methods of obtaining their solutions.

13.5.1 INSPECTION METHOD

Sometimes just by inspection a first order first degree differential equation can be written in the form

$$d\{f(x, y)\} + d\{g(x, y)\} = 0$$

or, $d\{f(x, y)\} = 0$,

where $d\{f(x, y)\}$ stands for the differential of $f(x, y)$.

In such cases the general solution of the differential equation is given by

$$f(x, y) + g(x, y) = C$$

or, $f(x, y) = C$

In order to solve a differential equation by inspection method the following results are very helpful :

(I) If the given differential equation contains $(x\,dy - y\,dx)$ as a term, then its multiplication by

(i) $\frac{1}{x^2}$ gives : $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(ii) $\frac{1}{y^2}$ gives : $\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$

(iii) $\frac{1}{xy}$ gives : $\frac{x\,dy - y\,dx}{xy} = \frac{1}{y}dy - \frac{1}{x}dx = d\left(\log\frac{y}{x}\right)$

(iv) $\frac{1}{x^2 + y^2}$ gives : $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(v) $\frac{1}{x\sqrt{x^2 - y^2}}$ gives : $\frac{x\,dy - y\,dx}{x\sqrt{x^2 - y^2}} = d\left\{\sin^{-1}\left(\frac{y}{x}\right)\right\}$

(II) If the given differential equation contains $(x\,dy + y\,dx)$ as a term, then its multiplication by

(i) 1 gives : $d(xy)$

(ii) $\frac{1}{xy}$ gives : $\frac{x\,dy + y\,dx}{xy} = \frac{d(xy)}{xy} = d\{\log(xy)\}$

(iii) $\frac{1}{(xy)^n}$ gives : $\frac{x\,dy + y\,dx}{(xy)^n}$

$$= \frac{d(xy)}{(xy)^n}$$

$$= d\left\{\frac{-1}{(n-1)(xy)^{n-1}}\right\}, n \neq 1$$

If the given differential equation contains $(x\,dx + y\,dy)$ as a term, then its multiplication by

(i) $\frac{1}{2}$ gives : $d(x^2+y^2)$

(ii) $\frac{2}{(x^2+y^2)}$ gives : $\frac{2x\,dx + 2y\,dy}{x^2+y^2}$

$$= \frac{d(x^2+y^2)}{x^2+y^2}$$

$$= d\{\log(x^2+y^2)\}$$

(iii) $\frac{2}{(x^2+y^2)^n}$ gives : $\frac{2x\,dx + 2y\,dx}{(x^2+y^2)^n}$

$$= \frac{d(x^2+y^2)}{(x^2+y^2)^n}$$

$$= d\left\{\frac{-1}{(n-1)(x^2+y^2)^{n-1}}\right\}, n \neq 1$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation :*

$$x\,dy + y\,dx + 2x^3\,dx = 0$$

SOLUTION We have,

$$x\,dy + y\,dx + 2x^3\,dx = 0$$

$$\Rightarrow \quad d(xy) + 2x^3\,dx = 0$$

$$\Rightarrow \quad d(xy) + \frac{1}{2}d(x^4) = 0$$

On integration, we get

$$xy + \frac{1}{2}x^4 = C$$

which is the required solution of the given differential equation.

EXAMPLE 2 *Solve the differential equation :*

$$x\,dx + y\,dy = a(x^2+y^2)\,dy$$

SOLUTION We have,

$$x\,dx + y\,dy = a(x^2+y^2)\,dy$$

$$\Rightarrow \quad \frac{2x\,dx + 2y\,dy}{x^2+y^2} = 2a\,dy$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{x^2+y^2} = 2a\,dy$$

On integrating, we obtain

$\log(x^2+y^2) = 2ay + C$, as the required solution.

EXAMPLE 3 *Solve :* $(1+xy)\,x\,dy + (1-xy)\,y\,dx = 0$

SOLUTION The given equation can be written as

$$(x\,dy + y\,dx) + xy(x\,dy - y\,dx) = 0$$

$$\Rightarrow \quad d(xy) + xy(x\,dy - y\,dx) = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + \frac{x\,dy - y\,dx}{xy} = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + \frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + d(\log y) - d(\log x) = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + d(\log y - \log x) = 0$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} + d\log\left(\frac{y}{x}\right) = 0$$

On integrating, we get

$$-\frac{1}{xy} + \log\left(\frac{y}{x}\right) = C$$

as the required solution.

EXAMPLE 4 *Solve :*

$$y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$$

SOLUTION The given differential equation is

$$y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$$

$$\Rightarrow \quad \frac{y\,dx - x\,dy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \sin y\,dy = 0$$

$$\Rightarrow \quad -\left(\frac{x\,dy - y\,dx}{x^2}\right) + \left(\frac{1}{x^2}+1\right)dx + \sin y\,dy = 0$$

$$\Rightarrow \quad -d\left(\frac{y}{x}\right) + d\left(-\frac{1}{x}+x\right) + d(-\cos y) = 0$$

On integrating, we get

$$-\frac{y}{x} + \left(-\frac{1}{x}+x\right) - \cos y = C$$

$$\Rightarrow \quad \frac{y}{x} + \frac{1}{x} - x + \cos y + C = 0,$$ which is the required solution of the given differential equation.

EXAMPLE 5 *Solve the differential equation :*

$$(1 + x\sqrt{x^2+y^2})\,dx + (-1 + \sqrt{x^2+y^2})\,y\,dy = 0$$

SOLUTION The given equation can be written as

$$dx - y\,dy + \sqrt{x^2+y^2}\,(x\,dx + y\,dy) = 0$$

$$\Rightarrow\ dx - y\,dy + \frac{1}{2}\sqrt{x^2+y^2}\,(2x\,dx + 2y\,dy) = 0$$

$$\Rightarrow\ dx - y\,dy + \frac{1}{2}\sqrt{x^2+y^2}\,d(x^2+y^2) = 0$$

$$\Rightarrow\ 2\,dx - 2y\,dy + \sqrt{x^2+y^2}\,d(x^2+y^2) = 0$$

$$\Rightarrow\ 2\,dx - d(y^2) + \sqrt{x^2+y^2}\,d(x^2+y^2) = 0$$

On integration, we get

$$2x - y^2 + \frac{(x^2+y^2)^{3/2}}{\frac{3}{2}} = C$$

$$\Rightarrow 2x - y^2 + \frac{2}{3}(x^2+y^2)^{3/2} = C\ ,$$

which is the solution of the given differential equation.

EXAMPLE 6 *Solve the differential equation*

$$\frac{x\,dy}{x^2+y^2} = \left(\frac{y}{x^2+y^2} - 1\right)dx$$

SOLUTION The given differential equation can be written as

$$\frac{x\,dy - y\,dx}{x^2+y^2} = -dx$$

$$\Rightarrow\ d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = -dx$$

On integrating, we obtain

$$\tan^{-1}\left(\frac{y}{x}\right) = -x + C, \text{ which is the required solution.}$$

EXAMPLE 7 *Solve the differential equation*

$$y\,e^{x/y}\,dx = (x\,e^{x/y} + y^2 \sin y)\,dy$$

SOLUTION We have,

$$y\,e^{x/y}\,dx = (x\,e^{x/y} + y^2 \sin y)\,dy$$

$$\Rightarrow\ (y\,dx - x\,dy)\,e^{x/y} = y^2 \sin y\,dy$$

$$\Rightarrow\ e^{x/y}\left\{\frac{y\,dx - x\,dy}{y^2}\right\} = \sin y\,dy$$

$$\Rightarrow\ e^{x/y}\,d\left(\frac{x}{y}\right) = -d(\cos y)$$

On integrating, we obtain

$$e^{x/y} = -\cos y + C, \text{ as the required solution.}$$

EXAMPLE 8 *Solve the differential equation*

$$\{y(1+x^{-1}) + \sin y\}\,dx + \{x + \log x + x\cos y\}\,dy = 0$$

SOLUTION The given differential equation can be written as

$$(y\,dx + x\,dy) + (\sin y\,dx + x\cos y\,dy) + \left(\frac{y}{x}\,dx + \log x\,dy\right) = 0$$

$$\Rightarrow\ d(xy) + d(x\sin y) + d(y\log x) = 0$$

On integrating, we get

$$xy + x\sin y + y\log x = C \text{ as the required solution.}$$

EXAMPLE 9 *Solve the differential equation*

$$x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

SOLUTION The given differential equation can be written as

$$y\,dx + x\,dy = \frac{y}{x}\tan\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

$$\Rightarrow\ d(xy) = \frac{y}{x}\tan\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

$$\Rightarrow\ \frac{d(xy)}{xy} = \tan\left(\frac{y}{x}\right)\left(\frac{x\,dy - y\,dx}{x^2}\right)$$

$$\Rightarrow\ \frac{d(xy)}{xy} = \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

On integrating, we get

$$\int \frac{1}{xy}\,d(xy) = \int \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

$$\Rightarrow\ \log(xy) = \log\sec\left(\frac{y}{x}\right) + \log C$$

$$\Rightarrow\ xy = C\sec\left(\frac{y}{x}\right), \text{ which is the required solution.}$$

EXAMPLE 10 *Solve the differential equation*

$$x\sin\left(\frac{y}{x}\right)dy = \left\{y\sin\left(\frac{y}{x}\right) - x\right\}dx$$

SOLUTION The given differential equation can be written as

$$x\sin\left(\frac{y}{x}\right)dy - y\sin\left(\frac{y}{x}\right)dx = -x\,dx$$

$$\Rightarrow\ (x\,dy - y\,dx)\sin\left(\frac{y}{x}\right) = -x\,dx$$

$$\Rightarrow\ \sin\left(\frac{y}{x}\right)\left(\frac{x\,dy - y\,dx}{x^2}\right) = -\frac{1}{x}\,dx$$

$$\Rightarrow\ \sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = -\frac{1}{x}\,dx$$

On integrating, we get

$$\int \sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = -\int\frac{1}{x}\,dx$$

$$\Rightarrow\ -\cos\left(\frac{y}{x}\right) = -\log x + \log C$$

$$\Rightarrow\ \log x - \cos\left(\frac{y}{x}\right) = \log C, \text{ which is the required solution.}$$

EXAMPLE 11 *Solve the differential equation*

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$$

SOLUTION The given differential equation can be written as

$$\frac{\sqrt{x^2+y^2}}{\sqrt{a^2-(\sqrt{x^2+y^2})^2}}(x\,dx + y\,dy) = x\,dy - y\,dx$$

$$\Rightarrow \quad \frac{d\,(x^2+y^2)}{2\sqrt{x^2+y^2}\,\sqrt{a^2-(\sqrt{x^2+y^2})^2}} = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{a^2-(\sqrt{x^2+y^2})^2}}\,d\,(\sqrt{x^2+y^2}) = d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

On integrating, we get

$$\int \frac{1}{\sqrt{a^2-(\sqrt{x^2+y^2})^2}}\,d\,(\sqrt{x^2+y^2}) = \int d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + C.$$

EXAMPLE 12 *Solve the differential equation*

$$\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 - y^2}}$$

SOLUTION The given differential equation can be written as

$$\frac{\sqrt{x^2-y^2}}{\sqrt{1+\{\sqrt{x^2-y^2}\}^2}}(x\,dx - y\,dy) = x\,dy - y\,dx$$

$$\Rightarrow \quad \frac{1}{2}\frac{\sqrt{x^2-y^2}}{\sqrt{1+\{\sqrt{x^2-y^2}\}^2}}\,d\,(x^2-y^2) = x\,(dy-dx) - y\,(dx-dy)$$

$$+ x\,dx - y\,dy$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x^2-y^2}\,\sqrt{1+\{\sqrt{x^2-y^2}\}^2}}\,d\,(x^2-y^2)$$

$$= -\frac{(x+y)\,(dx-dy)}{x^2-y^2} + \frac{1}{2}\frac{d\,(x^2-y^2)}{x^2-y^2}$$

$$\Rightarrow \frac{1}{\sqrt{1+(x^2-y^2)^2}}\,d\,\sqrt{(x^2-y^2)} = -\left(\frac{dx-dy}{x-y}\right) + \frac{1}{2}\frac{d\,(x^2-y^2)}{x^2-y^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+\{\sqrt{x^2-y^2}\}^2}}\,d\,(\sqrt{x^2-y^2}) = -\frac{d\,(x-y)}{x-y} + \frac{1}{2}\frac{d\,(x^2-y^2)}{x^2-y^2}$$

On integrating, we obtain

$$\log\left\{\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2}\right\} = -\log(x-y) + \frac{1}{2}\log(x^2-y^2)$$

$$+ \log C$$

$$\Rightarrow \sqrt{x^2+y^2} + \sqrt{1+x^2-y^2} = \left(\frac{\sqrt{x^2-y^2}}{x-y}\right)C$$

$$\Rightarrow \sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = C\sqrt{\frac{x+y}{x-y}},$$

which is the required solution of the given differential equation.

EXAMPLE 13 *Solve the differential equation*

$$(x^2\sin^3 y - y^2\cos x)\,dx + (x^3\cos y\sin^2 y - 2y\sin x)\,dy = 0$$

SOLUTION The given differential equation can be written as

$$x^2\sin^2 y\,(\sin y\,dx + x\cos y\,dy) - (y^2\cos x\,dx + 2y\sin x\,dy) = 0$$

$$\Rightarrow \quad (x\sin y)^2\,d\,(x\sin y) - d\,(y^2\sin x) = 0$$

On integrating, we get

$$\int (x\sin y)^2\,d\,(x\sin y) - \int 1\,.\,d\,(y^2\sin x) = 0$$

$$\Rightarrow \quad \frac{(x\sin y)^3}{3} - y^2\sin x = C, \text{ which is the required solution.}$$

EXAMPLE 14 *Solve the differential equation*

$$(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$$

SOLUTION The given differential equation can be written as

$$dx + e^{x/y}\left(dx + dy - \frac{x}{y}\,dy\right) = 0$$

$$\Rightarrow \quad dx + e^{x/y}\left(\frac{y\,dx - x\,dy}{y}\right) + e^{x/y}\,dy = 0$$

$$\Rightarrow \quad dx + y\,e^{x/y}\left(\frac{y\,dx - x\,dy}{y^2}\right) + e^{x/y}\,dy = 0$$

$$\Rightarrow \quad dx + y\,e^{x/y}\,d\left(\frac{x}{y}\right) + e^{x/y}\,dy = 0$$

$$\Rightarrow \quad dx + y\,d\,(e^{x/y}) + e^{x/y}\,dy = 0$$

$$\Rightarrow \quad dx + d\,(y\,e^{x/y}) = 0$$

On integrating, we get

$$x + y\,e^{x/y} = C$$

EXAMPLE 15 *Solve the differential equation*

$$\frac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = \frac{x\sin^2(x^2+y^2)}{y^3}$$

SOLUTION The given differential equation can be written as

$$\frac{x\,dx + y\,dy}{y\,dx - x\,dy} = \frac{x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \quad \frac{2x\,dx + 2y\,dy}{y\,dx - x\,dy} = \frac{2x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \quad \frac{d\,(x^2+y^2)}{\sin^2(x^2+y^2)} = \frac{2x}{y^3}\,(y\,dx - x\,dy)$$

$\Rightarrow \operatorname{cosec}^2(x^2+y^2)\,d(x^2+y^2) = 2\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$

On integrating, we get

$\int \operatorname{cosec}^2(x^2+y^2)\,d(x^2+y^2) = 2\int \left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$

$\Rightarrow -\cot(x^2+y^2) = \left(\frac{x}{y}\right)^2 + C$, which is the required solution.

EXAMPLE 16 *Solve the differential equation :*

$$(x\,dy + y\,dx)\sin(xy) + (x^2\,y\,dy + y^2\,x\,dx)\cos(xy) = 0$$

SOLUTION The given differential equation can be written as

$\sin(xy)\,d(xy) + xy\cos(xy)\,d(xy) = 0$

$\Rightarrow d(xy\,.\sin xy) = 0$

$\Rightarrow xy\sin xy = C$, which is the required solution.

EXAMPLE 17 *Solve the differential equation*

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{(1+\log x + \log y)^2}$$

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{[1+\log(xy)]^2}$$

$\Rightarrow [1+\log(xy)]^2\,(x\,dy + y\,dx) = x\,dx$

$\Rightarrow [1+\log(xy)]^2\,d(xy) = x\,dx$

On integrating, we obtain

$\int [1+\log(xy)]^2\,d(xy) = \int x\,dx$

$\Rightarrow \int (1+\log t)^2\,dt = \frac{x^2}{2} + C$, where $t = xy$

$\Rightarrow t(1+\log t)^2 - \int 2(1+\log t)\frac{1}{t}\,.\,t\,dt = \frac{x^2}{2} + C$

$\Rightarrow t(1+\log t)^2 - 2\int (1+\log t)\,dt = \frac{x^2}{2} + C$

$\Rightarrow t(1+\log t)^2 - 2[t + t\log t - t] = \frac{x^2}{2} + C$

$\Rightarrow xy(1+\log(xy)^2)^2 - 2xy\log(xy) = \frac{x^2}{2} + C$

$\Rightarrow xy(1+(\log xy)^2) = \frac{x^2}{2} + C$

EXAMPLE 18 *Solve the differential equation :*

$$\frac{dy}{dx} = \frac{3x^2y^4 + 2xy}{x^2 - 2x^3y^3}$$

SOLUTION The given differential equation can be written as

$$(3x^2y^4 + 2xy)\,dx = (x^2 - 2x^3y^3)\,dy$$

$\Rightarrow 3x^2y^4\,dx + 2x^3y^3\,dy + 2xy\,dx - x^2\,dy = 0$

$\Rightarrow y^2(3x^2y^2\,dx + 2x^3\,y\,dy) + 2xy\,dx - x^2\,dy = 0$

$\Rightarrow y^2[y^2\,d(x)^3 + x^3\,d(y)^2] + y\,d(x)^2 - x^2\,dy = 0$

$\Rightarrow y^2\,d(x)^3 + x^3\,d(y^2) + \frac{y\,d(x)^2 - x^2\,dy}{y^2} = 0$

$\Rightarrow d(x^3y^2) + d\left(\frac{x^2}{y}\right) = 0$

On integration, we obtain

$$x^3y^2 + \frac{x^2}{y} = C,$$

as the required solution.

EXAMPLE 19 *Solve the differential equation*

$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

SOLUTION The given differential equation can be written as

$$2x\,dx + 2y\,dy + 2\frac{(x\,dy - y\,dx)}{x^2+y^2} = 0$$

$\Rightarrow d(x^2+y^2) + 2\,d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = 0$

On integrating, we get

$$x^2 + y^2 + 2\tan^{-1}\left(\frac{y}{x}\right) = C$$

EXAMPLE 20 *Solve the differential equation :*

$$y(2xy+1)\,dx + x(1+2xy+x^2y^2)\,dy = 0$$

SOLUTION The given differential equation can be written as

$$(y\,dx + x\,dy) + 2xy(y\,dx + x\,dy) + x^3y^2\,dy = 0$$

$\Rightarrow (1+2xy)(y\,dx + x\,dy) + x^3y^2\,dy = 0$

$\Rightarrow (1+2xy)\,d(xy) + (xy)^3\frac{1}{y}\,dy = 0$

$\Rightarrow \left\{\frac{1+2xy}{(xy)^3}\right\}d(xy) + \frac{1}{y}\,dy = 0$

$\Rightarrow \left\{\frac{1}{(xy)^3} + \frac{2}{(xy)^2}\right\}d(xy) + \frac{1}{y}\,dy = 0$

On integrating, we obtain

$-\frac{1}{2(xy)^2} - \frac{2}{xy} + \log y = C$, which the required solution.

EXAMPLE 21 *Solve the differential equation :*

$$(3\tan x + 4\cot y - 7)\sin^2 y\,dx - (4\tan x + 7\cot y - 5)\cos^2 x\,dy = 0$$

SOLUTION Dividing throughout by $\cos^2 x \sin^2 y$, the given differential equation becomes

$$(3\tan x + 4\cot y - 7)\sec^2 x\,dx - (4\tan x + 7\cot y - 5)\operatorname{cosec}^2 y\,dy = 0$$

$$\Rightarrow 3\tan x\sec^2 x\,dx - 7\sec^2 x\,dx - 7\cot y\operatorname{cosec}^2 y\,dy + 5\operatorname{cosec}^2 y\,dy + 4\cot y\sec^2 x\,dx - 4\tan x\operatorname{cosec}^2 y\,dy = 0$$

$$\Rightarrow 3\tan x\,d(\tan x) - 7\,d(\tan x) + 7\cot y\,d(\cot y) - 5\,d(\cot y) + 4\,d(\tan x\cot y) = 0$$

On integrating, we obtain

$$\frac{3}{2}\tan^2 x - 7\tan x + \frac{7}{2}\cot^2 y - 5\cot y + 4\tan x\cot y = C.$$

EXAMPLE 22 *Solve the differential equation :*

$$\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx + (x + \log x - x\sin y)\,dy = 0$$

SOLUTION The given differential equation can be written as

$$y\,dx + x\,dy + \frac{y}{x}dx + \log x\,dy + \cos y\,dx - x\sin y\,dy = 0$$

$$\Rightarrow d(xy) + \{y\,d(\log x) + \log x\,dy\} + \{\cos y\,dx + x\,d(\cos y)\} = 0$$

$$\Rightarrow d(xy) + d(y\log x) + d(x\cos y) = 0$$

On integrating, we obtain

$$xy + y\log x + x\cos y = C,$$

as the required solution.

EXAMPLE 23 *Solve the differential equation :*

$$(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$$

SOLUTION The given differential equation can be written as

$$x^2\,dx + y^2\,dy - 4xy\,dx - 2y^2\,dx - 4xy\,dy - 2x^2\,dy = 0$$

$$\Rightarrow x^2\,dx + y^2\,dy - 2(2xy\,dx + x^2\,dy) - 2(2xy\,dy + y^2\,dx) = 0$$

$$\Rightarrow x^2\,dx + y^2\,dy - 2\,d(x^2y) - 2\,d(xy^2) = 0$$

On integrating, we get

$$\frac{x^3}{3} + \frac{y^3}{3} - 2x^2y - 2xy^2 = C,$$

as the required solution.

EXAMPLE 24 *Solve the differential equation :*

$$\cos\left(\frac{x}{y}\right)(y\,dx - x\,dy) = xy^3(x\,dy + y\,dx)$$

SOLUTION The given differential equation can be written as

$$\cos\left(\frac{x}{y}\right)\cdot\left(\frac{y\,dx - x\,dy}{y^2}\right) = (xy)(x\,dy + y\,dx)$$

$$\Rightarrow \cos\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) = (xy)\,d(xy)$$

On integrating, we get

$$\int \cos\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) = \int xy\,d(xy) + C$$

$$\Rightarrow \sin\left(\frac{x}{y}\right) = \frac{(xy)^2}{2} + C,$$ which is the required solution.

EXAMPLE 25 *Solve the differential equation :*

$$\frac{x\frac{dy}{dx} - y}{\sqrt{x^2 - y^2}} = mx^2$$

SOLUTION The given differential equation can be written as

$$\frac{x\,dy - y\,dx}{\sqrt{x^2 - y^2}} = m\,x^2\,dx$$

$$\Rightarrow \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}}\cdot\frac{x\,dy - y\,dx}{x^2} = mx\,dx$$

$$\Rightarrow \frac{1}{\sqrt{1^2 - \left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right) = mx\,dx$$

On integrating, we get

$$\sin^{-1}\left(\frac{y}{x}\right) = m\frac{x^2}{2} + C,$$ which is the required solution.

EXAMPLE 26 *Solve the differential equation :*

$$x\,dy - \{y + xy^3(1 + \log x)\}\,dx = 0$$

SOLUTION The given differential equation can be written as

$$x\,dy - y\,dx = xy^3(1 + \log x)\,dx$$

$$\Rightarrow -\left(\frac{y\,dx - x\,dy}{y^2}\right) = xy(1 + \log x)\,dx$$

$$\Rightarrow -\frac{x}{y}\left(\frac{y\,dx - x\,dy}{y^2}\right) = x^2(1 + \log x)\,dx$$

$$\Rightarrow -\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)\,dx$$

Integrating both sides, we get

$$-\int \left(\frac{x}{y}\right)d\left(\frac{x}{y}\right) = \int \underset{\text{II}}{x^2}\,\underset{\text{I}}{(1 + \log x)}\,dx$$

$$\Rightarrow -\frac{1}{2}\left(\frac{x}{y}\right)^2 = (1 + \log x)\frac{x^3}{3} - \frac{x^3}{9} + C,$$

which is the required solution.

EXERCISE 13.2

Solve the following differential equations by inspection method :

1. $(xy^4 + y)\,dx - x\,dy = 0$
2. $y\,dx + (x + x^2y)\,dy = 0$
3. $(x + y)\,dx - (x - y)\,dy = 0$
4. $y\,dx - (x - 2y^3)\,dy = 0$
5. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$
6. $y\,(axy + e^x)\,dx - e^x\,dy = 0$
7. $2xy^2\,dx = e^x\,(dy - y\,dx)$
8. $(y^2 e^x + 2xy)\,dx = x^2\,dy$
9. $x\,dx + y\,dy = y\,(x^2 + y^2)\,dy$
10. $y\,(2 + 2y + e^x)\,dx - (e^x + y^3)\,dy = 0$
11. $x\,dx = y\,(x^2 + y^2 - 1)\,dy$
12. $(x^3 e^x - my^2)\,dx + mxy\,dy = 0$
13. $y\,dx - x\,dy + \log x\,.\,dx = 0$
14. $(y\,dx - x\,dy)\cos\left(\frac{x}{y}\right) = xy^3\,(x\,dy + y\,dx)$
15. $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$
16. $\dfrac{x\,dx + y\,dy}{y\,dx - x\,dy} = \dfrac{x\cos^2(x^2 + y^2)}{y^3}$
17. $x\,(e^{x/y} + y^2 e^{xy})\,dy = (e^{x/y} - y^2 e^{xy})\,y\,dx$
18. $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$
19. $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$
20. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$
21. $(1 - x^2y^2)\,dx = y\,dx + x\,dy$
22. $x\,dy - y\,dx = (x^2 + y^2)\,dx$
23. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$
24. $\dfrac{dy}{dx} = \sin^2(x + 3y) + 5$
25. $y\,(x + y^3)\,dx = x\,(y^3 - x)\,dy$
26. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$
27. $(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$
28. $x\,(y^3 - x)\,dx = y\,(x + y^3)\,dy$

ANSWERS

1. $\dfrac{x^4}{4} + \dfrac{1}{3}\left(\dfrac{x}{y}\right)^3 = C$
2. $-\dfrac{1}{xy} + \log y = C$
3. $x^2 + y^2 = C\,e^{2\tan^{-1}\left(\frac{y}{x}\right)}$
4. $x + y^3 = Cy$
5. $\log\left(\dfrac{x}{y}\right) = C + \dfrac{1}{xy}$
6. $ax^2y - Cy + 2e^x = 0$
7. $e^x + (C + x^2)\,y = 0$
8. $y\,e^x + x^2 = Cy$
9. $\log(x^2 + y^2) = y^2 + C$
10. $4x^3y - 3y^3 + 6e^x = 6Cy$
11. $\log(x^2 + y^2) = y^2 + C$
12. $2x^2 e^x + my^2 = 2Cx^2$
13. $Cx + y + \log x + 1 = 0$
14. $\sin\left(\dfrac{x}{y}\right) = \dfrac{1}{2}x^2y^2 + C$
15. $\dfrac{y}{x} - \dfrac{1}{2(x^2 + y^2)} = C$
16. $\tan(x^2 + y^2) = \dfrac{x^2}{y^2} + C$
17. $xy = \log(e^{x/y} + C)$
18. $x^4 + y^4 + 6x^2y^2 = C$
19. $e^{xy^2} + x^4 - y^3 = C$
20. $x + y\,e^{x/y} = C$
21. $1 + xy = C\,e^{2x}(1 - xy)$
22. $y = x\tan(x + C)$
23. $\log\left|\dfrac{y}{x} + \sqrt{1 + y^2/x^2}\right| = \log x + C$
24. $x = \dfrac{1}{4\sqrt{19}}\tan^{-1}\left(\dfrac{\sqrt{19}\,u}{4}\right) + C$
25. $2x + y^3 + 2c\,x^2\,y = 0$
26. $y = x\tan\left(\dfrac{c - x^2 - y^2}{2}\right)$
27. $x^2\cos y + x^3\,y - \dfrac{y^2}{2} = C$
28. $\dfrac{y^2}{2x^2} + \dfrac{1}{xy} = C$

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13.5.2 EQUATIONS IN VARIABLE SEPARABLE FORM

If a first order first degree differential equation can be expressed in the form so that the coefficient of dx is only a function of x and that of dy is only a function of y, then we say that the variables are separable. The general form of such an equation is

$$f(x)\,dx + g(y)\,dy = 0$$

In order to solve this type of differential equations, we integrate both sides to obtain the general solution

$$\int f(x)\,dx + \int g(y)\,dy = C, \text{ where } C \text{ is an arbitrary constant.}$$

The following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$$

SOLUTION The given differential equation can be written as

$$\sqrt{(1+x^2)(1+y^2)}+xy\frac{dy}{dx}=0$$

$$\Rightarrow\ xy\,dy+\sqrt{1+x^2}\sqrt{1+y^2}\,dx=0$$

$$\Rightarrow\ \frac{y\,dy}{\sqrt{1+y^2}}+\frac{\sqrt{1+x^2}}{x}dx=0$$

On integrating, we get

$$\int\frac{y}{\sqrt{1+y^2}}dy+\int\frac{\sqrt{1+x^2}}{x}dx=C \qquad ...(i)$$

Let $I_1=\int\frac{y\,dy}{\sqrt{1+y^2}}$ and $I_2=\int\frac{\sqrt{1+x^2}}{x}dx$. Then,

$$I_1=\frac{1}{2}\int\frac{2y\,dy}{\sqrt{1+y^2}}=\frac{1}{2}\int\frac{d(1+y^2)}{\sqrt{1+y^2}}$$

$$=\frac{1}{2}\left\{\frac{(1+y^2)^{1/2}}{\frac{1}{2}}\right\}=\sqrt{1+y^2}$$

and, $I_2=\int\frac{\sqrt{1+x^2}}{x}dx$

$$=\int\frac{t^2}{t^2-1}dt \text{ where } t^2=1+x^2$$

$$=\int\frac{t^2-1+1}{t^2-1}dt=\int\left(1+\frac{1}{t^2-1}\right)dt$$

$$=t+\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$=\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|$$

Substituting the values of I_1 and I_2 in (i), we obtain

$$\sqrt{1+y^2}+\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|=C$$

as the required solution.

EXAMPLE 2 *Solve the differential equation*

$$xy^2\frac{dy}{dx}=1-x^2+y^2-x^2y^2$$

SOLUTION The given differential equation can be written as

$$xy^2\,dy=(1-x^2)(1+y^2)\,dx$$

$$\Rightarrow\ \frac{y^2\,dy}{1+y^2}=\left(\frac{1-x^2}{x}\right)dx$$

$$\Rightarrow\ \int\frac{y^2}{1+y^2}dy=\int\left(\frac{1}{x}-x\right)dx$$

$$\Rightarrow\ \int\frac{1+y^2-1}{1+y^2}dy=\int\left(\frac{1}{x}-x\right)dx$$

$$\Rightarrow\ \int\left(1-\frac{1}{1+y^2}\right)dy=\int\left(\frac{1}{x}-x\right)dx$$

$$\Rightarrow\ \int\left(1-\frac{1}{1+y^2}\right)dy=\int\left(\frac{1}{x}-x\right)dx$$

$\Rightarrow\ y-\tan^{-1}y=\log x-\frac{x^2}{2}+C$, which is the required solution of the given differential equation.

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13.5.3 EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORM

Differential equations of the form $dy/dx=f(ax+by+c)$ can be reduced to variable separable form by the substitution

$$ax+by+c=v.$$

The following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :*

(i) $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$

(ii) $\frac{dy}{dx}=\cos(x+y)$

(iii) $\frac{dy}{dx}=(4x+y+1)^2$

SOLUTION (i) We are given that

$$\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$$

$\Rightarrow \frac{dy}{dx} = \sin(x+y)$

Let $x+y = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So, the given differential equation becomes

$\therefore \frac{dv}{dx} - 1 = \sin v$

$\Rightarrow \frac{dv}{dx} = 1 + \sin v$

$\Rightarrow \frac{dv}{1+\sin v} = dx$

Integrating both sides, we get

$$\int \frac{1}{1+\sin v}\,dv = \int dx$$

$\Rightarrow \int dx = \int \frac{1-\sin v}{1-\sin^2 v}\,dv$

$\Rightarrow \int dx = \int \frac{1-\sin v}{\cos^2 v}$

$\Rightarrow \int dx = \int (\sec^2 v - \tan v \sec v)\,dv$

$\Rightarrow x = \tan v - \sec v + C$

$\Rightarrow x = \tan(x+y) - \sec(x+y) + C$,

which is the required solution.

(ii) We are given that

$$\frac{dy}{dx} = \cos(x+y)$$

Let $x+y = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So, the given equation becomes

$\frac{dv}{dx} - 1 = \cos v$

$\Rightarrow \frac{dv}{dx} = 1 + \cos v$

$\Rightarrow \frac{1}{1+\cos v}\,dv = dx$

$\Rightarrow \frac{1}{2}\sec^2\frac{v}{2}\,dv = dx.$

Integrating both sides, we get

$$\int \frac{1}{2}\sec^2\frac{v}{2}\,dv = \int 1 \cdot dx$$

$\Rightarrow \tan\frac{v}{2} = x + C$

$\Rightarrow \tan\left(\frac{x+y}{2}\right) = x + C$, which is the required solution.

(iii) We are given that

$$\frac{dy}{dx} = (4x+y+1)^2$$

Let $4x+y+1 = v$. Then,

$$4+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

So, the given equation becomes

$\frac{dv}{dx} - 4 = v^2$

$\Rightarrow \frac{dv}{dx} = v^2 + 4$

$\Rightarrow dv = (v^2+4)\,dx$

$\Rightarrow \frac{dv}{v^2+4} = dx$

Integrating both sides, we get

$$\int \frac{1}{v^2+4}\,dv = \int 1 \cdot dx$$

$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C$

$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + C$,

which is the required solution.

EXAMPLE 2 *Solve*: $(x+y)^2\frac{dy}{dx} = a^2$.

SOLUTION Let $x+y = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx}$$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1.$

So the given differential equation reduces to

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2$$

$\Rightarrow v^2\frac{dv}{dx} = a^2 + v^2$

$\Rightarrow v^2\,dv = (a^2+v^2)\,dx$

$\Rightarrow \frac{v^2}{v^2+a^2}\,dv = dx$

$\Rightarrow \left(1 - \frac{a^2}{v^2+a^2}\right)dv = dx$

On integrating, we get

$$\int 1 \cdot dv - a^2\int \frac{1}{v^2+a^2}\,dv = \int dx + C$$

$$\Rightarrow v - a\tan^{-1}\left(\frac{v}{a}\right) = x + C$$

$$\Rightarrow (x+y) - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + C,$$

which is the required solution

EXAMPLE 3 *Solve :* $\cos(x+y)\,dy = dx$

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{\cos(x+y)}$$

Let $x+y=v$. Then,

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dv}{dx}-1.$$

So the given differential equation reduces to

$$\frac{dv}{dx}-1 = \frac{1}{\cos v}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1+\cos v}{\cos v}$$

$$\Rightarrow \frac{\cos v}{1+\cos v}\,dv = dx$$

$$\Rightarrow \frac{\cos v\,(1-\cos v)}{1-\cos^2 v}\,dv = dx$$

$$\Rightarrow (\cot v \operatorname{cosec} v - \cot^2 v)\,dv = dx$$

$$\Rightarrow (\cot v \operatorname{cosec} v - \operatorname{cosec}^2 v + 1)\,dv = dx$$

On integration, we have

$$-\operatorname{cosec} v + \cot v + v = x + C$$

$$\Rightarrow -\operatorname{cosec}(x+y) + \cot(x+y) + x + y = x + C.$$

$$\Rightarrow -\operatorname{cosec}(x+y) + \cot(x+y) + y = C.$$

EXAMPLE 4 *Solve :* $\frac{dy}{dx} = \cos(x+y) + \sin(x+y).$

SOLUTION Let $x+y=v$. Then,

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dv}{dx}-1.$$

So the given differential equation reduces to

$$\frac{dv}{dx}-1 = \cos v + \sin v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \cos v + \sin v$$

By separating the variables, we have

$$\frac{1}{1+\cos v+\sin v}\,dv = dx$$

On integrating, we get

$$\int \frac{1}{1+\cos v+\sin v}\,dv = \int 1\cdot dx + C$$

$$\Rightarrow \int \frac{1}{1+\dfrac{1-\tan^2(v/2)}{1+\tan^2(v/2)}+\dfrac{2\tan(v/2)}{1+\tan^2(v/2)}}\,dv = x + C$$

$$\Rightarrow \int \frac{\sec^2(v/2)}{\{2\,(1+\tan(v/2)\}}\,dv = x + C$$

$$\Rightarrow \log|\,1+\tan(v/2)\,| = x + C$$

$$\Rightarrow \log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x + C,$$

which is the required solution.

EXAMPLE 5 *Solve the differential equation*

$$\sqrt{1+x+y}\,\frac{dy}{dx} = (x+y-1)$$

SOLUTION Let $1+x+y=v^2$. Then,

$$1+\frac{dy}{dx}=2v\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=2v\frac{dv}{dx}-1.$$

So, the given differential equation reduces to

$$v\left(2v\frac{dv}{dx}-1\right) = v^2-2$$

$$\Rightarrow \quad 2v^2\frac{dv}{dx} - v = v^2 - 2$$

$$\Rightarrow \quad 2v^2\frac{dv}{dx} = v^2 + v - 2$$

$$\Rightarrow \quad \frac{2v^2\,dv}{v^2+v-2} = dx$$

$$\Rightarrow \quad 2\left\{1-\frac{v-2}{v^2+v-2}\right\}dv = dx$$

$$\Rightarrow 2\left\{1-\frac{v-2}{(v+2)(v-1)}\right\}dv = dx$$

$$\Rightarrow \left\{1+\frac{1}{3}\left(\frac{1}{v-1}-\frac{4}{v+2}\right)\right\}dv = dx$$

On integrating, we get

$$2v+\frac{2}{3}\log(v-1)-\frac{8}{3}\log(v+2) = x + C$$

$$\Rightarrow 2\sqrt{x+y+1}+\frac{2}{3}\log(\sqrt{x+y+1}-1)-\frac{8}{3}\log(\sqrt{x+y+1}+2)$$

$$= x + C,$$

which is the required solution.

EXAMPLE 6 *Solve the differential equation :*

$$(x^2 + 4y^2 + 4xy)\, dy = (2x + 4y + 1)\, dx$$

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{\{2(x+2y)+1\}}{(x+2y)^2}$$

Let $x + 2y = v$. Then,

$$1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left\{\frac{dv}{dx} - 1\right\},$$

Putting the values of $x + 2y$ and $\frac{dy}{dx}$, we get,

$$\frac{1}{2}\left\{\frac{dv}{dx} - 1\right\} = \frac{2v+1}{v^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{4v+2}{v^2}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{4v+2}{v^2} + 1$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v^2 + 4v + 2}{v^2}$$

$$\Rightarrow \quad \frac{v^2}{v^2 + 4v + 2}\, dv = dx$$

$$\Rightarrow \quad \left\{1 - \frac{4v+2}{v^2+4v+2}\right\} dv = dx$$

$$\Rightarrow \quad \left\{1 - \frac{2(2v+4)-6}{v^2+4v+2}\right\} dv = dx$$

$$\Rightarrow \left\{1 - \frac{2(2v+4)}{v^2+4v+2} + \frac{6}{v^2+4v+2}\right\} dv = dx$$

On integrating, we get

$$\int 1 \,.\, dv - 2\int \frac{2v+4}{v^2+4v+2}\, dv + 6\int \frac{1}{(v+2)^2 - (\sqrt{2})^2}\, dv = \int dx + C$$

$$\Rightarrow \quad v - 2\log |\, v^2 + 4v + 2\,| + \frac{6}{2\sqrt{2}} \log \left|\frac{v+2-\sqrt{2}}{v+2+\sqrt{2}}\right| = x + C$$

$$\Rightarrow \quad (x+2y) - 2\log |\, x^2 + 4xy + 4y^2 + 4x + 8y + 2\,| + \frac{3}{\sqrt{2}} \log \left|\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right| = x + C$$

$$\Rightarrow \quad 2y - 2\log |\, x^2 + 4xy + 4y^2 + 4x + 8y + 2\,| + \frac{3}{\sqrt{2}} \log \left|\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right| = C,$$

which is the required solution.

EXAMPLE 7 *Solve the differential equation*

$$\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$$

SOLUTION Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Substituting these values in the given differential equation, we obtain

$$\left(\frac{v-a}{v-b}\right)\left(\frac{dv}{dx} - 1\right) = \frac{v+a}{v+b}$$

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{(v+a)(v-b)}{(v-a)(v+b)}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2(v^2 - ab)}{v^2 + (b-a)v - ab}$$

$$\Rightarrow \quad \frac{v^2 + (b-a)v - ab}{v^2 - ab}\, dv = 2\, dx$$

$$\Rightarrow \quad \left\{1 + \frac{(b-a)v}{v^2 - ab}\right\} dv = 2\, dx$$

$$\Rightarrow \quad \left\{1 + \left(\frac{b-a}{2}\right)\left(\frac{2v}{v^2-ab}\right)\right\} dv = 2\, dx$$

On integrating, we get

$$v + \left(\frac{b-a}{2}\right)\log |\, v^2 - ab\,| = 2x + C$$

$$\Rightarrow \quad x + y + \left(\frac{b-a}{2}\right)\log |\, (x+y)^2 - ab\,| = 2x + C$$

$$\Rightarrow \quad y + \left(\frac{b-a}{2}\right)\log \left|(x+y)^2 - ab\right| = x + C,$$

which is the required solution.

EXAMPLE 8 *Solve the differential equation*

$$\frac{dy}{dx} - x\tan(y - x) = 1.$$

SOLUTION Let $y - x = v$. Then,

$$\frac{dy}{dx} - 1 = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 + \frac{dv}{dx}.$$

Substituting these values in the given differential equation, we get

$$1 + \frac{dv}{dx} - x\tan v = 1$$

$$\Rightarrow \quad \frac{dv}{dx} = x\tan v$$

$$\Rightarrow \quad \cot v\, dv = x\, dx$$

$$\Rightarrow \quad \log |\sin v| = \frac{x^2}{2} + C$$

$\Rightarrow \log|\sin(y-x)| = \frac{x^2}{2} + C$, which is the required solution.

EXAMPLE 9 *Solve the differential equation :*

$$\frac{dy}{dx} = (x+y-1) + \frac{x+y}{\log(x+y)}$$

SOLUTION Let $x+y=v$. Then, $1+\frac{dy}{dx}=\frac{dv}{dx}$. The given differential equation reduces to

$$\frac{dv}{dx} - 1 = v + \frac{v}{\log v}$$

$$\Rightarrow \quad \frac{dv}{dx} = v + \frac{v}{\log v}$$

$$\Rightarrow \frac{\log v}{v(1+\log v)}\,dv = dx$$

On integrating, we get

$$\int \frac{\log v}{(1+\log v)}\frac{1}{v}\,dv = \int 1 \,.\, dx + C$$

$$\Rightarrow \int \frac{t-1}{t}\,dt = x + C, \text{ where } t = 1+\log v$$

$$\Rightarrow t - \log t = x + C$$

$$\Rightarrow (1+\log v) - \log\{1+\log v\} = x + C$$

EXERCISE 13.3

Solve the following differential equations by variable separable method :

1. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$
2. $(e^x+1)\,y\,dy + (y+1)\,dx = 0$
3. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
4. $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Solve the following differential equations by reducing them to variable separable form :

5. $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$
6. $(2x+y+1)\,dy + (4x+2y-1)\,dx = 0$
7. $(3x+4y-5)^2\frac{dy}{dx} = a^2$
8. $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \frac{x+y+1}{x+y+2}$
9. $\frac{dy}{dx} + 1 = e^{x+y}$

ANSWERS

1. $y = C(1-ay)(x+a)$
2. $e^{x+y} = C(y+1)(e^x+1)$
5. $y - x + \log|x+y| = C$
6. $(2x+y+1)^2 = 6x + C$
7. $\frac{1}{3}(3x+4y-5) - \frac{2a}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2a}(3x+4y-5) = x + C$
8. $y + \frac{1}{2}\log|(x+y)^2 - 2| = x + C$
9. $(x+C)\,e^{x+y} + 1 = 0$

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13.5.4 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A function $f(x, y)$ is called a homogeneous function of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

For example, $f(x, y) = x^2 - y^2 + 3xy$ is a homogeneous function of degree 2, because

$$f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 y^2 + 3\lambda x.\lambda y = \lambda^2 f(x, y).$$

A homogeneous function $f(x, y)$ of degree n can always be written as

$$f(x, y) = x^n f\left(\frac{y}{x}\right) \text{ or } f(x, y) = y^n f\left(\frac{x}{y}\right).$$

If a first-order first degree differential equation is expressible in the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation.

Such type of equations can be reduced to variable separable form by the substitution $y = vx$ as explained below :

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^n f(y/x)}{x^n g(y/x)} = \frac{f(y/x)}{g(y/x)} = F(y/x) \text{ (say)}$$

If $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting these values in $\frac{dy}{dx} = F(y/x)$, we get

$$v + x\frac{dv}{dx} = F(v)$$

$$\Rightarrow \quad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

On integrating, we get

$$\int \frac{1}{F(v) - v}\, dv = \int \frac{dx}{x} + C$$

where C is an arbitrary constant of integration.

After integration, v will be replaced by y/x to get the complete solution.

The following algorithm is helpful to solve homogeneous differential equaitons.

ALGORITHM

STEP I *Put the differential equation in the form* $\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)}$.

STEP II *Put* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *in the equation in Step I and cancel out x from the right hand side. The equation reduces to the form* $v + x\frac{dv}{dx} = F(v)$.

STEP III *Shift v on RHS and separate the variables v and x.*

STEP IV *Integrate both sides to obtain the solution in terms of v and x.*

STEP V *Replace v by* $\frac{y}{x}$ *in the solution obtained in Step IV to obtain the solution in terms of x and y.*

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve:* $(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$

SOLUTION We are given that

$$(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{3xy + y^2}{x^2 + xy}\right) \qquad \ldots(i)$$

Since each of the functions $(3xy + y^2)$ and $(x^2 + xy)$ is a homogeneous function of degree 2, the given equation is, therefore, a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -\left|\frac{3v\,x^2 + x^2 v^2}{x^2 + vx^2}\right|$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1 + v}\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1 + v} + v\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{2v^2 + 4v}{v + 1}\right\}$$

$$(v + 1)\, x\, dv = -(2v^2 + 4v)\, dx$$

$$\Rightarrow \quad \frac{v + 1}{2v^2 + 4v}\, dv = -\frac{dx}{x} \qquad \text{[By separating variables]}$$

$$\Rightarrow \quad \frac{(2v + 2)\, dv}{v^2 + 2v} = -4\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v + 2}{v^2 + 2v}\, dv = -4\int \frac{dx}{x}$$

$$\Rightarrow \quad \log |\, v^2 + 2v\, | = -4 \log x + \log C$$

$$\Rightarrow \quad \log |\, v^2 + 2v\, | = \log\left(\frac{C}{x^4}\right)$$

$$\Rightarrow \quad v^2 + 2v = \frac{C}{x^4}$$

$$\Rightarrow \quad \left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = \frac{C}{x^4} \qquad [\because\ v = y/x]$$

$$\Rightarrow \quad x^2(y^2 + 2xy) = C, \quad \text{which is the required solution.}$$

EXAMPLE 2 *Solve:* $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2 y)\, dy$.

SOLUTION We are given that

$$(x^3 - 3xy^2)\, dx = (y^3 - 3x^2 y)\, dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y}$$

Clearly, the given equation is a homogeneous equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3v^2 x^3}{v^3 x^3 - 3vx^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^4}{v^3-3v}$$

$$\Rightarrow x(v^3-3v)\,dv = (1-v^4)\,dx$$

$$\Rightarrow \frac{v^3-3v}{1-v^4}\,dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{v^3-3v}{1-v^4}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v^3}{1-v^4}\,dv - 3\int \frac{v}{1-v^4}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{4}\int \frac{-4v^3}{1-v^4}\,dv - \frac{3}{2}\int \frac{2v}{1-(v^2)^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{4}\int \frac{-4v^3}{1-v^4}\,dv - \frac{3}{2}\int \frac{dt}{1-t^2} = \int \frac{dx}{x}, \text{ where } t = v^2.$$

$$\Rightarrow -\frac{1}{4}\log|1-v^4| - \frac{3}{2}\times\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| = \log|x| + \log C$$

$$\Rightarrow -\frac{1}{4}\log|1-v^4| - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log|Cx|$$

$$\Rightarrow -\log|(1-v^4)| - 3\log\left|\frac{1+v^2}{1-v^2}\right| = 4\log|Cx|$$

$$\Rightarrow \log\left|(1-v^4)^{-1}\left(\frac{1+v^2}{1-v^2}\right)^{-3}\right| = \log|(Cx)^4|$$

$$\Rightarrow \frac{1}{1-v^4}\times\left(\frac{1-v^2}{1+v^2}\right)^3 = (Cx)^4$$

$$\Rightarrow (1-v^2)^2 = (1+v^2)^4\,(Cx)^4$$

$$\Rightarrow 1-v^2 = (1+v^2)^2\,(Cx)^2$$

$$\Rightarrow 1-y^2/x^2 = (1+y^2/x^2)^2\,C^2x^2 \qquad [\because v = y/x]$$

$$\Rightarrow x^2-y^2 = (x^2+y^2)^2\,C^2,$$

which is the required solution

EXAMPLE 3 *Solve :*

$$y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}dx - x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}dy = 0.$$

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}} \qquad ...(i)$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx\{x\cos v + vx\sin v\}}{x\{vx\sin v - x\cos v\}} = \frac{v\{\cos v + v\sin v\}}{\{v\sin v - \cos v\}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\frac{dx}{x}$$

Integrating both sides, we get

$$-\int \frac{\cos v - v\sin v}{v\cos v}\,dv = 2\int \frac{dx}{x}$$

$$\Rightarrow -\log|v\cos v| = 2\log|x| + \log C$$

$$\Rightarrow \log\frac{1}{|v\cos v|} = \log|x^2| + \log C$$

$$\Rightarrow \frac{1}{v\cos v} = Cx^2$$

$$\Rightarrow \frac{x}{y}\sec\left(\frac{y}{x}\right) = Cx^2$$

$$\Rightarrow xy\cos(y/x) = \frac{1}{C}$$

$$\Rightarrow xy\cos(y/x) = k, \text{ where } k = 1/C$$

This is the required solution.

ALITER The given differential equation can be written as

$$(y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right) = (x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow (y\,dx + x\,dy)\,x = (x\,dy - y\,dx)\,y\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{y\,dx + x\,dy}{y} = \left(\frac{x\,dy - y\,dx}{x}\right)\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{d(xy)}{xy} = \left(\frac{x\,dy - y\,dx}{x^2}\right)\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{d(xy)}{xy} = \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

On integrating, we get

$$\log|xy| = \log\left|\sec\left(\frac{y}{x}\right)\right| + \log C$$

$$\Rightarrow xy = C\sec\left(\frac{y}{x}\right)$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = C$$

EXAMPLE 4 *Solve the differential equation*

$$y\cos\left(\frac{y}{x}\right)(x\,dy - y\,dx) + x\sin\left(\frac{y}{x}\right)(x\,dy + y\,dx) = 0$$

given that $y(1) = \frac{\pi}{2}$.

SOLUTION The given differential equation can be written as

$$\left\{xy\cos\left(\frac{y}{x}\right)+x^2\sin\left(\frac{y}{x}\right)\right\}dy = \left\{y^2\cos\left(\frac{y}{x}\right)-xy\sin\left(\frac{y}{x}\right)\right\}dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left\{y\cos\left(\frac{y}{x}\right)-x\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\cos\left(\frac{y}{x}\right)+x\sin\left(\frac{y}{x}\right)\right\}}$$

Clearly, it is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{v\{v\cos v - \sin v\}}{\{v\cos v + \sin v\}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2\cos v - v\sin v - v^2\cos v - v\sin v}{v\cos v + \sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2v\sin v}{v\cos v + \sin v}$$

$$\Rightarrow \frac{v\cos v + \sin v}{v\sin v} = -\frac{2}{x}dx$$

$$\Rightarrow \int\frac{v\cos v + \sin v}{v\sin v}dv = -2\int\frac{1}{x}dx$$

$$\Rightarrow \log|v\sin v| = -2\log|x| + \log C$$

$$\Rightarrow \log|v\sin v| = \log\left(\frac{C}{x^2}\right)$$

$$\Rightarrow v\sin v = \frac{C}{x^2}$$

$$\Rightarrow xy\sin\left(\frac{y}{x}\right) = C \qquad ...(i)$$

It is given that $y = \frac{\pi}{2}$ when $x = 1$.

$$\therefore C = \frac{\pi}{2}$$

Putting $C = \frac{\pi}{2}$ in (i), we obtain

$xy\sin\left(\frac{y}{x}\right) = \frac{\pi}{2}$ as the desired solution.

ALITER The given differential equation can be written as

$$\frac{y}{x}\cot\left(\frac{y}{x}\right)(x\,dy - y\,dx) = -(x\,dy + y\,dx)$$

$$\Rightarrow \cot\left(\frac{y}{x}\right)\left(\frac{x\,dy - y\,dx}{x^2}\right) = -\left(\frac{x\,dy + y\,dx}{xy}\right)$$

$$\Rightarrow \cot\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = -\frac{d(xy)}{xy}$$

On integrating, we get

$$\log\left|\sin\left(\frac{y}{x}\right)\right| = -\log|xy| + \log C$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \frac{C}{xy}$$

$$\Rightarrow xy\sin\left(\frac{y}{x}\right) = C \qquad ...(i)$$

When $x = 1$, we have $y = \frac{\pi}{2}$. Putting these values in (i), we get

$$C = \frac{\pi}{2}$$

Putting $C = \frac{\pi}{2}$ in (i), we obtain

$xy\sin\left(\frac{y}{x}\right) = \frac{\pi}{2}$, as the desired solution.

EXAMPLE 5 *Solve the differential equation :*

$$\frac{dy}{dx} = \frac{x^2+y^2-2x-2y+2}{xy-x-y+1}$$

SOLUTION We have,

$$\frac{dy}{dx} = \frac{x^2+y^2-2x-2y+2}{xy-x-y+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x-1)^2+(y-1)^2}{(x-1)(y-1)}$$

Let $x - 1 = X$ and $y - 1 = Y$. Then,

$$\frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

So, the given differential equation reduces to

$$\frac{dv}{du} = \frac{X^2+Y^2}{XY}$$

Clearly, it is a homogeneous differential equation

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$, we get

$$V + X\frac{dV}{dX} = \frac{X^2+V^2X^2}{VX^2}$$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{1+V^2}{V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1}{V}$$

$$\Rightarrow V\,dV = \frac{dX}{X}$$

$$\Rightarrow \frac{V^2}{2} = \log|X| + C$$

$$\Rightarrow \frac{Y^2}{2X^2} = \log |X| + C$$

$$\Rightarrow \frac{(y-1)^2}{2(x-1)^2} = \log |x-1| + C$$

EXAMPLE 6 *Solve the differential equation :*

$$\frac{dy}{dx} = \frac{(x-1)^2 + (y-2)^2 \tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy-2x-y+2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$$

SOLUTION We have,

$$\frac{dy}{dx} = \frac{(x-1)^2 + (y-2)^2 \tan^{-1}\left(\frac{y-2}{x-1}\right)}{(x-1)\ (y-2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$$

Putting $x-1 = X, y-2 = Y, \frac{dy}{dx} = \frac{dY}{dX}$, we obtain

$$\frac{dY}{dX} = \frac{X^2 + Y^2 \tan^{-1}\left(\frac{Y}{X}\right)}{XY \tan^{-1}\left(\frac{Y}{X}\right)} \quad ...(i)$$

Clearly, it is a homogeneous differential equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$ in (i), we obtain

$$V + X\frac{dV}{dX} = \frac{1 + V^2 \tan^{-1} V}{V \tan^{-1} V}$$

$$\Rightarrow V + X\frac{dV}{dX} = V + \frac{1}{V \tan^{-1} V}$$

$$\Rightarrow V \tan^{-1} V\, dV = \frac{dX}{X}$$

$$\Rightarrow \int \underset{II}{V} \underset{I}{\tan^{-1} V}\, dV = \int \frac{1}{X} dX$$

$$\Rightarrow \frac{V^2}{2}\tan^{-1} V - \frac{1}{2}\int \frac{V^2}{1+V^2}\, dV = \log X + \log C$$

$$\Rightarrow \frac{V^2}{2}\tan^{-1} V - \frac{1}{2}\int \left(1 - \frac{1}{1+V^2}\right) dV = \log (CX)$$

$$\Rightarrow \frac{V^2}{2}\tan^{-1} V - \frac{1}{2}[V - \tan^{-1} V] = \log CX$$

$$\Rightarrow \frac{Y^2}{2X^2}\tan^{-1}\left(\frac{Y}{X}\right) - \frac{Y}{2X} + \frac{1}{2}\tan^{-1}\left(\frac{Y}{X}\right) = \log CX$$

$$\Rightarrow \left(\frac{Y^2 + X^2}{2X^2}\right)\tan^{-1}\left(\frac{Y}{X}\right) - \frac{Y}{2X} = \log CX$$

$$\Rightarrow [(x-1)^2 + (y-2)^2]\tan^{-1}\left(\frac{y-2}{x-1}\right) - 2(x-1)(y-2)$$

$$= \log C\{x-1\}$$

EXAMPLE 7 *Solve the differential equation :*

$$xy \log\left(\frac{x}{y}\right) dx + \left\{y^2 - x^2 \log\left(\frac{x}{y}\right)\right\} dy = 0$$

SOLUTION We have,

$$\frac{dx}{dy} = \frac{x^2 \log\left(\frac{x}{y}\right) - y^2}{xy \log\left(\frac{x}{y}\right)}$$

Clearly, it is a homogeneous differential equation.

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$, we get

$$v + y\frac{dv}{dy} = \frac{v^2 \log v - 1}{v \log v}$$

$$\Rightarrow v + y\frac{dv}{dy} = v - \frac{1}{v \log v}$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{1}{v \log v}$$

$$\Rightarrow v \log v\, dv = -\frac{dy}{y}$$

On integrating, we obtain

$$\int \underset{I}{\log v} \cdot \underset{II}{v}\, dv = -\int \frac{1}{y}\, dy$$

$$\Rightarrow (\log v)\frac{v^2}{2} - \int \frac{1}{v}\cdot\frac{v^2}{2}\, dv = -\log y + \log C$$

$$\Rightarrow \frac{v^2}{2}\log v - \frac{v^2}{4} = -\log y + \log C$$

$$\Rightarrow \frac{x^2}{2y^2}\log\left(\frac{x}{y}\right) - \frac{x^2}{4y^2} = -\log y + \log C$$

EXAMPLE 8 *Solve the differential equation*

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

SOLUTION The given differential equation can be written as

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}}$$

Putting $x = v\,y$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$, we get

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{-e^v - v}{1+e^v}$$

$$\Rightarrow \frac{e^v+1}{e^v+v}dv = -\frac{dy}{y}$$

$$\Rightarrow \log|e^v+v| = -\log|y| + \log C$$

$$\Rightarrow e^v + v = \frac{C}{y}$$

$$\Rightarrow e^{x/y} + \frac{x}{y} = \frac{C}{y}$$

EXERCISE 13.4

Solve each of the following homogeneous differential equations. If possible, also try to solve each of these by inspection method :

1. $x\sin\left(\frac{y}{x}\right)dy = \left\{y\sin\left(\frac{y}{x}\right) - x\right\}dx$
2. $x^3\,dx - y^3\,dy = 3xy\,(y\,dx - x\,dy)$
3. $x^2\,y\,dy + (x^3 + x^2y - 2xy^2 - y^3)\,dx = 0$
4. $\left(x\,e^{y/x} - y\sin\frac{y}{x}\right)dx + x\sin\left(\frac{y}{x}\right)dy = 0$
5. $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$
6. $\frac{dy}{dx} = \frac{2x-y}{x+y}$
7. $x^2\,y\,dx - (x^3+y^3)\,dy = 0$
8. $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2-x^2}$
9. $x^2y\,dy + (x^3 + x^2y - 2xy^2 - y^3)\,dx = 0$

ANSWERS

1. $\cos\left(\frac{y}{x}\right) = \log(Cx)$
2. $(x^2+y^2)^2 = C(y^2-x^2)$
3. $\log\left\{\frac{(y-x)\,C}{x^4(x+y)}\right\} = \frac{2x}{x+y}$
4. $\frac{1}{2}e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = C + \log x$
5. $\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = Cx$
6. $2x^2 - 2xy - y^2 = C$
7. $\frac{x^3}{3y^3} = \log Cy$
8. $y + \sqrt{y^2-x^2} = C\,xy$
9. $\log\left(\frac{y-x}{y+x}\right) - 4\log x - \log C = \frac{2x}{x+y}$

—❑❑❑—

13.5.5 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

In the previous section, we have studied homogeneous differential equations. In this section, we shall study those differential equations which are not homogeneous but they can be reduced to homogeneous differential equations by using some mathematical transformations.

Differential equations of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad ...(i)$$

are known as equations reducible to homogeneous form. We observe that had c_1 and c_2 been zero, the equations would have been homogeneous. So, our aim will be to remove the constant terms c_1 and c_2. For this we use the transformation

$$x = X + h \text{ and } y = Y + k \qquad ...(ii)$$

where h and k are constants which are to be determined.

Now, $\frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = 1.\frac{dY}{dX}.1 = \frac{dY}{dX} \qquad ...(iii) \quad \left[\because x = X+h \text{ and } y = Y+k \Rightarrow \frac{dx}{dX} = 1 \text{ and } \frac{dy}{dY} = 1\right]$$

Using (ii) and (iii) in equation (i), we get

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2} \qquad ...(iv)$$

$$\Rightarrow \frac{dY}{dX} = \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2k + c_2)} \qquad ...(v)$$

Let us choos h and k in such a way that

$a_1 h + b_1 k + c_1 = 0$ and, $a_2 h + b_2 k + c_2 = 0$ ...(vi)

Solving these two equations by cross-multiplication, we get

$$\frac{h}{b_1 c_2 - b_2 c_1} = \frac{k}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1},$$

provided that $a_1 b_2 - a_2 b_1 \neq 0$

$$\Rightarrow h = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, k = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

Using (vi), equation (v) reduces to

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$$

Clearly, this is a homogeneous differential equation in variables X and Y which can be solved by the substitution $Y = VX$. Replacing X and Y in the solution so obtained by $x - h$ and $y - k$ respectively, we obtain the solution of equation (i) in terms of x and y.

Following algorithm will be helpful to solve the differential equations reducible to homogeneous form.

ALGORITHM

STEP I *Obtain the differential equation. Let the diffrential equation be* $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$.

STEP II *Replace x by $X + h$, y by $Y + k$ and $\frac{dy}{dx}$ by $\frac{dY}{dX}$ in the equation obtained in step I to obtain the differential equation*

$$\frac{dY}{dX} = \frac{(a_1 X + b_1 Y) + (a_1 h + b_1 k + c_1)}{(a_2 X + b_2 Y) + (a_2 h + b_2 k + c_2)}$$

STEP III *Equate $a_1 h + b_1 k + c_1$ and $a_2 h + b_2 k + c_2$ to zero to obtain the following equations in h and k*

$$a_1 h + b_1 k + c_1 = 0$$

$$a_2 h + b_2 k + c_2 = 0$$

Solve these equations to obtain the values of h and k.

STEP IV *Substitute the values of h and k in the differential equation in step II to obtain the differential equation*

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$$

STEP V *Solve the homogeneous differential equation in step IV by substitution $Y = VX$*

STEP VI *Replace X by $x - h$ and Y by $y - k$ in the solution obtained in step V to get the required solution.*

The procedure explained above is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :* $\frac{dy}{dx} = \frac{2x - y + 3}{x + 2y + 4}$

SOLUTION. We have,

$$\frac{dy}{dx} = \frac{2x - y + 3}{x + 2y + 4} \quad ...(i)$$

Let $x = X + h$ and $y = Y + k$ Then, $\frac{dy}{dx} = \frac{dY}{dX}$.

Substituting the values of x, y and $\frac{dy}{dx}$ in (i), we get

$$\frac{dY}{dX} = \frac{(2X - Y) + (2h - k + 3)}{(X + 2Y) + (h + 2k + 4)} \quad ...(ii)$$

Let us choose h and k such that

$$2h - k + 3 = 0 \text{ and } h + 2k + 4 = 0$$

Solving these two equations, we get

$h = -2$ and $k = -1$

$\therefore$ $x = X - 2$ and $y = Y - 1$

$\Rightarrow$ $X = x + 2$ and $Y = y + 1$ ...(iii)

Putting the values of h and k in (ii), we get

$$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y} \quad ...(iv)$$

This is a homogeneous differential equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$ in (iv), we get

$$V + X\frac{dV}{dX} = \frac{2X - VX}{X + 2VX}$$

$$\Rightarrow \quad V + X\frac{dV}{dX} = \frac{2 - V}{1 + 2V}$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{2 - V}{1 + 2V} - V$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{2 - 2V - 2V^2}{2V + 1}$$

$$\Rightarrow \quad \frac{2V + 1}{2 - 2V - 2V^2}dV = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{2V + 1}{2V^2 + 2V - 2}dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{2V + 1}{V^2 + V - 1}dV = -2\frac{dX}{X}$$

Integrating both sides, we get

$$\int \frac{2V+1}{V^2+V-1}dV = -2\int \frac{dX}{X}$$

$$\Rightarrow \log(V^2+V-1) = -2\log X + \log C$$

$$\Rightarrow \log(V^2+V-1) = \log\left(\frac{C}{X^2}\right)$$

$$\Rightarrow V^2+V-1 = \frac{C}{X^2}$$

$$\Rightarrow \frac{Y^2}{X^2}+\frac{Y}{X}-1 = \frac{C}{X^2} \qquad [\because Y = VX \Rightarrow V = Y/X]$$

$$\Rightarrow Y^2+XY-X^2 = C$$

$$\Rightarrow (y+1)^2+(x+2)(y+1)-(x+2)^2 = C \qquad \text{[Using (iii)]}$$

$$\Rightarrow y^2-x^2+xy+4y-3x-1 = 0,$$

which is the required solution.

EXAMPLE 2 *Solve :* $\frac{dy}{dx} = \frac{x+y+1}{x-y-1}$

SOLUTION We have,

$$\frac{dy}{dx} = \frac{x+y+1}{x-y-1} \qquad ...(i)$$

Let $x = X+h$ and $y = Y+k$. Then, $\frac{dy}{dx} = \frac{dY}{dX}$.

Substituting the values of x, y and $\frac{dy}{dx}$ in (i), we get

$$\frac{dY}{dX} = \frac{(X+Y)+(h+k+1)}{(X-Y)+(h-k-1)} \qquad ...(ii)$$

Let us choose h and k such that

$$h+k+1 = 0 \text{ and } h-k-1 = 0$$

Solving these two equations, we get $h = 0, k = -1$.

$\therefore$ $x = X+0$ and $y = Y-1$

$\Rightarrow$ $X = x$ and $Y = y+1$ ...(iii)

Putting the values of h and k in (ii), we get

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \qquad ...(iv)$$

This is a homogeneous differential equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$ in (iv), we get

$$V+X\frac{dV}{dX} = \frac{X+VX}{X-VX}$$

$$\Rightarrow V+X\frac{dV}{dX} = \frac{1+V}{1-V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+V}{1-V}-V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+V-V+V^2}{1-V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+V^2}{1-V}$$

$$\Rightarrow \frac{1-V}{1+V^2}dV = \frac{dX}{X}$$

Integrating both sides, we get

$$\int \frac{1-V}{1+V^2}dV = \int \frac{dX}{X}$$

$$\Rightarrow \int \frac{1}{1+V^2}dV - \frac{1}{2}\int \frac{2V}{1+V^2}dV = \int \frac{dX}{X}$$

$$\Rightarrow \tan^{-1}V - \frac{1}{2}\log(1+V^2) = \log X + \log C$$

$$\Rightarrow \tan^{-1}\frac{Y}{X} = \frac{1}{2}\log\left(1+\frac{Y^2}{X^2}\right)+\log(XC) \qquad \left[\because Y = VX \atop \therefore V = Y/X\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{Y}{X}\right) = \log\left(\frac{X^2+Y^2}{X^2}\right)^{1/2}+\log(XC)$$

$$\Rightarrow \tan^{-1}\left(\frac{Y}{X}\right) = \log\left\{\frac{\sqrt{X^2+Y^2}}{X}.XC\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{Y}{X}\right) = \log\left\{C\sqrt{X^2+Y^2}\right\}$$

$$\Rightarrow C\sqrt{X^2+Y^2} = e^{\tan^{-1}(Y/X)}$$

$$\Rightarrow C\sqrt{x^2+(y+1)^2} = e^{\tan^{-1}\{(y+1)/x\}} \qquad \text{[Using (iii)]}$$

This is the required solution of the given differential equation.

EXAMPLE 3 *Solve :* $\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3}$.

SOLUTION We have,

$$\frac{dy}{dx} = \frac{2y-x-4}{y-3x+3} \qquad ...(i)$$

Let $x = X+h$ and $y = Y+k$. Then, $\frac{dy}{dx} = \frac{dY}{dX}$

Substituting the values of x, y and $\frac{dy}{dx}$ in (i), we get

$$\frac{dY}{dX} = \frac{(2Y-X)+(2k-h-4)}{(Y-3X)+(k-3h+3)} \qquad ...(ii)$$

Let us choose h and k such that

$$2k-h-4 = 0 \text{ and } k-3h+3 = 0$$

Solving these equations, we get

$h = 2$ and $k = 3$.

$\therefore$ $x = X+2$ and $y = Y+3$

$\Rightarrow$ $X = x-2$, $Y = y-3$ ...(iii)

Putting the values of h and k in (ii), we get

$$\frac{dY}{dX} = \frac{2Y - X}{Y - 3X} \quad ...(iv)$$

This is a homogeneous differential equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$ in (iv), we get

$$V + X\frac{dV}{dX} = \frac{2VX - X}{VX - 3X}$$

$$\Rightarrow \quad V + X\frac{dV}{dX} = \frac{2V - 1}{V - 3}$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{2V - 1}{V - 3} - V$$

$$\Rightarrow \quad X\frac{dV}{dX} = \frac{2V - 1 - V^2 + 3V}{V - 3}$$

$$\Rightarrow \quad \frac{V - 3}{5V - V^2 - 1}dV = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{V - 3}{V^2 - 5V + 1}dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{(2V - 5) - 1}{V^2 - 5V + 1}dV = -2\frac{dX}{X}$$

$$\Rightarrow \frac{2V - 5}{V^2 - 5V + 1}dV - \frac{1}{V^2 - 5V + 1}dV = -2\frac{dX}{X}$$

On integrating, we get

$$\int \frac{2V - 5}{V^2 - 5V + 1}dV - \int \frac{1}{V^2 - 5V + 1}dV = -2\int \frac{dX}{X}$$

$$\Rightarrow \int \frac{2V - 5}{V^2 - 5V + 1}dV - \int \frac{1}{(V - 5/2)^2 - (\sqrt{21}/2)^2}dV = -2\int \frac{dX}{X}$$

$$\Rightarrow \log\left|V^2 - 5V + 1\right| - \frac{1}{\sqrt{21}}\log\left|\frac{V - \frac{5}{2} - \frac{\sqrt{21}}{2}}{V - \frac{5}{2} + \frac{\sqrt{21}}{2}}\right|$$

$$= -2\log X + \log C$$

$$\Rightarrow \log\left|V^2 - 5V + 1\right| - \frac{1}{\sqrt{21}}\log\left|\frac{2V - 5 - \sqrt{21}}{2V - 5 + \sqrt{21}}\right| = \log\left(\frac{C}{X^2}\right)$$

$$\Rightarrow \log\left|\frac{Y^2 - 5XY + X^2}{X^2}\right| - \frac{1}{\sqrt{21}}\log\left|\frac{2Y - (5 + \sqrt{21})X}{2Y - (5 - \sqrt{21})X}\right| = \log\frac{C}{X^2}$$

$$\Rightarrow \log\left|\frac{(y-3)^2 - 5(x-2)(y-3) + (x-2)^2}{(x-2)^2}\right|$$

$$-\frac{1}{\sqrt{21}}\log\left|\frac{2(y-3) - (5 + \sqrt{21})(x-2)}{2(y-3) - (5 - \sqrt{21})(x-2)}\right| = \log\left\{\frac{C}{(x-2)^2}\right\}$$

This is the required solution of the given differential equation.

EXAMPLE 4 *Solve the differential equation :*

$$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$$

SOLUTION The given differential equation can be written as

$$\frac{y\,dy}{x\,dx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

$$\Rightarrow \frac{2y\,dy}{2x\,dx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

$$\Rightarrow \frac{dY}{dX} = \frac{2X + 3Y - 7}{3X + 2Y - 8}, \text{ where, } X = x^2 \text{ and } Y = y^2.$$

Let $X = U + h$ and $Y = V + k$ ...(i)

Then, $\frac{dX}{dU} = 1, \frac{dY}{dV} = 1$

$$\Rightarrow \frac{dY}{dX} = \frac{dY}{dV}\cdot\frac{dV}{dU}\cdot\frac{dU}{dX} = \frac{dV}{dU}$$

Substituting these values in the above differential equation, we get

$$\frac{dV}{du} = \frac{2U + 3V + 2h + 3k - 7}{3U + 2V + 3h + 2k - 8} \quad ...(ii)$$

Let us now choose h and k such that $2h + 3k - 7 = 0$ and $3h + 2k - 8 = 0$

Solving these two equations, we get $h = 2$ and $k = 1$

Substituting the values of h and k in (ii), we get

$$\frac{dV}{dU} = \frac{2U + 3V}{3U + 2V}$$

This is a homogeneous differential equation. Putting

$V = UW$ and $\frac{dV}{dU} = W + U\frac{dW}{dU}$, we get

$$W + U\frac{dW}{dU} = \frac{2 + 3W}{3 + 2W}$$

$$\Rightarrow \quad U\frac{dW}{dU} = \frac{2 - 2W^2}{3 + 2W}$$

$$\Rightarrow \frac{3 + 2W}{1 - W^2}dW = \frac{2}{U}dU$$

Integrating both sides, we get

$$\frac{3}{2}\log\left(\frac{1 + W}{1 - W}\right) - \log(1 - W^2) = 2\log U + \log C$$

$$\Rightarrow \log\left(\frac{1+W}{1-W}\right)^3 + \log(1-W^2)^{-2} = \log U^4 + \log c^2$$

$$\Rightarrow \left(\frac{1+W}{1-W}\right)^3 \times \frac{1}{(1-W^2)^2} = C^2U^4$$

$$\Rightarrow \frac{1+W}{(1-W)^5} = C^2U^4$$

$$\Rightarrow \frac{U+V}{(U-V)^5} = C^2$$

$$\Rightarrow \frac{(X-2)+(Y-1)}{(X-2-Y+1)^5} = C^2$$

$$\Rightarrow (X+Y-3) = C^2(X-Y-1)^5$$

$$\Rightarrow (x^2+y^2-3) = (x^2-y^2-1)^5C^2$$

EXAMPLE 5 *Solve the differential equation :*

$$\frac{dy}{dx} + \frac{\cos x(3\cos y - 7\sin x - 3)}{\sin y(3\sin x - 7\cos y + 7)} = 0$$

SOLUTION The given differential equation can be written as

$$\frac{\sin y}{\cos x}\frac{dy}{dx} + \frac{3\cos y - 7\sin x - 3}{3\sin x - 7\cos y + 7} = 0$$

$$\Rightarrow -\frac{\sin y}{\cos x}\cdot\frac{dy}{dx} = \frac{3\cos y - 7\sin x - 3}{3\sin x - 7\cos y + 7} \quad \ldots(i)$$

Let $\cos y = Y$ and $\sin x = X$. Then,

$$\frac{dY}{dX} = \frac{dY}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dX} = -\sin y\frac{dy}{dx}\times\frac{1}{\cos x} = -\frac{\sin y}{\cos x}\frac{dy}{dx}$$

Substituting these values in (i), we get

$$\frac{dY}{dX} = \frac{3Y - 7X - 3}{3X - 7Y + 7} \quad \ldots(ii)$$

Let $X = U + h$ and $Y = V + k$. Then,

$$\frac{dY}{dX} = \frac{dV}{dU}$$

Substituting these values in (ii), we get

$$\frac{dV}{dU} = \frac{3V - 7U - 7h + 3k - 3}{3U - 7V + 3h - 7k + 7} \quad \ldots(iii)$$

Let us choose h and k such that

$3h - 7k + 7 = 0$ and $-7h + 3k - 3 = 0$

Solving these equations, we get

$h = 0, k = 1$

$\therefore$ $X = U$ and $Y = V + 1$

Substituting the values of h and k in (iii), we get

$$\frac{dV}{dU} = \frac{3V - 7U}{3U - 7V} \quad \ldots(iv)$$

Let $V = UW$. Then, $\frac{dV}{dU} = W + U\frac{dW}{dU}$

Substituting these values in (iv), we get

$$W + U\frac{dW}{dU} = \frac{3W - 7}{3 - 7W}$$

$$\Rightarrow U\frac{dW}{dU} = \frac{7W^2 - 7}{3 - 7W}$$

$$\Rightarrow \frac{3 - 7W}{7W^2 - 7}dW = \frac{dU}{U}$$

$$\Rightarrow 7\frac{dU}{U} = \frac{-2}{W-1} - \frac{5}{W+1}$$

On integrating, we obtain

$7\log U = -2\log(W-1) - 5\log(W+1) + \log C$

$\Rightarrow U^7(W-1)^2(W+1)^5 = C$

$\Rightarrow (V-U)^2(V+U)^5 = C$

$\Rightarrow (Y-X-1)^2(X+Y-1)^5 = C$

$\Rightarrow (\cos y - \sin x - 1)^2(\sin x + \cos y - 1)^5 = C$

EXAMPLE 6 *Find the general solution of the differential equation :*

$$\frac{dy}{dx} = \left(\frac{x+2y-3}{2x+y+3}\right)^2$$

SOLUTION We have,

$$\frac{dy}{dx} = \left(\frac{x+2y-3}{2x+y+3}\right)^2$$

Let $x = X + h, y = Y + k$. Then, the above differential equation reduces to

$$\frac{dY}{dX} = \left(\frac{X+2Y+h+2k-3}{2X+Y+2h+k+3}\right)^2$$

Let us choose h, k such that $h + 2k - 3 = 0$ and $2h + k + 3 = 0$ i.e. $h = -3$ and $k = 3$. Putting $h = -3, k = 3$, the above equation reduces to

$$\frac{dY}{dX} = \left(\frac{X+2Y}{2X+Y}\right)^2$$

This is a homogeneous differential equation.

Putting $Y = VX$ and $\frac{dY}{dX} = V + X\frac{dV}{dX}$, it reduces to

$$V + X\frac{dV}{dX} = \left(\frac{1+2V}{2+V}\right)^2$$

$$\Rightarrow X\frac{dV}{dX} = \left(\frac{1+2V}{2+V}\right)^2 - V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1-V^3}{(2+V)^2}$$

$$\Rightarrow \quad \frac{(2+V)^2}{1-V^3}\,dV = \frac{dX}{X}$$

$$\Rightarrow \left(\frac{3}{1-V}+\frac{2V+1}{V^2+V+1}\right)dV = \frac{dX}{X}$$

$$\Rightarrow -3\log(1-V)+\log(V^2+V+1) = \log X + \log C$$

$$\Rightarrow \frac{V^2+V+1}{(1-V)^3} = CX$$

$$\Rightarrow X^2+XY+Y^2 = C(X-Y)^3$$

$$\Rightarrow X^3-Y^3 = C(X-Y)^4$$

$\Rightarrow (x+3)^3-(y-3)^3 = C(x-y+6)^4$, which is the general solution.

PARTICULAR CASE Consider the differential equation

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \qquad ...(i)$$

where $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ i.e. $a_1b_2-a_2b_1 = 0$.

For this type of differential equations, if we apply the method to reduce them to homogeneous form, we obtain the following equations to get the values of h and k.

$$a_1h+b_1k+c_1 = 0$$

$$a_2h+b_2k+c_2 = 0$$

These two equations are inconsistent, because $\frac{a_1}{a_2} = \frac{b_1}{b_2}$. This means that the values of h and k donot exist. So, the method of reducible to homogeneous form fails. In fact, if $a_1b_2-a_2b_1 = 0$, we have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda \text{ (say)} \Rightarrow a_1 = \lambda a_2 \text{ and } b_1 = \lambda b_2$$

Substituting the values of a_1 and b_1 in (i), we get

$$\frac{dy}{dx} = \frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}$$

This equation is in the reducible to variable separable form and can be reduced to that form by using the substitution $a_2x+b_2y = v$.

The procedure is illustrated in the following examples.

EXAMPLE 7 *Solve :* $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$

SOLUTION We have,

$$\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(3x-2y)+3}{(3x-2y)+1} \qquad ...(i)$$

Here, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ i.e. $\frac{6}{3} = \frac{-4}{-2}$. So, we make the substitution $3x-2y = v$.

Now,

$$3x-2y = v$$

$$\Rightarrow 3-2\frac{dy}{dx} = \frac{dv}{dx} \qquad \text{[Differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(3-\frac{dv}{dx}\right)$$

Substituting the values of $\frac{dy}{dx}$ and $3x-2y$ in (i), we get

$$\frac{1}{2}\left(3-\frac{dv}{dx}\right) = \frac{2v+3}{v+1}$$

$$\Rightarrow \quad 3-\frac{dv}{dx} = \frac{4v+6}{v+1}$$

$$\Rightarrow \quad \frac{dv}{dx} = 3-\frac{4v+6}{v+1}$$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{v+3}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{v+3}\,dv = -dx$$

On integrating, we get

$$\int \frac{v+1}{v+3}\,dv = -\int dx$$

$$\Rightarrow \int \frac{v+3-2}{v+3}\,dv = -\int dx$$

$$\Rightarrow \int\left(1-\frac{2}{v+3}\right)dv = -\int dx$$

$$\Rightarrow v-2\log|v+3| = -x+C$$

$$\Rightarrow (3x-2y)-2\log|3x-2y+3| = -x+C$$

[Putting $v = 3x-2y$]

$$\Rightarrow (4x-2y) = 2\log|3x-2y+3|+C$$

This is the required solution of the given differential equation.

EXAMPLE 8 *Solve :* $(2x+4y+3)\frac{dy}{dx} = 2y+x+1$

SOLUTION We have,

$$(2x+4y+3)\frac{dy}{dx} = 2y+x+1$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y+1}{2(x+2y)+3} \qquad ...(i)$$

Here, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ i.e. $\frac{1}{2} = \frac{2}{4}$.

So, we make the substitution $x+2y = v$.

Now, $x+2y = v$

$\Rightarrow 1+2\frac{dy}{dx} = \frac{dv}{dx}$ [Differentiating both sides w.r.t. x]

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{dv}{dx}-1\right)$

Substituting $x+2y=v$ and $\frac{dy}{dx}=\frac{1}{2}\left(\frac{dv}{dx}-1\right)$ in (i), we get

$\frac{1}{2}\left(\frac{dv}{dx}-1\right) = \frac{v+1}{2v+3}$

$\Rightarrow \frac{dv}{dx}-1 = \frac{2v+2}{2v+3}$

$\Rightarrow \frac{dv}{dx} = \frac{2v+2}{2v+3}+1$

$\Rightarrow \frac{dv}{dx} = \frac{4v+5}{2v+3}$

$\Rightarrow \frac{2v+3}{4v+5}dv = dx$

On integrating, we get

$\int \frac{2v+3}{4v+5}dv = \int dx$

$\Rightarrow \int \frac{4v+6}{4v+5}dv = 2\int dx$

$\Rightarrow \int\left(\frac{4v+5+1}{4v+5}\right)dv = 2\int dx$

$\Rightarrow \int\left(1+\frac{1}{4v+5}\right)dv = 2\int dx$

$\Rightarrow v+\frac{1}{4}\log(4v+5) = 2x+C$

$\Rightarrow x+2y+\frac{1}{4}\log|4(x+2y)+5| = 2x+C$ [Replacing v by $x+2y$]

$\Rightarrow 4x+8y+\log(4x+8y+5) = 8x+4C$

$\Rightarrow \log(4x+8y+5) = 4x-8y+C_1$, where $C_1 = 4C$.

This is the required solution of the given differential equation.

EXAMPLE 9 *Solve :* $(3x-2y+1)\,dy+(4y-6x+3)\,dx = 0$

SOLUTION We have,

$(3x-2y+1)\,dy+(4y-6x+3)\,dx = 0$

$\Rightarrow (3x-2y+1)\,dy = -(4y-6x+3)\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{6x-4y-3}{3x-2y+1}$ ...(i)

Here, $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ i.e. $\frac{6}{3}=\frac{-4}{-2}$.

So, we make the substitution $3x-2y=v$.

Now, $3x-2y=v$

$\Rightarrow 3-2\frac{dy}{dx} = \frac{dv}{dx}$ [Diff. both sides w.r.t. x]

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(3-\frac{dv}{dx}\right)$

Putting $3x-2y = v$ and $\frac{dy}{dx} = \frac{1}{2}\left(3-\frac{dv}{dx}\right)$ in (i), we get

$\frac{1}{2}\left(3-\frac{dv}{dx}\right) = \frac{2v-3}{v+1}$

$\Rightarrow 3-\frac{dv}{dx} = \frac{4v-6}{v+1}$

$\Rightarrow \frac{dv}{dx} = 3-\frac{4v-6}{v+1}$

$\Rightarrow \frac{dv}{dx} = \frac{-v+9}{v+1}$

$\Rightarrow \frac{dv}{dx} = -\left(\frac{v-9}{v+1}\right)$

$\Rightarrow \frac{v+1}{v-9}dv = -dx$

$\Rightarrow \frac{v-9+10}{v-9}dv = -dx$

$\Rightarrow \left(1+\frac{10}{v-9}\right)dv = -dx$

On integration, we get

$\int\left(1+\frac{10}{v-9}\right)dv = -\int dx$

$\Rightarrow v+10\log|v-9| = -x+C$

$\Rightarrow (3x-2y)+10\log|3x-2y-9| = -x+C$

$\Rightarrow 10\log|3x-2y-9| = 2y-4x+C$

This is the required solution of the given differential equation.

EXERCISE 13.5

Solve each of the following differential equations by reducing them to homogeneous form :

1. $(4x+3y+1)\,dx+(3x+2y+1)\,dy = 0$
2. $(3x+2y-5)\,dx+(2x+3y-5)\,dy = 0$
3. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$
4. $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$

5. $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

6. $(4x+6y+3)\,dx = (6x+9y+2)\,dy$

7. $(2x-3y+4)\frac{dy}{dx} = (2y-3x-1)$

8. $(x+y)\,dy + (5x-3y+8)\,dx = 0$

9. $(2x+3y+4)\,dx - (4x+6y+5)\,dy = 0$

10. $(x+y-1)\,dy = (x+y)\,dx$

11. $\frac{dy}{dx} = \frac{3y-x+7}{x-7y-3}$

12. $\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$

13. $\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$

14. $\frac{dy}{dx} = \frac{3y+2x+4}{4x+6y+5}$

15. $\frac{dy}{dx} = \frac{8x+6y+12}{4x+3y+2}$

16. $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$

17. $\frac{dy}{dx} = \frac{1-3x-3y}{2(x+y)}$

18. $\frac{dy}{dx} = \frac{x+y}{x+y-2}$

19. $\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$

20. $(x^5 + x^2y^3 + x^2)\,dx - (2y^5 + 2y^2x^3 + y^2)\,dy = 0$

21. $(x^2+y^3)(2x^2\,dx + 3y\,dy) = 12x\,dx + 18y^2\,dy$

ANSWERS

1. $2x^2 + 3xy + 2y^2 + x + y = C$

2. $3x^2 + 3y^2 + 4xy - 10x - 10y = C$

3. $(x+y-2) = C(x-y)^3$

4. $(x+2y-5) = C(2x-y)^2$

5. $\log X = -\frac{1}{2}\log(1-3V^2) + \frac{1}{3}\log\left(\frac{1+\sqrt{3}V}{1-\sqrt{3}V}\right) + C$,
where $V = \frac{Y}{X}$, $X = x-1$ and $Y = y+2$.

6. $(24x-36y) + 5\log(24x+6y+13) + C = 0$

7. $(x-y+1)(x+y-3)^5 = C$

8. $\log(5x^2 - 2xy + y^2 + 12x - 4y + 1) + \tan^{-1}\left(\frac{x+y-2}{2(x-1)}\right) = C$

9. $9\log(14x+21y-22) = 42y - 21x + C$

10. $2(y-x) - \log(2x+2y-1) = 2C$

11. $\frac{3}{4}\log\left|\frac{y-x+1}{y+x-1}\right| = \log\left\{k\sqrt{y^2-(x-1)^2}\right\}$.

12. $(y-x+1)^2(y+x-1)^5 = C$

13. $y - 2x + \frac{3}{8}\log(24y+16x+23) = C$

14. $2y - x - \frac{3}{7}\log(14x+21y+72) = C$

15. $5(3y-6x) - 12\log(20x+15y+22) = 10x + C$

16. $2y + x + \log(2x+y-1) = C$

17. $3x + 2y + C + 2\log(1-x-y) = 0$

18. $y - x = \log(x+y-1) + C$

19. $2y - x + \log(x+2y) = C$

20. $6x^3 + 6y^3 - \log|6x^3 + 6y^3 + 4| = 9x^3 + C$

21. $\frac{x^3}{9} + \frac{y^2}{4} = \log|x^2+y^3| + C$

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13.5.6 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree. The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q, \qquad \text{...(i)}$$

where P and Q are functions of x (or constants)

For example,

$$\frac{dy}{dx} + xy = x^3,\ x\frac{dy}{dx} + 2y = x^3,\ \frac{dy}{dx} + 2y = \sin x \text{ etc.}$$

are linear differential equations.

This type of differential equations are solved when they are multiplied by a factor, which is called integrating factor, because by multiplication of this factor the left hand side of the differential equation (i) becomes exact differential of some function.

Multiplying both sides of (i) by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Q\,e^{\int P\,dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left\{y\,e^{\int P\,dx}\right\} = Q\,e^{\int P\,dx}$$

On integrating both sides w.r.t. x, we get

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + C \qquad \text{...(ii)}$$

which is the required solution, where C is the constant of integration.

$e^{\int P\,dx}$ is called the integrating factor.

The solution (ii) in short may also be written as

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + C$$

To solve linear differential equqations we may use the following algorithm.

ALGORITHM

STEP I *Write the differential equation in the form* $\frac{dy}{dx} + Py = Q$ *and obtain P and Q.*

STEP II *Find integrating factor (I.F.) given by I.F.* $= e^{\int P dx}$

STEP III *Multiply both sides of equation in step I by I.F.*

STEP IV *Integrate both sides of the equation obtained in step III w.r.t. x to obtain*

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C$$

This gives the required solution

13.5.7 LINEAR DIFFERENTIAL EQUATIONS OF THE FORM

$$\frac{dx}{dy} + Rx = S$$

Sometimes a linear differential equation can be put in the form

$$\frac{dx}{dy} + Rx = S,$$

where R and S are functions of y or constants.

Note that y is independent variable and x is a dependent variable.

The following algorithm is used to solve these types of equations.

ALGORITHM

STEP I *Write the differential equation in the form* $\frac{dx}{dy} + Rx = S$ *and obtain R and S.*

STEP II *Find I.F. by using I.F.* $= e^{\int R\, dy}$

STEP III *Multiply both sides of the differential equation in Step I by I.F.*

STEP IV *Integrate both sides of the equation obtained is Step III w.r.t. y to obtain the solution given by*

$$x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C$$

where C is the constant of integration.

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :* $\frac{dy}{dx} + y = \cos x - \sin x$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \qquad \ldots(i)$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = \cos x - \sin x$.

$\therefore$ I.F. $= e^{\int P dx} = e^{\int 1\, dx} = e^x$

Multiplying both sides of (i) by I.F. $= e^x$, we get

$$e^x \frac{dy}{dx} + y = e^x (\cos x - \sin x)$$

Integrating both sides w.r.t. x, we get

$$ye^x = \int e^x (\cos x - \sin x)\, dx + C \quad \left[\begin{matrix}\text{Using :} \\ y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\end{matrix}\right]$$

$$\Rightarrow ye^x = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\cos x}\, dx - \int e^x \sin x\, dx + C$$

$$\Rightarrow ye^x = e^x \cos x - \int -\sin x\, e^x\, dx - \int e^x \sin x + C$$

$$\Rightarrow ye^x = e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx + C$$

$$\Rightarrow ye^x = e^x \cos x + C, \text{ which is the required solution.}$$

EXAMPLE 2 *Solve :* $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x \qquad \ldots(i)$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \tan x$ and $Q = 2x + x^2 \tan x$.

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

Multiplying both sides of (i) by I.F. $= \sec x$, we get

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2x \sec x + x^2 \sec x \tan x$$

Integrating both sides w.r.t. x, we get

$$y \sec x = \int (2x \sec x + x^2 \sec x \tan x)\, dx + C$$

$$\Rightarrow y \sec x = \int 2x \sec x\, dx + \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sec x \tan x}\, dx + C$$

$$\Rightarrow y \sec x = \int 2x \sec x\, dx + x^2 \sec x - \int 2x \sec x\, dx + C$$

$$\Rightarrow y \sec x = x^2 \sec x + C, \text{ which is the required solution.}$$

EXAMPLE 3 *Solve :* $\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = \cos x + \frac{\sin x}{x} \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{x} \text{ and } Q = \cos x + \frac{\sin x}{x}$$

$$\therefore \text{ I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$$

Multiplying both sides of (i) by I.F. = x, we get

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

Integrating both sides w.r.t. x, we get

$$yx = \int (x\cos x + \sin x)\, dx + C$$

$$\Rightarrow xy = \int \underset{\text{I}}{x}\underset{\text{II}}{\cos x}\, dx + \int \sin x\, dx + C$$

$$\Rightarrow xy = x\sin x - \int \sin x\, dx + \int \sin x\, dx + C$$

$$\Rightarrow xy = x\sin x + C, \text{ which is the required solution.}$$

EXAMPLE 4 *Solve:* $y\,dx - (x + 2y^2)\,dy = 0$

SOLUTION The given differential equation is

$$y\,dx - (x + 2y^2)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x + 2y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = -\frac{1}{y} \text{ and } S = 2y.$$

$$\therefore \text{ I.F.} = e^{\int R dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$$

Multiplying both sides of (i) by I.F. y^{-1}, we obtain

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2.$$

Integrating both sides w.r.t. y, we get

$$x\cdot\frac{1}{y} = \int 2\, dy + C \qquad [\text{Using : } x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C]$$

$$\Rightarrow \frac{x}{y} = 2y + C, \text{ which is the required solution.}$$

EXAMPLE 5 *Solve:* $y\,dx + (x - y^3)\,dy = 0$

SOLUTION The given differential equation is

$$y\,dx + (x - y^3)\,dy = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y^2 \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = \frac{1}{y} \text{ and } S = y^2.$$

$$\therefore \text{ I.F.} = e^{\int R dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$$

Multiplying both sides of (i) by I.F. = y, we obtain

$$y\frac{dx}{dy} + x = y^3$$

Integrating both sides w.r.t. y, we get

$$xy = \int y^3\, dy + C \qquad [\text{Using : } x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C]$$

$$\Rightarrow xy = \frac{y^4}{4} + C, \text{ which is the required solution.}$$

EXAMPLE 6 *Solve:* $(x + 2y^3)\,dy = y\,dx.$

SOLUTION The given differential equation can be written as

$$\frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y^2 \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = \frac{-1}{y} \text{ and } S = 2y^2.$$

We have

$$\text{I.F.} = e^{\int R dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

Multiplying both sides of (i) by I.F. = $\frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2y$$

Integrating both sides w.r.t. y, we get

$$x\left(\frac{1}{y}\right) = \int 2y\, dy + C \qquad [\text{Using : } x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C]$$

$$\Rightarrow \frac{x}{y} = y^2 + C, \text{ which is the required solution.}$$

EXAMPLE 7 *If y_1 and y_2 are the solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone and $y_2 = y_1 z$, then prove that $z = 1 + C\,e^{\int \frac{-Q}{y_1} dx}$, where C is an arbitrary constant.*

SOLUTION Since y_1 and y_2 are the solutions of the differential equation

$$\frac{dy}{dx} + Py = Q$$

$$\therefore \frac{dy_1}{dx} + Py_1 = Q \text{ and } \frac{dy_2}{dx} + Py_2 = Q$$

Now,

$$\frac{dy_2}{dx} + Py_2 = Q$$

$$\Rightarrow \frac{d}{dx}(y_1 z) + P y_1 z = Q \qquad [\because y_2 = y_1 z]$$

$$\Rightarrow \left(\frac{dy_1}{dx} + P y_1\right) z + y_1 \frac{dz}{dx} = Q$$

$$\Rightarrow Qz + y_1 \frac{dz}{dx} = Q \qquad \left[\because \frac{dy}{dx} + P y_1 = Q\right]$$

$$\Rightarrow \frac{dz}{dx} + \frac{Q}{y_1} z = \frac{Q}{y_1}$$

This is a linear differential equation with I.F. $= e^{\int Q/y_1\, dx}$

Multiplying both sides of this equation by I.F. and integrating both sides with respect to x, we get

$$z\, e^{\int Q/y_1\, dx} = \int \frac{Q}{y_1} e^{\int (Q/y_1)\, dx}\, dx + C$$

$$\Rightarrow z\, e^{\int Q/y_1\, dx} = e^{\int (Q/y_1)\, dx} + C$$

$$\Rightarrow z = 1 + C e^{-\int (Q/y_1)\, dx}$$

EXAMPLE 8 *If y_1, y_2 are two solutions of the differential equation $\frac{dy}{dx} + P(x) \cdot y = Q(x)$, then prove that $y = y_1 + C(y_1 - y_2)$ is the general solution of the equation where C is an arbitrary constant. For what relation between the constants a, b will the linear combination $a y_1 + b y_2$ also be a solution ?*

SOLUTION Since y_1 and y_2 are two solutions of the differential equation

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \qquad \ldots(i)$$

Therefore,

$$\frac{dy_1}{dx} + P(x) \cdot y_1 = Q(x) \qquad \ldots(ii)$$

and,

$$\frac{dy_2}{dx} + P(x) \cdot y_2 = Q(x) \qquad \ldots(iii)$$

Subtracting (ii) from (i) and (iii), from (ii), we obtain

$$\frac{d}{dx}(y - y_1) + P(x) \cdot (y - y_1) = 0 \qquad \ldots(iv)$$

and, $\frac{d}{dx}(y_1 - y_2) + P(x)(y_1 - y_2) = 0 \qquad \ldots(v)$

$$\Rightarrow \frac{d}{dx}(y - y_1) = -P(x) \cdot (y - y_1)$$

and, $\frac{d}{dx}(y_1 - y_2) = -P(x) \cdot (y_1 - y_2)$

$$\Rightarrow \frac{\frac{d}{dx}(y - y_1)}{\frac{d}{dx}(y_1 - y_2)} = \frac{y - y_1}{y_1 - y_2}$$

$$\Rightarrow \frac{d(y - y_1)}{y - y_1} = \frac{d(y_1 - y_2)}{y_1 - y_2}$$

On integrating, we get

$$\log(y - y_1) = \log(y_1 - y_2) + \log C$$

$$\Rightarrow y - y_1 = C(y_1 - y_2)$$

$$\Rightarrow y = y_1 + C(y_1 - y_2)$$

Hence, $y = y_1 + C(y_1 - y_2)$ is the general solution of the given differential equation.

Now, $y = a y_1 + b y_2$ will be a solution of the given differential equation, if

$$\frac{d}{dx}(a y_1 + b y_2) + P(x) \cdot (a y_1 + b y_2) = Q(x)$$

$$\Rightarrow a\left\{\frac{dy_1}{dx} + P(x) \cdot y_1\right\} + b\left\{\frac{dy_2}{dx} + P(x) \cdot y_2\right\} = Q(x)$$

$$\Rightarrow a \cdot Q(x) + b \cdot Q(x) = Q(x) \qquad \text{[From (ii) and (iii)]}$$

$$\Rightarrow a + b = 1$$

Hence, $y = a y_1 + b y_2$ will also be a solution, if $a + b = 1$.

EXAMPLE 9 *Let $u(x)$ and $v(x)$ satisfy the differential equations*

$$\frac{du}{dx} + p(x) \cdot u = f(x) \text{ and } \frac{dv}{dx} + p(x) \cdot v = g(x)$$

respectively where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y), where $x > x_1$, does not satisfy the equations $y = u(x)$ and $y = v(x)$.

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SOLUTION We have,

$$\frac{du(x)}{dx} + p(x) \cdot u(x) = f(x)$$

and, $\frac{dv(x)}{dx} + p(x)\, v(x) = g(x)$

$$\therefore \frac{d}{dx}\{u(x) - v(x)\} + p(x)\{u(x) - v(x)\} = f(x) - g(x)$$

$$\Rightarrow \frac{d}{dx}\{u(x) - v(x)\} \cdot e^{\int p\, dx} + p(x)\{u(x) - v(x)\} e^{\int p\, dx}$$

$$= \{f(x) - g(x)\} e^{\int p\, dx}$$

$$\Rightarrow \frac{d}{dx}\left[\{u(x) - v(x)\} e^{\int p\, dx}\right] = \{f(x) - g(x)\} e^{\int p\, dx}$$

Integrating both sides over the interval $[x_1, x]$, we get

$$\left[\{u(x) - v(x)\} e^{\int p\, dx}\right]_{x_1}^{x} = \int_{x_1}^{x} \left(\{f(x) - g(x)\} e^{\int p\, dx}\right) dx$$

$\Rightarrow \left[\{u(x)-v(x)\}\, e^{\int p\,dx}\right]_{x_1}^{x} > 0$ $\left[\because f(x) > g(x) \text{ for all } x > x_1 \text{ and } e^{\int p\,dx} > 0\right]$

$\Rightarrow \{u(x)-v(x)\}\lambda - \{u(x_1)-v(x_1)\}\mu > 0$, where

$\lambda = \left(e^{\int p\,dx}\right)_x$ and $\mu = \left(e^{\int p\,dx}\right)_{x_1}$.

$\Rightarrow \{u(x)-v(x)\}\lambda > \{u(x_1)-v(x_1)\}\mu$

$\Rightarrow \{u(x)-v(x)\}\lambda > 0$ for all $x > x_1$ $[\because u(x_1) > v(x_1)]$

$\Rightarrow \{u(x)-v(x)\} > 0$ for all $x > x_1$

$\Rightarrow u(x) \neq v(x)$ for $x > x_1$.

EXERCISE 13.6

1. $(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$
2. $\frac{dy}{dx} + y \tan x = \cos x$
3. $\frac{dy}{dx} + y \cot x = \cos x$
4. $\frac{dy}{dx} + y \tan x = x^2 \cos^2 x$
5. $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$
6. $x\,dy = (2y + 2x^4 + x^2)\,dx$
7. $(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$
8. $y^2\frac{dx}{dy} + x - \frac{1}{y} = 0$
9. $(2x - 10y^3)\frac{dy}{dx} + y = 0$
10. $(x + \tan y)\,dy = \sin 2y\,dx$
11. $dx + x\,dy = e^{-y}\sec^2 y\,dy$
12. $\frac{dy}{dx} = y \tan x - 2 \sin x$
13. $\frac{dy}{dx} + y \cos x = \sin x \cos x$
14. $(1+x^3)\frac{dy}{dx} + 6x^2 y = (1+x^2)$
15. $(\sin x)\frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$

ANSWERS

1. $y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x}$
2. $y \sec x = x + C$
3. $y \sin x = \frac{1}{2}\sin^2 x + C$
4. $y \sec x = x^2 \sin x + 2x \cos x - 2 \sin x + C$
5. $2ye^{\tan^{-1} x} = e^{2\tan^{-1} x} + C$
6. $y = x^4 + x^2 \log x + Cx^2$
7. $2xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + C$
8. $y = \left(\frac{y+1}{y}\right) + Ce^{1/y}$
9. $x = 2y^3 + Cy^{-2}$
10. $x = \tan y + C\sqrt{\tan y}$
11. $xe^y = \tan y + C$
12. $y = \sec x\,(-\sin^2 x + C)$
13. $y = \sin x - 1 + C\,e^{-\sin x}$
14. $y(1+x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$
15. $y \sin x = \frac{2}{3}\sin^3 x + C$

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13.5.7 EQUATIONS REDUCIBLE TO LINEAR FORM (BERNOULLI'S DIFFERENTIAL EQUATIONS

The equations of the form

$$\frac{dy}{dx} + Py = Q\,y^n \qquad \ldots(i)$$

where P and Q are constants or functions of x alone and n is a non-zero constant other than unity, are known as Bernoulli's equations.

ALGORITHM OT SOLVE BERNOULLI'S DIFFERENTIAL EQUAIONS

STEP I *Obtain the differential equation. Let it be*

$$\frac{dy}{dx} + Py = Q\,y^n, \text{ where } n \neq 1.$$

STEP II *Divide both sides by y^n to get*

$$\frac{1}{y^n}\frac{dy}{dx} + P.\frac{1}{y^{n-1}} = Q$$

$$\Rightarrow\ y^{-n}\frac{dy}{dx}+P\,y^{-n+1}=Q$$

STEP III *Put y^{-n+1} i.e. coefficient of P equal to v and obtain $\frac{dy}{dx}$ from it in terms of $\frac{dv}{dx}$. as follows :*

$$y^{-n+1}=v\ \Rightarrow\ (-n+1)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$$

STEP IV *Substitute the values of $\frac{dy}{dx}$ and v in the differential equation in step II, to obtain*

$$\frac{1}{1-n}\frac{dv}{dx}+Pv=Q$$

$$\Rightarrow\ \frac{dv}{dx}+(1-n)\,Pv=(1-n)\,Q$$

$$\Rightarrow\ \frac{dv}{dx}+P_1v=Q_1 \qquad ...(i)$$

This is a linear differential equation in v and x , where

$$P_1=(1-n)\,P \text{ and } Q_1=(1-n)\,Q.$$

STEP V *Find* I.F. $=e^{\int P_1\,dx}$

STEP VI *Multiply both sides of (i) in step IV by* I.F. $=e^{\int P_1\,dx}$ *to get*

$$\frac{dv}{dx}e^{\int P_1\,dx}+P_1\,v\,e^{\int P_1\,dx}=Q_1\,e^{\int P_1\,dx}$$

STEP VII *Integrate both sides of the differential equation in step VI w.r.to x, to get*

$$v\,e^{\int P_1\,dx}=\int(Q_1\,e^{\int P_1\,dx})\,dx+C$$

$$\Rightarrow\ v\,.\,(\text{I.F.})=\int Q_1\,.\,(\text{I.F.})\,dx+C$$

STEP VIII *Replace v by y^{-n+1} in the solution obtained in step VI to get the desired solution of the given differential equation.*

REMARK *For n = 1, the equation (i) reduces to variable separable form.*

The following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :* $x\frac{dy}{dx}+y=y^2\log x$

SOLUTION We have,

$$x\frac{dy}{dx}+y=y^2\log x$$

$$\frac{dy}{dx}+\frac{1}{x}y=\left(\frac{\log x}{x}\right)y^2$$

This is in the form of Bernoulli's differential equation. Dividing both sides by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx}+\left(\frac{1}{x}\right)\frac{1}{y}=\frac{\log x}{x}$$

Let $\frac{1}{y}=v$. Then,

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Substituting these in the given differential equation, we get

$$\Rightarrow\ -\frac{dv}{dx}+\frac{1}{x}v=\frac{\log x}{x}$$

$$\Rightarrow\ \frac{dv}{dx}+\left(-\frac{1}{x}\right)v=-\frac{\log x}{x} \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dv}{dx}+Pv=Q, \text{ where}$$

$$P=-\frac{1}{x} \text{ and } Q=-\frac{\log x}{x}.$$

We have,

$$\text{I.F.}=e^{\int P\,dx}=e^{\int-\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}.$$

Multiplying both sides of (i) by $\frac{1}{x}$, we get

$$\frac{1}{x}\frac{dv}{dx}+\left(-\frac{1}{x^2}\right)v=\frac{-\log x}{x^2}$$

Integrating both sides w.r.t. x, we get

$$v\left(\frac{1}{x}\right)=\int-\frac{\log x}{x^2}\,dx+C \quad [\text{Using : } v\,(I.F.)=\int Q\,(I.F.)\,dx+C]$$

$$\Rightarrow\ v\,.\,\frac{1}{x}=-\int \underset{I}{\log x}\,.\,\underset{II}{x^{-2}}\,dx+C$$

$$\Rightarrow\ v\,.\,\frac{1}{x}=-\left[\log x\left(-\frac{1}{x}\right)-\int\frac{1}{x}\left(-\frac{1}{x}\right)dx\right]+C$$

$$\Rightarrow\ v\,.\,\frac{1}{x}=-\left[-\frac{\log x}{x}+\int\frac{1}{x^2}\,dx\right]+C$$

$$\Rightarrow\ v\,.\,\frac{1}{x}=-\left[-\frac{\log x}{x}-\frac{1}{x}\right]+C$$

$$\Rightarrow\ v\,.\,\frac{1}{x}=\left(\frac{\log x+1}{x}\right)+C$$

$\Rightarrow \frac{1}{xy} = \frac{(\log x + 1)}{x} + C$ [Putting $v = 1/y$]

This is the required solution of the given differential equation.

EXAMPLE 2 *Solve :* $\frac{dy}{dx} = x^3 y^3 - xy$

SOLUTION We have,

$$\frac{dy}{dx} = x^3 y^3 - xy$$

$$\Rightarrow \frac{dy}{dx} + xy = x^3 y^3$$

This is the form of Bernouli's differential equation.

Dividing both sides by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + x\left(\frac{1}{y^2}\right) = x^3$$

Let $\frac{1}{y^2} = v$. Then,

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx},$$

Substituting these in the given differential equations we get

$$-\frac{1}{2}\frac{dv}{dx} + x \cdot v = x^3$$

$$\Rightarrow \frac{dv}{dx} + (-2x) = -2x^3 \quad ...(i)$$

This is a linear differential equation of the form,

$\frac{dv}{dx} + Pv = Q$, where $P = -2x$ and $Q = -2x^3$.

We have, I.F. $= e^{\int P\,dx} = e^{\int -2x\,dx} = e^{-x^2}$

Multiplying both sides of (i) by I.F $= e^{-x^2}$, we get

$$e^{-x^2}\frac{dv}{dx} + (-2x)\,v\,e^{-x^2} = -2x^3 e^{-x^2}$$

Integrating both sides w.r.t. x, we get

$$v e^{-x^2} = \int -2x^3 e^{-x^2}\,dx + C \qquad \left[\begin{array}{l}\text{Using :}\\ v\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\end{array}\right]$$

$$\Rightarrow v e^{-x^2} = \int -2x^3 e^t \left(\frac{dt}{2x}\right) + C, \text{ where } t = -x^2$$

$$\Rightarrow v e^{-x^2} = -\int \underset{I}{t}\,\underset{II}{e^t}\,dt + C$$

$$\Rightarrow v e^{-x^2} = -[t e^t - e^t] + C$$

$$\Rightarrow v e^{-x^2} = -e^t (t - 1) + C$$

$$\Rightarrow v e^{-x^2} = -e^{-x^2}(-x^2 - 1) + C \qquad [\because t = -x^2]$$

$$\Rightarrow v e^{-x^2} = e^{-x^2}(x^2 + 1) + C$$

$$\Rightarrow \frac{1}{y^2} e^{-x^2} = e^{-x^2}(x^2 + 1) + C \qquad [\because v = 1/y^2]$$

This is the required solution of the given differential equation.

EXAMPLE 3 *Solve :* $\tan y \frac{dy}{dx} + \tan x = \cos y \cdot \cos^3 x$

SOLUTION We have,

$$\tan y \frac{dy}{dx} + \tan x = \cos y \cdot \cos^3 x$$

Dividing both sides by $\cos y$, we get

$$\sec y \tan y \frac{dy}{dx} + \tan x \sec y = \cos^3 x$$

Putting $\sec y = v$ and $\sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$, we get

$$\frac{dv}{dx} + (\tan x)\,v = \cos^3 x \quad ...(i)$$

This is a linear differential equation of the form

$$\frac{dv}{dx} + Pv = Q, \quad \text{where } P = \tan x \text{ and } Q = \cos^3 x.$$

We have, I.F. $= e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$

Multiplying both sides of (i) by I.F. $= \sec x$, we get

$$\sec x \frac{dv}{dx} + (\sec x \tan x)\,v = \cos^2 x$$

Integrating both sides w.r.t., x we get

$$v \cdot \sec x = \int \cos^3 x \cdot \sec x\,dx + C$$

$$\Rightarrow v \sec x = \int \cos^2 x\,dx + C$$

$$\Rightarrow v \sec x = \frac{1}{2}\int (1 + \cos 2x) + dx + C$$

$$\Rightarrow v \sec x = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

$$\Rightarrow \sec y \cdot \sec x = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C \qquad [\because v = \sec y]$$

This is the required solution of the given differential equation.

EXAMPLE 4 *Solve :* $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

SOLUTION We have,

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

Dividing both sides by $\cos^2 y$, we get

$$\sec^2 y \frac{dy}{dx} + x\frac{\sin 2y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

Putting $\tan y = v$ and $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$, we get

$$\frac{dv}{dx} + (2x)\, v = x^3 \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dv}{dx} + Pv = Q, \text{ where } P = 2x \text{ and } Q = x^3.$$

Now, I.F. $= e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$.

Multiplying both sides of (i) by I.F. $= e^{x^2}$, we get

$$e^{x^2}\frac{dv}{dx} + 2x\, v \,.\, e^{x^2} = e^{x^2} \,.\, x^3$$

Integrating both sides w.r.t. x, we get

$$v\, e^{x^2} = \int x^3 e^{x^2}\, dx + C \quad [\text{Using} : v\,(\text{I.F.}) = \int Q.\,(\text{I.F.})\, dx + C]$$

$$\Rightarrow \quad v\, e^{x^2} = \int x^3 e^t \left(\frac{dt}{2x}\right) + C, \text{ where } t = x^2$$

$$\Rightarrow \quad v\, e^{x^2} = \frac{1}{2}\int \underset{\text{I}}{t}\ \underset{\text{II}}{e^t}\, dt + C$$

$$\Rightarrow \quad v\, e^{x^2} = \frac{1}{2}(t\, e^t - e^t) + C$$

$$\Rightarrow \quad v\, e^{x^2} = \frac{1}{2}(t-1)\, e^t + C$$

$$\Rightarrow \quad v\, e^{x^2} = \frac{1}{2}(x^2 - 1)\, e^{x^2} + C \qquad [\because t = x^2]$$

$$\Rightarrow \quad e^{x^2} \tan y \,.\, = \frac{1}{2}(x^2-1)\, e^{x^2} + C \qquad [\because \tan y = v]$$

This is the required solution of the given differential equation.

EXAMPLE 5 *Solve* : $3x(1-x^2)\, y^2 \frac{dy}{dx} + (2x^2 - 1)\, y^3 = ax^3$.

SOLUTION We have,

$$3x(1-x^2)\, y^2 \frac{dy}{dx} + (2x^2-1)\, y^3 = ax^3$$

Putting $y^3 = v$ and $3y^2 \frac{dy}{dx} = \frac{dv}{dx}$, we get

$$3x(1-x^2)\left(\frac{1}{3}\frac{dv}{dx}\right) + (2x^2-1)\, v = ax^3$$

$$\Rightarrow \quad x(1-x^2)\frac{dv}{dx} + (2x^2-1)\, v = ax^3$$

$$\Rightarrow \quad \frac{dv}{dx} + \frac{2x^2-1}{x(1-x^2)}\, v = \frac{ax^2}{(1-x^2)} \qquad \ldots(i)$$

This is a linear differential equation of the form

$$\frac{dv}{dx} + Pv = Q, \text{ where } P = \frac{2x^2-1}{x(1-x^2)} \text{ and } Q = \frac{ax^2}{(1-x^2)}$$

Now,

$$\text{I.F.} = e^{\int P\,dx}$$

$$= e^{\int \frac{2x^2-1}{x(1-x^2)}\,dx}$$

$$= e^{\int\left(-\frac{1}{x} + \frac{1}{2} - \frac{1}{1-x} - \frac{1}{2}\cdot\frac{1}{1+x}\right)dx} \qquad [\text{Using Partial fractions}]$$

$$= e^{-\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)}$$

$$= e^{-\frac{1}{2}\log x^2 (1-x^2)}$$

$$= e^{\log |x^2(1-x^2)|^{-1/2}}$$

$$= \frac{1}{\sqrt{x^2(1-x^2)}} = \frac{1}{x\sqrt{1-x^2}}.$$

Multiplying both sides of (i) by I.F. $= \frac{1}{x\sqrt{1-x^2}}$, we get

$$\frac{1}{x\sqrt{1-x^2}}\frac{dv}{dx} + \frac{2x^2-1}{x^2(1-x^2)^{3/2}}\, v = \frac{ax}{(1-x^2)^{3/2}}$$

Integrating both sides w.r.t. x, we get

$$v \,.\, \frac{1}{x\sqrt{1-x^2}} = \int \frac{ax}{(1-x^2)^{3/2}}\, dx + C$$

$$\Rightarrow \quad v \,.\, \frac{1}{x\sqrt{1-x^2}} = \frac{-a}{2}\int \frac{1}{t^{3/2}}\, dt + C, \text{ where } t = 1-x^2$$

$$\Rightarrow \quad v \,.\, \frac{1}{x\sqrt{1-x^2}} = -\frac{a}{2}\left(\frac{t^{-1/2}}{-1/2}\right) + C$$

$$\Rightarrow \quad v \,.\, \frac{1}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{t}} + C$$

$$\Rightarrow \quad v \,.\, \frac{1}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C \qquad [\because t = 1-x^2]$$

$$\Rightarrow \quad y^3 = ax + C \,.\, x\sqrt{1-x^2} \qquad [\because v = y^3]$$

This is the required solution of the given differential equation.

EXAMPLE 6 *Solve* : $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$.

SOLUTION We have,

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}$$

Putting $\frac{1}{\log y} = v$, and $-\frac{1}{(\log y)^2}\, y\frac{dy}{dx} = \frac{dv}{dx}$, we get

$$-\frac{dv}{dx}+\frac{1}{x}v = \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx}+\left(-\frac{1}{x}\right)v = -\frac{1}{x^2} \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dv}{dx}+Pv = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = -\frac{1}{x^2}.$$

Now, I.F. $= e^{\int P\,dx} = e^{\int -1/x\,dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$.

Multiplying both sides of (i) by I.F. $= \frac{1}{x}$, we get

$$\frac{1}{x}\frac{dv}{dx}+\left(\frac{-1}{x^2}\right)v = -\frac{1}{x^3}$$

Integrating both sides w.r.t. x, we get

$$v.\frac{1}{x} = \int -\frac{1}{x^3}\,dx + C \qquad [\text{Using : } v\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + C]$$

$$\Rightarrow \frac{v}{x} = \frac{1}{2x^2}+C$$

$$\Rightarrow \frac{1}{\log y}.\frac{1}{x} = \frac{1}{2x^2}+C$$

This is the required solution of the given differential equation.

EXAMPLE 7 *Solve the differential equation*

$$\frac{dz}{dx}+\frac{z}{x}(\log z) = \frac{z}{x^2}(\log z)^2$$

SOLUTION The given differential equation can be written as

$$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{x}\left(\frac{1}{\log z}\right) = \frac{1}{x^2}$$

Let $\frac{1}{\log z} = u$. Then, $\frac{1}{z(\log z)^2}\frac{dz}{dx} = -\frac{du}{dx}$.

Substituting these values, the above differential equation reduces to

$$-\frac{du}{dx}+\frac{u}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx}+\left(\frac{-1}{x}\right)u = -\frac{1}{x^2}$$

This is a linear differential equation with $I.F. = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$.

Multiplying the above differential equation by $\frac{1}{x}$ and integrating, we get

$$\frac{u}{x} = \int -\frac{1}{x^3}\,dx + C$$

$$\Rightarrow \frac{u}{x} = \frac{1}{2x^2}+C$$

$$\Rightarrow \frac{1}{x\log z} = \frac{1}{2x^2}+C.$$

EXERCISE 13.7

Solve each of the following differential equations :

1. $\frac{dy}{dx}+\frac{y}{x} = x^3y^3$
2. $x\frac{dy}{dx}+y = y^2\log x$
3. $(x^3y^2+xy)\,dx = dy$
4. $\frac{dy}{dx}+y\cos x = y^n\sin 2x$
5. $\frac{dy}{dx}+\frac{y}{x} = y^3$
6. $\cos x\frac{dy}{dx}+y\sin x = y^4\sin 2x$
7. $x\frac{dy}{dx}-y = y^3\log x$
8. $\frac{dy}{dx}+\frac{1}{x} = \frac{e^y}{x^2}$
9. $\frac{dy}{dx}+\frac{1}{x}\tan y = \frac{1}{x^2}\tan y\sin y$
10. $\frac{dy}{dx} = e^{x-y}(e^x-e^y)$
11. $xy-\frac{dy}{dx} = y^3e^{-x^2}$
12. $2\frac{dy}{dx}-y\sec x = y^3\tan x$
13. $2y\cos y^2\frac{dy}{dx}-\frac{2}{x+1}\sin y^2 = (x+1)^3$
14. $\frac{dy}{dx}+\frac{y}{x} = xe^xy^2$
15. $\frac{dy}{dx}+2xy+xy^4 = 0$
16. $(xy^2-e^{1/x^3})\,dx-x^2y\,dy = 0$

ANSWERS

1. $(1-x^2)\sqrt{y} = \frac{1}{2}\left(x-\frac{x^3}{3}\right)+C$
2. $(1+\log x)\,y - C\,xy = 1$
3. $y\,(2-x^2) - C\,y\,e^{-x^2/2} = 1$
4. $(y^{-n+1}-1)\,e^{\sin x} = C$
5. $1 = y^2\,(2x+Cx^2)$
6. $2\cos^3 x = (3\cos^4 x + C)\,y^3$.
7. $\frac{x^2}{y^2} = -x^2\log x + \frac{x^2}{2} + C$
8. $2x = 2\,C\,x^2\,e^y + e^y$
9. $2x = (\sin y)\,(1+2\,C\,x^2)$
10. $e^y = C\,e^{-e^x} + e^x - 1$
11. $e^{x^2} = (2x+C)\,y^2$
12. $-y^{-2} = 1 - x\,(\sec x - \tan x) + C\,(\sec x - \tan x)$
13. $\sin y^2 = \left(\frac{x^2}{2}+x+C\right)(x+1)^2$
14. $\frac{1}{xy} = -e^x + C$
15. $\frac{1}{y^3} = C\,e^{3x^2} - \frac{1}{2}$
16. $\frac{y^2}{x^2} = \frac{2}{3}\,e^{1/x^3+C}$

HINTS TO SELECTED PROBLEMS

1. Divide both sides by y^3 and then put $\frac{1}{y^2} = v$

2. Divide both sides by y^2 and then put $\frac{1}{y} = v$

3. We have,

$$\frac{dy}{dx} = x^3y^2 + xy$$

$$\Rightarrow \quad \frac{dy}{dx} - xy = x^3y^2$$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \left(-\frac{1}{y}\right)x = x^3$$

Now, put $-\frac{1}{y} = v$ and $\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$.

6. Put $1/y^3 = v$

7. Put $-1/y^2 = v$

8. We have, $e^{-y}\frac{dy}{dx} + e^{-y}.\frac{1}{x} = \frac{1}{x^2}$. Put $e^{-y} = v$

9. We have, $\operatorname{cosec} y \cot y \frac{dy}{dx} + \frac{1}{x}\operatorname{cosec} y = \frac{1}{x^2}$.

Put $\operatorname{cosec} y = v$

10. We have,

$$e^y\frac{dy}{dx} = e^x(e^x - e^y)$$

$$\Rightarrow e^y\frac{dy}{dx} + e^x.e^y = e^{2x}.$$

Now, put $e^y = v$ and $e^y\frac{dy}{dx} = \frac{dv}{dx}$.

11. We have,

$$\frac{dy}{dx} - xy = -y^3e^{-x^2}$$

$$\Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = e^{-x^2}.$$

Put $-\frac{1}{y^2} = v$ and $\frac{1}{y^3}\frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx}$.

12. We have, $\frac{2}{y^3}\frac{dy}{dx} + \left(-\frac{1}{y^2}\right)\sec x = \tan x$. Put $-\frac{1}{y^2} = v$

13. Put $\sin y^2 = v$

14. Divide throughout by y^2 and then put $\frac{1}{y} = v$

15. We have,

$$\frac{dy}{dx} + 2xy = -xy^4$$

$$\Rightarrow -\frac{1}{y^4}\frac{dy}{dx} + 2x\left(-\frac{1}{y^3}\right) = x.$$

Now, put $-\frac{1}{y^3} = v$ and $-\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dv}{dx}$

16. We have,

$$\frac{dy}{dx} = \frac{xy^2 - e^{1/x^3}}{x^2y}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{e^{1/x^3}}{x^2 y}$$

$$\Rightarrow y\frac{dy}{dx} = \frac{y^2}{x} - e^{1/x^3}$$

$$\Rightarrow y\frac{dy}{dx} - \frac{y^2}{x} = -e^{1/x^3}$$

Now, put $y^2 = v$ and solve it.

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13.6 DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

So far we have discussed differential equations in which the exponent of $\frac{dy}{dx}$ was unity. Now we shall deal with those differential equations in which $\frac{dy}{dx}$ has higher degree than one. These equations are known as differential equations of first order and higher degree. For the sake of convenience $\frac{dy}{dx}$ will be denoted by p.

The most general form of a first order and higher degree differential equation is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0,$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

A first order and higher degree differential equations may be one of the following types :

(i) Equations solvable for p

(ii) Equations solvable for y

(iii) Equations solvable for x

(iv) Clairaut's form of equation.

In this section, we shall discuss various methods of solving above types of differential equations.

13.6.1 EQUATIONS SOLVABLE FOR p

If a first order any degree equation can be resolved into linear factors involving $p\left(=\frac{dy}{dx}\right)$ i.e. it can be expressed as

$$\{p - f_1(x, y)\}\{p - f_2(x, y)\} \ldots\ldots \{p - f_n(x, y)\} = 0 \qquad \ldots(i)$$

In such a case we say that the equation is solvable for p.

Now, $\left(p - f_1(x, y)\right)\left(p - f_2(x, y)\right) \ldots \left(p - f_n(x, y)\right) = 0$

$\Rightarrow p = f_1(x, y), p = f_2(x, y), \ldots, p = f_n(x, y)$

Each of these is a first order first degree equation. Let their solutions be

$$g_1(x, y, c_1) = 0, g_2(x, y, c_2) = 0, \ldots, g_n(x, y, c_n) = 0,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Thus, the general solution is

$$g_1(x, y, c_1) \times g_2(x, y, c_2) \times \ldots \times g_n(x, y, c_n) = 0$$

But, it contains n arbitrary constants whereas the general solution of a first order first degree contains only one arbitrary constant. So, we take $c_1 = c_2 = \ldots = c_n = c$ (say) and the general solution is

$$g_1(x, y, c) \times g_2(x, y, c) \times \ldots \times g_n(x, y, c) = 0$$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$p^2 - 7p + 12 = 0, \text{ where } p = \frac{dy}{dx}$$

SOLUTION The given differential equation is

$$p^2 - 7p + 12 = 0$$

$$\Rightarrow (p - 4)(p - 3) = 0$$

$$\Rightarrow p - 4 = 0 \text{ or, } p - 3 = 0$$

Now,

$$p - 4 = 0$$

$$\Rightarrow p = 4 \Rightarrow \frac{dy}{dx} = 4 \Rightarrow y = 4x + C$$

and, $p - 3 = 0 \Rightarrow p = 3 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow y = 3x + C$

Hence, the general solution of the given differential equation is

$$(y - 4x - C)(y - 3x - C) = 0$$

EXAMPLE 2 *Solve the differential equation*

$xyp^2 - (x^2 - y^2)p - xy = 0$, where $p = \frac{dy}{dx}$

SOLUTION The given differential equation is

$xpy^2 - (x^2 - y^2)p - xy = 0$

$\Rightarrow (xyp^2 + y^2p) - (x^2p + xy) = 0$

$\Rightarrow yp(xp + y) - x(xp + y) = 0$

$\Rightarrow (xp + y)(yp - x) = 0$

$\Rightarrow xp + y = 0$ or, $yp - x = 0$

Now, $xp + y = 0$

$\Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow xdy + ydx = 0$

$\Rightarrow d(xy) = 0$

$\Rightarrow xy = C$

and,

$yp - x = 0$

$\Rightarrow y\frac{dy}{dx} - x = 0$

$\Rightarrow y\,dy - x\,dx = 0$

$\Rightarrow y^2 - x^2 = C$.

Hence, the general solution of the given differential equation is

$$(xy - C)(y^2 - x^2 - C) = 0$$

EXAMPLE 3 *Solve the differential equation :*

$$x^2(p^2 - y^2) + y^2 = x^4 + 2xyp$$

SOLUTION The given differential equation is

$(x^2p^2 - 2xyp + y^2) - x^2(x^2 + y^2) = 0$

$\Rightarrow (y - px)^2 = x^2(x^2 + y^2)$

$\Rightarrow y - px = \pm x\sqrt{x^2 + y^2}$

$\Rightarrow px = y \pm x\sqrt{x^2 + y^2}$

$\Rightarrow p = \frac{y}{x} \pm \sqrt{x^2 + y^2}$

Now,

$p = \frac{y}{x} \pm \sqrt{x^2 + y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \pm \sqrt{x^2 + y^2}$

$\Rightarrow \frac{x\,dy - y\,dx}{x} = \pm\sqrt{x^2 + y^2}\,dx$

$\Rightarrow \frac{x\,dy - y\,dx}{\sqrt{x^2 + y^2}} = \pm x\,dx$

$\Rightarrow \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} \times \frac{x\,dy - y\,dx}{x^2} = \pm dx$

$\Rightarrow \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} d\left(\frac{y}{x}\right) = \pm dx$

$\Rightarrow \log\left\{\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right\} = \pm x + C$

Hence, the general solution of the given differential equation is

$$\left\{\log\left(\frac{y + \sqrt{x^2 + y^2}}{x} - x - C\right)\right\}\left\{\log\left(\frac{y + \sqrt{x^2 + y^2}}{x}\right) + x - C\right\} = 0$$

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13.6.2 EQUATIONS SOLVABLE FOR *y*

If the given differential equation is expressible in the form

$$y = f(x, p) \qquad \ldots(i)$$

, then we say that it is solvable for y.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = f\left(x, p, \frac{dp}{dx}\right) \text{ or } p = f\left(x, p, \frac{dp}{dx}\right) \qquad \ldots(ii)$$

This equation contains two variables x and p. Solving this equation, we obtain

$$\phi(x, p, c) = 0 \qquad \ldots(iii)$$

The solution of differential equation (i) is obtained by eliminating p between (i) and (iii).

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation :*

$$y = 3x + \log p.$$

SOLUTION The given differential equation is

$$y = 3x + \log p \qquad \ldots(i)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p(p-3) = \frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{p(p-3)} = dx$$

$$\Rightarrow \left\{\frac{1}{p-3} - \frac{1}{p}\right\} dp = 3\,dx$$

$$\Rightarrow \log\left(\frac{p-3}{p}\right) = 3x + \log C$$

$$\Rightarrow \frac{p-3}{p} = Ce^{3x}$$

$$\Rightarrow 1 - \frac{3}{p} = Ce^{3x}$$

$$\Rightarrow p = \frac{3}{1 - Ce^{3x}}$$

Putting the value of p in (i), we obtain

$$y = 3x + \log\left(\frac{3}{1 - Ce^{3x}}\right) \text{ as the required solution.}$$

EXAMPLE 2 *Solve the differential equation*

$$y = (1+p)x + ap^2, \text{ where } p = \frac{dy}{dx}.$$

SOLUTION The given differential equation is

$$y = (1+p)x + ap^2 \qquad \ldots(i)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 1 + p + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow p = 1 + p + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow 0 = 1 + \frac{dp}{dx}(x + 2ap)$$

$$\Rightarrow \frac{dx}{dp} + x = -2ap \qquad \ldots(ii)$$

This is a linear differential equation having

$$\text{I.F.} = e^{\int 1\,.\,dp} = e^p$$

Multiplying both sides of (ii) by e^p and integrating, we get

$$xe^p = -2a\int p\,e^p\,dp$$

$$\Rightarrow xe^p = -2a(p-1)e^p + C \qquad \ldots(iii)$$

Eliminating p between (i) and (iii), we obtain the desired solution.

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13.6.3 EQUATIONS SOLVABLE FOR x

If a first order any degree differential equation is expressible in the form

$$x = f(y, p) \qquad \ldots(i)$$

then we say that it is solvable for x.

To find the general solution of this type of differential equation, we follow the following steps :

STEP I *Differentiate (i) w.r.t. y to obtain*

$$\frac{dx}{dy} = f\left(y, p, \frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1}{p} = f\left(y, p, \frac{dp}{dy}\right)$$

STEP II *Solve the above differential equation in p and y. Let its solution be*

$$g(y, p, c) = 0 \qquad \ldots(ii)$$

STEP III *Eliminate p between (i) and (ii) to get a relation between x, y and c which is the required solution of equation (i).*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation :*

$$4yp^2 + 2px - y = 0$$

SOLUTION The given differential equation can be written as

$$x = \frac{y}{2p} - 2yp \qquad \ldots(i)$$

Differentiating both sides of (i) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - 2\left\{p + y\frac{dp}{dy}\right\}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2p} - 2p - y\frac{dp}{dy}\left(\frac{1}{2p^2} + 2\right)$$

$$\Rightarrow \left(\frac{1}{2p} + 2p\right) = -y\frac{dp}{dy}\left(\frac{1}{2p^2} + 2\right)$$

$$\Rightarrow p\left(\frac{1}{2p^2}+2\right) = -y\frac{dp}{dy}\left(\frac{1}{2p^2}+2\right)$$

$$\Rightarrow p = -y\frac{dp}{dy}$$

$$\Rightarrow \frac{dp}{p}+\frac{dy}{y} = 0$$

$$\Rightarrow \log p + \log y = \log C \qquad \text{[On integrating]}$$

$$\Rightarrow py = C \qquad \ldots(ii)$$

Eliminating p between (i) and (ii), we get

$$4y\left(\frac{C^2}{y^2}\right)+2x\left(\frac{C}{y}\right)-y = 0$$

$$\Rightarrow \quad 4C^2+2xC-y^2 = 0$$

$\Rightarrow y^2 = 2Cx+4C^2$, which is the required solution.

EXAMPLE 2 *Solve the differential equation :*

$$x = yp+ap^2$$

SOLUTION The given differential equation is

$$x = yp+ap^2 \qquad \ldots(i)$$

Differentiating both sides of (i) w.r.to y, we get

$$\frac{dx}{dy} = p+y\frac{dp}{dy}+2ap\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = p+\frac{dp}{dy}(y+2ap)$$

$$\Rightarrow \frac{1}{p}-p = \frac{dp}{dy}(y+2ap)$$

$$\Rightarrow \frac{dy}{dp} = \frac{py}{1-p^2}+\frac{2ap^2}{1-p^2}$$

$$\Rightarrow \frac{dy}{dp}+\frac{p}{p^2-1}y = \frac{2ap^2}{1-p^2} \qquad \ldots(ii)$$

This is a linear differential equation in p and y having

$$\text{I.F.} = e^{\int \frac{p}{p^2-1}dp} = e^{(1/2)\log(p^2-1)} = \sqrt{p^2-1}.$$

Multiplying both sides of (ii) by I.F. and integrating w.r.to, we get

$$y\sqrt{p^2-1} = \int \frac{2ap^2}{1-p^2}\sqrt{p^2-1}\,dp$$

$$\Rightarrow y\sqrt{p^2-1} = -2a\int \frac{p^2}{\sqrt{p^2-1}}dp$$

$$\Rightarrow y\sqrt{p^2-1} = -2a\int \frac{p^2-1+1}{\sqrt{p^2-1}}dp$$

$$\Rightarrow y\sqrt{p^2-1} = -2a\int \sqrt{p^2-1}\,dp-2a\int \frac{1}{\sqrt{p^2-1}}dp$$

$$\Rightarrow y\sqrt{p^2-1} = -a\left[p\sqrt{p^2-1}+\log\left(p+\sqrt{p^2-1}\right)\right]$$
$$-2a\log\left(p+\sqrt{p^2-1}\right)+C$$

$$\Rightarrow y\sqrt{p^2-1} = -ap\sqrt{p^2-1}-3a\log\left(p+\sqrt{p^2-1}\right)+C \qquad \ldots(iii)$$

Eliminating p between (i) and (iii), we obtain the required solution.

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13.6.4 CLAIRAUT'S FORM OF DIFFERENTIAL EQUATIONS

If a first order any degree differential equation is first degree in x and y and is expressible in the form

$$y = px+f(p), \text{ where } p = \frac{dy}{dx} \qquad \ldots(i)$$

then it is said to be in Clairaut's form.

To solve this type of differential equation, we follow the following steps :

STEP I *Differentiate the equation with respect to x to obtain*

$$\frac{dy}{dx} = p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$\Rightarrow 0 = \frac{dp}{dx}\{x+f'(p)\}$$

$$\Rightarrow \text{either } \frac{dp}{dx} = 0 \text{ or, } x+f'(p) = 0$$

$$\text{Now, } \frac{dp}{dx} = 0 \Rightarrow p = C.$$

Putting $p = C$ in (i), we obtain

$$y = Cx+f(C)$$

The above solution containing one arbitrary constant is the general solution of the given differential equation.

Now, consider $x+f'(p) = 0$. If we eliminate p between this relation and the given differential equation, we will obtain a relation between x and y containing no arbitrary constant. This relation will satisfy the given differential equation. So, it is a solution of the given differential equation but it cannot be obtained by giving a particular value to the arbitrary constant in the general solution. Also, it does not contain any arbitrary constant. So, it is neither a general solution nor a particular solution. Such a solution is called a singular solution.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$y = xp + p - p^3.$$

SOLUTION The given differential equation is

$$y = px + p - p^3 \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 3p^2\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 3p^2\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}(x + 1 - 3p^2) = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \qquad ...(ii)$$

or, $x + 1 - 3p^2 = 0$ ...(iii)

Now, $\frac{dp}{dx} = 0 \Rightarrow p = C.$

Putting $p = C$ in (i), we obtain

$$y = Cx + C - C^3$$

which is the general solution of the given differential equation.

From (iii), we obtain

$$p = \pm\sqrt{\frac{x+1}{3}}$$

Putting the value of p in (i), we get

$$y = \pm\sqrt{\frac{x+1}{3}}\left\{x + 1 - \frac{x+1}{3}\right\}$$

$$\Rightarrow y = \pm\frac{x+1}{3} \times \frac{2}{3}(x+1)$$

$$\Rightarrow y^2 = \frac{4}{81}(x+1)^4$$

This is the singular solution of the given differential equation.

EXAMPLE 2 *Solve the differential equation :*

$$p^3x - p^2y - 1 = 0$$

SOLUTION The given differential equation can be written as

$$y = px - \frac{1}{p^2} \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{2}{p^3}\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + \frac{2}{p^3}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left\{x + \frac{2}{p^3}\right\} = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \qquad ...(ii)$$

or, $x + \frac{2}{p^3} = 0$ ...(iii)

Now, $\frac{dp}{dx} = 0 \Rightarrow p = C.$

Putting $p = C$ in (i), we obtain

$$y = Cx - \frac{1}{C^2}$$

This is the general solution of the given differential equation.

Now, consider $x + \frac{2}{p^3} = 0$

$$\Rightarrow p^3 = -\frac{2}{x}$$

Eliminating p between (i) and (iii), we obtain

$$y = \left(-\frac{2}{x}\right)^{1/3}x - \left(-\frac{x}{2}\right)^{2/3}$$

$$\Rightarrow y = -2\left(-\frac{x}{2}\right)^{2/3} - \left(-\frac{x}{2}\right)^{2/3}$$

$$\Rightarrow y = -3\left(-\frac{x}{2}\right)^{2/3}$$

$$\Rightarrow y^3 = \frac{-27x^2}{4}$$

$$\Rightarrow 4y^3 = -27x^2$$

This is the singular solution of the given differential equation.

EXERCISE 13.8

Solve the following differential equations :

1. $p^2 - 5p + 6 = 0$
2. $p^2 + (x - e^x)p - xe^x = 0$
3. $x^2p^2 + xyp - 6y^2 = 0$
4. $p(p + x) = y(x + y)$
5. $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$
6. $p(p - y) = x(x + y)$
7. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

8. $p^2y + 2px = y$
9. $y + px = p^2x^4$
10. $y = x + p^2 - 2p$
11. $ayp^2 + (2x - b)p - y = 0$
12. $y = 2px + y^2p^3$
13. $p = \log(px - y)$
14. $\sin px \cos y = \cos px \sin y + p$
15. $y = px + \sqrt{1 + p^2}$
16. $y = p(x - b) + \frac{a}{p}$

ANSWERS

1. $(y - 2x - C)(y - 3x - C) = 0$
2. $(2y + x^2 - C)(y - e^x - C - 1) = 0$
3. $(y - Cx^2)(x^3y - C) = 0$
4. $(y - Ce^x)(y + x - 1 - Ce^{-x}) = 0$
5. $(y - C)(y + x - C)(x^2 + xy + y^2 - C) = 0$
6. $(y + x + 1 - Ce^x)(2y + x^2 - C) = 0$
7. $(y - C)(y + x^2 - C)(xy + Cy + 1) = 0$
8. $y^2 = 2Cx + C^2$
9. $xy = C^2x - C$
10. $(x + C)^2 = 4(y + C)$
11. $aC^2 + (2x - b)C - y^2 = 0$
12. $y^2 = 2Cx + C^3$
13. $y = Cx - e^C$, $y = x \log x - x$
14. $y = Cx - \sin^{-1} C$
15. $y = Cx + \sqrt{1 + C^2}$
16. $y = C(x - b) + \frac{a}{C}$

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13.7 APPLICATIONS OF DIFFERENTIAL EQUATIONS

In this section, we shall discuss some applications of differential equations. In solving some geometrical problems, the following results are very helpful.

Consider the curve $y = f(x)$. Let $P(x, y)$ be a point on the curve as shown in Fig. 13.5 and let PT and PN be respectively the tangent and normal at P. If the tangent at P makes an angle θ with X-axis, then

Slope of the tangent at $P = \tan\theta = \left(\frac{dy}{dx}\right)_P$

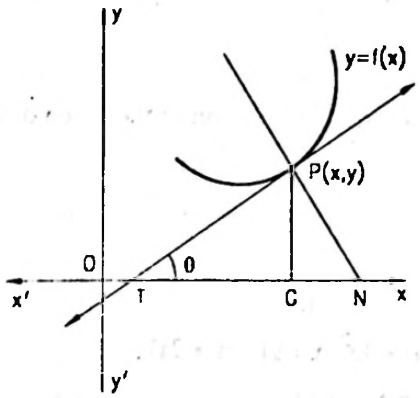


Fig. 13.5

Slope of the normal at $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$

Equation of the tangent at P is

$$Y - y = \left(\frac{dy}{dx}\right)_P (X - x)$$

Equation of the normal at P is

$$Y - y = -\frac{1}{\left(\frac{dy}{dx}\right)_P} (X - x)$$

PT = length of the tangent at P

$\Rightarrow PT = y \operatorname{cosec}\theta$

$\Rightarrow PT = y\frac{\sqrt{1 + \tan^2\theta}}{\tan\theta}$

$\Rightarrow PT = \frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}$

$\Rightarrow PN$ = Length of the normal at P

$\Rightarrow PN = y \sec\theta$

$\Rightarrow PN = y\sqrt{1 + \tan^2\theta}$

$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

TG = Length of the subtangent at P

$= y\cot\theta = \frac{y}{\frac{dy}{dx}}$

GN = Length of the sub-normal at P

$= y\tan\theta = y\frac{dy}{dx}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the curve for which the normal at every point passes through a fixed point is a circle.*

SOLUTION Let $P(x, y)$ be any point on the given curve. The equation of the normal to the given curve at (x, y) is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \qquad ...(i)$$

It is given that the normal at every point passes through a fixed point (a, b) (say). Therefore,

$$b - y = -\frac{dx}{dy}(a - x) \quad [\text{Putting } X = a \text{ and } Y = b \text{ in (i)}]$$

$\Rightarrow -(y-b)\,dy = (x-a)\,dx$

$\Rightarrow (x-a)\,dx + (y-b)\,dy = 0$

On integrating, we get

$$\frac{(x-a)^2}{2} + \frac{(y-b)^2}{2} = C$$

$\Rightarrow \quad (x-a)^2 + (y-b)^2 = 2C$

$\Rightarrow (x-a)^2 + (y-b)^2 = r^2$, where $r^2 = 2C$.

Clearly, this equation represents a circle.

EXAMPLE 2 *The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \cot\frac{y}{x}\cdot\cos\frac{y}{x}$, $(x > 0, y > 0)$ and the curve passes through the point $(1, \pi/4)$. Find the equation of the curve.*

SOLUTION Let $y = f(x)$ be the given curve. Then the slope of the tangent at $P(x, y)$ is $\frac{dy}{dx}$. But the slope of the tangent at P is given as $\frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x}$. Therefore,

$$\frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = v - \cot v\cos v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\cot v\cos v$$

$$\Rightarrow \sec v\tan v\,dv = -\frac{dx}{x}$$

On integrating, we get

$$\sec v = -\log|x| + \log C$$

$$\Rightarrow \quad \sec\left(\frac{y}{x}\right) = -\log|x| + C \qquad ...(i)$$

Since the curve passes through $(1, \pi/4)$. Therefore,

$\sec\pi/4 = -\log 1 + C \Rightarrow \sqrt{2} = C$

Putting $C = \sqrt{2}$ in (i), we get

$$\sec\left(\frac{y}{x}\right) = -\log|x| + \sqrt{2}$$

This is the equation of the required curve.

EXAMPLE 3 *If the tangent at any point P of a curve meets the axis of X in T. Find the curve for which $OP = PT$, O being the origin.*

SOLUTION Let $y = f(x)$ be the given curve and let $P(x, y)$ be a point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(i)$$

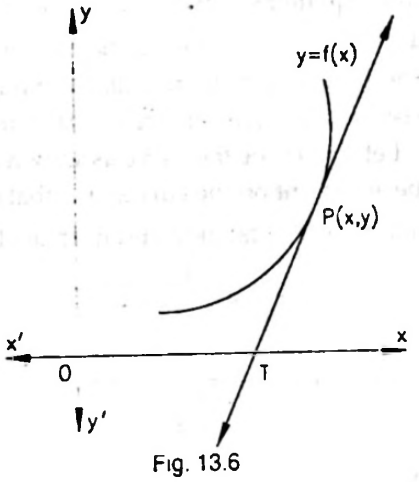


Fig. 13.6

This meets X-axis at T. So, the x-coordinate of T is obtained by putting $Y = 0$ in (i).

Putting $Y = 0$ in (i), we get

$$0 - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow X = x - y\frac{dx}{dy}$$

Thus, the coordinates of T are $\left(x - y\frac{dx}{dy}, 0\right)$

$$\therefore \quad PT = \sqrt{\left\{x - \left(x - y\frac{dx}{dy}\right)\right\}^2 + (y-0)^2}$$

$$= \sqrt{y^2\left(\frac{dx}{dy}\right)^2 + y^2}$$

Now,

$$PT = OP$$

$$\Rightarrow \sqrt{y^2 + y^2\left(\frac{dx}{dy}\right)^2} = \sqrt{x^2+y^2}$$

$$\Rightarrow \quad y^2 + y^2\left(\frac{dx}{dy}\right)^2 = x^2 + y^2$$

$$\Rightarrow \quad y^2\left(\frac{dx}{dy}\right)^2 = x^2$$

$$\Rightarrow \quad y\frac{dx}{dy} = \pm x$$

$$\Rightarrow \quad \frac{dx}{x} = \pm\frac{dy}{y}$$

On integrating, we get

$$\log x = \pm \log y + \log C$$

$$\Rightarrow \quad \log x = \log (C\,y)$$

$$\text{or,} \quad \log x = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow x = C\,y \text{ or, } x = \frac{C}{y}.$$

Hence, the equations of the curve are $x = Cy$ or, $xy = C$.

EXAMPLE 4 *Find the equation of the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance betweenthe origin and the normal at the same point.*

SOLUTION Let $y = f(x)$ be the curve as shown in Fig. 13.7 and let $P(x, y)$ be any point on the curve such that $OL = OM$.

The equations of the tangent and normal at P are

$$Y - y = \left(\frac{dy}{dx}\right)(X - x) \qquad \ldots(i)$$

and

$$Y - y = -\frac{1}{\left(\frac{dy}{dx}\right)}(X - x) \qquad \ldots(ii)$$

respectively.

It is given that the tangent and normal at P are equidistant from the origin.

$$\therefore \quad OL = OM$$

$$\Rightarrow \left|\frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right| = \left|\frac{x + y\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right) = \pm\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow (y \mp x) = \frac{dy}{dx}(x \pm y)$$

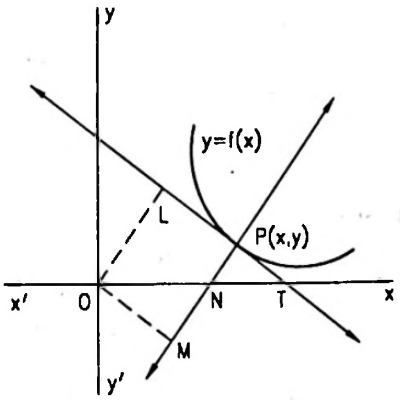


Fig. 13.7

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \mp x}{x \pm y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-x}{x+y} \text{ or, } \frac{dy}{dx} = \frac{y+x}{x-y}$$

Consider, $\frac{dy}{dx} = \frac{y-x}{x+y}$.

It is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we obtain

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{v^2+1}{v+1}\right)$$

$$\Rightarrow \quad \frac{v+1}{v^2+1}dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{2}\cdot\frac{2v}{v^2+1} + \frac{1}{v^2+1}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2v}{v^2+1}dv + \int\frac{1}{v^2+1}dv = -\int\frac{1}{x}dx$$

$$\Rightarrow \quad \frac{1}{2}\log(v^2+1) + \tan^{-1}v = -\log x + \log C$$

$$\Rightarrow \quad \log\left\{\frac{x(v^2+1)^{1/2}}{C}\right\} = -\tan^{-1}v$$

$$\Rightarrow \quad \frac{x(v^2+1)^{1/2}}{C} = e^{-\tan^{-1}v}$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = C\,e^{-\tan^{-1}(y/x)}$$

Now, consider $\frac{dy}{dx} = \frac{y+x}{x-y}$

Putting $y = v\,x$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we obtain

$$v + x\frac{dv}{dx} = \frac{v+1}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{v^2+1}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{v^2+1}dv - \frac{1}{2}\cdot\frac{2v}{v^2+1}dv = \frac{1}{x}dx$$

$$\Rightarrow \int\frac{1}{v^2+1}dv - \frac{1}{2}\int\frac{2v}{v^2+1}dv = \int\frac{1}{x}dx$$

$$\Rightarrow \tan^{-1}v - \frac{1}{2}\log(v^2+1) = \log x + \log C$$

$$\Rightarrow \tan^{-1}v = \log\left\{x(v^2+1)^{1/2}C\right\}$$

$$\Rightarrow C(x^2+y^2)^{1/2} = e^{\tan^{-1}(y/x)}$$

Hence, the equations of the curves are

$$\sqrt{x^2+y^2} = C\,e^{-\tan^{-1}(y/x)} \text{ or, } C\sqrt{x^2+y^2} = e^{\tan^{-1}(y/x)}$$

EXAMPLE 5 *Determine the equation of the curve passing through the origin in the form $y = f(x)$, which satisfies the differential equation* $\frac{dy}{dx} = \sin(10x+6y)$.

[IIT 1996]

SOLUTION Let $10x + 6y = v$. Then,

$$10 + 6\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{6}\left(\frac{dv}{dx} - 10\right)$$

Putting the value of $\frac{dy}{dx}$ and $10x + 6y$ in the given differential equation, we obtain

$$\frac{1}{6}\left(\frac{dv}{dx} - 10\right) = \sin v$$

$$\Rightarrow \frac{dv}{dx} = 10 + 6\sin v$$

$$\Rightarrow \frac{dv}{10+6\sin v} = dx$$

$$\Rightarrow \frac{1+\tan^2 v/2}{10\left(1+\tan^2\frac{v}{2}\right)+12\tan\frac{v}{2}} = dx$$

$$\Rightarrow \frac{\sec^2\frac{v}{2}}{10\tan^2\frac{v}{2}+12\tan\frac{v}{2}+10}dv = dx$$

On integrating, we obtain

$$\int\frac{2\,dt}{10t^2+12t+10} = \int dx + C, \text{ where } t = \tan\frac{v}{2}$$

$$\Rightarrow \frac{1}{5}\int\frac{1}{t^2+\frac{6}{5}t+1}dt = x + C$$

$$\Rightarrow \frac{1}{5}\int\frac{1}{\left(t+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}dt = x + C$$

$$\Rightarrow \frac{1}{5}\times\frac{1}{\frac{4}{5}}\tan^{-1}\left(\frac{t+3/5}{4/5}\right) = x + C$$

$$\Rightarrow \frac{1}{4}\tan^{-1}\left(\frac{5t+3}{4}\right) = x + C$$

$$\Rightarrow \frac{1}{4}\tan^{-1}\left\{\frac{5\tan(5x+3y)+3}{4}\right\} = x + C$$

$$\Rightarrow \frac{5\tan(5x+3y)+3}{4} = \tan(4x+4C) \quad \ldots(i)$$

The curve passes through the origin. Therefore, $y = 0$ when $x = 0$.

Putting $x = 0$, $y = 0$ in (i), we obtain

$$\frac{3}{4} = \tan 4C$$

Putting the value of tan 4C is (i), we get

$$\frac{5\tan(5x+3y)+3}{4} = \frac{\tan 4x + \frac{3}{4}}{1-\frac{3}{4}\tan 4x}$$

$$\Rightarrow \frac{5}{4}\tan(5x+3y) + \frac{3}{4} = \frac{4\tan 4x+3}{4-3\tan 4x}$$

$$\Rightarrow \frac{5}{4}\tan(5x+3y) = \frac{4\tan 4x+3}{4-3\tan 4x} - \frac{3}{4}$$

$$\Rightarrow \frac{5}{4}\tan(5x+3y) = \frac{25\tan 4x}{4(4-3\tan 4x)}$$

$$\Rightarrow \tan(5x+3y) = \frac{5\tan 4x}{4-3\tan 4x}$$

$$\Rightarrow y = \frac{1}{3}\left\{\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right) - 5x\right\}$$

This is the required equation of the curve.

EXAMPLE 6 *Find the curve the which the portion of the x-axis cut off between the origin and the tangent at a. point is twice the abscissa and which passes through the point (1, 2).*

SOLUTION Let $y = f(x)$ be the curve and $P(x, y)$ be any point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

T is the point of intersection of the X-axis and tangent at P. So, the coordinates of T are $\left(x - y\frac{dx}{dy}, 0\right)$.

$\therefore \quad OT = x - y\frac{dx}{dy}$.

It is given that $OT = 2\,OM$

$\Rightarrow \quad x - y\frac{dx}{dy} = 2x$

$\Rightarrow \quad -y\frac{dx}{dy} = x$

$\Rightarrow \quad -\frac{1}{x}dx = \frac{1}{y}dy$

$\Rightarrow \quad \log C - \log x = \log y$

$\Rightarrow \quad \frac{C}{x} = y$

$\Rightarrow \quad xy = C$ ...(i)

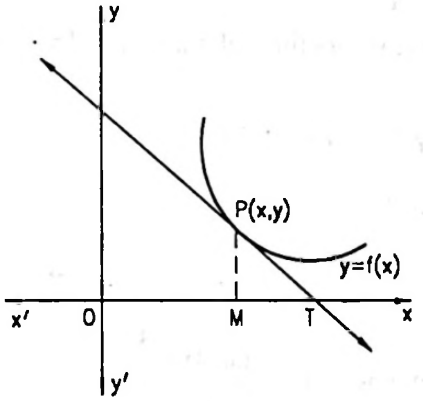


Fig. 13.8

It passes through (1, 2). Therefore, $2 = C$

Putting the value of C in (i), the equation of the required curve is

$$xy = 2.$$

EXAMPLE 7 *Find the curves for which the portion of the y-axis cut off between the origin and the tangent varies as the cube of the abscissae of the point of contact.*

SOLUTION Let $y = f(x)$ be the curve and $P(x, y)$ be any point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts y-axis at $T\left(0, y - x\frac{dy}{dx}\right)$.

$\therefore \quad OT = y - x\frac{dy}{dx}$

It is given that

$OT \propto x^3$

$\Rightarrow \quad OT = \lambda x^3, \lambda \in R$

$\Rightarrow \quad y - x\frac{dy}{dx} = \lambda x^3$

$\Rightarrow \quad x\frac{dy}{dx} - y = -\lambda x^3$

$\Rightarrow \quad \frac{x\,dy - y\,dx}{x^2} = -\lambda x\,dx$

$\Rightarrow \quad d\left(\frac{y}{x}\right) = -\lambda x\,dx$

On integrating, we get

$$\frac{y}{x} = -\lambda\frac{x^2}{2} + C$$

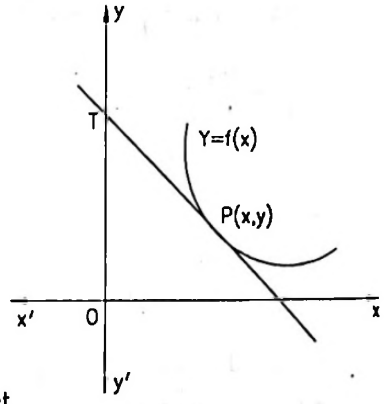


Fig. 13.9

$\Rightarrow 2y = -\lambda x^3 + 2Cx$, which is the equation of the required curve.

EXAMPLE 8 *Find the curve for which the length of the perpendicular from the foot of the ordinate of any point on the curve on the tangent is constant, equal to C and also given that the curve cuts the axis of y at right angles.*

SOLUTION Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on it. The equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \text{...(i)}$$

It is given that

LM = constant $(= C)$

$\Rightarrow$ Length of the perpendicular from $(x, 0)$ on (i) $= C$

$\Rightarrow \quad \left|\frac{y}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}\right| = C$

$\Rightarrow \quad y^2 = C^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{y^2 - C^2}{C^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{y^2 - C^2}}{C}$ ...(i)

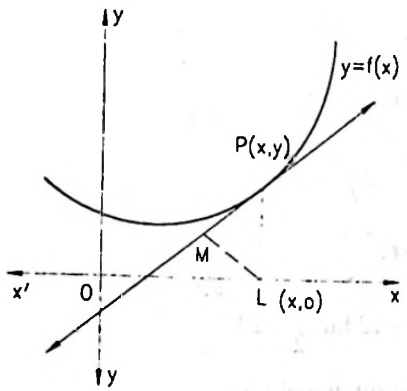


Fig. 13.10

$$\Rightarrow \quad \frac{dy}{\sqrt{y^2 - C^2}} = \frac{1}{C}\,dx$$

$$\Rightarrow \quad \int \frac{1}{\sqrt{y^2 - C^2}}\,dy = \frac{1}{C}\int 1 \,.\, dx$$

$$\Rightarrow \quad \log\left(y + \sqrt{y^2 - C^2}\right) = \frac{1}{C}x + \log k$$

$$\Rightarrow \quad \log\left\{\frac{y + \sqrt{y^2 - C^2}}{k}\right\} = \frac{x}{C}$$

$$\Rightarrow \quad y + \sqrt{y^2 - C^2} = k\,e^{x/C} \qquad \ldots(ii)$$

It is given that the curve (ii) cuts the y-axis at right angles.

Therefore, $\frac{dy}{dx} = 0$ when $x = 0$.

From (i), we have

$$\frac{\sqrt{y^2 - C^2}}{C} = 0$$

$$\Rightarrow \quad y = \pm C$$

Thus, $(0, \pm C)$ lies on (ii).

$\therefore \quad k = \pm C$.

Putting $k = \pm C$ in (ii), we obtain

$$y + \sqrt{y^2 - C^2} = \pm C\,e^{x/c}$$

as the equation of the required curve.

EXAMPLE 9 *Show that the curve such that the ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point (a, b) is equal to the constant $k\,(k > 0, k \neq 1)$ is a circle.*

SOLUTION Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on the curve. The equation of the normal at $P(x, y)$ to the given curve is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \qquad \ldots(i)$$

The distance of (i) from the origin is

$$d_1 = \left|\frac{y + \frac{x}{\frac{dy}{dx}}}{\sqrt{1 + \frac{1}{(dy/dx)^2}}}\right| = \left|\frac{x + y\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

The distance of (i) from the point (a, b) is

$$d_2 = \left|\frac{b - y + \frac{a - x}{\left(\frac{dy}{dx}\right)}}{\sqrt{1 + \left(\frac{1}{dy/dx}\right)^2}}\right| = \left|\frac{(a - x) + (b - y)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

It is given that

$$\frac{d_1}{d_2} = k$$

$$\Rightarrow \quad d_1 = kd_2$$

$$\Rightarrow \quad \left|\frac{x + y\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| = k\left|\frac{(a - x) + (b - y)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = \pm k\left[(a - x) + (b - y)\frac{dy}{dx}\right]$$

$$\Rightarrow \quad x\,dx + y\,dy = \pm k\,[(a - x)\,dx + (b - y)\,dy]$$

On integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \pm k\left[-\frac{(a - x)^2}{2} - \frac{(b - y)^2}{2}\right] + C$$

$$\Rightarrow \quad x^2 + y^2 = \mp\, k\left[(x - a)^2 + (y - b)^2\right] + C$$

$$\Rightarrow \quad x^2(1 \pm k) + y^2(1 \pm k) \mp 2k\,ax \mp 2k\,by \mp k(a^2 + b^2) - C = 0$$

$$\Rightarrow \quad x^2 + y^2 \mp \frac{2ak}{1 \pm k}x \mp \frac{2bk}{1 \pm k}y \pm \frac{k(a^2 + b^2) - C}{1 \pm k} = 0,$$

which represents a circle.

EXAMPLE 10 *Find the curve which is such that the product of the perpendiculars from the point $(\pm c, 0)$ on any tangent is equal to b^2.*

SOLUTION Let $y = f(x)$ be the required curve and $P(x, y)$ be any point on it. The equation of the tangent to the curve at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad X\frac{dy}{dx} - Y + y - x\frac{dy}{dx} = 0 \qquad \ldots(i)$$

Let p_1 and p_2 be the lengths of the perpendicular drawn from $(c, 0)$ and $(-c, 0)$ respectively on (i). Then,

$$p_1 = \left|\frac{c\frac{dy}{dx} + y - x\frac{dy}{dx}}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}\right| \text{ and } p_2 = \left|\frac{-c\frac{dy}{dx} + y - x\frac{dy}{dx}}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}\right|$$

It is given that

$$p_1 p_2 = b^2$$

$$\Rightarrow \quad \frac{y + \frac{dy}{dx}(c - x)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \times \frac{y - \frac{dy}{dx}(x + c)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \pm b^2$$

$$\Rightarrow \quad \left\{y + (c - x)\frac{dy}{dx}\right\}\left\{y - (x + c)\frac{dy}{dx}\right\} = \pm b^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

Taking positive sign, we obtain

$$y^2 - 2xy\frac{dy}{dx} + (x^2 - c^2)\left(\frac{dy}{dx}\right)^2 = b^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (b^2 + c^2 - x^2) + 2xy\frac{dy}{dx} + b^2 - y^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-xy \pm \sqrt{x^2y^2 - (b^2 - y^2)(b^2 + c^2 - x^2)}}{(b^2 + c^2 - x^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-xy \pm \sqrt{y^2(b^2 + c^2) - b^2(b^2 + c^2 - x^2)}}{(b^2 + c^2 - x^2)} \quad ...(i)$$

Let us consider the curve $y^2(b^2 + c^2) - b^2(b^2 + c^2 - x^2) = 0$ i.e.

$$\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1$$

We observe that for this curve

$$\frac{dy}{dx} = -\frac{b^2 x}{y(b^2 + c^2)} = \frac{-b^2 x^2 y}{y^2(b^2 + c^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 xy}{b^2(b^2 + c^2 - x^2)} \qquad \left[\because \frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2\left(\frac{b^2 + c^2 - x^2}{b^2 + c^2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-xy}{b^2 + c^2 - x^2} \quad ...(ii)$$

For the curve $\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1$

i.e. $y^2(b^2 + c^2) - b^2(b^2 + c^2 - x^2) = 0$

equation (i) reduces to

$$\frac{dy}{dx} = \frac{-xy}{b^2 + c^2 - x^2}$$

which is same as (ii)

Hence, the required curve is $\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1$.

If we take negative sign and proceed as above, then the equation of the required curve comes out $\frac{x^2}{-b^2 + c^2} - \frac{y^2}{b^2} = 1$, which is a hyperbola.

EXAMPLE 11 *The tangent at any point P of a curve C meets the x-axis at Q whose obscissa is positive and OP = OQ, O being the origin, show that C is a family of parabolas having a common axis.*

SOLUTION Let the equation of the curve C be $y = f(x)$. The equation of the tangent at any point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets x-axis at $Q\left(x - y\frac{dx}{dy}, 0\right)$

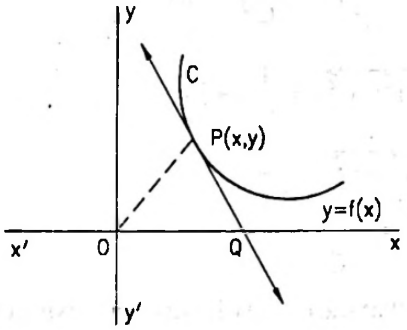


Fig. 13.11

It is given that

$$OP = OQ$$

$$\Rightarrow \sqrt{x^2 + y^2} = x - y\frac{dx}{dy}$$

$$\Rightarrow \sqrt{x^2 + y^2}\,dy = -y\,dx + x\,dy$$

$$\Rightarrow \frac{-y\,dx + x\,dy}{\sqrt{x^2 + y^2}} = dy$$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2\sqrt{1 + \frac{x^2}{y^2}}} = -\frac{1}{y}dy$$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2\sqrt{1 + \frac{x^2}{y^2}}} = -\frac{1}{y}dy$$

$$\Rightarrow \frac{1}{\sqrt{1 + \frac{x^2}{y^2}}}\left(\frac{y\,dx - x\,dy}{y^2}\right) = -\frac{1}{y}dy$$

$$\Rightarrow \frac{1}{\sqrt{1 + \left(\frac{x}{y}\right)^2}}d\left(\frac{x}{y}\right) = -\frac{1}{y}dy$$

On integrating, we obtain

$$\log\left\{\frac{x}{y} + \sqrt{1 + \left(\frac{x}{y}\right)^2}\right\} = -\log y + \log C$$

$$\Rightarrow \frac{x}{y} + \sqrt{1 + \frac{x^2}{y^2}} = \frac{C}{y}$$

$$\Rightarrow x + \sqrt{x^2 + y^2} = C$$

$$\Rightarrow x^2 + y^2 = (C - x)^2$$

$$\Rightarrow y^2 = C^2 - 2Cyx$$

$$\Rightarrow (y - 0)^2 = -2C\left(x - \frac{C}{2}\right),$$

which represents a family of parabolas having the common axis $y = 0$ i.e. x-axis.

EXAMPLE 12 *A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is*

$$y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

and the equation of such a curve passing through $(0, k)$.

[IIT 1994]

SOLUTION Let $y = f(x)$ be the curve such that the normal at $P(x, y)$ to this curve meets X-axis at Q. Then,

PQ = Length of the normal at P

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

But, $PQ = k$ (Given)

$$\therefore\ y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = k$$

$$\Rightarrow\ y^2 + y^2\left(\frac{dy}{dx}\right)^2 = k^2$$

$$\Rightarrow\ \left(y\frac{dy}{dx}\right)^2 = k^2 - y^2$$

$$\Rightarrow\ y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

$$\Rightarrow\ \frac{y\,dy}{\sqrt{k^2 - y^2}} = \pm dx$$

$$\Rightarrow\ \int\frac{y\,dy}{\sqrt{k^2 - y^2}} = \pm\int dx$$

$$\Rightarrow\ -\sqrt{k^2 - y^2} = \pm x + C \qquad \ldots(i)$$

Since the curve passes through $(0, k)$. Therefore,

$$0 = \pm 0 + C \Rightarrow C = 0$$

Putting $C = 0$ in (i), we obtain

$$-\sqrt{k^2 - y^2} = \pm x$$

$$\Rightarrow k^2 - y^2 = x^2$$

$$\Rightarrow x^2 + y^2 = k^2$$

Hence, the equation of the curve is $x^2 + y^2 = k^2$.

EXAMPLE 13 *A curve passing through the point (1, 1) has the property that the perpendicular distance of the normal at any point P on the curve from the origin is equal to the distance of P from x-axis. Determine the equation of the curve.*

[IIT 1999]

SOLUTION Let $P(x, y)$ be any point on the curve $y = f(x)$. Then the equation of the normal at P is

$$Y - y = \frac{-1}{\left(\frac{dy}{dx}\right)}(X - x)$$

or, $X + Y\frac{dy}{dx} - \left(y\frac{dy}{dx} + x\right) = 0$ ...(i)

It is given that

Distance of (i) from the origin = Distance of P from x-axis

$$\Rightarrow \left|\frac{0 - \left(y\frac{dy}{dx} + x\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right| = y$$

$$\Rightarrow \frac{y\frac{dy}{dx} + x}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \pm y$$

$$\Rightarrow \left(y\frac{dy}{dx} + x\right)^2 = y^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow x^2 + 2xy\frac{dy}{dx} = y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This is a homogeneous differential equation which can be solved by substituting $y = vx$. Let us solve it by inspection method. It can be written as

$$2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \frac{x\,d(y^2) - y^2\,dx}{x^2} = -dx$$

$$\Rightarrow d\left(\frac{y^2}{x}\right) = -dx$$

$$\Rightarrow \frac{y^2}{x} = -x + C \qquad \ldots(ii) \qquad \text{[On integration]}$$

This passes through (1, 1). Therefore,

$$1 = -1 + C \Rightarrow C = 2$$

Putting $C = 2$ in (ii), we obtain

$$\frac{y^2}{x} = -x + 2$$

$$\Rightarrow y^2 + x^2 = 2x$$

This is the equation of the required curve.

EXAMPLE 14 *Let $y = f(x)$ be a curve passing through (1, 1) such that the triangle formed by the coordinate axes and the tangent at any*

point of the curve lies in the first quadrant and has area 2. Form the differential equation and determine all such possible curves.

SOLUTION Let $P(x, y)$ be any point on the curve $y = f(x)$. Then, the equation of the tangent at $P(x, y)$ to the curve is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(i)$$

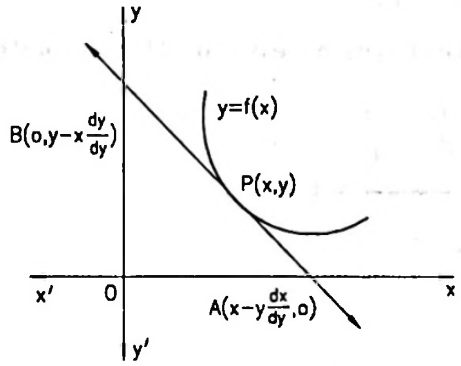


Fig. 13.12

The tangent at P meets the coordinate axes at

$$A\left(x - y\frac{dx}{dy}, 0\right) \text{ and } B\left(0, y - x\frac{dy}{dx}\right) \text{ respectively.}$$

Now,

Area of $\Delta OAB = 2$ sq. units

$$\Rightarrow \frac{1}{2}(OA)(OB) = 2$$

$$\Rightarrow \frac{1}{2}\left(x - y\frac{dx}{dy}\right)\left(y - x\frac{dy}{dx}\right) = 2$$

$$\Rightarrow \left(x - y\frac{dx}{dy}\right)\left(y - x\frac{dy}{dx}\right) = 4$$

$$\Rightarrow xy - x^2\frac{dy}{dx} - y^2\frac{dx}{dy} + xy = 4$$

$$\Rightarrow 2xy - x^2\frac{dy}{dx} - \frac{y^2}{\frac{dy}{dx}} = 4$$

$$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 + 4\frac{dy}{dx} = 0$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = -4\frac{dy}{dx}$$

$$\Rightarrow y - x\frac{dy}{dx} = 2\sqrt{-\frac{dy}{dx}} \qquad \left[\because \frac{dy}{dx} < 0\right]$$

$$\Rightarrow y - px = 2\sqrt{-p} \qquad ...(ii)$$

where, $p = \frac{dy}{dx}$

Differetiating w.r.t. x, we get

$$\frac{dy}{dx} - p - x\frac{dp}{dx} = \frac{2}{2\sqrt{-p}}\left(\frac{-dp}{dx}\right)$$

$$\Rightarrow p - p - x\frac{dp}{dx} = \frac{1}{\sqrt{-p}}\left(-\frac{dp}{dx}\right)$$

$$\Rightarrow -x\frac{dp}{dx} = -\frac{1}{\sqrt{-p}}\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{1}{\sqrt{-p}}\right)\frac{dp}{dx} = 0$$

$$\Rightarrow x - \frac{1}{\sqrt{-p}} = 0 \quad \text{or,} \quad \frac{dp}{dx} = 0$$

$$\Rightarrow x = \frac{1}{\sqrt{-p}} = 0 \quad \text{or,} \quad \frac{dp}{dx} = 0$$

$$\Rightarrow x^2 = -\frac{1}{p} \qquad \text{or,} \quad \frac{dp}{dx} = 0$$

$$\Rightarrow p = \frac{-1}{x^2} \qquad \text{or,} \quad \frac{dp}{dx} = 0$$

CASE I When $\frac{dp}{dx} = 0$:

In this case, we have

$$\frac{dp}{dx} = 0$$

$$\Rightarrow p = C$$

Putting $p = C$ in (ii), we obtain

$$y - Cx = 2\sqrt{-C} \qquad (iii)$$

It passes through (1, 1). Therefore,

$$1 - C = 2\sqrt{-C}$$

$$\Rightarrow (1 - C)^2 = -4C$$

$$\Rightarrow (1 + C)^2 = 0$$

$$\Rightarrow C = -1$$

Putting $C = -1$ in (iii), we get

$$y + x = 2$$

Thus, the equation of the curve is $y + x = 2$.

CASE II *When* $p = -\frac{1}{x^2}$:

In this case, we have

$$p = -\frac{1}{x^2}$$

Eliminating p between $p = -\frac{1}{x^2}$ and (ii), we obtain

$$y + \frac{1}{x} = \frac{2}{x}$$

$$\Rightarrow xy = 1.$$

Hence, the equations of the required curves are $x + y = 2$ and $xy = 1$.

EXAMPLE 15 *A curve C has the property that the area of the triangle formed by the x-axis, tangent to the curve and radius vector of the*

point of tangency is constant k^2. Find the equation of the curve C if the point (0, 1) lies on the curve.

SOLUTION Let $y = f(x)$ be the curve and $P(x, y)$ be any point on it. Then, the equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets X-axis at $A\left(x - y\frac{dx}{dy}, 0\right)$

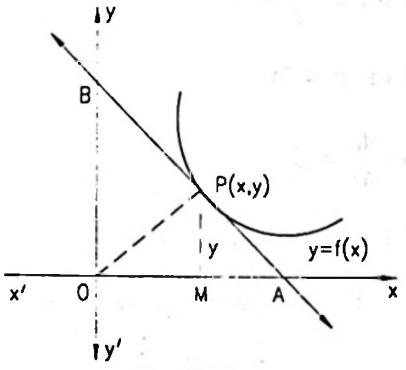


Fig. 13.13

Clearly, OP is the radius vector. It is given that

Area of $\Delta OAP = k^2$

$$\Rightarrow \frac{1}{2} OA \times PM = k^2$$

$$\Rightarrow \frac{1}{2}\left(x - y\frac{dx}{dy}\right) \times y = k^2$$

$$\Rightarrow x\,dy - y\,dx = -\frac{2k^2}{y}$$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = \frac{2k^2}{y^3}$$

$$\Rightarrow d\left(\frac{x}{y}\right) = -\frac{2k^2}{y^3}$$

On integrating, we get

$$\frac{x}{y} = \frac{k^2}{y^2} + C \qquad ...(i)$$

It passes through (0, 1). Therefore,

$$0 = k^2 + C \Rightarrow C = -k^2$$

Putting $C = -k^2$ in (i), we get

$$\frac{x}{y} = \frac{k^2}{y^2} - k^2 \Rightarrow y^2 x = k^2(1 - y^2)$$

EXAMPLE 16 *Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is a constant equal to a^2.*

SOLUTION Let $y = f(x)$ be the equation of the curve and let $P(x, y)$ be any point on it. Then the equation of the tangent at $P(x, y)$ is

$$Y - X = \frac{dy}{dx}(X - x)$$

It meets Y-axis at $T\left(0, y - x\frac{dy}{dx}\right)$.

Clearly, OT is the initial ordinate of the tangent at point P. It is given that

Area of rectangle $OLMT = a^2$

$$\Rightarrow OL \times OT = a^2$$

$$\Rightarrow \left|x\left(y - x\frac{dy}{dx}\right)\right| = a^2$$

$$\Rightarrow x\left(y - x\frac{dy}{dx}\right) = \pm a^2$$

$$\Rightarrow x(y\,dx - x\,dy) = \pm a^2\,dx$$

$$\Rightarrow y\,dx - x\,dy = \pm\frac{a^2}{x}\,dx$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \pm\frac{a^2}{x^3}dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = \pm\frac{a^2}{x^3}dx$$

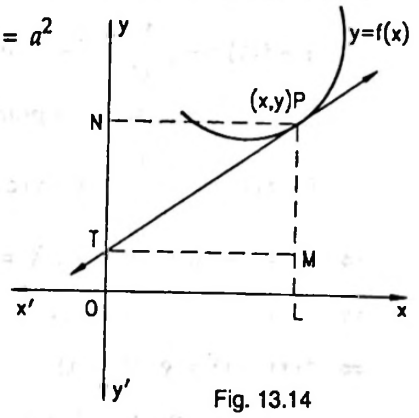


Fig. 13.14

Integrating both sides, we get

$$\frac{y}{x} = \pm\frac{a^2}{2x^2} + C$$

$$\Rightarrow y = \pm\frac{a^2}{2x} + Cx$$

This is the equation of the required curve.

EXAMPLE 17 *Find a pair of curves such that,*

(i) The tangents at points with equal abscissae intersecton on y-axis.

(ii) The normals drawn at points with equal abscissae intersect on x-axis.

(iii) One curve passes through (1, 1) and the other passes through (2, 3).

SOLUTION Let $y = f(x)$ and $y = g(x)$ be the required curves. The equation of the tangents to these two curves at points with equal abscissae x (say) are

$$Y - f(x) = f'(x)(X - x) \text{ and } Y - g(x) = g'(x)(X - x).$$

These two lines intersect at a point on Y-axis. Therefore,

$$Y - f(x) = -x f'(x) \text{ and } Y - g(x) = -x g'(x)$$

$$\Rightarrow Y = f(x) - x f'(x) \quad \text{and } Y = g(x) - x g'(x)$$

$$\Rightarrow f(x) - x f'(x) = g(x) - x g'(x)$$

$$\Rightarrow f(x) - g(x) = x\{f'(x) - g'(x)\}$$

$$\Rightarrow f(x) - g(x) = x\frac{d}{dx}\{f(x) - g(x)\}$$

$$\Rightarrow \frac{d\{f(x) - g(x)\}}{\{f(x) - g(x)\}} = \frac{dx}{x}$$

On integrating, we obtain

$$\log |f(x) - g(x)| = \log x + \log C$$

$$\Rightarrow \quad f(x) - g(x) = Cx \qquad \ldots(i)$$

The equations of the normals at points with equal abscissae x (say) to the two curves are

$$Y - f(x) = -\frac{1}{f'(x)}(X - x) \text{ and } Y - g(x) = -\frac{1}{g'(x)}(X - x)$$

These two intersect at a point on X-axis. Therefore,

$$0 - f(x) = -\frac{1}{f'(x)}(X - x) \text{ and } 0 - g(x) = -\frac{1}{g'(x)}(X - x)$$

$$\Rightarrow \quad X = x + f(x)f'(x) \text{ and } X = x + g(x)g'(x)$$

$$\Rightarrow \quad x + f(x)f'(x) = x + g(x)g'(x)$$

$$\Rightarrow \quad f(x)f'(x) = g(x)g'(x)$$

On integrating, we obtain

$$[f(x)]^2 = [g(x)]^2 + C_1$$

$$\Rightarrow \quad [f(x)]^2 - [g(x)]^2 = C_1$$

$$\Rightarrow \quad [f(x) + g(x)](Cx) = C_1 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad f(x) + g(x) = \frac{C_1}{Cx} \qquad \ldots(ii)$$

Solving (i) and (ii), we obtain

$$f(x) = \frac{1}{2}\left(Cx + \frac{C_1}{Cx}\right) \qquad \ldots(iii)$$

$$\text{and, } g(x) = \frac{1}{2}\left(\frac{C_1}{Cx} - Cx\right) \qquad \ldots(iv)$$

It is given that (iii) passes through (1, 1) and (iv) passes through (2, 3) therefore,

$$2 = C + \frac{C_1}{C} \text{ and } 6 = \frac{C_1}{2C} - 2C$$

$$\Rightarrow \quad C = -2 \text{ and } C_1 = -8$$

Substituting the values of C_1 and C in (iii) and (iv), we get

$$f(x) = \frac{1}{2}\left(-2x + \frac{4}{x}\right) \text{and } g(x) = \frac{1}{2}\left(\frac{4}{x} + 2x\right)$$

$$\text{or, } f(x) = \frac{2}{x} - x \qquad \text{and } g(x) = x + \frac{2}{x}.$$

EXAMPLE 18 *Find out all the curves passing through* $(0, -1)$ *satisfying the differential equation*

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2$$

SOLUTION We have,

$$y = xp - \frac{1}{4}p^2, \text{ where } p = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{2}p\frac{dp}{dx}. \qquad \text{[Differentiating w.r.t. x]}$$

$$\Rightarrow \quad p = p + x\frac{dp}{dx} - \frac{1}{2}p\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = \frac{dp}{dx}\left(x - \frac{p}{2}\right)$$

$$\Rightarrow \quad \frac{dp}{dx} = 0 \text{ or, } x - \frac{p}{2} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} = 0 \text{ or, } p = 2x$$

CASE I *When* $\frac{dp}{dx} = 0$:

$$\frac{dp}{dx} = 0 \Rightarrow p = C \text{ (Constant)}$$

Putting $p = C$ in the given equation, we obtain

$$y = Cx - \frac{1}{4}C^2 \qquad \ldots(i)$$

This curve passes through $(0, -1)$. Therefore,

$$-1 = 0 - \frac{C^2}{4} \Rightarrow C^2 = 4 \Rightarrow C = \pm 2$$

Putting the value of C in (i), we obtain

$$y = \pm 2x - 1$$

as the equations of the curves.

CASE II *When* $p = 2x$.

In this case, we have $p = 2x$

Substituting $p = 2x$ in the given differential equation, we get

$$y = 2x^2 - x^2 = x^2$$

Clearly, it does not pass through $(0, -1)$.

Hence, the equations of the required curves are

$$y = 2x - 1 \text{ and } y = -2x - 1$$

EXAMPLE 19 *Consider a curved mirror* $y = f(x)$ *passing through* $(8, 6)$ *having the property that all light rays emerging from the origin, after getting reflected from the mirror become parallel to X-axis. Find the equation of the curved mirror.*

SOLUTION Let the equation of the curved mirror be $y = f(x)$ with the property that any light ray OP emerging from the origin, after getting reflected becomes parallel to X-axis.

Let PN be the normal at P. Then,

$$\angle OPN = \angle NPQ = \theta$$

But, $\angle ONP = \angle NPQ$

$$\therefore \quad \angle OPN = \angle ONP = \theta$$

$$\Rightarrow \quad OP = ON$$

The equation of the normal at P is

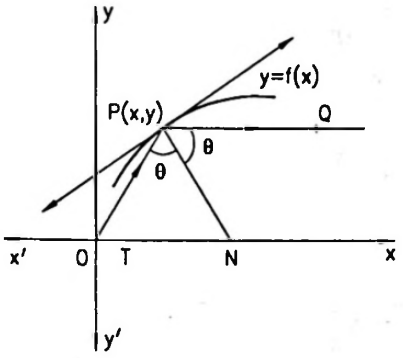


Fig. 13.15

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

It cuts X-axis at $N\left(x + y\frac{dy}{dx}, 0\right)$

$\therefore$ $ON = x + y\frac{dy}{dx}$

Now, $OP = ON$

$\Rightarrow \sqrt{x^2 + y^2} = x + y\frac{dy}{dx}$

$\Rightarrow x\,dx + y\,dy = \sqrt{x^2 + y^2}\,dx$

$\Rightarrow 2x\,dx + 2y\,dy = 2\sqrt{x^2 + y^2}\,dx$

$\Rightarrow \frac{d(x^2 + y^2)}{\sqrt{x^2 + y^2}} = 2\,dx$

$\Rightarrow 2\sqrt{x^2 + y^2} = 2x + C$ ...(i)

It passes through (8, 6). Therefore,

$20 = 16 + C \Rightarrow C = 4$

Putting $C = 4$ in (i), we obtain the equation of the curved mirror as

$2\sqrt{x^2 + y^2} = 2x + 4$

or, $\sqrt{x^2 + y^2} = x + 2$

or, $y^2 = 4x + 4$

EXAMPLE 20 *A curve is such that the segment of the normal, drawn at any point P on the curve, intercepted between the x and y axes, is bisected at the point P. Find the curve given that it passes thorugh (3, 1).*

SOLUTION Let the equation of the required curve be $y = f(x)$. Let $P(x, y)$ be any point on the curve. Then, the equation of the normal at P is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

This meets the coordinate axes at

$$A\left(x + y\frac{dy}{dx}, 0\right) \text{ and } B\left(0, y + \frac{x}{\frac{dy}{dx}}\right)$$

It is given that AB is bisected at P. Therefore,

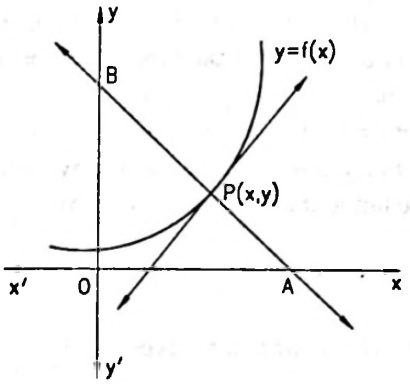


Fig. 13.16

$$x + y\frac{dy}{dx} = 2x \text{ and } y + \frac{x}{\frac{dy}{dx}} = 2y$$

$\Rightarrow y\frac{dy}{dx} = x$

$\Rightarrow y\,dy - x\,dx = 0$

$\Rightarrow y^2 - x^2 = C$ ...(i)

As this passes through (3, 1). Thereofre,

$9 - 1 = C \Rightarrow C = 8$

Putting $C = 8$ in (i), we obtain the equation of the required curve as $y^2 - x^2 = 8$.

EXAMPLE 21 *Prove that there exists an infinite number of points, placed at equal intervals along a line parallel to the y-axis, through which every member of the family of curves represented by the differential equation* $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)\,e^x \sec y$ *passes.*

SOLUTION The given differential equation is

$$\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)\,e^x \sec y$$

$\Rightarrow dy - \frac{\tan y}{1 + x}dx = (1 + x)\,e^x \sec y\,dx$

$\Rightarrow \cos y\,dy - \frac{\sin y}{1 + x}dx = (1 + x)\,e^x\,dx$

$\Rightarrow \frac{(1 + x)\cos y\,dy - \sin y\,dx}{(1 + x)^2} = e^x\,dx$

$\Rightarrow d\left(\frac{\sin y}{1 + x}\right) = e^x\,dx$

On integrating, we obtain

$\frac{\sin y}{1 + x} = e^x + C$

$\Rightarrow \sin y = (1 + x)(e^x + C)$

This gives a one parameter family of curves such that every member of the family passes through $(-1, n\pi)$, $n \in Z$ irrespec-

tive of the value of C. Clearly, these points are placed at equal intervals on the line $x = -1$.

EXAMPLE 22 *Find the differential equation of all curves with the property that the sum of the intercepts made by the tangent at any point on the curve on the coordinate axes is constant. Find all forms of the curve satisfying the given property.*

SOLUTION Let $y = f(x)$ be the equation of the curve satisfying the above property and let $P(x, y)$ be any point on it. The equation of the tangent to the curve at point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets the coordinate axes at $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$

Thus,

$$X\text{-intercept of the tangent} = x - \frac{y}{p}, \text{ where } p = \frac{dy}{dx}$$

and, Y-intercept of the tangent $= y - px$

It is given that the sum of the intercepts made by any tangent to the curve is constant. Therefore,

$$x - \frac{y}{p} + y - px = a \text{ (constant)}$$

$$\Rightarrow y\left(1 - \frac{1}{p}\right) = x(p-1) + a$$

$$\Rightarrow y = px + \frac{ap}{p-1} \quad \ldots(i)$$

This is the required differential equation.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{(p-1)^2}\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} - \frac{a}{(p-1)^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left(x - \frac{a}{(p-1)^2}\right) = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \text{ or, } x = \frac{a}{(p-1)^2}$$

CASE I *When* $\frac{dp}{dx} = 0$:

In this case,

$$\frac{dp}{dx} = 0 \Rightarrow p = C \text{ (constant)}$$

Putting $p = C$ in (i), we obtain

$$y = Cx + \frac{aC}{C-1} \quad \ldots(ii)$$

This is one form of the curve, which represents a family of straight lines.

CASE II *When* $x = \frac{a}{(p-1)^2}$:

In this case, we have

$$x = \frac{a}{(p-1)^2}$$

$$\Rightarrow (p-1)^2 = \frac{a}{x}$$

$$\Rightarrow p - 1 = \pm\sqrt{\frac{a}{x}}$$

$$\Rightarrow p = 1 \pm \sqrt{\frac{a}{x}}$$

Putting the value of p in (i), we obtain

$$y = \left(1 \pm \sqrt{\frac{a}{x}}\right)x \pm a\left(1 \pm \sqrt{\frac{a}{x}}\right)\sqrt{\frac{x}{a}}$$

$$\Rightarrow y = \left(1 \pm \sqrt{\frac{a}{x}}\right)(x \pm \sqrt{ax})$$

$$\Rightarrow y = \frac{(\sqrt{x} \pm \sqrt{a})(x \pm \sqrt{ax})}{\sqrt{x}}$$

$$\Rightarrow y = \frac{x\sqrt{x} \pm \sqrt{a}\,x \pm \sqrt{a}\,x + a\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow y = x + a \pm 2\sqrt{ax}$$

$$\Rightarrow (y - x - a)^2 = 4ax$$

$$\Rightarrow x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0 \quad \ldots(iii)$$

This equation represents a parabola as it satisfies $h^2 = ab$.

Hence, the required curves are given by (ii) and (iii).

EXAMPLE 23 *Form the differential equation of curves for which the length of the portion of the tangent intercepted between the coordinate X-axes is constant. Obtain the equation to such curves.*

SOLUTION Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on it. The equation of the tangent to the curve at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

or, $Y - y = p(X - x)$, where $p = \frac{dy}{dx}$.

This meets the coordinate axes at $A\left(x - \frac{y}{p}, 0\right)$ and $B(0, y - px)$.

It is given that

$$AB = a \text{ (constant)}$$

$$\Rightarrow AB^2 = a^2$$

$$\Rightarrow \left(x - \frac{y}{p}\right)^2 + (y - px)^2 = a^2$$

$$\Rightarrow \frac{(px - y)^2}{p^2} + (y - px)^2 = a^2$$

$$\Rightarrow (y - px)^2(1 + p^2) = a^2p^2$$

$$\Rightarrow y - px = \pm \frac{ap}{\sqrt{1+p^2}}$$

$$\Rightarrow y = px \pm \frac{ap}{\sqrt{1+p^2}} \quad \ldots(i)$$

This is the required differential equation.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} \pm a\left|\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} - \frac{p^2}{(1+p^2)^{3/2}}\frac{dp}{dx}\right|$$

$$\Rightarrow p = p + x\frac{dp}{dx} \pm \frac{a}{(1+p^2)^{3/2}}\frac{dp}{dx}$$

$$\Rightarrow 0 = \left[x \pm \frac{a}{(1+p^2)^{3/2}}\right]\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx} = 0 \text{ or } x \pm \frac{a}{(1+p^2)^{3/2}} = 0$$

CASE I When $\frac{dp}{dx} = 0$:

In this case, we have

$$\frac{dp}{dx} = 0 \Rightarrow p = C \text{ (constant)}$$

Putting $p = C$ in (i), we obtain

$$y = Cx \pm \frac{aC}{\sqrt{1+C^2}} \quad \ldots(ii)$$

as the required solution.

CASE II When $x \pm \frac{a}{(1+p^2)^{3/2}} = 0$:

In this case, we have

$$x = \pm \frac{a}{(1+p^2)^{3/2}}$$

$$\Rightarrow 1 + p^2 = \left(\frac{a^2}{x^2}\right)^{1/3}$$

$$\Rightarrow p^2 = \frac{a^{2/3} - x^{2/3}}{x^{2/3}}$$

Substituting $p^2 = \frac{a^{2/3} - x^{2/3}}{x^{2/3}}$ in (i), we obtain

$$y = px \pm \frac{ap}{(a/x)^{1/3}}$$

$$\Rightarrow y = px \pm a^{2/3}x^{1/3}p$$

$$\Rightarrow y^2 = p^2x^{2/3}(x^{2/3} \pm a^{2/3})^2$$

$$\Rightarrow y^2 = (a^{2/3} - x^{2/3})(x^{2/3} \pm a^{2/3})^2$$

$$\Rightarrow y^2 = (a^{2/3} - x^{2/3})^3 \text{ or, } y^2 = (a^{2/3} - x^{2/3})(x^{2/3} + a^{2/3})^2$$

$$\Rightarrow x^{2/3} + y^{2/3} = a^{2/3} \text{ or, } y^2 = (a^{2/3} - x^{2/3})(x^{2/3} + a^{2/3})^2 \quad \ldots(iii)$$

Hence, the required curves are given by (ii) and (iii).

EXAMPLE 24 *If the length of the subnormal at any point P on the curve is directly proportional to OP^2, where O is the origin, then form the differential equation of the family of curves and hence find the family of curves.*

SOLUTION Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on it. Then, the length of the subnormal at $P = y\frac{dy}{dx}$.

It is given that

Length of the subnormal at $P \propto OP^2$

$$\Rightarrow y\frac{dy}{dx} = \lambda OP^2, \lambda \text{ is the constant of proportionality}$$

$$\Rightarrow y\frac{dy}{dx} = \lambda(x^2 + y^2)$$

$$\Rightarrow 2y\frac{dy}{dx} - 2\lambda y^2 = 2\lambda x^2$$

$$\frac{d(y^2)}{dx} - 2\lambda y^2 = 2\lambda x^2$$

$$\Rightarrow \frac{dv}{dx} - 2\lambda v = 2\lambda x^2, \quad \ldots(i)$$

where $v = y^2$

This is a linear differential equation

$$\text{I.F.} = e^{-\int 2\lambda dx} = e^{-2\lambda x}$$

Multiplying (i) by $e^{-2\lambda x}$ and integrating, we obtain

$$ve^{-2\lambda x} = \int 2\lambda x^2 e^{-2\lambda x}dx$$

$$\Rightarrow ve^{-2\lambda x} = 2\lambda\left[-\frac{x^2}{2\lambda}e^{-2\lambda x} + \frac{2}{2\lambda}\int xe^{-2\lambda x}dx\right]$$

$$\Rightarrow ve^{-2\lambda x} = 2\lambda\left[-\frac{x^2}{2\lambda}e^{-2\lambda x} + \frac{1}{\lambda}\left|\frac{-xe^{-2\lambda x}}{2\lambda} - \frac{1}{4\lambda^2}e^{-2\lambda x}\right|\right] + C$$

$$\Rightarrow ve^{-2\lambda x} = -x^2e^{-2\lambda x} - \frac{x}{\lambda}e^{-2\lambda x} - \frac{1}{2\lambda^2}e^{-2\lambda x} + C$$

$$\Rightarrow y^2 = -x^2 - \frac{x}{\lambda} - \frac{1}{2\lambda^2} + Ce^{2\lambda x}$$

This is the required family of curves.

EXAMPLE 25 *The tangent at a point P of a curve meets the Y-axis at A, and the line parallel to the Y-axis through P meet the X-axis at B. If area of triangle OAB is constant (O being the origin), then form the differential equation of the curve and show that the curve is a hyperbola.*

SOLUTION Let the equation of the curve be $y = f(x)$ and let $P(x, y)$ be any point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets Y-axis at $A\left(0, y - x\frac{dy}{dx}\right)$.

The coordinates of B are $(x, 0)$.

It is given that

Area of $\Delta OAB = k$ (constant)

$\Rightarrow \frac{1}{2} OB \times OA = k$

$\Rightarrow \frac{1}{2} x\left(y - x\frac{dy}{dx}\right) = k$

$\Rightarrow x(y\,dx - x\,dy) = 2k\,dx$

$\Rightarrow y\,dx - x\,dy = \frac{2k}{x}dx$

$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = -\frac{2k}{x^3}dx$

$\Rightarrow d\left(\frac{y}{x}\right) = -\frac{2k}{x^3}dx$

On integrating, we get

$\frac{y}{x} = \frac{k}{x^2} + C$

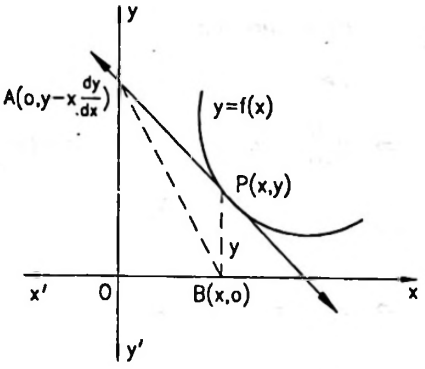


Fig. 13.17

$\Rightarrow xy = k + Cx^2$

$\Rightarrow Cx^2 - xy + k = 0$, which is the required equation of the curve.

For this curve, we have

$\Delta = -\frac{k}{4} \neq 0$ and $h^2 - ab = \frac{1}{4} > 0$.

So, it represents a hyperbola.

EXAMPLE 26 *A normal is drawn at a point $P(x, y)$ of a curve. It meets the X-axis and Y-axiz in the points A and B, respectively such that $\frac{1}{OA} + \frac{1}{OB} = 1$, where O is the origin. Find the equation of such a curve passing through (5, 4).*

SOLUTION Let the equation of the curve be $y = f(x)$ and let $P(x, y)$ be any point on it. The equation of the normal at $P(x, y)$ is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

It meets the coordinate axes at $A\left(x + y\frac{dy}{dx}, 0\right)$ and $B\left(0, y + x\frac{dx}{dy}\right)$ respectively.

$\therefore\ OA = x + y\frac{dy}{dx}$ and $OB = y + x\frac{dx}{dy}$.

Now, $\frac{1}{OA} + \frac{1}{OB} = 1$

$\Rightarrow \frac{1}{x + y\frac{dy}{dx}} + \frac{1}{y + x\frac{dx}{dy}} = 1$

$\Rightarrow 1 + \frac{dy}{dx} = x + y\frac{dy}{dx}$

$\Rightarrow (x - 1) + (y - 1)\frac{dy}{dx} = 0$

$\Rightarrow (x - 1)\,dx + (y - 1)\,dy = 0$

On integrating, we get

$\frac{(x-1)^2}{2} + \frac{(y-1)^2}{2} = C$

$\Rightarrow (x - 1)^2 + (y - 1)^2 = 2C$ ...(i)

It passes through (5, 4). Therefore,

$$16 + 9 = 2C \Rightarrow C = \frac{25}{2}$$

Putting the value of C in (i), we obtain the equation of the required curve as

$$(x - 1)^2 + (y - 1)^2 = 25.$$

EXAMPLE 27 *Let $y = f(x)$ be a curve passing through $(1, \sqrt{3})$ such that tangent at any point P on the curve lying in the first quadrant has positive slope and the tangent and the normal at the point P cuts the x-axis at A and B respectively so that the mid-point of AB is the origin. Find the differential equation of the curve and hence determine $f(x)$.*

SOLUTION The equations of the tagnent and normal at $P(x, y)$ are

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{...(i)}$$

and, $$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \quad \text{...(ii)}$$

respectively.

Lines (i) and (ii) cut X-axis at A and B respectively. So, the coordinates of A and B are

$$A\left(x - y\frac{dx}{dy}, 0\right), B\left(x + y\frac{dy}{dx}, 0\right)$$

It is given that the mid-point of AB is the origin. Therefore,

$$\frac{2x - y\frac{dx}{dy} + y\frac{dy}{dx}}{2} = 0$$

$\Rightarrow 2x - y\frac{dx}{dy} + y\frac{dy}{dx} = 0$

$\Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$

$$\Rightarrow \frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y} \qquad \left[\because \frac{dy}{dx} > 0\right]$$

$$\Rightarrow y\,dy + x\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow \frac{1}{2}\frac{d(x^2+y^2)}{\sqrt{x^2+y^2}} = dx$$

$$\Rightarrow \sqrt{x^2+y^2} = x + C \qquad ...(iii) \qquad [\text{On integration}]$$

As it passes through $(1, \sqrt{3})$. Therefore

$2 = 1 + C \Rightarrow C = 1.$

Putting $C = 1$ in (iii), we obtain the equation of the required curve as

$$\sqrt{x^2+y^2} = x+1 \quad \text{or,} \quad y^2 = 2x+1$$

EXAMPLE 28 *A curve passing through (1, 0) is such that the ratio of the square of the intercept cut by any tangent off the y-axis to the subnormal is equal to the ratio of the product of the coordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.*

SOLUTION Let $y = f(x)$ be the equation of the curve and let $P(x, y)$ be an arbitrary point on it. The equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets Y-axis at $\left(0, y - x\frac{dy}{dx}\right)$.

$\therefore$ Y-intercept $= y - x\frac{dy}{dx}$.

It is given that :

$$\frac{\text{Y- intercept}}{\text{Sub normal}} = \frac{\text{Product of the coordinates of } P}{(\text{Slope of the tangent})^2 \times (\text{Sub tangent})}$$

$$\frac{\left(y - x\frac{dy}{dx}\right)^2}{y\frac{dy}{dx}} = \frac{xy}{\left(\frac{dy}{dx}\right)^2\left(\frac{y}{dy/dx}\right)}$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = xy$$

$$\Rightarrow y - x\frac{dy}{dx} = \pm\sqrt{xy}$$

$$\Rightarrow x\frac{dy}{dx} - y = \pm\sqrt{xy}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \pm\sqrt{\frac{y}{x}}$$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} - v = \pm\sqrt{v}$$

$$\Rightarrow \frac{dv}{\sqrt{v}} = \pm\frac{dx}{x}$$

$$\Rightarrow 2\sqrt{v} = \pm\log x + \log C$$

$$\Rightarrow 2\sqrt{\frac{y}{x}} = \pm\log x + \log C \qquad ...(i)$$

It passes through (1, 0). Therefore,

$0 = \pm\log 1 + \log C \Rightarrow \log C = 0 \Rightarrow C = 1$

Putting $C = 1$ in (i), we obtain

$$2\sqrt{\frac{y}{x}} = \pm\log x$$

$$\Rightarrow \log x = \pm 2\sqrt{\frac{y}{x}}$$

$$\Rightarrow x = e^{\pm 2\sqrt{y/x}}.$$

EXAMPLE 29 *The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 minutes, when will the temperature be 295 K.*

SOLUTION Let T be the temperature of the substance at any time t.

Then,

$$\frac{dT}{dt} \propto -(T - 290)$$

$$\Rightarrow \frac{dT}{dt} = -k(T - 290)$$

$$\Rightarrow \frac{dT}{T-290} = -k\,dt$$

$$\Rightarrow \log(T - 290) = -kt + C \qquad ...(i)$$

When $t = 0$, $T = 370$ K and when $T = 10$, $t = 330$ K.

$\therefore \log(370 - 290) = C$ and $\log(330 - 290) = -10k + C$

$\Rightarrow C = \log 80$ and $\log 40 = -10k + C$

$\Rightarrow C = \log 80$ and $\log 40 = -10k + \log 80$

$$\Rightarrow C = \log 80 \text{ and } 10k = \log\left(\frac{80}{40}\right)$$

$$\Rightarrow C = \log 80 \text{ and } k = \frac{1}{10}\log 2.$$

Putting the values of C and k in (i), we obtain

$$\log(T - 290) = \left(\frac{1}{10}\log 2\right)t + \log 80.$$

When $T = 295$ K, we have

$$\log 5 = -\left(\frac{1}{10}\log 2\right)t + \log 80$$

$$\Rightarrow \left(\frac{1}{10}\log 2\right)t = \log\left(\frac{80}{5}\right)$$

$$\Rightarrow \left(\frac{1}{10}\log 2\right)t = 4\log 2$$

$$\Rightarrow t = 40$$

Hence, the temperature will be 295 K after 40 miniutes.

EXAMPLE 30 *Find the time required for a cylinderical tank of radius 2.5 m and height 3 m to empty through a round hole of radius 25 mm in the bottom of the tank, given that water flows from this hole with velocity $2.5\sqrt{h}$ m/sec, h being the depth of water in the tank.*

SOLUTION Let V be the volume of the water in the tank at any time t and h be the height of the water in the tank. Then,

$$\frac{dV}{dt} = -\pi\left(\frac{2.5}{100}\right)^2 (2.5\sqrt{h})$$

$$\Rightarrow \frac{d}{dt}(\pi \times 2.5)^2 \times h) = -\pi (2.5)^2 \times 10^{-4} \times 2.5\sqrt{h}$$

$$\Rightarrow \pi \times (2.5)^2 \frac{dh}{dt} = -\pi (2.5)^2 \times 10^{-4} \times 2.5\sqrt{h}$$

$$\Rightarrow \frac{dh}{dt} = -2.5 \times 10^{-4}\sqrt{h}$$

$$\Rightarrow h^{-1/2}\, dh = -2.5 \times 10^{-4}\, dt$$

On integrating, we get

$$2\sqrt{h} = -2.5 \times 10^{-4} t + C \qquad \ldots(i)$$

At $t = 0$, we have $h = 3$ m

$\therefore\ 2\sqrt{3} = C$

Putting the value of C in (i), we obtain

$$2\sqrt{h} = -2.5 \times 10^{-4} t + 2\sqrt{3}$$

$$t = \frac{2(\sqrt{3} - h)}{2.5 \times 10^{-4}} = 8(\sqrt{3} - h) \times 10^3$$

When $h = 0$, $t = 8\sqrt{3} \times 10^3$ sec.

EXAMPLE 31 *Suppose that a mothball loses volume by evaporation at a rate proportional to its instataneous area. If the diameter of the ball decreases from 2 cm to 1 cm in 3 months, how long will it take until the ball has practically gone.*

SOLUTION Let V be the volume, S the surface area and x the diameter of the mothball at any time t. Then,

$$V = \frac{\pi}{6}x^3 \text{ and } S = \pi x^2$$

It is given that

$$\frac{dV}{dt} \propto -S$$

$$\Rightarrow \frac{dV}{dt} = -\lambda S$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\pi}{6}x^3\right) = -\pi\lambda x^2$$

$$\Rightarrow \frac{dx}{dt} = -2\lambda$$

$$\Rightarrow x = -2\lambda t + C$$

At $t = 0$, we have, $x = 2$ and at $t = 3$, $x = 1$

$\therefore\ C = 2$ and $1 = -6\lambda + C$

$$\Rightarrow C = 2, \lambda = \frac{1}{6}$$

$$\therefore\ x = -\frac{1}{3}t + 2$$

When $x = 0$, we have, $-\frac{1}{3}t + 2 = 0 \Rightarrow t = 6$.

Thus, the mothball will vanish completely in 6 months.

EXAMPLE 32 *A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The out let is opened at some instant. The flow through the output is according to the law $v(t) = 0.6\sqrt{2g\,h(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank.*

[IIT 2001]

SOLUTION Let at any instant t the height of the water level above the bottom of the hemispherical tank be h metres and v be the volume of the water. Then,

$$\frac{dV}{dt} = -12 \times 10^{-4} v(t)$$

$$= -12 \times 10^{-4} \times 0.6\sqrt{2g\,h(t)}$$

$$= -72 \times 10^{-4}\sqrt{2gh}\ , \text{where } h = h(t) \qquad \ldots(i)$$

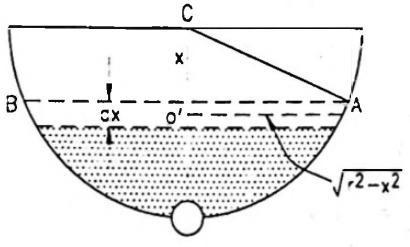


Fig. 13.18

In order to find the volume V of the water at time t, let us consider a water layer of thickness dx at a depth x below the centre O of the hemispherical tank. Then,

$$dV = \pi(r^2 - x^2)\, dx$$

$$\therefore V = \int_{(r-h)}^{r} dV$$

$$= \int_{(r-h)}^{r} \pi (r^2 - x^2)\, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left[r^2 (r - r + h) - \frac{1}{3} \{ r^3 - (r-h)^3 \} \right]$$

$$= \pi \left[r^2 h + \frac{1}{3} (3rh^2 - 3r^2 h - h^3) \right]$$

$$= \frac{\pi}{3} [3rh^2 - h^3]$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{3} \left[6r\, h \frac{dh}{dt} - 3h^2 \frac{dh}{dt} \right]$$

$$\Rightarrow \frac{dV}{dt} = \pi \left[2rh \frac{dh}{dt} - h^2 \frac{dh}{dt} \right]$$

$$= \pi (2rh - h^2) \frac{dh}{dt} \qquad \text{...(ii)}$$

From (i) and (ii)

$$\pi (2rh - h^2) \frac{dh}{dt} = -72 \times 10^{-4} \sqrt{2gh}$$

$$\Rightarrow \pi (4h - h^2) \frac{dh}{dt} = -72 \times 10^{-4} \sqrt{2gh} \qquad [\because r = 2m]$$

$$\Rightarrow \frac{\pi}{\sqrt{2g}} (4\sqrt{h} - h^{3/2})\, dh = -72 \times 10^{-4}\, dt$$

$$\Rightarrow 72 \times 10^{-4}\, dt = \frac{\pi}{\sqrt{2g}} (h^{3/2} - 4\sqrt{h})\, dh$$

$$\Rightarrow 72 \times 10^{-4} \int_{t=0}^{T} dt = \frac{\pi}{\sqrt{2g}} \int_{2}^{0} (h^{3/2} - 4\sqrt{h})\, dh$$

$$\Rightarrow 72 \times 10^{-4} T = \frac{\pi}{\sqrt{2g}} \left[\frac{2}{5} h^{5/2} - \frac{8}{3} h^{3/2} \right]_2^0$$

$$\Rightarrow 72 \times 10^{-4} T = \frac{\pi}{\sqrt{2g}} \left[-\frac{2}{5} \times 4\sqrt{2} + \frac{8}{3} \times 2\sqrt{2} \right]$$

$$\Rightarrow 72 \times 10^{-4} T = \frac{\pi}{\sqrt{2g}} \left(\frac{56\sqrt{2}}{15} \right)$$

$$\Rightarrow T = \frac{\pi}{\sqrt{2g}} \times \frac{56}{15} \sqrt{2} \times \frac{1}{72 \times 10^{-4}}$$

$$T = \frac{7\pi}{135\sqrt{g}} \times 10^4.$$

Hence, the hemispherical tank will be emptied out in time $T = \frac{7\pi}{135\sqrt{g}} \times 10^4$.

EXAMPLE 33 *A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservior B. Both the reservoirs are filled completely with water, their inlets are closed and then water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservior at that time. One hour after the water is released, the quantity of water in reservoir A is* $1\frac{1}{2}$ *times the quantity of water in reservoir B. After how many hours do both the resevoirs have the same quantity of water ?*

[IIT 1997]

SOLUTION Let v_0 and V_0 be the capacities of reservoirs A and B respectively. At any time t, let v and V be the volumes of the water in the reservoirs A and B respectively.

It is given that $v_0 = 2\, V_0$.

Also, $\frac{dv}{dt} = -kv$ and $\frac{dV}{dt} = -KV$

$\Rightarrow \frac{dv}{v} = -k\, dt$ and $\frac{dV}{V} = -K\, dt$

$\Rightarrow \log v = -k t + \log C_1$ and $\log V = -K t + \log C_2$

Initially i.e. at $t = 0$, $v = v_0$ and $V = V_0$.

$\therefore \log C_1 = \log v_0$ and $\log C_2 = \log V_0$.

Thus, $\log v = -kt + \log v_0$ and $\log V = -Kt + \log V_0$

$$\Rightarrow \log \left(\frac{v}{v_0} \right) = -kt \text{ and } \log \left(\frac{V}{V_0} \right) = -Kt$$

$$\Rightarrow \log \left(\frac{v}{v_0} \right) - \log \left(\frac{V}{V_0} \right) = -t\,(k - K)$$

$$\Rightarrow \log \left(\frac{v}{V} \times \frac{V_0}{v_0} \right) = -t\,(k - K)$$

$$\Rightarrow \log \left(\frac{v}{2V} \right) = -t\,(k - K) \qquad \text{...(i)} \qquad [\because v_0 = 2\, V_0]$$

At $t = 1$, it is given that $v = \frac{3}{2} V$. Therefore,

$$\log \left(\frac{3}{4} \right) = K - k$$

Suppose after T hours both the reservoirs have the same quantity of water i.e. $v = V$. Then,

$$\log \left(\frac{V}{2V} \right) = -T\,(k - K) \qquad [\text{Putting } v = V \text{ and } t = T \text{ in (i)}]$$

$$\Rightarrow \log \left(\frac{1}{2} \right) = T \left(\log \frac{3}{4} \right)$$

$$\Rightarrow T = \frac{\log (1/2)}{\log (3/4)} = \frac{\log 2}{\log (4/3)} = \frac{\log 2}{\log 4 - \log 3}$$

EXAMPLE 34 *According to Newton's law, the rate of cooling is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is 20°C and the body cools for 20 minutes from 100°C to 60°C, how long will it take for its temperature to drop to 30°C.*

SOLUTION At any time T, let the temperature of the body be T°C. Then,

$$\frac{dT}{dt} = -k\,(T-20)$$

$$\Rightarrow \frac{dT}{T-20} = -k\,dt$$

$$\Rightarrow \log(T-20) = -kt + \log C \qquad ...(i)$$

It is given that at

$t = 0, T = 100°\,C$ and at $t = 20 \times 60$ sec, $T = 60°\,C$

$$\therefore \log 80 = \log C \text{ and } \log 40 = -1200\,k + \log C$$

$$\Rightarrow \log 40 = -1200\,k + \log 80$$

$$\Rightarrow k = \frac{1}{1200}\log 2$$

Substituting the values of log C and k in (i), we get

$$\log(T-20) = \left(-\frac{1}{1200}\log 2\right)t + \log 80$$

Putting $T = 30$°C, we get

$$\log 10 = -\left(\frac{1}{1200}\log 2\right)t + \log 80$$

$$\Rightarrow \qquad t = \frac{1200}{\log 2}\log 8 = 3600 \text{ sec.} = 60 \text{ minutes.}$$

EXAMPLE 35 *Let $y = f(x)$ be a curve having property that distance of any tangent of the curve from the origin is always equal to 1 unit. Form the differential equation of the curve and hence determine all such curves.*

SOLUTION The equation of the tangent at any point $P(x, f(x))$ to the curve $y = f(x)$ is

$$Y - f(x) = f'(x)\,(X - x)$$

$$\Rightarrow Xf'(x) - Y + f(x) - xf'(x) = 0 \qquad ...(i)$$

It is given that the distance of any tangent of the curve $y = f(x)$ from the origin is always 1 unit. Therefore,

Length of the perpendicular from the origin to (i) = 1

$$\left|\frac{f(x) - xf'(x)}{\sqrt{[f'(x)]^2 + 1^2}}\right| = 1$$

$$\Rightarrow \left|\frac{y - xp}{\sqrt{p^2+1}}\right| = 1, \text{ where } p = f'(x) = \frac{dy}{dx}$$

$$\Rightarrow y - px = \pm\sqrt{p^2+1}$$

$$\Rightarrow y = px \pm \sqrt{p^2+1} \qquad ...(ii)$$

$$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} \pm \frac{p}{\sqrt{p^2+1}}\frac{dp}{dx}$$

$$\Rightarrow x\frac{dp}{dx} \pm \frac{p}{\sqrt{p^2+1}}\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{dp}{dx}\left\{x \pm \frac{p}{\sqrt{p^2+1}}\right\} = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \text{ or, } x \pm \frac{p}{\sqrt{p^2+1}} = 0$$

$$\Rightarrow p = c \text{ or, } x \pm \frac{p}{\sqrt{p^2+1}} = 0$$

CASE I *When $p = C$*

Putting $p = C$ in (ii), we get

$y = Cx \pm \sqrt{C^2+1}$, which is the family of equations of the required curves.

CASE II *When $x \pm \frac{p}{\sqrt{p^2+1}} = 0$*

In this case, we have to eliminate p from (ii) and $x \pm \frac{p}{\sqrt{p^2+1}} = 0$.

Now,

$$x \pm \frac{p}{\sqrt{p^2+1}} = 0$$

$$\Rightarrow x^2 = \frac{p^2}{p^2+1}$$

$$\Rightarrow \frac{p^2+1}{p^2} = \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{x^2} - 1$$

$$\Rightarrow \frac{1}{p^2} = \frac{1-x^2}{x^2}$$

$$\Rightarrow p = \pm\frac{x}{\sqrt{1-x^2}}$$

Putting this value of p in (ii), we get

$$y = \pm\frac{x^2}{\sqrt{1-x^2}} \pm \sqrt{\frac{x^2}{1-x^2}+1}$$

$$\Rightarrow y = \pm\frac{x^2}{\sqrt{1-x^2}} \pm \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y = \frac{\pm x^2 \pm 1}{\sqrt{1-x^2}}$$

$$\Rightarrow y = \frac{1+x^2}{\sqrt{1-x^2}}, y = \frac{x^2-1}{\sqrt{1-x^2}}, y = \frac{-x^2+1}{\sqrt{1-x^2}}, y = \frac{-1-x^2}{\sqrt{1-x^2}}$$

$$\Rightarrow y = \pm\frac{(1+x^2)}{\sqrt{1-x^2}}, y = \pm\sqrt{1-x^2}$$

$$\Rightarrow y = \pm \frac{1+x^2}{\sqrt{1-x^2}}, x^2 + y^2 = 1.$$

Hence, the equations of the curves are

$$y = Cx \pm \sqrt{C^2+1}, y = \pm \frac{1+x^2}{\sqrt{1-x^2}}, x^2 + y^2 = 1, \text{where } C \in R.$$

EXAMPLE 36 *Let $f(x, y)$ be a curve in the xy-plane having the property that distance from the origin of any tangent to the curve is equal to distance of point of contact from the y-axis. If $f(1, 2) = 0$, determine all such possible curves.*

SOLUTION Let $P(x, y)$ be any point on the curve. Then, the equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow X\frac{dy}{dx} - Y + y - x\frac{dy}{dx} = 0 \quad \ldots(i)$$

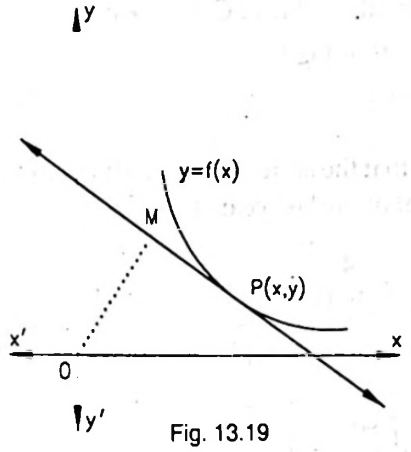


Fig. 13.19

It is given that the distance of (i) from the origin is equal to the distance of $P(x, y)$ from y-axis. Therefore,

$$\left|\frac{y - x\frac{dy}{dx}}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}\right| = |x|$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = x^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = x^2 + x^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2 - 2xy\frac{dy}{dx} = x^2$$

$$\Rightarrow y^2\,dx - 2xy\,dy = x^2\,dx$$

$$\Rightarrow 2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \frac{x\,d(y^2) - y^2\,dx}{x^2} = -dx$$

$$\Rightarrow d\left(\frac{y^2}{x}\right) = -dx$$

$$\Rightarrow \frac{y^2}{x} = -x + C$$

$$\Rightarrow y^2 = -x^2 + Cx \quad \ldots(ii)$$

This passes through (1, 2). Therefore,

$$4 = -1 + C \Rightarrow C = 5$$

Putting the value of C in (ii), we get

$$y^2 = -x^2 + 5x$$

This is the equation of the required curve.

EXAMPLE 37 *Given the curves $y = f(x)$ passing through the point $(0, 1)$ and $y = \int_{-\infty}^{x} f(t)\,dt$ passing through the point $\left(0, \frac{1}{2}\right)$. The tangents drawn to both the curves at the points with equal abscissae intersect on the x-axis. Find the curve $y = f(x)$.*

SOLUTION The equations of the tangents to the curves $y = f(x)$ and $y = \int_{-\infty}^{x} f(t)\,dt$ at arbitrary points on them are

$$Y - f(x) = f'(x)(X - x) \quad \ldots(i)$$

and, $Y - \int_{-\infty}^{x} f(t)\,dt = f(x)(X - x) \ldots(ii)$

$$\left[\begin{array}{l} y = \int_{-\infty}^{x} f(t)\,dt \\ \Rightarrow \frac{dy}{dx} = f(x) \end{array}\right]$$

respectively.

It is given that (i) and (ii) intersect at the same point on X-axis. Therefore, putting $y = 0$ and equating X-coordinates obtained from (i) and (ii), we get

$$x - \frac{f(x)}{f'(x)} = x - \frac{\int_{-\infty}^{x} f(t)\,dt}{f(x)}$$

$$\Rightarrow \frac{f(x)}{\int_{-\infty}^{x} f(t)\,dt} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \log\left|\int_{-\infty}^{x} f(t)\,dt\right| = \log f(x) + \log C \quad \text{[On intergration]}$$

$$\Rightarrow \int_{-\infty}^{x} f(t)\,dt = Cf(x) \quad \ldots(iii)$$

It is given that at $x = 0, f(0) = 1$ and $\int_{-\infty}^{0} f(t)\,dt = \frac{1}{2}$.

$$\therefore \frac{1}{2} = C \times 1 \Rightarrow C = \frac{1}{2}.$$

Putting $C = \frac{1}{2}$ in (iii), we get

$$\int_{-\infty}^{x} f(t)\,dt = \frac{1}{2}f(x)$$

Differentiating both sides w.r.to x, we get

$$f(x) = \frac{1}{2}f'(x)$$

$$\Rightarrow 2f(x) = f'(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 2$$

$$\Rightarrow \log |f(x)| = 2x + \log C_1$$

$$\Rightarrow f(x) = C_1 e^{2x}$$

At $x = 0$, we have $f(0) = 1$.

$$\therefore C_1 = 1.$$

Hence, $f(x) = e^{2x}$.

EXAMPLE 38 *A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the mid-point AB. The curve passes through the point (1, 1). Determine the equation of the curve.*

[IIT 1998]

SOLUTION Let $y = f(x)$ be the equation of the curve.

The equation of the tangent at any point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets the coordinate axes at $A\left(x - y\frac{dx}{dy}, 0\right)$ and $\left(0, y - x\frac{dy}{dx}\right)$.

It is given that $P(x, y)$ is the mid-point of AB. Therefore,

$$x - y\frac{dx}{dy} = 2x \text{ and } y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow x\frac{dy}{dx} = -y$$

$$\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

$$\Rightarrow \log x = -\log y + \log C$$

$$\Rightarrow xy = C$$

The curve passes through (1, 1). Therefore,

$$C = 1$$

Hence, the requation of the curve C is $xy = 1$.

EXAMPLE 39 *A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to*

$$\frac{\ln 10 - \ln 9}{\ln (1.04) - 0.03}.$$

[IIT 2000]

SOLUTION Let P_0 be the initial population of the country and Q_0 be its initial food production. Let P be the population of the country in year t and Q be its food production in year t. Then,

$$\frac{dP}{dt} = \text{rate of change of population}$$

$$\Rightarrow \frac{dP}{dt} = \frac{3P}{100} \quad \text{[Given]}$$

$$\Rightarrow \frac{dP}{P} = 0.03\,dt$$

$$\Rightarrow \log P = 0.03t + C \quad \text{...(i)}$$

At $t = 0$, we have $P = P_0$.

$$\therefore \log P_0 = C$$

Substituting the value of C in (i), we get

$$\log P = 0.03t + \log P_0$$

$$\Rightarrow P = P_0 e^{0.03t} \quad \text{...(ii)}$$

It is given that the annual food production every year is 4% more than that of the last year. Therefore,

$$Q = Q_0\left(1 + \frac{4}{100}\right)^t$$

Let the average consumption of food be k units per person. Then,

$$Q_0 = kP_0\left(\frac{90}{100}\right) = 0.9kP_0$$

$$\therefore Q = 0.9kP_0\left(1 + \frac{4}{100}\right)^t \quad \text{...(iii)}$$

This gives the quantity of food available in year t.

The population in year t is

$$P = P_0 e^{0.03t} \quad \text{[From (ii)]}$$

Therefore,

$$\text{Food consumption in year } t = kP_0 e^{0.03t} \quad \text{...(iv)}$$

The country will be self-sufficient, if

$$Q \geq P$$

$$\Rightarrow 0.9kP_0\left(1 + \frac{4}{100}\right)^t \geq kP_0 e^{0.03t}$$

$$\Rightarrow \frac{9}{10}(1.04)^t \geq e^{0.03t}$$

$$\Rightarrow (1.04)^t e^{-0.03t} \geq \frac{10}{9}$$

$$\Rightarrow t\ln(1.04) - 0.03t \geq \ln\left(\frac{10}{9}\right)$$

$$\Rightarrow t\{\ln(1.04) - 0.03\} \geq \ln 10 - \ln 9$$

$\Rightarrow t \geq \dfrac{\ln 10 - \ln 9}{\ln (1.04) - 0.03}$

Thus, the least integral value of the year n when the country becomes, self-sufficient is the smallest integer greater than or equal to

$$\frac{\ln 10 - \ln 9}{\ln (1.04) - 0.03}.$$

EXAMPLE 40 *A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = $k > 0$). Find the time after which the cone is empty.* [IIT 2003]

SOLUTION Let the semi-vertical angle of the cone be θ and let the height of the liquid at time 't' be 'h' from the vertex V and radius of the liquid cone be r.

Let V be the volume of liquid at time t. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi r^3 \cot\theta \qquad \left[\because \tan\theta = \frac{r}{h}\right]$$

Let S be the surface area of the liquid in contact with air at time t. Then, $S = \pi r^2$

It is given that

$$\frac{dV}{dt} \propto S$$

$$\Rightarrow \frac{dV}{dt} = -kS \text{, where } k \text{ is the constant of proportionality.}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{3}\pi r^3 \cot\theta\right) = -k\pi r^2$$

$$\Rightarrow \pi r^2 \frac{dr}{dt}\cot\theta = -k\pi r^2$$

$$\Rightarrow \frac{dr}{dt}\cot\theta = -k$$

$$\Rightarrow \cot\theta\, dr = -k\, dt$$

$$\Rightarrow \cot\theta \int_R^0 dr = -k\int_0^T dt, \text{ where } T \text{ is the required time}$$

$$\Rightarrow R\cot\theta = kT$$

$$\Rightarrow T = \frac{R\cot\theta}{k}$$

$$\Rightarrow T = \frac{H}{k} \qquad \left[\because \tan\theta = \frac{R}{H}\right]$$

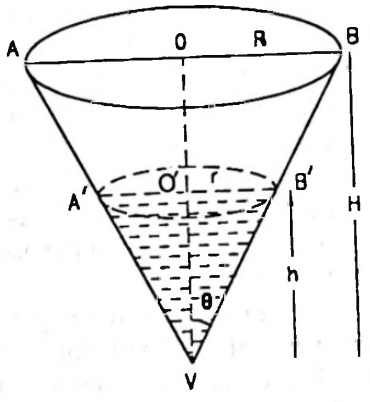


Fig. 13.20

EXERCISE 13.9

1. The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

2. Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year? [Use : $e^{-\frac{\log 2}{1590}} = 0.9996$]

3. The slope of the tangent at a point $P(x, y)$ on a curve is $\frac{-x}{y}$. If the curve passes through the point $(3, -4)$, find the equation of the curve.

4. Find the equation of the curve which passes through the point $(2, 2)$ and satisfies the differential equation

$$y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}.$$

5. Find the equation of the curve passing through the point $\left(1, \frac{\pi}{4}\right)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with x-axis.

6. Find the curve for which the intercept cut off by a tangent on x-axis is equal to four times the ordinate of the point of contact.

7. Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y + 2(x + 1) = 2e^{2x}$.

8. The tangent at any point (x, y) of a curve makes an angle $\tan^{-1}(2x + 3y)$ with x-axis. Find the equation of the curve if it passes through $(1, 2)$.

9. Find the equation of the curve such that the portion of the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point $(1, 2)$.

10. Find the equation to the curve satisfying $x(x+1)\frac{dy}{dx} - y = x(x+1)$ and passing through $(1, 0)$.

11. Find the equation of the curve which passes through the point $(3, -4)$ and has the slope $\frac{2y}{x}$ at any point (x, y) on it.

12. Find the equation of the curve which passes through the origin and has the slope $x + 3y - 1$ at any point (x, y) on it.

13. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0, 1)$. Find the equation of the curve.

14. A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

15. Find the equation of the curve which passes through the point $(1, 2)$ and the distance between the foot of the ordinate of the point of contact and the point of intersection of the tangent with x-axis is twice the abscissa of the point of contact.

16. The normal to a given curve at each point (x, y) on the curve passes through the point $(3, 0)$. If the curve contains the point $(3, 4)$, find its equation.

17. The rate of increase of bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 6 hours. Prove that the bacteria becomes 8 times at the end of 18 hours.

18. A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$ and hence find the equation of the curve.

19. A particle moves so that its velocities parallel to the axes of x and y are respectively $-ky$ and kx, where k is a constant different from zero. Find the path of the particle if it passes through $(3, 4)$.

20. Show that a particle $P(x, y)$ which moves so that $\frac{dy}{dt} = ax + hy + g$, $\frac{dx}{dt} = -(hx + by + f)$ always lies on a conic section.

21. The part of the normal between $P(x, y)$ of a curve and the x-axis is a constant k. Find the differential equation of the family and show that the curves are all circles.

22. The population of a country increases at a rate proportional to the number of inhabitants. If the population doubles in 30 years, find the number of years in which it will trible.

23. Show that all the members of the family of curves represented by $\frac{dy}{dx} + y\tan x = \sec x$ pass thorugh $(\pi/2, 1)$.

24. Show that the curves represented by the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ touch the x-axis at the origin.

25. For a certain curve $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has a local minimum value 5 when $x = 1$. Find the equation of the curve and also the global maximum and global minimum values of $f(x)$ given that $0 \le x \le 2$.

26. The tangent at any point $P(x, y)$ of a plane curve meets the X-axis in Q. If the area of ΔOPQ is constant equal to a^2 (where O is the origin), find the equation of the curve, if it passes through $(0, a)$.

27. Determine the equation of the curve passing through the point $(3a, a)$, $a > 0$ in the form $x = f(y)$ which satisfies the differential equation

$$\frac{a^2}{xy}\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x} - 2.$$

28. A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point $(1, 1)$. Determine the equation of the curve.

ANSWERS

1. $\frac{1}{k}\log 2$, k is the constant of proportionality

2. 0.04%

3. $x^2 + y^2 = 25$

5. $2xy - 2x - y - 2 = 0$

6. $\tan\left(\frac{y}{x}\right) = \log\left(\frac{e}{x}\right)$

7. $e^{-xy} = Cy^4$

8. $ye^{-3x} = \left(-\frac{2}{3}x - \frac{1}{9}\right)e^{-3x} + C$

9. $xy = 2$

10. $y = \frac{x}{x+1}(x - 1 + \log x)$

11. $9y + 4x^2 = 0$

12. $3(x + 3y) = 2(1 - e^{3x})$

13. $y + 1 = 2e^{x^2/2}$

14. $x^2 + y^2 = Cx$

15. $y^2 = 4x$

16. $x^2 + y^2 - 6x - 7 = 0$

18. $x^2 + y^2 = cx$

19. $x^2 + y^2 = 25$

22. $30\log_2 3 = 48$ years.

25. $y = x^3 - 2x^2 + x + 5$, Max. = 7, Min = 5

26. $a^3 + y^3 = 2axy^2$

27. $x = y + a\left(\frac{1 + e^{y-k}}{1 - e^{y-k}}\right)$, where $k = \frac{a}{2}(2 + \log 3)$

28. $xy = 1$

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13.8 ORTHOGONAL TRAJECTORIES OF CURVES

In this section, we shall discuss another interesting applications of differential equations for finding curves that intersect given curves at right angles.

ORTHOGONAL TRAJECTORY *Any curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family of curves.*

For example, $y = x$ is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$.

To find the orthogonal trajectories of a given family of curves, we follow the following steps :

ALGORITHM

STEP I *Obtain the given family of curves, say $f(x, y, c) = 0$ where c is an arbitrary constant.*

STEP II *Formulate the differential equation representing the family of curves in step I by eliminating the arbitrary constant (s).*

STEP III *Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the differential equation obtained in step II to obtain the differential equation repreenting orthogonal trajectories of the given family of curves.*

STEP IV *Solve the differential equation obtained in step III to obtain the required orthogonal trajectories.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the orthogonal trajectories of the curves $y = Cx^2$, where C is an arbitrary constant.*

SOLUTION The equation of the given family of curves is

$$y = Cx^2 \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2Cx \qquad ...(ii)$$

Eliminating C between (i) and (ii), we obtain

$$y = \left(\frac{1}{2x}\frac{dy}{dx}\right)x^2$$

$$\Rightarrow \quad 2y = x\frac{dy}{dx} \qquad ...(iii)$$

This is the differential equation of the family of curves given in (i).

The differential equation of the orthogonal trajectories of (i) is obtained by replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii).

Replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (iii), we obtain

$$2y = -x\frac{dx}{dy}$$

$$\Rightarrow \quad 2y\,dy = -x\,dx$$

On integrating, we obtain

$$y^2 = -\frac{x^2}{2} + C$$

$$\Rightarrow \quad x^2 + 2y^2 = 2C$$

This is the required family of orthogonal trajectories.

EXAMPLE 2 *Find the equation of all possible curves that will cut each member of the family of circles $x^2 + y^2 - 2Cx = 0$ (C is a parameter) at right angle.*

SOLUTION Here we have to find the orthogonal trajectories of the family of circles

$$x^2 + y^2 - 2Cx = 0 \qquad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2C = 0$$

$$\Rightarrow \quad C = x + y\frac{dy}{dx} \qquad ...(ii)$$

From (i) and (ii), we obtain

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0 \qquad \text{[By eliminating C]}$$

$$\Rightarrow \quad y^2 - x^2 - 2xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y^2 - x^2 = 2xy\frac{dy}{dx} \qquad ...(iii)$$

This is the differential equation representing the given family of circles.

To find the differential equation of the orthogonal trajectories, we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (iii).

Thus, the differential equation representing the orthogonal trajectories is

$$y^2 - x^2 = -2xy\frac{dx}{dy}$$

$$\Rightarrow \quad y^2\,dy = x^2\,dy - 2xy\,dx$$

$$\Rightarrow \quad -dy = \frac{y\,d(x^2) - x^2\,dy}{y^2}$$

$$\Rightarrow \quad -dy = d\left(\frac{x^2}{y}\right)$$

On integration, we obtain

$$-y = \frac{x^2}{y} + C$$

$$\Rightarrow -y^2 = x^2 + Cy$$

$$\Rightarrow x^2 + y^2 + Cy = 0$$

This is the required family of orthogonal trajectories.

EXAMPLE 3 *Find the orthogonal trajectories of the circles* $x^2 + y^2 - ay = 0$, *where a is a parameter.*

SOLUTION We have,

$$x^2 + y^2 - ay = 0 \qquad ...(i)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\Rightarrow a = \frac{2\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

Substituting the value of a in (i), we obtain

$$x^2 + y^2 - 2y\left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right) = 0$$

$$\Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

This is the differential equation of the family of circles given in (i). The differential equation representing the orthogonal trajectories is

$$-(x^2 - y^2)\frac{dx}{dy} - 2xy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x^2 - y^2}{2xy}\right)$$

$$\Rightarrow 2xy\,dy - y^2\,dx = -x^2\,dx$$

$$\Rightarrow \frac{x\,d(y^2) - y^2\,dx}{x^2} = -dx$$

$$\Rightarrow d\left(\frac{y^2}{x}\right) = -dx$$

$$\Rightarrow \frac{y^2}{x} = -x + C \qquad \text{[On integration]}$$

$$\Rightarrow y^2 + x^2 = Cx$$

REMARK *If the orthogonal trajectories of a family of curves is same as the given family of curves, then the given family of curves is called self-orthogonal.*

EXERCISE 13.10

Find the orthogonal trajectories of each of the following family of curves :

1. $y = Ce^{-x}$
2. $y = C\sqrt{x}$
3. $y = Cx^{3/2}$
4. $y = \frac{C}{x^2}$
5. $xy = C$
6. $x^2 + y^2 = a^2$
7. $y^2 = 4ax$
8. Prove that the family of curves $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, where λ is a parameter, is self orthogonal.

ANSWERS

1. $y^2 = 2x + C$
2. $2x^2 + y^2 = 2C$
3. $2x^2 + 3y^2 = C$
4. $x^2 - 2y^2 = C$
5. $x^2 - y^2 = C$
6. $y = Cx$
7. $2x^2 + y^2 = C^2$

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CHAPTER

14

ADDITION AND SUBTRACTION OF VECTORS

14.1 INTRODUCTION

Physical quantities are divided into two categories - scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly *scalars*. Examples of scalars are mass, volume, density work, temperature etc. To represent a scalar quantity we assign a real number to it which gives its magnitude in terms of a certain basic unit of a quantity a of the level type. Throughout this chapter by scalars we shall mean real numbers. Second kind of quantities are those which have both magnitude and direction. Such quantities are called *vectors*. Displacement, velocity, acceleration, momentum, weight, force etc. are examples of vector quantities.

NOTE *It is to note here that in addition to magnitude and direction, two vector quantities of the same kind should be capable of being compounded according to the parallelogram law of addition. Quantities having magnitude and direction but not obeying the parallelogram law of addition will not be treated as vectors. For example, the rotations of a rigid body through finite angles have both magnitudes and directions but do not satisfy the parallelogram law of addition.*

14.2 REPRESENTATION OF VECTORS

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end emphasizes the direction of the vector. A vector, denoted by $\overrightarrow{PQ}$, is determined by two points P, Q such that the magnitude of the vector is the length of the straight line PQ and its direction is that from P to Q. The point P is called the *initial point* of vector $\overrightarrow{PQ}$ and Q is called the *terminal point or tip*. Vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ etc.

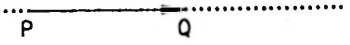


Fig. 14.1

The modulus, or module, or magnitude of a vector $\vec{a}$ is the positive number which is the measure of its length and is denoted by $|\vec{a}|$. The modulus $|\vec{a}|$ of a vector $\vec{a}$ is sometimes written as a.

NOTE *Every vector $\overrightarrow{PQ}$ has the following three characteristics:*

LENGTH *The length of $\overrightarrow{PQ}$ will be denoted by $|\overrightarrow{PQ}|$ or PQ.*

SUPPORT *The line of unlimited length of which PQ is segment is called the support of the vector $\overrightarrow{PQ}$.*

SENSE *The sense of $\overrightarrow{PQ}$ is from P to Q and that of $\overrightarrow{QP}$ is from Q to P. Thus, the sense of a directed line segment is from its initial point to the terminal point.*

14.3 TYPES OF VECTORS

ZERO OR NULL VECTOR *A vector whose initial and terminal points are coincident is called the zero or the null vector.*

Thus, the modulus of the null vector is zero but it can be thought of as having any line as its line of support. The null vector is denoted by $\vec{0}$.

Vectors other than the null vector are called *proper vectors.*

UNIT VECTOR *A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$, read 'a cap'. Thus, $|\hat{a}| = 1$.*

LIKE AND UNLIKE VECTORS *Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.*

EQUALITY OF VECTORS *Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have*

(i) *the same length*

(ii) *the same or parallel support, and*

(iii) *the same sense.*

COLLINEAR OR PARALLEL VECTORS *Vectors having the same or parallel supports are called collinear vectors.*

CO-INITIAL VECTORS *Vectors having the same initial point are called co-initial vectors.*

CO-PLANAR VECTORS *A system of vectors is said to be coplanar, if their supports are parallel to the same plane.*

Note that two vectors are always coplanar.

COTERMINOUS VECTORS *Vectors having the same terminal point are called coterminous vectors.*

NEGATIVE OF A VECTOR *The vector which has the same magnitude as the vector $\vec{a}$ but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus, if $\vec{PQ} = \vec{a}$, then $\vec{QP} = -\vec{a}$.*

RECIPROCAL OF A VECTOR *A vector having the same direction as that of a given vector $\vec{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\vec{a}^{-1}$.*

Thus, if $|\vec{a}| = a$, $|\vec{a}^{-1}| = 1/a$.

LOCALIZED AND FREE VECTORS *A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector. In this chapter we will be dealing with free vectors, unless otherwise stated. Thus a free vector can be taken anywhere in space by choosing an arbitrary initial point.*

14.4 PARALLELOGRAM LAW OF ADDITION OF VECTORS

DEFINITION *If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram which is coinitial with the given vectors.*

Symbolically, we have

$$\vec{OP} + \vec{OQ} = \vec{OR} \quad \text{or,} \quad \vec{a} + \vec{b} = \vec{c}$$

From Fig. 14.2 we have,

$$\vec{PR} = \vec{OQ} = \vec{b}.$$

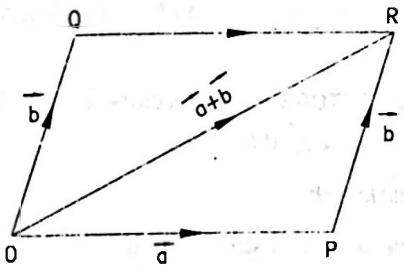


Fig. 14.2

Therefore, in triangle OPR, we have

$$\vec{OP} + \vec{PR} = \vec{OR}.$$

Thus, it follows from this that if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order. This is called the triangle law of addition of vectors.

NOTE *It should be noted that the magnitude of $\vec{a} + \vec{b}$ is not equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$.*

PROPERTIES OF ADDITION OF VECTORS Addition of vectors has the following properties :

(i) *Commutativity* : Vector additon is commutative i.e.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \text{ for any two vectors } \vec{a} \text{ and } \vec{b}.$$

(ii) *Associativity* : Vector addition is associative i.e.

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \text{ for any three vectors } \vec{a}, \vec{b} \text{ and } \vec{c}.$$

(iii) *Existence of additive identity* : For every vector $\vec{a}$, we have

$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}, \text{ where } \vec{0} \text{ is the null vector.}$$

The null vector $\vec{0}$ is called the additive identity.

(iv) *Existence of additive inverse* : For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$$

The vector $-\vec{a}$ is called the additive inverse of $\vec{a}$.

14.5 SUBTRACTION OF VECTORS

DEFINITION *If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and is denoted by $\vec{a} - \vec{b}$ i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.*

Thus, to subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ and add it to $\vec{a}$ as shown in Fig. 14.3.

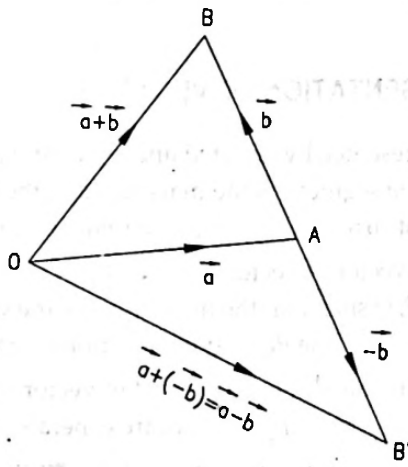


Fig. 14.3

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}, \vec{b}, \vec{c}$ be the vectors represented by the sides of a triangle, taken in order, then prove that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.*

SOLUTION Let ABC be a triangle such that $\vec{BC}=\vec{a}, \vec{CA}=\vec{b}$ and $\vec{AB}=\vec{c}$. Then,

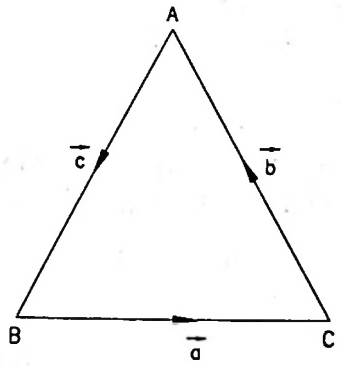


Fig. 14.4

$\vec{a}+\vec{b}+\vec{c}$

$= \vec{BC}+\vec{CA}+\vec{AB}$

$= \vec{BA}+\vec{AB}$ [$\because \vec{BC}+\vec{CA}=\vec{BA}$ by triangle law of addition of vectors]

$= \vec{BB}$ [By triangle law]

$= \vec{0}$ [By def. of null vector]

Hence, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

EXAMPLE 2 *If P_1, P_2, P_3, P_4 are points in a plane or space and O the origin of vectors, show that P_4 coincides with O iff $\vec{OP_1}+\vec{P_1P_2}+\vec{P_2P_3}+\vec{P_3P_4}=\vec{0}$.*

SOLUTION We have,

$\vec{OP_1}+\vec{P_1P_2}+\vec{P_2P_3}+\vec{P_3P_4}=\vec{0}$.

$\Rightarrow (\vec{OP_1}+\vec{P_1P_2})+\vec{P_2P_3}+\vec{P_3P_4} = \vec{0}$ [By assoc. of vector addition]

$\Rightarrow (\vec{OP_2}+\vec{P_2P_3})+\vec{P_3P_4}=\vec{0}$

$\Rightarrow \vec{OP_3}+\vec{P_3P_4}=\vec{0}$

$\Rightarrow \vec{OP_4}=\vec{0}$

$\Rightarrow P_4$ coincides with O [By def. of null vector]

EXAMPLE 3 *If $\vec{PO}+\vec{OQ}=\vec{QO}+\vec{OR}$, show that the points P, Q, R are collinear.*

SOLUTION We have,

$\vec{PO}+\vec{OQ}=\vec{QO}+\vec{OR}$

$\Rightarrow \vec{PQ}=\vec{QR}$ [By triangle law]

Thus, $\vec{PQ}$ and $\vec{QR}$ are either parallel or collinear. But Q is a point common to them. So, $\vec{PQ}$ and $\vec{QR}$ are collinear. Hence, points P, Q, R are collinear.

EXAMPLE 4 *If $\vec{a}, \vec{b}$ are any two vectors, then give the geometrical interpretation of the relation*

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|.$$

SOLUTION Let $\vec{OA}=\vec{a}$ and $\vec{AB}=\vec{b}$. Complete the parallelogram $OACB$. Then, $\vec{OC}=\vec{b}$ and $\vec{CB}=\vec{a}$.

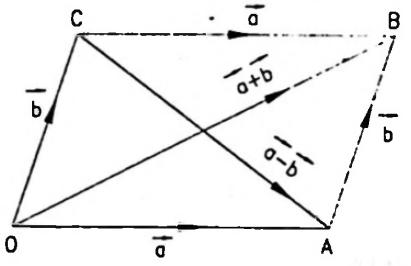


Fig. 14.5

From ΔOAB, we have

$\vec{OA}+\vec{AB} = \vec{OB}$

$\Rightarrow \vec{a}+\vec{b}=\vec{OB}$ ...(i)

From ΔOCA, we have,

$\vec{OC}+\vec{CA} = \vec{OA}$

$\Rightarrow \vec{b}+\vec{CA}=\vec{a}$

$\Rightarrow \vec{CA}=\vec{a}-\vec{b}$ ...(ii)

Now, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow |\vec{OB}| = |\vec{CA}|$

$\Rightarrow OB = CA$

$\Rightarrow$ Diagonals of parallelogram $OABC$ are equal

$\Rightarrow OABC$ is a rectangle

$\Rightarrow OA \perp OC \Rightarrow \vec{a} \perp \vec{b}$.

EXAMPLE 5 *If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order.*

SOLUTION Let $ABCDEF$ be a regular hexagon such that $\vec{AB}=\vec{a}$ and $\vec{BC}=\vec{b}$. Since AD is parallel to BC such that $AD = 2\,BC$. Therefore,

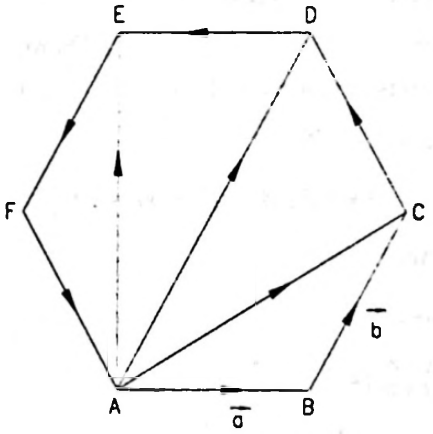


Fig. 14.6

$$\vec{AD} = 2\vec{BC} = 2\vec{b}.$$

In ΔABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{AC}$$

In ΔACD, we have

$$\vec{AC} + \vec{CD} = \vec{AD}$$

$$\Rightarrow \quad \vec{CD} = \vec{AD} - \vec{AC}$$

$$\Rightarrow \quad \vec{CD} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}.$$

Clearly, $\vec{DE} = -\vec{AB} = -\vec{a}$, $\vec{EF} = -\vec{BC} = -\vec{b}$

and, $\vec{FA} = -\vec{CD} = -(\vec{b} - \vec{a}) = \vec{a} - \vec{b}$.

EXAMPLE 6 *If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is* $\sqrt{3}$.

SOLUTION Let $\hat{a}$ and $\hat{b}$ be two unit vectors represented by sides OA and AB of a triangle OAB. Then ,

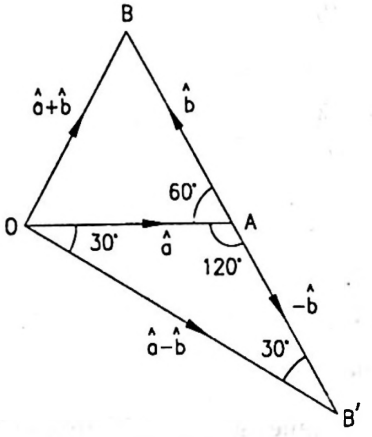


Fig. 14.7

$$\vec{OA} = \hat{a}, \vec{AB} = \hat{b},$$

$$\vec{OB} = \vec{OA} + \vec{AB} = \hat{a} + \hat{b} \qquad \text{[By triangle law of add.]}$$

It is given that

$$|\hat{a}| = |\hat{b}| = |\hat{a} + \hat{b}.| = 1$$

$$\Rightarrow \quad OA = AB = OB = 1$$

$\Rightarrow$ ΔOAB is an equilateral triangle.

Since $OA = |\hat{a}| = 1 = |-\hat{b}| = AB'$. Therefore,

$\Delta OAB'$ is an isosceles triangle such that $\angle OAB = 120°$

$$\Rightarrow \quad \angle AB'O = \angle AOB' = 30°$$

$$\Rightarrow \quad \angle BOB' = \angle BOA + \angle AOB' = 60° + 30° = 90°$$

In $\Delta BOB'$, we have

$$BB'^2 = OB^2 + OB'^2$$

$$\Rightarrow \quad 2^2 = |\hat{a} + \hat{b}|^2 + |\hat{a} - \hat{b}|^2$$

$$\Rightarrow \quad 2^2 = 1^2 + |\hat{a} - \hat{b}|^2$$

$$\Rightarrow \quad |\hat{a} - \hat{b}| = \sqrt{3}.$$

EXAMPLE 7 *For any two vectors* $\vec{a}$ and $\vec{b}$, *prove that*

(i) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

(ii) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$

(iii) $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$.

SOLUTION (i)

CASE I *When* $\vec{a}, \vec{b}$ *are non-collinear vectors:*

Let the vectors $\vec{a}$ and $\vec{b}$ be represented by sides $\vec{OA}$ and $\vec{AB}$ of a triangle OAB. Then

$$\vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law of add.]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB}.$$

Clearly, $|\vec{a}| = OA$, $|\vec{b}| = AB$ and $|\vec{a} + \vec{b}| = OB$.

Since the sum of two sides of a triangle is always greater than the third side, therefore

$$OA + AB > OB$$

$$\Rightarrow \quad |\vec{OB}| < |\vec{OA}| + |\vec{AB}|$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$$

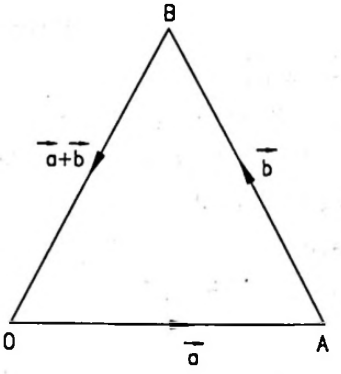


Fig. 14.8

CASE II *When* $\vec{a}, \vec{b}$ *are collinear vectors:*

Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$. Then,

$$\vec{a} + \vec{b} = \vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law of add.]}$$

Clearly, $OA = |\vec{a}|$, $AB = |\vec{b}|$

and, $OB = |\vec{a} + \vec{b}|$.

Now, $OB = OA + AB$

$$\Rightarrow \quad |\vec{OB}| = |\vec{OA}| + |\vec{AB}|$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$$

Hence, in general, $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.



Fig. 14.9

(ii) We have,

$$|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})|$$

$$\leq |\vec{a}| + |-\vec{b}| \quad \text{[Using (i)]}$$

$$\Rightarrow |\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}| \quad [\because |-\vec{b}| = |\vec{b}|]$$

(iii) We have,

$$|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$$

$$\leq |\vec{a}-\vec{b}| + |\vec{b}| \quad \text{[Using (i)]}$$

$$\Rightarrow |\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}| \text{ or, } |\vec{a}-\vec{b}| \geq |\vec{a}| - |\vec{b}|.$$

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14.6 MULTIPLICATION OF A VECTOR BY A SCALAR

Let m be a scalar and $\vec{a}$ be a vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|m|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.

From the above definition it is evident that

$$\vec{a} = |\vec{a}|\hat{a} \text{ or, } \hat{a} = \frac{1}{|\vec{a}|}\vec{a}.$$

It is also evident that two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel iff

$\vec{a} = m\vec{b}$ for some non-zero scalar m.

For any vector $\vec{a}$, we also define

$$1\vec{a} = \vec{a}, (-1)\vec{a} = -\vec{a} \text{ and } 0\vec{a} = \vec{0}.$$

For example, If $\vec{a}$, is a vector, then $3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. $-3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is opposite to that of $\vec{a}$.

PROPERTIES OF MULTIPLICATION OF A VECTOR BY A SCALAR

The following are properties of multiplication of vectors by scalars.

For vectors $\vec{a}, \vec{b}$ and scalars m, n, we have

(i) $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$

(ii) $(-m)(-\vec{a}) = m\vec{a}$

(iii) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

(iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(v) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

14.7 LINEAR COMBINATION OF VECTORS

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$... etc. if there exist scalars x, y, z etc., such that

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$$

Note that a linear combination of vectors involves two linear compositions of the addition of vectors and the multiplication of vectors with scalars.

In the following sections, the linear combinations of the form $x\vec{a}$, $x\vec{a}+y\vec{b}$ and $x\vec{a}+y\vec{b}+z\vec{c}$ will be of special interest to us.

14.8 COLLINEAR VECTORS

As we have discussed in earlier sections that vectors having the same or parallel supports are called collinear vectors. Collinear co-initial vectors have the same support.

Thus, if $\vec{a}$ is a vector and m is a scalar, then vectors $\vec{a}$ and $m\vec{a}$ are collinear vectors.

In other words, two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors iff there exists a scalar m such that $\vec{a} = m\vec{b}$. $\vec{a}$ and $\vec{b}$ are known as like or unlike parallel vectors according as $m > 0$ or $m < 0$.

We can also say that two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors iff $x\vec{a}+y\vec{b} = \vec{0}$ for some scalars x and y not both zero.

THEOREM If $\vec{a}, \vec{b}$ are any two non-zero non-collinear vectors and x, y are scalars, then

$$x\vec{a}+y\vec{b} = \vec{0} \Rightarrow x = y = 0$$

PROOF If possible, let $x \neq 0$. Then,

$$x\vec{a}+y\vec{b} = \vec{0}$$

$$\Rightarrow \quad x\vec{a} = -y\vec{b}$$

$$\Rightarrow \quad \vec{a} = \left(\frac{-y}{x}\right)\vec{b}$$

$\Rightarrow \vec{a}$ and $\vec{b}$ are collinear vectos.

This is a contradiction to the hypothesis that $\vec{a}$ and $\vec{b}$ are non-collinear vectors. Therefore, $x = 0$.

Similarly, $y = 0$.

Hence, $x\vec{a}+y\vec{b} = \vec{0} \Rightarrow x = y = 0$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of x for which the vectors $\vec{\alpha} = (2x+1)\vec{a} - \vec{b}$ and $\vec{\beta} = (x-2)\vec{a} + \vec{b}$ are collinear.*

SOLUTION Vectors $\vec{\alpha}$ and $\vec{\beta}$ will be collinear, if

$\vec{\alpha} = m\vec{\beta}$ for some scalar m

$\Rightarrow (2x+1)\vec{a} - \vec{b} = m\{(x-2)\vec{a} + \vec{b}\}$

$\Rightarrow \{(2x+1) - m(x-2)\}\vec{a} - (m+1)\vec{b} = \vec{0}$

$\Rightarrow (2x+1) - m(x-2) = 0$ and $-(m+1) = 0$

$\Rightarrow m = -1$ and $x = \frac{1}{3}$.

EXAMPLE 2 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-null vectors such that any two of them are non-collinear. If $\vec{a} + \vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then find $\vec{a} + \vec{b} + \vec{c}$.*

SOLUTION It is given that :

$\vec{a} + \vec{b}$ is collinear with $\vec{c}$

$\Rightarrow \vec{a} + \vec{b} = \lambda_1 \vec{c}$ for some scalar λ_1 ...(i)

and, $\vec{b} + \vec{c}$ is collinear with $\vec{a}$

$\Rightarrow \vec{b} + \vec{c} = \lambda_2 \vec{a}$ for some scalar λ_2 ...(ii)

From (i), we have

$$\vec{a} = \lambda_1 \vec{c} - \vec{b}$$

Substituting this value in (ii), we get

$\vec{b} + \vec{c} = \lambda_2 (\lambda_1 \vec{c} - \vec{b})$

$\Rightarrow (1+\lambda_2)\vec{b} + (1 - \lambda_1\lambda_2)\vec{c} = \vec{0}$

$\Rightarrow 1 + \lambda_2 = 0$ and $1 - \lambda_1\lambda_2 = 0$ [$\because \vec{b}$ and $\vec{c}$ are non-collinear]

$\Rightarrow \lambda_2 = -1, \lambda_1 = -1$

Substituting the values of λ_1 and λ_2 in (i) and (ii) respectively, we get

$$\vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

EXAMPLE 3 *Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that any two of them are non-collinear. If $\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$, then prove that $\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$.*

SOLUTION It is given that :

$\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$

$\Rightarrow \vec{a} + 2\vec{b} = \lambda\vec{c}$ for some scalar λ ...(i)

and, $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$

$\Rightarrow \vec{b} + 3\vec{c} = \mu\vec{a}$ for some scalar μ ...(ii)

From (i), we have

$$\vec{a} = \lambda\vec{c} - 2\vec{b}$$

Substituting this value of $\vec{a}$ in (ii), we get

$\vec{b} + 3\vec{c} = \mu(\lambda\vec{c} - 2\vec{b})$

$\Rightarrow (1+2\mu)\vec{b} + (3 - \mu\lambda)\vec{c} = \vec{0}$

$\Rightarrow 1 + 2\mu = 0$ and $3 - \mu\lambda = 0$ [$\because \vec{b}$ and $\vec{c}$ are non-collinear vectors]

$\Rightarrow \mu = -\frac{1}{2}$ and $\lambda = -6$

Substituting the values of λ and μ in (i) and (ii), respectively, we get

$$\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$$

EXAMPLE 4 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and vectors*

$\vec{\alpha} = (x+4y)\vec{a} + (2x+y+1)\vec{b}$

and, $\vec{\beta} = (-2x+y+2)\vec{a} + (2x-3y-1)\vec{b}$

are connected by the relation $3\vec{\alpha} = 2\vec{\beta}$, find x, y.

SOLUTION We have,

$3\vec{\alpha} = 2\vec{\beta}$

$\Rightarrow 3\{(x+4y)\vec{a} + (2x+y+1)\vec{b}\}$

$= 2\{(-2x+y+2)\vec{a} + (2x-3y-1)\vec{b}\}$

$\Rightarrow (3x+12y+4x-2y-4)\vec{a} + (6x+3y+3-4x+6y+2)\vec{b} = \vec{0}$

$\Rightarrow (7x+10y-4)\vec{a} + (2x+9y+5)\vec{b} = \vec{0}$

$\Rightarrow 7x+10y-4 = 0$ and $2x+9y+5 = 0$

$\Rightarrow x = 2, y = -1.$

EXAMPLE 5 *Let* $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$

and, $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0,1]$,

where f_1, f_2, g_1, g_2 are continuous functions.

If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all t and $\vec{A}(0) = 2\hat{i} + 3\hat{j}$, $\vec{A}(1) = 6\hat{i} + 2\hat{j}, \vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t. [IIT 2001]

SOLUTION In order to prove that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for some value of t, it is sufficient to show that

$\vec{A}(t) = \lambda\vec{B}(t)$ for some λ

$\Leftrightarrow \{f_1(t)\hat{i} + f_2(t)\hat{j}\} = \lambda\{g_1(t)\hat{i} + g_2(t)\hat{j}\}$

$\Leftrightarrow f_1(t) = \lambda g_1(t)$ and $f_2(t) = \lambda g_2(t)$

$\Leftrightarrow \frac{f_1(t)}{f_2(t)} = \frac{g_1(t)}{g_2(t)}$

$\Leftrightarrow f_1(t)g_2(t) - f_2(t)g_1(t) = 0$ for some $t \in [0,1]$

Let $F(t) = f_1(t)g_2(t) - g_1(t)f_2(t)$, $t \in [0,1]$

Since, $f_1, f_2 g_1$ and g_2 are continuous functions. Therefore, $F(t)$ is also a continuous function.

Also,

$$F(0) = f_1(0) g_2(0) - g_1(0) f_2(0)$$

$$= 2 \times 2 - 3 \times 3 = 4 - 9 = -5 < 0$$

and,

$$F(1) = f_1(1) g_2(1) - g_1(1) f_2(1)$$

$$= 6 \times 6 - 2 \times 21 = 32 > 0$$

Thus, $F(t)$ is a continuous function on $[0, 1]$ such that $F(0) F(1) < 0$. Therefore, by intermediate, value theorem there exists some $t \in (0, 1)$ such that

$$F(t) = 0$$

$$\Rightarrow f_1(t) g_2(t) - f_2(t) g_2(t) = 0$$

$$\Rightarrow \vec{A}(t) = \lambda \vec{B}(t) \text{ for some } \lambda.$$

Hence, $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors.

—❑❑❑—

14.9 COPLANAR VECTORS

A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

THEOREM *(Test of coplanarity of three vectors) Let $\vec{a}$ and $\vec{b}$ be two given non-zero non-collinear vectors. Then any vectors $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b}$, for some scalars x and y.*

PROOF Let $\vec{a}$ and $\vec{b}$ be two non-collinear non-zero vectors and let $\vec{r}$ be a vector coplanar with $\vec{a}$ and $\vec{b}$.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OP} = \vec{r}$. Complete the parallelogram $OLPM$ with OP as diagonal. Since vectors $\vec{OL}$ and $\vec{OM}$ are collinear with $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ respectively. Therefore,

$\vec{OL} = x\vec{a}$ and $\vec{OM} = y\vec{b}$ for some scalars x, y.

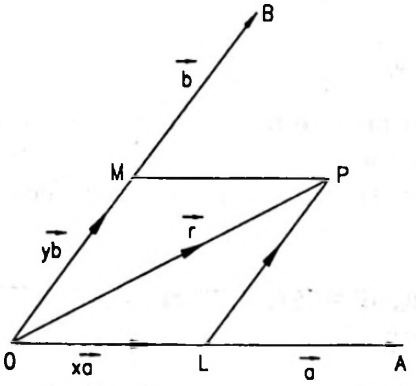


Fig. 14.10

Now, $\vec{OP} = \vec{OL} + \vec{LP}$ [By triangle law of add.]

$\Rightarrow$ $\vec{OP} = \vec{OL} + \vec{OM}$ $[\because \vec{LP} = \vec{OM}]$

$\Rightarrow$ $\vec{r} = x\vec{a} + y\vec{b}$.

Thus, $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars x and y.

To prove the uniqueness of this representation, let $\vec{r} = x_1\vec{a} + y_1\vec{b}$ for some scalars x_1 and y_1. Then,

$$x\vec{a} + y\vec{b} = x_1\vec{a} + y_1\vec{b}$$

$$\Rightarrow (x - x_1)\vec{a} + (y - y_1)\vec{b} = \vec{0}$$

$$\Rightarrow x - x_1 = 0 \text{ and } y - y_1 = 0 \quad [\because \vec{a} \text{ and } \vec{b} \text{ are non-collinear}]$$

$$\Rightarrow x = x_1 \text{ and } y = y_1.$$

Hence, the representation is unique.

Q.E.D.

The above theorem can also be expressed as under:

Three vectors are coplanar if one of them is expressible as a linear combination of the other two.

COMPONENTS AND COORDINATES IN A PLANE RELATIVE TO ANY TWO NON-COLLINEAR VECTORS

Let $\vec{a}, \vec{b}$ be two given non-collinear vectors and let $\vec{r}$ be a vector lying in the plane of $\vec{a}$ and $\vec{b}$. Then, by the above theorem there exist scalars x, y such that

$$\vec{r} = x\vec{a} + y\vec{b}.$$

The vectors $x\vec{a}$ and $y\vec{b}$ are said to be components of vector $\vec{r}$ in the directions of $\vec{a}$ and $\vec{b}$ respectively. The scalars x and y are known as the coordinates of the vector $\vec{r}$ relative to the non-collinear vectors $\vec{a}$ and $\vec{b}$.

We have,

$$\vec{r} = x\vec{a} + y\vec{b}$$

$$\Rightarrow \vec{r} = (x \mid \vec{a} \mid)\hat{a} + (y \mid \vec{b} \mid)\hat{b},$$

where $\hat{a}$ and $\hat{b}$ are unit vectors in the directions of $\vec{a}$ and $\vec{b}$ respectively.

$x \mid \vec{a} \mid$ and $y \mid \vec{b} \mid$ are known as projections or resolved parts or resolutes of $\vec{r}$ along $\vec{a}$ and $\vec{b}$ respectively.

The pair of non-collinear vectors $\vec{a}$ and $\vec{b}$ relative to which we decompose any vector $\vec{r}$ in their plane is called a base.

NOTE *In a plane, every pair of non-collinear vectors can be thought of as a base.*

ILLUSTRATION 1 *Show that the vectors* $\vec{a}-2\vec{b}+3\vec{c}, \vec{a}-3\vec{b}+5\vec{c}$ *and* $-2\vec{a}+3\vec{b}-4\vec{c}$ *are coplanar, where* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar.*

SOLUTION Recall that three vectors are coplanar if one of the given vectors is expressible as a linear combination of the other two.

Let $\vec{a}-2\vec{b}+3\vec{c}=x(\vec{a}-3\vec{b}+5\vec{c})+y(-2\vec{a}+3\vec{b}-4\vec{c})$, for some scalars x and y.

Then,

$$\vec{a}-2\vec{b}+3\vec{c}=(x-2y)\vec{a}+(-3x+3y)\vec{b}+(5x-4y)\vec{c}$$

$$\Rightarrow \quad 1=x-2y,\ -2=-3x+3y \text{ and } 3=5x-4y$$

Solving first two of these equations, we get $x=1/3$, $y=-1/3$.

Clearly, these values of x and y satisfy the third equation. Hence, the given vectors are coplanar.

ILLUSTRATION 2 *Show that the vectors* $2\vec{a}-\vec{b}+3\vec{c}, \vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ *are non-coplanar vectors.*

SOLUTION Let, if possible, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a}-\vec{b}+3\vec{c}=x(\vec{a}+\vec{b}-2\vec{c})+y(\vec{a}+\vec{b}-3\vec{c})$, for some scalars x and y.

$$\Rightarrow 2\vec{a}-\vec{b}+3\vec{c}=(x+y)\vec{a}+(x+y)\vec{b}+(-2x-3y)\vec{c}$$

$$\Rightarrow 2=x+y,\ -1=x+y \text{ and } 3=-2x-3y.$$

Solving, first and third of these equations, we get

$$x=9 \text{ and } y=-7.$$

Clearly, these values do not satisfy the third equation.

Hence, the given vectors are not coplanar.

ILLUSTRATION 3 *Prove that four points* $2\vec{a}+3\vec{b}-\vec{c}, \vec{a}-2\vec{b}+3\vec{c}, 3\vec{a}+4\vec{b}-2\vec{c}$ *and* $\vec{a}-6\vec{b}+6\vec{c}$ *are coplanar.*

SOLUTION Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So, let

$$\vec{PQ}=x\vec{PR}+y\vec{PS}$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=x(\vec{a}+\vec{b}-\vec{c})+y(-\vec{a}-9\vec{b}+7\vec{c})$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=(x-y)\vec{a}+(x-9y)\vec{b}+(-x+7y)\vec{c}$$

$$\Rightarrow \quad x-y=-1,\ x-9y=-5,\ -x+7y=4$$

Solving the first of these three equations, we get

$$x=-1/2, \qquad y=1/2.$$

These values also satisfy the third equation. Hence, the given four points are coplanar.

In the above discussion, we have learnt that if $\vec{a}$ and $\vec{b}$ are any two non-collinear vectos, then any vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as a linear combination of $\vec{a}$ and $\vec{b}$. It is known as oblique resolution of $\vec{r}$. The most important case of resolution of vectors is that in which the two directions are mutually perpendicular. This gives us the relation between vectors and coordinate geometry.

RECTANGULAR RESOLUTION OF A VECTOR IN A PLANE

Let OX and OY be rectangular cartesian coordinate axes in a plane as shown in Fig. 14.11 and O be the origin. The unit vectors parallel to OX and OY are denoted by $\hat{i}$ and $\hat{j}$ respectively. Let $\vec{r}$ be any vector lying in the plane of $\hat{i}$ and $\hat{j}$ and let $\vec{r}$ be represented by $\vec{OP}$. If (x, y) are the coordinates of P relative to OX and OY as the coordinate axes. Then, $OL=x$, $OM=y$. Also, $OL=x\hat{i}$ and $\vec{OM}=y\hat{j}$.

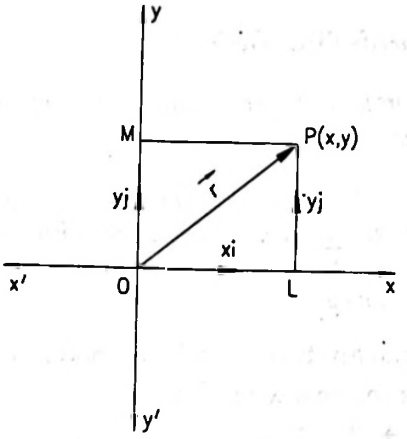


Fig. 14.11

In ΔOLP, we have

$$\vec{OP}=\vec{OL}+\vec{LP}$$

$$\Rightarrow \vec{OP}=\vec{OL}+\vec{OM} \qquad [\because \vec{LP}=\vec{OM}]$$

$$\Rightarrow \vec{OP}=x\hat{i}+y\hat{j}$$

$$\Rightarrow \quad \vec{r}=x\hat{i}+y\hat{j}$$

The component vectors $x\hat{i}, y\hat{j}$ are the orthogonal components of $\vec{r}$ in the directions of $\hat{i}$ and j. The scalars x and y are called the resolved parts or projections of $\vec{r}$ along OX and OY respectively.

14.10 OBLIQUE RESOLUTION OF A VECTOR IN SPACE

In this section, we shall see that any vector in space can be uniquely expressed as a linear combination of any three non-coplanar vectors. Such a resolution of a vector is known as the oblique resolution. We state and prove the above result as a theorem given below.

THEOREM 1 *If* $\vec{a}, \vec{b}, \vec{c}$ *are three given non-coplanar vectors, then every vector* $\vec{r}$ *in space can be uniquely expressed as* $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$ *for some scalars* x, y *and* z.

PROOF Let $\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ and $\vec{OC}=\vec{c}$. Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, therefore lines OA, OB and OC are also non-coplanar. Let $\vec{OP}=\vec{r}$. With OP as diagonal, construct a

parallelopiped having three coterminus edges OQ, OR and OS along OA, OB and OC respectively.

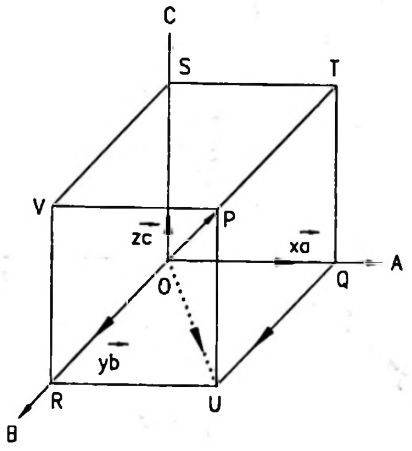


Fig. 14.12

Since $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$ are collinear with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ respectively. Therefore, there exist scalars x, y, z such that $\overrightarrow{OQ} = x\vec{a}$, $\overrightarrow{OR} = y\vec{b}$ and $\overrightarrow{OS} = z\vec{c}$.

Now, $\overrightarrow{OU} = \overrightarrow{OQ} + \overrightarrow{QU} = \overrightarrow{OQ} + \overrightarrow{OR} = x\vec{a} + y\vec{b}$

$\therefore \quad \overrightarrow{OP} = \overrightarrow{OU} + \overrightarrow{UP} = \overrightarrow{OU} + \overrightarrow{OS} = x\vec{a} + y\vec{b} + z\vec{c}$

Hence, $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ ...(i)

This is the required relation.

Now we will prove the uniqueness of this relation.

If possible, let $\vec{r} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$. Then,

$x\vec{a} + y\vec{b} + z\vec{c} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$ [Using (i)]

$\Rightarrow \quad (x - x_1)\vec{a} + (y - y_1)\vec{b} + (z - z_1)\vec{c} = \vec{0}$

$\Rightarrow \quad x - x_1 = 0,\ y - y_1 = 0,\ z - z_1 = 0$

$\Rightarrow \quad x = x_1,\ y = y_1,\ z = z_1.$

Hence, $\vec{r}$ is uniquely expressible as $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

Q.E.D

NOTE 1 *Here $\vec{r}$ may be called the resultant of three vectors $x\vec{a}, y\vec{b}, z\vec{c}$ which are the component vectors of $\vec{r}$ while x, y, z are the scalar components of $\vec{r}$ for the given directions.*

NOTE 2 *It follows from the above result that between any four non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ in space there exists a unique relation connecting them which may be expressed as*

$$x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}.$$

In the above theorem, we have proved that any vector $\vec{r}$ in space can be uniquely expressed as a linear combination of three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ i.e.

$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ for some scalars x, y, z.

or, $\vec{r} = (x \mid \vec{a} \mid)\hat{a} + (y \mid \vec{b} \mid)\hat{b} + (z \mid \vec{c} \mid)\hat{c}$,

where $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors in the direction of $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

The vectors $x\vec{a}, y\vec{b}, z\vec{c}$ are said to be components of $\vec{r}$ in the directions of $\vec{a}, \vec{b}$ and $\vec{c}$ respectively and the scalars x, y, z are known as the coordinates of the vector $\vec{r}$ relative to the triad of non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$. The triad of non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ relative to which we decompose any vector $\vec{r}$ is called a base. The scalars $x \mid \vec{a} \mid, y \mid \vec{b} \mid, z \mid \vec{c} \mid$ are known as the projections of $\vec{r}$ in the directions of $\vec{a}, \vec{b}, \vec{c}$ respectively.

NOTE *In space, every triad of non-coplanar vectors can be thought as a base.*

THEOREM 2 If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero non-coplanar vectors and y, z are three scalars, then

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$$

PROOF If possible, let $x \neq 0$. Then,

$x\vec{a} + y\vec{b} - z\vec{c} = \vec{0}$

$\Rightarrow x\vec{a} = -y\vec{b} - z\vec{c}$

$\Rightarrow \vec{a} = \left(\frac{-y}{x}\right)\vec{b} + \left(\frac{-z}{x}\right)\vec{c}$

$\Rightarrow \vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$.

This is contradiction to the fact that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

Therefore, $x = 0$.

Similarly, $y = z = 0$.

Hence, $x\vec{a} + y\vec{b} + z\vec{c} = 0 \Rightarrow x = y = z = 0$

Q.E.D.

14.11 RECTANGULAR RESOLUTION OF A VECTOR IN SPACE

In the previous section, we have discussed the resolution of a vector in space in the directions of three non-coplanar vectors. The most important case of resolution of vectors is that in which the three non-coplanar vectors are mutually perpendicular. The right handed system of three mutually perpendicular directions OX, OY, OZ is the most important and suitable sys-

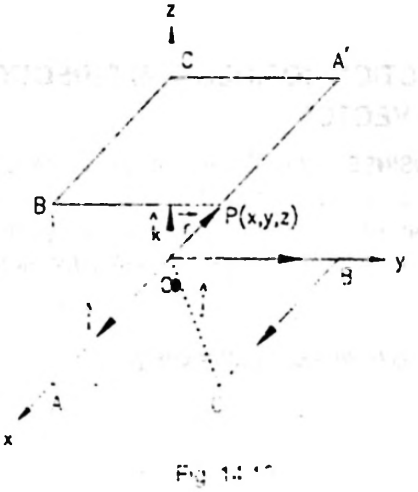


Fig. 14.13

tem as shown in Fig.14.13. The unit vectors $\hat{i}, \hat{j}, \hat{k}$ parallel to OX, OY and OZ respectively are known as an *orthonormal triad of vectors.*

Let P be any point in space. With reference to OX, OY and OZ as the coordinate axes, let the coordinates of P be (x, y, z). Then,

$$OA = x, OB = y, OC = z$$

and, $\vec{OA} = x\hat{i}, \vec{OB} = y\hat{j}, \vec{OC} = z\hat{k}$

From Fig. 14.13, we have

$$\vec{BC} = \vec{OA} = x\hat{i}, \vec{C'P} = \vec{OC} = z\hat{k}$$

Now, $\vec{OP} = \vec{OC'} + \vec{C'P} = \vec{OB} + \vec{BC'} + \vec{C'P}$

$$= \vec{OB} + \vec{OA} + \vec{OC}$$

$$= \vec{OA} + \vec{OB} + \vec{OC}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}.$$

Thus, if $P(x, y, z)$ is any point in space. Then, $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now,

$$OP^2 = OC^2 + C'P^2$$

$$= (OB^2 + BC'^2) + C'P^2$$

$$= (OB^2 + OA^2) + OC^2$$

$$= OA^2 + OB^2 + OC^2 = x^2 + y^2 + z^2.$$

$$\therefore \quad OP = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \quad |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Thus, if a point P in space has coordinates (x, y, z), then $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$.

The vectors $x\hat{i}, y\hat{j}, z\hat{k}$ are known as the components of vector $\vec{OP}$ along OX, OY and OZ respectively. The scalars x, y, z are also known as the projections of $\vec{OP}$ along OX, OY and OZ respectively.

14.12 DIRECTION COSINES AND DIRECTION RATIOS OF A VECTOR

DIRECTION COSINES *If α, β, γ are the angles which a vector $\vec{OP}$ makes with the positive directions of the coordinates axes OX, OY, OZ respctively, then $\cos\alpha\ \cos\beta\ \cos\gamma$ are known as the direction cosines of $\vec{OP}$ and are generally denoted by the letters l, m, n respectively.*

Thus, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Clearly, $\vec{PO}$ makes angles $\pi - \alpha$, $\pi - \beta, \pi - \gamma$ with OX, OY, OZ respectively. So, the direction cosines of $\vec{PO}$ are

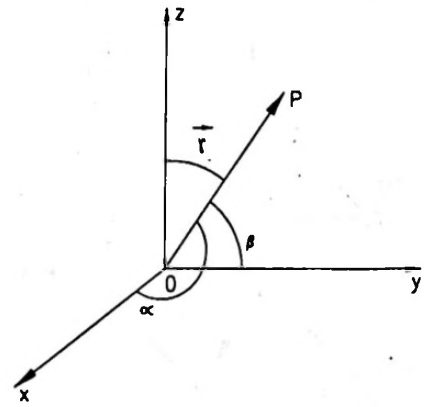


Fig. 14.14

$$\cos(\pi - \alpha) = -\cos\alpha = -l,$$

$$\cos(\pi - \beta) = -\cos\beta = -m,$$

$$\cos(\pi - \gamma) = -\cos\gamma = -n.$$

As the x-axis makes angles $0, \pi/2, \pi/2$ with OX, OY and OZ respectively. So, the direction cosines of OX are $\cos 0 = 1$, $\cos \pi/2 = 0$, $\cos \pi/2 = 0$. Similarly, direction cosines of OY and OZ are 0, 1, 0 and 0, 0, 1 respectively.

If l, m, n are direction cosines of $\vec{OP} = \vec{r}$ and the point P has (x, y, z) its coordinates, then we have the following results.

(i) $l^2 + m^2 + n^2 = 1$

(ii) $x = l\,|\vec{r}|, y = m\,|\vec{r}|, z = n\,|\vec{r}|$

(iii) $\vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$

(iv) $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

DIRECTION RATIOS *Let l, m, n be the direction cosines of a vector $\vec{r}$ and a, b, c be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$. Then, a, b, c are known as direction ratios or direction numbers of $\vec{r}$.*

If $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$ are direction cosines of a vector $\vec{r}$, then its direction ratios are 2, – 2, 1 or, – 2, 2, – 1 or, 4, – 4, 2 because

$$\frac{2/3}{2} = \frac{-2/3}{-2} = \frac{1/3}{1}; \frac{2/3}{2} = \frac{-2/3}{2} = \frac{1/3}{-1};$$

$$\frac{2/3}{4} = \frac{-2/3}{-4} = \frac{1/3}{2}$$

If l, m, n are the direction cosines of a vector $\vec{r}$ and its direction ratios are proportional to a, b, c, then

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the signs should be taken all positive or all negative.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *A vector $\vec{A}$ has components $\vec{A}_1, \vec{A}_2, \vec{A}_3$ in a right handed rectangular cartesian coordinate system OX, OY, OZ. The coordinate system is rotated about the z-axis through an angle $\pi/2$. Find the components of $\vec{A}$ in the new coordinate system in terms of $\vec{A}_1, \vec{A}_2, \vec{A}_3$.*

[IIT 1983]

SOLUTION When the coordinate system is rotated about the z-axis through an angle $\pi/2$ in anticlockwise sense, then the new x-axis is along the old y-axis new y-axis is along the old OX' axis and z-axis remains same. So, the components of $\vec{A}$ in the new coordinate system are

$$\vec{A}_2, -\vec{A}_1, \vec{A}_3$$

Thus, we have

$\vec{A} = |\vec{A}_1|\hat{i} + |\vec{A}_2|\hat{j} + |\vec{A}_3|\hat{k}$ with reference to old sytem

and,

$\vec{A} = |A_2|\hat{i}| - |\vec{A}_1|\hat{j} + |\vec{A}_3|\hat{k}$ with reference to new system.

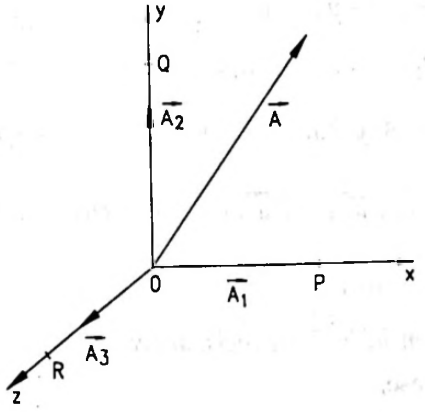


Fig. 14.15

EXAMPLE 2 *ABCDEF is a regular hexagon. M is the mid-point of $\vec{DE}$, N is the mid-point of $\vec{AM}$, P is the mid-point of $\vec{BC}$. Resolve the vector $\vec{NP}$ into components with respect to the vectors $\vec{AB}$ and $\vec{AF}$.*

SOLUTION Let $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.

Then,

$$\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}.$$

Also,

$$\vec{AD} = 2\vec{BC} = 2\vec{b}$$

In ΔACD, we have

$$\vec{AC} + \vec{CD} = \vec{AD}$$

$$\Rightarrow \quad \vec{CD} = \vec{AD} - \vec{AC} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}.$$

In ΔBCD, we have

$$\vec{BD} = \vec{BC} + \vec{CD} = \vec{b} + \vec{b} - \vec{a} = 2\vec{b} - \vec{a}.$$

$$\therefore \quad \vec{AE} = 2\vec{b} - \vec{a}.$$

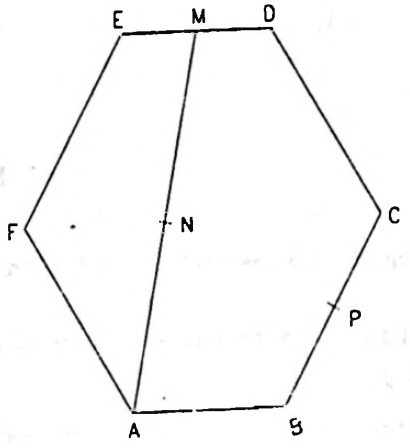


Fig. 14.16

Now, M is the mid-point of DE.

$$\therefore \quad \vec{AM} = \frac{1}{2}(\vec{AE} + \vec{AD}) = \frac{1}{2}(2\vec{b} - \vec{a} + 2\vec{b}) = 2\vec{b} - \frac{1}{2}\vec{a}.$$

N is the mid-point of AM.

$$\therefore \quad \vec{AN} = \frac{1}{2}\vec{AM} = \frac{1}{2}\left(2\vec{b} - \frac{\vec{a}}{2}\right) = \vec{b} - \frac{1}{4}\vec{a}.$$

Now,

$$\vec{AN} + \vec{NP} = \vec{AP}$$

$$\Rightarrow \quad \vec{NP} = \vec{AP} - \vec{AN}$$

$$\Rightarrow \quad \vec{NP} = \frac{1}{2}(\vec{AB} + \vec{AC}) - \vec{AN} \qquad \left[\because P \text{ is the mid-point of } BC \;\; \therefore \vec{AP} = \frac{1}{2}(\vec{AB} + \vec{AC})\right]$$

$$\Rightarrow \quad \vec{NP} = \frac{1}{2}(\vec{a} + \vec{a} + \vec{b}) - \left(\vec{b} - \frac{1}{4}\vec{a}\right)$$

$$\Rightarrow \quad \vec{NP} = \frac{5}{4}\vec{a} - \frac{1}{2}\vec{b} \qquad \ldots(i)$$

We have,

$$\vec{AB} = \vec{a} \text{ and } \vec{AF} = \vec{CD} = \vec{b} - \vec{a}$$

$$\therefore \quad \vec{b} = \vec{AF} + \vec{a} = \vec{AF} + \vec{AB}$$

Substituting these in (i), we get

$$\vec{NP} = \frac{5}{4}\vec{AB} - \frac{1}{2}(\vec{AF} + \vec{AB})$$

$$\Rightarrow \quad \vec{NP} = \frac{3}{4}\vec{AB} - \frac{1}{2}\vec{AF}$$

—❑❑❑—

14.13 LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS

LINEARLY INDEPENDENT VECTORS *A set of non-zero vectors* $\vec{a_1}, \vec{a_2}, \ldots \vec{a_n}$ *is said to be linearly independent, if*

$$x\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = 0 \Rightarrow x_1 = x_2 = \ldots = x_n = 0$$

LINEARLY DEPENDENT VECTORS *A set of vectors* $\vec{a_1}, \vec{a_2}, \ldots \vec{a_n}$ *is said to be linearly dependent if there exist scalars* $x_1, x_2, \ldots, x_n$ *not all zero such that*

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{0}.$$

It follows from this definition that a set of vectors is linearly dependent iff one of them is expressible as a linear combination of others.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Two non-zero, non-collinear vectors are linearly independent.*

SOLUTION Let $\vec{a}$ and $\vec{b}$ be two non-zero non-collinear vectors such that $x\vec{a} + y\vec{b} = \vec{0}$.

In order to prove that $\vec{a}, \vec{b}$ are linearly independent, we must show that $x = y = 0$.

If possible, let $x \neq 0$. Then,

$$x\vec{a} + y\vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{a} = -\frac{y}{x}\vec{b} \qquad [\because x \neq 0]$$

This shows that $\vec{a}$ and $\vec{b}$ are collinear vectors. This is a contradiction. So, $x = 0$. Similarly, we have $y = 0$.

Thus, $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$

Hence, $\vec{a}$ and $\vec{b}$ are linearly independent vectors.

EXAMPLE 2 *Any two collinear vectors are linearly dependent.*

SOLUTION Let $\vec{a}, \vec{b}$ be two collinear vectors. Then,

$$\vec{b} = x\vec{a} \text{ for some scalar } x.$$

$$\Rightarrow x\vec{a} + (-1)\vec{b} = \vec{0}$$

$$\Rightarrow x\vec{a} + y\vec{b} = 0, \text{ where } y = -1 \neq 0.$$

This shows that the vectors $\vec{a}, \vec{b}$ are linearly dependent.

EXAMPLE 3 *Any three non-coplanar vectors are linearly independent.*

SOLUTION Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and let $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ for some scalars x, y, z. In order to prove that $\vec{a}, \vec{b}, \vec{c}$ are linearly independent we must show that

$$x = y = z = 0.$$

If possible, let $x \neq 0$. Then,

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} = \left(-\frac{y}{x}\right)\vec{b} + \left(-\frac{z}{x}\right)\vec{c} \qquad [\because x \neq 0]$$

$\Rightarrow \vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar vectors

This is a contradiction. So, $x = 0$.

Similarly, we can show that $y = z = 0$.

Hence, $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors.

NOTE *Since* $\hat{i}, \hat{j}, \hat{k}$ *are non-coplanar vectors, therefore* $\hat{i}, \hat{j}, \hat{k}$ *are linearly independent unit vectors.*

EXAMPLE 4 *Any three coplanar vectors are linearly dependent.*

SOLUTION Let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors. Then,

$\vec{a} = x\vec{b} + y\vec{c}$ for some scalars x and y.

$$\Rightarrow (-1)\vec{a} + x\vec{b} + y\vec{c} = \vec{0}$$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

EXAMPLE 5 *Any four vectors in 3-dimensional space are linearly dependent.*

SOLUTION Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be four vectors in 3-dimensional space.

Two cases arise.

CASE I *When* $\vec{a}, \vec{b}, \vec{c}$ *are coplanar vectors*

In this case,

$\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow$ there exist scalars x, y such that $\vec{a} = x\vec{b} + y\vec{c}$.

$$\Rightarrow \vec{a} + (-x)\vec{b} + (-y)\vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} + (-x)\vec{b} + (-y)\vec{c} + 0\,\vec{d} = \vec{0}$$

$$\Rightarrow x_1\vec{a} + x_2\vec{b} + x_3\vec{c} + x_4\vec{d} = 0,$$

where $x_1 = 1, x_2 = -x, x_3 = -y$ and $x_4 = 0$.

$\Rightarrow \vec{a}, \vec{b}, \vec{c}, \vec{d}$ are linearly dependent vectors $[\because x_1 = 1 \neq 0]$

CASE II *When* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar vectors*

In this case,

$\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and $\vec{d}$ is any vector in space

$\Rightarrow$ There exist scalars x, y, z such that $\vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$.

$$\Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + (-1)\vec{d} = 0$$

$$\Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = 0, \text{ where } u = -1 \neq 0$$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors.

EXAMPLE 6 *If $\vec{a}, \vec{b}, \vec{c}$ is a system of linearly independent vectors, show that the system of vectors*

$$\vec{a} - 2\vec{b} + \vec{c},\ 2\vec{a} - \vec{b} + \vec{c},\ 3\vec{a} + \vec{b} + 2\vec{c}$$

is a linearly independent system of vectors.

SOLUTION Let x, y, z be scalars such that

$$x(\vec{a} - 2\vec{b} + \vec{c}) + y(2\vec{a} - \vec{b} + \vec{c}) + z(3\vec{a} + \vec{b} + 2\vec{c}) = \vec{0}$$

$$\Rightarrow (x + 2y + 3z)\vec{a} + (-2x - y + z)\vec{b} + (x + y + 2z)\vec{c} = \vec{0}$$

$$\Rightarrow x + 2y + 3z = 0, -2x - y + z = 0, x + y + 2z = 0$$

[$\because \vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors]

This is a homogeneous system of equations. The determinant of the coeffcient matrix is

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -3 + 10 - 3 \neq 0$$

So, the system of equations has only the trivial solution given by $x = 0 = y = z = 0$.

Hence, the given system of vectors is a linearly indpendent system of vectors.

EXAMPLE 7 *If $\vec{u}, \vec{v}, \vec{w}$ is a linearly independent system of vectors, examine the system $\vec{p}, \vec{q}, \vec{r}$, where*

$$\vec{p} = (\cos a)\vec{u} + (\cos b)\vec{v} + (\cos c)\vec{w}$$

$$\vec{q} = (\sin a)\vec{u} + (\sin b)\vec{v} + (\sin c)\vec{w}$$

$$\vec{r} = \sin(x + a)\vec{u} + \sin(x + b)\vec{v} + \sin(x + c)\vec{w}$$

for linear independence.

SOLUTION Let l, m, n be scalars such that

$$l\vec{p} + m\vec{q} + n\vec{r} = \vec{0}$$

$$\Rightarrow l\{(\cos a)\vec{u} + (\cos b)\vec{v} + (\cos c)\vec{w}\} + m\{(\sin a)\vec{u} + (\sin b)\vec{v} + (\sin c)\vec{w}\} + n\{\sin(x + a)\vec{u} + \sin(x + b)\vec{v} + \sin(x + c)\}\vec{w} = \vec{0}$$

$$\Rightarrow \{l\cos a + m\sin a + n\sin(x + a)\}\vec{u} + \{l\cos b + m\sin b + n\sin(x + b)\}\vec{v} + \{\cos c + m\sin c + n\sin(x + c)\}\vec{w} = \vec{0}$$

$$\Rightarrow l\cos a + m\sin a + n\sin(x + a) = 0$$

[$\because \vec{u}, \vec{v}, \vec{w}$ are linearly independence vectors]

$$l\cos b + m\sin b + n\sin(x + b) = 0$$

$$l\cos c + m\sin c + n\sin(x + c) = 0$$

This is a homogeneous system of linear equations in l, m, n. The determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} \cos a & \sin a & \sin(x + a) \\ \cos b & \sin b & \sin(x + b) \\ \cos c & \sin c & \sin(x + c) \end{vmatrix}$$

$$= \begin{vmatrix} \cos a & \sin a & 0 \\ \cos b & \sin b & 0 \\ \cos c & \sin c & 0 \end{vmatrix} = 0 \quad \left[\begin{array}{c}\text{Applying} \\ C_3 \to C_3 - \sin x\, C_1 - \cos x\, C_2\end{array}\right]$$

So, the above system of equations has non-trivial solutions also. This means that l, m, n may attain non-zero values also.

Hence, the given system of vectors is a linearly dependent system of vectors.

EXAMPLE 8 *Show that the vectors $\hat{i} - 3\hat{j} + 2\hat{k}$, $2\hat{i} - 4\hat{j} - 4\hat{k}$, $3\hat{i} + 2\hat{j} - \hat{k}$ are linearly independent.*

SOLUTION Let x, y, z be scalars such that

$$x(\hat{i} - 3\hat{j} + 2\hat{k}) + y(2\hat{i} - 4\hat{j} - 4\hat{k}) + z(3\hat{i} + 2\hat{j} - \hat{k}) = \vec{0}$$

$$\Rightarrow (x + 2y + 3z)\hat{i} + (-3x - 4y + 2z)\hat{j} + (2x - 4y - z)\hat{k} = \vec{0}$$

$$\Rightarrow x + 2y + 3z = 0, -3x - 4y + 2z = 0 \text{ and } 2x - 4y - z = 0$$

[$\because \hat{i}, \hat{j}, \hat{k}$ are linearly independent]

This is a homogeneous system of equations. The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -4 & -1 \end{vmatrix} = 78 \neq 0$$

Hence, the given set of vectors is linearly independent.

EXAMPLE 9 *Show that the vectors $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j} - \hat{k}$ and $-\hat{i} - 2\hat{j} + 2\hat{k}$ are linearly dependent.*

SOLUTION Let x, y, z be scalars such that

$$x(-\hat{i} - 2\hat{j} + 2\hat{k}) + y(2\hat{i} + 3\hat{j} - \hat{k}) + z(\hat{i} + \hat{j} + \hat{k}) = \vec{0}$$

$$\Rightarrow (-x + 2y + z)\hat{i} + (-2x + 3y + z)\hat{j} + (2x - y + z)\hat{k} = \vec{0}$$

$$\Rightarrow -x + 2y + z = 0, -2x + 3y + z = 0 \text{ and } 2x - y + z = 0$$

[$\because \hat{i}, \hat{j}, \hat{k}$ are linear independent]

This is a homogeneous system of equations. The determinant of the coefficient matrix is

$$\begin{vmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0.$$

So, the system of equations has infinitely many non-trivial solutions.

Thus, x, y, z can also assume non-zero values. Hence, the given set of vectors is linearly dependent.

EXAMPLE 10 *A particle in equilibrium is subjected to four forces*

$$\vec{F}_1 = -10\hat{k},$$

$$\vec{F}_2 = u\left(\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right)$$

$$\vec{F}_3 = v\left(-\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right)$$

and, $\vec{F}_4 = w(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

Find the values of u, v, w in terms of θ.

SOLUTION Since the particle is in equilibrium. Therefore,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0}$$

$$\Rightarrow -10\hat{k} + u\left(\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right) + v\left(-\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right) + w(\cos\theta\,\hat{i} + \sin\theta\hat{j}) = \vec{0}$$

$$\Rightarrow \left(\frac{4u}{13} - \frac{4v}{13} + w\cos\theta\right)\hat{i} + \left(\frac{-12u}{13} - \frac{12v}{13} + w\sin\theta\right)\hat{j} + \left(\frac{3u}{13} + \frac{3v}{13} - 10\right)\hat{k} = \vec{0}$$

$$\Rightarrow \frac{4u}{13} - \frac{4v}{13} + w\cos\theta = 0 \quad \text{...(i)}$$

$$-\frac{12u}{13} - \frac{12v}{13} + w\sin\theta = 0 \quad \text{...(ii)} \quad \left[\because \hat{i}, \hat{j}, \hat{k} \text{ are linearly in depdendent}\right]$$

$$\frac{3u}{13} + \frac{3v}{13} - 10 = 0 \quad \text{...(iii)}$$

From (iii), we get :

$$u + v = \frac{130}{3}$$

From (ii), we have

$$-\frac{12}{13}(u+v) + w\sin\theta = 0$$

$$\Rightarrow -\frac{12}{13} \times \frac{130}{3} + w\sin\theta = 0 \qquad \left[\because u + v = \frac{130}{3}\right]$$

$$\Rightarrow w = \frac{40}{\sin\theta} = 40\,\text{cosec}\,\theta$$

Substituting the value of w in (i) and (ii), we get

$$u - v = -130\cot\theta \text{ and, } u + v = \frac{130}{3}$$

Solving these two equations, we get

$$u = \frac{65}{3} - 65\cot\theta \text{ and } v = \frac{65}{3} + 65\cot\theta$$

Hence,

$$u = \frac{65}{3} - 65\cot\theta,\ v = \frac{65}{3} + 65\cot\theta \text{ and } w = 40\,\text{cosec}\,\theta.$$

EXAMPLE 11 *If $\vec{a} = 2\vec{p} + 3\vec{q} - \vec{r}$, $\vec{b} = \vec{p} - 2\vec{q} + 2\vec{r}$, $\vec{c} = -2\vec{p} + \vec{q} - 2\vec{r}$ and $\vec{R} = 3\vec{p} - \vec{q} + 2\vec{r}$, express $\vec{R}$ in terms of $\vec{a}, \vec{b}, \vec{c}$ when $\vec{p}, \vec{q}, \vec{r}$ are non-coplanar vectors.*

SOLUTION Let $\vec{R} = x\vec{a} + y\vec{b} + z\vec{c}$. Then,

$$3\vec{p} - \vec{q} + 2\vec{r} = (2x + y - 2z)\vec{p} + (3x - 2y + z)\vec{q} + (-x + 2y - 2z)\vec{r}$$

$$\Rightarrow 2x + y - 2z = 3$$

$$3x - 2y + z = -1 \qquad [\because \vec{p}, \vec{q}, \vec{r} \text{ are non-coplanar}]$$

$$-x + 2y - 2z = 2$$

Solving these equations, we get

$$x = 2, y = 5, z = 3$$

$$\therefore \vec{R} = 2\vec{a} + 5\vec{b} + 3\vec{c}$$

EXERCISE 14.1

Show that the following sets of vectors are coplanar :

1. (i) $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - 4\vec{c}, -\vec{b} + 2\vec{c}$

   (ii) $5\vec{a} + 6\vec{b} + 7\vec{c}, -7\vec{a} - 8\vec{b} + 9\vec{c}, 3\vec{a} + 20\vec{b} + 5\vec{c}$

   where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors in space.

2. If the vectors $\vec{a}, \vec{b}, \vec{c}$ form a system of linearly independent vectors, show that the following sets of vectors are also independent :

(i) $\vec{a} - \vec{b} + \vec{c}, \vec{b} + \vec{c} - \vec{a}, 2\vec{a} - 3\vec{b} + 4\vec{c}$

(ii) $2\vec{a} - \vec{b} + 3\vec{c}, \vec{a} + \vec{b} - 2\vec{c}, \vec{a} + \vec{b} - 3\vec{c}$

3. If the vectors $\vec{a}, \vec{b}, \vec{c}$ form a linearly indpendent system of vectors, show that the vectors :

   (i) $\vec{a} + 2\vec{b} + \vec{c}, 2\vec{a} - \vec{b} + \vec{c}, 3\vec{a} + \vec{b} + 2\vec{c}$ form a linearly dependent system of vectors.

   (ii) $2\vec{a} + \vec{b} + \vec{c}, \vec{a} - \vec{b} + 2\vec{c}, -\vec{a} + \vec{b} - \vec{c}$ form a linearly independent system of vectors

   (iii) $m\vec{a} + \vec{b} + \vec{c}, \vec{a} + m\vec{b} + \vec{c}, \vec{a} + \vec{b} + m\vec{c}$ form a linearly independent system of vectors if and only if $m \neq -2$.

4. If $\vec{a}, \vec{b}, \vec{c}$ are any three non-coplanar vectors, find the linear relations between the following systems of vectors ;

(i) $\vec{r}_1 = \vec{a} - \vec{b} + \vec{c}, \vec{r}_2 = \vec{b} + \vec{c} - \vec{a}, \vec{r}_3 = \vec{c} + \vec{a} + \vec{b}, \vec{r}_4 = 2\vec{a} - 3\vec{b} + 4\vec{c}$

(ii) $\vec{r}_1 = \vec{a} - 2\vec{b} + \vec{c}, \vec{r}_2 = -\vec{b} + 2\vec{c} + \vec{a}, \vec{r}_3 = \vec{c} - \vec{a} + 3\vec{b}, \vec{r}_4 = \vec{a} + \vec{b} + \vec{c}$

5. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the following vectors are coplanar vectors :

(i) $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$, $\vec{a}-20\vec{b}+5\vec{c}$

(ii) $3\vec{a}-7\vec{b}-4\vec{c}$, $\vec{a}+\vec{b}+2\vec{c}$, $3\vec{a}-2\vec{b}+\vec{c}$

(iii) $\vec{a}-3\vec{b}+5\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$

(iv) $3\vec{a}+20\vec{b}+5\vec{c}$, $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$

6. Find all values of λ such that $(x, y, z) \neq (0, 0, 0)$ and $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+z(-4\hat{i}+5\hat{j}) = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$

7. If $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}-2\hat{k}$, $\vec{c} = 2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{d} = 3\hat{i}+2\hat{j}+5\hat{k}$ are four vectors such that $\vec{d} = x\vec{a}+y\vec{b}+z\vec{c}$, then find the scalars x, y, z.

ANSWERS

4. (i) $\vec{r}_4 = \frac{7}{2}\vec{r}_1+\vec{r}_2-\frac{1}{2}\vec{r}_3$ (ii) $\vec{r}_4 = -8\vec{r}_1+6\vec{r}_2-3\vec{r}_3$ 6. $\lambda = 0, -1$ 7. $x=-2, y=1, z=-3$

—❑❑❑—

14.15 POSITION VECTOR

The concept of position vector is very useful in the applications of vector methods to geometry. In this section, we shall introduce the same. For this, we take an arbitrary point, O, to be called the origin of reference.

POSITION VECTOR *If a point O is fixed as the origin in space (or a plane) and P is any point, then $\vec{OP}$ is called the position vector of P with respect to O.*

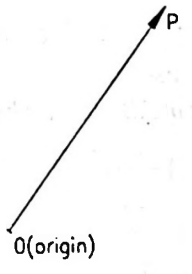


Fig. 14.17

Whenever we say that P is the point $\vec{r}$, then it will mean that the position vector of P is $\vec{r}$ with respect to some origin O.

Thus, with the choice of any point O, as the origin of reference, we can always associate a vector to every point P and, conversely, to every given vector, $\vec{r}$, there corresponds a piont P such that $\vec{OP} = \vec{r}$.

A VECTOR EXPRESSED IN TERMS OF THE POSITION VECTORS OF ITS END POINTS

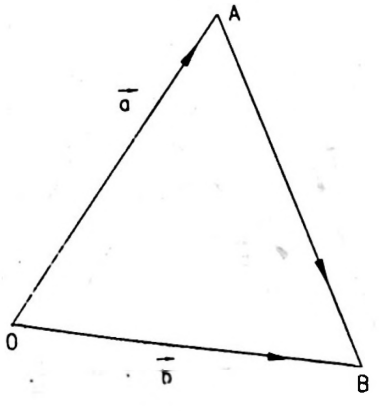


Fig. 14.18

Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively.

Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

In ΔOAB, we have,

$\vec{OA}+\vec{AB} = \vec{OB}$

$\Rightarrow \quad \vec{a}+\vec{AB} = \vec{b}$

$\Rightarrow \quad \vec{AB} = \vec{b}-\vec{a}$

or, $\vec{AB}$ = (Position vector of B) – (Position vector of A)

ILLUSTRATION *Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ be the position vectors of points $P_1, P_2, P_3, \ldots, P_n$ relative to an origin O. Show that if the vector equation $a_1\vec{r}_1+a_2\vec{r}_2+a_3\vec{r}_3+\ldots+a_n\vec{r}_n = \vec{0}$ holds, then a similar equation will also hold good with respect to any other origin O', if $a_1+a_2+a_3+\ldots+a_n = 0$.*

SOLUTION Let the position vector of O' with reference to O as the origin be $\vec{\alpha}$. Then,

$\vec{OO'} = \vec{\alpha}$

Now,

$\vec{O'P_i}$ = Position vector of P_i – Position vector of O'

$= \vec{r}_i-\vec{\alpha}, \qquad i = 1, 2, \ldots, n \qquad \ldots(i)$

Let the position vectors of $P_1, P_2, P_3, \ldots, P_n$ with respect to O' as the origin be $\vec{R}_1, \vec{R}_2, \ldots, \vec{R}_n$ respectively. Then,

$\vec{R}_i = \vec{O'P_i} = \vec{r}_i-\vec{\alpha}, \; i = 1, 2, \ldots, n$ [Using (i)]

Now,

$a_1\vec{R}_1+a_2\vec{R}_2+\ldots+a_n\vec{R}_n = \vec{0}$

$\Rightarrow \sum_{i=1}^{n} a_i\vec{R}_i = \vec{0}$

$$\Rightarrow \sum_{i=1}^{n} a_i(\vec{r_i} - \vec{\alpha}) = \vec{0}$$

$$\Rightarrow \sum_{i=1}^{n} a_i \vec{r_i} - \sum_{i=1}^{n} a_i \vec{\alpha} = \vec{0}$$

$$\Rightarrow \vec{0} - \alpha\left(\sum_{i=1}^{n} a_i\right) = \vec{0} \qquad \left[\because \sum_{i=1}^{n} a_i \vec{r_i} = \vec{0} \text{ (given)}\right]$$

$$\Rightarrow \sum_{i=1}^{n} a_i = 0$$

Thus, $\sum_{i=1}^{n} a_i \vec{R_i} = \vec{0}$ will hold good, if $\sum_{i=1}^{n} a_i = 0$.

14.15 SECTION FORMULAE

(I) INTERNAL DIVISION *Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively, and let C be a point dividing AB internally in the ratio m : n. Then the position vector of C is given by*

$$\vec{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

PROOF Let O be the origin. Then $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio $m : n$. Then,

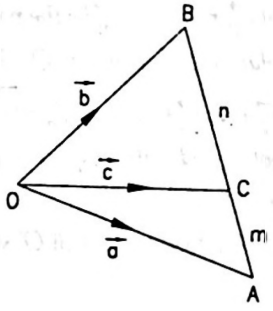


Fig. 14.19

$$\frac{AC}{CB} = \frac{m}{n}$$

$$\Rightarrow n \cdot AC = m \cdot CB$$

$$\Rightarrow n\vec{AC} = m\vec{CB}$$

$$\Rightarrow n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } B - \text{P.V. of } C)$$

$$\Rightarrow n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$$

$$\Rightarrow n\vec{c} + m\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow (m+n)\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n} \text{ or, } \vec{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

REMARK 1 *If C is the mid point of AB, then it divides AB in the ratio 1 : 1. Therefore, position vector of C is*

$$\frac{1 \cdot \vec{a} + 1 \cdot \vec{b}}{1+1} = \frac{\vec{a} + \vec{b}}{2}.$$

Thus, the position vector of the mid point of AB is $\frac{1}{2}(\vec{a} + \vec{b})$

REMARK 2 *We have,*

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n} = \frac{m}{m+n}\vec{b} + \frac{n}{m+n}\vec{a}$$

$$\Rightarrow \vec{c} = \left(\frac{n}{m+n}\right)\vec{a} + \left(\frac{m}{m+n}\right)\vec{b}$$

$$\Rightarrow \vec{c} = \lambda\vec{a} + \mu\vec{b}, \text{ where } \lambda = \frac{n}{m+n},\ \mu = \frac{m}{m+n}.$$

Thus, position ovector of any point C on AB can always be taken as $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, where $\lambda + \mu = 1$.

REMARK 3 *We have*

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

$$\Rightarrow (m+n)\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow n \cdot \vec{OA} + m \cdot \vec{OB} = (n+m)\vec{OC}$$

Thus, we have

$$n \cdot \vec{OA} + m \cdot \vec{OB} = (n+m)\vec{OC},$$

where C is a point on AB dividing it in the ratio m : n

This result is often useful.

If C is the mid point of AB, then we have

$$1 \cdot \vec{OA} + 1 \cdot \vec{OB} = (1+1)\vec{OC}$$

$$\Rightarrow \vec{OA} + \vec{OB} = 2 \cdot \vec{OC}$$

(II) EXTERNAL DIVISION *Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let C be a point dividing AB externally in the ratio m : n. Then the position vector of C is given by*

$$\vec{OC} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

PROOF Let O be the origin. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of a point C dividing AB externally in the ratio $m : n$. Then,

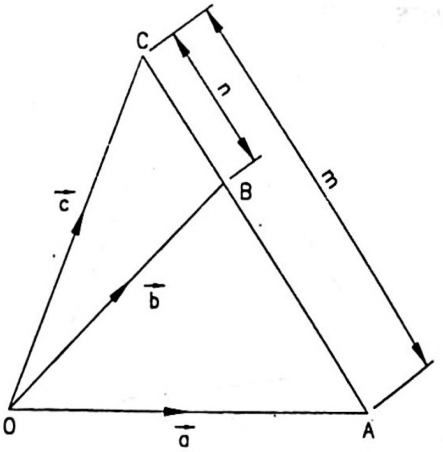


Fig. 14.20

$$\frac{AC}{BC} = \frac{m}{n}$$

$\Rightarrow n \cdot AC = m \cdot BC$

$\Rightarrow n\,\vec{AC} = m\,\vec{BC}$

$\Rightarrow n$ (P.V. of C – P.V of A) $= m$ (P.V. of C – P.V. of B)

$\Rightarrow n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$

$\Rightarrow m\vec{c} - n\vec{c} = m\vec{b} - n\vec{a}$

$\Rightarrow (m - n)\vec{c} = m\vec{b} - n\vec{a}$

$\Rightarrow \vec{c} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$

$\Rightarrow \vec{OC} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the position vectors of the points which divide the join of the points $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ internally and externally in the ratio 2 : 3.*

SOLUTION Let A and B be the given points with position vectors $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ respectively. Let P and Q be the points dividing AB in the ratio 2 : 3 internally and externally respectively. Then,

$$\text{Position vector of } P = \frac{3(2\vec{a} - 3\vec{b}) + 2(3\vec{a} - 2\vec{b})}{3 + 2}$$

$$= \frac{12\vec{a}}{5} - \frac{13\vec{b}}{5}$$

$$\text{Position vector of } Q = \frac{3(2\vec{a} - 3\vec{b}) - 2(3\vec{a} - 2\vec{b})}{3 - 2} = -5\vec{b}$$

EXAMPLE 2 *Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of three distinct points A,B,C. If there exist scalars x, y, z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ and $x + y + z = 0$, then show that A, B and C lie on a line.*

SOLUTION It is given that x, y, z are not all zero. So, let z be non-zero.

Then,

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$

$\Rightarrow z\vec{c} = -(x\vec{a} + y\vec{b})$

$\Rightarrow \vec{c} = -\dfrac{(x\vec{a} + y\vec{b})}{z}$

$\Rightarrow \vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x + y}$ $\quad [\because x + y + z = 0 \ \therefore z = -(x + y)]$

This shows that the point C divides the line joining the points A and B in the ratio $y : x$. Hence, A, B and C lie on the same line.

EXAMPLE 3 *If ABCD is a quadrilateral and E and F are the mid-points of AC and BD respectively, prove that*

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\vec{EF}.$$

SOLUTION Since F is the mid-point of BD, therefore in triangle ABD, we have

$$1 \cdot \vec{AB} + 1 \cdot \vec{AD} = (1 + 1)\vec{AF}$$

$$\Rightarrow \vec{AB} + \vec{AD} = 2\vec{AF} \qquad \ldots(i)$$

In triangle BCD, we have

$$1 \cdot \vec{CB} + 1 \cdot \vec{CD} = (1 + 1)\vec{CF}$$

$$\Rightarrow \vec{CB} + \vec{CD} = 2\vec{CF} \qquad \ldots(ii)$$

Adding (i) and (ii), we obtain

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 2\vec{AF} + 2\vec{CF}$$

$$= -2\vec{FA} - 2\vec{FC}$$

$$= -2(\vec{FA} + \vec{FC})$$

$$= -2(2\vec{FE}) \quad [\because E \text{ is the mid-point of AC}]$$

$$= 4\vec{EF}.$$

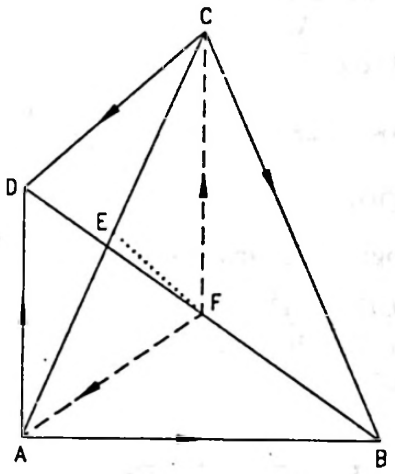


Fig. 14.21

EXAMPLE 4 *If O is the circumcentre and O′ the orthocentre of a triangle ABC, prove that*

(i) $\vec{SA} + \vec{SB} + \vec{SC} = 3\vec{SG}$,

where S is any point in the plane of triangle ABC and G is its centroid

(ii) $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OO'}$

(iii) $\vec{O'A} + \vec{O'B} + \vec{O'C} = 2\vec{O'O}$

(iv) $\vec{AO'} + \vec{O'B} + \vec{O'C} = \vec{AP}$,

where $\vec{AP}$ is a diameter of the circumcircle.

SOLUTION Let G be the centroid of triangle ABC. First we will show that the circumcentre O, orthocentre O' and centroid G are collinear and $O'G = 2OG$.

Let AL and BM be perpendiculars on the sides BC and CA respectively. Let AD be the median and OD be the perpendicular from O on side BC. If R is the circumradius of circumcircle of ΔABC, then $OB = OC = R$.

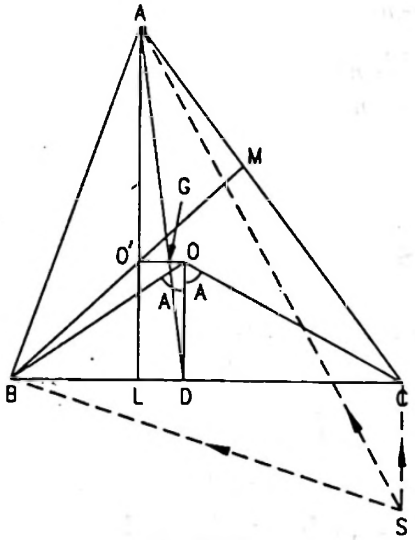


Fig. 14.22

From ΔOBD, we have

$$OD = R\cos A \qquad ...(i)$$

From ΔABM, we have

$$AM = AB\cos A = c\cos A \qquad ...(ii)$$

From $\Delta AO'M$, we have

$AO' = AM \sec \angle O'AM$ [Using (ii)]

$= c\cos A \operatorname{cosec} C$

$$= \frac{c}{\sin C}\cos A = 2R\cos A \quad \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$$

$\therefore\ AO' = 2(OD)$ ...(iii) [Using (i)]

Since, triangles AGO' and OGD are similar

$$\therefore \quad \frac{OG}{O'G} = \frac{GD}{GA} = \frac{OD}{AO'} = \frac{1}{2} \qquad \text{[Using (iii)]}$$

$$\Rightarrow 2\cdot OG = O'G \qquad ...(iv)$$

(i) We have,

$$\vec{SA} + \vec{SB} + \vec{SC} = \vec{SA} + (\vec{SB} + \vec{SC})$$
$$= \vec{SA} + 2\vec{SD}$$
$$= (1+2), \vec{SG}$$
$$= 3\,\vec{SG}.$$

(ii) Replacing S by O in (i), we get

$$\vec{OA} + \vec{OB} + \vec{OC} = 3\vec{OG}$$
$$= 2\vec{OG} + \vec{OG}$$
$$= \vec{GO'} + \vec{OG} \qquad [\because\ 2\cdot OG = GO']$$
$$= \vec{OG} + \vec{GO'} = \vec{OO'}$$

(iii) $\vec{O'A} + \vec{O'B} + \vec{O'C} = 3\vec{O'G}$

$$= 2\vec{O'G} + \vec{O'G} \qquad \text{[From (i)]}$$
$$= 2\vec{O'G} + 2\vec{GO} \qquad [\because\ 2\cdot OG = GO']$$
$$= 2\vec{O'O}$$

(iv) $\vec{AO'} + \vec{O'B} + \vec{O'C} = 2\vec{AO'} + (\vec{O'A} + \vec{O'B} + \vec{O'C})$

$$= 2\vec{AO'} + 2\vec{O'O} \qquad \text{(From (iii))}$$
$$= 2(\vec{AO'} + \vec{O'O})$$
$$= 2\vec{AO} = \vec{AP} \quad \left[\because AO \text{ is the circumradius of } \Delta ABC\right]$$

EXAMPLE 5 *Prove using vectors: Medians of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let D, E, F be the midpoints of its sides BC, CA and AB respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B and C respectively. Then position vectors of D, E and F are $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{a}}{2}$ and $\frac{\vec{a}+\vec{b}}{2}$ respectively.

The position vector of a point dividing AD in the ratio $2:1$ is

$$\frac{1\cdot\vec{a} + 2\left(\frac{\vec{b}+\vec{c}}{2}\right)}{1+2} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}.$$

Similarly, position vectors of points dividing BE and CF in the ratio $2:1$ are each equal to $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

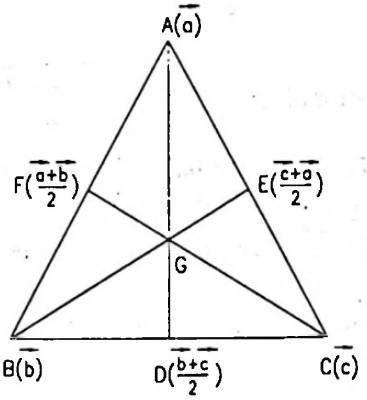


Fig. 14.23

Thus, the point dividing AD in the ratio $2:1$ also divides BE and CF in the same ratio. Hence, the medians of a triangle are concurrent.

The position vector of the centroid is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

NOTE *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle, then the position vector of its centroid is* $\frac{(\vec{a}+\vec{b}+\vec{c})}{3}$.

EXAMPLE 6 *If G is the centroid of a triangle ABC, prove that*

$$\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}.$$

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C respectively. Then, the position vector of the centroid G is

$$\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Now,

$$\vec{GA} + \vec{GB} + \vec{GC}$$

$$= \vec{a} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) + \vec{b} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) + \vec{c} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$= (\vec{a}+\vec{b}+\vec{c}) - 3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$= (\vec{a}+\vec{b}+\vec{c}) - (\vec{a}+\vec{b}+\vec{c}) = \vec{0}.$$

EXAMPLE 7 *If D and E are the mid-points of sides AB and AC of a triangle ABC respectively, show that* $\vec{BE} + \vec{DC} = \frac{3}{2}\vec{BC}$.

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Since D and E are the mid-points of AB and AC. Therefore, the position vectors of D and E are $\frac{\vec{b}}{2}$ and $\frac{\vec{c}}{2}$ respectively.

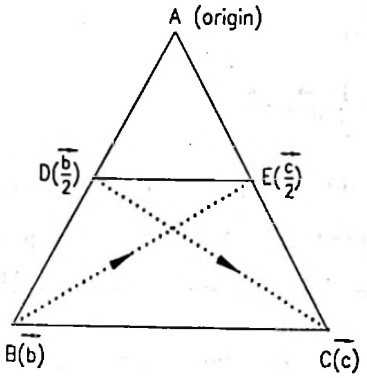


Fig. 14.24

Now,

$$\vec{BE} + \vec{DC}$$

$$= (\text{P.v. of } E - \text{P.v. of } B) + (\text{P.v. of } C - \text{P.v. of } D)$$

$$= \left(\frac{\vec{c}}{2} - \vec{b}\right) + \left(\vec{c} - \frac{\vec{b}}{2}\right)$$

$$= \frac{3}{2}(\vec{c} - \vec{b}) = \frac{3}{2}\vec{BC}.$$

EXAMPLE 8 *If ABC and A'B'C' are two triangles and G, G' be their centroids, prove that*

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\vec{GG'}.$$

SOLUTION Let the position vectors of A, B, C and A', B', C' with reference to some origin be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ respectively.

Then, $\vec{AA'} + \vec{BB'} + \vec{CC'}$

$$= (\vec{a'} - \vec{a}) + (\vec{b'} - \vec{b}) + (\vec{c'} - \vec{c})$$

$$= (\vec{a'} + \vec{b'} + \vec{c'}) - (\vec{a} + \vec{b} + \vec{c}) \quad \ldots(i)$$

The position vectors of the centroids G and G' of triangles ABC $A'B'C'$ are $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ and $\frac{\vec{a'}+\vec{b'}+\vec{c'}}{3}$ respectively. Therefore,

$$\vec{GG'} = \frac{\vec{a'}+\vec{b'}+\vec{c'}}{3} - \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$= \frac{1}{3}[(\vec{a'}+\vec{b'}+\vec{c'}) - (\vec{a}+\vec{b}+\vec{c})]$$

$$\Rightarrow 3\vec{GG'} = (\vec{a'}+\vec{b'}+\vec{c'}) - (\vec{a}+\vec{b}+\vec{c}) \quad \ldots(ii)$$

From (i) and (ii), we have

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\vec{GG'}.$$

EXAMPLE 9 *Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and equal to half of it.*

SOLUTION Let ABC be a triangle and let D and E be the mid points of its sides AB and AC respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of vertices A, B and C respectively. Then position vectors of D and E are $\frac{\vec{a}+\vec{b}}{2}$ and $\frac{\vec{a}+\vec{c}}{2}$ respectively.

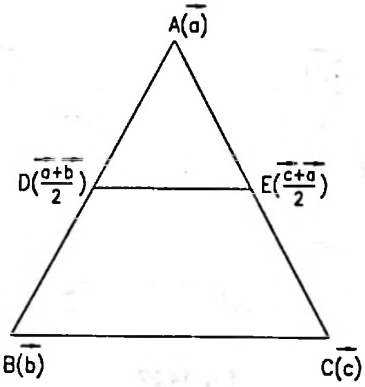


Fig. 14.25

Now $\vec{DE} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}(\vec{c} - \vec{b}) = \frac{1}{2}\vec{BC}$

$\therefore \quad DE \parallel BC.$

Also, $DE = |\vec{DE}| = \frac{1}{2}|\vec{BC}| = \frac{1}{2}BC.$

Hence, $DE \parallel BC$ and $DE = \frac{1}{2}BC.$

EXAMPLE 10 *Prove that the sum of the vectors directed from the vertices to the mid-points of opposite sides of a triangle is zero.*

SOLUTION Let ABC be a triangle and let the position vectors of vertices A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Let D, E, F be mid points of sides BC, CA and AB respectively. Then position vectors of D, E and F are $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{a}}{2}$ and $\frac{\vec{a}+\vec{b}}{2}$ respectively.

Now,

$$\vec{AD} + \vec{BE} + \vec{CF}$$

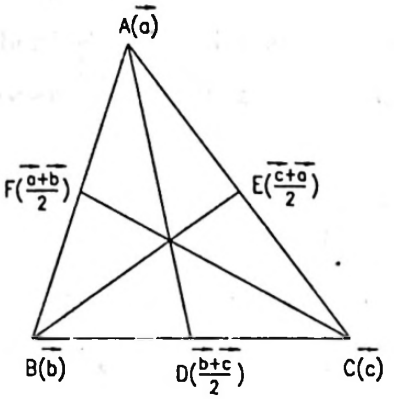


Fig. 14.26

$$= \left(\frac{\vec{b}+\vec{c}}{2}-\vec{a}\right)+\left(\frac{\vec{c}+\vec{a}}{2}-\vec{b}\right)+\left(\frac{\vec{a}+\vec{b}}{2}-\vec{c}\right)$$

$$= \frac{1}{2}(\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}) = \vec{0}.$$

EXAMPLE 11 *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is trisected thereat.*

SOLUTION Let $OABC$ be a parallelogram. Taking O as the origin let the position vectors of A and C be $\vec{a}$ and $\vec{b}$ respectively.

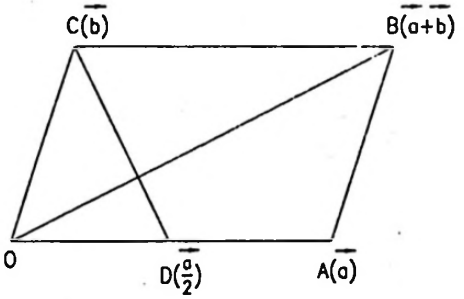


Fig. 14.27

In ΔOAB, we have

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \vec{OA} + \vec{OC} = \vec{OB}$$

$$\Rightarrow \vec{OB} = \vec{a}+\vec{b}$$

So position vector of B is $\vec{a}+\vec{b}$.

Let D be the mid-point of OA. Then position vector of D is $\frac{\vec{a}}{2}$. Position vector of a point dividing CD in the ratio 2 : 1 is

$$\frac{2\cdot(\vec{a}/2)+1\cdot\vec{b}}{2+1} = \frac{\vec{a}+\vec{b}}{3}.$$

Position vector of a point dividing OB in the ratio 1 : 2 is

$$\frac{1\cdot(\vec{a}+\vec{b})+2\cdot\vec{0}}{1+2} = \frac{\vec{a}+\vec{b}}{3}$$

Thus, the position vectors of points trisecting OB and DC are same. Hence, DC trisects OB and DC is trisected thereat.

EXAMPLE 12 *Prove using vectors: The diagonals of a quadrilateral bisect each other iff it is a parallelogram.*

SOLUTION First, let us assume that $ABCD$ is a parallelogram. Then we have to prove that its diagonals bisect each other. Let the position vectors of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

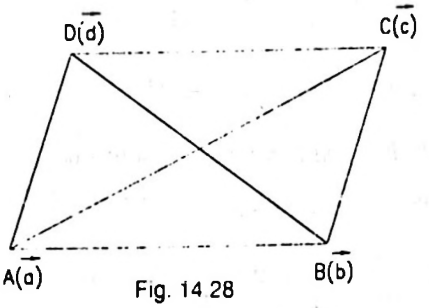


Fig. 14.28

Since $ABCD$ is a parallelogram, therefore

$$\vec{AB} = \vec{DC}$$

$$\Rightarrow \vec{b}-\vec{a} = \vec{c}-\vec{d}$$

$$\Rightarrow \vec{b}+\vec{d} = \vec{a}+\vec{c}$$

$$\Rightarrow \frac{1}{2}(\vec{b}+\vec{d}) = \frac{1}{2}(\vec{a}+\vec{c})$$

$\Rightarrow$ P.V. of the mid-point of BD = P.V. of the mid-point of AC.

Thus, the point which bisects AC also bisects BD.

Hence, diagonals of parallelogram $ABCD$ bisect each other.

Conversely, let $ABCD$ be a quadrilateral such that its diagonals bisect each other. Then we have to prove that it is a parallelogram. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of its vertices A, B, C and D respectively.

Since diagonals AC and BD bisect each other, therefore

P.V. of the mid-point of AC = P.V. of the mid-point of BD

$$\Rightarrow \frac{1}{2}(\vec{a}+\vec{c}) = \frac{1}{2}(\vec{b}+\vec{d})$$

$$\Rightarrow \vec{a}+\vec{c} = \vec{b}+\vec{d}. \quad \text{...(i)}$$

$$\Rightarrow \vec{b}-\vec{a} = \vec{c}-\vec{d}$$

$$\Rightarrow \vec{AB} = \vec{DC}$$

Again, from (i), we have

$$\vec{d}-\vec{a} = \vec{c}-\vec{b} \Rightarrow \vec{AD} = \vec{BC}$$

Hence, $ABCD$ is a parallelogram.

EXAMPLE 13 *Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram.*

SOLUTION Let $ABCD$ be a quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD and DA respectively.

Then, the position vectors of P, Q, R, S are $\frac{\vec{a}+\vec{b}}{2}$, $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{d}}{2}$ and $\frac{\vec{d}+\vec{a}}{2}$ respectively.

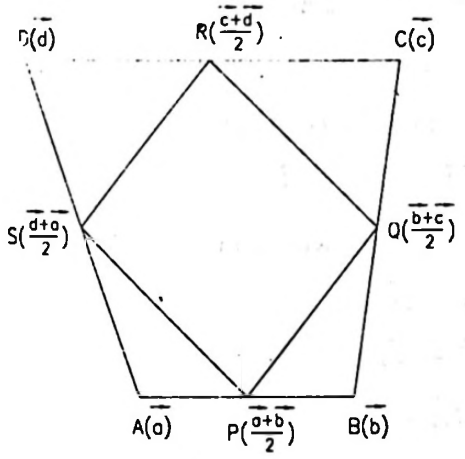


Fig. 14.29

In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that $\vec{PQ} = \vec{SR}$.

We have,

$$\vec{PQ} = \text{Position Vector of } Q - \text{Position Vector of } P$$

$$= \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$$

and,

$$\vec{SR} = \text{Position Vector of } R - \text{Position Vector of } S$$

$$= \left(\frac{\vec{c}+\vec{d}}{2}\right) - \left(\frac{\vec{d}+\vec{a}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$$

So, $\vec{PQ} = \vec{SR}$.

Consequently, $PQ \parallel SR$ and $PQ = SR$.

Hence, $PQRS$ is a parallelogram.

EXAMPLE 14 *Prove that the segment joining the middle points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.*

SOLUTION Let $ABCD$ be the given trapezium. Let the position vectors of A, B, C and D with reference to some origin O be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

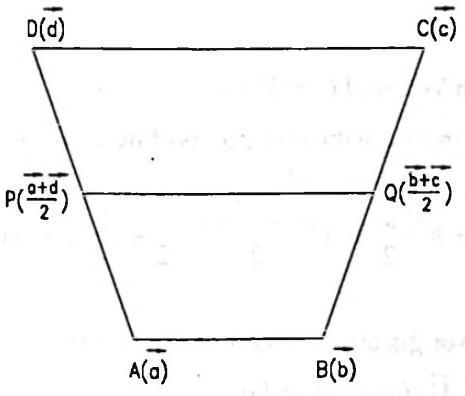


Fig. 14.30

Let P and Q be the mid-points of AD and BC respectively. Then the position vectors of P and Q are $\frac{\vec{a}+\vec{d}}{2}$ and $\frac{\vec{b}+\vec{c}}{2}$ respectively.

We have, $\vec{AB} = \vec{b} - \vec{a}$ and $\vec{DC} = \vec{c} - \vec{d}$.

Since $\vec{DC}$ is parallel to $\vec{AB}$. Therefore, there exists a scalar λ such that

$$\vec{DC} = \lambda \vec{AB} \Rightarrow (\vec{c}-\vec{d}) = \lambda(\vec{b}-\vec{a}) \quad \text{...(i)}$$

Now,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$= \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{d}}{2}\right)$$

$$= \frac{1}{2}[(\vec{b}-\vec{a}) + (\vec{c}-\vec{d})]$$

$$= \frac{1}{2}[(\vec{b}-\vec{a}) + \lambda(\vec{b}-\vec{a})] \quad \text{[Using (i)]}$$

$$= \frac{1}{2}(\lambda+1)(\vec{b}-\vec{a})$$

$$= \frac{1}{2}(\lambda+1)\vec{AB} \quad \text{...(ii)}$$

This shows that PQ is parallel to AB. But, AB is parallel to CD. Therefore, PQ is parallel to CD.

We also have

$$|\vec{AB}| + |\vec{DC}|$$

$$= |\vec{AB}| + |\lambda\vec{AB}| \quad [\because \vec{DC} = \lambda\vec{AB} \text{ (from (i))}]$$

$$= |\vec{AB}| + \lambda|\vec{AB}| \quad [\because \lambda > 0]$$

$$= (1+\lambda)|\vec{AB}|$$

$$\therefore \frac{1}{2}[|\vec{AB}| + |\vec{DC}|]$$

$$= \frac{1}{2}(\lambda+1)|\vec{AB}|$$

$$= |\vec{PQ}| \quad \text{[From (ii)]}$$

This shows that PQ is half of the sum of the lengths of AB and CD.

EXAMPLE 15 *Prove by vector method that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.*

SOLUTION Let $ABCD$ be a trapezium and let the position vectors of A, B, C and D with reference to some origin O be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

Let P and Q be the mid-points of diagonals BD and AC respectively. Then the position vectors of P and Q are $\frac{\vec{b}+\vec{d}}{2}$ and $\frac{\vec{a}+\vec{c}}{2}$ respectively.

$\therefore \quad \vec{PQ}$ = Position Vector of Q – Position Vector of P

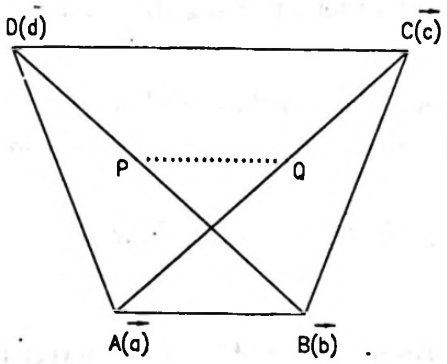


Fig. 14.31

$$= \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{b}+\vec{d}}{2}\right)$$

$$= \frac{1}{2}[(\vec{c}-\vec{d}) + (\vec{a}-\vec{b})]$$

$$= \frac{1}{2}[(\vec{c}-\vec{d}) - (\vec{b}-\vec{a})]$$

$$= \frac{1}{2}[\vec{DC} - \vec{AB}]$$

Now, $\vec{DC} \parallel \vec{AB}$

$\Rightarrow \vec{DC} = \lambda \vec{AB}$ for some scalar λ ...(i)

$$\therefore \vec{PQ} = \frac{1}{2}[\lambda \vec{AB} - \vec{AB}] = \frac{1}{2}(\lambda - 1)\vec{AB} \qquad ...(ii)$$

This shows that PQ is parallel to AB. But, AB is parallel to DC. Therefore, PQ is parallel to DC also.

Now, $|\vec{DC}| - |\vec{AB}| = \lambda |\vec{AB}| - |\vec{AB}|$ [Using (i)]

$$= (\lambda - 1)|\vec{AB}|$$

$$\Rightarrow \frac{1}{2}(|\vec{DC}| - |\vec{AB}|) = \frac{1}{2}(\lambda - 1)|\vec{AB}| = |\vec{PQ}| \text{ [Using (ii)]}$$

This shows that PQ is half of the difference of parallel sides.

EXAMPLE 16 *If $ABCD$ is quadrilateral and E and F are the mid-points of AC and BD respectively, prove that*

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\vec{EF}.$$

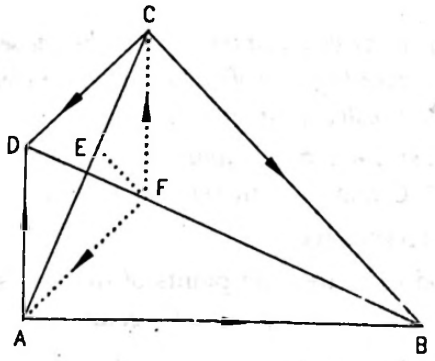


Fig. 14.32

SOLUTION Since F is the mid-point of BD, therefore in triangle ABD, we have

$$1 \cdot \vec{AB} + 1 \cdot \vec{AD} = (1+1)\vec{AF}$$

$$\Rightarrow \quad \vec{AB} + \vec{AD} = 2\vec{AF} \qquad ...(i)$$

Similarly, in triangle BCD, we have

$$1 \cdot \vec{CB} + 1 \cdot \vec{CD} = (1+1)\vec{CF}$$

$$\Rightarrow \quad \vec{CB} + \vec{CD} = 2\vec{CF} \qquad ...(ii)$$

Adding (i) and (ii), we obtain

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD}$$

$$= 2\vec{AF} + 2\vec{CF}$$

$$= -2\vec{FA} - 2\vec{FC}$$

$$= -2(\vec{FA} + \vec{FC})$$

$$= -2(2\vec{FE}) \qquad [\because E \text{ is the mid-point of } AC]$$

$$= 4\vec{EF}.$$

EXAMPLE 17 *$ABCD$ is a parallelogram. E, F are mid-points of BC, CD respectively. AE, AF meet the diagonal BD at Q and P respectively. Show that P, Q trisect DB.*

SOLUTION Let A be the origin and let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then,

$$\vec{AB} = \vec{b}, \vec{AD} = \vec{d}.$$

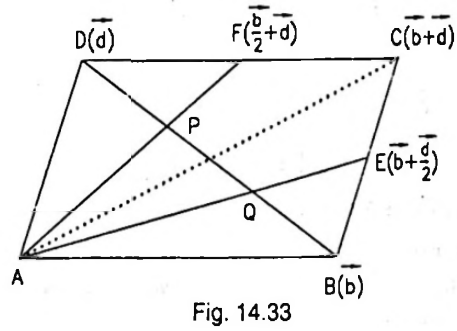


Fig. 14.33

In ΔABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \vec{AB} + \vec{AD} = \vec{AC} \qquad [\because \vec{BC} = \vec{AD}]$$

$$\Rightarrow \vec{b} + \vec{d} = \vec{AC}$$

$\Rightarrow$ Position Vector of $C = \vec{b} + \vec{d}$.

Since E and F are the mid-points of BC and CD respectively. Therefore, position vectors of E and F are $\frac{\vec{b}+(\vec{b}+\vec{d})}{2} = \vec{b} + \frac{\vec{d}}{2}$ and $\frac{(\vec{b}+\vec{d})+\vec{d}}{2} = \frac{\vec{b}}{2} + \vec{d}$ respectively.

Now,

A is the origin and P lies on AF. Therefore,

$\vec{AP} = \lambda \vec{AF}$ for some scalar λ

$\Rightarrow \vec{AP} = \lambda\left(\frac{\vec{b}}{2} + \vec{d}\right)$ ...(i)

Suppose P divides DB in the ratio $\mu : 1$. Then the position vector of P is

$$\frac{\mu\vec{b} + 1\cdot\vec{d}}{\mu + 1}$$

$\Rightarrow \vec{AP} = \frac{\mu\vec{b} + \vec{d}}{\mu + 1}$ ...(ii) [$\because$ A is the origin]

From (i) and (ii), we have

$$\lambda\left(\frac{\vec{b}}{2} + \vec{d}\right) = \frac{\mu\vec{b} + \vec{d}}{\mu + 1}$$

$$\Rightarrow \left(\frac{\lambda}{2} - \frac{\mu}{\mu+1}\right)\vec{b} + \left(\lambda - \frac{1}{\mu+1}\right)\vec{d} = 0$$

$\Rightarrow \frac{\lambda}{2} - \frac{\mu}{\mu+1} = 0$ and $\lambda - \frac{1}{\mu+1} = 0$ [$\because \vec{b}$ and $\vec{d}$ are non-collinear]

$\Rightarrow \mu = \frac{1}{2}$ and $\lambda = \frac{2}{3}$.

Thus, P divides DB in the ratio $\frac{1}{2} : 1$ i.e. $1 : 2$.

Similarly, Q divides DB in the ratio $2 : 1$. Hence, P and Q trisect DB.

EXAMPLE 18 *$ABCD$ is a parallelogram. If L and M are the mid-points of BC and DC respectively, then express $\vec{AL}$ and $\vec{AM}$ in terms of $\vec{AB}$ and $\vec{AD}$. Also, prove that $\vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$.*

SOLUTION Taking A as the origin, let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then,

$$\vec{AB} = \vec{b} \text{ and } \vec{AD} = \vec{d}.$$

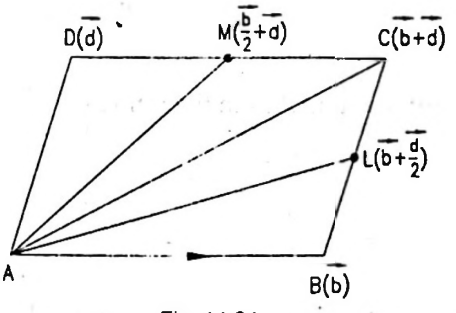


Fig. 14.34

In triangle ABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$\Rightarrow \vec{AC} = \vec{AB} + \vec{AD}$ [$\because \vec{BC} = \vec{AD}$]

$\Rightarrow \vec{AC} = \vec{b} + \vec{d}$

$\Rightarrow$ Position vector of $C = \vec{b} + \vec{d}$.

Since L and M are mid-points of BC and CD respectively. Therefore,

Position vector of $L = \frac{\vec{b} + (\vec{b} + \vec{d})}{2} = \vec{b} + \frac{1}{2}\vec{d}$

Position vector of $M = \frac{(\vec{b} + \vec{d}) + \vec{d}}{2} = \frac{\vec{b}}{2} + \vec{d}$.

$\therefore$ $\vec{AL}$ = Position vector of L – Position vector of A

$$= \vec{b} + \frac{1}{2}\vec{d} - \vec{0}$$

$$= \vec{b} + \frac{1}{2}\vec{d}$$

$$= \vec{AB} + \frac{1}{2}\vec{AD}$$

and,

$\vec{AM}$ = Position vector of M – Position vector of A

$$= \frac{\vec{b}}{2} + \vec{d} - \vec{0} = \frac{\vec{b}}{2} + \vec{d}$$

$$= \frac{1}{2}\vec{AB} + \vec{AD}$$

Now,

$$\vec{AL} + \vec{AM} = \left(\vec{b} + \frac{1}{2}\vec{d}\right) + \left(\frac{1}{2}\vec{b} + \vec{d}\right)$$

$$= \frac{3}{2}\vec{b} + \frac{3}{2}\vec{d}$$

$$= \frac{3}{2}(\vec{b} + \vec{d})$$

$$= \frac{3}{2}\vec{AC}.$$

EXAMPLE 19 *If P and Q are the mid-points of the sides AB and CD of a parallelogram $ABCD$, prove that DP and BQ cut the diagonal AC in its points of trisection which are also the points of trisection of DP and BQ respectively.*

SOLUTION Taking O as the origin, let the position vector B and D be $\vec{b}$ and $\vec{d}$ respectively. Then, position vector of C is $\vec{b} + \vec{d}$ (see Example 17).

Since P and Q are the mid-points of AB and CD respectively, therefore position vectors of P and Q are $\frac{\vec{b}}{2}$ and $\frac{\vec{b}}{2} + \vec{d}$ respectively.

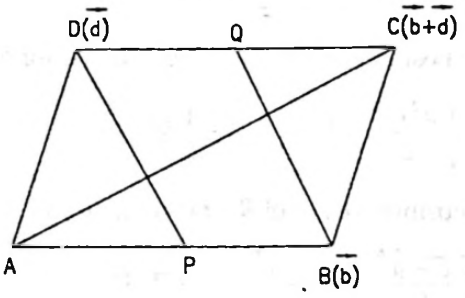


Fig. 14.35

The position vector of a point dividing AC in the ratio 1 : 2 is

$$\frac{1\cdot(\vec{b}+\vec{d})+2\cdot\vec{0}}{1+2}=\frac{\vec{b}+\vec{d}}{3}$$

Also, the position vector of the point dividing DP in the ratio 2 : 1 is

$$\frac{2\left(\frac{\vec{b}}{2}\right)+1\cdot\vec{d}}{2+1}=\frac{\vec{b}+\vec{d}}{3}.$$

Thus, the point of trisection of AC coincides with the point of trisection of DP.

Hence, DP cuts the diagonal AC in its point of trisection, which is also the point of trisection of DP. Similarly, BQ cuts the diagonal AC in its point of trisection, which is also the point of trisection of BQ.

EXAMPLE 20 *"The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram". Prove using vectors.*

SOLUTION Let $ABCD$ be a quadrilateral and let P, R be the mid-points of the sides AB and CD respectively. Let Q and S be the mid-points of diagonals AC and BD respectively.

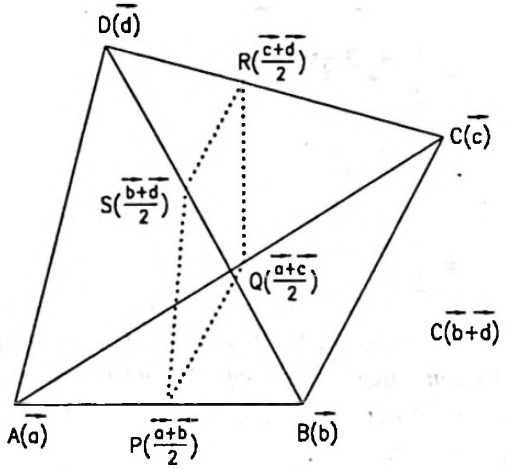


Fig. 14.36

With reference to some origin O, let the position vectors of A, B, C and D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively

Since, P, Q, R and S are the mid-points of AB, AC, CD and BD respectively. Therefore, position vectors of P, Q, R and S are $\frac{\vec{a}+\vec{b}}{2}$, $\frac{\vec{a}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{d}}{2}$ and $\frac{\vec{b}+\vec{d}}{2}$ respectively.

Now, $\vec{PQ}$ = Position vector of Q – Position vector of P

$$=\left(\frac{\vec{a}+\vec{c}}{2}\right)-\left(\frac{\vec{a}+\vec{b}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

and, $\vec{SR}$ = Position vector of R – Position vector of S

$$=\left(\frac{\vec{c}+\vec{d}}{2}\right)-\left(\frac{\vec{b}+\vec{d}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

$\therefore\ \vec{PQ}=\vec{SR}$.

Hence, $PQRS$ is a parallelogram.

EXAMPLE 21 *The lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.*

SOLUTION Let $OABC$ be a tetrahedron. Taking O as the origin, let the position vectors of the vertices A, B, C be $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively. Let G, G_1, G_2, G_3 be the centroids of the faces ABC, OAB, OBC and OCA respectively. Then,

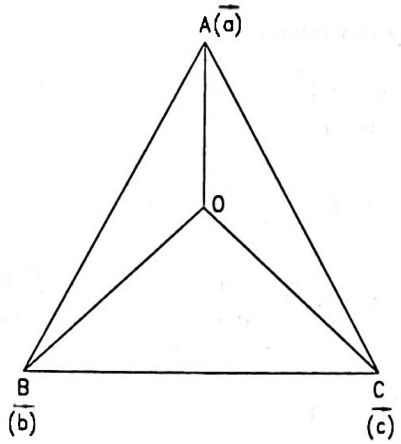


Fig. 14.37

Position vector of $G=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$

Position vector of $G_1=\frac{\vec{a}+\vec{b}}{3}$

Position vector of $G_2=\frac{\vec{b}+\vec{c}}{3}$

Position vector of $G_3=\frac{\vec{c}+\vec{a}}{3}$

Now, P.V. of a point dividing OG in the ratio 3 : 1

$$=\frac{3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+1.\vec{0}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

P.V. of a point dividing AG_2 in the ratio 3 : 1

$$=\frac{3\left(\frac{\vec{b}+\vec{c}}{3}\right)+1.\vec{a}}{2+1}=\frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

P.V. of a point dividing BG_3 in the ratio 3 : 1

$$=\frac{3\left(\frac{\vec{c}+\vec{a}}{3}\right)+1.\vec{b}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

P.V. of a point dividing CG_1 in the ratio 3 : 1

$$=\frac{3\left(\frac{\vec{a}+\vec{b}}{3}\right)+1.\vec{c}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

Thus, the point having position vector $\frac{\vec{a}+\vec{b}+\vec{c}}{4}$ is common to OG, AG_2, BG_3 and CG_1. Hence, the line joining the vertices of a tetrahedrun of the centroids of opposite faces are concurrent.

EXAMPLE 22 *Find a point P within a quadrilateral such that* $\vec{PA}, \vec{PB}, \vec{PC}$ *and* $\vec{PD}$ *have the sum equal to the null vector.*

SOLUTION Let E and F be the mid-pionts of diagonals AC and BD respectively.

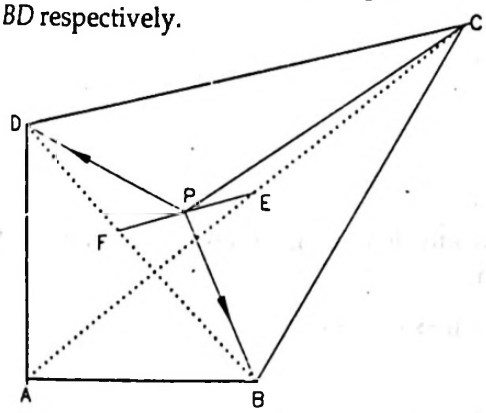


Fig. 14.38

Now,

$\vec{PA}+\vec{PB}+\vec{PC}+\vec{PD} = \vec{0}$ [Given]

$\Rightarrow (\vec{PA}+\vec{PC})+(\vec{PB}+\vec{PD}) = \vec{0}$

$\Rightarrow 2\vec{PE}+2\vec{PF} = \vec{0}$

$\Rightarrow 2(\vec{PE}+\vec{PF}) = \vec{0}$

$\Rightarrow \vec{PE}+\vec{PF} = \vec{0}$

$\Rightarrow \vec{PE} = -\vec{PF}$

$\Rightarrow$ P is the mid-point of EF.

Hence, the required point P is the mid-point of the line segment joining the mid-pionts of diagonals AC and BD of quadrilateral $ABCD$.

EXAMPLE 23 *If I be the centre of the circle inscribed in the* ΔABC, *then prove that*

$$a.\vec{IA}+b.\vec{IB}+c.\vec{IC} = \vec{0},$$

where $a = BC, b = CA$ and $c = AB$

SOLUTION Let $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be the position vectors of the vertices A, B, C respectively.

Then,

Position vector of $I = \frac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

$a.\vec{IA}+b.\vec{IB}+c.\vec{IC}$

$= a\left(\alpha-\frac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}\right)+b\left(\vec{\beta}-\frac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}\right)$

$+c\left(\vec{\gamma}-\frac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}\right)$

$= \frac{1}{a+b+c}\{ab(\vec{\alpha}-\vec{\beta})+ac(\vec{a}-\vec{\gamma})+ab(\vec{\beta}-\vec{\alpha})+bc(\vec{\beta}-\vec{\gamma})$

$+ca(\vec{\gamma}-\vec{\alpha})+bc(\vec{r}-\vec{\beta})]$

$= \vec{0}.$

EXAMPLE 24 *Show, by vector methods, that the angular bisectors of a triangle are concurrent and find the position vector of the point of concurrency in terms of the position vectors of the vertices.*

[IIT 2001]

SOLUTION Let ABC be a triangle such that the position vectors of its vertices A, B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Let AD, BE, and CF be the bisectors of angles A, B and C respectively. Since AD divides BC in the ratio $AB : AC = c : b$. So, the position vector of D is

$$\frac{b\vec{b}+c\vec{c}}{b+c}.$$

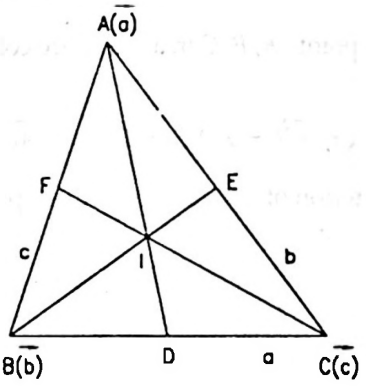


Fig. 14.39

Let I be the point of intersection of AD and BE. Then,

BI is the angle bisector of $\angle DBA$ of ΔABD.

$\therefore$ I divides AD in the ratio $BA : BD$.

But,

$$\frac{DC}{BD} = \frac{b}{c}$$

$$\Rightarrow \frac{BD+DC}{BD} = \frac{c+b}{c}$$

$$\Rightarrow \frac{BC}{BD} = \frac{c+b}{c}$$

$$\Rightarrow \frac{a}{BD} = \frac{c+b}{c}$$

$\Rightarrow BD = \frac{ac}{b+c}$

Thus,

$BA : BD = c : \frac{ac}{b+c} = b+c : a.$

So, the position vector of I is

$$\frac{a\vec{a} + (b+c)\left(\frac{b\vec{b}+c\vec{c}}{b+c}\right)}{a+b+c} = \frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}.$$

By symmetry of the result, we observe that I lies on the angle bisectors AD, BE and CF.

Thus, AD, BE and CF are concurrent at I and the position vector of I is

$$\frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$$

ALITER See Example 2 on page 18.14.

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14.16 COLLINEARITY OF POINTS

As we know that any two points are always collinear. So, in this section, we shall discuss collinearity of three or more points.

Let A, B, C be three points. Then,

A, B, C are collinear $\Leftrightarrow$ $\vec{AB}$ and $\vec{BC}$ are collinear

$\Leftrightarrow \vec{AB} = \lambda \vec{BC}$ for some scalar λ

Thus, three points A, B, C in a plane are collinear if and only if

$\vec{AB} = \lambda \vec{BC}$ or, $\vec{AB} = \lambda \vec{AC}$ or, $\vec{BC} = \lambda \vec{AC}$ for some scalar λ.

Another criterion of collinearity of three points is given by the following theorem.

THEOREM *(Test of collinearity of three points) Three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear iff there exist scalars x, y, z not all zero such that*

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } x + y + z = 0.$$

PROOF First, let us assume that the three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ be collinear. Then one of the points divides the line joining the other two in some ratio. Suppose C divides AB in the ratio $m : n$. Then, by section formula, we have

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

$$\Rightarrow m\vec{b} + n\vec{a} - (m+n)\vec{c} = \vec{0}$$

$$\Rightarrow x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

where $x = n$, $y = m$ and $z = -(m+n)$.

$\therefore\ x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = [n + m - (m+n)] = 0$.

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B and C respectively such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } x + y + z = 0.$$

We have to prove that the points A, B, C are collinear.

Without any loss of generality, we may assume that $z \neq 0$. Then,

$x + y + z = 0 \Rightarrow x + y = -z$.

Now,

$\therefore\ x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$

$\Rightarrow x\vec{a} + y\vec{b} = -z\vec{c}$

$\Rightarrow \vec{c} = \frac{x\vec{a} + y\vec{b}}{-z}$

$\Rightarrow \vec{c} = \frac{x\vec{a} + y\vec{b}}{x+y}$ $[\because -z = x + y]$

$\Rightarrow \vec{c}$ is the position vector of a point which divides the join of the points A and B in the ratio $y : x$.

Hence, the points A, B, C are collinear.

Q.E.D.

REMARK *Four or more points are collinear, if every group of three points in it form a set of collinear points.*

ILLUSTRATION *Prove that the points $2\vec{a} + \vec{b} - \vec{c}$, $5\vec{a} - \vec{b} + 2\vec{c}$ and $8\vec{a} - 3\vec{b} + 5\vec{c}$ are collinear.*

SOLUTION Let x, y, z be scalars such that

$$x(2\vec{a} + \vec{b} - \vec{c}) + y(5\vec{a} - \vec{b} + 2\vec{c}) + z(8\vec{a} - 3\vec{b} + 5\vec{c}) = \vec{0}$$

$$\Rightarrow (2x + 5y + 8z)\vec{a} + (x - y - 3z)\vec{b} + (-x + 2y + 5z)\vec{c} = \vec{0}$$

$$\Rightarrow 2x + 5y + 8z = 0,\ x - y - 3z = 0 \text{ and } -x + 2y + 5z = 0$$

Solving first two of these equations by cross-multiplication, we get

$$\frac{x}{-7} = \frac{y}{14} = \frac{z}{-7}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = \lambda \text{ (say)}$$

$$\Rightarrow x = \lambda, y = -2\lambda \text{ and } z = \lambda.$$

These values of x, y, z satisfy the third equation and $x + y + z = 0$.

Hence, the given points are collinear.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* $\vec{AO}+\vec{OB}=\vec{BO}+\vec{OC}$, *prove that the points A, B, C are collinear.*

SOLUTION We have,

$$\vec{AO}+\vec{OB}=\vec{BO}+\vec{OC}$$

$$\Rightarrow \quad \vec{AB}=\vec{BC} \qquad \text{[By addition of vectors]}$$

$\Rightarrow \vec{AB}, \vec{BC}$ are collinear vectors.

But, the point where $\vec{AB}$ terminates, $\vec{BC}$ starts from the same point.

Hence, points A, B, C are collinear.

EXAMPLE 2 *If* $\vec{a}$ and $\vec{b}$ *are two non-collinear vectors, prove that the points* $x_1\vec{a}+y_1\vec{b}, x_2\vec{a}+y_2\vec{b}, x_3\vec{a}+y_3\vec{b}$ *are collinear if*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

SOLUTION We know that three ponits with position vectors $\vec{u}, \vec{v}, \vec{w}$ are collinear iff there exist scalars x, y, z such that

$$x\vec{u}+y\vec{v}+z\vec{w}=\vec{0}, \text{ where } x+y+z=0,$$

Therefore, the given points will be collinear iff there exist scalars l, m, n such that

$$l(x_1\vec{a}+y_1\vec{b})+m(x_2\vec{a}+y_2\vec{b})+n(x_3\vec{a}+y_3\vec{b})=\vec{0}$$

where $l+m+n=0$

$$\Rightarrow (lx_1+mx_2+nx_3)\vec{a}+(ly_1+my_2+ny_3)\vec{b}=\vec{0}$$

$$\Rightarrow lx_1+mx_2+nx_3=0,$$

and,

$ly_1+my_2+ny_3=0$ [$\because \vec{a}$ and $\vec{b}$ are non-collinear vectors]

Thus, we have

$$lx_1+mx_2+nx_3=0$$

$$ly_1+my_2+ny_3=0$$

and, $$l+m+n=0$$

Eliminating l, m, n from these equations, we get

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

EXAMPLE 3 *Let* $\vec{a}, \vec{b}, \vec{c}$ *be three non-coplanar vectors. Prove that the points* $\vec{a}-2\vec{b}+3\vec{c}, 2\vec{a}+3\vec{b}-4\vec{c}, -7\vec{b}+10\vec{c}$ *are collinear.*

SOLUTION Let the given points be P, Q and R respectively. Then,

$$\vec{PQ} = \text{P.V. of } Q - \text{P.V. of } P$$

$$= (2\vec{a}+3\vec{b}-4\vec{c})-(\vec{a}-2\vec{b}+3\vec{c})$$

$$= \vec{a}+5\vec{b}-7\vec{c}$$

and,

$$\vec{QR} = \text{P.V. of } R - \text{P.V. of } Q$$

$$= (-7\vec{b}+10\vec{c})-(2\vec{a}+3\vec{b}-4\vec{c})$$

$$= -2\vec{a}-10\vec{b}+14\vec{c}$$

Clearly, $\vec{QR}=-2\vec{PQ}$.

Hence, points P, Q, R are collinear points.

EXAMPLE 4 *If the points with position vectors* $10\hat{i}+3\hat{j}, 12\hat{i}-5\hat{j}$ *and* $a\hat{i}+11\hat{j}$ *are collinear, find the value of a.*

SOLUTION Let the given points be P, Q and R respectively. Then,

$$\vec{PQ} = \text{P.V. of } Q - \text{P.V. of } P$$

$$= (12\hat{i}-5\hat{j})-(10\hat{i}+3\hat{j})=2\hat{i}-8\hat{j}$$

and,

$$\vec{QR} = \text{P.V. of } R - \text{P.V. of } Q$$

$$= (a\hat{i}+11\hat{j})-(12\hat{i}-5\hat{j})$$

$$= (a-12)\hat{i}+16\hat{j}$$

Since points P, Q, R are collinear points. Therefore,

$$\vec{PQ}=\lambda\vec{QR} \quad \text{for some scalar } \lambda$$

$$\Rightarrow 2\hat{i}-8\hat{j}=\lambda\{(a-12)\hat{i}+16\hat{j}\}$$

$$\Rightarrow \{\lambda(a-12)-2\}\hat{i}+(16\lambda+8)\hat{j}=\vec{0}$$

$\Rightarrow \lambda(a-12)-2=0$ and $16\lambda+8=0$ [$\because \hat{i}, \hat{j}$ are non-collinear]

$$\Rightarrow \lambda=-\frac{1}{2} \text{ and } a=8$$

EXAMPLE 5 *Prove that the points with position vectors* $\vec{a}+\vec{b}, \vec{a}-\vec{b}$ *and* $\vec{a}+\lambda\vec{b}$ *are collinear for all real values of* λ.

SOLUTION Let the given points be A, B and C respectively. Then,

$$\vec{AB} = \text{P.V. of } B - \text{P.V. of } A = (\vec{a}-\vec{b})-(\vec{a}+\vec{b}) = -2\vec{b}$$

and,

$$\vec{BC} = \text{P.V. of C} - \text{P.V. of } B = (\vec{a}+\lambda\vec{b})-(\vec{a}-\vec{b}) = (\lambda+1)\vec{b}$$

Clearly, $\vec{BC}=-\left(\frac{\lambda+1}{2}\right)\vec{AB}$ for all real values of λ

Hence, the given points are collinear for all real values of λ.

EXAMPLE 6 *Let OABCD be a pentagon in which the sides OA and CB are parallel and the sides OD and AB are parallel as shown in Fig. 14.40. Also, OA : CB = 2 : 1 and OD : AB = 1 : 3. If the diagonals OC and AD meet at X, find OX : OC.*

SOLUTION Let O be the origin of reference. Let the position vectors of A, B, C, D be $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively.

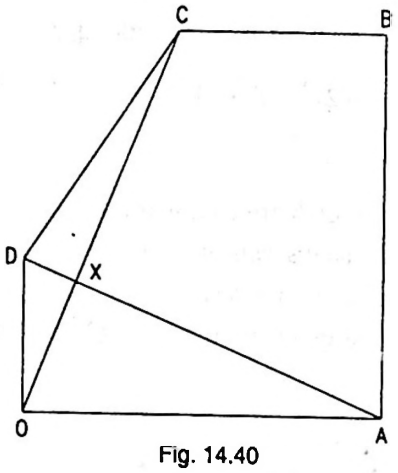


Fig. 14.40

Then,

$OA : CB = 2 : 1$

$\Rightarrow \frac{OA}{CB} = \frac{2}{1}$

$\Rightarrow OA = 2\,CB$

$\Rightarrow \vec{OA} = 2\,\vec{CB}$

$\Rightarrow \vec{a} = 2(\vec{b} - \vec{c})$ ...(i)

and,

$OD : AB = 1 : 3$

$\Rightarrow \frac{OD}{AB} = \frac{1}{3}$

$\Rightarrow 3\,OD = AB$

$\Rightarrow 3\,\vec{OD} = \vec{AB}$

$\Rightarrow 3\vec{d} = \vec{b} - \vec{a}$

$\Rightarrow 3\vec{d} = \vec{b} - 2(\vec{b} - \vec{c})$ [Using (i)]

$\Rightarrow 3\vec{d} = -\vec{b} + 2\vec{c}$ ...(ii)

Let $OX : XC = \lambda : 1$ and $AX : XD = \mu : 1$

Now, X divides OC in the ratio $\lambda : 1$. Therefore,

$$\text{P.V. of } X = \frac{\lambda\vec{c}}{\lambda + 1} \quad \text{...(iii)}$$

X also dividies AD in the ratio $\mu : 1$.

$$\text{P.V. of } X = \frac{\mu\vec{d} + \vec{a}}{\mu + 1} \quad \text{...(iv)}$$

From (iii) and (iv), we get

$$\frac{\lambda\vec{c}}{\lambda + 1} = \frac{\mu\vec{d} + \vec{a}}{\mu + 1}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda + 1}\right)\vec{c} = \left(\frac{\mu}{\mu + 1}\right)\vec{d} + \left(\frac{1}{\mu + 1}\right)\vec{a}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda + 1}\right)\vec{c} = \left(\frac{\mu}{\mu + 1}\right)\left(\frac{-\vec{b} + 2\vec{c}}{3}\right) + \left(\frac{1}{\mu + 1}\right)2(\vec{b} - \vec{c})$$

[Using (i) and (iv)]

$$\Rightarrow \left(\frac{\lambda}{\lambda + 1}\right)\vec{c} = \left(\frac{2}{\mu + 1} - \frac{\mu}{3(\mu + 1)}\right)\vec{b} + \left(\frac{2\mu}{3(\mu + 1)} - \frac{2}{\mu + 1}\right)\vec{c}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda + 1}\right)\vec{c} = \left(\frac{6 - \mu}{3(\mu + 1)}\right)\vec{b} + \left(\frac{2\mu - 6}{3(\mu + 1)}\right)\vec{c}$$

$$\Rightarrow \left(\frac{6 - \mu}{3(\mu + 1)}\right)\vec{b} + \left(\frac{2\mu - 6}{3(\mu + 1)} - \frac{\lambda}{\lambda + 1}\right)\vec{c} = \vec{0}$$

$$\Rightarrow \frac{6 - \mu}{3(\mu + 1)} = 0 \text{ and } \frac{2\mu - 6}{3(\mu + 1)} - \frac{\lambda}{\lambda + 1} = 0$$

[$\because \vec{b}$ and $\vec{c}$ are non-collinear vectors]

$\Rightarrow \mu = 6$ and $\lambda = \frac{2}{5}$

$\Rightarrow OX : XC = 2 : 5. \Rightarrow OX : OC = 2 : 7$

EXAMPLE 7 *In a quadrialteral PQRS, $\vec{PQ} = \vec{a}$, $\vec{QR} = \vec{b}$, $\vec{SP} = \vec{a} - \vec{b}$. If M is the mid-point of QR and X is a point of SM such that $SX = \frac{4}{5}SM$, then prove that the points P, X and R are collinear.*

SOLUTION If we take point P as the origin, then the position vectors of Q and S are $\vec{a}$ and $\vec{b} - \vec{a}$ respectively.

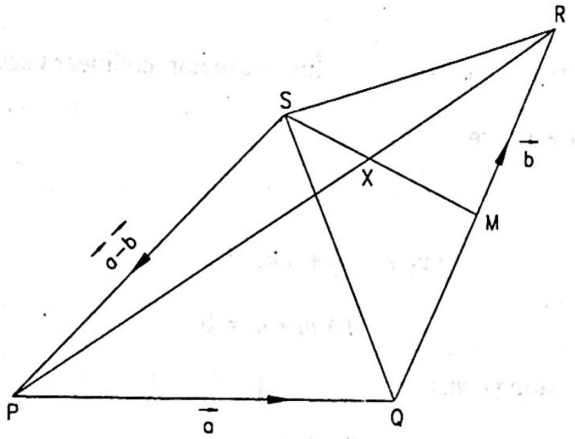


Fig. 14.41

In ΔPQR, we have

$\vec{PR} = \vec{PQ} + \vec{QR}$

$\Rightarrow \vec{PR} = \vec{a} + \vec{b}$

$\therefore$ Position vector of $R = \vec{a} + \vec{b}$

$\Rightarrow$ P.V of $M = \frac{\vec{a} + (\vec{a} + \vec{b})}{2} = \left(\vec{a} + \frac{1}{2}\vec{b}\right)$

Now,

$$SX = \frac{4}{5} SM$$

$$\Rightarrow XM = SM - SX = SM - \frac{4}{5} SM = \frac{1}{5} SM$$

$$\therefore SX : XM = 4 : 1$$

$$\Rightarrow \text{P.V. of } X = \frac{4\left(\vec{a} + \frac{1}{2}\vec{b}\right) + 1 \cdot (\vec{b} - \vec{a})}{4+1} = \frac{3\vec{a} + 3\vec{b}}{5}$$

$$\Rightarrow \vec{PX} = \frac{3}{5}(\vec{a} + \vec{b})$$

$$\Rightarrow \vec{PX} = \frac{3}{5}\vec{PR} \qquad [\because \vec{PR} = \vec{a} + \vec{b}]$$

$\Rightarrow$ P, X and R are collinear points.

EXAMPLE 8 *A transversal cuts the sides OL, OM and diagonal ON of a parallelogram at A, B, C respectively. Prove that*

$$\frac{OL}{OA} + \frac{OM}{OB} = \frac{ON}{OC}$$

SOLUTION Let $\vec{OL} = x\,\vec{OA}$, $\vec{OM} = y\vec{OB}$ and $\vec{ON} = z\,\vec{OC}$. Then,

$$|\vec{OL}| = x\,|\vec{OA}|, |\vec{OM}| = y\,|\vec{OB}| \text{ and } |\vec{ON}| = z\,|\vec{OC}|$$

$$\Rightarrow x = \frac{OL}{OA}, y = \frac{OM}{OB}, z = \frac{ON}{OC}$$

Taking O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

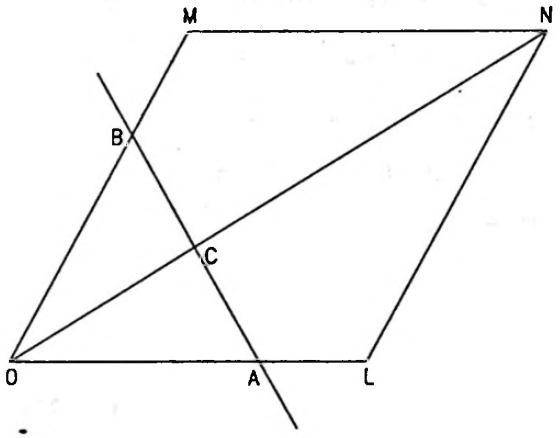


Fig. 14.42

In ΔOLN, we have

$$\vec{OL} + \vec{LN} = \vec{ON}$$

$$\Rightarrow \vec{ON} = \vec{OL} + \vec{OM}$$

$$\Rightarrow z \cdot \vec{OC} = x\,\vec{OA} + y\,\vec{OB}$$

$$\Rightarrow x\,\vec{OA} + y\,\vec{OB} - z\,\vec{OC} = \vec{0}$$

$$\Rightarrow x\vec{a} + y\vec{b} + (-z)\vec{c} = \vec{0}$$

But, points A, B, C are collinear. Therefore,

$$x\vec{a} + y\vec{b} + (-z)\vec{c} = \vec{0}$$

$$\Rightarrow x + y - z = 0$$

$$\Rightarrow x + y = z$$

$$\Rightarrow \frac{OL}{OA} + \frac{OM}{OB} = \frac{ON}{OC}$$

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14.17 COPLANARITY OF POINTS

In section 14.9, we have learnt about coplanar vectors. As we have learnt that three non-zero vectors are coplanar iff one of them is expressible as a linear combination of other two. In this section, we shall learn about the coplanarity of four points. Four points P, Q, R, S are coplanar if three vectors obtained by joining these four points are coplanar. Another criterion of coplanarity of four points is given in the following theorem.

THEOREM *(Test of coplanarity of four points) Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff there exist scalars x, y, z, u not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where $x + y + z + u = 0$.*

PROOF Let A, B, C, and D be four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

First, let A, B, C, D be four coplanar points. Then $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ are coplanar vectors. So, there exist scalars λ and μ such that

$$\vec{AB} = \lambda\,\vec{AC} + \mu\,\vec{AD}$$

$$\Rightarrow [\text{P.V of } B - \text{P.V. of } A = \lambda(\text{P.V. of } C - \text{P.V. of } A) + \mu(\text{P.V. of } D - \text{P.V. of } A)$$

$$\Rightarrow \vec{b} - \vec{a} = \lambda(\vec{c} - \vec{a}) + \mu(\vec{d} - \vec{a})$$

$$\Rightarrow (\lambda + \mu - 1)\vec{a} + \vec{b} + (-\lambda)\vec{c} + (-\mu)\vec{d} = \vec{0}$$

$$\Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}, \text{ where } x = \lambda + \mu - 1, y = 1, z = -\lambda \text{ and } u = -\mu$$

$$\Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}, \text{ where } x + y + z + u = \lambda + \mu - 1 + 1 - \lambda - \mu = 0.$$

Conversely, let there be scalars x, y, z, u not all zero such that

$$x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = 0 \qquad \ldots(i)$$

where

$$x + y + z + u = 0 \qquad \ldots(ii)$$

Putting $x = -(y+z+u)$ from (ii) into (i), we get

$$-(y+z+u)\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=\vec{0}$$

$$\Rightarrow \quad y(\vec{b}-\vec{a})+z(\vec{c}-\vec{a})+u(\vec{d}-\vec{a})=\vec{0}$$

$$\Rightarrow \quad y\vec{AB}+z\vec{AC}+u\vec{AD}=\vec{0}$$

Let $y \neq 0$. Then,

$$\vec{AB}=\left(\frac{-z}{y}\right)\vec{AC}+\left(\frac{-u}{y}\right)\vec{AD}.$$

This shows that $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar vectors. Hence, points A, B, C, D are coplanar points.

Q.E.D.

Let us now discuss some examples to illustrate the concept of coplanarity of four points.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that four points* $2\vec{a}+3\vec{b}-\vec{c}, \vec{a}-2\vec{b}+3\vec{c}, 3\vec{a}+4\vec{b}-2\vec{c}$ *and* $\vec{a}-6\vec{b}+6\vec{c}$ *are coplanar.*

SOLUTION Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So, let

$$\vec{PQ}=x\vec{PR}+y\vec{PS}$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=x(\vec{a}+\vec{b}-\vec{c})+y(-\vec{a}-9\vec{b}+7\vec{c})$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=(x-y)\vec{a}+(x-9y)\vec{b}+(-x+7y)\vec{c}$$

$$\Rightarrow \quad x-y=-1,\ x-9y=-5,\ -x+7y=4$$

[Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first of these three equations, we get $x=-\frac{1}{2}, y=\frac{1}{2}$. These values also satisfy the third equation. Hence, the given four points are coplanar.

EXAMPLE 2 *If* $\vec{a}, \vec{b}, \vec{c}$ *be any three non-coplanar vectors, then prove that the points* $l_1\vec{a}+m_1\vec{b}+n_1\vec{c}$, $l_2\vec{a}+m_2\vec{b}+n_2\vec{c}$, $l_3\vec{a}+m_3\vec{b}+n_3\vec{c}$ *and* $l_4\vec{a}+m_4\vec{b}+n_4\vec{c}$ *are coplanar if*

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

SOLUTION We know that four points having position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if there exist scalars x, y, z, t such that

$$x\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=\vec{0}, \text{ where } x+y+z+t=0$$

Therefore, the given points will be coplanar if there exist scalars x, y, z, t such that

$$x(l_1\vec{a}+m_1\vec{b}+n_1\vec{c})+y(l_2\vec{a}+m_2\vec{b}+n_2\vec{c})$$

$$+z(l_3\vec{a}+m_3\vec{b}+n_3\vec{c})+t(l_4\vec{a}+m_4\vec{p}+n_4\vec{c})=\vec{0}$$

, where $x+y+z+t=0$

$$\Rightarrow (l_1x+l_2y+l_3z+l_4t)\vec{a}+(m_1x+m_2y+m_3z+m_4t)\vec{b}$$

$$+(n_1x+n_2y+n_3z+n_4t)\vec{c}=0, \text{ where } x+y+z+t=0$$

$$\Rightarrow \quad l_1x+l_2y+l_3z+l_4t=0$$

$$m_1x+m_2y+m_3z+m_4t=0 \quad [\because \vec{a}, \vec{b}, \vec{c} \text{ are non-coplanar}]$$

$$n_1x+n_2y+n_3z+n_4t=0$$

$$x+y+z+t=0$$

Eliminating x, y, z and t from these equations, we get

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

EXAMPLE 3 *If any point O within or without a tetrahedron ABCD is joined to the vertices and AO, BO, CO, DO are produced to cut the planes of the opposite faces in P, Q, R, S respectively, then prove that*

$$\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1$$

SOLUTION Taking point O as the origin, let the position vectors of the vertices A, B, C, D be $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively. AO, BO, CO and DO are produced to meet the planes of the opposite faces in P, Q, R and S respectively. Since any set of three vectors out of four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ is a set of non-coplanar vectors. Therefore, any one of the vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ may be expressed as a linear combination of the other three.

Consequently, there must exist a linear relation between them, say

$$l\vec{a}+m\vec{b}+n\vec{c}+p\vec{d}=\vec{0} \qquad \ldots(i)$$

Now, $\vec{AO}$ and $\vec{OP}$ are collinear vectors, therefore

$$\vec{OP}=\lambda\vec{AO} \text{ for some scalar } \lambda>0$$

$$\Rightarrow \quad \vec{OP}=-\lambda\vec{a} \qquad [\because \vec{OA}=\vec{a}]$$

Let $\vec{OP}=\vec{r}$. Then,

$\vec{r} = -\lambda \vec{a}$

$\Rightarrow \vec{r} = \frac{\lambda}{l}(m\vec{b} + n\vec{c} + p\vec{d})$ $\left[\text{From (i) } \vec{a} = -\frac{(m\vec{b} + n\vec{c} + p\vec{d})}{l}\right]$

$\Rightarrow l\vec{r} - \lambda m\vec{b} - \lambda n\vec{c} - \lambda p\vec{d} = \vec{0}$

Clearly, this is a relation between the position vectors of non-coplanar points P, B, C and D.

$\therefore l - \lambda m - \lambda n - \lambda p = 0$

$\Rightarrow \lambda = \frac{l}{m+n+p}$

Now, $\vec{AP} = \vec{AO} + \vec{OP}$

$\Rightarrow \vec{AP} = -\vec{a} - \lambda\vec{a} = -(\lambda+1)\vec{a}$

$\Rightarrow \vec{AP} = -(\lambda+1)\vec{OA}$

$\Rightarrow |\vec{AP}| = (\lambda+1)|\vec{OA}|$

Also, $\vec{OP} = -\lambda\vec{a}$

$\Rightarrow |\vec{OP}| = \lambda|\vec{OA}|$

$\therefore \frac{OP}{AP} = \frac{\lambda}{\lambda+1} = \frac{l/m+n+p}{\frac{l}{m+n+p}+1} = \frac{l}{l+m+n+p}$

Similarly, we have

$\frac{OQ}{BQ} = \frac{m}{l+m+n+p}, \frac{OR}{CR} = \frac{n}{l+m+n+p}$

and,

$\frac{OS}{DS} = \frac{p}{l+m+n+p}$

$\therefore \frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} + \frac{OS}{DS} = \frac{l+m+n+p}{l+m+n+p} = 1$

EXERCISE 14.2

1. Prove that the following points are collinear :

(i) $\vec{a}, \vec{b}, 3\vec{a} - 2\vec{b}$ (ii) $\vec{a} + \vec{b} + \vec{c}, 4\vec{a} + 3\vec{b}, 10\vec{a} + 7\vec{b} - 2\vec{c}$

(iii) $\vec{a} + 2\vec{b} + 3\vec{c}, 7\vec{a} - \vec{c}, -2\vec{a} + 3\vec{b} + 5\vec{c}$

2. If the points with position vectors $60\hat{i} + 3\hat{j}, 40\hat{i} - 8\hat{j}, a\hat{i} - 52\hat{j}$ are collinear, find a.

[IIT 1983]

3. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then prove that in each of the following sets of four points, the points are collinear :

(i) $6\vec{a} - 4\vec{b} + 10\vec{c}, -5\vec{a} + 3\vec{b} - 10\vec{c}, 4\vec{a} - 6\vec{b} - 10\vec{c}, 2\vec{b} + 10\vec{c}$

(ii) $6\vec{a} + 2\vec{b} - \vec{c}, 2\vec{a} - \vec{b} + 3\vec{c}, -\vec{a} + 2\vec{b} - 4\vec{c}, -12\vec{a} - \vec{b} - 3\vec{c}$

(iii) $\vec{a} - 6\vec{b} + 6\vec{c}, 2\vec{a} + 3\vec{b} - \vec{c}, \vec{a} - 2\vec{b} + 3\vec{c}, 3\vec{a} + 4\vec{b} - 2\vec{c}$

4. If O and H are the circumcentre and the orthocentre of the ΔABC, prove that

(i) $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OH}$

(ii) $\vec{HA} + \vec{HB} + \vec{HC} = 2(\vec{AO} + \vec{BO} + \vec{CO})$

5. In a triangle ABC, D is on AC and divides AC in the ratio 2 : 1. BD is produced to F such that $DF = 2BD$. Prove, by vector methods, that AF is parallel to BC and is equal to $2\,BD$ and CF is parallel to the median and is twice the median.

ANSWERS

2. -40

—❑❑❑—

CHAPTER

15

SCALAR OR DOT PRODUCT OF VECTORS

15.1 DEFINITION

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined as the scalar $|\vec{a}|\,|\vec{b}|\cos\theta$

Thus, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

REMARK *If $\vec{a}$ or $\vec{b}$ or both are equal to the null vector, θ is not defined as $\vec{O}$ has no direction and in such a case their dot product is defined as the scalar zero.*

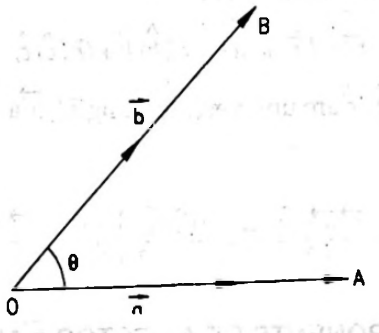


Fig. 15.1

15.2 GEOMETRICAL INTERPRETATION OF SCALAR PRODUCT

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ respectively. Let θ be the angle between $\vec{OA}$ and $\vec{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.

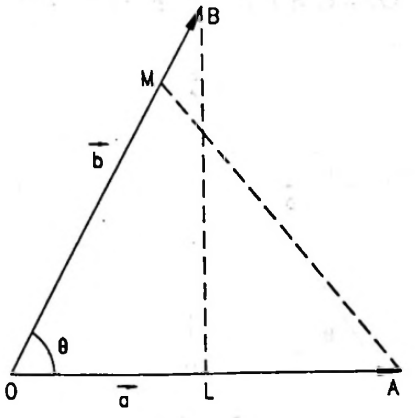


Fig. 15.2

From triangles OBL and OAM, we have

$OL = OB\cos\theta$ and $OM = OA\cos\theta$.

Here, OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

Now,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$
$$= |\vec{a}|\,(OB\cos\theta)$$
$$= |\vec{a}|\,(OL)$$
$$= (\text{Magnitude of } \vec{a})\,(\text{Projection of } \vec{b} \text{ on } \vec{a}) \quad \ldots(i)$$

Again,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$
$$= |\vec{b}|\,||\vec{a}|\cos\theta|$$
$$= |\vec{b}|\,(OA\cos\theta)$$
$$= |\vec{b}|\,(OM)$$
$$= (\text{Magnitude of } \vec{b})\,(\text{Projection of } \vec{a} \text{ on } \vec{b}) \quad \ldots(ii)$$

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

REMARK *From (i) and (ii) we have*

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$$

and,

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$

Thus, the projection of $\vec{a}$ on $\vec{b}$ is the dot product of $\vec{a}$ with the unit vector along $\vec{b}$ and the projection of $\vec{b}$ on $\vec{a}$ is the dot product of $\vec{b}$ with the unit vector along $\vec{a}$.

15.3 PROPERTIES OF SCALAR PRODUCT

In this section, we list some properties (without proof) of scalar product :

(i) *Commutativity* : The scalar product of two vectors is commutative, i.e.

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

(ii) *Associativity*. The scalar prodcut of vectors is distributive over vector addition i.e.

(i) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$ (Left distributivity)

(ii) $(\vec{b}+\vec{c}).\vec{a} = \vec{b}.\vec{a}+\vec{c}.\vec{a}$ (Right distributivity)

(iii) The dot product of two non-zero vectors is zero iff the vectors are perpendicular i.e.

$\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$.

$\therefore \hat{i}.\hat{j} = \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = \hat{k}.\hat{j} = 0$ and $\hat{i}.\hat{k} = \hat{k}.\hat{i} = 0$

(iv) For any vector $\vec{a}$, we have

$\vec{a}.\vec{a} = |\vec{a}|^2$

$\therefore \hat{i}.\hat{i} = |\hat{i}|^2 = 1, \hat{j}.\hat{j} = |\hat{j}|^2$ and $\hat{k}.\hat{k} = |\hat{k}|^2 = 1$

(v) If m is a scalar and $\vec{a}, \vec{b}$ are two vectors, then

$$m\vec{a}.\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.m\vec{b}$$

(vi) If m, n are scalars and $\vec{a}, \vec{b}$ are two vectors, then

$$m\vec{a}.n\vec{b} = mn(\vec{a}.\vec{b}) = (mn\vec{a}).\vec{b} = \vec{a}.(mn\vec{b})$$

(vii) For any two vectors $\vec{a}$ and $\vec{b}$, we have

(a) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}.\vec{b})$

(b) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})$

(c) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(d) $\vec{a}.\vec{b} < 0$ iff $\vec{a}$ and $\vec{b}$ are inclined at an obtuse angle

(e) $\vec{a}.\vec{b} > 0$ iff $\vec{a}$ and $\vec{b}$ are inclined at an acute angle

(f) $\vec{a}.\vec{b} = 0$ iff $\vec{a}$ and $\vec{b}$ are at right angle.

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(ix) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are inclined at an angle θ, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

(x) If $\vec{a}, \vec{b}, \vec{c}$ are threee vectors, then

$$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$$

(x) If $\vec{r}$ is a unit vector making angles α, β, γ with OX, OY and OZ respectively, then

$$\cos\alpha = \vec{r}.\hat{i},\ \cos\beta = \vec{r}.\hat{j},\ \cos\gamma = \vec{r}.\hat{k}$$

$$\vec{r} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$$

(xi) A vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ makes an acute or obtuse angles with OX, OY, OZ respectively according as x, y, z are respectively positive or negative.

(xii) If a vector $\vec{r}$ makes angles α, β, γ with OX, OY and OZ respectively, then

$$\vec{r} = (|\vec{r}|\cos\alpha)\hat{i} + (|\vec{r}|\cos\beta)\hat{j} + (|\vec{r}|\cos\gamma)\hat{k}$$

(xiii) If $\vec{r}$ is a non-zero vector coplanar to two given vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{r}.\vec{a} = \vec{r}.\vec{b} = 0 \Rightarrow \vec{a} \text{ and } \vec{b} \text{ are collinear}$$

(xiv) If $\vec{r}$ is a non-zero vector in space and $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then,

$$\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are coplanar.}$$

(xv) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors in space, any vector $\vec{r}$ in space can be written as

$$\vec{r} = (\vec{r}.\hat{a})\hat{a} + (\vec{r}.\hat{b})\hat{b} + (\vec{r}.\hat{c})\hat{c},$$

where $\hat{a}, \hat{b}, \hat{c}$ are unit vectors along $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

Also,

$$\vec{r} = \left\{\frac{(\vec{r}.\vec{a})}{|\vec{a}|^2}\right\}\vec{a} + \left\{\frac{(\vec{r}.\vec{b})}{|\vec{b}|^2}\right\}\vec{b} + \left\{\frac{(\vec{r}.\vec{c})}{|\vec{c}|^2}\right\}\vec{c}$$

15.4 COMPONENTS OF A VECTOR $\vec{b}$ ALONG AND PERPENDICULAR TO VECTOR $\vec{a}$

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

Draw $BM \perp OA$.

In ΔOBM, we have

$$\vec{OB} = \vec{OM} + \vec{MB} \Rightarrow \vec{b} = \vec{OM} + \vec{MB}$$

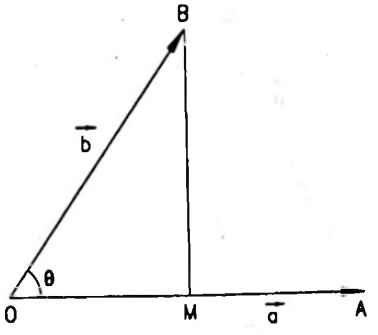


Fig. 15.3

Thus, $\vec{OM}$ and $\vec{MB}$ are components of $\vec{b}$ along $\vec{a}$ and perpendicular to $\vec{a}$ respectively.

Now,

$$\vec{OM} = (OM)\hat{a}$$

$$= (OB\cos\theta)\hat{a}$$

$$= (|\vec{b}|\cos\theta)\hat{a}$$

$$= \left\{|\vec{b}|\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|\,|\vec{b}|}\right\}\hat{a}$$

$$= \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right\}\hat{a} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right\}\frac{\vec{a}}{|\vec{a}|} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$$

$\therefore\ \vec{b} = \vec{OM} + \vec{MB}$

$\Rightarrow \vec{MB} = \vec{b} - \vec{OM} = \vec{b} - \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are

$\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$ and $\vec{b} - \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$ respectively.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 For any vector $\vec{r}$, prove that

$$\vec{r} = (\vec{r}\cdot\hat{i})\hat{i} + (\vec{r}\cdot\hat{j})\hat{j} + (\vec{r}\cdot\hat{k})\hat{k}$$

SOLUTION Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Then,

$$\vec{r}\cdot\hat{i} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{i} = x,$$

$$\vec{r}\cdot\hat{j} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{j} = y,$$

and, $$\vec{r}\cdot\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{k} = z$$

Substituting the values of x, y, z in $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$\vec{r} = (\vec{r}\cdot\hat{i})\hat{i} + (\vec{r}\cdot\hat{j})\hat{j} + (\vec{r}\cdot\hat{k})\hat{k}$$

EXAMPLE 2 *If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$.*

SOLUTION Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0 \qquad ...(i)$$

Now,

$$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \qquad \text{[Using (i)]}$$

$$= 3\lambda^2 \qquad [\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$$

$\therefore\ |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ...(ii)

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}} \qquad \text{[Using (ii)]}$$

$\therefore\ \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly, we have

$\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and, $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

$\therefore\ \theta_1 = \theta_2 = \theta_3$.

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclinded with $\vec{a}, \vec{b}$ and $\vec{c}$.

EXAMPLE 3 *Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.*

SOLUTION The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Now,

θ is obtuse $\Rightarrow \cos\theta < 0$

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} < 0$$

$$\Rightarrow \vec{a}\cdot\vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$$

$$\Rightarrow 14x^2 - 8x + x < 0$$

$$\Rightarrow 7x(2x-1) < 0$$

$$\Rightarrow x(2x-1) < 0 \Rightarrow 0 < x < \frac{1}{2}$$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$.

EXAMPLE 4 *Find the values of c for which the vectors* $\vec{a} = (c \log_2 x)\,\hat{i} - 6\hat{j} + 3\hat{k}$ *and* $\vec{b} = (\log_2 x)\,\hat{i} + 2\hat{j} + (2c \log_2 x)\,\hat{k}$ *make an obtuse angle for any* $x \in (0, \infty)$.

SOLUTION Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Now,

θ is obtuse

$\Rightarrow \cos\theta < 0$ for all $x \in (0, \infty)$

$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} < 0$ for all $x \in (0, \infty)$

$\Rightarrow \vec{a}\cdot\vec{b} < 0$ for all $x \in (0, \infty)$

$\Rightarrow c(\log_2 x)^2 - 12 + 6c(\log_2 x) < 0$ for all $x \in (0, \infty)$

$\Rightarrow cy^2 + 6cy - 12 < 0$ for all $y \in R$, where $y = \log_2 x$

$[\because x > 0 \Rightarrow y = \log_2 x \in R]$

$\Rightarrow c < 0$ and $36c^2 + 48c < 0$ $\left[\because ax^2 + bx + c < 0 \text{ for all } x \Rightarrow a < 0 \text{ and Disc} < 0\right]$

$\Rightarrow c < 0$ and $c(3c + 4) < 0$

$\Rightarrow c < 0$ and $-\dfrac{4}{3} < c < 0$

$\Rightarrow c \in (-4/3, 0)$

EXAMPLE 5 *Find the values of 'a' for which the vector* $\vec{r} = (a^2 - 4)\,\hat{i} + 2\hat{j} - (a^2 - 9)\,\hat{k}$ *makes acute angles with the coordinate axes.*

SOLUTION For vector $\vec{r}$ to be inclined with acute angles with the coordinate axes, we must have

$$\vec{r}\cdot\hat{i} > 0, \vec{r}\cdot\hat{j} > 0 \text{ and } \vec{r}\cdot\hat{k} > 0$$

$\Rightarrow \vec{r}\cdot\hat{i} > 0$ and $\vec{r}\cdot\hat{k} > 0$ $[\because \vec{r}\cdot\hat{j} = 2 > 0]$

$\Rightarrow (a^2 - 4) > 0$ and $-(a^2 - 9) > 0$ $\left[\because \vec{r}\cdot\hat{i} = a^2 - 4 \text{ and } \vec{r}\cdot\hat{k} = -(a^2 - 9)\right]$

$\Rightarrow (a - 2)(a + 2) > 0$ and $(a + 3)(a - 3) < 0$

$\Rightarrow a < -2$ or $a > 2$ and $-3 < a < 3$

$\Rightarrow a \in (-3, -2) \cup (2, 3)$.

EXAMPLE 6 *Let* $\vec{a}, \vec{b}, \vec{c}$ *be three vectors of magnitudes 3, 4 and 5 respectively. If each one is perpendicular to the sum of the other two vectors, prove that* $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$.

SOLUTION We have,

$$|\vec{a}| = 3, |\vec{b}| = 4 \text{ and } |\vec{c}| = 5.$$

It is given that:

$$\vec{a} \perp (\vec{b} + \vec{c}), \vec{b} \perp (\vec{c} + \vec{a}) \text{ and } \vec{c} \perp (\vec{a} + \vec{b})$$

$\Rightarrow \vec{a}\cdot(\vec{b} + \vec{c}) = 0, \vec{b}\cdot(\vec{c} + \vec{a}) = 0$ and $\vec{c}\cdot(\vec{a} + \vec{b}) = 0$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0, \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} = 0$ and $\vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0$

Adding all these, we get

$$2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = 0$ ...(i)

Now,

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$= 3^2 + 4^2 + 5^2 + 0 \quad \text{[Using (i)]}$$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = \sqrt{50} = 5\sqrt{2}$.

EXAMPLE 7 *If with reference to a right handed system of mutually perpendicular unit vectors* $\hat{i}, \hat{j}, \hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$. *Express* $\vec{\beta}$ *in the form* $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, *where* $\vec{\beta}_1$ *is parallel to* $\vec{\alpha}$ *and* $\vec{\beta}_2$ *is perpendicular to* $\vec{\alpha}$.

SOLUTION Since $\vec{\beta}_1$ is parallel to $\vec{\alpha}$. Therefore,

$$\vec{\beta}_1 = \lambda\vec{\alpha} \text{ for some scalar } \lambda \quad \text{...(i)}$$

Let $\vec{\beta}_2 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ...(ii)

Then, $\vec{\beta}_2 \perp \vec{\alpha} \Rightarrow \vec{\beta}_2 \cdot \vec{\alpha} = 0$

$\Rightarrow (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (3\hat{i} - \hat{j}) = 0$

$\Rightarrow 3a_1 - a_2 = 0$ ...(iii)

Now,

$$\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$$

$\Rightarrow \vec{\beta} = \lambda\vec{\alpha} + \vec{\beta}_2$

$\Rightarrow (2\hat{i} + \hat{j} - 3\hat{k}) = \lambda(3\hat{i} - \hat{j}) + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$\Rightarrow 2\hat{i} + \hat{j} - 3\hat{k} = (3\lambda + a_1)\hat{i} + (a_2 - \lambda)\hat{j} + a_3\hat{k}$

$\Rightarrow 2 = 3\lambda + a_1,\ 1 = a_2 - \lambda,\ -3 = a_3$ [On equating coeff. of $\hat{i}, \hat{j}, \hat{k}$]

$\Rightarrow a_1 = 2 - 3\lambda,\ a_2 = 1 + \lambda,\ a_3 = -3$ ...(iv)

Putting the values of a_1 and a_2 in (iii), we get

$$3(2 - 3\lambda) - (1 + \lambda) = 0$$

$\Rightarrow -10\lambda + 5 = 0 \Rightarrow \lambda = \dfrac{1}{2}$

Substituting the value of λ in (iv), we get

$$a_1 = 1/2 \text{ and } a_2 = 3/2.$$

Putting the values of a_1, a_2, and a_3 in (ii), we get

$$\vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$

Putting the value of λ in (i), we get

$$\vec{\beta_1} = \frac{1}{2}(3\hat{i} - \hat{j}).$$

Hence, $\vec{\beta_1} = \frac{1}{2}(3\hat{i} - \hat{j})$ and $\vec{\beta_2} = \frac{1}{2}(\hat{i} + 3\hat{j} - 6\hat{k})$

EXAMPLE 8 *Show that the angle between two diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let a be the length of an edge of the cube and let one corner be the origin as shown in Fig. 15.4. OP, AR, BS and CQ are the diagonals of the cube. Consider the diagonals OP and AR.

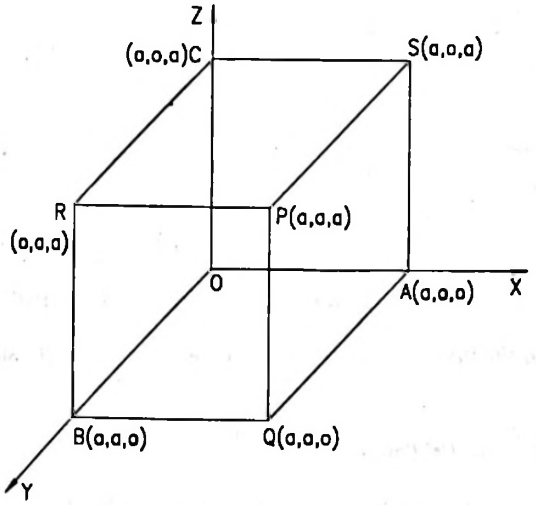


Fig. 15.4

We have,

$$\vec{OP} = a\hat{i} + a\hat{j} + a\hat{k}$$

$\vec{AR}$ = Position vector of R – Position vector of A

$$= (0\hat{i} + a\hat{j} + a\hat{k}) - (a\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= -a\hat{i} + a\hat{j} + a\hat{k}.$$

Let θ be the angle between $\vec{OP}$ and $\vec{AR}$. Then,

$$\cos\theta = \frac{\vec{OP} \cdot \vec{AR}}{|\vec{OP}|\,|\vec{AR}|}$$

$$= \frac{(a\hat{i} + a\hat{j} + a\hat{k}) \cdot (-a\hat{i} + a\hat{j} + a\hat{k})}{\sqrt{a^2 + a^2 + a^2}\ \sqrt{(-a)^2 + a^2 + a^2}}$$

$$= \frac{-a^2 + a^2 + a^2}{\sqrt{3}a\ \sqrt{3}a} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Similarly, the angle θ between the other pairs of diagonals is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 9 *The length of the sides a, b, c of a triangle ABC are related as $a^2 + b^2 = 5c^2$. Prove, using vector methods, that the medians drawn to the sides a and b are perpendicular.*

SOLUTION *Let ABC be a triangle such that $BC = a$, $CA = b$ and $AB = c$.*

Taking A as the orgin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

$$|\vec{b}| = AB = c \text{ and } |\vec{c}| = AC = b$$

Let AD and BE be two medians of ΔABC. Then, D and E are the mid-points of BC and AC respectively. We have to prove that $AD \perp BE$.

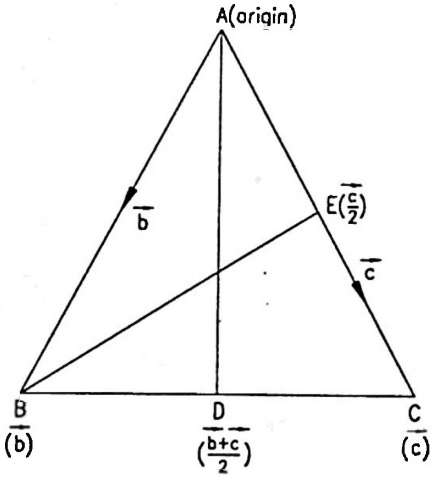


Fig. 15.5

We have,

$$a^2 + b^2 = 5c^2$$

$$\Rightarrow |\vec{BC}|^2 + |\vec{CA}|^2 = 5\,|\vec{AB}|^2$$

$$\Rightarrow |\vec{c} - \vec{b}|^2 + |\vec{c}|^2 = 5\,|\vec{b}|^2$$

$$\Rightarrow |\vec{c}|^2 + |\vec{b}|^2 - 2(\vec{b}.\vec{c}) + |\vec{c}|^2 = 5\,|\vec{b}|^2$$

$$\Rightarrow 2\,|\vec{c}|^2 - 2(\vec{b}.\vec{c}) - 4\,|\vec{b}|^2 = 0$$

$$\Rightarrow |\vec{c}|^2 - (\vec{b}.\vec{c}) - 2\,|\vec{b}|^2 = 0 \qquad \ldots(i)$$

Now,

$$\vec{AD}.\vec{BE} = \left(\frac{\vec{b} + \vec{c}}{2} - \vec{0}\right).\left(\frac{\vec{c}}{2} - \vec{b}\right)$$

$$= \frac{1}{4}(\vec{b} + \vec{c}).(\vec{c} - 2\vec{b})$$

$$= \frac{1}{4}[\vec{b}.\vec{c} - 2\vec{b}.\vec{b} + \vec{c}.\vec{c} - 2\vec{b}.\vec{c}]$$

$$= \frac{1}{4}[|\vec{c}|^2 - (\vec{b}.\vec{c}) - 2\,|\vec{b}|^2] \qquad \text{[Using (i)]}$$

$\therefore \vec{AD} \perp \vec{BE}$

Hence, $AD \perp BE$.

EXAMPLE 10 *$P_1, P_2, P_3, \ldots, P_n$ are n-points on a circle of radius one unit. Prove, using vector methods, that the sum of the squares of their mutual distance is not greater than n^2.*

SOLUTION Let O be the centre of the circle of radius one unit. Taking O as the origin, let the position vectors of $P_1, P_2, P_3, \ldots, P_n$ be $\vec{r_1}, \vec{r_2}, \vec{r_3}, \ldots, \vec{r_n}$ respectively.

Then, $|\vec{OP_1}| = |\vec{OP_2}| = \ldots = |\vec{OP_n}| = 1$

$\Rightarrow \quad |\vec{r_i}| = 1, i = 1, 2, \ldots, n.$

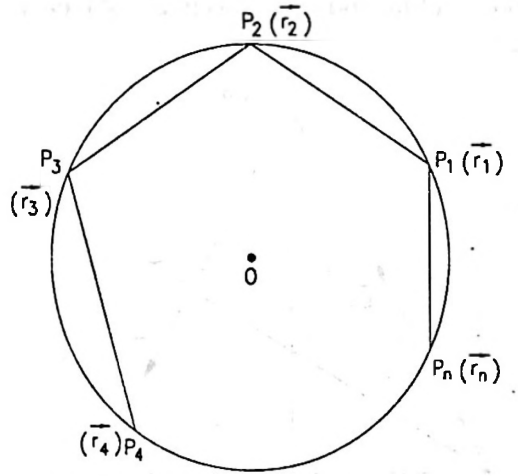


Fig. 15.6

Now,

Sum of the squares of the mutal distances of points $P_1, P_2, \ldots, P_n$

$$= \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} P_i P_j^2$$

$$= \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} |\vec{r_j} - \vec{r_i}|^2$$

$$= \left\{ |\vec{r_1} - \vec{r_2}|^2 + |\vec{r_1} - \vec{r_3}|^2 + \ldots + |\vec{r_1} - \vec{r_n}|^2 \right\}$$

$$+ \left\{ |\vec{r_2} - \vec{r_3}| + \ldots + |\vec{r_2} - \vec{r_n}| \right\}$$

$$+ \left\{ |\vec{r_3} - \vec{r_4}|^2 + |\vec{r_3} - \vec{r_5}|^2 + |\vec{r_3} - \vec{r_6}|^2 + .. + |\vec{r_3} - \vec{r_n}|^2 \right\}$$

$$+ \ldots + |\vec{r_n} - \vec{r_{n-1}}|^2 \}$$

$$= (n-1) \{ |\vec{r_1}|^2 + |\vec{r_2}|^2 + \ldots + |\vec{r_n}|^2 \} - 2 \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} (\vec{r_i} . \vec{r_j})$$

$$= n \{ |\vec{r_1}|^2 + |\vec{r_2}|^2 + \ldots + |\vec{r_n}|^2 \} - |\vec{r_1} + \vec{r_2} + .. + \vec{r_n}|^2$$

$$= n \times n - |\vec{r_1} + \vec{r_2} + \ldots + \vec{r_n}|^2 \leq n^2$$

EXAMPLE 11 *If two vectors $\vec{a}$ and $\vec{b}$ act at such an angle that $|\vec{a} + \vec{b}| = |\vec{a}|$, show that if $\vec{a}$ is doubled then the new vector sum is at right angle to $\vec{b}$.*

SOLUTION We have,

$$|\vec{a} + \vec{b}| = |\vec{a}|$$

$$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{a}|^2$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} . \vec{b}) = |\vec{a}|^2$$

$$\Rightarrow |\vec{b}|^2 + 2(\vec{a} . \vec{b}) = 0$$

$$\Rightarrow \vec{b} . \vec{b} + 2(\vec{a} . \vec{b}) = 0$$

$$\Rightarrow (\vec{b} + 2\vec{a}) . \vec{b} = 0$$

$$\Rightarrow \vec{b} + 2\vec{a} \perp \vec{b}$$

$$\Rightarrow 2\vec{a} + \vec{b} \perp \vec{b}.$$

EXAMPLE 12 *The vector sum of $\vec{a}$ and $\vec{b}$ trisects the angle between them. If $|\vec{a}| = a$, $|\vec{b}| = b$ such that $a > b$. Show that the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $3 \cos^{-1}\left(\frac{a}{2b}\right)$ and the sum vector $\vec{a} + \vec{b}$ has the magnitude $\frac{a^2 - b^2}{b}$.*

SOLUTION Let 3θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} . \vec{b} = ab \cos 3\theta.$$

It is given that the angle between $\vec{a} + \vec{b}$ and $\vec{b}$ is 2θ. This is due to the reason that $\vec{a} + \vec{b}$ is closer to the vector larger in magnitude.

Now,

$$(\vec{a} + \vec{b}) . \vec{a} = |\vec{a} + \vec{b}| \, |\vec{a}| \cos\theta$$

$$\Rightarrow \vec{a} . \vec{a} + \vec{b} . \vec{a} = |\vec{a} + \vec{b}| \, |\vec{a}| \cos\theta$$

$$\Rightarrow |\vec{a}|^2 + |\vec{a}| \, |\vec{b}| \cos 3\theta = |\vec{a} + \vec{b}| \, |\vec{a}| \cos\theta \quad \ldots(i)$$

And,

$$(\vec{a} + \vec{b}) . \vec{b} = (\vec{a} + \vec{b}) \, |\vec{b}| \cos 2\theta$$

$$\Rightarrow \vec{a} . \vec{b} + \vec{b} . \vec{b} = |\vec{a} + \vec{b}| \, |\vec{b}| \cos 2\theta$$

$$\Rightarrow |\vec{a}| \, |\vec{b}| \cos 3\theta + |\vec{b}|^2 = |\vec{a} + \vec{b}| \, |\vec{b}| \cos 2\theta \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{|\vec{a}|^2 + |\vec{a}| \, |\vec{b}| \cos 3\theta}{|\vec{a}| \, |\vec{b}| \cos 3\theta + |\vec{b}|^2} = \frac{|\vec{a}| \cos\theta}{|\vec{b}| \cos 2\theta}$$

$$\Rightarrow \frac{a + b \cos 3\theta}{b + a \cos 3\theta} = \frac{\cos\theta}{\cos 2\theta} \qquad [\because |\vec{a}| = a, |\vec{b}| = b]$$

$$\Rightarrow a(\cos 2\theta - \cos 3\theta \cos\theta) = b(\cos\theta - \cos 3\theta \cos 2\theta)$$

$\Rightarrow a(2\cos 2\theta - \cos 4\theta - \cos 2\theta) = b(2\cos\theta - \cos 5\theta - \cos\theta)$

$\Rightarrow a(\cos 2\theta - \cos 4\theta) = b(\cos\theta - \cos 5\theta)$

$\Rightarrow a(2\sin 3\theta \sin\theta) = b(2\sin 3\theta \sin 2\theta)$

$\Rightarrow a\sin\theta = b\sin 2\theta$

$\Rightarrow a = 2b\cos\theta$

$\Rightarrow \cos\theta = \frac{a}{2b}.$

$\Rightarrow \theta = \cos^{-1}\left(\frac{a}{2b}\right).$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is

$$3\theta = 3\cos^{-1}\left(\frac{a}{2b}\right).$$

Now,

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos 3\theta$$

$$= a^2 + b^2 + 2ab(4\cos^3\theta - 3\cos\theta)$$

$$= a^2 + b^2 + 2ab\left(\frac{4a^2}{8b^3} - \frac{3a}{2b}\right)$$

$$= a^2 + b^2 + \frac{a^4}{b^2} - 3a^2$$

$$= \frac{b^4 + a^4 - 2a^2b^2}{b^2}$$

$$= \left(\frac{a^2 - b^2}{b}\right)^2$$

$$\therefore |\vec{a}+\vec{b}| = \frac{a^2 - b^2}{b}.$$

EXAMPLE 13 *If $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors and $\vec{a}$ is not parallel to $\vec{b}$, prove that*

$$\left(|\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2\right)\vec{c} = \begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}\vec{b}$$

SOLUTION Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Therefore, there exist scalars x, y such that

$$\vec{c} = x\vec{a} + y\vec{b} \quad \ldots(i)$$

Taking dot products on both sides successively by $\vec{a}$ and $\vec{b}$, we get

$$\vec{c}.\vec{a} = x(\vec{a}.\vec{a}) + y(\vec{b}.\vec{a})$$

$$\vec{c}.\vec{b} = x(\vec{a}.\vec{b}) + y(\vec{b}.\vec{b})$$

Solving these two equations by Cramer's rule, we have

$$x = \frac{\begin{vmatrix} \vec{c}.\vec{a} & \vec{b}.\vec{a} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}}{\begin{vmatrix} \vec{a}.\vec{a} & \vec{b}.\vec{a} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}}, y = \frac{\begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}}{\begin{vmatrix} \vec{a}.\vec{a} & \vec{b}.\vec{a} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}}.$$

$$\Rightarrow x = \frac{\begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2}, y = \frac{\begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2}$$

$$\Rightarrow x = \frac{\begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}}{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2}, y = \frac{\begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}}{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2}$$

Substituting the values of x and y in (i), we get

$$\vec{c} = \frac{\begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}\vec{b}}{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2}$$

$$\Rightarrow \left(|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2\right)\vec{c} = \begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}\vec{b}$$

EXAMPLE 14 *Show that the vectors*

$$\vec{\alpha} = (\log a)\hat{i} + (\log b)\hat{j} + (\log c)\hat{k}$$

and, $$\vec{\beta} = (q-r)\hat{i} + (r-p)\hat{j} + (p-q)\hat{k}$$

are perpendicular if a, b, c are the pth, qth and rth terms of a G.P.

SOLUTION Let a, b, c be the pth, qth and rth terms of a G.P. with first term A and common ratio R. Then,

$$a = AR^{p-1}, b = AR^{q-1}, c = AR^{r-1}$$

Now,

$$\vec{\alpha}.\vec{\beta} = (q-r)\log a + (r-p)\log b + (p-q)\log c$$

$$= \log\{a^{q-r}b^{r-p}c^{p-q}\}$$

$$= \log\left[\{AR^{p-1}\}^{q-r}\{AR^{q-1}\}^{r-p}\{AR^{r-1}\}^{p-q}\right]$$

$$= \log\left[A^{q-r+r-p+p-q}R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}\right]$$

$$= \log(A^0R^0) = \log 1 = 0$$

$\Rightarrow \vec{\alpha} \perp \vec{\beta}.$

EXAMPLE 15 *If $\vec{r}.\vec{a} = 0, \vec{r}.\vec{b} = 0, \vec{r}.\vec{c} = 0$ for some non-zero vector $\vec{r}$, then prove that $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.*

[IIT 1983]

SOLUTION We have,

$$\left.\begin{array}{l} \vec{r}.\vec{a} = 0 \Rightarrow \vec{r} \perp \vec{a} \\ \vec{r}.\vec{b} = 0 \Rightarrow \vec{r} \perp \vec{b} \\ \vec{r}.\vec{c} = 0 \Rightarrow \vec{r} \perp \vec{c} \end{array}\right] \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are coplanar vectors}$$

EXAMPLE 16 *If $\vec{a}, \vec{b}, \vec{c}$ are non-zero non-coplanar vectors such that $\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = 0$, then prove that $\vec{r}$ is a null vector.*

SOLUTION If possible, let $\vec{r}$ be a non-null vector. Then,

$$\left.\begin{array}{l}\vec{r}.\vec{a} = 0 \Rightarrow \vec{r} \perp \vec{a} \\ \vec{r}.\vec{b} = 0 \Rightarrow \vec{r} \perp \vec{b} \\ \vec{r}.\vec{c} = 0 \Rightarrow \vec{r} \perp \vec{c}\end{array}\right] \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are coplanar vectors}$$

This is a contradiction to the fact that $\vec{a}, \vec{b}, \vec{c}$ are given to be non-coplanar.

Hence, $\vec{r} = \vec{0}$.

EXAMPLE 17 *A vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ makes equal angles with the vectors $\vec{b} = y\hat{i} - 2z\hat{j} + 3x\hat{k}$ and $\vec{c} = 2z\hat{i} + 3x\hat{j} - y\hat{k}$ and is perpendicular to the vector $\vec{d} = \hat{i} - \hat{j} + 2\hat{k}$. If $|\vec{a}| = 2\sqrt{3}$ and it makes an obtuse angle with the positive direction of y-axis, find $\vec{a}$.*

SOLUTION We have,

$$|\vec{a}| = 2\sqrt{3} \qquad ...(i)$$

$$\Rightarrow x^2 + y^2 + z^2 = 12$$

Also,

$$\vec{a} \perp \vec{d} \Rightarrow \vec{a}.\vec{d} = 0$$

$$\Rightarrow x - y + 2z = 0 \qquad ...(ii)$$

Since $\vec{a}$ makes equal angles with $\vec{b}$ and $\vec{c}$. Therefore,

$$\frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\vec{a}.\vec{c}}{|\vec{a}|\,|\vec{c}|}$$

$$\Rightarrow \quad \vec{a}.\vec{b} = \vec{a}.\vec{c} \quad [\because |\vec{b}| = |\vec{c}| = \sqrt{9x^2 + y^2 + 4z^2}]$$

$$\Rightarrow \quad xy - 2yz + 3xz = 2xz + 3xy - yz$$

$$\Rightarrow \quad 2xy + yz - zx = 0 \qquad ...(iii)$$

Since $\vec{a}$ makes an obtuse angle with y-axis. Therefore,

$$\vec{a}.\hat{j} < 0 \Rightarrow y < 0 \qquad ...(iv)$$

From equations (ii) and (iii), we have

$$2(y - 2z)\,y + yz - z(y - 2z) = 0 \quad [\text{Putting } x = y - 2z \text{ in (iii)}]$$

$$\Rightarrow 2y^2 - 4yz + 2z^2 = 0$$

$$\Rightarrow (y - z)^2 = 0$$

$$\Rightarrow y = z$$

Putting $y = z$ in (ii), we get

$$x + z = 0 \Rightarrow x = -z$$

Putting $x = -z, y = z$ in (i), we get

$$3z^2 = 12 \Rightarrow z = \pm 2$$

Since $y = z$ and $y < 0$. Therefore, $z < 0$.

Hence, $z = -2, y = -2$ and $x = 2$

So, $\vec{a} = 2\hat{i} - 2\hat{j} - 2\hat{k}$

EXAMPLE 18 *AD, BE and CF are three medians of a triangle ABC. Find the value of $\vec{BC}.\vec{AD} + \vec{CA}.\vec{BE} + \vec{AB}.\vec{CF}$.*

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of

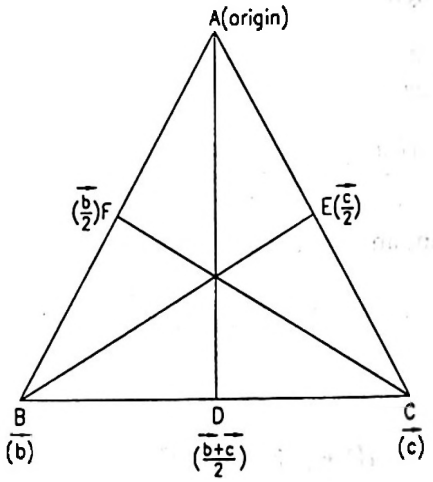


Fig. 15.7

D, E and F are respectively $\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}}{2}$ and $\frac{\vec{b}}{2}$.

$$\therefore \ \vec{BC} = \vec{c} - \vec{b}, \ \vec{AD} = \frac{\vec{b}+\vec{c}}{2}, \vec{CA} = -\vec{c},$$

$$\vec{BE} = \frac{\vec{c}}{2} - \vec{b}, \ \vec{AB} = \vec{b}$$

and, $\vec{CF} = \frac{\vec{b}}{2} - \vec{c}$

Now,

$$\vec{BC}.\vec{AD} + \vec{CA}.\vec{BE} + \vec{AB}.\vec{CF}$$

$$= (\vec{c} - \vec{b}).\left(\frac{\vec{b}+\vec{c}}{2}\right) + (-\vec{c}).\left(\frac{\vec{c}}{2} - \vec{b}\right) + \vec{b}.\left(\frac{\vec{b}}{2} - \vec{c}\right)$$

$$= \frac{1}{2}(\vec{c} - \vec{b}).(\vec{c} + \vec{b}) - \frac{1}{2}\{\vec{c}.(\vec{c} - 2\vec{b})\} + \frac{1}{2}\{\vec{b}.(\vec{b} - 2\vec{c})\}$$

$$= \frac{1}{2}(|\vec{c}|^2 - |\vec{b}|^2) - \frac{1}{2}\{|\vec{c}|^2 - 2\vec{c}.\vec{b}\} + \frac{1}{2}\{|\vec{b}|^2 - 2\vec{b}.\vec{c}\}$$

$$= \frac{1}{2}\left[|\vec{c}|^2 - |\vec{b}|^2 - |\vec{c}|^2 + 2(\vec{c}.\vec{b}) + |\vec{b}|^2 - 2(\vec{b}.\vec{c})\right]$$

$$= 0$$

EXAMPLE 19 *Find a unit vector which makes equal angles with 3 vectors*

$$\vec{a} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}), \ \vec{b} = \frac{1}{5}(-4\hat{i} - 3\hat{k}) \text{ and } \vec{c} = \hat{j}.$$

SOLUTION Let the required unit vector be

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Then,

$$x^2 + y^2 + z^2 = 1 \qquad ...(i)$$

Suppose $\vec{r}$ makes equal angle θ with each of the vectors $\vec{a}, \vec{b}, \vec{c}$. Then,

$\cos\theta = \vec{a}.\vec{r} = \vec{b}.\vec{r} = \vec{c}.\vec{r}$ [$\because \vec{a}, \vec{b}, \vec{c}$ are unit vectors]

$\Rightarrow \vec{a}.\vec{r} = \vec{b}.\vec{r} = \vec{c}.\vec{r}$

Now,

$\vec{a}.\vec{r} = \vec{b}.\vec{r}$

$\Rightarrow \dfrac{x - 2y + 2z}{3} = \dfrac{-4x - 3z}{5}$

$\Rightarrow 17x - 10y + 19z = 0 \qquad ...(ii)$

$\vec{b}.\vec{r} = \vec{c}.\vec{r}$

$\Rightarrow \dfrac{-4x - 3z}{5} = y$

$\Rightarrow 4x + 5y + 3z = 0 \qquad ...(iii)$

and,

$\vec{a}.\vec{r} = \vec{c}.\vec{r}$

$\Rightarrow \dfrac{x - 2y + 2z}{3} = y$

$\Rightarrow x - 5y + 2z = 0 \qquad ...(iv)$

Adding (iii) and (iv), we get

$$5x + 5z = 0 \Rightarrow x = -z$$

Putting $x = -z$ in (ii), we get

$-10y + 2z = 0 \Rightarrow z = 5y \Rightarrow y = z/5$

Putting $x = -z$ and $y = z/5$ in (i), we get

$$z^2 + \frac{z^2}{25} + z^2 = 1$$

$$\Rightarrow z = \pm \frac{5}{\sqrt{51}}$$

$$\therefore x = \mp \frac{5}{\sqrt{51}} \text{ and } y = \pm \frac{1}{\sqrt{51}}$$

Hence, required unit vector is

$$\vec{r} = \pm \frac{1}{\sqrt{51}} (-5x + y + 5z).$$

EXAMPLE 20 *In ΔABC, D is the mid-point of side AB and E is the centroid of triangle CDA. If $\vec{OE} \cdot \vec{CD} = 0$, where O is the circumcentre of ΔABC, using vectors, prove that $AB = AC$.*

SOLUTION Taking the circumcentre O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Since O is the circumcentre of ΔABC, therefore

$$OA = OB = OC$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$$

Also, $\angle AOB = 2C$, $\angle BOC = 2A$ and $\angle COA = 2B$

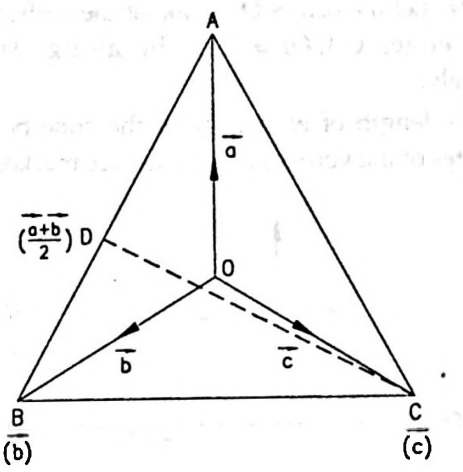


Fig. 15.8

D is the mid-point of AB. Therefore,

Position vector of $D = \dfrac{\vec{a} + \vec{b}}{2}$

$$\therefore \vec{CD} = \frac{\vec{a} + \vec{b}}{2} - \vec{c} = \frac{\vec{a} + \vec{b} - 2\vec{c}}{2}$$

The position vector of the centroid E of ΔCDA is

$$\frac{\vec{a} + \frac{\vec{a} + \vec{b}}{2} + \vec{c}}{3} = \frac{3\vec{a} + \vec{b} + 2\vec{c}}{6}$$

$$\therefore \vec{OE} = \frac{3\vec{a} + \vec{b} + 2\vec{c}}{6}$$

Now,

$\vec{CD} \cdot \vec{OE} = 0$

$$\Rightarrow \left(\frac{3\vec{a} + \vec{b} + 2\vec{c}}{6}\right) \cdot \left(\frac{\vec{a} + \vec{b} - 2\vec{c}}{2}\right) = 0$$

$$\Rightarrow (3\vec{a} + \vec{b} + 2\vec{c}) \cdot (\vec{a} + \vec{b} - 2\vec{c}) = 0$$

$$\Rightarrow 3\vec{a}.\vec{a} + \vec{b}.\vec{b} - 2\vec{c}.\vec{c} + 3\vec{a}.\vec{b} - 6\vec{a}.\vec{c} + \vec{b}.\vec{a} - 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a} + 2\vec{c}.\vec{b} = 0$$

$$\Rightarrow 3|\vec{a}|^2 + |\vec{b}|^2 - 4|\vec{c}|^2 + 4\vec{a}.\vec{b} - 4\vec{a}.\vec{c} = 0$$

$$\Rightarrow 4(\vec{a}.\vec{b} - \vec{a}.\vec{c}) = 0 \qquad [\because |\vec{a}| = |\vec{b}| = |\vec{c}|]$$

$$\Rightarrow \vec{a}.\vec{b} = \vec{a}.\vec{c}$$

$$\Rightarrow |\vec{a}|\,|\vec{b}| \cos 2C = |\vec{a}|\,|\vec{c}| \cos 2B$$

$$\Rightarrow \cos 2C = \cos 2B \qquad [\because |\vec{a}| = |\vec{b}| = |\vec{c}|]$$

$$\Rightarrow B = C$$

$$\Rightarrow AC = AB.$$

EXAMPLE 21 *A line makes angles α, β, γ and δ with the diagonals of a cube. Prove that :*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

SOLUTION Let the origin O be one of the vertices of the cube and the edges OA, OB and OC be along OX, OY and OZ respectively.

Let the length of each edge of the cube be a units. The coordinates of the vertices of the cube are marked in Fig. 15.9.

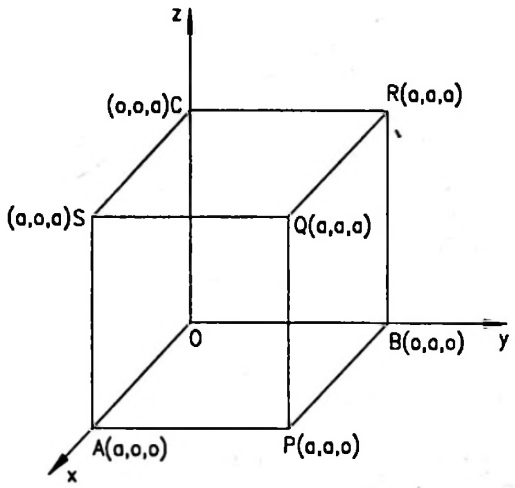


Fig. 15.9

Clearly, OQ, AR, CP and BS are the four diagonals of the cube.

We have,

$$\vec{OQ} = a\hat{i}+a\hat{j}+a\hat{k}, \qquad \vec{AR} = -a\hat{i}+a\hat{j}+a\hat{k},$$

$$\vec{CP} = a\hat{i}+a\hat{j}-a\hat{k} \text{ and } \vec{BS} = a\hat{i}-a\hat{j}+a\hat{k}$$

Let $\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$ be the unit vector along the line which makes angles α, β, γ and δ with OQ, AR, CP and BS respectively. Then,

$$\cos\alpha = \frac{\hat{n}\,.\,\vec{OQ}}{|\,\hat{n}\,|\,|\,\vec{OQ}\,|} = \frac{ax+ay+az}{a\sqrt{3}} = \frac{x+y+z}{\sqrt{3}}$$

$$\cos\beta = \frac{\hat{n}\,.\,\vec{AR}}{|\,\vec{n}\,|\,|\,\vec{AR}\,|} = \frac{-ax+ay+az}{a\sqrt{3}} = \frac{-x+y+z}{\sqrt{3}}$$

$$\cos\gamma = \frac{\hat{n}\,.\,\vec{CP}}{|\,\vec{n}\,|\,|\,\vec{CP}\,|} = \frac{ax+ay-az}{a\sqrt{3}} = \frac{x+y-z}{\sqrt{3}}$$

$$\cos\delta = \frac{\vec{n}\,.\,\vec{BS}}{|\,\vec{n}\,|\,|\,\vec{BS}\,|} = \frac{ax-ay+az}{a\sqrt{3}} = \frac{x-y+z}{\sqrt{3}}$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$

$$= \left(\frac{x+y+z}{\sqrt{3}}\right)^2+\left(\frac{-x+y+z}{\sqrt{3}}\right)^2+\left(\frac{x+y-z}{\sqrt{3}}\right)^2+\left(\frac{x-y+z}{\sqrt{3}}\right)^2$$

$$= \frac{4}{3}(x^2+y^2+z^2) \qquad \left[\because \vec{n} = x\hat{i}+y\hat{j}+z\hat{k} \text{ is a unit vector} \Rightarrow x^2+y^2+z^2 = 1\right]$$

$$= \frac{4}{3}$$

EXAMPLE 22 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to the plane of $\vec{b}$, $\vec{c}$ and the angle between $\vec{b}$, $\vec{c}$ is $\pi/3$, find* $|\,\vec{a}+\vec{b}+\vec{c}\,|$.

SOLUTION We have,

$$|\,\vec{a}\,| = |\,\vec{b}\,| = |\,\vec{c}\,| = 1.$$

Since $\vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Therefore,

$$\vec{a}\perp\vec{b}, \vec{a}\perp\vec{c} \Rightarrow \vec{a}.\vec{b} = 0 = \vec{a}.\vec{c}$$

Also, $\vec{b}.\vec{c} = |\,\vec{b}\,|\,|\,\vec{c}\,|\cos\frac{\pi}{3} = \frac{1}{2}$

$$\therefore |\,\vec{a}+\vec{b}+\vec{c}\,|^2$$

$$= |\,\vec{a}\,|^2+|\,\vec{b}\,|^2+|\,\vec{c}\,|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$= 1+1+1+2\left(0+0+\frac{1}{2}\right) = 4$$

$$\Rightarrow |\,\vec{a}+\vec{b}+\vec{c}\,| = 2.$$

EXAMPLE 23 *If a, b, c are the lengths of the sides opposite respectively to the angles A, B, C of a triangle ABC, using vector methods prove that :*

(i) $a^2 = b^2+c^2-2bc\cos A$

(ii) $a = b\cos C + c\cos B$

(iii) $a\cos B - b\cos A = \dfrac{a^2-b^2}{c}$

(iv) $2(bc\cos A + ca\cos B + ab\cos C) = a^2+b^2+c^2$

(v) $a(1+\cos A)+b(1+\cos B)+c(1+\cos C)$

$$= (a+b+c)(\cos A+\cos B+\cos C)$$

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then,

$$|\,\vec{a}\,| = BC = a, \; |\,\vec{b}\,| = CA = b \text{ and } |\,\vec{c}\,| = AB = c.$$

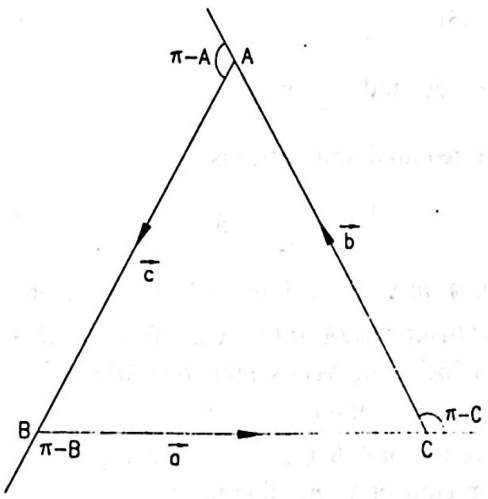


Fig. 15.10

In ΔABC, we have

$$\vec{BC}+\vec{CA} = \vec{BA}$$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

(i) We have,

$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow |\vec{b}+\vec{c}| = |-\vec{a}|$

$\Rightarrow |\vec{b}+\vec{c}|^2 = |-\vec{a}|^2$

$\Rightarrow |\vec{b}|^2+|\vec{c}|^2+2(\vec{b}.\vec{c}) = |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2+|\vec{c}|^2+2|\vec{b}||\vec{c}|\cos(\pi-A) = |\vec{a}|^2$

$\Rightarrow b^2+c^2-2bc\cos A = a^2$

(ii) We have,

$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}=-(\vec{b}+\vec{c})$

$\Rightarrow \vec{a}.\vec{a}=-(\vec{b}+\vec{c}).\vec{a}$

$\Rightarrow \vec{a}.\vec{a}=-(\vec{b}.\vec{a}+\vec{c}.\vec{a})$

$\Rightarrow |\vec{a}|^2 = -\{|\vec{b}||\vec{a}|\cos(\pi-C)+|\vec{c}||\vec{a}|\cos(\pi-B)\}$

$\Rightarrow a^2 = -(-ab\cos C-ca\cos B)$

$\Rightarrow a = b\cos C+c\cos B$

(iii) We have,

$\vec{a}.\vec{c}-\vec{b}.\vec{c}$

$= |\vec{a}||\vec{c}|\cos(\pi-B)-|\vec{b}||\vec{c}|\cos(\pi-A)$

$= -ac\cos B+bc\cos A$

$= -c(a\cos B-b\cos A)$ ...(i)

Also,

$\vec{a}.\vec{c}-\vec{b}.\vec{c} = (\vec{a}-\vec{b}).\vec{c}$

$= -(\vec{a}-\vec{b}).(\vec{a}+\vec{b})$ $\left[\because \vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{c}=-(\vec{a}+\vec{b})\right]$

$= -(|\vec{a}|^2-|\vec{b}|^2)$

$= -(a^2-b^2)$

$\therefore -c(a\cos B-b\cos A) = -(a^2-b^2)$ [Using (i)]

$\Rightarrow a\cos B-b\cos A = \dfrac{a^2-b^2}{c}$

(iv) We have,

$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(|\vec{a}||\vec{b}|\cos(\pi-C)$
$+|\vec{b}||\vec{c}|\cos(\pi-A)+|\vec{c}||\vec{a}|\cos(\pi-B))=0$

$\Rightarrow a^2+b^2+c^2-2(ab\cos C+bc\cos A+ca\cos B) = 0$

$\Rightarrow 2(bc\cos A+ca\cos B+ab\cos C) = a^2+b^2+c^2.$

(v) From (i), we have

$a = b\cos C+c\cos B$

Similarly, we can obtain

$b = c\cos A+a\cos C$ and $c = a\cos B+b\cos A$

$\therefore (a+b+c)(\cos A+\cos B+\cos C)$

$= a\cos A+a(\cos B+\cos C)+b\cos B+b(\cos A+\cos C)$
$+c\cos C+c(\cos A+\cos B)$

$= \{a\cos A+(b\cos C+c\cos B)\}+\{b\cos B+(c\cos A+a\cos C)\}$
$+\{c\cos C+(a\cos B+b\cos A)\}$

$= (a\cos A+a)+(b\cos B+b)+(c\cos C+c)$

$= a(1+\cos A)+b(1+\cos B)+c(1+\cos C).$

EXAMPLE 24 *Using vector methods, prove that in a* ΔABC

$$\cos 2A+\cos 2B+\cos 2C \geq -\frac{3}{2}$$

SOLUTION Let O be the circumcentre of ΔABC. Taking O as the origin, let the position vectors of vertices A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

$\because OA = OB = OC = R$ [Circum-radius]

$\therefore |\vec{a}| = |\vec{b}| = |\vec{c}| = R.$

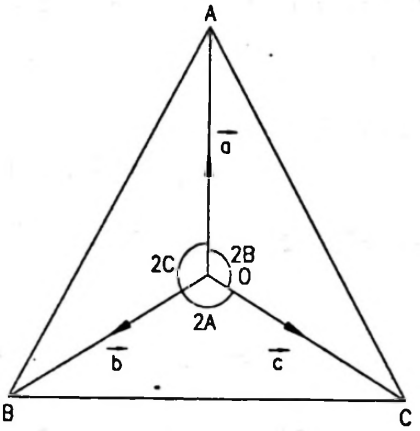


Fig. 15.11

Now,

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$
$+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 3R^2 + 2[|\vec{a}|\,|\vec{b}|\cos 2C + |\vec{b}| + |\vec{c}|\cos 2A + |\vec{c}|\,|\vec{a}|\cos 2B]$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C)$$

But, $|\vec{a}+\vec{b}+\vec{c}|^2 \geq 0$

$\therefore\ 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \geq 0$

$\Rightarrow \cos 2A + \cos 2B + \cos 2C \geq -3/2$

EXAMPLE 25 *Let I be the incentre of ΔABC. Using vectors, prove that for any point P,*

$$a(PA)^2 + b(PB)^2 + c(PC)^2 = a(IA)^2 + b(IB)^2 + c(IC)^2 + (a+b+c)(IP)^2$$

where a, b, c are the lengths of sides opposite to angle A, B, C respectively.

SOLUTION Taking I as the origin, let the position vectors of the vertices A, B, C of ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then, the position vector of I is

$$\frac{a.\vec{a}+b.\vec{b}+c.\vec{c}}{a+b+c}$$

But, I is taken as the origin. Therefore,

$$\frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c} = \vec{0} \Rightarrow a\vec{a}+b\vec{b}+c\vec{c} = \vec{0}$$

Let $\vec{r}$ be the position vector of any point P. Then,

$$a(PA)^2 + b(PB)^2 + c(PC)^2$$

$$= a|\vec{PA}|^2 + b|\vec{PB}|^2 + c|\vec{PC}|^2$$

$$= a|\vec{r}-\vec{a}|^2 + b|\vec{r}-\vec{b}|^2 + c|\vec{r}-\vec{c}|^2$$

$$= [a(|\vec{r}|^2 - 2(\vec{r}.\vec{a}) + |\vec{a}|^2) + b(|\vec{r}|^2 - 2(\vec{r}.\vec{b}) + |\vec{b}|^2) + c(|\vec{r}|^2 - 2(\vec{r}.\vec{c}) + |\vec{c}|^2)]$$

$$= (a+b+c)|\vec{r}|^2 - 2\vec{r}.(a\vec{a}+b\vec{b}+c\vec{c}) + (a|\vec{a}|^2 + b|\vec{b}|^2 + c|\vec{c}|^2)$$

$$= (a+b+c)|\vec{r}|^2 - 2\vec{r}.\vec{0} + (a|\vec{a}|^2 + b|\vec{b}|^2 + c|\vec{c}|^2) \quad \text{[Using (i)]}$$

$$= (a+b+c)|\vec{r}|^2 + a|\vec{a}|^2 + b|\vec{b}|^2 + c|\vec{c}|^2$$

$$= (a+b+c)|\vec{IP}|^2 + a|\vec{IA}|^2 + b|\vec{IB}|^2 + c|IC|^2$$

$$= a(IA)^2 + b(IB)^2 + c(IC)^2 + (a+b+c)(IP)^2$$

EXAMPLE 26 *If P is the mid-point of the side BC of a triangle ABC, prove that*

$$AB^2 + AC^2 = 2(AP^2 + BP^2)$$

SOLUTION With referecne to vertex A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The position vector of P is $\frac{\vec{b}+\vec{c}}{2}$.

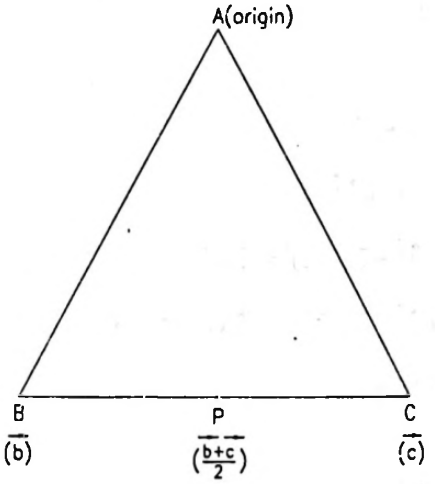


Fig. 15.12

Now,

$$2(AP^2 + BP^2)$$

$$= 2\{|\vec{AP}|^2 + |\vec{BP}|^2\}$$

$$= 2\left[\left|\frac{\vec{b}+\vec{c}}{2}\right|^2 + \left|\frac{\vec{b}+\vec{c}}{2} - \vec{b}\right|^2\right]$$

$$= 2\left[\frac{1}{4}|\vec{b}+\vec{c}|^2 + \frac{1}{4}|\vec{c}-\vec{b}|^2\right]$$

$$= \frac{1}{2}\{|\vec{b}+\vec{c}|^2 + |\vec{c}-\vec{b}|^2\}$$

$$= \frac{1}{2}\{|\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{b}.\vec{c}) + |\vec{c}|^2 + |\vec{b}|^2 - 2(\vec{b}.\vec{c})\}$$

$$= |\vec{b}|^2 + |\vec{c}|^2$$

$$= |\vec{AB}|^2 + |\vec{AC}|^2$$

$$= AB^2 + AC^2.$$

Hence, $AB^2 + AC^2 = 2(AP^2 + BP^2)$.

EXAMPLE 27 *In a quadrilateral $ABCD$, prove that*

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4PQ^2,$$

where P and Q are the mid-points of the diagonals AC and BD respectively.

SOLUTION Taking D as the origin, let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

$\therefore\ \vec{AB} = \vec{b}-\vec{a},\ \vec{BC} = \vec{c}-\vec{b},\ \vec{CD} = -\vec{c},\ \vec{DA} = \vec{a},\ \vec{AC} = \vec{c}-\vec{a}$ and, $\vec{BD} = -\vec{b}$.

Since P and Q are the mid-points of AC and BD respectively. Therefore,

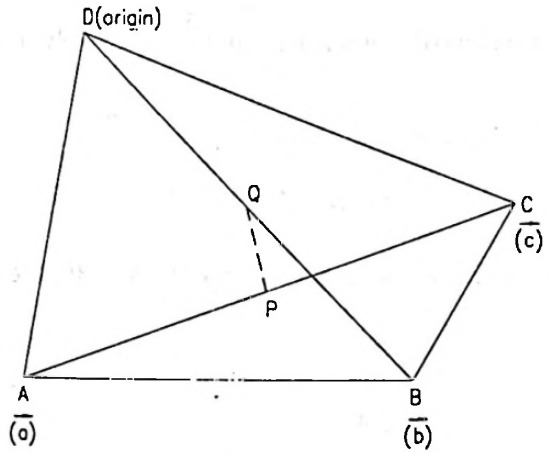


Fig. 15.13

Position vector of $P = \frac{\vec{a}+\vec{c}}{2}$, Position vector of $Q = \frac{\vec{b}}{2}$.

Now,

$$AB^2 + BC^2 + CD^2 + DA^2$$
$$= |\vec{b}-\vec{a}|^2 + |\vec{c}-\vec{b}|^2 + |-\vec{c}|^2 + |\vec{a}|^2$$
$$= |\vec{b}|^2 + |\vec{a}|^2 - 2(\vec{b}.\vec{a}) + |\vec{c}|^2 + |\vec{b}|^2 - 2(\vec{c}.\vec{b}) + |\vec{c}|^2 + |\vec{a}|^2$$
$$= 2\,|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - \vec{a}.\vec{b} - \vec{b}.\vec{c}| \quad ...(i)$$

$$AC^2 + BD^2 + 4PQ^2$$
$$= |\vec{c}-\vec{a}|^2 + |-\vec{b}|^2 + 4\left|\frac{\vec{b}}{2} - \frac{\vec{a}+\vec{c}}{2}\right|^2$$
$$= |\vec{c}-\vec{a}|^2 + |\vec{b}|^2 + |\vec{b}-\vec{a}-\vec{c}|^2$$
$$= |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a} + |\vec{b}|^2 + |\vec{b}|^2 + |\vec{a}|^2 + |\vec{c}|^2 - 2(\vec{b}.\vec{a}) + 2(\vec{a}.\vec{c}) - 2(\vec{b}.\vec{c})$$
$$= 2\,|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - \vec{a}.\vec{b} - \vec{b}.\vec{c}| \quad ...(ii)$$

From (i) and (ii), we get

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

EXAMPLE 28 *Prove that in a rectangular parallelopiped the sum of the squares of the diagonals is equal to four times the sum of the squares of three conterminous edges.*

SOLUTION Let one vertex of the parallelopiped be at the origin O. Taking O as the origin, let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. The position vectors of P, Q, R and S are $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$ and $\vec{a}+\vec{c}$ respectively.

Clearly, OP, AR, CQ and BS are four diagonals of the parallelopiped.

Now,

$$\vec{OP} = \vec{OQ} + \vec{QP} = \vec{a}+\vec{b}+\vec{c}$$

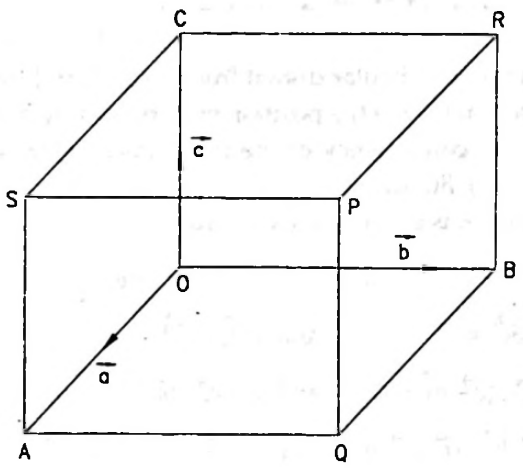


Fig. 15.14

$$\vec{AR} = \vec{OR} - \vec{OA} = \vec{b}+\vec{c}-\vec{a} \quad [\because \vec{OR} = \vec{OB} + \vec{BR}]$$
$$\vec{CQ} = \vec{OQ} - \vec{OC} = \vec{a}+\vec{b}-\vec{c} \quad [\because \vec{OQ} = \vec{OB} + \vec{BQ}]$$
and, $\vec{BS} = \vec{OS} - \vec{OB} = \vec{c}+\vec{a}-\vec{b} \quad [\because \vec{OS} = \vec{OA} + \vec{AS}]$

$$\therefore\ OP^2 + AR^2 + CQ^2 + BS^2$$
$$= |\vec{OP}|^2 + |\vec{AR}|^2 + |\vec{CQ}|^2 + |BS|^2$$
$$= |\vec{a}+\vec{b}+\vec{c}|^2 + |\vec{b}+\vec{c}-\vec{a}|^2 + |\vec{a}+\vec{b}-\vec{c}|^2 + |\vec{c}+\vec{a}-\vec{b}|^2$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}) + 2(\vec{b}.\vec{c}) + (\vec{c}.\vec{a}) + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{a}|^2 + 2(\vec{b}.\vec{c}) - 2(\vec{c}.\vec{a}) - 2(\vec{a}.\vec{b}) + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}) - 2(\vec{b}.\vec{c}) - 2(\vec{c}.\vec{a}) + |\vec{c}|^2 + |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{c}.\vec{a}) - 2(\vec{a}.\vec{b}) - 2(\vec{b}.\vec{c})$$
$$= 4(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)$$
$$= 4(OA^2 + OB^2 + OC^2)$$

EXAMPLE 29 *Let ABC and PQR be two triangles in the same plane. Assume that the perpendicular from the points A, B, C to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB respectively are also concurrent.*

[IIT 2000]

SOLUTION It is given that the perpendiculars drawn from the vertices A, B, C of the ΔABC to the sides QR, RP, PQ respectively are concurrent. Taking this point of concurrency as the origin O, let the position vectors of A, B, C, P, Q, R be $\vec{a}, \vec{b}, \vec{c}, \vec{p}, \vec{q}$ and $\vec{r}$ respectively.

Now,

$\vec{OA} \perp \vec{QR}$, $\vec{OB} \perp \vec{RP}$ and $\vec{OC} \perp \vec{PQ}$

$\Rightarrow \vec{OA}.\vec{QR} = 0,\ \vec{OB}.\vec{RP} = 0, \vec{OC}.\vec{PQ} = 0$

$\Rightarrow \vec{a}.(\vec{r}-\vec{q}) = 0, \vec{b}.(\vec{p}-\vec{r}) = 0, \vec{c}.(\vec{q}-\vec{p}) = 0$

$\Rightarrow \vec{a}.\vec{r} = \vec{a}.\vec{q},\ \vec{b}.\vec{p} = \vec{b}.\vec{r},\ \vec{c}.\vec{q} = \vec{c}.\vec{p}$ ...(i)

Let the perpendicular drawn from P to BC and from Q to CA intersect at O'. Let the position vectors of O' be $\vec{\alpha}$. In order to prove the concurrency of the perpendicular drawn from P, Q and R to BC, CA and AB respectively, it is sufficient to prove that $\vec{O'R}$ is perpendicular to $\vec{AB}$.

Since $O'P \perp \vec{BC}$ and $\vec{O'Q} \perp \vec{CA}$. Therefore,

$\vec{O'P}.\vec{BC} = 0$ and $\vec{O'Q}.\vec{CA} = 0$

$\Rightarrow (\vec{p}-\vec{\alpha}).(\vec{c}-\vec{b}) = 0$ and $(\vec{q}-\vec{\alpha}).(\vec{a}-\vec{c}) = 0$

$\Rightarrow \vec{p}.(\vec{c}-\vec{b}) = \vec{\alpha}.(\vec{c}-\vec{b})$ and $\vec{q}.(\vec{a}-\vec{c}) = \vec{\alpha}.(\vec{a}-\vec{c})$

$\Rightarrow \vec{p}.\vec{c}-\vec{p}.\vec{b} = \vec{\alpha}.(\vec{c}-\vec{b})$ and $\vec{q}.\vec{a}-\vec{q}.\vec{c} = \vec{\alpha}.(\vec{a}-\vec{c})$

$\Rightarrow (\vec{p}.\vec{c}-\vec{p}.\vec{b})+(\vec{q}.\vec{a}-\vec{q}.\vec{c}) = \vec{\alpha}.(\vec{c}-\vec{b})+\vec{\alpha}.(\vec{a}-\vec{c})$

$\Rightarrow \vec{q}.\vec{a}-\vec{p}.\vec{b} = \vec{\alpha}.\vec{a}-\vec{\alpha}.\vec{b}$

$\Rightarrow \vec{a}.\vec{r}-\vec{b}.\vec{r} = \vec{\alpha}.\vec{a}-\vec{\alpha}.\vec{b}$ [Using (i)]

$\Rightarrow (\vec{a}-\vec{b}).\vec{r} = \vec{\alpha}.(\vec{a}-\vec{b})$

$\Rightarrow (\vec{a}-\vec{b}).\vec{r}-(\vec{a}-\vec{b}).\vec{\alpha} = 0$

$\Rightarrow (\vec{a}-\vec{b}).(\vec{r}-\vec{\alpha}) = 0$

$\Rightarrow \vec{BA}.\vec{O'R} = 0$

$\Rightarrow \vec{O'R}.\vec{AB} = 0$

$\Rightarrow \vec{O'R} \perp \vec{AB}$.

Hence, perpendiculars drawn from P, Q, R to BC, CA and AB are also concurrent.

EXAMPLE 30 *Find the obtuse angle between the medians of an isosceles right angled triangle drawn from the vertices of the acute angles.*

SOLUTION Let OAB be an isosceles right triangle, right, angled at O. Taking O as the origin, let the position vectors of the vertices A and B be $\vec{a}$ and $\vec{b}$ respectively. Let AD and BE be two medians drawn from the vertices of the acute angles. The position vectors of D and E are $\frac{\vec{b}}{2}$ and $\frac{\vec{a}}{2}$ respectively. It is given that

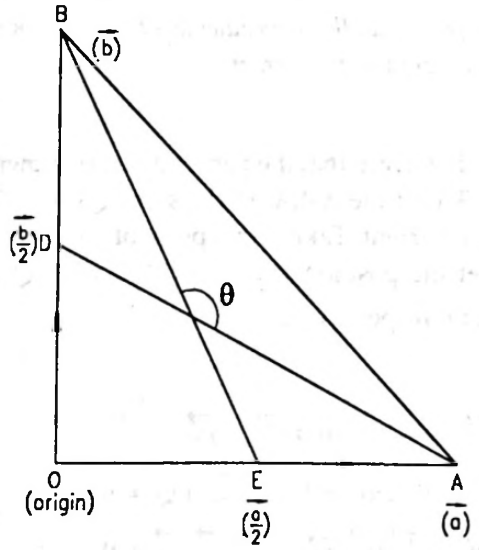


Fig. 15.15

$$OA = OB \Rightarrow |\vec{a}| = |\vec{b}|$$

Now, $\vec{AD} = \frac{\vec{b}}{2}-\vec{a}$ and $\vec{BE} = \frac{\vec{a}}{2}-\vec{b}$.

Let θ be the obtuse angle between AD and BE. Then,

$$\cos\theta = \frac{\vec{AD}.\vec{BE}}{|\vec{AD}|\,|\vec{BE}|} \quad ...(i)$$

We have, $OB \perp OA$

$\Rightarrow \vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$

Also, $OA = OB$

$\Rightarrow |\vec{a}| = |\vec{b}| = \lambda$ (say).

Now,

$$\vec{AD}.\vec{BE} = \left(\frac{\vec{b}}{2}-\vec{a}\right).\left(\frac{\vec{a}}{2}-\vec{b}\right)$$

$$= \frac{1}{4}(\vec{b}-2\vec{a}).(\vec{a}-2\vec{b})$$

$$= \frac{1}{4}\left\{-2|\vec{a}|^2-2|\vec{b}|^2+5(\vec{a}.\vec{b})\right\}$$

$$= \frac{1}{4}(-2\lambda^2-2\lambda^2+5\times 0) = -\lambda^2$$

$$|\vec{AD}|^2 = \left|\frac{\vec{b}}{2}-\vec{a}\right|^2 = \frac{1}{4}|\vec{b}-2\vec{a}|^2$$

$$= \frac{1}{4}\left\{|\vec{b}|^2+4|\vec{a}|^2-4\vec{a}.\vec{b}\right\}$$

$$= \frac{1}{4}(\lambda^2+4\lambda^2-4\times 0) = \frac{5}{4}\lambda^2$$

$$\therefore\ |\vec{AD}| = \sqrt{\frac{5}{4}}\lambda$$

and,

$$|\vec{BE}|^2 = \left|\frac{\vec{a}}{2}-\vec{b}\right|^2 = \frac{1}{4}|\vec{a}-2\vec{b}|^2$$

$$= \frac{1}{4}(|\vec{a}|^2-4\vec{a}.\vec{b}+4|\vec{b}|^2)$$

$$= \frac{1}{4}(\lambda^2-4\times 0+4\lambda^2) = \frac{5\lambda^2}{4}$$

$$\Rightarrow |\vec{BE}| = \sqrt{\frac{5}{4}}\lambda$$

Substituting these values in (i), we get

$$\cos\theta = \frac{\vec{AD}.\vec{BE}}{|\vec{AD}|.|\vec{BE}|} = \frac{-\lambda^2}{\sqrt{\frac{5}{4}}\lambda.\sqrt{\frac{5}{4}}\lambda} = \frac{-4}{5}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{-4}{5}\right)$

EXAMPLE 31 *Given four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. The vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar but not collinear pair by pair and the vector $\vec{d}$ is not coplanar with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. If it is known that the angle between $\vec{a}$ and $\vec{b}$ is equal to that between $\vec{b}$ and $\vec{c}$, each being equal to 60°. The angle between $\vec{d}$ and $\vec{a}$ is α and between $\vec{d}$ and $\vec{b}$ is β. Find the angle between the vectors $\vec{d}$ and $\vec{c}$.*

SOLUTION Let $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ be unit vectors in the directions of vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively. Then,

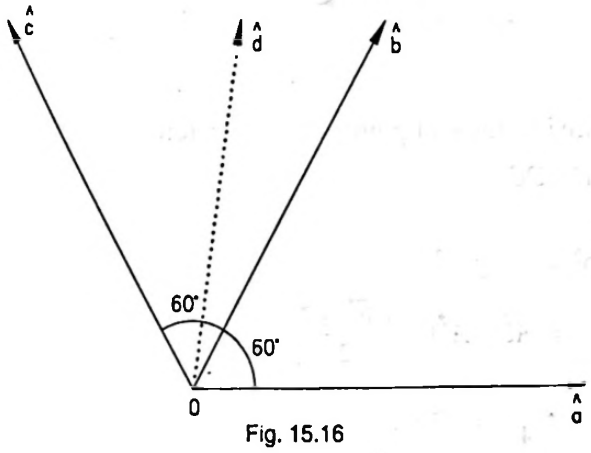


Fig. 15.16

$\hat{a}.\hat{b} = \cos 60° = \frac{1}{2}$ and $\hat{b}.\hat{c} = \cos 60° = \frac{1}{2}$

Clearly, the angle between $\hat{a}$ and $\hat{c}$ is 120°. Therefore,

$\hat{a}.\hat{c} = \cos 120° = -\frac{1}{2}$

It is given that the vector $\vec{d}$ makes angles α and β with vectors $\vec{a}$ and $\vec{b}$ respectively. Therefore,

$\hat{d}.\hat{a} = \cos\alpha$ and $\hat{d}.\hat{b} = \cos\beta$ ...(i)

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Therefore,

$\hat{c} = x\hat{a} + y\hat{b}$ ...(ii)

$\Rightarrow \hat{c}.\hat{a} = (x\hat{a} + y\hat{b}).\hat{a}$ and $\hat{c}.\hat{b} = (x\hat{a} + y\hat{b}).\hat{b}$

[Taking dot product of with $\hat{a}$ and $\hat{b}$]

$\Rightarrow -\frac{1}{2} = x + \frac{y}{2}$ and $\frac{1}{2} = \frac{x}{2} + y$ $\left[\because \hat{a}.\hat{b} = \hat{b}.\hat{c} = \frac{1}{2}\right.$ and $\left.\hat{a}.\hat{c} = -\frac{1}{2}\right]$

$\Rightarrow x = -1, y = 1$

Substituting the values of x and y in (ii), we get

$\hat{c} = \hat{b} - \hat{a}$

$\Rightarrow \hat{d}.\hat{c} = \hat{d}.(\hat{b} - \hat{a})$ [Taking dot product with $\hat{d}$]

$\Rightarrow \cos\theta = \hat{d}.\hat{b} - \hat{d}.\hat{a}$, where θ is the angle between $\vec{c}$ and $\vec{d}$

$\Rightarrow \cos\theta = \cos\beta - \cos\alpha$ [Using (i)]

$\Rightarrow \theta = \cos^{-1}(\cos\beta - \cos\alpha)$

EXAMPLE 32 *Prove using vectors: The median to the base of an isosceles triangle is perpendicular to the base.*

SOLUTION Let ABC be an isosceles triangle in which $AB = AC$. Taking A as the origin, let the position vectors of AB and AC be $\vec{b}$ and $\vec{c}$ respectively. Then the position vector of D is $\frac{\vec{b}+\vec{c}}{2}$ and $\vec{AB} = \vec{b}, \vec{AC} = \vec{c}$.

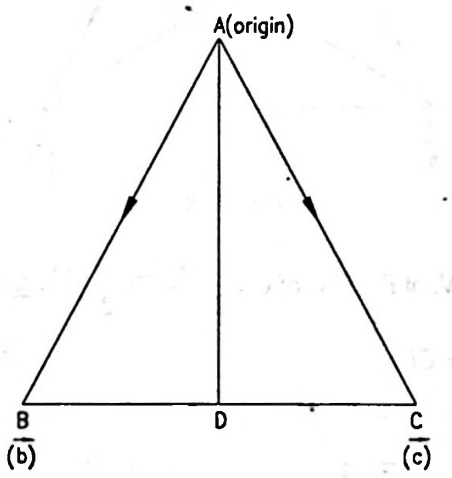


Fig. 15.17

Now, $\vec{AD}$ = Position vector of D – Position vector of A

$= \frac{\vec{b}+\vec{c}}{2} - \vec{0} = \frac{1}{2}(\vec{b}+\vec{c})$

and, $\vec{BC}$ = Position vector of C – Position vector of B

$= \vec{c} - \vec{b}$

$\therefore \vec{AD} \cdot \vec{BC} = \frac{1}{2}(\vec{b}+\vec{c}) \cdot (\vec{c}-\vec{b})$

$= \frac{1}{2}|(\vec{c}+\vec{b}) \cdot (\vec{c}-\vec{b})|$

$= \frac{1}{2}[|\vec{c}|^2 - |\vec{b}|^2]$

$= \frac{1}{2}(AC^2 - AB^2)$

$= \frac{1}{2}(AC^2 - AC^2) = 0$ $[\because AB = AC]$

$\Rightarrow \vec{AD} \perp \vec{BC} \Rightarrow AD \perp BC$.

EXAMPLE 33 *Prove using vectors: If two medians of a triangle are equal, then it is isosceles.*

SOLUTION Let ABC be a triangle and let BE and CF be two equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

P.V of $E = \frac{1}{2}\vec{c}$ and, P.V. of $F = \frac{1}{2}\vec{b}$

$\therefore \vec{BE}$ = P.V. of E – P.V. of $B = \frac{1}{2}\vec{c} - \vec{b} = \frac{1}{2}(\vec{c} - 2\vec{b})$

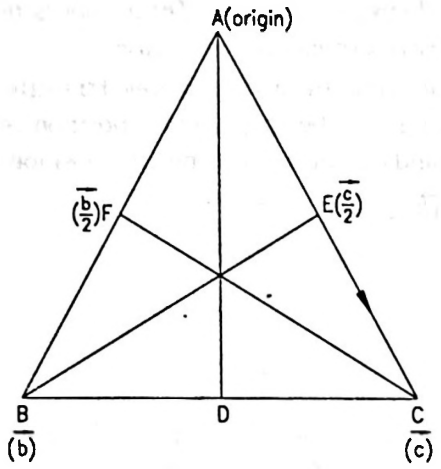


Fig. 15.18

$$\vec{CF} = \text{P.V. of } F - \text{P.V. of } C = \frac{1}{2}\vec{b} - \vec{c} = \frac{1}{2}(\vec{b} - 2\vec{c})$$

Now, $BE = CF$

$$\Rightarrow |\vec{BE}| = |\vec{CF}|$$

$$\Rightarrow |\vec{BE}|^2 = |\vec{CF}|^2$$

$$\Rightarrow \left|\frac{1}{2}(\vec{c} - 2\vec{b})\right|^2 = \left|\frac{1}{2}(\vec{b} - 2\vec{c})\right|^2$$

$$\Rightarrow \frac{1}{4}|\vec{c} - 2\vec{b}|^2 = \frac{1}{4}|\vec{b} - 2\vec{c}|^2$$

$$\Rightarrow |\vec{c} - 2\vec{b}|^2 = |\vec{b} - 2\vec{c}|^2$$

$$\Rightarrow (\vec{c} - 2\vec{b})\cdot(\vec{c} - 2\vec{b}) = (\vec{b} - 2\vec{c})\cdot(\vec{b} - 2\vec{c})$$

$$\Rightarrow \vec{c}\cdot\vec{c} - 4\vec{b}\cdot\vec{c} + 4\vec{b}\cdot\vec{b} = \vec{b}\cdot\vec{b} - 4\vec{b}\cdot\vec{c} + 4\vec{c}\cdot\vec{c}$$

$$\Rightarrow |\vec{c}|^2 - 4\vec{b}\cdot\vec{c} + 4|\vec{b}|^2 = |\vec{b}|^2 - 4\vec{b}\cdot\vec{c} + 4|\vec{c}|^2$$

$$\Rightarrow 3|b|^2 = 3|\vec{c}|^2$$

$$\Rightarrow |\vec{b}|^2 = |\vec{c}|^2$$

$$\Rightarrow |\vec{b}| = |\vec{c}|$$

$$\Rightarrow AB = AC$$

Hence, triangle ABC is an isosceles triangle.

EXAMPLE 34 *Prove that the mid-point of the hypotenuse of a right angled triangle is equidistant from its vertices.*

SOLUTION Let ABC be a right angled triangle, right angled at A and let D be the mid-point of hypotenuse BC. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vector of D is $\frac{\vec{b} + \vec{c}}{2}$.

Since $AB \perp AC$. Therefore,

$$\vec{AB} \perp \vec{AC}$$

$$\Rightarrow \vec{b} \perp \vec{c}$$

$$\Rightarrow \vec{b}\cdot\vec{c} = 0 \qquad \ldots(i)$$

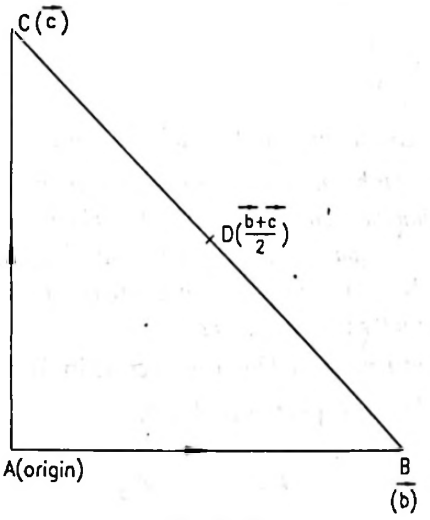


Fig. 15.19

Since D is the mid-point of BC. Therefore,

$$BD = DC.$$

Now,

$$AD^2 = |\vec{AD}|^2$$

$$= \vec{AD}\cdot\vec{AD} = \left(\frac{\vec{b}+\vec{c}}{2}\right)\cdot\left(\frac{\vec{b}+\vec{c}}{2}\right)$$

$$= \frac{1}{4}(|\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c})$$

$$= \frac{1}{4}\{|\vec{b}|^2 + |\vec{c}|^2\} \qquad [\because \vec{b}\cdot\vec{c} = 0 \text{ from (i)}]$$

$$= \frac{1}{4}(AB^2 + AC^2) = \frac{1}{4}BC^2$$

$$\therefore AD = \frac{1}{2}BC.$$

Hence, $AD = BD = DC$.

EXAMPLE 35 *Prove that the altitudes of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let AD, BE be its two altitudes intersecting at O. In order to prove that the altitudes are concurrent it is sufficient to prove that CO produced is perpendicular to AB. Taking O as the origin, let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then

$$\vec{OA} = \vec{a},\ \vec{OB} = \vec{b},\ \vec{OC} = \vec{c}.$$

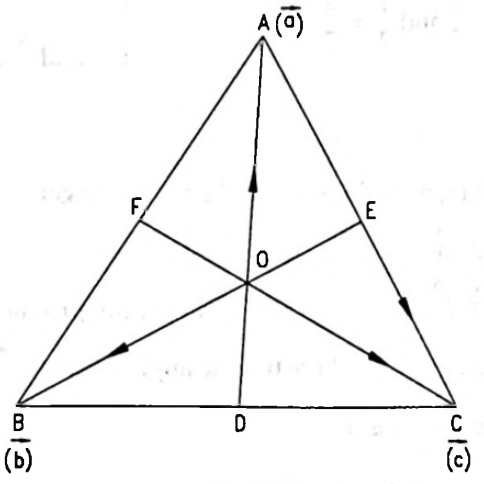


Fig. 15.20

Now, $AD \perp BC$

$\Rightarrow \vec{OA} \perp \vec{BC}$

$\Rightarrow \vec{OA} \cdot \vec{BC} = \vec{0}$

$\Rightarrow \vec{a} \cdot (\vec{c} - \vec{b}) = 0$

$\Rightarrow \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0$ ...(i)

and, $BE \perp CA$

$\Rightarrow \vec{OB} \perp \vec{CA}$

$\Rightarrow \vec{OB} \cdot \vec{CA} = 0$

$\Rightarrow \vec{b} \cdot (\vec{a} - \vec{c}) = 0$

$\Rightarrow \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{c} = 0$ ...(ii)

Adding (i) and (ii), we get

$\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} = 0$

$\Rightarrow (\vec{a} - \vec{b}) \cdot \vec{c} = 0$

$\Rightarrow \vec{BA} \cdot \vec{OC} = 0$

$\Rightarrow \vec{OC} \perp \vec{AB}$

$\Rightarrow CF \perp AB.$

Hence, the three altitudes are concurrent.

EXAMPLE 36 *Prove that the perpendicular bisectors of the sides of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let DO and EO be perpendicular bisectors of BC and CA intersecting at O. Join O to the mid-point F of AB. In order to prove that the perpendicular bisectors are concurrent it is sufficient to prove that OF is perpendicular to AB. Taking O as the origin, let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then the position vectors of D, E and F are $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{a}}{2}$ and $\frac{\vec{a}+\vec{b}}{2}$ respectively.

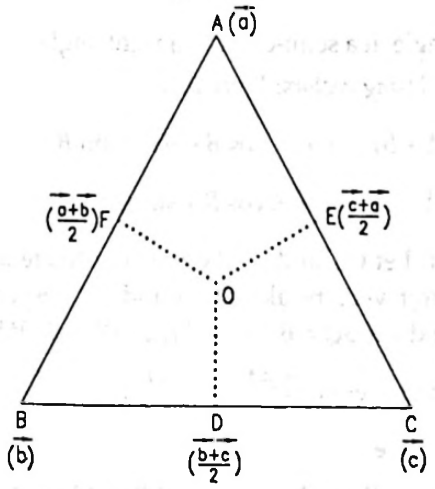


Fig. 15.21

Now,

$\vec{OD} \perp \vec{BC}$

$\Rightarrow \vec{OD} \cdot \vec{BC} = 0$

$\Rightarrow \left(\frac{\vec{b}+\vec{c}}{2}\right) \cdot (\vec{c} - \vec{b}) = 0$

$\Rightarrow (\vec{b} + \vec{c}) \cdot (\vec{c} - \vec{b}) = 0$

$\Rightarrow |\vec{b}|^2 - |\vec{c}|^2 = 0$ ...(i)

and, $\vec{OE} \perp \vec{CA}$

$\Rightarrow \vec{OE} \cdot \vec{CA} = 0$

$\Rightarrow \left(\frac{\vec{c}+\vec{a}}{2}\right) \cdot (\vec{a} - \vec{c}) = 0$

$\Rightarrow (\vec{c} + \vec{a}) \cdot (\vec{a} - \vec{c}) = 0$

$\Rightarrow |\vec{c}|^2 - |\vec{a}|^2 = 0$ ...(ii)

Adding (i) and (ii), we get

$|\vec{a}|^2 - |\vec{b}|^2 = 0$

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow \frac{1}{2}(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow \vec{OF} \cdot \vec{BA} = 0$

$\Rightarrow \vec{OF} \perp \vec{BA} \Rightarrow OF \perp AB.$

Hence, the perpendicular bisectors of the sides of a triangle are concurrent.

EXAMPLE 37 *Prove using vectors : If the diagonals of a prallelogram are equal in length, then it is a rectangle.*

SOLUTION Let $ABCD$ be a parallelogram such that its diagonals AC and BD are equal. Taking A as the origin, let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then, $\vec{AB} = \vec{b}$, $\vec{AD} = \vec{d}$.

By triangle law of addition of vectors, we have

$\vec{AB} + \vec{BC} = \vec{AC}$

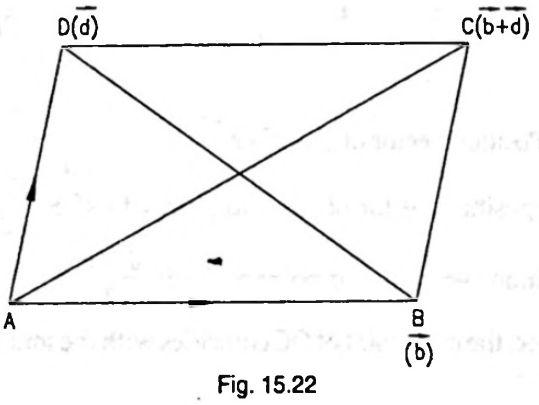


Fig. 15.22

$\Rightarrow \vec{AB} + \vec{AD} = \vec{AC}$ $[\because \vec{BC} = \vec{AD}]$

$\Rightarrow \vec{b} + \vec{d} = \vec{AC}$.

and, $\vec{BD} = \vec{d} - \vec{b}$.

Now, $AC = BD$

$\Rightarrow |\vec{AC}| = |\vec{BD}|$

$\Rightarrow |\vec{AC}|^2 = |\vec{BD}|^2$

$\Rightarrow |\vec{b} + \vec{d}|^2 = |\vec{d} - \vec{b}|^2$

$\Rightarrow |\vec{b}|^2 + |\vec{d}|^2 + 2(\vec{b} \cdot \vec{d}) = |\vec{d}|^2 + |\vec{b}|^2 - 2(\vec{b} \cdot \vec{d})$

$\Rightarrow 4(\vec{b} \cdot \vec{d}) = 0$

$\Rightarrow \vec{b} \cdot \vec{d} = 0$

$\Rightarrow \vec{b} \perp \vec{d}$

$\Rightarrow \vec{AB} \perp \vec{AD}$

Hence, $ABCD$ is a rectangle.

EXAMPLE 38 *Show that the diagonals of a rhombus bisect each other at right angles.*

SOLUTION Let $OACB$ be a rhombus. Taking O as the origin let the position vectors of A and B be $\vec{a}$ and $\vec{b}$ respectively. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

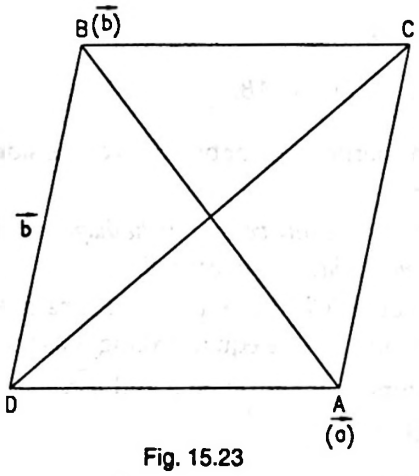


Fig. 15.23

In ΔOAC, we have

$$\vec{OC} = \vec{OA} + \vec{AC}$$
$$= \vec{OA} + \vec{OB} \quad [\because \vec{OB} = \vec{AC}]$$
$$= \vec{a} + \vec{b}$$

$\therefore$ Position vector of C is $\vec{a} + \vec{b}$.

The position vector of the mid-point of OC is $\frac{\vec{a} + \vec{b}}{2}$, and, the position vector of mid-point of AB is $\frac{\vec{a} + \vec{b}}{2}$.

Hence, the mid-point of OC coincides with the mid-point of AB.

Now, $\vec{OC} \cdot \vec{AB} = (\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a})$

$$= |\vec{b}|^2 - |\vec{a}|^2$$
$$= OB^2 - OA^2 = 0 \quad [\because OB = OA]$$

$\therefore \vec{OC} \perp \vec{AB}$.

Hence, the diagonals of a rhombus bisect each other at right angles.

EXAMPLE 39 *Using vector method, prove that the angle in a semi-circle is a right angle.*

SOLUTION Let O be the centre of the semi-circle and AA' be the diameter. Let P be any point on the circumference of the semi-circle.

Taking O as the origin, let the position vectors of A and P be $\vec{a}$ and $\vec{r}$ respectively. Then the position vector of A' is $-\vec{a}$.

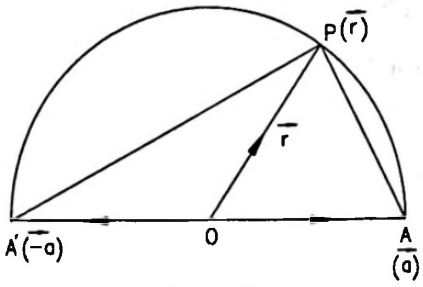


Fig. 15.24

Now,

$\vec{AP}$ = Position vector of P – Position vector of $A = \vec{r} - \vec{a}$

and,

$\vec{A'P}$ = Position vector of P – Position vector of A'
$= \vec{r} + \vec{a}$

$\therefore \vec{AP} \cdot \vec{A'P} = (\vec{r} - \vec{a}) \cdot (\vec{r} + \vec{a})$

$$= |\vec{r}|^2 - |\vec{a}|^2$$
$$= OP^2 - OA^2$$
$$= (\text{radius})^2 - (\text{radius})^2 = 0$$

$\Rightarrow \vec{AP} \perp \vec{A'P} \Rightarrow \angle APA' = \frac{\pi}{2}$.

Hence, angle in a semi-circle is a right angle.

EXAMPLE 40 *Using vectors: Prove that*

(i) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

SOLUTION (i) Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively. Let $\angle XOP = A$ and $\angle XOQ = B$. Draw $PL \perp OX$ and $QM \perp OX$.

Clearly angle between $\vec{OP}$ and $\vec{OQ}$ is $A + B$.

In ΔOLP, we have

$OL = OP \cos A$ and $LP = OP \sin A$.

Therefore $\vec{OL} = (OP \cos A)\hat{i}$ and $\vec{LP} = (OP \sin A)(-\hat{j})$.

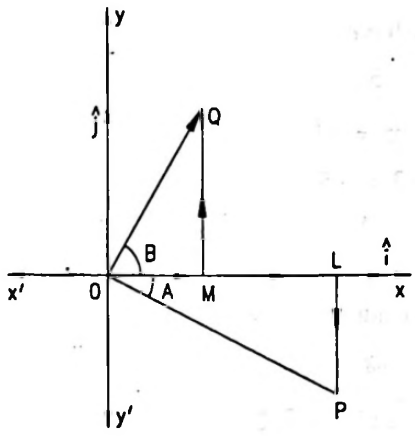


Fig. 15.25

Now,

$$\vec{OP} = \vec{OL} + \vec{LP}$$

$$\Rightarrow \vec{OP} = (OP\cos A)\hat{i} + (OP\sin A)(-\hat{j})$$

$$\Rightarrow \vec{OP} = OP[(\cos A)\hat{i} - (\sin A)\hat{j}] \quad ...(i)$$

In ΔOMQ,

$$OM = OQ\cos B \text{ and } MQ = OQ\sin B.$$

Therefore,

$$\vec{OM} = (OQ\cos B)\hat{i},\ \vec{MQ} = (OQ\sin B)\hat{j}.$$

Now,

$$\vec{OQ} = \vec{OM} + \vec{MQ}$$

$$\Rightarrow \vec{OQ} = (OQ\cos B)\hat{i} + (OQ\sin B)\hat{j}$$

$$\Rightarrow \vec{OQ} = OQ[(\cos B)\hat{i} + (\sin B)\hat{j}] \quad ...(ii)$$

From (i) and (ii), we get

$$\vec{OP}\cdot\vec{OQ} = OP[(\cos A)\hat{i} - (\sin A)\hat{j}]\cdot OQ[(\cos B)\hat{i} + (\sin B)\hat{j}]$$

$$= OP\cdot OQ[\cos A\cos B - \sin A\sin B]$$

But,

$$\vec{OP}\cdot\vec{OQ} = |\vec{OP}|\,|\vec{OQ}|\cos(A+B)$$

$$= OP\cdot OQ\cos(A+B)$$

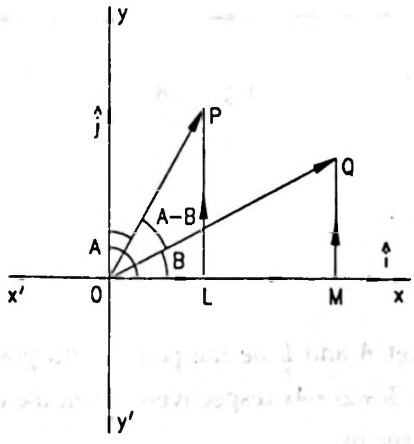


Fig. 15.26

$$\therefore OP\cdot OQ\cos(A+B) = OP\cdot OQ[\cos A\cos B - \sin A\sin B]$$

$$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$$

(ii) Proceeding as in (i), we have

$$\vec{OP} = OP[(\cos A)\hat{i} + (\sin A)\hat{j}]$$

and,

$$\vec{OQ} = OQ[(\cos B)\hat{i} + (\sin B)\hat{j}]$$

$$\therefore \vec{OP}\cdot\vec{OQ} = OP[(\cos A)\hat{i} + (\sin A)\hat{j}]\cdot OQ[(\cos B)\hat{i} + (\sin B)\hat{j}]$$

$$= OP\cdot OQ[\cos A\cos B + \sin A\sin B]$$

But,

$$\vec{OP}\cdot\vec{OQ} = |\vec{OP}|\,|\vec{OQ}|\cos(A-B)$$

$$= OP\cdot OQ\cos(A-B)$$

$$\therefore OP\cdot OQ\cos(A-B) = OP\cdot OQ[\cos A\cos B + \sin A\sin B]$$

$$\Rightarrow \cos(A-B) = \cos A\cos B + \sin A\sin B.$$

EXAMPLE 41 *Prove that the cosine formula for triangles is equivalent to the definition of the scalar product.*

SOLUTION Let OAB be a triangle such that $\angle AOB = \theta$. Then by cosine formula

$$AB^2 = OA^2 + OB^2 - 2\,OA\cdot OB\cos\theta \quad ...(i)$$

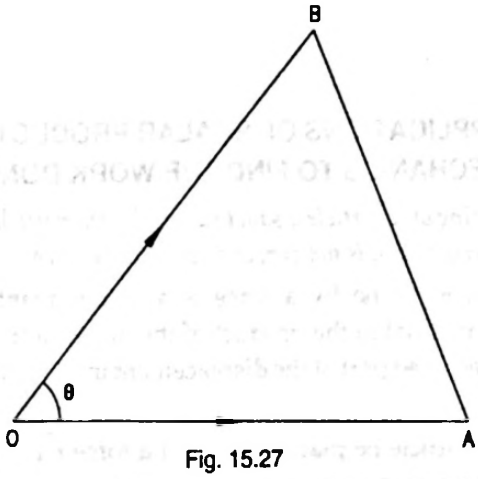


Fig. 15.27

But,

$$AB^2 = |\vec{AB}|^2 = |\vec{OB} - \vec{OA}|^2$$

$$= |\vec{OB}|^2 + |\vec{OA}|^2 - 2\,\vec{OB}\cdot\vec{OA}$$

$$= OB^2 + OA^2 - 2\,\vec{OB}\cdot\vec{OA} \quad ...(ii)$$

From (i) and (ii), we get

$$OA^2 + OB^2 - 2\,OA\cdot OB\cos\theta = OB^2 + OA^2 - 2\,\vec{OB}\cdot\vec{OA}$$

$$\Rightarrow \vec{OB}\cdot\vec{OA} = OB\cdot OA\cos\theta,$$

which is the definition of scalar product.

EXAMPLE 42 *Find three dimensional vectors* $\vec{v_1}, \vec{v_2}, \vec{v_2}$ *satisfying* $\vec{v_1}\cdot\vec{v_1} = 4$, $\vec{v_1}\cdot\vec{v_2} = -2$, $\vec{v_1}\cdot\vec{v_3} = 6$, $\vec{v_2}\cdot\vec{v_2} = 2$, $\vec{v_2}\cdot\vec{v_3} = 5$ *and* $\vec{v_3}\cdot\vec{v_3} = 29$.

[IIT 2001]

SOLUTION We observe that three unknown vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ contain 9 unknown quantities. So, it is not possible to determine $\vec{v_1}$ $\vec{v_2}$ and $\vec{v_3}$ uniquely. There can be infinitely many sets of three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ satisfying the given conditions.

Let us take

$\vec{v_2} = 2\hat{k}$ $[\because \vec{v_1} \cdot \vec{v_1} = 4]$

$\vec{v_2} = a\hat{i} + b\hat{j} + c\hat{k}$

and, $\vec{v_3} = x\hat{i} + y\hat{j} + z\hat{k}$

Then, $\vec{v_1} \cdot \vec{v_2} = -2$ and $\vec{v_1} \cdot \vec{v_3} = 6$

$\Rightarrow 2c = -2$ and $2z = 6$

$\Rightarrow c = -1$ and $z = 3$

Now, $\vec{v_2} \cdot \vec{v_2} = 2$ and $\vec{v_3} \cdot \vec{v_3} = 29$

$\Rightarrow a^2 + b^2 + c^2 = 2$ and $x^2 + y^2 + z^2 = 29$

$\Rightarrow a^2 + b^2 + 1 = 2$ and $x^2 + y^2 + 9 = 29$

$\Rightarrow a^2 + b^2 = 1$ ...(i)

and, $x^2 + y^2 = 20$ ...(ii)

Also, we have,

$\vec{v_2} \cdot \vec{v_3} = -5$

$\Rightarrow ax + by + cz = -5$

$\Rightarrow ax + by - 3 = -5$

$\Rightarrow ax + by = -2$ ...(iii)

Equations (i), (ii) and (iii) give three equations connecting 4 unknowns. Taking $b = 0$, we get

$a = \pm 1, \quad ax = -2.$

$\Rightarrow a = \pm 1$ and $x = \mp 2.$

$\therefore x^2 + y^2 = 20 \Rightarrow y = \pm 4.$

Thus, we have $a = 1, b = 0, c = -1$ and $x = -2, y = 4, z = 3.$

OR

$a = -1, b = 0, c = -1$ and $x = 2, y = -4, z = 3.$

Hence, we have

$\vec{v_1} = 2\hat{k}, \vec{v_2} = \hat{i} - \hat{k}$ and $\vec{v_3} = -2\hat{i} + 4\hat{j} + 3\hat{k}$

OR,

$\vec{v_1} = 2\hat{k}, \vec{v_2} = -\hat{i} - \hat{k}$ and $\vec{v_3} = 2\hat{i} - 4\hat{j} + 3\hat{k}.$

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15.5 APPLICATIONS OF SCALAR PRODUCT IN MECHANICS TO FIND THE WORK DONE

A force acting on a particle is said to do work if the particle is displaced in a direction which is not perpendicular to the force.

The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of the force.

Let a particle be placed at O and a force $\vec{F}$ represented by $\vec{OB}$ be acting on the particle at O. Due to the application of force $\vec{F}$ the particle is displaced in the direction of $\vec{OA}$. Let $\vec{OA}$ be the displacement. Then the component of $\vec{OA}$ in the direction of the force $\vec{F}$ is $|\vec{OA}| \cos\theta$.

$\therefore$ Work done $= |\vec{F}|\,|\vec{OA}| \cos\theta$

$= \vec{F} \cdot \vec{OA} = \vec{F} \cdot \vec{d}$, where $\vec{d} = \vec{OA}$.

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

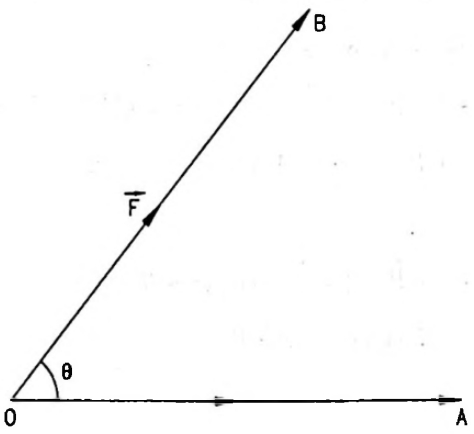


Fig. 15.28

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the work done by the force* $\vec{F} = \hat{i} + 2\hat{j} + \hat{k}$ *acting on a particle, if the particle is displaced from the point with position vector* $2\hat{i} + \hat{j} + \hat{k}$ *to the point with position vector* $3\hat{i} + 2\hat{j} + 4\hat{k}$.

SOLUTION Let A and B be the points with position vectors $2\hat{i} + \hat{j} + \hat{k}$ and $3\hat{i} + 2\hat{j} + 4\hat{k}$ respectively. Then the displacement vector $\vec{d}$ is given by

$\vec{d} = \vec{AB}$ = Position vector of B – Position vector of A

$= (3\hat{i} + 2\hat{j} + 4\hat{k}) - (2\hat{i} + \hat{j} + \hat{k})$

$= \hat{i} + \hat{j} + 3\hat{k}$

$\therefore$ Work done by the force $\vec{F} = \vec{F} \cdot \vec{d}$

$= (\hat{i} + 2\hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} + 3\hat{k})$

$= 1 + 2 + 3 = 6$ units.

EXAMPLE 2 *Constant forces $2\hat{i} - 5\hat{j} + 6\hat{k}$ and $-\hat{i} + 2\hat{j} - \hat{k}$ act on a particle. Determine the work done when the particle is displaced from a point A with position vector $4\hat{i} - 3\hat{j} - 2\hat{k}$ to a point B with position vector is $6\hat{i} + \hat{j} - 3\hat{k}$.*

SOLUTION Let $\vec{F}$ be the resultant force and $\vec{d}$ be the displacement vector. Then,

$$\vec{F} = (2\hat{i} - 5\hat{j} + 6\hat{k}) + (-\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} - 3\hat{j} + 5\hat{k}$$

and,

$\vec{d} = \vec{AB}$ = Position vector of B – Position vector of A

$= (6\hat{i} + \hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} - 2\hat{k})$

$= 2\hat{i} + 4\hat{j} - \hat{k}$

$\therefore$ Work done $= \vec{F} \cdot \vec{d}$

$= (\hat{i} - 3\hat{j} + 5\hat{k}) \cdot (2\hat{i} + 4\hat{j} - \hat{k})$

$= 2 - 12 - 5 = -15$ units.

EXAMPLE 3 *Forces of magnitudes 5 and 3 units acting in the directions $6\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + 6\hat{k}$ respectively act on a particle which is displaced from the point (2, 2, – 1) to (4, 3, 1). Find the work done by the forces.*

SOLUTION Let $\vec{F}$ be the resultant force and $\vec{d}$ be the displacement vector. Then,

$$\vec{F} = 5\frac{(6\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{36 + 4 + 9}} + 3\frac{(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{9 + 4 + 36}}$$

$$= \frac{1}{7}(39\hat{i} + 4\hat{j} + 33\hat{k})$$

and,

$$\vec{d} = (4\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ Total work done

$= \vec{F} \cdot \vec{d}$

$= \frac{1}{7}(39\hat{i} + 4\hat{j} + 33\hat{k}) \cdot (2\hat{i} + \hat{j} + 2\hat{k})$

$= \frac{1}{7}(78 + 4 + 66) = \frac{148}{7}$ units.

EXERCISE 15.1

1. If $\hat{a}$ and $\hat{b}$ be unit vectors and θ be the angle between them, prove that

(i) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$ (ii) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

2. If $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = \sqrt{3}$, then evaluate $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

3. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

4. If $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, prove that the angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$.

5. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, prove that

(i) $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \end{vmatrix} = 0$ (ii) $\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} = 0$

[IIT 1989]

6. Determine the lengths of the diagonals of a parallelogram constructed on the vectors $\vec{a} = 2\hat{\alpha} + \hat{\beta}$ and $\vec{b} = \hat{\alpha} - 2\hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are unit vectors forming an angle of 60°.

7. A point $A(x_1, y_1)$ with abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with ordinate $y_2 = 11$ are given on the part of the curve $y = x^2 - 2x + 3$ in the first quadrant. Find $\vec{OA} \cdot \vec{OB}$.

8. Two points A and B are given on the curve $y = x^2$ such that $\vec{OA} \cdot \hat{i} = 1$ and $\vec{OB} \cdot \hat{i} = -2$. Find $|2\vec{OA} - 3\vec{OB}|$.

9. In a right triangle right angled at A, prove by vector methods, $AB^2 + AC^2 = BC^2$. Also, prove that the mid-point of the hypotenuse is equidistant from the vertices.

10. Find the value of x for which the angle between the vectors $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ is acute and the angle between $\vec{b}$ and the axis of x is obtuse.

11. If a, b, c are the pth, qth, rth terms of a HP then find the angle between the vectors $\vec{u} = (q - r)\hat{i} + (r - p)\hat{j} + (p - q)\hat{k}$ and $\vec{v} = a^{-1}\hat{i} + b^{-1}\hat{j} + c^{-1}\hat{k}$.

12. Find the acute angle between the medians of an isoscles right angled triangle drawn from the vertices of the acute angles.

13. Determine the values of c so that for all real x, the vectors $cx\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other. [IIT 1991]

14. P is any point on the circumcircle of triangle ABC, other than the vertices. H is the ortho-centre of ΔABC, M is the mid-point of PH and D is the mid-point of BC. Prove by vector method that the line DM is perpendicular to $\vec{AP}$.

(Hint: Take circumcentre as the origin)

ANSWERS

2. $\frac{-11}{2}$ 3. $\frac{-3}{2}$ 6. $\sqrt{7}, \sqrt{13}$ 8. $2\sqrt{41}$ 11. $\frac{\pi}{2}$ 12. $\cos^{-1}\left(\frac{4}{5}\right)$

7. 26 13. $c \in (-4/3, 0)$

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CHAPTER

16

VECTOR (CROSS) PRODUCT OF VECTORS

16.1 DEFINITION

VECTOR (CROSS) PRODUCT *Let $\vec{a}, \vec{b}$ be two non-zero non-parallel vectors. Then the vector product $\vec{a} \times \vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}|\,|\vec{b}| \sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}$ and this direction constitute a right handed system.*

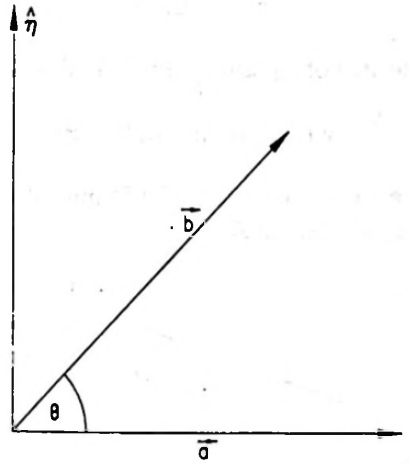


Fig. 16.1

In other words,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

When we say that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system it means that if we rotate vector $\vec{a}$ into the vector $\vec{b}$, then $\hat{\eta}$ will point in the direction perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in which a right handed screw will move if it is turned in the same manner.

NOTE 1 *If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$, then θ is not defined as $\vec{0}$ has no direction and so $\hat{\eta}$ is not defined. In this case we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 2 *If $\vec{a}$ and $\vec{b}$ are collinear i.e. if $\theta = 0$ or π, then the direction of $\hat{\eta}$ is not well defined and so in this case also we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 3 *$\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$. Since we are putting cross between $\vec{a}$ and $\vec{b}$ that is why it is called cross-product. As the resulting quantity is a vector so it is also known as the vector product.*

16.2 GEOMETRICAL INTERPRETATION

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors represented by $\vec{OA}$ and $\vec{OB}$ respectively and let θ be the angle between them. Complete the parallelogram $OACB$. Draw $BL \perp OA$.

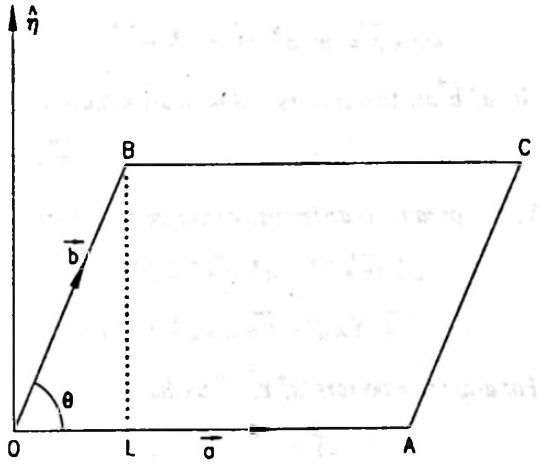


Fig. 16.2

In ΔOBL, we have

$$\sin\theta = \frac{BL}{OB}$$

$$\Rightarrow BL = OB \sin\theta = |\vec{b}| \sin\theta \qquad \ldots(i)$$

Now,

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$= (OA)(BL)\,\hat{\eta} \qquad \text{[From (i)]}$$

$$= (\text{Base}\times\text{height})\,\hat{\eta}$$

$$= (\text{area of parallelogram } OACB)\,\hat{\eta}$$

$$= \text{Vector area of the parallelogram } OACB.$$

Thus, $\vec{a}\times\vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{\eta}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system. Hence, $\vec{a}\times\vec{b}$ represents the vector area of the parallelogram having adjacent sides along $\vec{a}$ and $\vec{b}$.

Thus, area of parallelogram $OACB = |\vec{a}\times\vec{b}|$.

Also, area of $\Delta OAB = \frac{1}{2}$ Area of parallelogram $OACB$

$$= \frac{1}{2}|\vec{a}\times\vec{b}|$$

$$= \frac{1}{2}|\vec{OA}\times\vec{OB}|.$$

NOTE *By the term vector area of a plane figure we mean that a vector of magnitude equal to the area of the plane figure and direction normal to the plane of the figure in the sense of right handed rotation.*

16.3 PROPERTIES OF VECTOR PRODUCT

The vector product has the following properties :

(i) *Vector product is not commutative i.e. if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$, however $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.*

(ii) *If $\vec{a}$ and $\vec{b}$ are two vectors and m is a scalar, then*

$$m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\vec{b}$$

(iii) *If $\vec{a}, \vec{b}$ are two vectors and m, n are scalars, then*

$$m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b}) = m(\vec{a}\times n\vec{b}) = n(m\vec{a}\times\vec{b})$$

(iv) *Vector product is distributive over vector addition i.e.*

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

and, $$(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a} \text{ for all } \vec{a},\vec{b},\vec{c}$$

(v) *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have*

$$\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$$

(vi) *The vector product of two non-zero vectors is the null vector iff they are collinear or parallel i.e.*

$$\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\parallel\vec{b}, \text{ where } \vec{a},\vec{b} \text{ are non-null vectors}$$

(vii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$, $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{i} = -\hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{k}\times\hat{i} = \hat{j}, \hat{i}\times\hat{k} = -\hat{j}$.

(viii) *If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then*

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

16.4 SOME USEFUL RESULTS

RESULT I If $\vec{a}$ and $\vec{b}$ are two non-zero, non-parallel vectors, then unit vectors normal to the plane of $\vec{a}$ and $\vec{b}$ are

$$\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

RESULT II Vectors of magnitude λ normal to the plane of $\vec{a}$ and $\vec{b}$, are

$$\pm\lambda\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$$

RESULT III The area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is

$$|\vec{a}\times\vec{b}|$$

RESULT IV The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is

$$\frac{1}{2}|\vec{a}\times\vec{b}|$$

RESULT V Area of $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$$= \frac{1}{2}|\vec{BC}\times\vec{BA}|$$

$$= \frac{1}{2}|\vec{CA}\times\vec{CB}|$$

RESULT VI The area of a plane quadrilateral $ABCD$ is

$\frac{1}{2}|\vec{AC}\times\vec{BD}|$, where AC and BD are its diagonals.

PROOF Let the diagonals AC and BD intersect at O. Then, vector area of quadrilateral $ABCD$

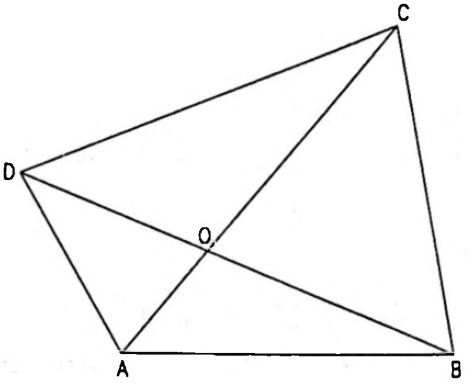


Fig. 16.3

= (Vector area of triangle ABC) + (Vector area of triangle ADC)

$$= \frac{1}{2}(\vec{AB}\times\vec{AC}) + \frac{1}{2}(\vec{AC}\times\vec{AD})$$

$$= -\frac{1}{2}(\vec{AC}\times\vec{AB}) + \frac{1}{2}(\vec{AC}\times\vec{AD})$$

$= \frac{1}{2}[\vec{AC} \times (\vec{AD} - \vec{AB})]$

$= \frac{1}{2}[\vec{AC} \times (\vec{BA} + \vec{AD})]$ $[\because -\vec{AB} = \vec{BA}]$

$= \frac{1}{2}[\vec{AC} \times \vec{BD}]$ $[\because \vec{BA} + \vec{AD} = \vec{BD}]$

So, area of quadrilateral $ABCD = \frac{1}{2} | \vec{AC} \times \vec{BD} |$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}, \vec{b}$ are two vectors, then prove that*

$$| \vec{a} \times \vec{b} |^2 + (\vec{a} . \vec{b})^2 = | \vec{a} |^2 | \vec{b} |^2$$

or

$$| \vec{a} \times \vec{b} |^2 = | \vec{a} |^2 | \vec{b} |^2 - (\vec{a} . \vec{b})^2 = \begin{vmatrix} \vec{a} . \vec{a} & \vec{a} . \vec{b} \\ \vec{b} . \vec{a} & \vec{b} . \vec{b} \end{vmatrix}$$

SOLUTION We have,

$| \vec{a} \times \vec{b} | = | \vec{a} | | \vec{b} | \sin\theta$ and $\vec{a} . \vec{b} = | \vec{a} | | \vec{b} | \cos\theta$,

where θ is the angle between $\vec{a}$ and $\vec{b}$.

$\therefore | \vec{a} \times \vec{b} |^2 + (\vec{a} . \vec{b})^2$

$= | a |^2 | \vec{b} |^2 \sin^2\theta + | \vec{a} |^2 | \vec{b} |^2 \cos^2\theta$

$= | \vec{a} |^2 | \vec{b} |^2 (\sin^2\theta + \cos^2\theta)$

$= | \vec{a} |^2 | \vec{b} |^2$

$\Rightarrow | \vec{a} \times \vec{b} |^2 = | \vec{a} |^2 | \vec{b} |^2 - (\vec{a} . \vec{b})^2$

$= (\vec{a} . \vec{a})(\vec{b} . \vec{b}) - (\vec{a} . \vec{b})(\vec{b} . \vec{a})$

$= \begin{vmatrix} \vec{a} . \vec{a} & \vec{a} . \vec{b} \\ \vec{b} . \vec{a} & \vec{b} . \vec{b} \end{vmatrix}$

EXAMPLE 2 *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, show that the area of triangle ABC is $\frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.*

Deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be collinear.

SOLUTION We have,

Area of $\Delta ABC = \frac{1}{2} | \vec{AB} \times \vec{AC} |$.

Now, $\vec{AB} = \vec{b} - \vec{a}$, and $\vec{AC} = \vec{c} - \vec{a}$

$\therefore \vec{AB} \times \vec{AC}$

$= (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$

$= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$ [By distributivity]

$= \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0}$ $[\because \vec{a} \times \vec{a} = \vec{0}]$

$= \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$

Hence, area of $\Delta ABC = \frac{1}{2} | \vec{AB} \times \vec{AC} |$

$= \frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} |$

If the points, A, B, C are collinear, then

Area of $\Delta ABC = 0$

$\Rightarrow \frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} | = 0$

$\Rightarrow | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} | = 0$

$\Rightarrow \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

Thus, $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$ is the required condition of collinearity of three points $\vec{a}, \vec{b}, \vec{c}$.

EXAMPLE 3 *Show that the perpendicular distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is*

$$\frac{| \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b} |}{| \vec{b} - \vec{a} |}$$

SOLUTION Let ABC be a triangle and let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of its vertices A, B, C respectively. Let CM be the perpendicular from C on AB. Then,

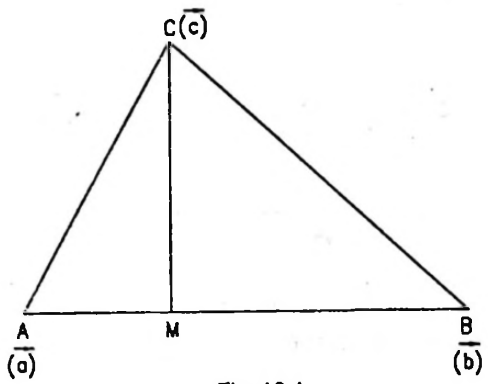


Fig. 16.4

Area of $\Delta ABC = \frac{1}{2} (AB) \cdot CM = \frac{1}{2} | \vec{AB} | (CM)$

Also,

Area of $\Delta ABC = \frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} |$ [See Example 2]

$\therefore \frac{1}{2} | \vec{AB} | (CM) = \frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} |$

$\Rightarrow CM = \frac{| \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} |}{| \vec{AB} |}$

$$\Rightarrow CM = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{a}|}$$

EXAMPLE 4 *If A, B, C, D be any four points in space, prove that*

$$|\vec{AB}\times\vec{CD}+\vec{BC}\times\vec{AD}+\vec{CA}\times\vec{BD}| = 4 \text{ (Area of triangle } ABC).$$

[IIT 1987]

SOLUTION Taking A as the origin, let the position vectors of B, C and D be $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Then,

$$\vec{AB} = \vec{b}, \vec{CD} = \vec{d}-\vec{c}, \quad \vec{BC} = \vec{c}-\vec{b}, \vec{AD} = \vec{d}, \quad \vec{CA} = -\vec{c}$$

and, $\vec{BD} = \vec{d}-\vec{b}$

$$\therefore |\vec{AB}\times\vec{CD}+\vec{BC}\times\vec{AD}+\vec{CA}\times\vec{BD}|$$

$$= |\vec{b}\times(\vec{d}-\vec{c})+(\vec{c}-\vec{b})\times\vec{d}+(-\vec{c})\times(\vec{d}-\vec{b})|$$

$$= |\vec{b}\times\vec{d}-\vec{b}\times\vec{c}+\vec{c}\times\vec{d}-\vec{b}\times\vec{d}-\vec{c}\times\vec{d}+\vec{c}\times\vec{b}|$$

$$= |-\vec{b}\times\vec{c}+\vec{c}\times\vec{b}|$$

$$= |-2(\vec{b}\times\vec{c})|$$

$$= 2|\vec{b}\times\vec{c}|$$

$$= 2|\vec{AB}\times\vec{AC}|$$

$$= 4 \text{ (Area of } \Delta ABC) \quad [\because \text{ Area of } \Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|]$$

EXAMPLE 5 *P and Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD, show that* $\Delta APD = \Delta CQB$.

SOLUTION Taking A as the origin, let the position vectors of B, C, D be $\vec{b}, \vec{c}, \vec{d}$ respectively.

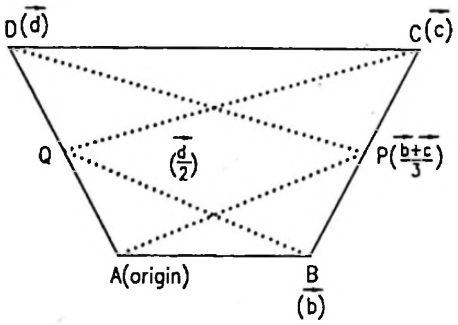


Fig. 16.5

Since $AB \parallel DC$. Therefore,

$$\vec{AB} \parallel \vec{DC}$$

$$\Rightarrow \vec{AB} = \lambda\vec{DC}$$

$$\Rightarrow \vec{b} = \lambda(\vec{c}-\vec{d})$$

Now,

$$\Delta_1 = \text{Area of } \Delta APD$$

$$= \frac{1}{2}|\vec{AP}\times\vec{AD}|$$

$$= \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{2}\times\vec{d}\right|$$

$$= \frac{1}{4}|\vec{b}\times\vec{d}+\vec{c}\times\vec{d}|$$

$$= \frac{1}{4}|\lambda(\vec{c}-\vec{d})\times\vec{d}+\vec{c}\times\vec{d}|$$

$$= \frac{1}{4}|\lambda(\vec{c}\times\vec{d}-\vec{d}\times\vec{d})+\vec{c}\times\vec{d}|$$

$$= \frac{1}{4}|(\lambda+1)(\vec{c}\times\vec{d})|$$

$$= \left(\frac{\lambda+1}{4}\right)|\vec{c}\times\vec{d}| \quad \text{...(i)}$$

and,

$$\Delta_2 = \text{Area of } \Delta CQB$$

$$= \frac{1}{2}|\vec{CQ}\times\vec{CB}|$$

$$= \frac{1}{2}\left|\left(\frac{\vec{d}}{2}-\vec{c}\right)\times(\vec{b}-\vec{c})\right|$$

$$= \frac{1}{4}\left|(\vec{d}-2\vec{c})\times(\vec{b}-\vec{c})\right|$$

$$= \frac{1}{4}\left|\vec{d}\times\vec{b}-\vec{d}\times\vec{c}-2\vec{c}\times\vec{b}+2\vec{c}\times\vec{c}\right|$$

$$= \frac{1}{4}\left|\vec{d}\times\vec{b}-\vec{d}\times\vec{c}-2(\vec{c}\times\vec{b})\right|$$

$$= \frac{1}{4}\left|\vec{d}\times\lambda(\vec{c}-\vec{d})-\vec{d}\times\vec{c}-2[\vec{c}\times\lambda(\vec{c}-\vec{d})]\right|$$

$$= \frac{1}{4}\left|\lambda(\vec{d}\times\vec{c})-\vec{d}\times\vec{c}+2\lambda(\vec{c}\times\vec{d})\right|$$

$$= \frac{1}{4}\left|-\lambda(\vec{c}\times\vec{d})+\vec{c}\times\vec{d}+2\lambda(\vec{c}\times\vec{d})\right|$$

$$= \frac{1}{4}\left|(\lambda+1)(\vec{c}\times\vec{d})\right|$$

$$= \left(\frac{\lambda+1}{4}\right)|\vec{c}\times\vec{d}| \quad \text{...(ii)}$$

From (i) and (ii), we have

$\Delta_1 = \Delta_2$ i.e. Area of ΔAPD = Area of ΔCQB.

EXAMPLE 6 *Let ABC be a triangle. Points M, N and P are taken on the sides AB, BC and CA respectively such that* $\frac{AM}{AB} = \frac{BN}{BC} = \frac{CP}{CA} = \alpha$. *Prove that the vectors* $\vec{AN}$, $\vec{BP}$ and $\vec{CM}$ *form a triangle. Also, find* α *for which the area of the triangle formed by these vectors is the least.*

SOLUTION We have,

$$\frac{AM}{AB} = \alpha$$

$$\Rightarrow AM = AB\,\alpha$$

$$\Rightarrow BM = AB - AM$$

$$\Rightarrow BM = AB - AB\,\alpha$$

$\Rightarrow BM = AB(1-\alpha)$

$\therefore \frac{AM}{BM} = \frac{AB\,\alpha}{AB(1-\alpha)} = \frac{\alpha}{1-\alpha}$

$\Rightarrow AM : BM = \alpha : (1-\alpha)$

Similarly, $BN : NC = \alpha : 1-\alpha$ and $CP : AP = \alpha : 1-\alpha$.

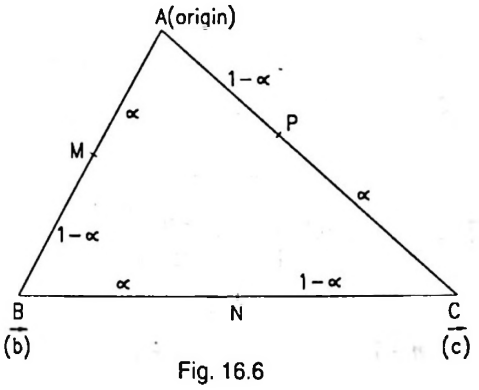


Fig. 16.6

Let A be the origin and let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

Position vector of $N = (1-\alpha)\vec{b} + \alpha\vec{c}$

Position vector of $P = (1-\alpha)\vec{c}$

Position vector of $M = \alpha\vec{b}$

Now,

$\vec{AN} = (1-\alpha)\vec{b} + \alpha\vec{c}$, $\vec{BP} = (1-\alpha)\vec{c} - \vec{b}$, $\vec{CM} = \alpha\vec{b} - \vec{c}$

$\therefore \vec{AN} + \vec{BP} + \vec{CM}$

$= (1-\alpha)\vec{b} + \alpha\vec{c} + (1-\alpha)\vec{c} - \vec{b} + \alpha\vec{b} - \vec{c}$

$= \vec{0}$

$\Rightarrow \vec{AN}, \vec{BP}, \vec{CM}$ form a triangle.

Now,

Δ = Area of the triangle formed by $\vec{AN}$, $\vec{BP}$ and $\vec{CM}$

$= \frac{1}{2}\left|\vec{AN} \times \vec{BP}\right|$

$= \frac{1}{2}\left|\{(1-\alpha)\vec{b} + \alpha\vec{c}\} \times \{(1-\alpha)\vec{c} - \vec{b}\}\right|$

$= \frac{1}{2}\left|(1-\alpha)^2(\vec{b}\times\vec{c}) - \alpha(\vec{c}\times\vec{b})\right|$

$= \frac{1}{2}\left|(1-\alpha+\alpha^2)(\vec{b}\times\vec{c})\right|$

$= \frac{1}{2}(\alpha^2-\alpha+1)\left|\vec{b}\times\vec{c}\right| \quad [\because \alpha^2-\alpha+1 > 0 \text{ for all } \alpha]$

$= \frac{1}{2}\left\{\left(\alpha-\frac{1}{2}\right)^2 + \frac{3}{4}\right\}|\vec{b}\times\vec{c}|$

Thus, Δ will be least if $\alpha = 1/2$.

EXAMPLE 7 *Let A_r $(r = 1, 2, 3, 4)$ be the areas of the faces of a tetrahedron. Let $\vec{n_r}$ be the outward drawn normals to the respective faces with magnitudes equal to corresponding areas. Prove that $\vec{n_1} + \vec{n_2} + \vec{n_3} + \vec{n_4} = 0$.*

SOLUTION Let the one vertex of the tetrahedron be at the origin O and the other vertices be A, B and C. Let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$.

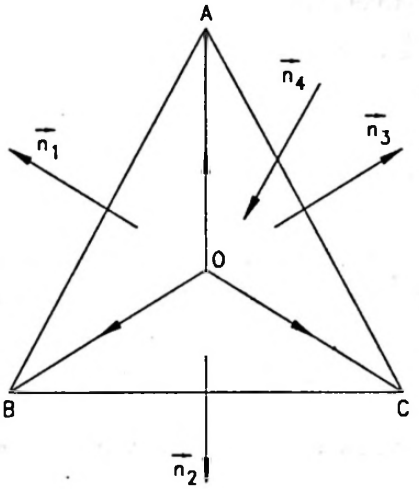


Fig. 16.7

Now,

$\vec{n_1}$ = vector area of $\Delta OAB = \frac{1}{2}(\vec{OA}\times\vec{OB}) = \frac{1}{2}(\vec{a}\times\vec{b})$

$\vec{n_2}$ = vector area of $\Delta OBC = \frac{1}{2}(\vec{OB}\times\vec{OC}) = \frac{1}{2}(\vec{b}\times\vec{c})$

$\vec{n_3}$ = vector area of $\Delta OCA = \frac{1}{2}(\vec{OC}\times\vec{OA}) = \frac{1}{2}(\vec{c}\times\vec{a})$

$\vec{n_4}$ = vector area of ΔABC

$= \frac{1}{2}(\vec{AC}\times\vec{AB})$ [Taken outwards from the tetrahedron]

$= \frac{1}{2}\{(\vec{c}-\vec{a})\times(\vec{b}-\vec{a})\}$

$= \frac{1}{2}(\vec{b}\times\vec{a} + \vec{c}\times\vec{b} + \vec{a}\times\vec{c})$

$\therefore \vec{n_1} + \vec{n_2} + \vec{n_3} + \vec{n_4} = 0.$

EXAMPLE 8 *Let $\vec{OA} = \vec{a}$, $\vec{OB} = 10\vec{a} + 2\vec{b}$, and $\vec{OC} = \vec{b}$ where O is origin. Let p denote the area of the quadrilateral $OABC$ and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that $p = 6q$.* [IIT 1997]

SOLUTION We have,

p = Area of the quadrilateral $OABC$

$= \frac{1}{2}|\vec{OB}\times\vec{AC}|$

$= \frac{1}{2}|\vec{OB}\times(\vec{OC}-\vec{OA})| \quad [\because \vec{AC} = \vec{OC} - \vec{OA}]$

$= \frac{1}{2}|(10\vec{a} + 2\vec{b})\times(\vec{b}-\vec{a})| \quad [\because \vec{OB} = 10\vec{a} + 2\vec{b}]$

$= \frac{1}{2}|10(\vec{a}\times\vec{b}) - 10(\vec{a}\times\vec{a}) + 2(\vec{b}\times\vec{b}) - 2(\vec{b}\times\vec{a})|$

$$= \frac{1}{2} \mid 10(\vec{a} \times \vec{b}) - 0 + 0 + 2(\vec{a} \times \vec{b}) \mid$$

$$= \frac{1}{2} \mid 12(\vec{a} \times \vec{b}) \mid = 6 \mid \vec{a} \times \vec{b} \mid \quad \ldots(i)$$

and,

q = Area of the parallelogram with OA and OC as adjacent sides

$$= \mid \vec{OA} \times \vec{OC} \mid$$

$$= \mid \vec{a} \times \vec{b} \mid \quad \ldots(ii)$$

From (i) and (ii), we get

$$p = 6q$$

EXAMPLE 9 *$ABCD$ is a quadrilateral such that $\vec{AB} = \vec{b}, \vec{AD} = \vec{d}$, $\vec{AC} = m\vec{b} + p\vec{d}$. Show that the area of the quadrilateral $ABCD$ is*

$$\frac{1}{2} \mid m + p \mid \mid \vec{b} \times \vec{d} \mid$$

SOLUTION We have, $\vec{AB} = \vec{b}, \vec{AD} = \vec{d}$ and $\vec{AC} = m\vec{b} + p\vec{d}$.

Now,

$$\vec{AB} + \vec{BD} = \vec{AD}$$

$$\Rightarrow \vec{BD} = \vec{AD} - \vec{AB}$$

$$\Rightarrow \vec{BD} = \vec{d} - \vec{b}.$$

$\therefore$ Δ = Area of quadrilateral $ABCD$

$$= \frac{1}{2} \mid \vec{AC} \times \vec{BD} \mid$$

$$= \frac{1}{2} \mid (m\vec{b} + p\vec{d}) \times (\vec{d} - \vec{b}) \mid$$

$$= \frac{1}{2} \mid m(\vec{b} \times \vec{d}) - p(\vec{d} \times \vec{b}) \mid$$

$$= \frac{1}{2} \mid m(\vec{b} \times \vec{d}) + p(\vec{b} \times \vec{d}) \mid$$

$$= \frac{1}{2} \mid (m + p)(\vec{b} \times \vec{d}) \mid$$

$$= \frac{1}{2} \mid m + p \mid \mid \vec{b} \times \vec{d} \mid.$$

EXAMPLE 10 *In a ΔABC, points D, E, F are taken on the sides BC, CA and AB respectively such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = n$, prove that*

$$\Delta DEF = \frac{n^2 - n + 1}{(n+1)^2} \Delta ABC$$

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The position vectors of D, E and F are given by

$$\frac{\vec{b} + n\vec{c}}{n+1}, \frac{\vec{c}}{n+1} \text{ and } \frac{n\vec{b}}{n+1} \text{ respectively.}$$

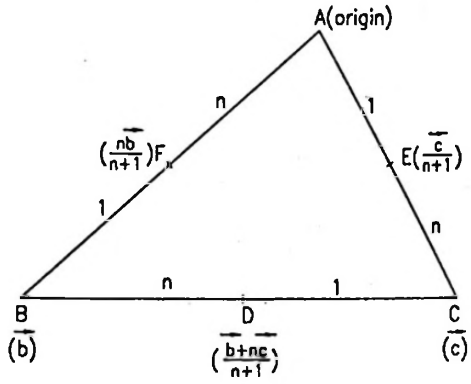


Fig. 16.8

$$\therefore \quad \vec{DE} = \frac{\vec{c}}{n+1} - \frac{\vec{b} + n\vec{c}}{n+1} = \frac{(1-n)\vec{c} - \vec{b}}{n+1}$$

and, $\vec{DF} = \frac{n\vec{b}}{n+1} - \frac{\vec{b} + n\vec{c}}{n+1} = \frac{(n-1)\vec{b} - n\vec{c}}{n+1}$

Now,

Area of ΔDEF

$$= \frac{1}{2} \mid \vec{DE} \times \vec{DF} \mid$$

$$= \frac{1}{2} \left| \frac{(1-n)\vec{c} - \vec{b}}{n+1} \times \frac{(n-1)\vec{b} - n\vec{c}}{n+1} \right|$$

$$= \frac{1}{2(n+1)^2} \mid \{(1-n)\vec{c} - \vec{b}\} \times \{(n-1)\vec{b} - n\vec{c}\} \mid$$

$$= \frac{1}{2(n+1)^2} \mid (1-n)(n-1)(\vec{c} \times \vec{b}) + n(\vec{b} \times \vec{c}) \mid$$

$$= \frac{1}{2(n+1)^2} \mid (n-1)^2(\vec{b} \times \vec{c}) + n(\vec{b} \times \vec{c}) \mid$$

$$= \frac{1}{2(n+1)^2} (n^2 - n + 1) \mid \vec{b} \times \vec{c} \mid$$

$$= \frac{(n^2 - n + 1)}{(n+1)^2} \frac{1}{2} (\vec{b} \times \vec{c})$$

$$= \frac{n^2 - n + 1}{(n+1)^2} \cdot \frac{1}{2} \mid \vec{AB} \times \vec{AC} \mid$$

$$= \frac{n^2 - n + 1}{(n+1)^2} \text{ Area of } \Delta ABC$$

EXAMPLE 11 *The internal bisector of the angles A, B and C of a triangle meet the opposite sides at D, E and F. Using vector methods, show that the area of ΔDEF cannot exceed one-fourth of the area of ΔABC.*

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The position vectors of D, E and F are given by

$$\frac{b\vec{b} + c\vec{c}}{b+c}, \frac{c\vec{c}}{c+a} \text{ and } \frac{b\vec{b}}{a+b} \text{ respectively, where}$$

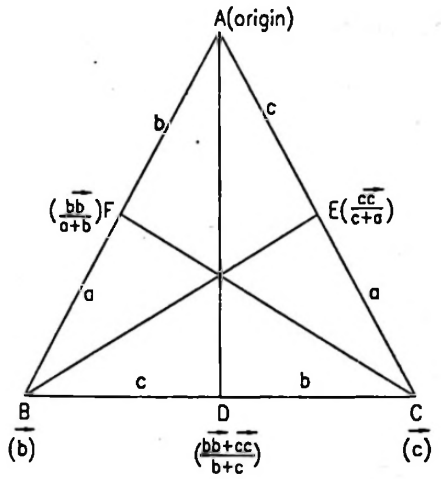


Fig. 16.9

Now,

$BC = a$, $CA = b$ and $AB = c$

Δ = Area of ΔDEF

$$= \frac{1}{2}\left|\frac{\vec{bb}+\vec{cc}}{b+c}\times\frac{\vec{cc}}{c+a}+\frac{\vec{cc}}{c+a}\times\frac{\vec{bb}}{a+b}+\frac{\vec{bb}}{a+b}\times\frac{\vec{bb}+\vec{cc}}{b+c}\right|$$

$$\frac{1}{2}\left|\frac{bc\,(\vec{b}\times\vec{c})}{(b+c)\,(c+a)}+\frac{bc\,(\vec{c}\times\vec{b})}{(c+a)\,(a+b)}+\frac{bc\,(\vec{b}\times\vec{c})}{(a+b)\,(b+c)}\right|$$

$$= \frac{1}{2}bc\,|\,\vec{b}\times\vec{c}\,|\left|\frac{1}{(b+c)\,(c+a)}-\frac{1}{(c+a)\,(a+b)}+\frac{1}{(a+b)\,(b+c)}\right|$$

$$= \frac{1}{2}bc\,|\,\vec{b}\times\vec{c}\,|\left\{\frac{a+b-(b+c)+(c+a)}{(a+b)\,(b+c)\,(c+a)}\right\}$$

$$= \frac{1}{2}\,|\,\vec{b}\times\vec{c}\,|\;bc\left\{\frac{2a}{(a+b)\,(b+c)\,(c+a)}\right\}$$

$$= \frac{2abc}{(a+b)\,(b+c)\,(c+a)}\cdot\frac{1}{2}\,|\,\vec{b}\times\vec{c}\,|$$

$$= \frac{2abc}{(a+b)\,(b+c)\,(c+a)}\ \text{(Area of } \Delta ABC)$$

Now, $A.M. > G.M.$

$$\frac{a+b}{2} > \sqrt{ab},\ \frac{b+c}{2} > \sqrt{bc} \text{ and } \frac{c+a}{2} > \sqrt{ca} \qquad [\because\ AM > GM]$$

$$\Rightarrow (a+b) > 2\sqrt{ab},\ b+c > 2\sqrt{bc} \text{ and } c+a > 2\sqrt{ca}$$

$$\Rightarrow (a+b)\,(b+c)\,(c+a) > 8\sqrt{ab}\,\sqrt{bc}\,\sqrt{ca}$$

$$\Rightarrow (a+b)\,(b+c)\,(c+a) > 8abc$$

$$\Rightarrow \frac{1}{4} > \frac{2abc}{(a+b)\,(b+c)\,(c+a)}$$

$$\Rightarrow \frac{2abc}{(a+b)\,(b+c)\,(c+a)} < \frac{1}{4}$$

$$\therefore\quad \Delta_1 < \frac{1}{4}\ \text{(Area of } \Delta ABC)$$

$$\text{Area of } \Delta DEC < \frac{1}{4}\ \text{(Area of } \Delta ABC)$$

EXAMPLE 12 *For any two vectors $\vec{u}$ and $\vec{v}$, prove that*

$$\left(1+|\,\vec{u}\,|^2\right)\left(1+|\,\vec{v}\,|^2\right) = \left(1-\vec{u}.\vec{v}\right)^2 + \left|\,\vec{u}+\vec{v}+(\vec{u}\times\vec{v})^2\,\right|$$

[IIT 1998]

SOLUTION : Using $|\,\vec{a}+\vec{b}\,|^2 = |\,\vec{a}\,|^2 + |\,\vec{b}\,|^2 + 2\,(\vec{a}.\vec{b})$, we have

$$\text{RHS} = (1-\vec{u}.\vec{v})^2 + |\,(\vec{u}+\vec{v})+(\vec{u}\times\vec{v})\,|^2$$

$$= 1-2\,(\vec{u}.\vec{v})+(\vec{u}.\vec{v})^2 + |\,\vec{u}+\vec{v}\,|^2 + |\,\vec{u}\times\vec{v}\,|^2 + 2\,(\vec{u}+\vec{v})\,.\,(\vec{u}\times\vec{v})$$

$$= 1+(\vec{u}.\vec{v})^2 + |\,\vec{u}\times\vec{v}\,|^2 - 2\,(\vec{u}.\vec{v}) + |\,\vec{u}+\vec{v}\,|^2 + 2\,[\vec{u}.(\vec{u}\times\vec{v}) + \vec{v}.(\vec{u}\times\vec{v})]$$

$$= 1+|\,\vec{u}\,|^2\,|\,\vec{v}\,|^2 - 2\,(\vec{u}.\vec{v}) + |\,\vec{u}\,|^2 + |\,\vec{v}\,|^2 + 2\,(\vec{u}.\vec{v}) + 2\times 0 \qquad [\because\ \vec{u}\times\vec{v}\perp\vec{u}, \vec{u}\times\vec{v}\perp\vec{v}]$$

$$= 1+|\,\vec{u}\,|^2\,|\,\vec{v}\,|^2 + |\,\vec{u}\,|^2 + |\,\vec{v}\,|^2$$

$$= (1+|\,\vec{u}\,|^2)\,(1+|\,\vec{v}\,|^2$$

$$= \text{LHS}$$

Hence,

$$\left(1+|\,\vec{u}\,|^2\right)\left(1+|\,\vec{v}\,|^2\right) = \left(1-\vec{u}.\vec{v}\right)^2 + \left|\,\vec{u}+\vec{v}+(\vec{u}\times\vec{v})^2\,\right|$$

EXAMPLE 13 *For any vector $\vec{a}$, prove that*

$$|\,\vec{a}\times\hat{i}\,|^2 + |\,\vec{a}\times\hat{j}\,|^2 + |\,\vec{a}\times\hat{k}\,|^2 = 2\,|\,\vec{a}\,|^2$$

SOLUTION Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then,

$$\vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i}$$

$$= a_1(\hat{i}\times\hat{i}) + a_2(\hat{j}\times\hat{i}) + a_3(\hat{k}\times\hat{i})$$

$$= -a_2\hat{k}+a_3\hat{i}$$

$$\Rightarrow\quad |\,\vec{a}\times\hat{i}\,|^2 = a_2^2+a_3^2$$

$$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j}$$

$$= a_1(\hat{i}\times\hat{j}) + a_2(\hat{j}\times\hat{j}) + a_3(\hat{k}\times\hat{j})$$

$$= a_1\hat{k}-a_3\hat{i}$$

$$\Rightarrow\quad |\,\vec{a}\times\hat{j}\,|^2 = a_1^2+a_3^2$$

and,

$$\vec{a}\times\hat{k} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{k}$$

$$= a_1(\hat{i}\times\hat{k}) + a_2(\hat{j}\times\hat{k}) + a_3(\hat{k}+\hat{k})$$

$$\Rightarrow\quad |\,\vec{a}\times\hat{k}\,|^2 = a_1^2+a_2^2$$

$$\therefore\quad |\,\vec{a}\times\hat{i}\,|^2 + |\,\vec{a}\times\hat{j}\,|^2 + |\,\vec{a}\times\hat{k}\,|^2$$

$$= a_2^2+a_3^2+a_1^2+a_3^2+a_1^2+a_2^2$$

$$= 2\,(a_1^2+a_2^2+a_3^2) = 2\,|\,\vec{a}\,|^2$$

EXAMPLE 14 *If $A_1, A_2, A_3, ..., A_n$ are the vertices of a regular plane polygon with n sides and O is its centre, show that*

$$\sum_{i=1}^{n-1} (\vec{OA_i} \times \vec{OA_{i+1}}) = (n-1)(\vec{OA_1} \times \vec{OA_2})$$

[IIT 1983]

SOLUTION Let r be the radius of the circumcircle of the polygon $A_1 A_2 .. A_n$. Then,

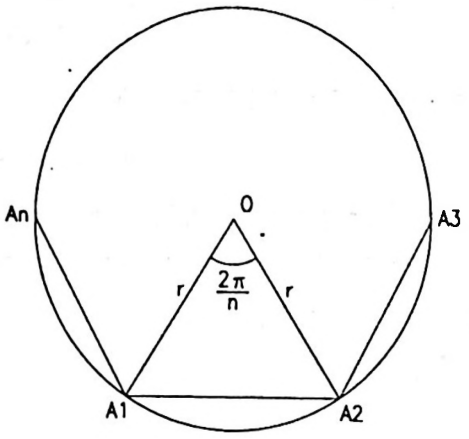


Fig. 16.10

$| \vec{OA_i} | = r, i = 1, 2, ..., n.$

Now,

$$\sum_{i=1}^{n-1} (\vec{OA_i} \times \vec{OA_{i+1}})$$

$$= \sum_{i=1}^{n-1} | \vec{OA_i} | \, | OA_{i+1} | \sin \frac{2\pi}{n} \hat{n}$$, where $\hat{n}$ is a unit vector perpendicular to the plane of the polygon.

$$= \sum_{i=1}^{n-1} \left(r^2 \sin \frac{2\pi}{n} \right) \hat{n}$$

$$= (n-1) r^2 \sin \frac{2\pi}{n} \hat{n}$$

$$= (n-1) \left\{ | \vec{OA_1} | \, | \vec{OA_2} | \sin \frac{2\pi}{n} \hat{n} \right\} \quad [\because OA_1 = OA_2 = r]$$

$$= (n-1)(\vec{OA_1} \times \vec{OA_2})$$

EXAMPLE 15 *Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ be three vectors. Find a vector $\vec{r}$ which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} . \vec{a} = 0$.*

[IIT 1990]

SOLUTION We have,

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

$$\Rightarrow \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$$

$$\Rightarrow (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$$

$$\Rightarrow \vec{r} - \vec{c} \text{ is parallel to } \vec{b}$$

$$\Rightarrow \vec{r} - \vec{c} = \lambda \vec{b} \text{ for some scalar } \lambda.$$

$$\Rightarrow \vec{r} = \vec{c} + \lambda \vec{b}.$$

Now,

$$\vec{r} . \vec{a} = 0$$

$$\Rightarrow (\vec{c} + \lambda \vec{b}) . \vec{a} = 0$$

$$\Rightarrow (\vec{c} . \vec{a}) + \lambda (\vec{b} . \vec{a}) = 0$$

$$\Rightarrow \lambda = -\left(\frac{\vec{c} . \vec{a}}{\vec{b} . \vec{a}} \right)$$

$$\Rightarrow \lambda = -\left(\frac{8+7}{2+0+1} \right) = \frac{-15}{3} = -5$$

$$\therefore \vec{r} = \vec{c} - 5\vec{b} = (4\hat{i} - 3\hat{j} + 7\hat{k}) - 5(\hat{i} + \hat{j} + \hat{k}) = -\hat{i} - 8\hat{j} + 2\hat{k}$$

EXAMPLE 16 *If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the vertices of a cyclic quadrilateral ABCD, then show that*

$$\frac{| \vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a} |}{(\vec{b} - \vec{a}) . (\vec{d} - \vec{a})} + \frac{| \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b} |}{(\vec{b} - \vec{c}) . (\vec{d} - \vec{c})} = 0$$

SOLUTION We have,

$$(\vec{b} - \vec{a}) \times (\vec{d} - \vec{a}) = \vec{b} \times \vec{d} - \vec{b} \times \vec{a} - \vec{a} \times \vec{d} + \vec{a} \times \vec{a}$$

$$= \vec{b} \times \vec{d} + \vec{a} \times \vec{b} + \vec{d} \times \vec{a}$$

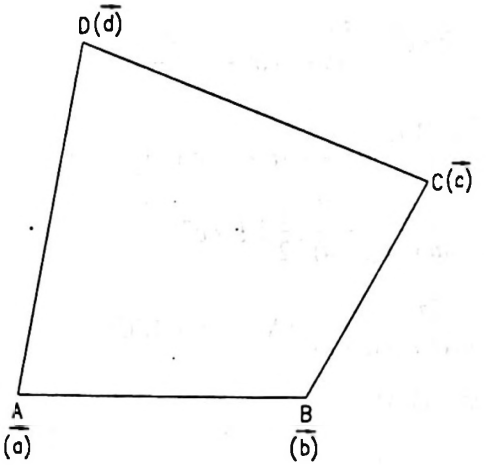


Fig. 16.11

and,

$$(\vec{b} - \vec{c}) \times (\vec{d} - \vec{c}) = \vec{b} \times \vec{d} - \vec{b} \times \vec{c} - \vec{c} \times \vec{d} + \vec{c} \times \vec{c}$$

$$= \vec{b} \times \vec{d} + \vec{c} \times \vec{b} + \vec{d} \times \vec{c}$$

$$\therefore \frac{| \vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a} |}{(\vec{b} - \vec{a}) . (\vec{d} - \vec{a})} + \frac{| \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b} |}{(\vec{b} - \vec{c}) . (\vec{d} - \vec{c})}$$

$$= \frac{| (\vec{b} - \vec{a}) \times (\vec{d} - \vec{a}) |}{(\vec{b} - \vec{a}) . (\vec{d} - \vec{a})} + \frac{| -(\vec{b} - \vec{c}) \times (\vec{d} - \vec{c}) |}{(\vec{b} - \vec{c}) . (\vec{d} - \vec{c})}$$

$$= \frac{| \vec{AB} \times \vec{AD} |}{\vec{AB} . \vec{AD}} + \frac{| \vec{CB} \times \vec{CD} |}{\vec{CB} . \vec{CD}}$$

$$= \tan A + \tan C$$

$= \tan A + \tan(180 - A)$ $\left[\because ABCD \text{ is a cyclic quadrilateral} \Rightarrow A + C = 180° \Rightarrow C = 180° - A\right]$

$= \tan A - \tan A$

$= 0.$

EXAMPLE 17 *Given that the vectors $\vec{a}, \vec{b}, \vec{c}$ form a triangle such that $\vec{b} + \vec{c} = \vec{a}$. Find a, b, c, d such that the area of the triangle is $5\sqrt{6}$, where $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{b} = d\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$.*

SOLUTION We have,

$\vec{b} + \vec{c} = \vec{a}$

$\Rightarrow (d\hat{i} + 3\hat{j} + 4\hat{k}) + (3\hat{i} + \hat{j} - 2\hat{k}) = a\hat{i} + b\hat{j} + c\hat{k}$

$\Rightarrow (d+3)\hat{i} + 4\hat{j} + 2\hat{k} = a\hat{i} + b\hat{j} + c\hat{k}$

$\Rightarrow d + 3 = a, 4 = b, 2 = c$ ...(i)

$\therefore \vec{a} = (d+3)\hat{i} + 4\hat{j} + 2\hat{k}$

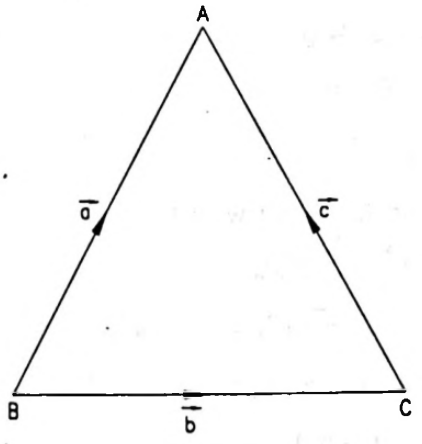


Fig. 16.12

Now,

Area of $\Delta ABC = 5\sqrt{6}$.

$\Rightarrow \frac{1}{2} |\vec{BC} \times \vec{BA}| = 5\sqrt{6}$ ...(ii)

We have,

$$\vec{BC} \times \vec{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d & 3 & 4 \\ d+3 & 4 & 2 \end{vmatrix} = -10\hat{i} + (2d+12)\hat{j} + (d-9)\hat{k}$$

$\therefore |\vec{BC} \times \vec{BA}| = \sqrt{100 + (2d+12)^2 + (d-9)^2}$

$= \sqrt{5d^2 + 30d + 325}$

Substituting this in (ii), we get

$\frac{1}{2}\sqrt{5d^2 + 30d + 325} = 5\sqrt{6}$

$\Rightarrow 5d^2 + 30d + 325 = 600$

$\Rightarrow d^2 + 6d - 55 = 0$

$\Rightarrow (d+11)(d-5) = 0$

$\Rightarrow d = -11, 5$

$\Rightarrow a = -8, 8$ $[\because a = d + 3]$

Hence, $a = -8, b = 4, c = 2, d = -11$ or, $a = 8, b = 4, c = 2, d = 5$

EXAMPLE 18 *If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$.*

SOLUTION Let $\vec{b} = x\hat{i} + y\hat{j} + 3\hat{k}$. Then,

$$\vec{a} \times \vec{b} = \vec{c} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = (\hat{j} - \hat{k})$$

$\Rightarrow (z-y)\hat{i} - (z-x)\hat{j} + (y-x)\hat{k} = \hat{j} - \hat{k}$

$\Rightarrow z - y = 0, -(z-x) = 1, y - x = -1$

$\Rightarrow y = z, x - z = 1, x - y = 1$

$\Rightarrow x - z = 1, x - y = 1$ $\left[\because \text{These two equations are equivalent to } y = z\right]$

$\Rightarrow z = x - 1$ and $y = x - 1$.

Now,

$\vec{a} \cdot \vec{b} = 3$

$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\Rightarrow x + y + z = 3$

$\Rightarrow x + x - 1 + x - 1 = 3$

$\Rightarrow 3x = 5$

$\Rightarrow x = \frac{5}{3}$

$\therefore y = x - 1 \Rightarrow y = \frac{5}{3} - 1 = \frac{2}{3}$.

Also, $y = z \Rightarrow z = \frac{2}{3}$.

Hence, $\vec{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$.

EXAMPLE 19 *Find a vector $\vec{r}$ satisfying the following conditions :*

(i) $\vec{r}$ is perpendicular to $\vec{a}$ and $\vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = 18\hat{i} - 22\hat{j} - 5\hat{k}$

(ii) $\vec{r}$ makes an acute angle with $\hat{i} + \hat{j} + \hat{k}$

(iii) $|\vec{r}| = 14$.

SOLUTION Since $\vec{r}$ is perpendicular to $\vec{a}$ and $\vec{b}$, so it is parallel to $\vec{a} \times \vec{b}$.

$\therefore \vec{r} = \lambda(\vec{a} \times \vec{b})$ for some scalar λ

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 2 \\ 18 & -22 & -5 \end{vmatrix} = 34\hat{i} + 51\hat{j} - 102\hat{k}$$

$\therefore \vec{r} = \lambda(34\hat{i} + 51\hat{j} - 102\hat{k}) = 17\lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$ ...(i)

Since $\vec{r}$ makes an acute angle with $(\hat{i} + \hat{j} + \hat{k})$. Therefore,

$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) > 0 \Rightarrow 17\lambda(2 + 3 - 6) > 0$

$\Rightarrow -17\lambda > 0 \Rightarrow \lambda < 0$

Now,

$|\vec{r}| = 14 \Rightarrow 17|\lambda|\sqrt{4 + 9 + 36} = 14$

$\Rightarrow |\lambda| = \frac{2}{17} \Rightarrow \lambda = \frac{-2}{17}$

Now, putting the value of λ in (i), we get

$$\vec{r} = -2(2\hat{i} + 3\hat{j} - 6\hat{k})$$

Hence, $\vec{r} = -4\hat{i} - 6\hat{j} + 12\hat{k}$.

EXAMPLE 20 *Find all possible vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a} \times \vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{a} + \vec{b} = \hat{i} - 3\hat{j} - 4\hat{k}$.*

SOLUTION Let $\vec{a} = x\hat{i} + y\hat{j} + 3\hat{k}$. Then,

$$\vec{a} + \vec{b} = \hat{i} - 3\hat{j} - 4\hat{k}$$

$$\Rightarrow \vec{b} = (\hat{i} - 3\hat{j} - 4\hat{k}) - (x\hat{i} + y\hat{j} + 3\hat{k})$$

$$\Rightarrow \vec{b} = (1 - x)\hat{i} - (y + 3)\hat{j} - (z + 4)\hat{k}$$

Again,

$$\vec{a} + \vec{b} = \hat{i} - 3\hat{j} - 4\hat{k}$$

$$\Rightarrow \vec{a} \times (\vec{a} + \vec{b}) = \vec{a} \times (\hat{i} - 3\hat{j} - 4\hat{k})$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{a} \times (\hat{i} - 3\hat{j} - 4\hat{k})$$

$$\Rightarrow 2\hat{i} + 2\hat{j} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & -3 & -4 \end{vmatrix}$$

$$\Rightarrow 2\hat{i} + 2\hat{j} - \hat{k} = (-4y + 3z)\hat{i} + (4x + z)\hat{j} + (-3x - y)\hat{k}$$

$$\Rightarrow -4y + 3z = 2, 4x + z = 2, 3x + y = 1$$

Out of these three equations, only two equations are independent. So, we cosider only two equations.

$$4x + z = 2 \text{ and } 3x + y = 1$$

$$\Rightarrow z = 2 - 4x \text{ and } y = 1 - 3x$$

$$\therefore \vec{a} = x\hat{i} + (1 - 3x)\hat{j} + (2 - 4x)\hat{k}$$

and, $\vec{b} = (1 - x)\hat{i} - (4 - 3x)\hat{j} - (6 - 4x)\hat{k}$, where $x \in R$.

EXAMPLE 21 *In a triangle ABC, prove by vector method that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$.

Then $|\vec{a}| = a$, $|\vec{b}| = b$, $|\vec{c}| = c$.

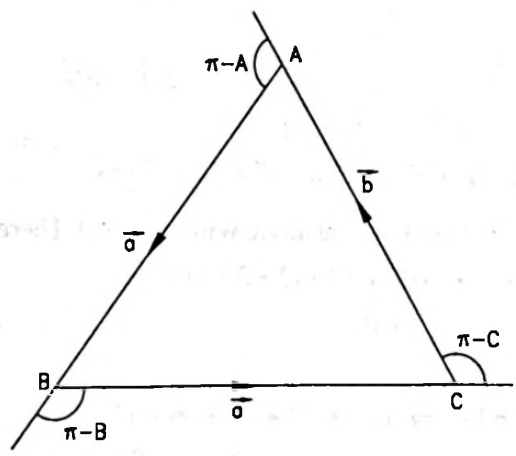


Fig. 16.13

We have, $\vec{BC} + \vec{CA} = \vec{BA}$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

$$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{c} \qquad [\because \vec{a} \times \vec{a} = \vec{0}]$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \qquad \text{...(i)}$$

Again,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$$

$$\Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = 0 \qquad [\because \vec{b} \times \vec{b} = \vec{0}]$$

$$\Rightarrow \vec{b} \times \vec{c} = -\vec{b} \times \vec{a}$$

$$\Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \qquad \text{...(ii)}$$

Thus, from (i) and (ii), we get

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B)$$

$$\Rightarrow ab \sin C = bc \sin A = ca \sin B$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{[Dividing by } abc\text{]}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

EXAMPLE 22 *Prove by vector method that*

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

SOLUTION Let OX and OY be two mutually perpendicular lines taken as axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively.

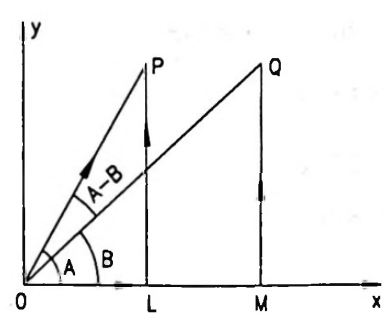


Fig. 16.14

(i) Let $\vec{OP}$ and $\vec{OQ}$ be two vectors such that $\angle XOP = A$ and $\angle XOQ = B$. Then, $\angle POQ = A - B$. Draw $PL \perp OX$ and $QM \perp OX$. Let $\hat{k}$ be the unit vector along Z-axis.

In ΔOPL, we have,

$OL = OP\cos A$ and $PL = OP\sin A$.

$\therefore \quad \vec{OP} = \vec{OL} + \vec{LP}$

$\Rightarrow \quad \vec{OP} = (OP\cos A)\hat{i} + (OP\sin A)\hat{j}$ ...(i)

In ΔOQM, we have

$OM = OQ\cos B$ and $QM = OQ\sin B$

$\therefore \quad \vec{OQ} = \vec{OM} + \vec{MQ}$

$\Rightarrow \quad \vec{OQ} = (OQ\cos B)\hat{i} + (OQ\sin B)\hat{j}$ ...(ii)

From (i) and (ii), we have

$\vec{OP} \times \vec{OQ}$

$= [(OP\cos A)\hat{i} + (OP\sin A)\hat{j}] \times [(OQ\cos B)\hat{i} + (OQ\sin B)\hat{j}]$

$= OP \cdot OQ\,[(\cos A)\hat{i} + (\sin A)\hat{j}] \times [(\cos B)\hat{i} + (\sin B)\hat{j}]$

$= OP \cdot OQ\,[\cos A\sin B\,(\hat{i}\times\hat{j}) + \sin A\cos B\,(\hat{j}\times\hat{i})]$

$= OP \cdot OQ\,[\cos A\sin B\,\hat{k} - \sin A\cos B\,\hat{k}]$

$= OP \cdot OQ\,[\cos A\sin B - \sin A\cos B]\hat{k}$

But, $\vec{OP} \times \vec{OQ} = |\vec{OP}|\,|\vec{OQ}|\sin(A-B)(-\hat{k})$

$= -OP \cdot OQ\sin(A-B)\,\hat{k}$

$\therefore \quad -OP \cdot OQ\sin(A-B)\hat{k}$

$= -OP \cdot OQ\,[\sin A\cos B - \cos A\sin B]\hat{k}$

$\Rightarrow \sin(A-B) = \sin A\cos B - \cos A\sin B.$

(ii) Let $\vec{OP}$ and $\vec{OQ}$ be two vectors such that $\angle XOP = A$ and $\angle XOQ = B$. Then, $\angle QOP = A + B$. Draw $PL \perp OX$ and $QM \perp OX$.

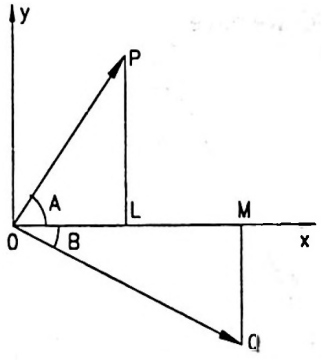


Fig. 16.15

From ΔOPL, we have

$OL = OP\cos A$ and $PL = OP\sin A$

$\therefore \quad \vec{OP} = \vec{OL} + \vec{LP}$

$\Rightarrow \quad \vec{OP} = (OP\cos A)\hat{i} + (OP\sin A)\hat{j}$ ...(i)

In ΔOQM, we have

$OM = OQ\cos B$ and $MQ = OQ\sin B$

$\therefore \quad \vec{OQ} = \vec{OM} + \vec{MQ}$

$\Rightarrow \quad \vec{OQ} = (OQ\cos B)\hat{i} - (OQ\sin B)\hat{j}$ ...(ii)

From (i) and (ii), we have

$\vec{OP} \times \vec{OQ}$

$= [(OP\cos A)\hat{i} + (OP\sin A)\hat{i}] \times [(OQ\cos B)\hat{i} - (OQ\sin B)\hat{j}]$

$= OP \cdot OQ\,[(\cos A)\hat{i} + (\sin A)\hat{j}] \times [(\cos B)\hat{i} - (\sin B)\hat{j}]$

$= OP \cdot OQ\,[-\cos A\sin B\,(\hat{i}\times\hat{j}) + \sin A\cos B\,(\hat{j}\times\hat{i})]$

$= OP \cdot OQ\,[-(\cos A\sin B)\hat{k} - (\sin A\cos B)\hat{k}]$

$= -OP \cdot OQ\,[\cos A\sin B + \sin A\cos B]\hat{k}$ ...(i)

But, $\vec{OP} \times \vec{OQ} = |\vec{OP}|\,|\vec{OQ}|\sin(A+B)(-\hat{k})$

$= -OP \cdot OQ\sin(A+B)\hat{k}$

$\therefore \quad -OP \cdot OQ\sin(A+B)\hat{k} = -OP \cdot OQ\,[\cos A\sin B + \sin A\cos B]\hat{k}$

$\Rightarrow \quad \sin(A+B) = \sin A\cos B + \cos A\sin B.$

EXAMPLE 23 *Prove by vector method that the parallelogram on the same base and between the same parallels are equal in area.*

SOLUTION Let $OACB$ and $OAED$ be parallelograms on the same base OA and between the same parallels.

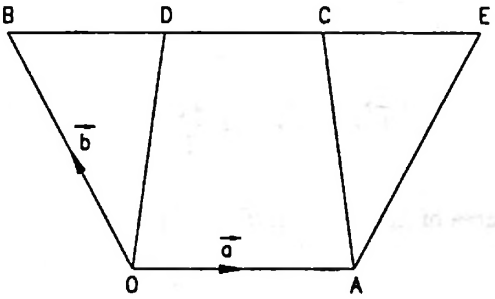


Fig. 16.16

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

Since BD is parallel to $\vec{OA}$, therefore $\vec{BD} = x\vec{OA} = x\vec{a}$ for some scalar x.

In ΔOBD, we have

$\vec{OB} + \vec{BD} = \vec{OD}$

$\Rightarrow \quad \vec{OD} = \vec{b} + x\vec{a}$ ...(i)

Vector area of parallelogram $OACB = \vec{a} \times \vec{b}$

Vector area of parallelogram $OAED$

$= \vec{OA} \times \vec{OD}$

$= \vec{a} \times (\vec{b} + x\vec{a})$ [Using (i)]

$= \vec{a} \times \vec{b} + x(\vec{a} \times \vec{a})$

$= \vec{a} \times \vec{b}$ $[\because \vec{a} \times \vec{a} = \vec{0}]$

= Vector area of parallelogram $OACB$.

EXAMPLE 24 *If D, E, F are the mid-points of the sides of a triangle ABC, prove by vector method that*

$$\text{Area of } \Delta DEF = \frac{1}{4}(\text{Area of } \Delta ABC).$$

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of D, E and F are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

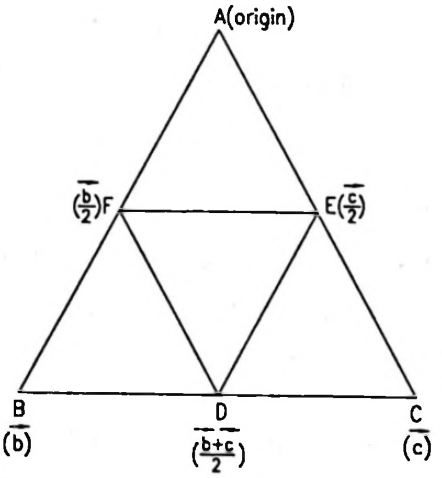


Fig. 16.17

Now, $\vec{DE}$ = P.V. of E – P.V. of D

$$= \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{b}}{2}$$

and, $\vec{DF}$ = P.V of F – P.V. of D

$$= \frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{c}}{2}$$

$\therefore$ Vector area of $\Delta DEF = \frac{1}{2}(\vec{DE} \times \vec{DF})$

$$= \frac{1}{2}\left(\frac{-\vec{b}}{2} \times \frac{-\vec{c}}{2}\right)$$

$$= \frac{1}{8}(\vec{b} \times \vec{c})$$

$$= \frac{1}{4}\left\{\frac{1}{2}(\vec{AB} \times \vec{AC})\right\}$$

$$= \frac{1}{4}(\text{vector area of } \Delta ABC)$$

Hence, area of $\Delta DEF = \frac{1}{4}$ area of ΔABC.

EXAMPLE 25 *Using vectors : prove that if a, b, c are the lengths of three sides of a triangle, then its area is*

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } 2s = a+b+c.$$

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$.

Then, $|\vec{BC}| = |\vec{a}| = a$, $|\vec{CA}| = |\vec{b}| = b$

and, $|\vec{AB}| = |\vec{c}| = c$.

By triangle law of addition of vectors, we have,

$$\vec{BC} + \vec{CA} = \vec{BA}$$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

Now, $\Delta = \frac{1}{2}|\vec{BC} \times \vec{BA}|$

$$\Delta = \frac{1}{2}|\vec{b} \times -\vec{c}| = \frac{1}{2}|\vec{b} \times \vec{c}|$$

$$\Rightarrow 2\Delta = |\vec{b} \times \vec{c}|$$

$$\Rightarrow 4\Delta^2 = |\vec{b} \times \vec{c}|^2$$

$$\Rightarrow 16\Delta^2 = 4|\vec{b} \times \vec{c}|^2$$

$$\Rightarrow 16\Delta^2 = 4[-(\vec{b}\cdot\vec{c})^2 + |\vec{b}|^2 |\vec{c}|^2] \quad [\text{Using Lagrange's identity}]$$

$$\Rightarrow 16\Delta^2 = 4|\vec{b}|^2 |\vec{c}|^2 - 4(\vec{b}\cdot\vec{c})^2$$

$$= 4|\vec{b}|^2 |\vec{c}|^2 - [|\vec{b}|^2 + |\vec{c}|^2 - |\vec{b}+\vec{c}|^2]^2$$

$$= 4|\vec{b}|^2 |\vec{c}|^2 - [|\vec{b}|^2 + |\vec{c}|^2 - |-\vec{a}|^2]^2$$

$$[\because \vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow \vec{b}+\vec{c} = -\vec{a}]$$

$$= 4|\vec{b}|^2 |\vec{c}|^2 - [|\vec{b}|^2 + |\vec{c}|^2 - |\vec{a}|^2]^2$$

$$= 4b^2c^2 - [b^2 + c^2 - a^2]^2$$

$$= (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)$$

$$= \{[(b+c)^2 - a^2]\,[a^2 - (b-c)^2]\}$$

$$= (b+c+a)(b+c-a)(a+b-c)(a-b+c)$$

$$= (2s)(2s-2a)(2s-2c)(2s-2b)$$

$$= 16s(s-a)(s-b)(s-c)$$

$$\Rightarrow \Delta^2 = s(s-a)(s-b)(s-c)$$

$$\Rightarrow \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

EXAMPLE 26 *Prove vectorially that the area of a triangle ABC is given by*

$$\Delta = \frac{1}{2}\frac{a^2 \sin B \sin C}{\sin A}$$

SOLUTION In ΔABC, we have

$$\vec{BC} + \vec{CA} = \vec{BA}$$

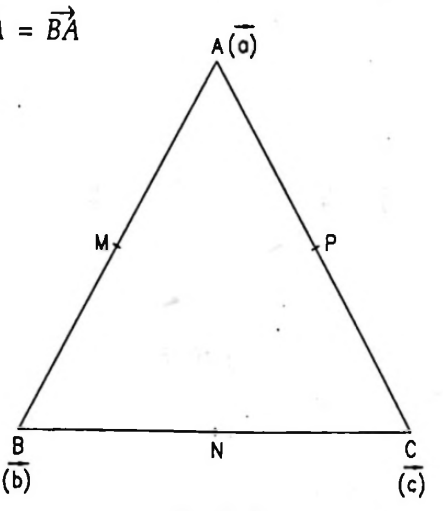


Fig. 16.18

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$\Rightarrow \vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{0}$

$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

Similarly, we have

$\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

$\therefore \vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

$\Rightarrow |(\vec{b}\times\vec{c})|\,|\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|\,.\,|\vec{a}\times\vec{b}|$

$\Rightarrow (bc\sin A)(2\Delta) = (ca\sin B)(ab\sin C) \quad \left[\because \Delta=\frac{1}{2}|\vec{b}\times\vec{c}|\right]$

$\Rightarrow \Delta = \dfrac{a^2\sin B\sin C}{2\sin A}$

—❑❑❑—

16.5 APPLICATIONS OF VECTOR PRODUCT IN MECHANICS TO FIND THE MOMENT OF A FORCE

(I) ABOUT A POINT Let a force $\vec{F}$ be applied at point P of a rigid body. Then the moment of $\vec{F}$ about a point O measures the tendency (amount) of $\vec{F}$ to turn the body about point O. If this tendency of rotation about O is in anticlockwise direction the moment is positive, otherwise it is negative.

Let $\vec{r}$ be the position vector of P relative to O. Then the moment or torque of $\vec{F}$ about the point O is defined as the vector

$$\vec{M} = \vec{r}\times\vec{F}$$

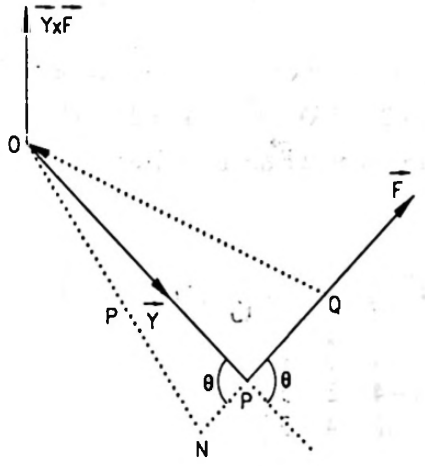


Fig. 16.19

This vector is perpendicular to the plane containing O and the line of action of the force. The vector $\vec{M}$ about O is independent of the choice of the point P on the line of action of the force. To prove this, let Q be any other point on the line of action of force $\vec{F}$. Then,

$$\vec{OQ}\times\vec{F} = (\vec{OP}+\vec{PQ})\times\vec{F} \qquad [\because \vec{OQ}=\vec{OP}+\vec{PQ}]$$
$$= \vec{OP}\times\vec{F}+\vec{PQ}\times\vec{F}$$
$$= \vec{OP}\times\vec{F}+\vec{0} \qquad [\because \vec{PQ} \text{ and } \vec{F} \text{ are collinear}]$$
$$= \vec{r}\times\vec{F}.$$

If several forces are acting through the same point P, then the vector sum of the moments of the separate forces about O is equal to the moment of their resultant force about O.

(II) ABOUT A LINE The moment of a force $\vec{F}$ acting at a point P about a line L is a scalar given by

$$(\vec{r}\times\vec{F})\cdot\hat{a}$$

where $\hat{a}$ is a unit vector in the direction of the line, and $\vec{OP}=\vec{r}$ where O is any point on the line.

Thus, the moment of a force $\vec{F}$ about a line is the resolve part (component) along this line of the moment of $\vec{F}$ about any point on the line.

NOTE *It is to note here that the moment of a force about a point is a vector while the moment about a straight line is a scalar quantity.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the moment about the point* $\hat{i}+2\hat{j}+3\hat{k}$ *of a force represented by* $\hat{i}+\hat{j}+\hat{k}$ *acting through the point* $-2\hat{i}+3\hat{j}+\hat{k}$.

SOLUTION Let O be the point $\hat{i}+2\hat{j}+3\hat{k}$ and P the point $-2\hat{i}+3\hat{j}+\hat{k}$.

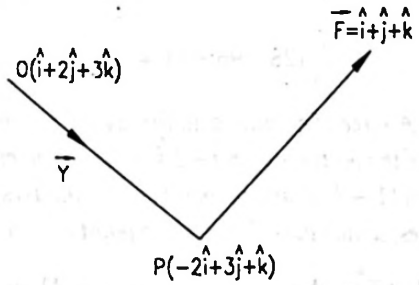


Fig. 16.20

Then, $\vec{OP}$ = P.V. of P − P.V. of O

$\Rightarrow \vec{r} = -3\hat{i}+\hat{j}-2\hat{k}$

Let $\vec{M}$ be the vector moment of $\vec{F}$ acting at P about point O. Then,

$$\vec{M} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 3\hat{i}+\hat{j}-4\hat{k}.$$

EXAMPLE 2 *A force is represented in mgnitude and direction by the line joining the point* $A(1,-2,4)$ *to the point* $B(5,2,3)$. *Find its moment about the point* $(-2,3,5)$.

SOLUTION We have,

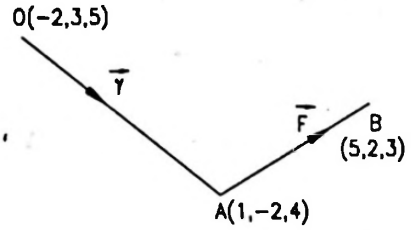


Fig. 16.21

$$\vec{F} = \vec{AB} = \text{P.V. of } B - \text{P.V. of } A$$
$$= (5\hat{i} + 2\hat{j} + 3\hat{k}) - (\hat{i} - 2\hat{j} + 4\hat{k})$$
$$= 4\hat{i} + 4\hat{j} - \hat{k}$$

Let O be the point $(-2, 3, 5)$. Then,

$$\vec{r} = \vec{OA} = \text{P.V. of } A - \text{P.V. of } O$$
$$= (\hat{i} - 2\hat{j} + 4\hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k})$$
$$= 3\hat{i} - 5\hat{j} - \hat{k}$$

Let $\vec{M}$ be the moment of $\vec{F}$ about O. Then,

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & -1 \\ 4 & 4 & -1 \end{vmatrix} = 9\hat{i} - \hat{j} + 32\hat{k}$$

EXAMPLE 3 *Forces $2\hat{i} + 7\hat{j}$, $2\hat{i} - 5\hat{j} + 6\hat{k}$, $-\hat{i} + 2\hat{j} - \hat{k}$ act at a point P whose position vector is $4\hat{i} - 3\hat{j} - 2\hat{k}$. Find the vector moment of the resultant of three forces acting at P about the point Q, whose position vector is $6\hat{i} + \hat{j} - 3\hat{k}$.*

SOLUTION Let $\vec{F_1} = 2\hat{i} + 7\hat{j}$, $\vec{F_2} = 2\hat{i} - 5\hat{j} + 6\hat{k}$, $\vec{F_3} = -\hat{i} + 2\hat{j} - \hat{k}$. Then the resultant force $\vec{F}$ is given by $\vec{F} = 3\hat{i} + 4\hat{j} + 5\hat{k}$.

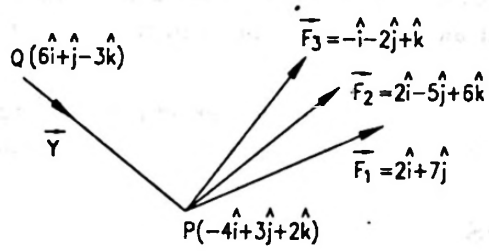


Fig. 16.22

Let $\vec{r} = \vec{QP}$. Then,

$$\vec{r} = \text{P.V. of } P - \text{P.V. of } Q$$
$$= (4\hat{i} - 3\hat{j} - 2\hat{k}) - (6\hat{i} + \hat{j} - 3\hat{k})$$
$$= -2\hat{i} - 4\hat{j} + \hat{k}$$

Let $\vec{M}$ be the moment of the resultant force $\vec{F}$ about Q. Then,

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -4 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -24\hat{i} + 13\hat{j} + 4\hat{k}.$$

EXAMPLE 4 *Find the moment about a line through the origin having the direction of $2\hat{i} - 2\hat{j} + \hat{k}$ due to a 30 kg force acting at a point $(-4, 2, 5)$ in the direction of $12\hat{i} - 4\hat{j} - 3\hat{k}$.*

SOLUTION Let $\vec{F}$ be the force. Then,

$$\vec{F} = \frac{30(12\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{12^2 + (-4)^2 + (-3)^2}} = \frac{30}{13}(12\hat{i} - 4\hat{j} - 3\hat{k}).$$

Suppose the force $\vec{F}$ acts at point $P(-4, 2, 5)$. The moment of $\vec{F}$ acting at P about a line in the direction $2\hat{i} - 2\hat{j} + \hat{k}$ is equal to the resolve part along the line of the moment of $\vec{F}$ about a point on the line. It is given that the line passes through the origin O. So we choose O as a point on the line. Let $\vec{OP} = \vec{r}$. Then,

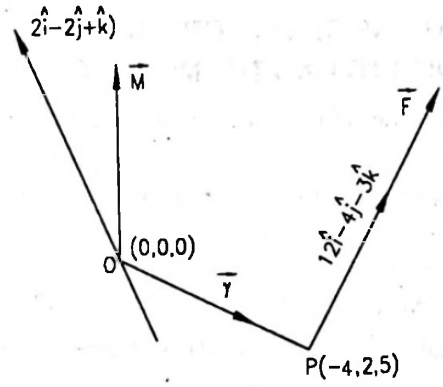


Fig. 16.23

$$\vec{r} = \vec{OP}$$
$$= \text{Position vector of } P - \text{Position vector of } O$$
$$= (-4\hat{i} + 2\hat{j} + 5\hat{k}) - \vec{0} = -4\hat{i} + 2\hat{j} + 5\hat{k}.$$

Let $\vec{M}$ be the moment of $\vec{F}$ about O. Then,

$$\vec{M} = \vec{r} \times \vec{F}$$
$$= (-4\hat{i} + 2\hat{j} + 5\hat{k}) \times \frac{30}{13}(12\hat{i} - 4\hat{j} - 3\hat{k})$$
$$= \frac{30}{13}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 2 & 5 \\ 12 & -4 & -3 \end{vmatrix}$$
$$= \frac{30}{13}(14\hat{i} + 48\hat{j} - 8\hat{k})$$

Let $\vec{a}$ be unit vector in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$. Then,

$$\vec{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4 + 4 + 1}} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$$

Thus, the moment of $\vec{F}$ about the given line

$$= \vec{M} \cdot \vec{a}$$
$$= \frac{30}{13}(14\hat{i} + 48\hat{j} - 8\hat{k}) \cdot \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$$
$$= \frac{10}{13}(28 - 96 - 8) = -\frac{760}{13}.$$

EXAMPLE 5 *A force of 6 units acts through the point $(4, -1, 7)$ in the direction of the vector $9\hat{i} + 6\hat{j} - 2\hat{k}$. Find the moment of the force about the point $(1, -3, 2)$ and moments about the axes, parallel to the coordinate axes, which pass through the point $(1, -3, 2)$.*

SOLUTION Let $\vec{F}$ be the given force vector. Then,

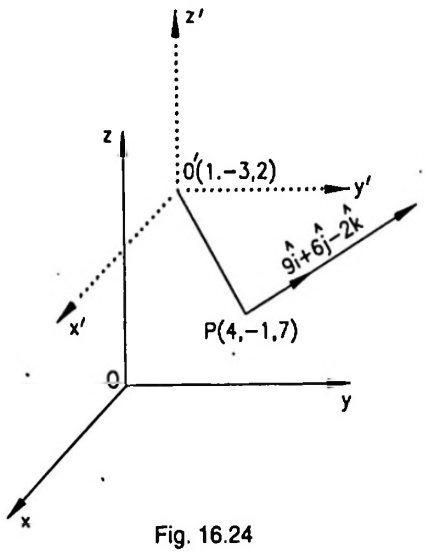


Fig. 16.24

$$\vec{F} = \frac{6(9\hat{i}+6\hat{j}-2\hat{k})}{\sqrt{81+36+4}} = \frac{6}{11}(9\hat{i}+6\hat{j}-2\hat{k}),$$

Suppose the force $\vec{F}$ acts at $P(4,-1,7)$ and let O' be the point $(1,-3,2)$. Then,

$$\vec{r} = \vec{O'P} = (4\hat{i}-\hat{j}+7\hat{k}) - (\hat{i}-3\hat{j}+2\hat{k})$$
$$= 3\hat{i}+2\hat{j}+5\hat{k}$$

$\therefore$ The moment of $\vec{F}$ about the point $O'(1,-3,2)$ is given by

$$\vec{M} = \vec{r}\times\vec{F} = \frac{6}{11}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 9 & 6 & -2 \end{vmatrix} = \frac{6}{11}(-34\hat{i}+51\hat{j}).$$

The unit vectors along $O'X', O'Y'$ and $O'Z'$ are $\hat{i}, \hat{j}$ and $\hat{k}$ respectively. Therefore, the moments of $\vec{F}$ about these axes are $\vec{M}\cdot\hat{i}, M\cdot\hat{j}$ and $\vec{M}\cdot\hat{k}$ i.e. $-\frac{204}{11}, \frac{306}{11}$ and 0 respectively.

EXERCISE 16.1

1. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then prove that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$.
2. If A, B, C, D are four points in space, prove that

$$\text{Area of } \Delta ABC = \frac{1}{4}\left|\vec{AB}\times\vec{CD} + \vec{BC}\times\vec{AD} + \vec{CA}\times\vec{BD}\right|$$

3. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that

$$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$$

4. If $\vec{a} = \hat{i}+\hat{j}-\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{c} = -\hat{i}+2\hat{j}-\hat{k}$, find a unit vector normal to the vectors $\vec{a}+\vec{b}$ and $\vec{b}-\vec{c}$.
5. Find a vector $\vec{r}$ which is normal to both $\vec{a} = 4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$ and $\vec{r}\cdot\vec{c} = 21$, where $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$.
6. The position vectors of four points A, B, C, D on a plane are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c}) = 0 = (\vec{b}-\vec{d})(\vec{c}-\vec{a})$. Show that the point D is the orthocentre of ΔABC.
7. If G is the centroid of the triangle ABC, using vector methods show that

$$\text{Area of } \Delta AGB = \text{Area of } \Delta BGC$$
$$= \text{Area of } \Delta CGA$$
$$= \frac{1}{3}\text{ Area of } \Delta ABC$$

8. Prove that the area of a triangle formed by joining the mid-point of one of the non-parallel sides of a trapezoid to the extremeties of the opposite side is half of the trapezoid.
9. If A, B, C, D are any four points in space, show that

$$\vec{AB}\times\vec{CD} + \vec{BC}\times\vec{AD} + \vec{CA}\times\vec{BD}$$

is independent of D.

10. If G is the centroid of ΔABC, such that $\vec{GB}$ and $\vec{GC}$ are inclined at an obtuse angle, prove that $5a^2 > b^2 + c^2$.

ANSWERS

4. $\hat{i}$

5. $\frac{18}{25}(3\hat{j}+4\hat{k})$

—❑❑❑—

CHAPTER

17

PRODUCT OF THREE VECTORS

17.1 INTRODUCTION

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. By inserting dot and cross between $\vec{a}, \vec{b}, \vec{c}$ in the same alphabetical order we introduce the following products :

$(\vec{a} \cdot \vec{b}) \cdot \vec{c}$, $(\vec{a} \cdot \vec{b}) \times \vec{c}$, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c}$

Consider the product $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$. Since $\vec{a} \cdot \vec{b}$ is a scalar quantity and dot product is defined between two vector quantities, therefore the product $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not meaningful. Similarly, $(\vec{a} \cdot \vec{b}) \times \vec{c}$ is not meaningful. But, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is meaningful, because $\vec{a} \times \vec{b}$ is a vector and its dot product with $\vec{c}$ i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar quantity. This product is known as the *scalar triple product* of $\vec{a}, \vec{b}, \vec{c}$. The product $(\vec{a} \times \vec{b}) \times \vec{c}$ is also meaningful, because $\vec{a} \times \vec{b}$ is a vector and its cross-product with $\vec{c}$ i.e. $(\vec{a} \times \vec{b}) \times \vec{c}$ is also a vector. This product is known as the *vector triple product* of $\vec{a}, \vec{b}, \vec{c}$.

17.2 SCALAR TRIPLE PRODUCT

DEFINITION *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the scalar $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.*

Thus, $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$

GEOMETRICAL INTERPRETATION Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Consider a parallelopiped having coterminous edges OA, OB and OC such that $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Then, $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Let ϕ be the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$. If $\hat{\eta}$ is a unit vector along $\vec{a} \times \vec{b}$, then ϕ is also the angle between $\hat{\eta}$ and $\vec{c}$.

Now,

$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$

= (Area of the parallelogram $OADB$) $\hat{\eta} \cdot \vec{c}$

= (Area of the parallelogram $OADB$) $(\hat{\eta} \cdot \vec{c})$

= (Area of the parallelogram $OADB$) $|\hat{\eta}|\,|\vec{c}| \cos\phi$

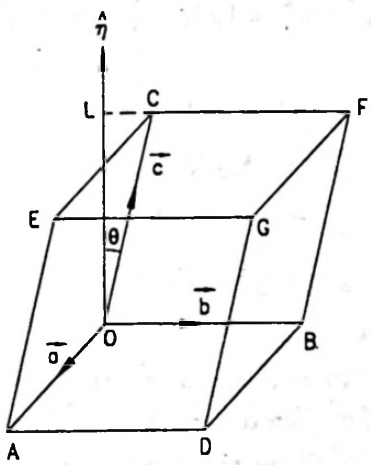


Fig. 17.1

= (Area of the parallelogram $OADB$) $(|\vec{c}| \cos\phi)$ $[\because |\hat{\eta}| = 1]$

= (Area of the parallelogram $OADB$) (OL) $[\because OC \cos\phi = OL]$

= (Area of the base of the parallelopiped) × (height)

= Volume of the parallelopiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$

Thus , *the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of the parallelopiped whose coterminous edges $\vec{a}, \vec{b}, \vec{c}$ form a right handed system of vectors.*

17.3 PROPERTIES OF SCALAR TRIPLE PRODUCT

PROPERTY I *If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same.*

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

or, $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

PROPERTY II *The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.*

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{c}\,\vec{b}\,\vec{a}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

PROPERTY III *In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same i.e.*

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$$

PROPERTY IV *The scalar triple product of three vectors is zero if any two of them are equal.*

PROPERTY V *For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ,*

$$[\lambda\vec{a}\,\vec{b}\,\vec{c}] = \lambda[\vec{a}\,\vec{b}\,\vec{c}].$$

PROPERTY VI *The scalar triple product of three vectors is zero if any two of them are parallel or collinear.*

PROPERTY VII If $\vec{a}\,\vec{b}\,\vec{c}\,\vec{d}$ *are four vectors, then*

$$[\vec{a}+\vec{b}\,\vec{c}\,\vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}].$$

PROOF We have,

$$\begin{aligned}[\vec{a}+\vec{b}\,\vec{c}\,\vec{d}] &= [(\vec{a}+\vec{b})\times\vec{c}]\cdot\vec{d}\\ &= (\vec{a}\times\vec{c}+\vec{b}\times\vec{c})\cdot\vec{d}\\ &= (\vec{a}\times\vec{c})\cdot\vec{d}+(\vec{b}\times\vec{c})\cdot\vec{d}\\ &= [\vec{a}\,\vec{c}\,\vec{d}]+[\vec{b}\,\vec{c}\,\vec{d}]\end{aligned}$$

PROPERTY VIII *The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that $[\vec{a}\,\vec{b}\,\vec{c}] = 0$. i.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$.*

PROOF First, $\vec{a}, \vec{b}, \vec{c}$ be three non-zero, non-collinear coplanar vectors. Then we have to prove that their scalar triple product is zero.

Now,

$\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow$ $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$ $\quad[\because \vec{a}, \vec{b}, \vec{c}$ are coplanar]

$\Rightarrow (\vec{a}\times\vec{b})\cdot\vec{c} = 0$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

Thus,

$\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$.

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero, non-collinear vectors such that $[\vec{a}\,\vec{b}\,\vec{c}] = 0$. Then,

$[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$\Rightarrow (\vec{a}\times\vec{b})\cdot\vec{c} = 0$

$\Rightarrow \vec{a}\times\vec{b} = \vec{0}$ or, $\vec{c} = \vec{0}$ or, $(\vec{a}\times\vec{b}) \perp \vec{c}$

$\Rightarrow (\vec{a}\times\vec{b}) \perp \vec{c}$ $\quad[\because \vec{c} \neq \vec{0}$ and $\vec{a}\times\vec{b} \neq 0$ because $\vec{a}, \vec{b}$ are non–zero non–collinear vectors]

But, $\vec{a}\times\vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Therefore, $(\vec{a}\times\vec{b})$ is perpendicular to $\vec{c}$

$\Rightarrow$ $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors

Thus, $[\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

PROPERTY IX *For points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ will be coplanar if*

$$[\vec{d}\,\vec{b}\,\vec{c}] + [\vec{d}\,\vec{c}\,\vec{a}] + [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]$$

PROPERTY X *If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ are three vectors, then*

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find* $[\vec{a}\,\vec{b}\,\vec{c}]$, when $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{c} = 3\hat{i}-\hat{j}+2\hat{k}$.

SOLUTION We have,

$$\begin{aligned}[\vec{a}\,\vec{b}\,\vec{c}] &= \begin{vmatrix} 2 & -3 & 4\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}\\ &= 2(4-1)-(-3)(2+3)+4(-1-6)\\ &= 6+15-28 = -7\end{aligned}$$

EXAMPLE 2 *Find the volume of a parallelopiped whose sides are given by* $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

SOLUTION Let $\vec{a} = -3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}$. We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is $\left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$.

We have,

$$\begin{aligned}[\vec{a}\,\vec{b}\,\vec{c}] &= \begin{vmatrix} -3 & 7 & 5\\ -5 & 7 & -3\\ 7 & -5 & -3\end{vmatrix}\\ &= -3(-21-15)-7(15+21)+5(25-49)\\ &= 108-252-120\\ &= -264\end{aligned}$$

So, required volume of the parallelopiped

$$= \left|[\vec{a}\,\vec{b}\,\vec{c}]\right| = |-264| = 264 \text{ cubic units}$$

EXAMPLE 3 *Find λ so that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.*

SOLUTION We know that vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar iff $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow 2(10 + 3\lambda) + 1(5 + 9) + (\lambda - 6) = 0$$

$$\Rightarrow 7\lambda + 28 = 0 \Rightarrow \lambda = -4.$$

EXAMPLE 4 *Find the value of λ for which the four points with position vectors $3\hat{i} - 2\hat{j} - \hat{k}, 2\hat{i} + 3\hat{j} - 4\hat{k}, -\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$ are coplanar.*

SOLUTION Let A, B, C, D be the given points. Then,

$\vec{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i} + 3\hat{j} - 4\hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k}) = -\hat{i} + 5\hat{j} - 3\hat{k}$$

$\vec{AC}$ = Position vector of C – Position vector of A

$$= (-\hat{i} + \hat{j} + 2\hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k}) = -4\hat{i} + 3\hat{j} + 3\hat{k}$$

$\vec{AD}$ = Position vector of D – Position vector of A

$$= (4\hat{i} + 5\hat{j} + \lambda\hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k}) = \hat{i} + 7\hat{j} + (\lambda + 1)\hat{k}$$

The given points are coplanar iff vectors $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are coplanar.

$\therefore$ Points A, B, C, D are coplanar

$\Leftrightarrow \vec{AB}, \vec{AC}, \vec{AD}$ are coplanar

$$\Leftrightarrow \quad [\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$$

$$\Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda + 1 \end{vmatrix} = 0$$

$$\Leftrightarrow -1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-28 - 3) = 0$$

$$\Leftrightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0$$

$$\Leftrightarrow 17\lambda + 146 = 0$$

$$\Rightarrow \lambda = \frac{-146}{17}.$$

EXAMPLE 5 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$[\vec{a} + \vec{b}\ \vec{b} + \vec{c}\ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

and hence prove that the vectors $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are coplanar iff $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

SOLUTION We have,

$$[\vec{a} + \vec{b}\ \vec{b} + \vec{c}\ \vec{c} + \vec{a}]$$

$$= [(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})] \cdot (\vec{c} + \vec{a})$$

$$= [\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c}] \cdot (\vec{c} + \vec{a})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{a} \times \vec{b}) \cdot \vec{a} + (\vec{a} \times \vec{c}) \cdot \vec{c} + (\vec{a} \times \vec{c}) \cdot \vec{a} + (\vec{b} \times \vec{c}) \cdot \vec{c} + (\vec{b} \times \vec{c}) \cdot \vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{c}] + [\vec{a}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= 2[\vec{a}\ \vec{b}\ \vec{c}]$$

Now,

$\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Leftrightarrow 2[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Leftrightarrow [\vec{a} + \vec{b}\ \vec{b} + \vec{c}\ \vec{c} + \vec{a}] = 0$$

$\Leftrightarrow \vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$ are coplanar.

EXAMPLE 6 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$[\vec{a} - \vec{b}\ \vec{b} - \vec{c}\ \vec{c} - \vec{a}] = 0$$

SOLUTION We have :

$$[\vec{a} - \vec{b}\ \vec{b} - \vec{c}\ \vec{c} - \vec{a}]$$

$$= [(\vec{a} - \vec{b}) \times (\vec{b} - \vec{c})] \cdot (\vec{c} - \vec{a})$$

$$= (\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{b} + \vec{b} \times \vec{c}) \cdot (\vec{c} - \vec{a})$$

$$= (\vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{b} \times \vec{c}) \cdot (\vec{c} - \vec{a})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{b}) \cdot \vec{a} + (\vec{c} \times \vec{a}) \cdot \vec{c} - (\vec{c} \times \vec{a}) \cdot \vec{a} + (\vec{b} \times \vec{c}) \cdot \vec{c} - (\vec{b} \times \vec{c}) \cdot \vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{c}\ \vec{a}\ \vec{c}] - [\vec{c}\ \vec{a}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= 0 \qquad \left[\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]\right]$$

ALITER We have,

$$(\vec{a} - \vec{b}) + (\vec{b} - \vec{c}) + (\vec{c} - \vec{a}) = \vec{0}$$

$\Rightarrow \vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}$ are coplanar vectors

$$\Rightarrow [\vec{a} - \vec{b}\ \vec{b} - \vec{c}\ \vec{c} - \vec{a}] = 0$$

EXAMPLE 7 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that*

$$[\vec{a}\ \vec{b} + \vec{c}\ \vec{a} + \vec{b} + \vec{c}] = 0$$

SOLUTION We have,

$$[\vec{a}\ \vec{b} + \vec{c}\ \vec{a} + \vec{b} + \vec{c}]$$

$$= [\vec{a} \times (\vec{b} + \vec{c})] \cdot (\vec{a} + \vec{b} + \vec{c})$$

$= (\vec{a}\times\vec{b}+\vec{a}\times\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$= (\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{b}).\vec{b}+(\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{c}).\vec{a}$

$+(\vec{a}\times\vec{c}).\vec{b}+(\vec{a}\times\vec{c}).\vec{c}$

$= 0+0+[\vec{a}\ \vec{b}\ \vec{c}]+0+[\vec{a}\ \vec{c}\ \vec{b}]+0$

$= [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{c}] \quad = 0$

ALITER Since $\vec{a}+\vec{b}+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ *i.e.* $\vec{a}+\vec{b}+\vec{c}$ is expressible as the linear combination of the other two vectors.

Therefore,

$\vec{a},\ \vec{a}+\vec{b},\ \vec{a}+\vec{b}+\vec{c}$ are coplanar vectors.

Hence, $[\vec{a}\ \vec{a}+\vec{b}\ \vec{a}+\vec{b}+\vec{c}] = 0$

EXAMPLE 8 If $\vec{a},\ \vec{b},\ \vec{c}$ *are three non-coplanar vectors, prove that*

$$[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{a}+\vec{c}] = -[\vec{a}\ \vec{b}\ \vec{c}]$$

SOLUTION We have,

$[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \vec{a}+\vec{c}]$

$= (\vec{a}+\vec{b}+\vec{c}).\{(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})\}$

$= (\vec{a}+\vec{b}+\vec{c}).\{\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}\}$

$= (\vec{a}+\vec{b}+\vec{c}).\{-\vec{a}\times\vec{b}+\vec{b}\times\vec{c}-\vec{c}\times\vec{a}\}$

$= -\vec{a}.(\vec{a}\times\vec{b})+\vec{a}.(\vec{b}\times\vec{c})-\vec{a}.(\vec{c}\times\vec{a})-\vec{b}.(\vec{a}\times\vec{b})$

$+\vec{b}.(\vec{b}\times\vec{c})-\vec{b}.(\vec{c}\times\vec{a})-\vec{c}.(\vec{a}\times\vec{b})+\vec{c}.(\vec{b}\times\vec{c})-\vec{c}.(\vec{c}\times\vec{a})$

$= -[\vec{a}\ \vec{a}\ \vec{b}]+[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{c}\ \vec{a}]-[\vec{b}\ \vec{a}\ \vec{b}]+[\vec{b}\ \vec{b}\ \vec{c}]$

$-[\vec{b}\ \vec{c}\ \vec{a}]-[\vec{c}\ \vec{a}\ \vec{b}]+[\vec{c}\ \vec{b}\ \vec{c}]-[\vec{c}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{b}\ \vec{c}\ \vec{a}]-[\vec{c}\ \vec{a}\ \vec{b}]$

$= -[\vec{a}\ \vec{b}\ \vec{c}]$

EXAMPLE 9 *If the vectors* $\vec{\alpha}=a\hat{i}+\hat{j}+\hat{k}, \vec{\beta}=\hat{i}+b\hat{j}+\hat{k}$ *and* $\vec{\gamma}=\hat{i}+\hat{j}+c\hat{k}$ *are coplanar, then prove that* $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$, *where* $a\neq 1, b\neq 1$ and $c\neq 1$.

[IIT 1987]

SOLUTION Since $\vec{\alpha},\ \vec{\beta},\ \vec{\gamma}$ are coplanar vectors. Therefore,

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow abc - a - c + 1 + 1 - b = 0$$

$$\Rightarrow abc = a+b+c-2 \qquad \ldots(i)$$

Now,

$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$

$= \frac{(1-b)(1-c)+(1-c)(1-a)+(1-a)(1-b)}{(1-a)(1-b)(1-c)}$

$= \frac{3-2(a+b+c)+(ab+bc+ca)}{1-(a+b+c)+(ab+bc+ca)-abc}$

$= \frac{3-2(a+b+c)+(ab+bc+ca)}{1-(a+b+c)+ab+bc+ca-(a+b+c-2)}$ [Using (i)

$= \frac{3-2(a+b+c)+ab+bc+ca}{3-2(a+b+c)+ab+bc+ca} = 1$

EXAMPLE 10 *If the vectors* $\vec{\alpha}=a\hat{i}+a\hat{j}+c\hat{k}$, $\vec{\beta}=\hat{i}+\hat{k}$ *and* $\vec{\gamma}=c\hat{i}+c\hat{j}+b\hat{k}$ *are coplanar, then prove that c is the geometric mean of a and b.*

[IIT 1993]

SOLUTION Since $\vec{\alpha},\ \vec{\beta}$ and $\vec{\gamma}$ are coplanar vectors, therefore

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0 \Rightarrow \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$$\Rightarrow a(0-c)-a(b-c)+c(c-0) = 0$$

$$\Rightarrow -ac-ab+ac+c^2 = 0$$

$$\Rightarrow c^2 = ab$$

$\Rightarrow c$ is the geometric mean of a and b

EXAMPLE 11 *If a is a non-zero real number, then prove that the vectors* $\vec{\alpha}=a\hat{i}+2a\hat{j}-3a\hat{k}$, $\vec{\beta}=(2a+1)\hat{i}+(2a+3)\hat{j}+(a+1)\hat{k}$ *and* $\vec{\gamma}=(3a+5)\hat{i}+(a+5)\hat{j}+(a+2)\hat{k}$ *are never coplanar.*

SOLUTION We have,

$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$

$$= \begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$= a\{(2a+3)(a+2)-(a+1)(a+5)\}-2a\{(2a+1)(a+2)$

$-(a+1)(3a+5)\}-3a\{(2a+1)(a+5)-(2a+3)(3a+5)\}$

$= a(2a^2+7a+6-a^2-6a-5)-2a(2a^2+5a+2-3a^2-8a-5)$

$-3a(2a^2+11a+5-6a^2-19a-15)$

$= a(a^2+a+1)-2a(-a^2-3a-3)-3a(-4a^2-8a-10)$

$= a(a^2+a+1)+a(2a^2+6a+6)+a(12a^2+24a+30)$

$= a(15a^2+31a+37)$

$= 15a\left\{\left(a+\frac{31}{30}\right)^2+\frac{1259}{900}\right\} \neq 0$ for all non-zero a.

Hence, the given vectors are non-coplanar.

EXAMPLE 12 *Let* $\vec{a},\ \vec{b},\ \vec{c}$ *be three non-zero vectors such that* $\vec{c}$ *is a unit vector perpendicular to both* $\vec{a}$ and $\vec{b}$. *If the angle between* $\vec{a}$ *and* $\vec{b}$ *is* $\pi/6$, *prove that* $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.

[IIT 1986]

SOLUTION Since $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore, it is parallel to $\vec{a} \times \vec{b}$.

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}]^2$$
$$= |(\vec{a} \times \vec{b}) . \vec{c}|^2$$
$$= |\,|\vec{a} \times \vec{b}|\ |\vec{c}| \cos 0°|^2 \qquad [\because \vec{c} \parallel (\vec{a} \times \vec{b})]$$
$$= |\vec{a} \times \vec{b}|^2\ |\vec{c}|^2$$
$$= (|\vec{a}|\ |\vec{b}| \sin \pi/6)^2 \qquad \left[\because |\vec{c}| = 1 \text{ and the angle between } \vec{a} \text{ and } \vec{b} \text{ is } \pi/6\right]$$
$$= \frac{1}{4} |\vec{a}|^2\ |\vec{b}|^2.$$

EXAMPLE 13 *If* $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ *and the vectors* $\vec{A} = \hat{i} + a\hat{j} + a^2\hat{k}$, $\vec{B} = \hat{i} + b\hat{j} + b^2\hat{k}$, $\vec{C} = \hat{i} + c\hat{j} + c^2\hat{k}$ *are non-coplanar, then prove that* $abc = -1$.

[IIT 1985]

SOLUTION Since $\vec{A}, \vec{B}, \vec{C}$ are non-coplanar vectors. Therefore,

$$[\vec{A}\ \vec{B}\ \vec{C}] \neq 0$$
$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \qquad \ldots(i)$$

Now,

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$
$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$
$$\Rightarrow -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ c & c & c^2 \end{vmatrix} = 0 \qquad \left[\begin{array}{r}\text{Applying } C_2 \leftrightarrow C_3 \text{ in} \\ \text{Firstdet. and taking} \\ a, b, c \text{ common from} \\ C_1, C_2, C_3 \text{ of 2nd det.}\end{array}\right]$$
$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \qquad \left[\begin{array}{r}\text{Applying } C_2 \Leftrightarrow C_1 \\ \text{in first det.}\end{array}\right]$$
$$\Rightarrow (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$
$$\Rightarrow (1+abc)\ [\vec{A}\ \vec{B}\ \vec{C}] = 0$$
$$\Rightarrow 1 + abc = 0 \qquad [\because [\vec{A}\ \vec{B}\ \vec{C}] \neq 0]$$

EXAMPLE 14 *If* $\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 0$ *and* $\vec{X} = (x^2, x, 1)$, $\vec{Y} = (y^2, y, 1)$, $\vec{Z} = (z^2, z, 1)$ *are non-coplanars, then prove that* $\vec{a} = (a^2, a, 1)$, $\vec{b} = (b^2, b, 1)$, $\vec{c} = (c^2, c, 1)$ *are coplanar.*

SOLUTION Since $\vec{X} = (x^2, x, 1)$, $\vec{Y} = (y^2, y, 1)$ and $\vec{Z} = (z^2, z, 1)$ are non-coplanar vectors. Therefore,

$$[\vec{X}\ \vec{Y}\ \vec{Z}] \neq 0 \Rightarrow \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \neq 0.$$

Now,

$$\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 0$$
$$\Rightarrow \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2a & a^2 \\ 1 & -2b & b^2 \\ 1 & -2c & c^2 \end{vmatrix} = 0$$
$$\Rightarrow \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$
$$\Rightarrow [\vec{X}\ \vec{Y}\ \vec{Z}]\ [\vec{a}\ \vec{b}\ \vec{c}] = 0$$
$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \qquad [\because [\vec{X}\ \vec{Y}\ \vec{Z}] \neq 0]$$
$$\Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are coplanar.}$$

EXAMPLE 15 *Consider the vectors :*

$$\vec{A} = \hat{i} + \cos(\beta - \alpha)\hat{j} + \cos(\gamma - \alpha)\hat{k},$$
$$\vec{B} = \cos(\alpha - \beta)\hat{i} + \hat{j} + \cos(\gamma - \beta)\hat{k}$$
and, $$\vec{C} = \cos(\alpha - \gamma)\hat{i} + \cos(\beta - \gamma)\hat{j} + a\hat{k}$$

where α, β *and* γ *are different angles in* $(0, \pi/2)$. *If* $\vec{A}, \vec{B}, \vec{C}$ *are coplanar vectors, show that a is independent of* α, β *and* γ.

SOLUTION It is given that $\vec{A}, \vec{B}, \vec{C}$ are coplanar vectors. Therefore,

$$[\vec{A}\ \vec{B}\ \vec{C}] = 0$$
$$\Rightarrow \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha \\ \cos(\alpha-\beta & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & a \end{vmatrix} = 0$$
$$\Rightarrow \begin{vmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos^2\beta + \sin^2\beta & \cos\beta\cos\gamma + \sin\beta\sin\gamma \\ \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma + \sin\beta\sin\gamma & a \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & a-1 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix} = 0$$

$$\Rightarrow (a-1)(\cos\alpha\sin\beta - \sin\alpha\cos\beta)(\cos\alpha\sin\beta - \sin\alpha\cos\beta) = 0$$

$$\Rightarrow (a-1)\sin^2(\beta-\alpha) = 0$$

$$\Rightarrow a - 1 = 0$$

$$\Rightarrow a = 1 \qquad [\because \alpha \neq \beta \therefore \sin(\beta-\alpha) \neq 0]$$

Hence, a in dependent of α, β, γ.

EXAMPLE 16 *Show that the vectors :*

$$\vec{a} = 2\hat{i} + (\alpha+\beta+\gamma+\delta)\hat{j} + (\alpha\beta+\gamma\delta)\hat{k}$$

$$\vec{b} = (\alpha+\beta+\gamma+\delta)\hat{i} + 2(\alpha+\beta)(\gamma+\delta)\hat{j} + [\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta)]\hat{k}$$

$$\vec{c} = (\gamma\delta+\alpha\beta)\hat{i} + [\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta)]\hat{j} + 2\alpha\beta\gamma\delta\hat{k}$$

are coplanar.

SOLUTION We have,

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \gamma\delta+\alpha\beta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix}$$

$$= 0$$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

EXAMPLE 17 *If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{l}, \vec{m}, \vec{n}$ are two triads of vectors, prove that*

$$[\vec{a}\,\vec{b}\,\vec{c}][\vec{l}\,\vec{m}\,\vec{n}] = \begin{vmatrix} \vec{a}\cdot\vec{l} & \vec{b}\cdot\vec{l} & \vec{c}\cdot\vec{l} \\ \vec{a}\cdot\vec{m} & \vec{b}\cdot\vec{m} & \vec{c}\cdot\vec{m} \\ \vec{a}\cdot\vec{n} & \vec{b}\cdot\vec{n} & \vec{c}\cdot\vec{n} \end{vmatrix}$$

SOLUTION Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \quad \vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k},$$

$$\vec{m} = m_1\hat{i} + m_2 = \hat{j} + \hat{k}$$

and, $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

Then,

$$[\vec{a}\,\vec{b}\,\vec{c}][\vec{l}\,\vec{m}\,\vec{n}]$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 l_1 + a_2 l_2 + a_3 l_3 & a_1 m_1 + a_2 m_2 + a_3 m_3 & a_1 n_1 + a_2 n_2 + a_3 \\ b_1 l_1 + b_2 l_2 + b_3 l_3 & b_1 m_1 + b_2 m_2 + b_3 m_3 & b_1 n_1 + b_2 n_2 + b_3 \\ c_1 l_1 + c_2 l_2 + c_3 l_3 & c_1 m_1 + c_2 m_2 + c_3 m_3 & c_1 n_1 + c_2 n_2 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{l} & \vec{a}\cdot\vec{m} & \vec{a}\cdot\vec{n} \\ \vec{b}\cdot\vec{l} & \vec{b}\cdot\vec{m} & \vec{b}\cdot\vec{n} \\ \vec{c}\cdot\vec{l} & \vec{c}\cdot\vec{m} & \vec{c}\cdot\vec{n} \end{vmatrix} = \begin{vmatrix} \vec{a}\cdot\vec{l} & \vec{b}\cdot\vec{l} & \vec{c}\cdot\vec{l} \\ \vec{a}\cdot\vec{m} & \vec{b}\cdot\vec{m} & \vec{c}\cdot\vec{m} \\ \vec{a}\cdot\vec{n} & \vec{b}\cdot\vec{n} & \vec{c}\cdot\vec{n} \end{vmatrix}$$

EXAMPLE 18 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that*

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

SOLUTION Replace $\vec{l}, \vec{m}, \vec{n}$ by $\vec{a}, \vec{b}, \vec{c}$ respectively in Ex. 17.

EXAMPLE 19 *If $\vec{a}, \vec{b}, \vec{c}, \vec{l}, \vec{m}$ are vectors, prove that*

$$[\vec{a}\,\vec{b}\,\vec{c}]\,(\vec{l}\times\vec{m}) = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{l} & \vec{b}\cdot\vec{l} & \vec{c}\cdot\vec{l} \\ \vec{a}\cdot\vec{m} & \vec{b}\cdot\vec{m} & \vec{c}\cdot\vec{m} \end{vmatrix}$$

SOLUTION We have,

$$[\vec{a}\,\vec{b}\,\vec{c}]\,(\vec{l}\times\vec{m})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} i & j & k \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\hat{i}+a_2\hat{j}+a_3\hat{k} & a_1 l_1 + a_2 l_2 + a_3 l_3 & a_1 m_1 + a_2 m_2 + a_3 m_3 \\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k} & b_1 l_1 + b_2 l_2 + b_3 l_3 & b_1 m_1 + b_2 m_2 + b_3 m_3 \\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k} & c_1 l_1 + c_2 l_2 + c_3 l_3 & c_1 m_1 + c_2 m_2 + c_3 m_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a} & \vec{a}\cdot\vec{l} & \vec{a}\cdot\vec{m} \\ \vec{b} & \vec{b}\cdot\vec{l} & \vec{b}\cdot\vec{m} \\ \vec{c} & \vec{c}\cdot\vec{l} & \vec{c}\cdot\vec{m} \end{vmatrix} = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{l} & \vec{b}\cdot\vec{l} & \vec{c}\cdot\vec{l} \\ \vec{a}\cdot\vec{m} & \vec{b}\cdot\vec{m} & \vec{c}\cdot\vec{m} \end{vmatrix}$$

EXAMPLE 20 *If $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar unit vectors equally inclined to one another at an angle θ such that*

$$\vec{a}\times\vec{b} + \vec{b}\times\vec{c} = p\vec{a} + q\vec{b} + r\vec{c},$$

find p, q, r in terms of θ. Also, prove that $p^2 + \frac{q^2}{\cos\theta} + r^2 = 2$.

[IIT 1997]

SOLUTION We have,

$$p\vec{a} + q\vec{b} + r\vec{c} = \vec{a}\times\vec{b} + \vec{b}\times\vec{c}$$

Taking dot products with $\vec{a}, \vec{b}$ and $\vec{c}$ successively, we get

$$p(\vec{a}\cdot\vec{a}) + q(\vec{a}\cdot\vec{b}) + r(\vec{a}\cdot\vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}]$$

$$p(\vec{b}\cdot\vec{a}) + q(\vec{b}\cdot\vec{b}) + r(\vec{b}\cdot\vec{c}) = 0$$

$$p(\vec{c}\cdot\vec{a}) + q(\vec{c}\cdot\vec{b}) + r(\vec{c}\cdot\vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}]$$

Solving this system of euations by Cramer's rule, we get

$$p = \frac{D_1}{D}, q = \frac{D_2}{D}, r = \frac{D_3}{D}, \text{ where}$$

$$D_1 = \begin{vmatrix} [\vec{a}\,\vec{b}\,\vec{c}] & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ 0 & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ [\vec{a}\,\vec{b}\,\vec{c}] & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \vec{a}.\vec{a} & [\vec{a}\,\vec{b}\,\vec{c}] & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & 0 & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & [\vec{a}\,\vec{b}\,\vec{c}] & \vec{c}.\vec{c} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & [\vec{a}\,\vec{b}\,\vec{c}] \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & 0 \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & [\vec{a}\,\vec{b}\,\vec{c}] \end{vmatrix},$$

$$D = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

We have,

$$\vec{a}.\vec{a} = |\vec{a}|^2 = 1,\ \vec{b}.\vec{b} = |\vec{b}|^2 = 1,\ \vec{c}.\vec{c} = |\vec{c}|^2 = 1$$

$$\vec{a}.\vec{b} = \vec{b}.\vec{a} = \cos\theta,\ \vec{a}.\vec{c} = \vec{c}.\vec{a} = \cos\theta,\ \vec{b}.\vec{c} = \vec{c}.\vec{b} = \cos\theta$$

$$\therefore\ D = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1-\cos^2\theta) - \cos\theta(\cos\theta - \cos^2\theta) + \cos\theta(\cos^2\theta - \cos\theta)$$

$$= 1 - 3\cos^2\theta + 2\cos^3\theta$$

$$= (\cos\theta - 1)^2(2\cos\theta + 1)$$

$$= (1-\cos\theta)^2(2\cos\theta + 1)$$

We know that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

$$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]^2 = D$$

$$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = \sqrt{D}$$

$$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = (1-\cos\theta)\sqrt{2\cos\theta + 1} \qquad \ldots(i)$$

Now,

$$D_1 = \begin{vmatrix} [\vec{a}\,\vec{b}\,\vec{c}] & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ 0 & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ [\vec{a}\,\vec{b}\,\vec{c}] & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] \begin{vmatrix} 1 & 1 & \cos\theta \\ \cos\theta & 0 & 0 \\ \cos\theta & 1 & \cos\theta \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[(1-\cos^2\theta) + \cos^2\theta - \cos\theta]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,(1-\cos\theta)$$

$$\Rightarrow D_1 = (1-\cos\theta)^2\sqrt{2\cos\theta + 1} \qquad \text{[Using (i)]}$$

$$D_2 = \begin{vmatrix} \vec{a}.\vec{a} & [\vec{a}\,\vec{b}\,\vec{c}] & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & 0 & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & [\vec{a}\,\vec{b}\,\vec{c}] & \vec{c}.\vec{c} \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] \begin{vmatrix} 1 & 1 & \cos\theta \\ \cos\theta & 0 & 0 \\ \cos\theta & 1 & \cos\theta \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[-\cos\theta - (\cos\theta - \cos^2\theta) + \cos^2\theta]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,(-2\cos\theta + 2\cos^2\theta)$$

$$= -2\cos\theta(1-\cos\theta)(1-\cos\theta)\sqrt{2\cos\theta + 1} \qquad \text{[Using (i)]}$$

$$= -2\cos\theta(1-\cos\theta)^2\sqrt{2\cos\theta + 1}$$

$$D_3 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & [\vec{a}\,\vec{b}\,\vec{c}] \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & 0 \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & [\vec{a}\,\vec{b}\,\vec{c}] \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] \begin{vmatrix} 1 & \cos\theta & 1 \\ \cos\theta & 1 & 0 \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,(1 - \cos^2\theta + \cos^2\theta - \cos\theta)$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,(1-\cos\theta)$$

$$= (1-\cos\theta)^2\sqrt{2\cos\theta + 1}$$

$$\therefore\ p = \frac{D_1}{D} = \frac{(1-\cos\theta)^2\sqrt{2\cos\theta+1}}{(1-\cos\theta)^2(2\cos\theta+1)} = \frac{1}{\sqrt{2\cos\theta+1}}$$

$$q = \frac{D_2}{D} = \frac{-2\cos\theta(1-\cos\theta)^2\sqrt{2\cos\theta+1}}{(1-\cos\theta)^2(2\cos\theta+1)} = -\frac{2\cos\theta}{\sqrt{2\cos\theta+1}}$$

$$r = \frac{D_3}{D} = \frac{(1-\cos\theta)^2\sqrt{(2\cos\theta+1)}}{(1-\cos\theta)^2(2\cos\theta+1)} = \frac{1}{\sqrt{2\cos\theta+1}}$$

Now,

$$p^2 + \frac{q^2}{\cos\theta} + r^2 = \frac{1}{1+2\cos\theta} + \frac{4\cos\theta}{2\cos\theta+1} + \frac{1}{2\cos\theta+1}$$

$$= \frac{2(2\cos\theta+1)}{2\cos\theta+1} = 2$$

Hence, $p^2 + \dfrac{q^2}{\cos\theta} + r^2 = 2$.

EXAMPLE 21 *Show that the vectors* $\alpha\hat{i} + 3\hat{j} - \hat{k}$, $\hat{i} + (\alpha-1)\hat{j} + 2\hat{k}$ *and* $3\hat{i} + 5\hat{j} + 2\hat{k}$ *give rise to two distinct planes for suitable value of* α. *Find the angle between the planes.*

SOLUTION Let $\vec{a} = \alpha\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} + (\alpha-1)\hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 5\hat{j} + 2\hat{k}$ be the given vectors. These vectors will be coplanar, if

$$[\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow \begin{vmatrix} \alpha & 3 & -1 \\ 1 & \alpha-1 & 2 \\ 3 & 5 & 2 \end{vmatrix} = 0$$

$\Rightarrow \alpha(2\alpha-12)-3(2-6)-(5-3\alpha+3)=0$

$\Rightarrow \alpha(2\alpha-12)+12-(8-3\alpha)=0$

$\Rightarrow 2\alpha^2-9\alpha+4=0$

$\Rightarrow (2\alpha-1)(\alpha-4)=0$

$\Rightarrow \alpha = 4, \frac{1}{2}.$

For $\alpha = 4$, we have

$\vec{a}=4\hat{i}+3\hat{j}-\hat{k}, \vec{b}=\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{c}=3\hat{i}+5\hat{j}+2\hat{k}$

A vector normal to the plane containing these vectors is

$$\vec{n_1}=\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&3&2\\3&5&2\end{vmatrix}=-4\hat{i}+4\hat{j}-4\hat{k}$$

For $\alpha = \frac{1}{2}$, we have

$\vec{a}=\frac{1}{2}\hat{i}+3\hat{j}-\hat{k}, \vec{b}=\hat{i}-\frac{1}{2}\hat{j}+2\hat{k}$ and $\vec{c}=3\hat{i}+5\hat{j}+2\hat{k}.$

Let $\vec{n_2}$ be a normal vector perpendicular to the plane containing these vectors. Then,

$$\vec{n_2}=\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1/2&2\\3&5&2\end{vmatrix}=-11\hat{i}+4\hat{j}+\frac{13}{2}\hat{k}$$

Let θ be the angle between the planes. Then,

$$\cos\theta=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}=\hat{n}_1\cdot\hat{n}_2$$

$$=\frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}-\hat{k})\cdot\frac{(-22\hat{i}+8\hat{j}+13\hat{k})}{\sqrt{717}}$$

$$=\frac{(-22+8-13)}{3\sqrt{239}}=\frac{17}{3\sqrt{239}}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{17}{3\sqrt{239}}\right)$$

EXAMPLE 22 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, prove that any vector $\vec{r}$ is expressible as*

$$\vec{r}=\left|\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{a}+\left|\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{b}+\left|\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{c}.$$

SOLUTION Since $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors. Therefore, there exist scalars x, y, z such that

$$\vec{r}=x\vec{a}+y\vec{b}+z\vec{c} \qquad ...(i)$$

Taking dot products with $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ successively, we get

$$\vec{r}\cdot(\vec{b}\times\vec{c})=(x\vec{a}+y\vec{b}+z\vec{c})\cdot(\vec{b}\times\vec{c})$$

$$\vec{r}\cdot(\vec{c}\times\vec{a})=(x\vec{a}\times y\vec{b}+z\vec{c})\cdot(\vec{c}\times\vec{a})$$

$$\vec{r}\cdot(\vec{a}\times\vec{b})=(x\vec{a}+y\vec{b}+z\vec{c})\cdot(\vec{a}\times\vec{b})$$

$$\Rightarrow \quad [\vec{r}\,\vec{b}\,\vec{c}]=x[\vec{a}\,\vec{b}\,\vec{c}]$$

$$[\vec{r}\,\vec{c}\,\vec{a}]=y[\vec{b}\,\vec{c}\,\vec{a}]$$

and, $[\vec{r}\,\vec{a}\,\vec{b}]=z[\vec{c}\,\vec{a}\,\vec{b}]$

$$\Rightarrow \quad x=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, y=\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ and } z=\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Substituting the values of x, y, z in (i), we get

$$\vec{r}=\left|\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{a}+\left|\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{b}+\left|\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right|\vec{c}$$

EXAMPLE 23 *Determine α such that a vector $\vec{r}$ is at right angles to each of the vectors*

$\vec{a}=\alpha\hat{i}+\hat{j}+3\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\alpha\hat{k}, \vec{c}=-2\hat{i}+\alpha\hat{j}+3\hat{k}.$

SOLUTION Since $\vec{r}$ is at right angles to each of the vectors $\vec{a}, \vec{b}, \vec{c}$. Therefore, $\vec{a}, \vec{b}, \vec{c}$ must be coplanar vectors.

$$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}]=0 \Rightarrow \begin{vmatrix}\alpha&1&3\\2&1&-\alpha\\-2&\alpha&3\end{vmatrix}=0$$

$$\Rightarrow \alpha^3+11\alpha=0 \Rightarrow \alpha=0.$$

EXAMPLE 24 *If $\vec{A}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}, \vec{B}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}, \vec{C}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$, prove that*

$$[\vec{A}\,\vec{B}\,\vec{C}]=\begin{vmatrix}x_1&z_1&y_1\\x_2&z_2&y_2\\x_3&z_3&y_3\end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}]$$

SOLUTION Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then,

$$\vec{A}=(x_1a_1+y_1b_1+z_1c_1)\hat{i}+(x_1a_2+y_1b_2+z_1c_2)\hat{j}+(x_1a_3+y_1b_3+z_1c_3)\hat{k},$$

$$\vec{B}=(x_2a_1+y_2b_1+z_2c_1)\hat{i}+(x_2a_2+y_2b_2+z_2c_2)\hat{j}+(x_2a_3+y_2b_3+z_2c_3)\hat{k}$$

and,

$$\vec{C}=(x_3a_1+y_3b_1+z_3c_1)\hat{i}+(x_3a_2+y_3b_2+z_3c_2)\hat{j}+(x_3a_3+y_3b_3+z_3c_3)\hat{k}.$$

$$\therefore \quad [\vec{A}\,\vec{B}\,\vec{C}]$$

$$=\begin{vmatrix}x_1a_1+y_1b_1+z_1c_1&x_1a_2+y_1b_2+z_1c_2&x_1a_3+y_1b_3+z_1c_3\\x_2a_1+y_2b_1+z_2c_1&x_2a_2+y_2b_2+z_2c_2&x_2a_3+y_2b_3+z_2c_3\\x_3a_1+y_3b_1+z_3c_1&x_3a_2+y_3b_2+z_3c_2&x_3a_3+y_3b_3+z_3c_3\end{vmatrix}$$

$$=\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

EXAMPLE 25 *If* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar vectors, express* $\vec{a}, \vec{b}, \vec{c}$ *in terms of* $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$. *Also express* $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$ *in terms of* $\vec{a}, \vec{b}, \vec{c}$.

SOLUTION We know that

$$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$$

$$\Leftrightarrow [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] \neq 0$$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$\Leftrightarrow \vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are non-coplanar

We also know that any vector in space can be expressed as a linear combination of any three non-coplanar vectors. So, let

$$\vec{a} = l(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b})$$

Taking dot product on both sides successively by $\vec{a}, \vec{b}, \vec{c}$, we get

$$\vec{a}.\vec{a} = l\{\vec{a}.(\vec{b}\times\vec{c})\}, \vec{a}.\vec{b} = m\{(\vec{c}\times\vec{a}).\vec{b}\}$$

and, $\vec{c}.\vec{a} = n\{\vec{c}.(\vec{a}\times\vec{b})\}$

$$\Rightarrow \quad l = \frac{\vec{a}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, m = \frac{\vec{a}.\vec{b}}{[\vec{c}\ \vec{a}\ \vec{b}]} = \frac{\vec{a}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

and, $\quad n = \frac{\vec{c}.\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]} = \frac{\vec{c}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$\therefore \quad \vec{a} = \frac{\vec{a}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{a}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{c}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

Similarly,

$$\vec{b} = \frac{\vec{b}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{b}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{b}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

and,

$$\vec{c} = \frac{\vec{c}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{c}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{c}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

In order to express $\vec{b}\times\vec{c}$ in terms of $\vec{a}, \vec{b}, \vec{c}$, let us assume that

$$\vec{b}\times\vec{c} = x\vec{a} + y\vec{b} + z\vec{c}$$

Taking dot product on both sides successively by $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ and $(\vec{a}\times\vec{b})$, we get

$$(\vec{b}\times\vec{c}).(\vec{b}\times\vec{c}) = x\{\vec{a}.(\vec{b}\times\vec{c})\}$$

$$(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a}) = y\{\vec{b}.(\vec{c}\times\vec{a})\}$$

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{b}) = z\{\vec{c}.(\vec{a}\times\vec{b})\}$$

$$\Rightarrow x = \frac{(\vec{b}\times\vec{c}).(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}, y = \frac{(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]},$$

$$z = \frac{(\vec{b}\times\vec{c}).(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\therefore \quad \vec{b}\times\vec{c} = \frac{(\vec{b}\times\vec{c}).(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a} + \frac{(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b} + \frac{(\vec{b}\times\vec{c}).(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}$$

Similarly,

$$\vec{c}\times\vec{a} = \frac{(\vec{c}\times\vec{a}).(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a} + \frac{(\vec{c}\times\vec{a}).(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b} + \frac{(\vec{c}\times\vec{a}).(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}$$

and,

$$\vec{a}\times\vec{b} = \frac{(\vec{a}\times\vec{b}).(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a} + \frac{(\vec{a}\times\vec{b}).(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b} + \frac{(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}$$

EXAMPLE 26 *If* $\vec{a}, \vec{b}, \vec{c}$ *are three given non-coplanar vectors, show that any arbitrary vector* $\vec{r}$ *in space is expressible in the form*

$$\vec{r} = \frac{\Delta_1}{\Delta}\vec{a} + \frac{\Delta_2}{\Delta}\vec{b} + \frac{\Delta_3}{\Delta}\vec{c},$$

where

$$\Delta_1 = \begin{vmatrix} \vec{r}.\vec{a} & \vec{b}.\vec{a} & \vec{c}.\vec{a} \\ \vec{r}.\vec{b} & \vec{b}.\vec{b} & \vec{c}.\vec{b} \\ \vec{r}.\vec{c} & \vec{b}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{r}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{r}.\vec{b} & \vec{c}.\vec{b} \\ \vec{a}.\vec{c} & \vec{r}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{b}.\vec{a} & \vec{r}.\vec{a} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} & \vec{r}.\vec{b} \\ \vec{a}.\vec{c} & \vec{b}.\vec{c} & \vec{r}.\vec{c} \end{vmatrix},$$

$$\Delta = \begin{vmatrix} \vec{a}.\vec{a} & \vec{b}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} & \vec{c}.\vec{b} \\ \vec{a}.\vec{c} & \vec{b}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix}$$

SOLUTION Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, therefore

$$[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]^2 \neq 0$$

$$\Rightarrow \Delta = \begin{vmatrix} \vec{a}.\vec{a} & \vec{b}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} & \vec{c}.\vec{b} \\ \vec{a}.\vec{c} & \vec{b}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix} = [\vec{a}\ \vec{b}\ \vec{c}]^2 \neq 0$$

Since any vector $\vec{r}$ in space can be expressed as a linear combination of three non-coplanar vectors. So, let

$$\vec{r} = l\vec{a} + m\vec{b} + n\vec{c} \quad \text{...(i)}$$

Taking dot products on both sides successively by $\vec{a}, \vec{b}, \vec{c}$, we get

$$\vec{r}\cdot\vec{a} = l(\vec{a}\cdot\vec{a}) + m(\vec{b}\cdot\vec{a}) + n(\vec{c}\cdot\vec{a}) \quad \text{...(ii)}$$

$$\vec{r}\cdot\vec{b} = l(\vec{a}\cdot\vec{b}) + m(\vec{b}\cdot\vec{b}) + n(\vec{c}\cdot\vec{b}) \quad \text{...(iii)}$$

and, $$\vec{r}\cdot\vec{c} = l(\vec{a}\cdot\vec{c}) + m(\vec{b}\cdot\vec{c}) + n(\vec{c}\cdot\vec{c}) \quad \text{...(iv)}$$

Eliminating l, m, n from these four relations, we get

$$\begin{vmatrix} \vec{r} & \vec{a} & \vec{b} & \vec{c} \\ \vec{r}\cdot\vec{a} & \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} & \vec{c}\cdot\vec{a} \\ \vec{r}\cdot\vec{b} & \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{c}\cdot\vec{b} \\ \vec{r}\cdot\vec{c} & \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix} = 0$$

On expanding this determinant along first row, we obtain the desired result.

EXAMPLE 27 *Let V be the volume of the parallelopiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_i\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r, c_r, where $r = 1, 2, 3$ are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$.*

[IIT 2002]

SOLUTION We have,

V = Absolute value of $[\vec{a}\,\vec{b}\,\vec{c}]$

$$= \text{Absolute value of } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= |(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)|$$

$$= \pm\{(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)\}$$

Without any loss of generality, let us assume that

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \ge a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1$$

$$\therefore\ V = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$$

$$\Rightarrow V \le a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \quad \text{...(i)} \quad [\because a_i, b_i, c_i \ge 0]$$

Now,

$$\text{A.M} \ge \text{G.M.}$$

$$\Rightarrow \frac{a_1 + b_2 + c_3}{3} \ge (a_1b_2c_3)^{1/3},$$

$$\frac{a_2 + b_3 + c_1}{3} \ge (a_2b_3c_1)^{1/3}$$

and, $$\frac{a_3 + b_1 + c_2}{3} \ge (a_3b_1c_2)^{1/3}$$

$$\Rightarrow (a_1b_2c_3) \le \left\{\frac{a_1 + b_2 + c_3}{3}\right\}^3$$

$$(a_2b_3c_1) \le \left\{\frac{a_2 + b_3 + c_1}{3}\right\}^3$$

and, $$(a_3b_1c_2) \le \left\{\frac{a_3 + b_1 + c_2}{3}\right\}^3$$

$$\Rightarrow a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \le \frac{1}{27}\left\{(a_1 + b_2 + c_3)^3 + (a_2 + b_3 + c_1)^3 + (a_3 + b_1 + c_2)^3\right\}$$

$$\Rightarrow a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \le \frac{1}{27}\left\{(a_1 + b_2 + c_3) + (a_2 + b_3 + c_1) + (a_3 + b_1 + c_2)^3\right\}^3$$

$$\left[\because x^3 + y^3 + z^3 \le (x + y + z)^3 \text{ for all } x, y, z \ge 0\right]$$

$$\Rightarrow a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \le \frac{1}{27}\{(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3)\}^3$$

$$\Rightarrow a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \le \frac{1}{27}(3L)^3 \qquad \left[\because \sum_{r=1}^{3}(a_r + b_r + c_r) = 3L\right]$$

$$\Rightarrow a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \le L^3 \quad \text{...(ii)}$$

From (i) and (ii), we get

$V \le L^3$.

EXAMPLE28 *The vector $\vec{OP} = 5\hat{i} + 12\hat{j} + 13\hat{k}$ turns through an angle $\pi/2$ about O passing through the positive side of the $\hat{j}$ axis on its way. Find the vector in the new position.*

SOLUTION Let $\vec{OQ} = x\hat{i} + y\hat{j} + z\hat{k}$ be the vector in the new position. Then,

$$|\vec{OP}| = |\vec{OQ}| \Rightarrow x^2 + y^2 + z^2 = 338 \quad \text{...(i)}$$

Also, $\vec{OP} \perp \vec{OQ}$.

$$\therefore\ \vec{OP}\cdot\vec{OQ} = 0 \Rightarrow 5x + 12y + 13z = 0 \quad \text{...(ii)}$$

Clearly, $\vec{OP}, \hat{j}$ and $\vec{OQ}$ are coplanar.

$$\therefore\ [\vec{OP}\ \hat{j}\ \vec{OQ}] = 0$$

$$\Rightarrow \begin{vmatrix} 5 & 12 & 13 \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = 0$$

$\Rightarrow -13x+5z=0 \Rightarrow z=\frac{13}{5}x$ ...(iii)

From (ii) and (iii), we get $y=-\frac{97}{30}x$.

Putting $z=\frac{13}{5}x, y=-\frac{97}{30}x$ in (i), we get

Putting $z=\frac{13}{5}x, y=-\frac{97}{30}x$ in (i), we get $x=\pm\sqrt{\frac{2}{97}}$

Since the vector in its new position makes an obtuse angle with X-axis. Therefore, $x=-30\sqrt{\frac{2}{97}}$.

Hence, $\overrightarrow{OQ}=\frac{2}{\sqrt{97}}(-30\hat{i}+97\hat{j}-78\hat{k})$.

—❑❑❑—

17.6 VECTOR TRIPLE PRODUCT

DEFINITION *Let $\vec{a},\vec{b},\vec{c}$ be any three vectors, then the vectors $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple products of $\vec{a},\vec{b},\vec{c}$.*

THEOREM *For any three vectors $\vec{a},\vec{b},\vec{c}$, we have*

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}$$

PROOF Let $\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$. Since cross-product of two vectors is a vector perpendicular to both the vectors. Therefore,

$\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$

$\Rightarrow \vec{r}\perp\vec{a}$ and $\vec{r}\perp\vec{b}\times\vec{c}$ ...(i)

But, $(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Therefore,

$\vec{r}\perp(\vec{b}\times\vec{c})$

$\Rightarrow \vec{r}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow \vec{r}$ is expressible as a linear combination of $\vec{b}$ and $\vec{c}$

$\Rightarrow$ There exist scalars x, y such that

$\vec{r}=x\vec{b}+y\vec{c}$ ...(ii)

Now,

$\vec{r}\perp\vec{a}$

$\Rightarrow \vec{r}\cdot\vec{a}=0$

$\Rightarrow (x\vec{b}+y\vec{c})\cdot\vec{a}=0$ $[\because \vec{r}=x\vec{b}+y\vec{c}]$

$\Rightarrow x(\vec{b}\cdot\vec{a})+y(\vec{c}\cdot\vec{a})=0$

$\Rightarrow x(\vec{a}\cdot\vec{b})+y(\vec{a}\cdot\vec{c})=0$

$\Rightarrow x(\vec{a}\cdot\vec{b})=-y(\vec{a}\cdot\vec{c})$

$\Rightarrow \frac{x}{(\vec{a}\cdot\vec{c})}=\frac{y}{-(\vec{a}\cdot\vec{b})}=\lambda$ (say)

$\Rightarrow x=\lambda(\vec{a}\cdot\vec{c}), y=-\lambda(\vec{a}\cdot\vec{b})$.

Substituting the values of x, y in (ii), we get

$\vec{r}=\lambda(\vec{a}\cdot\vec{c})\vec{b}-\lambda(\vec{a}\cdot\vec{b})\vec{c}$

$\Rightarrow \vec{r}=\lambda\{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\}$

$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c})=\lambda[(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}]$ ...(iii)

This is a vector identity and so it is true for all $\vec{a},\vec{b},\vec{c}$. Taking $\vec{a}=\hat{i},\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$, we have

$\vec{a}\times(\vec{b}\times\vec{c})$

$=\hat{i}\times\{(\hat{i}+\hat{j})\times(\hat{i}+\hat{j}+\hat{k})\}$

$=\hat{i}\times\{\hat{i}\times\hat{i}+\hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{i}+\hat{j}\times\hat{j}+\hat{j}\times\hat{k}\}$

$=\hat{i}\times\{\hat{k}-\hat{j}-\hat{k}+\hat{i}\}=\hat{i}\times(\hat{i}-\hat{j})=-\hat{i}\times\hat{j}=-\hat{k}$

and, $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$=\{\hat{i}\cdot(\hat{i}+\hat{j}+\hat{k})\}(\hat{i}+\hat{j})-\{\hat{i}\cdot(\hat{i}+\hat{j})\}(\hat{i}+\hat{j}+\hat{k})$

$=(\hat{i}+\hat{j})-(\hat{i}+\hat{j}+\hat{k})=-\hat{k}$

Substituting these values in (iii), we obtain

$-\hat{k}=\lambda[-\hat{k}]\Rightarrow\lambda=1.$

Putting $\lambda=1$ in (iii), we obtain

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}$$

AID TO MEMORY: *$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ can also be written as $I\times(II\times III)=(I\cdot III)\,II-(I\cdot II)\,III$.*

REMARK 1 *The vector triple product $\vec{a}\times(\vec{b}\times\vec{c})$ is a linear combination of those two vectors which are within brackets.*

REMARK 2 *The vector $\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and lies in the plane of $\vec{b}$ and $\vec{c}$.*

REMARK 3 The formula $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ *is true only when the vector outside the bracket is on the left-most side. If it is not we first shift on left by using the properties of cross-product and then apply the same formula. For example,*

$(\vec{b}\times\vec{c})\times\vec{a}=-\{\vec{a}\times(\vec{b}\times\vec{c})\}$

$=-\{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\}$

$=-(\vec{a}\cdot\vec{c})\vec{b}+(\vec{a}\cdot\vec{b})\vec{c}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that :*

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}.$$

SOLUTION We have,

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})\\&=\{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\}+\{(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}\}\\&\qquad+\{(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}\}\\&=\{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\}+\{(\vec{a}\cdot\vec{b})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}\}\\&\qquad+\{(\vec{b}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\vec{b}\}\\&=\vec{0}\end{aligned}$$

EXAMPLE 2 *Show that the vectors* $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ *and* $\vec{c}\times(\vec{a}\times\vec{b})$ *are coplanar.*

SOLUTION Let $\vec{p}=\vec{a}\times(\vec{b}\times\vec{c}),\ \vec{q}=\vec{b}\times(\vec{c}\times\vec{a})$

and, $\vec{r}=\vec{c}\times(\vec{a}\times\vec{b}).$

Then,

$$\vec{p}+\vec{q}+\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$$

$$\Rightarrow\quad \vec{p}=(-1)\vec{q}+(-1)\vec{r}$$

$\Rightarrow$ $\vec{p}, \vec{q}, \vec{r}$ are coplanar

Hence, $\vec{a}\times(\vec{b}\times\vec{c}),\ \vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.

EXAMPLE 3 *For any vector* $\vec{a}$, *prove that*

$$\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}.$$

SOLUTION Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then,

$$\begin{aligned}&\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})\\&=\{(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}\}+\{(\hat{j}\cdot\hat{j})\vec{a}-(\hat{j}\cdot\vec{a})\hat{j}\}+\{(\hat{k}\cdot\hat{k})\vec{a}-(\hat{k}\cdot\vec{a})\hat{k}\}\\&=\{\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}\}+\{\vec{a}-(\hat{j}\cdot\vec{a})\hat{j}\}+\{\vec{a}-(\hat{k}\cdot\vec{a})\hat{k}\}\\&=3\vec{a}-\{(\hat{i}\cdot\vec{a})\hat{i}+(\hat{j}\cdot\vec{a})\hat{j}+(\hat{k}\cdot\vec{a})\hat{k}\}\\&=3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\quad[\because\ \hat{i}\cdot\vec{a}=a_1,\hat{j}\cdot\vec{a}=a_2\text{ and }\hat{k}\cdot\vec{a}=a_3]\\&=3\vec{a}-\vec{a}=2\vec{a}\end{aligned}$$

EXAMPLE 4 *Prove that* $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]^2$.

SOLUTION We have,

$$\begin{aligned}&[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]\\&=\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\cdot(\vec{c}\times\vec{a})\\&=\{\vec{d}\times(\vec{b}\times\vec{c})\}.(\vec{c}\times\vec{a}),\text{ where }\vec{d}=\vec{a}\times\vec{b}\\&=\{(\vec{d}\cdot\vec{c})\vec{b}-(\vec{d}\cdot\vec{b})\vec{c}\}\cdot(\vec{c}\times\vec{a})\\&=\{((\vec{a}\times\vec{b})\cdot\vec{c})\vec{b}-((\vec{a}\times\vec{b})\cdot\vec{b})\vec{c}\}\cdot(\vec{c}\times\vec{a})\\&=\{[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}-[\vec{a}\,\vec{b}\,\vec{b}]\vec{c}\}\cdot(\vec{c}\times\vec{a})\\&=\{[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}-(0)\vec{c}\}\cdot(\vec{c}\times\vec{a})\qquad[\because\ [\vec{a}\,\vec{b}\,\vec{b}]=0]\\&=[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}\cdot(\vec{c}\times\vec{a})\\&=[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{b}\cdot\{\vec{c}\times\vec{a}\}\}\\&=[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{b}\,\vec{c}\,\vec{a}]\\&=[\vec{a}\,\vec{b}\,\vec{c}]^2\end{aligned}$$

EXAMPLE 5 *Show that* $\vec{a}, \vec{b}, \vec{c}$ *are coplanar iff* $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ *are coplanar.*

PROOF $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}]=0$$

$$\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}]^2=0$$

$$\Leftrightarrow [\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}]=0$$

$\Leftrightarrow$ $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are coplanar

EXAMPLE 6 *Prove that:*

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}.$$

SOLUTION We have,

$$\begin{aligned}&(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\\&=\vec{r}\times(\vec{c}\times\vec{d}),\text{ where }\vec{r}=\vec{a}\times\vec{b}\\&=(\vec{r}\cdot\vec{d})\vec{c}-(\vec{r}\cdot\vec{c})\vec{d}\\&=\{(\vec{a}\times\vec{b})\cdot\vec{d}\}\vec{c}-\{(\vec{a}\times\vec{b})\cdot\vec{c}\}\vec{d}\\&=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}.\end{aligned}$$

EXAMPLE 7 *Prove that :*

$$\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}=(\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c})-(\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d}).$$

SOLUTION We have,

$$\begin{aligned}&\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}\\&=\vec{a}\times\{(\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}\}\\&=\vec{a}\times\{(\vec{b}\cdot\vec{d})\vec{c}\}-\vec{a}\times\{(\vec{b}\cdot\vec{c})\vec{d}\}\\&=(\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c})-(\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d})\end{aligned}$$

EXAMPLE 8 *Prove that :*

$$(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})\ \ iff\ \ (\vec{c}\times\vec{a})\times\vec{b}=\vec{0}.$$

SOLUTION We have,

$$\begin{aligned}&(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})\\\Leftrightarrow\quad&-\vec{c}\times(\vec{a}\times\vec{b})=\vec{a}\times(\vec{b}\times\vec{c})\\\Leftrightarrow\quad&-\{(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}\}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\\\Leftrightarrow\quad&-(\vec{c}\cdot\vec{b})\vec{a}+(\vec{a}\cdot\vec{c})\vec{b}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\end{aligned}$$

$\Leftrightarrow \quad -(\vec{c}\cdot\vec{b})\vec{a} = -(\vec{a}\cdot\vec{b})\vec{c}$

$\Leftrightarrow \quad (\vec{b}\cdot\vec{c})\vec{a} = (\vec{b}\cdot\vec{a})\vec{c}$

$\Leftrightarrow \quad (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} = \vec{0}$

$\Leftrightarrow \quad \vec{b}\times(\vec{c}\times\vec{a}) = \vec{0}$

$\Leftrightarrow \quad -(\vec{b}\times(\vec{c}\times\vec{a})) = \vec{0}$

$\Leftrightarrow \quad (\vec{c}\times\vec{a})\times\vec{b} = \vec{0}$

EXAMPLE 9 *Show that :*

$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ iff $\vec{a}$ and $\vec{c}$ are collinear.

SOLUTION We have,

$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$$

$\Leftrightarrow \quad -\{(\vec{c}\times(\vec{a}\times\vec{b})\} = \vec{a}\times(\vec{b}\times\vec{c})$

$\Leftrightarrow \quad -\{(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\Leftrightarrow \quad -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{c})\vec{b} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\Leftrightarrow \quad -(\vec{c}\cdot\vec{b})\vec{a} = -(\vec{a}\cdot\vec{b})\vec{c}$

$\Leftrightarrow \quad (\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{b})\vec{c}$

$\Leftrightarrow \quad \vec{a} = \dfrac{(\vec{a}\cdot\vec{b})}{(\vec{c}\cdot\vec{b})}\vec{c}$

$\Leftrightarrow \quad \vec{a} = \lambda\vec{c}$, where $\lambda = \dfrac{\vec{a}\cdot\vec{b}}{\vec{c}\cdot\vec{b}}$

$\Leftrightarrow \vec{a}$ and $\vec{c}$ are collinear.

EXAMPLE 10 *If $\vec{p}, \vec{q}, \vec{r}$ denote the vectors $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$ respectively, show that $\vec{a}$ is parallel to $\vec{q}\times\vec{r}$, $\vec{b}$ to $\vec{r}\times\vec{p}$, $\vec{c}$ to $\vec{p}\times\vec{q}$ respectively.*

SOLUTION We have,

$\vec{a}\times(\vec{q}\times\vec{r})$

$= (\vec{a}\cdot\vec{r})\vec{q} - (\vec{a}\cdot\vec{q})\vec{r}$

$= \{\vec{a}\cdot(\vec{a}\times\vec{b})\}\vec{q} - \{\vec{a}\cdot(\vec{c}\times\vec{a})\}\vec{r}$

$= [\vec{a}\ \vec{a}\ \vec{b}]\vec{q} - [\vec{a}\ \vec{c}\ \vec{a}]\vec{r}$

$= 0\vec{q} - 0\vec{r} = \vec{0}$

So, $\vec{a}$ is parallel to $\vec{q}\times\vec{r}$.

Similarly, $\vec{b} \parallel \vec{r}\times\vec{p}$ and $\vec{c} \parallel \vec{p}\times\vec{q}$.

EXAMPLE 11 *If $\vec{a}, \vec{b}, \vec{c}$ be three non-parallel unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.*

SOLUTION We have,

$\vec{a}\times(\vec{b}\times\vec{c}) = \dfrac{1}{2}\vec{b}$

$\Rightarrow \quad (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \dfrac{1}{2}\vec{b}$

$\Rightarrow \quad \left(\vec{a}\cdot\vec{c} - \dfrac{1}{2}\right)\vec{b} = (\vec{a}\cdot\vec{b})\vec{c}$

$\Rightarrow \quad \vec{a}\cdot\vec{c} - \dfrac{1}{2} = 0$ and $\vec{a}\cdot\vec{b} = 0$ $\quad [\because \vec{b}$ and $\vec{c}$ are non-parallel]

$\Rightarrow \quad \vec{a}\cdot\vec{c} = \dfrac{1}{2}$ and $\vec{a}\perp\vec{b}$.

Suppose $\vec{a}$ makes angle θ with $\vec{c}$. Then,

$\vec{a}\cdot\vec{c} = \dfrac{1}{2}$

$\Rightarrow \quad |\vec{a}|\ |\vec{c}|\cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{2}$ $\quad [\because |\vec{a}| = |\vec{c}| = 1]$

$\Rightarrow \quad \theta = \pi/3$

Thus, $\vec{a}$ is perpendicular to $\vec{b}$ and makes an angle $\dfrac{\pi}{3}$ with $\vec{c}$.

EXAMPLE 12 *Prove that* $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$.

SOLUTION We have,

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

$= (\vec{a}\times\vec{b})\cdot\vec{r}$, where $\vec{r} = \vec{c}\times\vec{d}$

$= \vec{a}\cdot(\vec{b}\times\vec{r})$ [Interchanging dot and cross]

$= \vec{a}\cdot\{\vec{b}\times(\vec{c}\times\vec{d})\}$ $\quad [\because \vec{r} = \vec{c}\times\vec{d}]$

$= \vec{a}\cdot\{(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}\}$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

EXAMPLE 13 *If $\vec{a} = -\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 0\hat{j} + \hat{k}$, then find the vector $\vec{X}$ satisfying the conditions:*

(i) that it is coplanar with $\vec{a}$ and $\vec{b}$ (ii) that is perpendicular to $\vec{b}$, (iii) that $\vec{a}\cdot\vec{X} = 7$.

SOLUTION Since $\vec{X}$ is in the plane of $\vec{a}$ and $\vec{b}$ and is perpendicular to $\vec{b}$. Therefore,

$\vec{X} = \lambda\{\vec{b}\times(\vec{a}\times\vec{b})\}$

$\Rightarrow \quad \vec{X} = \lambda[(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}]$

$\Rightarrow \quad \vec{X} = \lambda[5(-\hat{i} + \hat{j} + \hat{k}) - (-1)(2\hat{i} + 0\hat{j} + \hat{k})]$

$\Rightarrow \quad \vec{X} = \lambda[-5\hat{i} + 5\hat{j} + 5\hat{k} + 2\hat{i} + 0\hat{j} + \hat{k}]$

$\Rightarrow \quad \vec{X} = \lambda[-3\hat{i} + 5\hat{j} + 6\hat{k}]$

Now,

$\vec{a}\cdot\vec{X} = 7$

$\Rightarrow \quad (-\hat{i} + \hat{j} + \hat{k})\cdot\lambda(-3\hat{i} + 5\hat{j} + 6\hat{k}) = 7$

$\Rightarrow \lambda(3+5+6) = 7$

$\Rightarrow \lambda = \frac{1}{2}.$

Hence, $\vec{X} = \frac{1}{2}(-3\hat{i}+5\hat{j}+6\hat{k})$

EXAMPLE 14 *If $\vec{a}$, $\vec{b}$ $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.*

[IIT 1995]

SOLUTION We have,

$$\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$\Rightarrow \left\{\vec{a}.\vec{c} - \frac{1}{\sqrt{2}}\right\}\vec{b} - \left\{\vec{a}.\vec{b} + \frac{1}{\sqrt{2}}\right\}\vec{c} = \vec{0}$$

$$\Rightarrow \vec{a}.\vec{c} - \frac{1}{\sqrt{2}} = 0 \text{ and } \vec{a}.\vec{b} + \frac{1}{\sqrt{2}} = 0 \quad \left[\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \vec{a}.\vec{c} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \vec{a}.\vec{b} = -\frac{1}{\sqrt{2}}$$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}$ and $\cos\beta = -\frac{1}{\sqrt{2}}$, where α and β are the angles made by $\vec{a}$ with $\vec{b}$ and $\vec{c}$ respectively.

$$\Rightarrow \alpha = \frac{\pi}{4} \text{ and } \beta = \frac{3\pi}{4}$$

EXAMPLE 15 *Let $\hat{a}$ be a unit vector and $\vec{b}$ be a non-zero vector not parallel to $\hat{a}$. Find the angles of the triangle, two sides of which are represented by the vectors $\sqrt{3}(\hat{a}\times\vec{b})$ and $\vec{b} - (\hat{a}.\vec{b})\hat{a}$.*

SOLUTION Let $\vec{\alpha} = \sqrt{3}(\hat{a}\times\vec{b})$ and $\vec{\beta} = \vec{b} - (\hat{a}.\vec{b})\hat{a}$. Then,

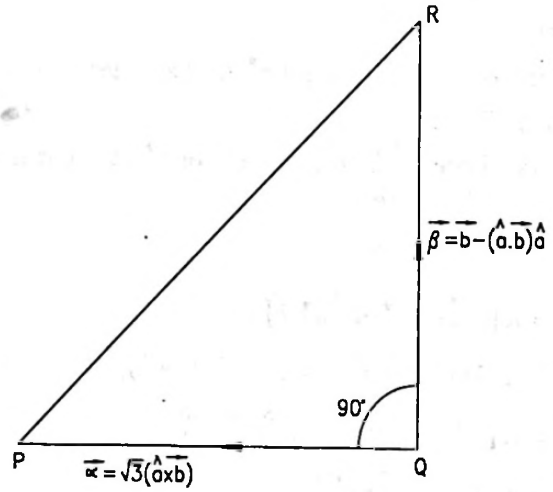


Fig. 17.2

$$\vec{\beta} = (\hat{a}.\hat{a})\vec{b} - (\hat{a}.\vec{b})\hat{a} \qquad [\because \hat{a}.\hat{a} = |\hat{a}|^2 = 1]$$
$$= \hat{a}\times(\vec{b}\times\hat{a})$$
$$= -\hat{a}\times(\hat{a}\times\vec{b})$$

Clearly, $\vec{\beta}$ is a vector lying in the plane of $\hat{a}$ and $\vec{b}$.

But, $\vec{\alpha} = \sqrt{3}(\hat{a}\times\vec{b})$

$\Rightarrow$ $\vec{\alpha}$ is perpendicular to the plane of $\hat{a}$ and $\vec{b}$.

$\therefore$ $\vec{\beta}$ is perpendicular to $\vec{\alpha}$

$\Rightarrow$ Angle between $\vec{\alpha}$ and $\vec{\beta}$ is $\pi/2$

$$\Rightarrow \angle PQR = \frac{\pi}{2}$$

Now,

$$PQ = |\vec{\alpha}| = \sqrt{3}\,|\hat{a}\times\vec{b}|$$

and,

$$QR = |\vec{\beta}| = |\hat{a}\times(\vec{b}\times\hat{a})|$$
$$= |\hat{a}|\,|\vec{b}\times\hat{a}|\sin\frac{\pi}{2}$$
$$= |\vec{b}\times\hat{a}| = |\hat{a}\times\vec{b}|$$

$\therefore$ $PQ = \sqrt{3}\,QR$

$$\Rightarrow \frac{PQ}{QR} = \sqrt{3} \Rightarrow \tan\angle PRQ = \sqrt{3} \Rightarrow \angle PRQ = 60°$$

Thus, we have, $\angle PQR = 90°$, $\angle PRQ = 60°$ and $\angle QPR = 30°$

EXAMPLE 16 *Find the scalars α and β, if*

$$\vec{a}\times(\vec{b}\times\vec{c}) + (\vec{a}.\vec{b})\vec{b} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$$

and $(\vec{c}.\vec{c})\vec{a} = \vec{c}$ where $\vec{b}$ and $\vec{c}$ are non-collinear vectors and α, β re scalars.

SOLUTION We have,

$$(\vec{c}.\vec{c})\,\vec{a} = \vec{c}$$

$$\Rightarrow (\vec{c}.\vec{c})(\vec{a}.\vec{c}) = (\vec{c}.\vec{c}) \qquad \text{[Taking dot product with } \vec{c}\text{]}$$

$$\Rightarrow |\vec{c}|^2(\vec{a}.\vec{c}) = |\vec{c}|^2$$

$$\Rightarrow \vec{a}.\vec{c} = 1$$

Now,

$$\vec{a}\times(\vec{b}\times\vec{c}) + (\vec{a}.\vec{b})\vec{b} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$$

$$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{a}.\vec{b})\vec{b} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$$

$$\Rightarrow (1 + \vec{a}.\vec{b})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (4 - 2\beta - \sin\alpha)\vec{b} + (\beta^2 - 1)\vec{c}$$

$$\Rightarrow 1 + \vec{a}.\vec{b} = 4 - 2\beta - \sin\alpha$$

and,

$$-(\vec{a}.\vec{b}) = \beta^2 - 1 \qquad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow 1 + 1 - \beta^2 = 4 - 2\beta - \sin\alpha$$

$$\Rightarrow \beta^2 - 2\beta + 2 - \sin\alpha = 0$$

$$\Rightarrow (\beta - 1)^2 + (1 - \sin\alpha) = 0$$

$\Rightarrow \beta - 1 = 0$ and $1 - \sin\alpha = 0$

$\Rightarrow \beta = 1$ and $\sin\alpha = 1$

$\Rightarrow \beta = 1$ and $\sin\alpha = \sin\frac{\pi}{2}$

$\Rightarrow \beta = 1$ and $\alpha = 2n\pi + \frac{\pi}{2}, n \in Z.$

EXAMPLE 17 *Show that :*

$$[\vec{a}\times\vec{p}\ \vec{b}\times\vec{q}\ \vec{c}\times\vec{r}] + [\vec{a}\times\vec{q}\ \vec{b}\times\vec{r}\ \vec{c}\times\vec{p}] + [\vec{a}\times\vec{r}\ \vec{b}\times\vec{p}\ \vec{c}\times\vec{q}] = 0$$

SOLUTION We have,

$$[\vec{a}\times\vec{p}\ \vec{b}\times\vec{q}\ \vec{c}\times\vec{r}]$$

$$= (\vec{a}\times\vec{p}).\{(\vec{b}\times\vec{q})\times(\vec{c}\times\vec{r})\}$$

$$= (\vec{a}\times\vec{p}).\{[\vec{b}\ \vec{c}\ \vec{r}]\vec{q} - [\vec{q}\ \vec{c}\ \vec{r}]\vec{b}\}$$

$$= [\vec{a}\ \vec{p}\ \vec{q}]\ [\vec{b}\ \vec{c}\ \vec{r}] - [\vec{a}\ \vec{p}\ \vec{b}]\ [\vec{q}\ \vec{c}\ \vec{r}]$$

$$[\vec{a}\times\vec{q}\ \vec{b}\times\vec{r}\ \vec{c}\times\vec{p}]$$

$$= (\vec{a}\times\vec{q}).\{(\vec{b}\times\vec{r})\times(\vec{c}\times\vec{p})\}$$

$$= (\vec{a}\times\vec{q}).\{[\vec{b}\ \vec{r}\ \vec{p}]\vec{c} - [\vec{b}\ \vec{r}\ \vec{c}]\vec{p}\}$$

$$= [\vec{a}\ \vec{q}\ \vec{c}]\ [\vec{b}\ \vec{r}\ \vec{p}] - [\vec{a}\ \vec{q}\ \vec{p}]\ [\vec{b}\ \vec{r}\ \vec{c}]$$

and,

$$[\vec{a}\times\vec{r}\ \vec{b}\times\vec{p}\ \vec{c}\times\vec{q}]$$

$$= \{(\vec{a}\times\vec{r})\times(\vec{b}\times\vec{p})\}.(\vec{c}\times\vec{q})$$

$$= \{[\vec{a}\ \vec{b}\ \vec{p}]\vec{r} - [\vec{r}\ \vec{b}\ \vec{p}]\vec{a}\}.(\vec{c}\times\vec{q})$$

$$= [\vec{a}\ \vec{b}\ \vec{p}]\ [\vec{r}\ \vec{c}\ \vec{q}] - [\vec{r}\ \vec{b}\ \vec{p}]\ [\vec{a}\ \vec{c}\ \vec{q}]$$

Adding these three, we get

$$[\vec{a}\times\vec{p}\ \vec{b}\times\vec{q}\ \vec{c}\times\vec{r}] + [\vec{a}\times\vec{q}\ \vec{b}\times\vec{r}\ \vec{c}\times\vec{p}] + [\vec{a}\times\vec{r}\ \vec{b}\times\vec{p}\ \vec{c}\times\vec{q}] = 0$$

EXAMPLE 18 *If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $|\vec{b}| = |\vec{c}|$, prove that*

$$\{(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})\}\times(\vec{b}\times\vec{c}).(\vec{b}+\vec{c}) = 0$$

[IIT 1997]

SOLUTION We have,

$$\{(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})\}\times(\vec{b}\times\vec{c}).(\vec{b}+\vec{c})$$

$$= (\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c})\times(\vec{b}\times\vec{c}).(\vec{b}+\vec{c})$$

$$= \{(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a})\times(\vec{b}+\vec{c}) + (\vec{b}\times\vec{c})\times(\vec{b}\times\vec{c})\}.(\vec{b}+\vec{c})$$

$$= \{[\vec{a}\ \vec{c}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{c}\ \vec{b}]\vec{c} + [\vec{b}\ \vec{a}\ \vec{c}]\vec{b} - [\vec{b}\ \vec{a}\ \vec{b}]\vec{c}\}.(\vec{b}+\vec{c})$$

$$= \{-[\vec{a}\ \vec{c}\ \vec{b}]\vec{c} + [\vec{b}\ \vec{a}\ \vec{c}]\vec{b}\}.(\vec{b}+\vec{c})$$

$$= -[\vec{a}\ \vec{c}\ \vec{b}]\{\vec{c}.(\vec{b}+\vec{c})\} + [\vec{b}\ \vec{a}\ \vec{c}]\{\vec{b}.(\vec{b}+\vec{c})\}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{c}.\vec{b} + |\vec{c}|^2) - [\vec{a}\ \vec{b}\ \vec{c}]\ (|\vec{b}|^2 + \vec{b}.\vec{c})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]\ (|\vec{c}|^2 - |\vec{b}|^2)$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]\times 0 \qquad [\because |\vec{b}| = |\vec{c}|]$$

$$= 0$$

EXAMPLE 19 *If the vectors $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the vector.*

$(\vec{r}\times\vec{a})\times(\vec{b}\times\vec{c}) + (\vec{r}\times\vec{b})\times(\vec{c}\times\vec{a}) + (\vec{r}\times\vec{c})\times(\vec{a}\times\vec{b})$ *is expressible as $m\vec{r}$, where m is a scalar.*

SOLUTION Since any vector in space is expressible as the linear combination of three non-coplanar vectors. So, let

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$$

$$\Rightarrow \vec{r}.(\vec{b}\times\vec{c}) = (x\vec{a} + y\vec{b} + z\vec{c}).(\vec{b}\times\vec{c})$$

$$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] = x[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow x = \frac{[\vec{r}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Similarly, we have

$$y = \frac{[\vec{r}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} \text{ and } z = \frac{[\vec{r}\ \vec{a}\ \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\therefore \vec{r} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\{[\vec{r}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\vec{c}\} \quad ...(i)$$

Now,

$$(\vec{r}\times\vec{a})\times(\vec{b}\times\vec{c})$$

$$= -\{(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})\}$$

$$= -\{[\vec{b}\ \vec{c}\ \vec{a}]\vec{r} - [\vec{b}\ \vec{c}\ \vec{r}]\vec{a}\}$$

$$= [\vec{b}\ \vec{c}\ \vec{r}]\vec{a} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

$$(\vec{r}\times\vec{b})\times(\vec{c}\times\vec{a})$$

$$= -\{(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})\}$$

$$= -\{[\vec{c}\ \vec{a}\ \vec{b}]\vec{r} - [\vec{c}\ \vec{a}\ \vec{r}]\vec{b}\}$$

$$= [\vec{c}\ \vec{a}\ \vec{r}]\vec{b} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

and,

$$(\vec{r}\times\vec{c})\times(\vec{a}\times\vec{b})$$

$$= -\{(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})\}$$

$$= -\{[\vec{a}\ \vec{b}\ \vec{c}]\vec{r} - [\vec{a}\ \vec{b}\ \vec{r}]\vec{c}\}$$

$$= [\vec{a}\ \vec{b}\ \vec{r}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

$$\therefore\ (\vec{r}\times\vec{a})\times(\vec{b}\times\vec{c})+(\vec{r}\times\vec{b})\times(\vec{c}\times\vec{a})+(\vec{r}\times\vec{c})\times(\vec{a}\times\vec{b})$$

$$=[\vec{b}\ \vec{c}\ \vec{r}]\vec{a}+[\vec{c}\ \vec{a}\ \vec{r}]\vec{b}+[\vec{a}\ \vec{b}\ \vec{r}]\vec{c}-3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}-3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

$$=-2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$$

$$=m\vec{r}, \text{ where } m=-2[\vec{a}\ \vec{b}\ \vec{c}].$$

EXAMPLE 20 *Let $\hat{a},\hat{b},\hat{c}$ be three non-coplanar unit vectors. The angle between $\vec{b}$ and $\hat{c}$ be α, angle between $\hat{c}$ and $\hat{a}$ be β and between $\hat{a}$ and $\hat{b}$ be γ. If A $(\hat{a}\cos\alpha)$, B $(\hat{b}\cos\beta)$ and C $(\hat{c}\cos\gamma)$, then show that*

$$\frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}$$

$$=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1|},$$

where $\hat{n}_1=\dfrac{(\hat{b}\times\hat{c})}{|\hat{b}\times\hat{c}|}$, $\hat{n}_2=\dfrac{\hat{c}\times\hat{a}}{|\hat{c}\times\hat{a}|}$ and $\hat{n}_3=\dfrac{\hat{a}\times\hat{b}}{|\hat{a}\times\hat{b}|}$

SOLUTION We have,

$$\hat{a}\times(\hat{b}\times\hat{c})=(\hat{a}.\hat{c})\vec{b}-(\hat{a}.\hat{b})\hat{c}$$

$$=(\cos\beta)\hat{b}-(\cos\gamma)\hat{c}$$

$$=\vec{CB}$$

$$\hat{b}\times(\hat{c}\times\hat{a})=(\hat{b}.\hat{a})\hat{c}-(\hat{b}.\hat{c})\hat{a}$$

$$=(\cos\gamma)\hat{c}-(\cos\alpha)\hat{a}$$

$$=\vec{AC}$$

and,

$$\hat{c}\times(\hat{a}\times\hat{b})=(\hat{c}.\hat{b})\hat{a}-(\hat{c}.\hat{a})\hat{b}$$

$$=(\cos\alpha)\hat{a}-(\cos\beta)\hat{b}$$

$$=\vec{BA}$$

$$\therefore\quad AB=|\vec{AB}|=|\hat{c}\times(\hat{a}\times\hat{b})|,$$

$$BC=|\vec{BC}|=|\hat{a}\times(\hat{b}\times\hat{c})|$$

and, $CA=|\vec{AC}|=|\vec{b}\times(\hat{c}\times\hat{a})|$

Using sine rule in ΔABC, we have

$$\frac{BC}{\sin A}=\frac{CA}{\sin B}=\frac{AB}{\sin C}=2R, \qquad \text{...(i)}$$

where R is the circum radius.

Also, $R=\dfrac{BC.CA.AB}{4\Delta}$, where Δ is the area of ΔABC

Now,

$$\Delta=\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|$$

$$=\frac{1}{2}\left|(\hat{b}\cos\beta-\hat{a}\cos\alpha)\times(\hat{c}\cos\gamma-\hat{a}\cos\alpha)\right|$$

$$=\frac{1}{2}\left|(\hat{b}\times\hat{c})\cos\beta\cos\gamma+(\hat{a}\times\hat{b})\cos\alpha\cos\beta+(\hat{c}\times\hat{a})\cos\alpha\cos\gamma\right|$$

$$=\frac{1}{2}\left|\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1+\cos\alpha\cos\beta\sin\gamma\,\hat{n}_3+\cos\alpha\sin\beta\cos\gamma\,\hat{n}_2\right|$$

$$=\frac{1}{2}\left|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1\right|$$

$$\therefore\quad R=\frac{BC.CA.AB}{4\Delta}$$

$$\Rightarrow\quad 2R=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1|}$$

Substituting these values in (i), we get

$$\frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}$$

$$=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1|}$$

EXAMPLE 21 *If $\vec{a},\vec{b},\vec{c},\vec{d}$ are non-zero vectors and $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})-(\vec{b}\times\vec{c})\times(\vec{d}\times\vec{a})=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$, show that $\vec{a},\vec{b},\vec{c}$ are necessarily coplanar.*

SOLUTION We have,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})-(\vec{b}\times\vec{c})\times(\vec{d}\times\vec{a})=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$\Rightarrow\ -(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})-(\vec{b}\times\vec{c})\times(\vec{d}\times\vec{a})=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$\Rightarrow\ -[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}+[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{b}\ \vec{c}\ \vec{a}]\vec{d}+[\vec{b}\ \vec{c}\ \vec{d}]\vec{a}=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$\Rightarrow\ -[\vec{b}\ \vec{c}\ \vec{a}]\vec{d}=\vec{0}$$

$$\Rightarrow\ [\vec{b}\ \vec{c}\ \vec{a}]=0 \qquad [\because\ \vec{d}\neq\vec{0}]$$

$\Rightarrow$ $\vec{a},\vec{b},\vec{c}$ are coplanar vectors.

EXAMPLE 22 *Find a vector in the plane of $\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$, orthogonal to $-2\hat{i}+\hat{j}+\hat{k}$ and with its projection along $\hat{i}-\hat{j}+\hat{k}$ equal to $6\sqrt{3}$.*

SOLUTION Let $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$, $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{c}=-2\hat{i}+\hat{j}+\hat{k}$ and $\vec{d}=-\hat{i}+\hat{j}+\hat{k}$.

Required vector $\vec{r}$ is parallel to $\vec{c}\times(\vec{a}\times\vec{b})$. Therefore,

$$\vec{r}=\lambda\{\vec{c}\times(\vec{a}\times\vec{b})\}$$

$$=\lambda\{(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}\}$$

$$=\lambda\{(-2-2+1)(\hat{i}+\hat{j}-2\hat{k})-(-2+1-2)(\hat{i}-2\hat{j}+\hat{k})\}$$

$$=\lambda(-3\hat{i}-3\hat{j}+6\hat{k}+3\hat{i}-6\hat{j}+3\hat{k})$$

$= \lambda(-9\hat{j}+9\hat{k})$

Now,

Projection of $\vec{r}$ along $\vec{d}$ is $6\sqrt{3}$

$\Rightarrow \vec{r}.\hat{d} = 6\sqrt{3}$

$\Rightarrow \lambda(-9\hat{j}+9\hat{k}).\dfrac{(\hat{i}-\hat{j}+\hat{k})}{\sqrt{3}} = 6\sqrt{3}$

$\Rightarrow \lambda\dfrac{(9+9)}{\sqrt{3}} = 6\sqrt{3} \Rightarrow \lambda = 1$

Hence, $\vec{r} = -9\hat{j}+9\hat{k}$.

EXAMPLE 23 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, prove that*

$$\vec{p}\times\vec{q} = \frac{1}{\sqrt{[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}]}}\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}.\vec{p} & \vec{b}.\vec{p} & \vec{c}.\vec{p}\\ \vec{a}.\vec{q} & \vec{b}.\vec{q} & \vec{c}.\vec{q}\end{vmatrix}$$

SOLUTION Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, $\vec{p} = p_1\hat{i}+p_2\hat{j}+p_3\hat{k}$ and $\vec{q} = q_1\hat{i}+q_2\hat{j}+q_3\hat{k}$.

Then,

$$\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}.\vec{p} & \vec{b}.\vec{p} & \vec{c}.\vec{p}\\ \vec{a}.\vec{q} & \vec{b}.\vec{q} & \vec{c}.\vec{q}\end{vmatrix}$$

$$= \begin{vmatrix} a_1\hat{i}+a_2\hat{j}+a_3\hat{k} & b_1\hat{i}+b_2\hat{j}+b_3\hat{k} & c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\\ a_1p_1+a_2p_2+a_3p_3 & b_1p_1+b_2p_2+b_3p_3 & c_1p_1+c_2p_2+c_3p_3\\ a_1q_1+a_2q_2+a_3q_3 & b_1q_1+b_2q_2+b_3q_3 & c_1q_1+c_2q_2+c_3q_3\end{vmatrix}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$= (\vec{p}\times\vec{q})\ [\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{p}\times\vec{q})$

$= \sqrt{[\vec{a}\ \vec{b}\ \vec{c}]^2}\ (\vec{p}\times\vec{q})$

$= \sqrt{[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}]}\ (\vec{p}\times\vec{q})$

$$\therefore\ \vec{p}\times\vec{q} = \frac{1}{\sqrt{[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}]}}\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}.\vec{p} & \vec{b}.\vec{p} & \vec{c}.\vec{p}\\ \vec{a}.\vec{q} & \vec{b}.\vec{q} & \vec{c}.\vec{q}\end{vmatrix}$$

EXAMPLE 24 *The vectors $\vec{x}, \vec{y}, \vec{z}$ each is of magnitude $\sqrt{2}$, make angles of 60° with each other. If $\vec{x}\times(\vec{y}\times\vec{z}) = \vec{a}$, $\vec{y}\times(\vec{z}\times\vec{x}) = \vec{b}$ and $\vec{x}\times\vec{y} = \vec{c}$, find $\vec{x}, \vec{y}, \vec{z}$ in terms of $\vec{a}, \vec{b}, \vec{c}$.*

SOLUTION We have, $|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt{2}$ and $\vec{x}, \vec{y}, \vec{z}$ make angles of 60° with each other. Therefore,

$$\vec{x}.\vec{y} = |\vec{x}|\,|\vec{y}|\cos 60° = \sqrt{2}\times\sqrt{2}\times\frac{1}{2} = 1,$$

$$\vec{y}.\vec{z} = |\vec{y}|\,|\vec{z}|\cos 60° = \sqrt{2}\times\sqrt{2}\times\frac{1}{2} = 1,$$

and, $$\vec{x}.\vec{z} = |\vec{x}|\,|\vec{z}|\cos 60° = \sqrt{2}\times\sqrt{2}\times\frac{1}{2} = 1$$

Also, $\vec{x}.\vec{x} = |\vec{x}|^2 = 2$, $\vec{y}.\vec{y} = |\vec{y}|^2 = 2$

and, $\vec{z}.\vec{z} = |z|^2 = 2$.

Now, $\vec{x}\times(\vec{y}\times\vec{z}) = \vec{a}$ and $\vec{y}\times(\vec{z}\times\vec{x}) = \vec{b}$ [Given]

$\Rightarrow (\vec{x}.\vec{z})\vec{y} - (\vec{x}.\vec{y})\vec{z} = \vec{a}$

and, $(\vec{y}.\vec{x})\vec{z} - (\vec{y}.\vec{z})\vec{x} = \vec{b}$

$\Rightarrow \vec{y}-\vec{z} = \vec{a}$ and $\vec{z}-\vec{x} = \vec{b}$.

$\Rightarrow (\vec{y}-\vec{z})+(\vec{z}-\vec{x}) = \vec{a}+\vec{b}$

$\Rightarrow \vec{y}-\vec{x} = \vec{a}+\vec{b}$

Thus, we have

$$\vec{y}-\vec{z} = \vec{a} \quad ...(i)$$

$$\vec{z}-\vec{x} = \vec{b} \quad ...(ii)$$

$$\vec{y}-\vec{x} = \vec{a}+\vec{b} \quad ...(iii)$$

Now,

$\vec{x}\times\vec{y} = \vec{c}$ [Given]

$\Rightarrow \vec{x}\times(\vec{x}\times\vec{y}) = \vec{x}\times\vec{c}$ [Taking cross-product with $\vec{x}$]

$\Rightarrow (\vec{x}.\vec{y})\times\vec{x} - (\vec{x}.\vec{x})\vec{y} = \vec{x}\times\vec{c}$

$\Rightarrow \vec{x}-2\vec{y} = \vec{x}\times\vec{c}$ ...(iv)

Again,

$\vec{x}\times\vec{y} = \vec{c}$

$\Rightarrow \vec{y}\times(\vec{x}\times\vec{y}) = \vec{y}\times\vec{c}$ [Taking cross-product with $\vec{y}$]

$\Rightarrow (\vec{y}.\vec{y})\vec{x} - (\vec{y}.\vec{x})\vec{y} = \vec{y}\times\vec{c}$

$\Rightarrow 2\vec{x}-\vec{y} = \vec{y}\times\vec{c}$ ...(v)

On subtracting (iv) and (v), we get

$\vec{x}+\vec{y} = (\vec{y}\times\vec{c})-(\vec{x}\times\vec{c})$

$\Rightarrow \vec{x}+\vec{y} = (\vec{y}-\vec{x})\times\vec{c}$

$\Rightarrow \vec{x}+\vec{y} = (\vec{a}+\vec{b})\times\vec{c}$ ...(vi) [Using (iii)]

Adding (iii) and (vi), we get

$2\vec{y} = (\vec{a}+\vec{b})+(\vec{a}+\vec{b})\times\vec{c}$

$\Rightarrow \vec{y} = \dfrac{1}{2}[(\vec{a}+\vec{b})+(\vec{a}+\vec{b})\times\vec{c}]$

Substituting the value of $\vec{y}$ in (iii) and (i) we get

$$\vec{x} = \frac{1}{2}\left[(\vec{a}+\vec{b}) + (\vec{a}+\vec{b})\times\vec{c}\right] - (\vec{a}+\vec{b})$$

$$= \frac{1}{2}\left[(\vec{a}+\vec{b})\times\vec{c} - (\vec{a}+\vec{b})\right]$$

$$\vec{z} = \frac{1}{2}\left[(\vec{a}+\vec{b}) + (\vec{a}+\vec{b})\times\vec{c}\right] - \vec{a}$$

$$= \frac{1}{2}\left[(\vec{b}-\vec{a}) + (\vec{a}+\vec{b})\times\vec{c}\right]$$

EXAMPLE 25 *Let $\vec{x}, \vec{y}, \vec{z}$ be unit vectors such that $\vec{x}+\vec{y}+\vec{z}=\vec{a}$, $\vec{x}\times(\vec{y}\times\vec{z})=\vec{b}$, $(\vec{x}\times\vec{y})\times\vec{z}=\vec{c}$, $\vec{a}.\vec{x}=\frac{3}{2}$, $\vec{a}.\vec{y}=\frac{7}{4}$ and $|\vec{a}|=2$. Find $\vec{x}$, $\vec{y}$ and $\vec{z}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.*

SOLUTION We have,

$$\vec{x}+\vec{y}+\vec{z}=\vec{a} \qquad ...(i)$$

$$\Rightarrow |\vec{x}+\vec{y}+\vec{z}| = |\vec{a}|$$

$$\Rightarrow |\vec{x}+\vec{y}+\vec{z}|^2 = |\vec{a}|^2$$

$$\Rightarrow |\vec{x}|^2+|\vec{y}|^2+|\vec{z}|^2+2(\vec{x}.\vec{y}+\vec{y}.\vec{z}+\vec{z}.\vec{x}) = |\vec{a}|^2$$

$$\Rightarrow 1+1+1+2(\vec{x}.\vec{y}+\vec{y}.\vec{z}+\vec{z}.\vec{x}) = 4$$

$$\Rightarrow 2(\vec{x}.\vec{y}+\vec{y}.\vec{z}+\vec{z}.\vec{x}) = 4-3$$

$$\Rightarrow \vec{x}.\vec{y}+\vec{y}.\vec{z}+\vec{z}.\vec{x} = \frac{1}{2} \qquad ...(ii)$$

Now,

$$\vec{a}.\vec{x} = \frac{3}{2}$$

$$\Rightarrow (\vec{x}+\vec{y}+\vec{z}).\vec{x} = \frac{3}{2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \vec{x}.\vec{x}+\vec{y}.\vec{x}+\vec{z}.\vec{x} = \frac{3}{2}$$

$$\Rightarrow 1+\vec{x}.\vec{y}+\vec{z}.\vec{x} = \frac{3}{2}$$

$$\Rightarrow \vec{x}.\vec{y}+\vec{z}.\vec{x} = \frac{1}{2} \qquad ...(iii)$$

$$\vec{a}.\vec{y} = \frac{7}{4}$$

$$\Rightarrow (\vec{x}+\vec{y}+\vec{z}).\vec{y} = \frac{7}{4}$$

$$\Rightarrow \vec{x}.\vec{y}+\vec{y}.\vec{y}+\vec{z}.\vec{y} = \frac{7}{4}$$

$$\Rightarrow \vec{x}.\vec{y}+1+\vec{y}.\vec{z} = \frac{7}{4}$$

$$\Rightarrow \vec{x}.\vec{y}+\vec{y}.\vec{z} = \frac{3}{4} \qquad ...(iv)$$

From (ii) and (iii), we obtain

$$\vec{y}.\vec{z} = 0$$

From (ii) and (iv), we get

$$\vec{z}.\vec{x} = \frac{-1}{4}$$

Putting the value of $\vec{z}.\vec{x}$ in (iii), we get

$$\vec{x}.\vec{y} = \frac{3}{4}$$

Now,

$$\vec{x}\times(\vec{y}\times\vec{z}) = \vec{b} \text{ and } (\vec{x}\times\vec{y})\times\vec{z} = \vec{c}$$

$$\Rightarrow (\vec{x}.\vec{z})\vec{y} - (\vec{x}.\vec{y})\vec{z} = \vec{b} \text{ and } (\vec{z}.\vec{x})\vec{y} - (\vec{z}.\vec{y})\vec{x} = \vec{c}$$

$$\Rightarrow -\frac{1}{4}\vec{y} - \frac{3}{4}\vec{z} = \vec{b} \text{ and } -\frac{1}{4}\vec{y} = \vec{c}$$

$$\Rightarrow \vec{y} = -4\vec{c},\ \vec{z} = \frac{4}{3}(\vec{c}-\vec{b})$$

Substituting these values in (i), we get

$$\vec{x} = \frac{1}{3}(3\vec{a}+4\vec{b}+8\vec{c})$$

EXAMPLE 26 *If $\vec{x}\times\vec{y}=\vec{a}$, $\vec{y}\times\vec{z}=\vec{b}$, $\vec{x}.\vec{b}=\gamma$, $\vec{x}.\vec{y}=1$ and $\vec{y}.\vec{z}=1$, then find $\vec{x}$, $\vec{y}$, $\vec{z}$ in terms of $\vec{a}$, $\vec{b}$ and scalar γ.*

SOLUTION We have,

$$\vec{x}\times\vec{y} = \vec{a} \text{ and } \vec{y}\times\vec{z} = \vec{b}$$

$\Rightarrow \vec{a}$ is perpendicular to $\vec{x}$ and $\vec{y}$ and $\vec{b}$ is perpendicular to $\vec{y}$ and $\vec{z}$.

$$\Rightarrow \vec{x}.\vec{a}=0=\vec{y}.\vec{a},\ \vec{y}.\vec{b}=0=\vec{z}.\vec{b}$$

Thus, we have

$$\left.\begin{aligned}\vec{x}.\vec{a}=0,\ \vec{x}.\vec{b}=\gamma,\ \vec{x}.\vec{y}=1\\ \vec{y}.\vec{a}=\vec{y}.\vec{b}=0,\ \vec{y}.\vec{z}=1\end{aligned}\right\} \qquad ...(i)$$

and, $\vec{z}.\vec{b}=0$

Now,

$$\vec{x}\times\vec{y} = \vec{a}$$

$$\Rightarrow \vec{b}\times(\vec{x}\times\vec{y}) = \vec{b}\times\vec{a}$$

$$\Rightarrow (\vec{b}.\vec{y})\vec{x} - (\vec{b}.\vec{x})\vec{y} = \vec{b}\times\vec{a}$$

$$\Rightarrow -\gamma\vec{y} = \vec{b}\times\vec{a} \qquad [\because \vec{b}.\vec{y}=0 \text{ and } \vec{b}.\vec{x}=\gamma]$$

$$\Rightarrow \vec{y} = \frac{1}{\gamma}(\vec{a}\times\vec{b})$$

Again,

$$\vec{x}\times\vec{y} = \vec{a}$$

$$\Rightarrow (\vec{x}\times\vec{y})\times\vec{y} = \vec{a}\times\vec{y}$$

$\Rightarrow (\vec{y}.\vec{x})\vec{y}-(\vec{y}.\vec{y})\vec{x}=\vec{a}\times\vec{y}$

$\Rightarrow \vec{y}-|\vec{y}|^2\vec{x}=\vec{a}\times\vec{y}$

$\Rightarrow |\vec{y}|^2\vec{x}=\vec{y}-\vec{a}\times\vec{y}$

$\Rightarrow \left|\frac{1}{\gamma}(\vec{a}\times\vec{b})\right|^2\vec{x}=\frac{1}{\gamma}(\vec{a}\times\vec{b})-\frac{1}{\gamma}\{\vec{a}\times(\vec{a}\times\vec{b})\}$

$\Rightarrow \vec{x}=\frac{\gamma^2}{|\vec{a}\times\vec{b}|^2}\left[\frac{1}{\gamma}(\vec{a}\times\vec{b})-\frac{1}{\gamma}\{\vec{a}\times(\vec{a}\times\vec{b})\}\right]$

$\Rightarrow \vec{x}=\frac{\gamma}{|\vec{a}\times\vec{b}|^2}\{(\vec{a}\times\vec{b})-\vec{a}\times(\vec{a}\times\vec{b})\}$

Now,

$\vec{y}\times\vec{z}=\vec{b}$

$\Rightarrow \vec{y}\times(\vec{y}\times\vec{z})=\vec{y}\times\vec{b}$

$\Rightarrow (\vec{y}.\vec{z})\vec{y}-(\vec{y}.\vec{y})\vec{z}=\vec{y}\times\vec{b}$

$\Rightarrow \vec{y}-|\vec{y}|^2\vec{z}=\vec{y}\times\vec{b}$

$\Rightarrow \vec{z}=\frac{1}{|\vec{y}|^2}[\vec{y}-(\vec{y}\times\vec{b})]$

$\Rightarrow \vec{z}=\frac{\gamma^2}{|\vec{a}\times\vec{b}|^2}\left[\frac{1}{\gamma}(\vec{a}\times\vec{b})-\frac{1}{\gamma}\{(\vec{a}\times\vec{b})\times\vec{b}\}\right]$

$\Rightarrow \vec{z}=\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}-(\vec{a}\times\vec{b})\times\vec{b}]$

EXAMPLE 27 *If $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar vectors, then prove that the vector*

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ *is parallel to* $\vec{a}$. [IIT 1997]

SOLUTION We have,

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

$=-(c\times\vec{d})\times(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})-(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})$

$=-[\vec{c}\,\vec{d}\,\vec{b}]\vec{a}+[\vec{c}\,\vec{d}\,\vec{a}]\vec{b}+[\vec{a}\,\vec{c}\,\vec{b}]\vec{d}$

$-[\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}+[\vec{b}\,\vec{c}\,\vec{a}]\vec{d}$

$=-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

$=-2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

$=\lambda\vec{a}$, where $\lambda=-2[\vec{b}\,\vec{c}\,\vec{d}]$

Hence, the given vector is parallel to $\vec{a}$.

EXAMPLE 28 *Let $\vec{a}, \vec{b}, \vec{c}$ be three mutually perpendicular vectors of equal magnitude. Then, find a vector $\vec{r}$ satisfying the equation*

$\vec{a}\times\{(\vec{r}-\vec{b})\times\vec{a}\}+\vec{b}\times\{(\vec{r}-\vec{c})\times\vec{b}\}+\vec{c}\times\{(\vec{r}-\vec{a})\times\vec{c}\}=0$

SOLUTION Let $|\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$ (say).

We have,

$\vec{a}\times\{(\vec{r}-\vec{b})\times\vec{a}\}=(\vec{a}.\vec{a})(\vec{r}-\vec{b})-\{\vec{a}.(\vec{r}-\vec{b})\}\vec{a}$

$=|\vec{a}|^2(\vec{r}-\vec{b})-(\vec{a}.\vec{r}-\vec{a}.\vec{b})\vec{a}$

$=\lambda^2(\vec{r}-\vec{b})-(\vec{a}.\vec{r})\vec{a}$ $[\because \vec{a}\perp\vec{b}\therefore \vec{a}.\vec{b}=0]$

Similarly, we have

$\vec{b}\times\{(\vec{r}-\vec{c})\times\vec{b}\}=\lambda^2(\vec{r}-\vec{c})-(\vec{b}.\vec{r})\vec{b}$

and,

$\vec{c}\times\{(\vec{r}-\vec{a})\times\vec{c}\}=\lambda^2(\vec{r}-\vec{a})-(\vec{c}.\vec{r})\vec{c}$

$\therefore \vec{a}\times\{(\vec{r}-\vec{b})\times\vec{a}\}+\vec{b}\times\{(\vec{r}-\vec{c})\times\vec{b}\}+\vec{c}\times\{(\vec{r}-\vec{a})\times\vec{c}\}=\vec{0}$

$\Rightarrow \lambda^2(\vec{r}-\vec{b})-(\vec{a}.\vec{r})\vec{a}+\lambda^2(\vec{r}-\vec{c})-(\vec{b}.\vec{r})\vec{b}+\lambda^2(\vec{r}-\vec{a})$

$-(\vec{c}.\vec{r})\vec{c}=\vec{0}$

$\Rightarrow \lambda^2\{(\vec{r}-\vec{b})+(\vec{r}-\vec{c})+(\vec{r}-\vec{a})\}$

$-[(\vec{a}.\vec{r})\vec{a}+(\vec{b}.\vec{r})\vec{b}+(\vec{c}.\vec{r})\vec{c}]=\vec{0}$

$\Rightarrow \lambda^2[3\vec{r}-(\vec{a}+\vec{b}+\vec{c})]-[|\vec{a}|^2(\vec{r}.\hat{a})\hat{a}+|\vec{b}|^2(\vec{r}.\hat{b})\hat{b}$

$+|\vec{c}|^2(\vec{r}.\hat{c})\hat{c}]=0$

$\Rightarrow \lambda^2[3\vec{r}-(\vec{a}+\vec{b}+\vec{c})]-\lambda^2[(\vec{r}.\hat{a})\hat{a}+(\vec{r}.\hat{b})\hat{b}+(\vec{r}.\hat{c})\hat{c}]=0$

$[\because |\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda]$

$\Rightarrow [3\vec{r}-(\vec{a}+\vec{b}+\vec{c})]-\vec{r}=0$ $\left[\begin{array}{l}\because \lambda^2\neq 0 \\ \text{and} \\ \vec{r}=(\vec{r}.\hat{a})\hat{a}+(\vec{r}.\hat{b})\hat{b}+(\vec{r}.\hat{c})\hat{c}\end{array}\right]$

$\Rightarrow 2\vec{r}-(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow \vec{r}=\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$

EXAMPLE 29 *If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ and $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$, then prove that*

$$2(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\begin{vmatrix}-\vec{a} & -\vec{b} & \vec{c} & \vec{d}\\ a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\end{vmatrix}$$

SOLUTION We have,

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$=[\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

$=\begin{vmatrix}a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix}\vec{b}-\begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix}\vec{a}$ ...(i)

and also

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{d}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{c} - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\vec{d} \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{a} + \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{b}$$

$$+\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{c} - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\vec{d}$$

$$= \begin{vmatrix} -\vec{a} & -\vec{b} & \vec{c} & \vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

EXAMPLE 30 *If* $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ *are four vectors, prove that*

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0$$

Use this result to prove that :

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\sin(A+B)\sin(A-B) = \cos^2 B - \sin^2 A$.

SOLUTION We have,

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})$$

$$= \vec{b}\cdot\{\vec{c}\times(\vec{a}\times\vec{d})\}$$

$$= \vec{b}\cdot\{(\vec{c}\cdot\vec{d})\vec{a} - (\vec{c}\cdot\vec{a})\vec{d}\}$$

$$= (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d}) - (\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a}) \qquad ...(i)$$

Similarly,

$$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = (\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d}) - (\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b}) \qquad ...(ii)$$

and,

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \qquad ...(iii)$$

Adding (i), (ii) and (iii), we obtain

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0 \qquad ...(iv)$$

In order to prove the results (i) and (ii), let us assume that $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors such that the angle between the vectors $\vec{a}$ and $\vec{c}$ is A, $\vec{b}$ and $\vec{d}$ is A, $\vec{a}$ and $\vec{b}$ is A and $\vec{c}$ and $\vec{d}$ is also B.

Let $\hat{n}$ be a unit vector perpendicular to the plane containing vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. Then,

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) = \{bc\sin(A-B)\hat{n}\}\cdot\{ad\sin(A+B)\hat{n}\}$$

$$= \{abcd\sin(A+B)\sin(A-B)\}(\hat{n}\cdot\hat{n})$$

$$= abcd\sin(A+B)\sin(A-B)\,|\,\hat{n}\,|^2$$

$$= abcd\sin(A+B)\sin(A-B)$$

$$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = (-ca\sin A)\hat{n}\cdot(bd\sin A)\hat{n}$$

$$= -abcd\sin^2 A\,(\hat{n}\cdot\hat{n})$$

$$= -abcd\sin^2 A$$

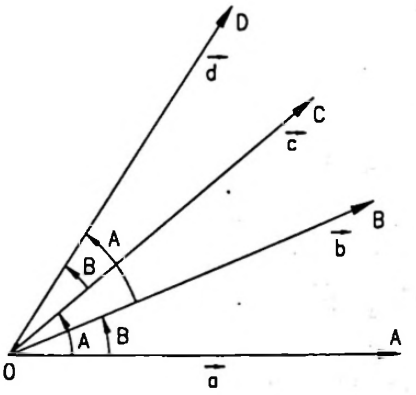


Fig. 17.3

and,

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (ab\sin B)\hat{n}\cdot(cd\sin B)\hat{n}$$

$$= (abcd\sin^2 B)(\hat{n}\cdot\hat{n})$$

$$= (abcd\sin^2 B)\,|\,\hat{n}\,|^2$$

$$= abcd\sin^2 B$$

Substituting these values in (iv), we get

$$abcd\sin(A+B)\sin(A-B) - abcd\sin^2 A + abcd\sin^2 B = 0$$

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$$

From (ii) and (iii), we get

$$(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$$

$$= (ac\cos A)(bd\cos A) - (cd\cos B)(ab\cos B)$$

$$= abcd(\cos^2 A - \cos^2 B)$$

Substituting the values of $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})$ and $(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ in (i), we get

$$abcd\sin(A+B)\sin(A-B) + abcd(\cos^2 A - \cos^2 B) = 0$$

$$\Rightarrow \sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$$

EXAMPLE 31 *AB, AC and AD are three adjacent edges of a parallelo-piped. The diagonal of the parallelopiped passing through A and directed away from it is a vector* $\vec{a}$. *The vector area of the faces*

containing vertices A, B, C and A, B, D are $\vec{b}$ *and* $\vec{c}$ *respectively. If projection of each edge AB and AC on diagonal vector* $\vec{a}$ *is* $\frac{|\vec{a}|}{3}$, *then find the vectors* $\vec{AB}, \vec{AC}$ *and* $\vec{AD}$ *in terms of* $\vec{a}, \vec{b}, \vec{c}$ *and* $|\vec{a}|$.

SOLUTION Let $\vec{AB} = \vec{x}$, $\vec{AC} = \vec{y}$ and $\vec{AD} = \vec{z}$.

In ΔACQ, we have

$\vec{AC} + \vec{CQ} = \vec{AQ}$

$\Rightarrow \quad \vec{y} + \vec{x} = \vec{AQ}$

Now,

In ΔAQP, we have

$\vec{AQ} + \vec{QP} = \vec{AP}$

$\Rightarrow \vec{AQ} + \vec{AD} = \vec{AP}$

$\Rightarrow \vec{y} + \vec{x} + \vec{z} = \vec{d}$

$\Rightarrow \vec{x} + \vec{y} + \vec{z} = \vec{a}$ ...(i)

It is given that the vector areas of faces *ABQC* and *ABSD* are $\vec{b}$ and $\vec{c}$ respectively.

$\therefore \quad \vec{x} \times \vec{y} = \vec{b}$ ...(ii)

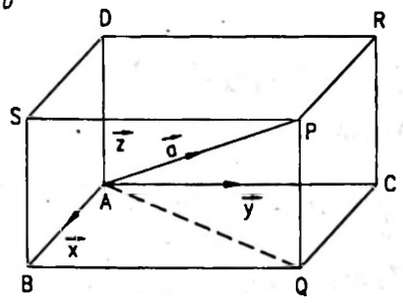


Fig. 17.4

and,

$\vec{z} \times \vec{x} = \vec{c}$ ...(iii)

It is given that the projection of edges $\vec{AB}$ and $\vec{AC}$ on the diagonal $\vec{AP}$ $(= \vec{a})$ is $\frac{|\vec{a}|}{3}$. Therefore,

$$\vec{x} \cdot \hat{a} = \frac{|\vec{a}|}{3} \text{ and } \vec{y} \cdot \hat{a} = \frac{|\vec{a}|}{3} \quad \text{...(iv)}$$

Taking dot product on both sides of (i) by $\hat{a}$, we get

$(\vec{x} + \vec{y} + \vec{z}) \cdot \hat{a} = \hat{a} \cdot \hat{a}$

$\Rightarrow \vec{x} \cdot \hat{a} + \vec{y} \cdot \hat{a} + \vec{z} \cdot \hat{a} = |\vec{a}|$

$\Rightarrow \vec{z} \cdot \hat{a} = \frac{|\vec{a}|}{3}$ ...(v) [Using (iv)]

Taking cross product on the sides of (ii) and (iii) by $\hat{a}$, we get

$a \times (\vec{x} \times \vec{y}) = \hat{a} \times \vec{b}$ and $\vec{a} \times (\vec{z} \times \vec{x}) = \hat{a} \times \vec{c}$

$\Rightarrow (\hat{a} \cdot \vec{y})\vec{x} - (\hat{a} \cdot \vec{x})\vec{y} = \hat{a} \times \vec{b}$ and $(\hat{a} \cdot \vec{x})\vec{z} - (\hat{a} \cdot \vec{z})\vec{x} = \hat{a} \times \vec{c}$

$\Rightarrow \frac{|\vec{a}|}{3}(\vec{x} - \vec{y}) = \hat{a} \times \vec{b}$ and $\frac{|\vec{a}|}{3}(\vec{z} - \vec{x}) = \hat{a} \times \vec{c}$ [Using (iv)&(v)]

$\Rightarrow \vec{x} - \vec{y} = \frac{3}{|\vec{a}|}(\hat{a} \times \vec{b})$ and $\vec{z} - \vec{x} = \frac{3}{|\vec{a}|}(\hat{a} \times \vec{c})$

Solving these two equations with (i), we get

$$\vec{x} = \frac{\vec{a}}{3} - \frac{\hat{a} \times (2\vec{b} + \vec{c})}{|\vec{a}|}, \quad \vec{y} = \frac{\vec{a}}{3} + \frac{\hat{a} \times (\vec{b} - \vec{c})}{|\vec{a}|}$$

$$\vec{z} = \frac{\vec{a}}{3} + \frac{\vec{a} \times (\vec{b} + 2\vec{c})}{|\vec{a}|}$$

EXAMPLE 32 *If* $\vec{u}, \vec{v}, \vec{w}$, *are three non-coplanar-unit vectors and* α, β, γ *are the angles between* $\vec{u}$ *and* $\vec{v}$, $\vec{v}$ *and* $\vec{w}$, $\vec{w}$ *and* $\vec{u}$ *respectively and* $\vec{x}, \vec{y}, \vec{z}$ *are unit vectors along the bisectors of the angles* α, β, γ, *respectively. Prove that*

$$[\vec{x} \times \vec{y}\ \ \vec{y} \times \vec{z}\ \ \vec{z} \times \vec{x}] = \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2} \sec^2\frac{\beta}{2} \sec^2\frac{\gamma}{2}.$$

[IIT 2003]

SOLUTION We have,

$$\vec{x} = \frac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|}, \quad \vec{y} = \frac{\vec{v} + \vec{w}}{|\vec{v} + \vec{w}|} \text{ and } \vec{z} = \frac{\vec{w} + \vec{u}}{|\vec{w} + \vec{u}|}$$

Now,

$|\vec{u} + \vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + 2(\vec{u} \cdot \vec{v})$

$= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}|\ |\vec{v}| \cos\alpha \ [\because |\vec{u}| = |\vec{v}| = 1]$

$= 2 + 2\cos\alpha = 4\cos^2\alpha/2$

$\therefore \quad |\vec{u} + \vec{v}| = 2\cos\frac{\alpha}{2},$

Similarly, we have

$|\vec{v} + \vec{w}| = 2\cos\frac{\beta}{2}$ and $|\vec{w} + \vec{u}| = 2\cos\frac{\gamma}{2}.$

$\therefore \quad \vec{x} = \frac{1}{2}\sec\frac{\alpha}{2}(\vec{u} + \vec{v}), \ \vec{y} = \frac{1}{2}\sec\frac{\beta}{2}(\vec{v} + \vec{w})$ and

$\vec{z} = \frac{1}{2}\sec\frac{\gamma}{2}(\vec{w} + \vec{u})$

$\Rightarrow [\vec{x}\vec{y}\vec{z}] = \frac{1}{8}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}[\vec{u} + \vec{v}\ \vec{v} + \vec{w}\ \vec{w} + \vec{u}]$

But, $[\vec{u} + \vec{v}\ \ \vec{v} + \vec{w}\ \ \vec{w} + \vec{u}] = 2[\vec{u}\vec{v}\vec{w}]$

$\therefore [\vec{x}\vec{y}\vec{z}] = \frac{2}{8}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}[\vec{u}\vec{v}\vec{w}]$

$\Rightarrow [\vec{x}\vec{y}\vec{z}] = \frac{1}{4}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}[\vec{u}\vec{v}\vec{w}]$

Now,

$\Rightarrow [\vec{x} \times \vec{y}\ \ \vec{y} \times \vec{z}\ \ \vec{z} \times \vec{x}]$

$= [\vec{x}\vec{y}\vec{z}]^2$ [see Ex. 4 on page 17.12]

$= \frac{1}{16}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}[\vec{u}\vec{v}\vec{w}]^2$

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17.5 RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors, so that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$. We define another set of three vectors $\vec{a'}, \vec{b'}, \vec{c'}$ as given below:

$$\vec{a'} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b'} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},$$

$$\vec{c'} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

These three vectors are said to form a reciprocal system of vectors for the vectors $\vec{a}, \vec{b}, \vec{c}$. The reason for the name lies in the relations between $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ as given in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors and $\vec{a'}, \vec{b'}, \vec{c'}$ form a reciprocal system of vectors, then prove that*

(i) $\vec{a}.\vec{a'} = \vec{b}.\vec{b'} = \vec{c}.\vec{c'} = 1$

(ii) $\vec{a}.\vec{b'} = \vec{a}.\vec{c'} = 0; \vec{b}.\vec{c'} = \vec{b}.\vec{a'} = 0; \vec{c}.\vec{a'} = \vec{c}.\vec{b'} = 0$

(iii) $[\vec{a'}\ \vec{b'}\ \vec{c'}] = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

(iv) $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar iff so are* $\vec{a'}, \vec{b'}, \vec{c'}$.

SOLUTION We have,

$$\vec{a'} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b'} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \text{ and } \vec{c'} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}.$$

or, $\vec{a'} = \lambda(\vec{b}\times\vec{c}), \vec{b'} = \lambda(\vec{c}\times\vec{a})$ and $\vec{c'} = \lambda(\vec{a}\times\vec{b})$,

where $\lambda = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

(i) $\vec{a}\cdot\vec{a'} = \vec{a}\cdot\lambda(\vec{b}\times\vec{c}) = \lambda\{\vec{a}\cdot(\vec{b}\times\vec{c})\}$

$$= \lambda[\vec{a}\ \vec{b}\ \vec{c}] = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1$$

Similarly, we have $\vec{b}\cdot\vec{b'} = 1$ and $\vec{c}\cdot\vec{c'} = 1$.

(ii) $\vec{a}\cdot\vec{b'} = \vec{a}\cdot\lambda(\vec{c}\times\vec{a}) = \lambda\{\vec{a}\cdot(\vec{c}\times\vec{a})\}$

$$= \lambda[\vec{a}\ \vec{c}\ \vec{a}] = \lambda 0 = 0$$

Similarly, we have

$$\vec{a}\cdot\vec{c'} = 0, \vec{b}\cdot\vec{c'} = \vec{b}\cdot\vec{a'} = 0, \vec{c}\cdot\vec{a'} = \vec{c}\cdot\vec{b'} = 0.$$

(iii) $[\vec{a'}\ \vec{b'}\ \vec{c'}] = (\vec{a'}\times\vec{b'})\cdot\vec{c'}$

$$= \{\lambda(\vec{b}\times\vec{c})\times\lambda(\vec{c}\times\vec{a})\}\cdot\vec{c'}$$

$$= \lambda^2\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}\cdot\vec{c'}$$

$$= \lambda^2\left[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c} - \{(\vec{b}\times\vec{c})\cdot\vec{c}\}\vec{a}\right]\cdot\vec{c'}$$

$$= \lambda^2\left\{[\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - [\vec{b}\ \vec{c}\ \vec{c}]\vec{a}\right\}\cdot\vec{c'}$$

$$= \lambda^2\left\{[\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - 0\,\vec{a}\right\}\cdot\vec{c'} = \lambda^2\left\{[\vec{b}\ \vec{c}\ \vec{a}](\vec{c}\cdot\vec{c'})\right\}$$

$$= \lambda^2[\vec{b}\ \vec{c}\ \vec{a}] \qquad [\because \vec{c}\cdot\vec{c'} = 1]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^2}[\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

(iv) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$$\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$$

$$\Leftrightarrow \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]} \neq 0$$

$$\Leftrightarrow [\vec{a'}\ \vec{b'}\ \vec{c'}] \neq 0 \qquad \left[\because [\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{[\vec{a'}\ \vec{b'}\ \vec{c'}]}\right]$$

$$\Leftrightarrow \vec{a'}, \vec{b'}, \vec{c'} \text{ are non-coplanar.}$$

EXAMPLE 2 *If $\vec{a}\ \vec{b}\ \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ be the reciprocal system of vectors, prove that*

(i) $\vec{a}\cdot\vec{a'} + \vec{b}\cdot\vec{b'} + \vec{c}\cdot\vec{c'} = 3$

(ii) $\vec{a}\times\vec{a'} + \vec{b}\times\vec{b'} + \vec{c}\times\vec{c'} = \vec{0}$

SOLUTION We have,

$$\vec{a'}c = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \vec{b'} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} \text{ and } \vec{c'} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

or, $\vec{a'} = \lambda(\vec{b}\times\vec{c}), \vec{b'} = \lambda(\vec{c}\times\vec{a})$ and $\vec{c'} = \lambda(\vec{a}\times\vec{b})$,

$$\text{where } \lambda = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

(i) By Property I, we have

$$\vec{a}\cdot\vec{a'} = \vec{b}\cdot\vec{b'} = \vec{c}\cdot\vec{c'} = 1$$

$$\therefore \vec{a}\cdot\vec{a'} + \vec{b}\cdot\vec{b'} + \vec{c}\cdot\vec{c'} = 1+1+1 = 3.$$

(ii) We have,

$$\vec{a}\times\vec{a'} = \vec{a}\times\lambda(\vec{b}\times\vec{c}) = \lambda\{\vec{a}\times(\vec{b}\times\vec{c})\} = \lambda\{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\}$$

$$\vec{b}\times\vec{b'} = \vec{b}\times\lambda(\vec{c}\times\vec{a}) = \lambda\{\vec{b}\times(\vec{c}\times\vec{a})\} = \lambda\{(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}\}$$

and,

$$\vec{c}\times\vec{c'} = \vec{c}\times\lambda(\vec{a}\times\vec{b}) = \lambda\{\vec{c}\times(\vec{a}\times\vec{b})\} = \lambda\{(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\}$$

$$\therefore \vec{a}\times\vec{a'} + \vec{b}\times\vec{b'} + \vec{c}\times\vec{c'}$$

$$= \lambda\{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\} + \lambda\{(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}\} + \lambda\{(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\}$$

$$= \lambda[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}]$$

$$= \lambda[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}]$$

$= \lambda \vec{0} = \vec{0}$.

EXAMPLE 3 *If $\vec{e_1}, \vec{e_2}, \vec{e_3}$ and $\vec{e_1}', \vec{e_2}', \vec{e_3}'$ are two sets of non-coplanar vectors such that for $i = 1, 2, 3$, we have*

$$\vec{e_i} \cdot \vec{e_j}' = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases},$$

then show that $[\vec{e_1}\ \vec{e_2}\ \vec{e_3}]\ [\vec{e_2}'\ \vec{e_2}'\ \vec{e_3}'] = 1$

SOLUTION We have,

$$\vec{e_i} \cdot \vec{e_j}' = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases} \text{ for } i, j = 1, 2, 3$$

$\Rightarrow \vec{e_1} \cdot \vec{e_2}' = 0, \vec{e_1} \cdot \vec{e_3}' = 0$

$\Rightarrow \vec{e_1} \perp \vec{e_2}', \vec{e_1} \perp \vec{e_3}'$

$\Rightarrow \vec{e_1} \parallel \vec{e_2}' \times \vec{e_3}'$,

$\Rightarrow \vec{e_1} = \lambda\ (\vec{e_2}' \times \vec{e_3}')$ ...(i)

$\Rightarrow \vec{e_1} \cdot \vec{e_1}' = \lambda\ \{\vec{e_1}' \cdot (\vec{e_2}' \times \vec{e_3}')\}$

$\Rightarrow 1 = \lambda\ [\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']$ $[\because \vec{e_1} \cdot \vec{e_1}' = 1]$

$\therefore \Rightarrow \lambda = \dfrac{1}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']}$

$\therefore$ From (i), we have

$$\vec{e_1} = \frac{\vec{e_2}' \times \vec{e_3}'}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']}$$

Similarly,

$$\vec{e_2} = \frac{\vec{e_3}' \times \vec{e_1}'}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']} \text{ and } \vec{e_3} = \frac{\vec{e_1}' \times \vec{e_2}'}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']}$$

$$\therefore [\vec{e_1}\ \vec{e_2}\ \vec{e_3}] = \frac{[\vec{e_2}' \times \vec{e_3}'\ \ \vec{e_3}' \times \vec{e_1}'\ \ \vec{e_1}' \times \vec{e_2}']}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']^3}$$

$$= \frac{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']^2}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']^3} \quad \left[\because [\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2\right]$$

$$= \frac{1}{[\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}']}$$ Hence, $[\vec{e_1}\ \vec{e_2}\ \vec{e_3}]\ [\vec{e_1}'\ \vec{e_2}'\ \vec{e_3}'] = 1$

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17.6 SOLUTIONS OF VECTOR EQUATIONS

Solving a vector equation means determining an unknown vector or a number of vectors satisfying the given conditions. Generally, to solve a vector equation, we express the unknown vector as a linear combination of three known non-coplanar vectors and then we determine the coefficients from the given conditions. The following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the vector equation $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$, where $\vec{a}, \vec{b}$ are two given vectors.*

SOLUTION We have, $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$

$\Rightarrow \vec{r} \times \vec{b} - \vec{a} \times \vec{b} = \vec{0} \Rightarrow (\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$

$\Rightarrow \vec{r} - \vec{a}$ is parallel to $\vec{b} \Rightarrow \vec{r} - \vec{a} = \lambda \vec{b}$ for some scalar λ

$\Rightarrow \vec{r} = \vec{a} + \lambda \vec{b}$ is the required solution.

ALITER Since $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are three non-coplaner vectors and any vector can be uniquely expressed as a linear combination of three non-coplanar vectors. So, let

$\vec{r} = x\vec{a} + y\vec{b} + z\ (\vec{a} \times \vec{b})$ ...(i)
where x, y, z are scalars

Substituting the expression for $\vec{r}$ in $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$, we get

$\{x\vec{a} + y\vec{b} + z\ (\vec{a} \times \vec{b})\} \times \vec{b} = \vec{a} \times \vec{b}$

$\Rightarrow x\ (\vec{a} \times \vec{b}) + y\ (\vec{b} \times \vec{b}) + z\ \{(\vec{a} \times \vec{b}) \times \vec{b}\} = \vec{a} \times \vec{b}$

$\Rightarrow x\ (\vec{a} \times \vec{b}) - z\ \{(\vec{b} \cdot \vec{b})\vec{a} - (\vec{b} \cdot \vec{a})\vec{b}\} = \vec{a} \times \vec{b}$

$\Rightarrow (x - 1)\ (\vec{a} \times \vec{b}) - z\ (\vec{b} \cdot \vec{b})\vec{a} + z\ (\vec{a} \cdot \vec{b})\vec{b} = \vec{0}$

$\Rightarrow x - 1 = 0, z\ (\vec{b} \cdot \vec{b}) = 0$ and $z\ (\vec{a} \cdot \vec{b}) = 0$

$[\because \vec{a} \times \vec{b}, \vec{a}, \vec{b}$ are non–coplanar vectors]

$\Rightarrow x = 1, z = 0.$

Putting $x = 1, z = 0$ in (i), we get

$\vec{r} = \vec{a} + y\vec{b}$ as the required solution.

EXAMPLE 2 *If $\vec{a} \cdot \vec{b} \neq 0$, find the vector $\vec{r}$ which satisfies the equations*

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b} \text{ and } \vec{r} \cdot \vec{a} = 0$$

SOLUTION We have, $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0} \Rightarrow (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$

$\Rightarrow \vec{r} - \vec{c}$ is parallel to $\vec{b} \Rightarrow \vec{r} - \vec{c} = \lambda \vec{b}$ for some scalar λ

$\Rightarrow \vec{r} = \vec{c} + \lambda \vec{b}$ ...(i)

Now, $\vec{r} \cdot \vec{a} = 0$

$\Rightarrow (\vec{c} + \lambda \vec{b}) \cdot \vec{a} = 0$

$\Rightarrow \vec{c} \cdot \vec{a} + \lambda \vec{b} \cdot \vec{a} = 0$

$\Rightarrow \lambda = -\left(\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)$

Substituting the value of λ in (i), we get

$\vec{r} = \vec{c} - \left(\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)\vec{b}$ as the desired solution.

ALITER Let $\vec{r} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$ ...(i)

where, λ, μ, ν scalars.

Substituting this value of $\vec{r}$ in $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$, we get

$\Rightarrow (\lambda\vec{a} + \mu\vec{b} + \nu\vec{c})\times\vec{b} = \vec{c}\times\vec{b}$

$\Rightarrow \lambda(\vec{a}\times\vec{b}) + \mu(\vec{b}\times\vec{b}) + \nu(\vec{c}\times\vec{b}) = \vec{c}\times\vec{b}$

$\Rightarrow \lambda(\vec{a}\times\vec{b}) + (\nu - 1)(\vec{c}\times\vec{b}) = \vec{0}$

$\Rightarrow \lambda = 0, \nu - 1 = 0$ [$\because \vec{a}\times\vec{b}$ and $\vec{c}\times\vec{b}$ are non–collinear vectors]

$\Rightarrow \lambda = 0, \nu = 1$

Now, substituting the value of $\vec{r}$ in $\vec{r}\cdot\vec{a} = 0$, we get

$(\lambda\vec{a} + \mu\vec{b} + \nu\vec{c})\cdot\vec{a} = 0$

$\Rightarrow \lambda(\vec{a}\cdot\vec{a}) + \mu(\vec{b}\cdot\vec{a}) + \nu(\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \mu(\vec{b}\cdot\vec{a}) + (\vec{c}\cdot\vec{a}) = 0$ [$\because \lambda = 0, \nu = 1$]

$\Rightarrow \mu = -\left(\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)$

Substituting the values of λ, μ and ν in (i), we get

$\vec{r} = \vec{c} - \left(\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)\vec{b}$ as the required solution.

EXAMPLE 3 *Solve the vector equation $\vec{r}\times\vec{a} = \vec{b}$ given that $\vec{a}$ and $\vec{b}$ are given vectors such that $\vec{a}\cdot\vec{b} = 0$*

SOLUTION Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar vectors. So, let

$$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b}) \quad ...(i)$$

for some scalars x, y, z.

Substituting this value of $\vec{r}$ in $\vec{r}\times\vec{a} = \vec{b}$, we obtain

$\{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}\times\vec{a} = \vec{b}$

$\Rightarrow x(\vec{a}\times\vec{a}) + y(\vec{b}\times\vec{a}) + z\{(\vec{a}\times\vec{b})\times\vec{a}\} = \vec{b}$

$\Rightarrow y(\vec{b}\times\vec{a}) + z\{-(\vec{a}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{a})\vec{b}\} = \vec{b}$

$\Rightarrow 0\vec{a} + \{z|\vec{a}|^2 - 1\}\vec{b} + y(\vec{b}\times\vec{a}) = \vec{0}$ [$\because \vec{a}\cdot\vec{b} = 0$]

$\Rightarrow \{z|\vec{a}|^2 - 1\} = 0, y = 0$ [$\because \vec{a}, \vec{b}, \vec{a}\times\vec{b}$ are non–collinear vectors]

$\Rightarrow y = 0, z = \frac{1}{|\vec{a}|^2}$

Putting the values of y and z in (i), we get

$\vec{r} = x\vec{a} + \frac{1}{|\vec{a}|^2}(\vec{a}\times\vec{b})$, where $x \in R$

as the required solution.

EXAMPLE 4 *Solve the vector equation $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$, where $\vec{a}, \vec{b}$ are two given vectors and k is a given scalar.*

SOLUTION Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are two non-coplanar vectors. Therefore,

$$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b}) \quad ...(i)$$

for some scalars x, y, z

Putting this value of $\vec{r}$ in $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$, we get

$\{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}\times\vec{a} + k\{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\} = \vec{b}$

$\Rightarrow y(\vec{b}\times\vec{a}) + z\{(\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\} + k\{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\} = \vec{b}$

$\Rightarrow \{kx - z(\vec{a}\cdot\vec{b})\}\vec{a} + \{ky + z(\vec{a}\cdot\vec{a}) - 1\}\vec{b} + (-y + zk)(\vec{a}\times\vec{b}) = \vec{0}$

$\Rightarrow kx - z(\vec{a}\cdot\vec{b}) = 0, ky + z(\vec{a}\cdot\vec{a}) - 1 = 0$ and $-y + zk = 0$

$\Rightarrow z = \frac{1}{|\vec{a}|^2 + k^2}, y = \frac{k}{k^2 + |\vec{a}|^2}$ and $x = \frac{\vec{a}\cdot\vec{b}}{k\left(|\vec{a}|^2 + k^2\right)}$

Putting the values of x, y and z in (i), we get

$\vec{r} = \frac{1}{|\vec{a}|^2 + k^2}\left[\frac{1}{k}(\vec{a}\cdot\vec{b})\vec{a} + k\vec{b} + (\vec{a}\times\vec{b})\right]$

EXAMPLE 5 *Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the unit vectors such that $\vec{a}\times\vec{b} = \vec{c}$. Also, $\vec{X}$ is any vector such that*

$$[\vec{X}\ \vec{b}\ \vec{c}] = 3, [\vec{X}\ \vec{c}\ \vec{a}] = 4 \text{ and } [\vec{X}\ \vec{a}\ \vec{b}] = 2$$

find $\vec{X}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

SOLUTION We have,

$\vec{a}\times\vec{b} = \vec{c}$

$\Rightarrow \vec{c}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.

Again,

$(\vec{a}\times\vec{b}) = \vec{c}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{c}\cdot\vec{c}$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = |\vec{c}|^2 = 1$

Since any vector in space can be expressed as a linear combination of three non-coplanar vectors. So, let

$$\vec{X} = l\vec{a} + m\vec{b} + n\vec{c} \quad ...(i)$$

Taking dot product on both sides by $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ successively, we get

$[\vec{X}\ \vec{a}\ \vec{b}] = n[\vec{c}\ \vec{a}\ \vec{b}], [\vec{X}\ \vec{b}\ \vec{c}] = l[\vec{a}\ \vec{b}\ \vec{c}]$

and, $[\vec{X}\ \vec{c}\ \vec{a}] = m[\vec{b}\ \vec{c}\ \vec{a}]$

$\Rightarrow 2 = n[\vec{a}\ \vec{b}\ \vec{c}], 3 = l[\vec{a}\ \vec{b}\ \vec{c}]$ and $4 = m[\vec{a}\ \vec{b}\ \vec{c}]$

$$\Rightarrow l = \frac{3}{[\vec{a}\,\vec{b}\,\vec{c}]}, m = \frac{4}{[\vec{a}\,\vec{b}\,\vec{c}]}, n = \frac{2}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\Rightarrow l = 3, m = 4, n = 2 \qquad [\because [\vec{a}\,\vec{b}\,\vec{c}] = 1]$$

Substituting these values in (i), we get

$$\vec{X} = (3\vec{a} + 4\vec{b} + 2\vec{c}).$$

EXAMPLE 6 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors; solve the vector equation*

$$\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = 1.$$

SOLUTION Since $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, therefore $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are also non-coplanar vectors.

Let $\vec{r} = x(\vec{a}\times\vec{b}) + y(\vec{b}\times\vec{c}) + z(\vec{c}\times\vec{a})$. ...(i)

Then,

$$\vec{r}.\vec{a} = 1$$

$$\Rightarrow 1 = y\{(\vec{b}\times\vec{c}).\vec{a}\}$$

$$\Rightarrow y = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly, $\vec{r}.\vec{b} = 1$ and $\vec{r}.\vec{c} = 1$ give

$$z = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ and } x = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Substituting the values of x, y and z in (i), we get

$$\vec{r} = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$$

as the required solution.

EXAMPLE 7 *If $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, solve the vector equations $\vec{a}.\vec{r} = 4$, $\vec{b}.\vec{r} = -3$ and $(\vec{a}\times\vec{b}).\vec{r} = 10$.*

SOLUTION Let $\vec{r} = l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})$ ...(i)

Then,

$$\vec{a}.\vec{r} = 4$$

$$\Rightarrow \{l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})\}.\vec{a} = 4$$

$$\Rightarrow l(\vec{a}.\vec{a}) + m(\vec{b}.\vec{a}) + n\{(\vec{a}\times\vec{b}).\vec{a}\} = 4$$

$$\Rightarrow 6l - 2m = 4 \qquad \text{...(ii)} \qquad [\because (\vec{a}\times\vec{b}).\vec{a} = 0]$$

$$\vec{b}.\vec{r} = -3$$

$$\Rightarrow \vec{b}.\{l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})\} = -3$$

$$\Rightarrow l(\vec{b}.\vec{a}) + m(\vec{b}.\vec{b}) + n\{\vec{b}.(\vec{a}\times\vec{b})\} = -3$$

$$\Rightarrow -2l + 9m = -3 \qquad \text{...(iii)}$$

and,

$$(\vec{a}\times\vec{b}).\vec{r} = 10$$

$$\Rightarrow (\vec{a}\times\vec{b}).\{l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})\} = 10$$

$$\Rightarrow l\{(\vec{a}\times\vec{b}).\vec{a}\} + m\{(\vec{a}\times\vec{b}).\vec{b}\} + n\{(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})\} = 10$$

$$\Rightarrow l \times 0 + m \times 0 + n\,|\vec{a}\times\vec{b}|^2 = 10.$$

$$\Rightarrow n(9 + 16 + 25) = 10 \qquad [\because \vec{a}\times\vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}]$$

$$\Rightarrow n = \frac{1}{5}.$$

Solving (ii) and (iii), we get

$$l = \frac{3}{5}, m = -\frac{1}{5}$$

Substituting the values of l, m, n in (i), we get

$$\vec{r} = \frac{3}{5}\vec{a} - \frac{1}{5}\vec{b} + \frac{1}{5}(\vec{a}\times\vec{b})$$

$$= \frac{1}{5}\{3(\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k}) + 3\hat{i} + 4\hat{j} + 5\hat{k}\}$$

$$= \frac{1}{5}(4\hat{i} - 3\hat{j} + 10\hat{k})$$

EXAMPLE 8 *Solve the following simultaneous vector equations for $\vec{X}$ and $\vec{Y}$:*

$$\vec{X} + \vec{Y} = \vec{a}, \vec{X}\times\vec{Y} = \vec{b} \text{ and } \vec{X}.\vec{a} = 1$$

SOLUTION Let

$$\vec{X} = l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b}) \qquad \text{...(i)}$$

where l, m, n are scalars.

Then,

$$\vec{X} + \vec{Y} = \vec{a}$$

$$\Rightarrow \vec{Y} = \vec{a} - \vec{X}$$

$$\Rightarrow \vec{Y} = (1 - l)\vec{a} - m\vec{b} - n(\vec{a}\times\vec{b})$$

Now,

$$\vec{X}.\vec{a} = 1$$

$$\Rightarrow \{l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})\}.\vec{a} = 1$$

$$\Rightarrow l(\vec{a}.\vec{a}) + m(\vec{b}.\vec{a}) + n\{(\vec{a}\times\vec{b}).\vec{a}\} = 1$$

$$\Rightarrow l\,|\vec{a}|^2 + m(\vec{a}.\vec{b}) = 1 \qquad \text{...(ii)}$$

$$\vec{x}\times\vec{y} = \vec{b}$$

$$\Rightarrow \vec{X}\times(\vec{a} - \vec{X}) = \vec{b}$$

$$\vec{X}\times\vec{a} = \vec{b}$$

$$\Rightarrow \{l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})\}\times\vec{a} = \vec{b}$$

$$\Rightarrow l(\vec{a}\times\vec{a}) + m(\vec{b}\times\vec{a}) + n(\vec{a}\times\vec{b})\times\vec{a} = \vec{b}$$

$$\Rightarrow m(\vec{b}\times\vec{a}) - n\{(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}\} = \vec{b}$$

$$\Rightarrow m(\vec{b}\times\vec{a}) - n(\vec{a}.\vec{b})\vec{a} + \{n\,|\vec{a}|^2 - 1\}\vec{b} = \vec{0}$$

$\Rightarrow m=0, -n(\vec{a}.\vec{b})=0,\ n\,|\vec{a}|^2-1=0$

[$\because \vec{a},\vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar]

$\Rightarrow n=\dfrac{1}{|\vec{a}|^2}$ and $m=0$

Putting $m=0$ in (ii), we get

$$l\,|\vec{a}|^2=1 \Rightarrow l=\frac{1}{|\vec{a}|^2}$$

Putting the values of l, m, n in (i), we get

$$\vec{X}=\frac{1}{|\vec{a}|^2}\vec{a}+\frac{1}{|\vec{a}|^2}(\vec{a}\times\vec{b})$$

$$\Rightarrow \vec{X}=\frac{1}{|\vec{a}|^2}(\vec{a}+\vec{a}\times\vec{b})$$

$$\therefore \vec{Y}=\vec{a}-\vec{X}$$

$$\Rightarrow \vec{Y}=\vec{a}-\frac{1}{|\vec{a}|^2}(\vec{a}+\vec{a}\times\vec{b}).$$

EXAMPLE 9 *If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}.\vec{b}\neq 0$, then solve the vector equations*

$$\vec{r}.\vec{a}=0, \vec{r}.\vec{b}=1, [\vec{r}\,\vec{a}\,\vec{b}]=1.$$

SOLUTION Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar vectors. So, let

$$\vec{r}=l\vec{a}+m\vec{b}+n(\vec{a}\times\vec{b}) \qquad ...(i)$$

where l, m, n are scalars.

Now,

$$\vec{r}.\vec{a}=0$$

$$\Rightarrow \{l\vec{a}+m\vec{b}+n(\vec{a}\times\vec{b})\}.\vec{a}=0$$

$$\Rightarrow l(\vec{a}.\vec{a})+m(\vec{b}.\vec{a})+n\{(\vec{a}\times\vec{b}).\vec{a}\}=0$$

$$\Rightarrow l\,|\vec{a}|^2+m(\vec{a}.\vec{b})+n\times 0=0 \qquad [\because [\vec{a}\,\vec{b}\,\vec{a}]=0]$$

$$\Rightarrow l\,|\vec{a}|^2+m(\vec{a}.\vec{b})=0 \qquad ...(ii)$$

$$\vec{r}.\vec{b}=1$$

$$\Rightarrow \{l\vec{a}+m\vec{b}+n(\vec{a}\times\vec{b})\}.\vec{b}=1$$

$$\Rightarrow l(\vec{a}.\vec{b})+m(\vec{b}.\vec{b})+n\{(\vec{a}\times\vec{b}).\vec{b}\}=1$$

$$\Rightarrow l(\vec{a}.\vec{b})+m\,|\vec{b}|^2=1 \qquad ...(iii)$$

and,

$$[\vec{r}\,\vec{a}\,\vec{b}]=1$$

$$\Rightarrow \vec{r}.(\vec{a}\times\vec{b})=1$$

$$\Rightarrow \{l\vec{a}+m\vec{b}+n(\vec{a}\times\vec{b})\}.(\vec{a}\times\vec{b})=1$$

$$\Rightarrow l[\vec{a}.(\vec{a}\times\vec{b})]+m[\vec{b}.(\vec{a}\times\vec{b})]+n(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})=1$$

$$\Rightarrow n\,|\vec{a}\times\vec{b}|^2=1$$

$$\Rightarrow n=\frac{1}{|\vec{a}\times\vec{b}|^2}$$

Solving (ii) and (iii), we get

$$l=\frac{-(\vec{a}.\vec{b})^2}{|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2},\quad m=\frac{|\vec{a}|^2}{|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2}$$

$$\Rightarrow l=\frac{-(\vec{a}.\vec{b})^2}{|\vec{a}\times\vec{b}|^2},\qquad m=\frac{|\vec{a}|^2}{|\vec{a}\times\vec{b}|^2}$$

Substituting these values in (i), we get

$$\vec{r}=\frac{1}{|\vec{a}\times\vec{b}|^2}\left\{-(\vec{a}.\vec{b})^2\vec{a}+|\vec{a}|^2\vec{b}+(\vec{a}\times\vec{b})\right\}.$$

EXAMPLE 10 *If $\vec{r}\times\vec{a}+(\vec{r}.\vec{b})\vec{c}=\vec{d}$, then prove that*

$$\vec{r}=\lambda\vec{a}+\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}.\vec{c})\,|\vec{a}|^2}\right\},\ \textit{where } \lambda \textit{ is a scalar}$$

SOLUTION We have,

$$\vec{r}\times\vec{a}+(\vec{r}.\vec{b})\vec{c}=\vec{d}$$

$$\Rightarrow \{\vec{r}\times\vec{a}+(\vec{r}.\vec{b})\vec{c}\}\times\vec{c}=\vec{d}\times\vec{c}$$

$$\Rightarrow (\vec{r}\times\vec{a})\times\vec{c}=\vec{d}\times\vec{c}$$

$$\Rightarrow (\vec{r}.\vec{c})\vec{a}-(\vec{a}.\vec{c})\vec{r}=\vec{d}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\{(\vec{r}.\vec{c})\vec{a}-(\vec{a}.\vec{c})\vec{r}\}=\vec{a}\times(\vec{d}\times\vec{c})$$

[Taking cross product with $\vec{a}$]

$$\Rightarrow -(\vec{a}.\vec{c})(\vec{a}\times\vec{r})=\vec{a}\times(\vec{d}\times\vec{c})$$

$$\Rightarrow \vec{r}\times\vec{a}=\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}.\vec{c})}$$

$$\Rightarrow \vec{a}\times(\vec{r}\times\vec{a})=\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}.\vec{c}}\right\}$$ [Taking cross-product with $\vec{a}$]

$$\Rightarrow (\vec{a}.\vec{a})\vec{r}-(\vec{a}.\vec{r})\vec{a}=\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}.\vec{c})}\right\}$$

$$\Rightarrow |\vec{a}|^2\vec{r}=(\vec{a}.\vec{r})\vec{a}+\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}.\vec{c}}\right\}$$

$$\Rightarrow \vec{r}=\frac{(\vec{a}.\vec{r})\vec{a}}{|\vec{a}|^2}+\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}.\vec{c})\,|\vec{a}|^2}\right\}$$

$$\Rightarrow \vec{r}=\lambda\vec{a}+\vec{a}\times\left\{\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}.\vec{c})\,|\vec{a}|^2}\right\},\ \text{where } \lambda=\frac{\vec{a}.\vec{r}}{|\vec{a}|^2}.$$

EXAMPLE 11 *If $\vec{x}+\vec{y}+\vec{z}=\vec{a}$, $\vec{x}\times\vec{y}=\vec{b}$, $\vec{y}\times\vec{z}=\vec{c}$, $\vec{a}.\vec{x}=\vec{a}.\vec{y}=1$ and $|\vec{a}|=\sqrt{3}$, then find vectors $\vec{x}, \vec{y}, \vec{z}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$*

SOLUTION We have,

$$\vec{x}+\vec{y}+\vec{z}=\vec{a}$$

$\Rightarrow \vec{a}.(\vec{x}+\vec{y}+\vec{z}) = \vec{a}.\vec{a}$

$\Rightarrow \vec{a}.\vec{x}+\vec{a}.\vec{y}+\vec{a}.\vec{z} = |\vec{a}|^2$

$\Rightarrow 1+1+\vec{a}.\vec{z} = 3$

$\Rightarrow \vec{a}.\vec{z} = 1$

Now,

$\vec{x}\times\vec{y} = \vec{b}$

$\Rightarrow \vec{a}\times(\vec{x}\times\vec{y}) = \vec{a}\times\vec{b}$ [Taking cross product with $\vec{a}$]

$\Rightarrow (\vec{a}.\vec{y})\vec{x}-(\vec{a}.\vec{x})\vec{y} = \vec{a}\times\vec{b}$

$\Rightarrow \vec{x}-\vec{y} = \vec{a}\times\vec{b}$ ...(i) $[\because \vec{a}.\vec{x} = \vec{a}.\vec{y} = 1]$

It is given that

$\vec{y}\times\vec{z} = \vec{c}$

$\Rightarrow \vec{a}\times(\vec{y}\times\vec{z}) = \vec{a}\times\vec{c}$

$\Rightarrow (\vec{a}.\vec{z})\vec{y}-(\vec{a}.\vec{y})\vec{z} = \vec{a}\times\vec{c}$

$\Rightarrow \vec{y}-\vec{z} = \vec{a}\times\vec{c}$ ...(ii) $[\because \vec{a}.\vec{y} = \vec{a}.\vec{z} = 1]$

It is given that

$\vec{x}+\vec{y}+\vec{z} = \vec{a}$ ...(iii)

Solving (i), (ii) and (iii), we get

$\Rightarrow \vec{x} = \dfrac{\vec{a}+\vec{a}\times\vec{c}+2(\vec{a}\times\vec{b})}{3}, \vec{y} = \dfrac{\vec{a}+\vec{a}\times\vec{c}-\vec{a}\times\vec{b}}{3}$

and,

$\vec{z} = \dfrac{\vec{a}-2\vec{a}\times\vec{c}-\vec{a}\times\vec{b}}{3}$

EXAMPLE 12 *Find vectors $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}, p, q$ and r, if*

$$p\vec{x}+q\vec{y} = \vec{a},\ \vec{x}.\vec{y} = r.$$

SOLUTION We have,

$p\vec{x}+q\vec{y} = \vec{a}$ ...(i)

and, $\vec{x}.\vec{y} = r$ ...(ii)

We know that

$|\vec{p}\vec{x}+\vec{q}\vec{y}|^2-|p\vec{x}-q\vec{y}|^2 = 4pq(\vec{x}.\vec{y})$

$\Rightarrow |p\vec{x}-q\vec{y}|^2 = |p\vec{x}+q\vec{y}|^2-4pq(\vec{x}.\vec{y})$

$\Rightarrow |p\vec{x}-q\vec{y}|^2 = |\vec{a}|^2-4pqr$

$\Rightarrow |p\vec{x}-q\vec{y}| = \sqrt{|\vec{a}|^2-4pqr}$

$\Rightarrow p\vec{x}-q\vec{y} = \sqrt{|\vec{a}|^2-4pqr}\,\hat{n}$,

where $\hat{n}$ is arbitrary unit vector.

Thus, we have,

$p\vec{x}+q\vec{y} = \vec{a}$ and $p\vec{x}-q\vec{y} = \sqrt{|\vec{a}|^2-4pqr}\,\hat{n}$

$\Rightarrow \vec{x} = \dfrac{\vec{a}+\sqrt{|\vec{a}|^2-4pqr}\,\hat{n}}{2p}$ and $\vec{y} = \dfrac{\vec{a}-\sqrt{|\vec{a}|^2-4pqr}\,\hat{n}}{2q}$

EXAMPLE 13 *Let $\vec{a},\vec{b},\vec{c}$ be non-coplanar vectors. Given equations $\vec{r}\times\vec{a} = \vec{b}, \vec{r}\times\vec{c} = \vec{d}$, obtain condition for these equations to be consistent and then solve the equations.*

SOLUTION We have,

$\vec{r}\times\vec{a} = \vec{b}$ ...(i)

and, $\vec{r}\times\vec{c} = \vec{d}$ ...(ii)

Since $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors. Therefore, there exist scalars x, y, z such that

$\vec{r} = x\vec{a}+y\vec{b}+z\vec{c}$ ...(iii)

Now,

$\vec{r}\times\vec{a} = \vec{b}$

$\Rightarrow (x\vec{a}+y\vec{b}+z\vec{c})\times\vec{a} = \vec{b}$

$\Rightarrow y(\vec{b}\times\vec{a})+z(\vec{c}\times\vec{a}) = \vec{b}$ ...(iv)

$\Rightarrow \{y(\vec{b}\times\vec{a})+z(\vec{c}\times\vec{a})\}.\vec{c} = \vec{b}.\vec{c}$

$\Rightarrow y(\vec{b}\times\vec{a}).\vec{c} = \vec{b}.\vec{c}$

$\Rightarrow y[\vec{b}\ \vec{a}\ \vec{c}] = \vec{b}.\vec{c}$

$\Rightarrow y = -\dfrac{\vec{b}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

and,

$\vec{r}\times\vec{c} = \vec{d}$

$\Rightarrow (x\vec{a}+y\vec{b}+z\vec{c})\times\vec{c} = \vec{d}$

$\Rightarrow x(\vec{a}\times\vec{c})+y(\vec{b}\times\vec{c}) = \vec{d}$ ...(v)

$\Rightarrow \{x(\vec{a}\times\vec{c})+y(\vec{b}\times\vec{c})\}.\vec{a} = \vec{d}.\vec{a}$

$\Rightarrow y[\vec{b}\ \vec{c}\ \vec{a}] = \vec{d}.\vec{a}$

$\Rightarrow y = \dfrac{\vec{d}.\vec{a}}{[\vec{b}\ \vec{c}\ \vec{a}]} = \dfrac{\vec{d}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

For the consistency of the system of equations, these two values of y must be equal.

$\therefore -\dfrac{\vec{b}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \dfrac{\vec{d}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$\Rightarrow -(\vec{b}.\vec{c}) = \vec{a}.\vec{d}$

$\Rightarrow \vec{a}.\vec{d}+\vec{b}.\vec{c} = 0$

To find the value of x, let us take dot product on both sides of (v) by $\vec{b}$. This gives

$\{x(\vec{a}\times\vec{c})+y(\vec{b}\times\vec{c})\}.\vec{b} = \vec{d}.\vec{b}$

$\Rightarrow x[\vec{a}\ \vec{c}\ \vec{b}] = \vec{d}.\vec{b}$

$\Rightarrow x = \dfrac{\vec{d}.\vec{b}}{[\vec{a}\ \vec{c}\ \vec{b}]} = \dfrac{\vec{d}.\vec{b}}{-[\vec{c}\ \vec{a}\ \vec{b}]} = \dfrac{\vec{d}.\vec{b}}{-[\vec{a}\ \vec{b}\ \vec{c}]} = -\dfrac{\vec{b}.\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Taking dot product on both sides of (iv) by $\vec{b}$, we get

$$\{y(\vec{b}\times\vec{a})+z(\vec{c}\times\vec{a})\}.\vec{b}=\vec{b}.\vec{b}$$

$$\Rightarrow \quad z[\vec{c}\,\vec{a}\,\vec{b}]=\vec{b}.\vec{b}$$

$$\Rightarrow \quad z=\frac{\vec{b}.\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Substituting the values of x, y, z in (i), we get

$$\vec{r}=\frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}\left\{-(\vec{b}.\vec{d})\vec{a}+(\vec{a}.\vec{d})\vec{b}+(\vec{b}.\vec{b})\vec{c}\right\}$$

EXAMPLE 14 *If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, solve the vector equation*

$$x\vec{a}+y\vec{b}+z\vec{c}=\vec{d}$$

SOLUTION We have,

$$x\vec{a}+y\vec{b}+z\vec{c}=\vec{d}$$

Taking dot products on both sides by $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ successively, we get

$$x\vec{a}.(\vec{b}\times\vec{c})=\vec{d}.(\vec{b}\times\vec{c})$$

$$y\vec{b}.(\vec{c}\times\vec{a})=\vec{d}.(\vec{c}\times\vec{a})$$

$$z\vec{c}.(\vec{a}\times\vec{b})=\vec{d}.(\vec{a}\times\vec{b})$$

$$\Rightarrow \quad x[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{d}\,\vec{b}\,\vec{c}],$$

$$y[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{d}\,\vec{c}\,\vec{a}],$$

$$z[\vec{c}\,\vec{a}\,\vec{b}]=[\vec{d}\,\vec{a}\,\vec{b}]$$

$$\Rightarrow \quad x=\frac{[\vec{d}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, y=\frac{[\vec{d}\,\vec{c}\,\vec{a}]}{[\vec{b}\,\vec{c}\,\vec{a}]}, z=\frac{[\vec{d}\,\vec{a}\,\vec{b}]}{[\vec{c}\,\vec{a}\,\vec{b}]}$$

$$\Rightarrow \quad x=\frac{[\vec{d}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, y=\frac{[\vec{d}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, z=\frac{[\vec{d}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\Rightarrow \quad x=\frac{[\vec{d}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, y=\frac{[\vec{a}\,\vec{d}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}, z=\frac{[\vec{a}\,\vec{b}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

REMARK *This is known as the vector form of the Cramer's rule.*

EXAMPLE 15 *Let $\vec{a}$ and $\vec{b}$ be two given non-zero and non-collinear vectors and $\vec{c}$ be a vector such that $\vec{c}\times\vec{a}=\vec{b}-\vec{c}$. Express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.*

SOLUTION Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar vectors. So, let

$$\vec{c}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b}) \qquad ...(i)$$

$$\Rightarrow \quad \vec{c}\times\vec{a}=\{x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})\}\times\vec{a}$$

$$\Rightarrow \quad \vec{c}\times\vec{a}=y(\vec{b}\times\vec{a})+z\{(\vec{a}.\vec{a})\vec{b}-(\vec{a}.\vec{b})\vec{a}\}$$

$$\Rightarrow \quad \vec{c}\times\vec{a}=z(\vec{a}.\vec{a})\vec{b}-z(\vec{a}.\vec{b})\vec{a}+y(\vec{b}\times\vec{a})$$

and,

$$\vec{b}-\vec{c}=\vec{b}-\{x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})\}$$

$$\vec{b}\times\vec{c}=-x\vec{a}+(1-y)\vec{b}-z(\vec{a}\times\vec{b})$$

$$\therefore \quad \vec{c}\times\vec{a}=\vec{b}-\vec{c}$$

$$\Rightarrow \quad z(\vec{a}.\vec{a})\vec{b}-z(\vec{a}.\vec{b})\vec{a}+y(\vec{b}\times\vec{a})=-x\vec{a}+(1-y)\vec{b}-z(\vec{a}\times\vec{b})$$

$$\Rightarrow \quad \{x-z(\vec{a}.\vec{b})\}\vec{a}+\{z(\vec{a}.\vec{a})-1+y\}\vec{b}+(z-y)(\vec{a}\times\vec{b})=\vec{0}$$

$$\Rightarrow \quad x-z(\vec{a}.\vec{b})=0, z(\vec{a}.\vec{a})-1+y=0, z-y=0$$

$[\because \vec{a}, \vec{b}, \vec{a}\times\vec{b}$ are non-coplanar$]$

$$\Rightarrow \quad y=z=\frac{1}{1+|\vec{a}|^2} \text{ and } x=\frac{\vec{a}.\vec{b}}{1+|\vec{a}|^2}$$

Substituting the values of x, y, z in (i), we get

$$\vec{c}=\left\{\frac{\vec{a}.\vec{b}}{1+|\vec{a}|^2}\right\}\vec{a}+\frac{1}{1+|\vec{a}|^2}(\vec{b}+\vec{a}\times\vec{b})$$

EXAMPLE 16 *Let $\vec{u}$ and $\vec{v}$ be two given non-collinear unit vectors and $\vec{w}$ be a vector such that $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$. Then, prove that*

$$|(\vec{u}\times\vec{v}).\vec{w}|\le\frac{1}{2}$$

[IIT 1999]

SOLUTION Since $\vec{u}, \vec{v}$ and $\vec{u}\times\vec{v}$ are non-coplanar vectors. So, any vector in space can be written as the linear combination of these vectors.

$$\therefore \quad \vec{w}=x\vec{u}+y\vec{v}+z(\vec{u}\times\vec{v}) \qquad ...(i)$$

$$\Rightarrow \quad \vec{w}\times\vec{u}=(x\vec{u}+y\vec{v}+z(\vec{u}\times\vec{v}))\times\vec{u}$$

$$\Rightarrow \quad \vec{w}\times\vec{u}=y(\vec{v}\times\vec{u})+z(\vec{u}.\vec{u})\vec{v}-(\vec{u}.\vec{v})\vec{u})$$

$$\Rightarrow \quad \vec{w}\times\vec{u}=-y(\vec{u}\times\vec{v})+z(\vec{u}.\vec{u})\vec{v}-z(\vec{u}.\vec{v})\vec{u}$$

Now,

$$\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$$

$$\Rightarrow \quad x\vec{u}+y\vec{v}+z(\vec{u}\times\vec{v})-y(\vec{u}\times\vec{v})+z(\vec{u}.\vec{u})\vec{v}-z(\vec{u}.\vec{v})\vec{u}=\vec{v}$$

$$\Rightarrow \quad \{x-z(\vec{u}.\vec{v})\}\vec{u}+\{y+z(\vec{u}.\vec{u})-1\}\vec{v}+(z-y)(\vec{u}\times\vec{v})=\vec{0}$$

$$\Rightarrow \quad x-z(\vec{u}.\vec{v})=0, y+z(\vec{u}.\vec{u})-1=0,$$

$$z-y=0 \qquad [\because \vec{u}, \vec{v}, \vec{u}\times\vec{v} \text{ are non coplanar}]$$

$$\Rightarrow \quad x-z(\vec{u}.\vec{v})=0, y+z|\vec{u}|^2-1=0, z-y=0$$

$$\Rightarrow \quad x-z(\vec{u}.\vec{v})=0, y+z-1=0, z=y \qquad [\because |\vec{u}|=1]$$

$$\Rightarrow \quad y=z=\frac{1}{2} \text{ and } x=\frac{1}{2}(\vec{u}.\vec{v})$$

$$\therefore \quad \vec{w}=\frac{1}{2}(\vec{u}.\vec{v})\vec{u}+\frac{1}{2}\vec{v}+\frac{1}{2}(\vec{u}\times\vec{v}) \qquad [\text{Substituting the value of } x, y, z \text{ in (i)}]$$

$$\Rightarrow (\vec{u}\times\vec{v}).\vec{w} = \frac{1}{2}(\vec{u}.\vec{v})\{(\vec{u}\times\vec{v}).\vec{u}\} + \frac{1}{2}\{(\vec{u}\times\vec{v}).\vec{v}\} + \frac{1}{2}(\vec{u}\times\vec{v}).(\vec{u}\times\vec{v})$$

$$\Rightarrow (\vec{u}\times\vec{v}).\vec{w} = \frac{1}{2}|\vec{u}\times\vec{v}|^2$$

$$\Rightarrow (\vec{u}\times\vec{v}).\vec{w} = \frac{1}{2}\sin^2\theta, \text{ where } \theta \text{ is the angle between } \vec{u} \text{ and } \vec{v}.$$

$$\Rightarrow |(\vec{u}\times\vec{v}).\vec{w}| = \frac{1}{2}\sin^2\theta \le \frac{1}{2} \qquad [\because \sin^2\theta \le 1]$$

EXAMPLE 17 *Prove that the problem of finding a vector $\vec{X}$ for given vectors $\vec{a}, \vec{b}$ such that $\vec{a}.\vec{X} = p$, $\vec{b}.\vec{X} = q$ and $(\vec{a}\times\vec{b}).(\vec{a}\times\vec{X}) = r$ has no solution unless p, q and r are related and that then also there is no unique solution.*

SOLUTION We have,

$$\vec{a}.\vec{X} = p, \vec{b}.\vec{X} = q \text{ and } (\vec{a}\times\vec{b}).(\vec{a}\times\vec{X}) = r$$

Now,

$$(\vec{a}\times\vec{b}).(\vec{a}\times\vec{X}) = r$$

$$\Rightarrow (\vec{a}.\vec{a})(\vec{b}.\vec{X}) - (\vec{a}.\vec{b})(\vec{a}.\vec{X}) = r$$

$$\Rightarrow (\vec{a}.\vec{a})q - (\vec{a}.\vec{b})p = r$$

Thus, the given system of simultaneous equation has no solution unless p, q and r are related by the above relation.

Let us now assume that the above relation between p, q and r holds good.

Let $\vec{X} = l\vec{a} + m\vec{b} + n(\vec{a}\times\vec{b})$. Then,

$$\vec{a}.\vec{X} = p \text{ and } \vec{b}.\vec{X} = q$$

$$\Rightarrow l(\vec{a}.\vec{a}) + m(\vec{a}.\vec{b}) = p \text{ and } l(\vec{a}.\vec{b}) + m(\vec{b}.\vec{b}) = q$$

$$\Rightarrow l = \frac{p(\vec{b}.\vec{b}) - q(\vec{a}.\vec{b})}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2} \text{ and } m = \frac{q(\vec{a}.\vec{a}) - p(\vec{a}.\vec{b})}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2}$$

$$\therefore \vec{X} = \left\{\frac{p(\vec{b}.\vec{b}) - q(\vec{a}.\vec{b})}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2}\right\}\vec{a} + \left\{\frac{q(\vec{a}.\vec{a}) - p(\vec{a}.\vec{b})}{(\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2}\right\}\vec{b} + n(\vec{a}\times\vec{b}).$$

Clearly, n can assume infinitely many values. So, $\vec{X}$ also attains infinitely many values.

Hence, there is no unique solution.

EXAMPLE 18 *Given that vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, find vector $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations*

$$\vec{r}.\vec{a} = 0 = \vec{r}.\vec{b} = 1 \text{ and } [\vec{r}\,\vec{a}\,\vec{b}] = 1$$

SOLUTION Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar vectors. So, $\vec{r}$ can be expressed as a linear combination of $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$.

Let

$$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b}) \qquad \ldots(i)$$

Now,

$$\vec{r}.\vec{a} = 0$$

$$\Rightarrow \{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}.\vec{a} = 0$$

$$\Rightarrow x(\vec{a}.\vec{a}) + y(\vec{b}.\vec{a}) + z\{(\vec{a}\times\vec{b}).\vec{a}\} = 0$$

$$\Rightarrow x|\vec{a}|^2 = 0 \qquad [\because \vec{a} \perp \vec{b} \therefore \vec{a}.\vec{b} = 0]$$

$$\Rightarrow x = 0.$$

$$\vec{r}.\vec{b} = 1$$

$$\Rightarrow \{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}.\vec{b} = 1$$

$$\Rightarrow x(\vec{a}.\vec{b}) + y(\vec{b}.\vec{b}) + z\{(\vec{a}\times\vec{b}).\vec{b}\} = 1$$

$$\Rightarrow y|\vec{b}|^2 = 1$$

$$\Rightarrow y = \frac{1}{|\vec{b}|^2} \qquad [\because \vec{a}.\vec{b} = 0 \text{ and } (\vec{a}\times\vec{b}).\vec{b} = 0]$$

and,

$$[\vec{r}\,\vec{a}\,\vec{b}] = 1$$

$$\Rightarrow \vec{r}.(\vec{a}\times\vec{b}) = 1$$

$$\Rightarrow \{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}.(\vec{a}\times\vec{b}) = 1$$

$$\Rightarrow x\{\vec{a}.(\vec{a}\times\vec{b})\} + y\{\vec{b}.(\vec{a}\times\vec{b})\} + z(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b}) = 1$$

$$\Rightarrow z = \frac{1}{|\vec{a}\times\vec{b}|^2}$$

Substituting the values of x, y, z in (i), we get

$$\vec{r} = \frac{1}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{a}\times\vec{b}|^2}(\vec{a}\times\vec{b}).$$

EXAMPLE 19 *Express a non-zero vector $\vec{r}$ as a linear combination of a non-zero vector $\vec{a}$ which is non-collinear with $\vec{r}$ and another vector perpendicular to $\vec{a}$ and coplanar with $\vec{r}$ and $\vec{a}$.*

SOLUTION We know that the vector $\vec{a}\times(\vec{a}\times\vec{r})$ is perpendicular to $\vec{a}$ and coplanar with $\vec{a}$ and $\vec{r}$. So, let

$$\vec{r} = x\vec{a} + y\{\vec{a}\times(\vec{a}\times\vec{r})\} \qquad \ldots(i)$$

Taking dot product with $\vec{a}$ on both sides, we get

$$\vec{r}.\vec{a} = x(\vec{a}.\vec{a}) + y\{\vec{a}\times(\vec{a}\times\vec{r})\}.\vec{a}$$

$$\Rightarrow \vec{r}.\vec{a} = x(\vec{a}.\vec{a})$$

$$\Rightarrow x = \frac{\vec{r}.\vec{a}}{\vec{a}.\vec{a}}$$

Taking cross product with $\vec{a}$ on both sides of (i), we get

$$\vec{r}\times\vec{a} = x(\vec{a}\times\vec{a}) + y\{\vec{a}\times(\vec{a}\times\vec{r})\}\times\vec{a}$$

$$\Rightarrow \vec{r}\times\vec{a} = y\{(\vec{a}.\vec{r})\vec{a} - (\vec{a}.\vec{a})\vec{r}\}\times\vec{a}$$

$$\Rightarrow \vec{r}\times\vec{a} = -y(\vec{a}.\vec{a})(\vec{r}\times\vec{a})$$

$$\Rightarrow\ y = -\frac{-1}{\vec{a}.\vec{a}}$$

Substituting the values of x and y in (i), we get

$$\vec{r} = \left\{\frac{\vec{r}.\vec{a}}{\vec{a}.\vec{a}}\right\}\vec{a} - \frac{1}{\vec{a}.\vec{a}}\{\vec{a}\times(\vec{a}\times\vec{r})\}.$$

EXAMPLE 20 *If* $\vec{X}\times\vec{Y} = \vec{a}$; $\vec{Y}\times\vec{Z} = \vec{b} = k$; $\vec{X}.\vec{Y} = 1$ *and* $\vec{Y}.\vec{Z} = 1$, *then find* $\vec{X}$. $\vec{Y}$ *and* $\vec{Z}$ *in terms of* $\vec{a}$, $\vec{b}$ *and* k.

SOLUTION We have,

$$\vec{X}\times\vec{Y} = \vec{a},\ \vec{Y}\times\vec{Z} = \vec{b},\ \vec{X}.\vec{b} = k,\ \vec{X}.\vec{Y} = 1 \text{ and } \vec{Y}.\vec{Z} = 1.$$

Now,

$$\vec{Y}\times\vec{Z} = \vec{b}$$

$$\Rightarrow\ \vec{Y}.(\vec{Y}\times\vec{Z}) = \vec{Y}.\vec{b} \qquad \text{[Taking cross-product with } \vec{Y}\text{]}$$

$$\Rightarrow\ 0 = \vec{Y}.\vec{b}$$

$$\Rightarrow\ \vec{Y}.\vec{b} = 0 \qquad \ldots(i)$$

and,

$$\vec{X}\times\vec{Y} = \vec{a}$$

$$\Rightarrow\ \vec{b}\times(\vec{X}\times\vec{Y}) = \vec{b}\times\vec{a} \qquad \text{[Taking cross-product with } \vec{b}\text{]}$$

$$\Rightarrow\ (\vec{b}.\vec{Y})\vec{X} - (\vec{b}.\vec{X})\vec{Y} = \vec{b}\times\vec{a}$$

$$\Rightarrow\ -k\vec{Y} = \vec{b}\times\vec{a} \qquad [\because \vec{b}.\vec{X} = k \text{ (given) and } \vec{b}.\vec{Y} = 0 \text{ [from(i)]}$$

$$\Rightarrow\ \vec{Y} = \frac{1}{k}(\vec{a}\times\vec{b}) \qquad \ldots(ii)$$

Again,

$$\vec{X}\times\vec{Y} = \vec{a}$$

$$\Rightarrow\ (\vec{X}\times\vec{Y})\times\vec{Y} = \vec{a}\times\vec{Y} \qquad \text{[Taking cross-product with } \vec{Y}\text{]}$$

$$\Rightarrow\ (\vec{X}.\vec{Y})\vec{Y} - (\vec{Y}.\vec{Y})\vec{X} = \vec{a}\times\vec{Y}$$

$$\Rightarrow\ \vec{Y} - |\vec{Y}|^2\vec{X} = \vec{a}\times\vec{Y} \qquad [\because \vec{X}.\vec{Y} = 1 \text{ (given)}]$$

$$\Rightarrow\ \frac{1}{k}(\vec{a}\times\vec{b}) - \frac{1}{k^2}|\vec{a}\times\vec{b}|^2\vec{X} = \frac{1}{k}\{\vec{a}\times(\vec{a}\times\vec{b})\} \qquad \text{[Using (ii)]}$$

$$\Rightarrow\ k(\vec{a}\times\vec{b}) - |\vec{a}\times\vec{b}|^2\vec{X} = k\{(\vec{a}.\vec{b})\vec{a} - |\vec{a}|^2\vec{b}\}$$

$$\Rightarrow\ |\vec{a}\times\vec{b}|^2\vec{X} = k(\vec{a}\times\vec{b}) - k(\vec{a}.\vec{b})\vec{a} + k|\vec{a}|^2\vec{b}$$

$$\Rightarrow\ \vec{X} = \frac{k}{|\vec{a}\times\vec{b}|^2}\{\vec{a}\times\vec{b} - (\vec{a}.\vec{b})\vec{a} + |\vec{a}|^2\vec{b}\}$$

Now,

$$\vec{Y}\times\vec{Z} = \vec{b}$$

$$\Rightarrow\ (\vec{Y}\times\vec{Z})\times\vec{Y} = \vec{b}\times\vec{Y} \qquad \text{[Taking cross-product with } \vec{y}\text{]}$$

$$\Rightarrow\ (\vec{Y}.\vec{Y})\vec{Z} - (\vec{Y}.\vec{Z})\vec{Y} = \vec{b}\times\vec{Y}$$

$$\Rightarrow\ |\vec{Y}|^2\vec{Z} - \vec{Y} = \vec{b}\times\vec{Y} \qquad [\because \vec{Y}.\vec{Z} = 1 \text{ (given)}]$$

$$\Rightarrow\ \vec{Z} = \frac{1}{|\vec{Y}|^2}(\vec{Y} + \vec{b}\times\vec{Y})$$

$$\Rightarrow\ \vec{Z} = \frac{k^2}{|\vec{a}\times\vec{b}|^2}\left\{\frac{1}{k}(\vec{a}\times\vec{b}) + \vec{b}\times\frac{1}{k}(\vec{a}\times\vec{b})\right\} \qquad \text{[Using (ii)]}$$

$$\Rightarrow\ \vec{Z} = \frac{k}{|\vec{a}\times\vec{b}|^2}\{\vec{a}\times\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})\}$$

—❑❑❑—

17.7 VOLUME OF A TETRAHEDRON

TETRAHEDRON *A tetrahedron is a three dimensional figure formed by four triangles.*

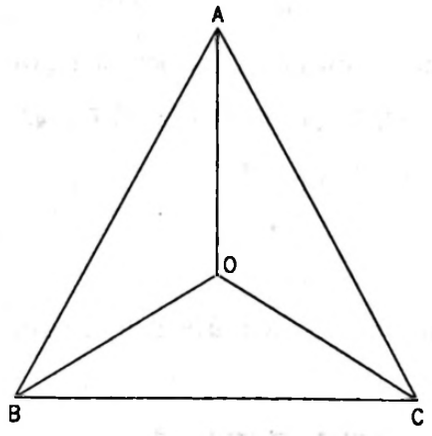


Fig. 17.5

In Fig. 17.5 $OABC$ is a tetrahedron with ΔABC as the base and OAB, OBC, OCA as other three faces.

OA, AB, BC and CA are known as the edges of the tetrahedron $OABC$.

OA, BC; OB, CA and OC, AB are known as the pairs of opposite edges of the tetrahedron $OABC$.

REGULAR TETRAHEDRON *A tetrahedron in which all edges are equal is called a regular tetrahedron.*

THEOREM *(i) If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.*

(ii) *The sum of the squares of two opposite edges is the same for each pair of opposite edges.*

(iii) *Any two opposite edges in a regular tetrahedron are perpendicular.*

PROOF (i) *Let OABC be a tetrahedron with OA, BC; OB, CA and OC, AB as the pairs of opposite edges (See Fig. 17.6). Let OA ⊥ BC and OB ⊥ CA. Then we have to prove that OC ⊥ AB.*

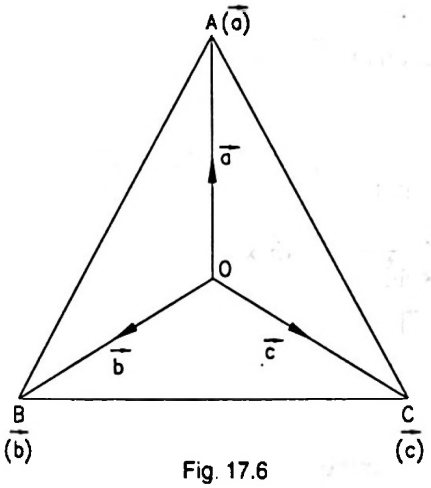


Fig. 17.6

Let O be the origin and let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$\vec{OA} \perp \vec{BC}$ and $\vec{OB} \perp \vec{CA}$

$\Rightarrow \vec{OA} \cdot \vec{BC} = 0$ and $\vec{OB} \cdot \vec{CA} = 0$

$\Rightarrow \vec{a} \cdot (\vec{c} - \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} - \vec{c}) = 0$

$\Rightarrow \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0$ and $\vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \cdot \vec{c} = \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$ ...(i)

$\Rightarrow \vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$

$\Rightarrow \vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{c} = 0$

$\Rightarrow (\vec{b} - \vec{a}) \cdot \vec{c} = 0$

$\Rightarrow \vec{AB} \cdot \vec{OC} = 0$

$\Rightarrow \vec{OC} \perp \vec{AB}$

(ii) We have,

$OA^2 + BC^2$

$= |\vec{OA}|^2 + |\vec{BC}|^2$

$= |\vec{a}|^2 + |\vec{c} - \vec{b}|^2$

$= |\vec{a}|^2 + |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b} \cdot \vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2\vec{b} \cdot \vec{c}$

$OB^2 + CA^2$

$= |\vec{OB}|^2 + |\vec{CA}|^2$

$= |\vec{b}|^2 + |\vec{a} - \vec{c}|^2$

$= |\vec{b}|^2 + |\vec{a}|^2 + |\vec{c}|^2 - 2\vec{a} \cdot \vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2(\vec{b} \cdot \vec{c})$ $\left[\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}\right]$

and,

$OC^2 + AB^2$

$= |\vec{OC}|^2 + |\vec{AB}|^2$

$= |\vec{c}|^2 + |\vec{b} - \vec{a}|^2$

$= |\vec{c}|^2 | + |\vec{b}|^2 + |\vec{a}|^2 - 2\vec{a} \cdot \vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2(\vec{b} \cdot \vec{c})$ $\left[\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}\right]$

Hence, $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2\vec{b} \cdot \vec{c}$

(iii) In this case, we have

$OA = OB = OC = AB = BC = CA.$

Now,

$AB = BC = CA$

$\Rightarrow |\vec{AB}| = |\vec{BC}| = |\vec{CA}|$

$\Rightarrow |\vec{AB}|^2 = |\vec{BC}|^2 = |\vec{CA}|^2$

$\Rightarrow |\vec{b} - \vec{a}|^2 = |\vec{c} - \vec{b}|^2 = |\vec{a} - \vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$

$= |\vec{b}|^2 + |\vec{c}|^2 - 2\vec{b} \cdot \vec{c}$

$= |\vec{a}|^2 + |\vec{c}|^2 - 2\vec{a} \cdot \vec{c}$

$\Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$ $[\because |\vec{a}| = |\vec{b}| = |\vec{c}|]$

Now,

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$

$\Rightarrow \vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{a}$ $[\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

$\Rightarrow \vec{b} \cdot (\vec{c} - \vec{a}) = 0$

$\Rightarrow \vec{OB} \cdot \vec{AC} = 0$

$\Rightarrow \vec{OB} \perp \vec{AC}.$

Similarly, $\vec{OC} \perp \vec{AB}$ and $\vec{OA} \perp \vec{BC}$.

Hence, any two opposite edges are perpendicular.

CENTROID OF A TETRAHEDRON *In a tetrahedron OABC, let G_1, G_2, G_3 and G_4 be the centroids of faces OBC, OCA, OAB and ABC respectively. Then AG_1, BG_2, CG_3 and OG_4 are concurrent. The point of their concurrence is the centroid or centre of tetrahedron OABC.*

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of the vertices of a tetrahdron, then the position vector of its centroid is

$$\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

The position vectors of the centroids of four faces are

$$\frac{\vec{a}+\vec{b}+\vec{c}}{3}, \frac{\vec{a}+\vec{b}+\vec{d}}{3}, \frac{\vec{a}+\vec{c}+\vec{d}}{3} \text{ and } \frac{\vec{b}+\vec{c}+\vec{d}}{3}.$$

THEOREM *The volume V of a tetrahedron whose three coterminous edges in the right-handed system are* $\vec{a}, \vec{b}, \vec{c}$ *is given by*

$$V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

PROOF Let ABC be a tetrahedron. Taking O as the origin, let the position vectors of vertices A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then, $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{OC} = \vec{c}$.

Let $\hat{n}$ be the unit vector perpendicular to the plane ABC to ΔABC. Then,

$$\hat{n} = \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$$

$$\Rightarrow \quad \hat{n} = \frac{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})}{|\vec{AB} \times \vec{AC}|}$$

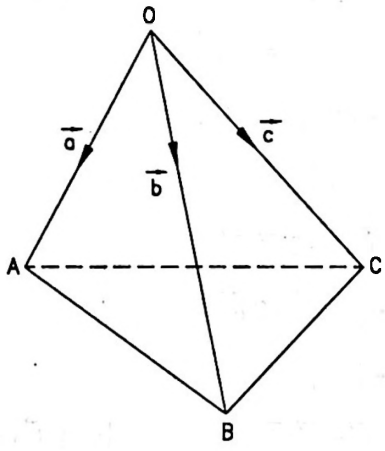


Fig. 17.7

$$\Rightarrow \quad \hat{n} = \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{AB} \times \vec{AC}|}$$

We have, length of the perpendicular from the vertex O on the plane ABC

$$= \text{Projection of } \vec{OA} \text{ on } \vec{n}$$

$$= \vec{a} \cdot \hat{n}$$

$$= \frac{1}{|\vec{AB} \times \vec{AC}|} |\vec{a} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})|$$

$$= \frac{1}{|\vec{AB} \times \vec{AC}|} |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{\vec{AB} \times \vec{AC}}$$

Now,

Volume of tetrahedron

$$= \frac{1}{3}(\text{Area of the base} \times \text{height})$$

$$= \frac{1}{3}(\text{Area of } \Delta ABC) \times (\text{Projection of } \vec{OA} \text{ on } \vec{n})$$

$$= \frac{1}{3}\left[\frac{1}{2}|\vec{AB} \times \vec{AC}|\right] \times \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{AB} \times \vec{AC}|}$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

REMARK 1 *It follows from the above result that the volume of tetrahedron* $OABC = \frac{1}{6}[\vec{OA}\ \vec{OB}\ \vec{OC}]$

REMARK 2 *If A, B, C, D are the vertices of a tetrahedron ABCD, then its volume is numerically equal to*

$$\frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}] \text{ or } \frac{1}{6}[\vec{BC}\ \vec{BD}\ \vec{BA}] \text{ or } \frac{1}{6}[\vec{CD}\ \vec{CB}\ \vec{CA}]$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that each of the four faces of a tetrahedron subtends the same volume at the centroid.*

SOLUTION Let $ABCD$ be a tetrahedron with centroid. Taking the centroid O as the origin, let the position vectors of the four vertices A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively. Then,

$$\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4} = \vec{0}$$

$$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c}+\vec{d} = \vec{0}$$

Now,

V_1 = Volume of the tetrahedron $OABC$

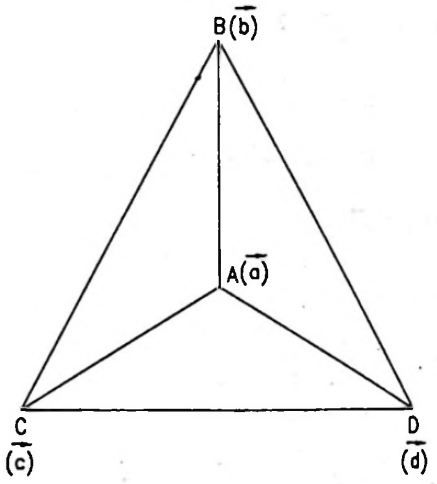


Fig. 17.8

$$= \frac{1}{6}[\vec{OA}\ \vec{OB}\ \vec{OC}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

V_2 = Volume of the tetrahedoron $OACD$

$$= \frac{1}{6}[\vec{OA}\ \vec{OC}\ \vec{OD}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{c}\ \vec{d}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{c}\ (-\vec{a}-\vec{b}-\vec{c})]$$

$$= \frac{1}{6}\left\{[\vec{a}\ \vec{c}-\vec{a}]+[\vec{a}\ \vec{c}-\vec{b}]+[\vec{a}\ \vec{c}-\vec{c}]\right\}$$

$$= -\frac{1}{6}[\vec{a}\ \vec{c}\ \vec{b}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

V_3 = Volume of the tetrahedron $OABD$

$$= \frac{1}{6}[\vec{OA}\ \vec{OD}\ \vec{OB}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{d}\ \vec{b}]$$

$$= \frac{1}{6}[\vec{a}\ (-\vec{a}-\vec{b}-\vec{c})\ \vec{b}]$$

$$= \frac{1}{6}\left\{[\vec{a}-\vec{a}\ \vec{b}]+[\vec{a}-\vec{b}\ \vec{b}]+[\vec{a}-\vec{c}\ \vec{b}]\right\}$$

$$= -\frac{1}{6}[\vec{a}\ \vec{c}\ \vec{b}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

and,

V_4 = Volume of the tetrahedron $OBCD$

$$= \frac{1}{6}[\vec{OB}\ \vec{OD}\ \vec{OC}]$$

$$= [\vec{b}-(\vec{a}+\vec{b}+\vec{c})\ \vec{c}]$$

$$= \frac{1}{6}\left\{[\vec{b}-\vec{a}\ \vec{c}]+[\vec{b}-\vec{b}\ \vec{c}]+[\vec{b}-\vec{c}\ \vec{c}]\right\}$$

$$= -\frac{1}{6}[\vec{b}\ \vec{a}\ \vec{c}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

Clearly, $V_1 = V_2 = V_3 = V_4 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

EXAMPLE 2 *If G_1, G_2, G_3 are the centroids of the triangular faces OBC, OCA, OAB of a tetrahedron OABC. Prove that the ratio of the volume of the tetrahedron to the volume of the parallelopiped with OG_1, OG_2, OG_3 as coterminus edges is 9 : 4.*

SOLUTION With referecne to the origin O, let the position vectors of the vertices A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of G_1, G_2, G_3 are $\frac{\vec{b}+\vec{c}}{3}, \frac{\vec{c}+\vec{a}}{3}, \frac{\vec{a}+\vec{b}}{3}$ respectively.

Let V_1 be the volume of the tetrahedron $OABC$. Then,

$$V_1 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

Let V_2 be the volume of the parallelopiped with OG_1, OG_2, OG_3 as conterminus edges. Then,

$$V_2 = [\vec{OG}_1\ \vec{OG}_2\ \vec{OG}_3]$$

$$= \left[\frac{\vec{b}+\vec{c}}{3}\ \frac{\vec{c}+\vec{a}}{3}\ \frac{\vec{a}+\vec{b}}{3}\right]$$

$$= \frac{1}{27}[\vec{b}+\vec{c}\ \vec{c}+\vec{a}\ \vec{a}+\vec{b}]$$

$$= \frac{2}{27}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]}{\frac{2}{27}[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{9}{4}$$

Hence, $V_1 : V_2 = 9 : 4$

EXAMPLE 3 *Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of the points A, B, C respectively. Let α, β, γ be the inclinations between $\vec{b}, \vec{c}$; $\vec{a}, \vec{b}$ and $\vec{a}, \vec{c}$. Then, prove that the volume V of the tetrahedron OABC is given by*

$$V^2 = \frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\beta & \cos\gamma \\ \cos\beta & 1 & \cos\alpha \\ \cos\gamma & \cos\alpha & 1 \end{vmatrix}$$

SOLUTION Let $OABC$ be the tetrahedron. Taking O as the origin, let the position vectors of the vertices A, B, C be $\vec{a}, \vec{b}, \vec{c}$ respectively. It is given that α, β, γ are the inclinations between $\vec{b}, \vec{c}$; $\vec{a}, \vec{b}$ and $\vec{a}, \vec{c}$. Therefore,

$$\vec{a}.\vec{b} = ab\cos\beta, \quad \vec{b}.\vec{c} = bc\cos\alpha \text{ and } \vec{a}.\vec{c} = ac\cos\gamma.$$

The volume V of the tetrahedron $OABC$ is given by

$$V = \frac{1}{6}[\vec{OA}\ \vec{OB}\ \vec{OC}]$$

$$= \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow \quad V^2 = \frac{1}{36}[\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$\Rightarrow V^2 = \frac{1}{36}\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

$$\Rightarrow V^2 = \frac{1}{36}\begin{vmatrix} a^2 & ab\cos\beta & ac\cos\gamma \\ ba\cos\beta & b^2 & bc\cos\gamma \\ ca\cos\gamma & bc\cos\alpha & c^2 \end{vmatrix}$$

$$\Rightarrow V^2 = \frac{abc}{36}\begin{vmatrix} a & b\cos\beta & c\cos\gamma \\ a\cos\beta & b & c\cos\alpha \\ a\cos\gamma & b\cos\alpha & c \end{vmatrix}$$

$$\Rightarrow V^2 = \frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\beta & \cos\gamma \\ \cos\beta & 1 & \cos\alpha \\ \cos\gamma & \cos\alpha & 1 \end{vmatrix}$$

EXAMPLE 4 *Show that in a regular tetrahedron, the angle between any two faces is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let $OABC$ be a regular tetrahedron. Then, OAB, OBC, OAC and ABC are equilateral triangles of equal size. Thus, the angle between any two concurrent edges is equal to 60°.

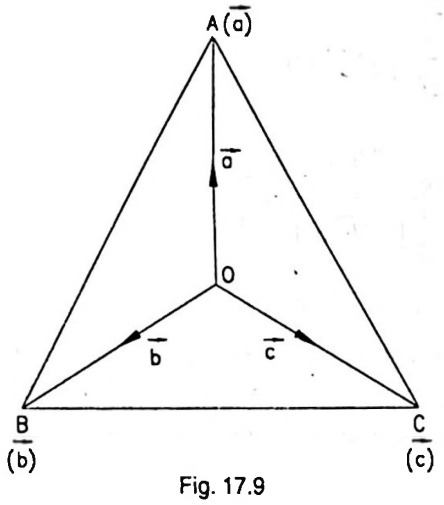


Fig. 17.9

Taking O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}, \vec{c}$ respectively. Since $OABC$ is a regular tetrahedron. Therefore,

$$OA = OB = OC = AB = BC = CA$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{b}-\vec{a}| = |\vec{c}-\vec{b}| = |\vec{a}-\vec{c}| = \lambda \text{ (say)}$$

We know that the angle between two planes is the angle between their normals. So, the angle between faces OAB and OBC is the angle between their normals. The vectors $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are respectively normals to the planes OBC and OCA. Therefore, angle θ between these faces is given by

$$\cos\theta = \left|\frac{(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a})}{|\vec{b}\times\vec{c}|\,|\vec{c}\times\vec{a}|}\right|$$

Now,

$$(\vec{b}\times\vec{c}).(\vec{c}\times\vec{a})$$

$$= (\vec{b}.\vec{c})(\vec{c}.\vec{a}) - (\vec{b}.\vec{a})(\vec{c}.\vec{c})$$

$$= (|\vec{b}|\,|\vec{c}|\cos 60°)(|\vec{c}|\,|\vec{a}|\cos 60°) - (|\vec{b}|\,|\vec{a}|\cos 60°)(|\vec{c}|)^2$$

$$= \frac{|\vec{b}|\,|\vec{c}|^2|\vec{a}|}{4} - \frac{|\vec{b}|\,|\vec{a}|\,|\vec{c}|^2}{2}$$

$$= -\frac{1}{4}|\vec{a}|\,|\vec{b}|\,|\vec{c}|^2 = -\frac{1}{4}\lambda^4$$

and,

$$|\vec{b}\times\vec{c}|\,|\vec{c}\times\vec{a}|$$

$$= (|\vec{b}|\,|\vec{c}|\sin 60°)(|\vec{c}|\,|\vec{a}|\sin 60°)$$

$$= \frac{3}{4}|\vec{b}|\,|\vec{c}|^2|\vec{a}| = \frac{3}{4}\lambda^4.$$

$$\therefore \cos\theta = \left|\frac{-1/4\,\lambda^4}{3/4\,\lambda^4}\right| = \frac{1}{3}.$$

Hence, the acute angle between any two faces is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 5 *A tetrahedron has three of its vertices at A, B and C where* $\vec{OA} = 3\hat{i}+2\hat{j}$, $\vec{OB} = \hat{i}+3\hat{j}-\hat{k}$; $\vec{OC} = 2\hat{j}$. *Find the unit vector perpendicular to the face ABC. The fourth vertex D is such that* $\vec{DA}.\vec{AB} = 0 = \vec{DA}.\vec{AC}$. *Find the vector equation of AD. If the volume of the tetrahedron is* $3\sqrt{2}$ *cubic units and D is on the same side as the origin, find the coordinates of D.*

SOLUTION We have,

$$\vec{OA} = 3\hat{i}+2\hat{j},\ \vec{OB} = \hat{i}+3\hat{j}-\hat{k} \text{ and } \vec{OC} = 2\hat{j}.$$

$$\therefore \vec{AB} = \vec{OB}-\vec{OA} = -2\hat{i}+\hat{j}-\hat{k}.$$

$$\vec{AC} = \vec{OC}-\vec{OA} = -3\hat{i}$$

$$\Rightarrow \vec{AB}\times\vec{AC} = 3(\hat{j}+\hat{k}) \quad \text{...(i)}$$

The unit vector perpendicular to the face ABC is given by

$$\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|} = \frac{3(\hat{j}+\hat{k})}{\sqrt{9+9}} = \frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$$

Now,

$$\vec{DA}.\vec{AB} = 0 \text{ and } \vec{DA}.\vec{AC} = 0$$

$$\Rightarrow \vec{DA} \perp \vec{AB} \quad \text{and} \quad \vec{DA} \perp \vec{AC}$$

$$\Rightarrow \vec{DA} \parallel \vec{AB}\times\vec{AC}$$

Line AD passes through A and is parallel to $\vec{DA}$. So, vector equation of AD is

$$\vec{r} = (3\hat{i}+2\hat{j}) + \lambda(\hat{j}+\hat{k}) \quad \text{...(ii)}$$

Let p be the altitude of the tetrahedron. Then,

Volume $= 3\sqrt{2}$ cubic units

$\Rightarrow \frac{1}{3}$(Area of the base $\times p$) $= 3\sqrt{2}$

$\Rightarrow$ (Area of the face ABC) $\times p = 9\sqrt{2}$

$\Rightarrow \frac{1}{2} | \vec{AB} \times \vec{AC} | \times p = 9\sqrt{2}$

$\Rightarrow \frac{1}{2} \times \sqrt{9+9} \times p = 9\sqrt{2}$ [Using (i)]

$\Rightarrow p = 6$

Clearly, point D is on the line AD at a distance of 6 units from vertex A.

Let the position vector of D be $3\hat{i} + 2\hat{j} + \lambda(\hat{j} + \hat{k})$ [Using (ii)]

or, $3\hat{i} + (\lambda + 2)\hat{j} + \lambda\hat{k}$

$\therefore \vec{AD} = \{3\hat{i} + (\lambda + 2)\hat{j} + \lambda\hat{k}\} - (3\hat{i} + 2\hat{j})$

$\Rightarrow \vec{AD} = \lambda\hat{j} + \lambda\hat{k}$

$\Rightarrow | \vec{AD} | = \sqrt{\lambda^2 + \lambda^2} = | \lambda | \sqrt{2}$

$\therefore | \vec{AD} | = 6$

$\Rightarrow | \lambda | \sqrt{2} = 6$

$\Rightarrow | \lambda | = 3\sqrt{2}$

$\Rightarrow \lambda = -3\sqrt{2}$ [$\because D$ is on the same side as the origin]

Thus, the position vector of D is $(3\hat{i} + 2\hat{j}) - 3\sqrt{2}(\hat{j} + \hat{k})$

Hence, the coordinates of D are $(3, -3\sqrt{2} + 2, -3\sqrt{2})$

EXAMPLE 6 *The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$ respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$ cubic units, find the position vector of the point E for all its possible positions.*

[IIT 1996]

SOLUTION We have,

$\vec{AB} = \hat{i} - (\hat{i} + \hat{j} + \hat{k}) = -\hat{j} - \hat{k}$

$\vec{AC} = 3\hat{i} - (\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} - \hat{j} - \hat{k}$

$\therefore \vec{AB} \times \vec{AC} = (-\hat{j} - \hat{k}) \times (2\hat{i} - \hat{j} - \hat{k})$

$= -2\hat{j} + 2\hat{k}$

$\Rightarrow | \vec{AB} \times \vec{AC} | = 2\sqrt{2}$

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2} | \vec{AB} \times \vec{AC} | = \frac{1}{2}(2\sqrt{2}) = \sqrt{2}$

Now,

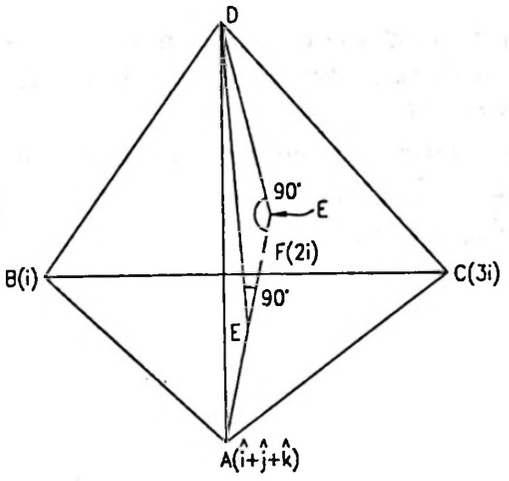


Fig. 17.10

Volume of the tetrahedron $= \frac{2\sqrt{2}}{3}$ cubic units

$\Rightarrow \frac{1}{3}$(Area of the base $\times$ Height) $= \frac{2\sqrt{2}}{3}$

$\Rightarrow \frac{1}{3} \times \sqrt{2} \times DE = \frac{2\sqrt{2}}{3}$

$\Rightarrow DE = 2$

In right angled triangle ADE, we have

$AD = 4$ and $DE = 2$

$\therefore AD^2 = AE^2 + DE^2$

$\Rightarrow 4^2 = AE^2 + 2^2$

$\Rightarrow AE = \sqrt{16 - 4} = 2\sqrt{3}$

Suppose E divides the median AF in the ratio $\lambda : 1$.

$\therefore$ P.V. of $E = \frac{\lambda \cdot 2\hat{i} + (\hat{i} + \hat{j} + \hat{k})}{\lambda + 1} = \frac{(2\lambda + 1)\hat{i} + \hat{j} + \hat{k}}{\lambda + 1}$

$\therefore \vec{AE} = \frac{(2\lambda + 1)\hat{i} + \hat{j} + \hat{k}}{\lambda + 1} - (\hat{i} + \hat{j} + \hat{k})$

$\Rightarrow \vec{AE} = \frac{\lambda}{\lambda + 1}(\hat{i} - \hat{j} - \hat{k})$

$\Rightarrow | \vec{AE} |^2 = 3\left(\frac{\lambda}{\lambda + 1}\right)^2$

$\Rightarrow 12 = 3\left(\frac{\lambda}{\lambda + 1}\right)^2$

$\Rightarrow \frac{\lambda}{\lambda + 1} = \pm 2$

$\Rightarrow \lambda = -2, -\frac{2}{3}$.

Thus, the possible position vectors of E are $3\hat{i} - \hat{j} - \hat{k}$ and $-\hat{i} + 3\hat{j} + 3\hat{k}$.

EXAMPLE 7 *OABC is a regular tetrahedron. D is the circumcentre of ΔOAB and E is the middle point of the edge AC. Use vector methods to find distance DE.*

SOLUTION Taking O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Since $OABC$ is a regular tetrahedron, therefore

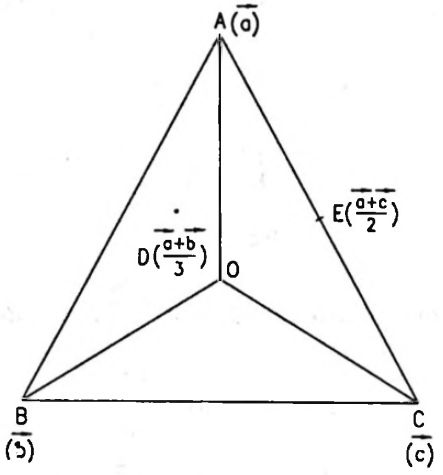


Fig. 17.11

$$OA = OB = OC = AB = BC = AC$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{b}-\vec{a}| = |\vec{c}-\vec{b}| = |\vec{c}-\vec{a}|$$

Also, faces OAB, OBC, OCA, ABC are equilateral triangles and the angle between any two edges is 60°.

Since ΔOAB is equilateral triangle. Therefore, its circumcentre and centroid coincide. Thus,

The position vector of D is $\frac{\vec{a}+\vec{b}}{3}$.

E is the mid-point of AC. Therefore,

P.V. of E is $\frac{\vec{a}+\vec{c}}{2}$

$$\therefore \quad \vec{DE} = \text{P.V. of } E - \text{P.V. of } D$$

$$\Rightarrow \vec{DE} = \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{3}$$

$$\Rightarrow \vec{DE} = \frac{1}{6}(\vec{a}-2\vec{b}+3\vec{c})$$

$$\Rightarrow |\vec{DE}| = \frac{1}{6}|\vec{a}-2\vec{b}+3\vec{c}|$$

We have, $|\vec{a}| = |\vec{b}| = |\vec{c}|$. Therefore,

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos 60° = \frac{1}{2}|\vec{a}|^2$$

$$\vec{b}.\vec{c} = |\vec{b}|\,|\vec{c}| \cos 60° = \frac{1}{2}|\vec{a}|^2$$

and, $\vec{a}.\vec{c} = |\vec{a}|\,|\vec{c}| \cos 60° = \frac{1}{2}|\vec{a}|^2$

$$\therefore \quad |\vec{a}-2\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + 4|\vec{b}|^2 + 9|\vec{c}|^2 - 4(\vec{a}.\vec{b}) - 12(\vec{b}.\vec{c}) + 6(\vec{a}.\vec{c})$$

$$= |\vec{a}| + 4|\vec{a}|^2 + 9|\vec{a}|^2 - 4\times\frac{1}{2}|\vec{a}|^2 - 12\times\frac{1}{2}|\vec{a}|^2 + 6\times\frac{1}{2}|\vec{a}|$$

$$= 9|\vec{a}|^2$$

$$\Rightarrow |\vec{a}-2\vec{b}+3\vec{c}| = 3|\vec{a}|$$

$$\Rightarrow \vec{DE} = \frac{1}{6}|\vec{a}-2\vec{b}+3\vec{c}|$$

$$\Rightarrow \vec{DE} = \frac{1}{2}|\vec{a}|$$

Hence, $DE = \frac{1}{2}(OA)$ = Half of an edge of the tetrahedron.

EXAMPLE 8 *A pyramid with vertex at the point P whose position vector is $4\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}$ has a regular hexagonal base ABCDEF. The points A and B have position vectors $\hat{i}$ and $\hat{i}+2\hat{j}$ respectively. The centre of hexagon has position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Given that the volume of the pyramid is $6\sqrt{3}$ and the perpendicular from the vertex meets the diagonal AD, locate the position vectors of the foot of this perpendicular.*

SOLUTION Let G be centre of the hexagon $ABCDEF$.

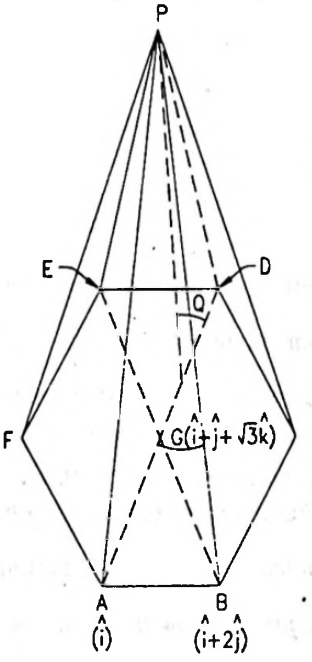


Fig. 17.12

We have,

$$\vec{AB} = \text{P.V. of } B - \text{P.V. of } A$$

$$= \hat{i}+2\hat{j}-\hat{i} = 2\hat{j}$$

$$\therefore \quad AB = |\vec{AB}| = 2.$$

Now,

Area of the hexagon

$$= 6 \text{ (Area of } \Delta AGB)$$

$= 6\left(\frac{\sqrt{3}}{4} \times 2^2\right)$ sq. units

$= 6\sqrt{3}$ sq. units.

Let p be length of the perpendicular from vertex P to the base of the pyramid. Then,

Volume of the pyramid $= 6\sqrt{3}$ sq. units

$\Rightarrow \frac{1}{3}$ (Area of the base $\times p$) $= 6\sqrt{3}$

$\Rightarrow \frac{1}{3} \times (6\sqrt{3} \times p) = 6\sqrt{3}$

$\Rightarrow p = 3$

Suppose the perpendicular from the vertex P meets the diagonal AD at Q. Then,

$AP^2 = AQ^2 + PQ^2$

Now,

$\vec{AP}$ = P.V. of P – P.V. of $A = 3\hat{i} + 2\hat{j} + 2\sqrt{3}\hat{k}$

$\Rightarrow AP = \sqrt{9+4+12} = 5$

and, $PQ = p = .3$

$\therefore AP^2 = AQ^2 + PQ^2$

$\Rightarrow 5^2 = AQ^2 + 3^2$

$\Rightarrow AQ = 4.$

Also, $AD = 2AG = 2AB = 4.$

$\therefore$ Q coincides with D.

Hence, the foot of the perpendicular from vertex P coincides with D.

Now,

$\vec{AD} = 2\vec{AG}$

$\vec{AD} = 2(\hat{i} + \hat{j} + \sqrt{3}\hat{k} - \hat{i}) = 2(\hat{j} + \sqrt{3}\hat{k})$

$\Rightarrow$ P.V. of D – P.V. of $A = 2(\hat{j} + \sqrt{3}\hat{k})$

$\Rightarrow$ P.V. of D = P.V. of $A + 2(\hat{j} + \sqrt{3}\hat{k})$

$\Rightarrow$ P.V. of $D = \hat{i} + 2\hat{j} + 2\sqrt{3}\hat{k}$.

Hence, the position vector of the foot of the perpendicular from P is $\hat{i} + 2\hat{j} + 2\sqrt{3}\hat{k}$.

The other position of the foot of the perpendicular from P on the base is D_1 on DA produced (outside the hexagon) such that DD_1 has its mid-point at A.

Let the position vector of D_1 be $\vec{d_1}$. Then,

$\frac{\vec{d_1} + \text{P.V. of } D}{2} = \text{P.V. of } A$

$\Rightarrow \frac{\vec{d_1} + (\hat{i} + 2\hat{j} + 2\sqrt{3}\hat{k})}{2} = \hat{i}$

$\Rightarrow \vec{d_1} = \hat{i} - 2\hat{j} - 2\sqrt{3}\hat{k}$

EXAMPLE 9 *Find the volume of a prism of triangular base, the three sides of the prism meeting on a vertex are given below :*

$\vec{OA} = 3\hat{i} + 4\hat{j} + 12\hat{k}$, $\vec{OB} = 12\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{OC} = 4\hat{i} + 12\hat{j} + 3\hat{k}$.

SOLUTION Clearly,

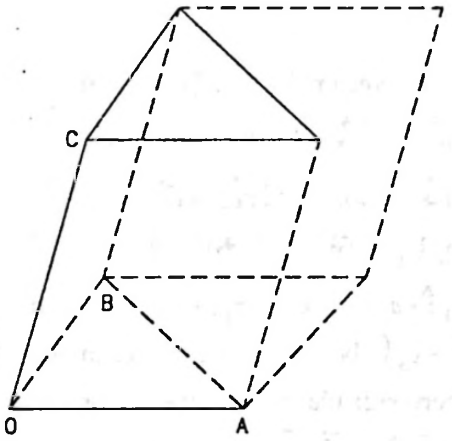


Fig. 17.13

Volume of the prism

$= \frac{1}{2}$ (Volume of the tetrahedron formed by $\vec{OA}, \vec{OB}, \vec{OC}$)

$= \frac{1}{12}[\vec{OA}\ \vec{OB}\ \vec{OC}]$

$= \frac{1}{12}\begin{vmatrix} 3 & 4 & 12 \\ 12 & 3 & 4 \\ 4 & 12 & 3 \end{vmatrix} = \frac{1387}{12}.$

EXAMPLE 10 *Prove that the circum-centre of the tetrahedron $OABC$ where $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$, is given by*

$$\frac{|\vec{a}|^2(\vec{b} \times \vec{c}) + |\vec{b}|^2(\vec{c} \times \vec{a}) + |\vec{c}|^2(\vec{a} \times \vec{b})}{2[\vec{a}\ \vec{b}\ \vec{c}]}$$

SOLUTION Let P be circum-centre of the tetrahedron $OABC$. Let the position vector of P be $\vec{r}$. Then,

$OP = PA = PB = PC$

$\Rightarrow |\vec{OP}| = |\vec{PA}| = |\vec{PB}| = |\vec{PC}|$

$\Rightarrow |\vec{r}|^2 = |\vec{a} - \vec{r}|^2 = |\vec{b} - \vec{r}|^2 = |\vec{c} - \vec{r}|^2$

$\Rightarrow |\vec{a}|^2 - 2(\vec{a}.\vec{r}) = |\vec{b}|^2 - 2(\vec{b}.\vec{r}) = |\vec{c}|^2 - 2(\vec{c}.\vec{r}) = 0$

$\Rightarrow 2(\vec{a}.\vec{r}) = |\vec{a}|^2, 2(\vec{b}.\vec{r}) = |\vec{b}|^2, 2(\vec{c}.\vec{r}) = |\vec{c}|^2$ ...(i)

Clearly, $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors. So, $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are also non-coplanar vectors.

We know that any vector in space can be written on the linear combination of three non-coplanar vectors. So, let

$$\vec{r} = x(\vec{b}\times\vec{c}) + y(\vec{c}\times\vec{a}) + z(\vec{a}\times\vec{b}) \quad ...(ii)$$

Taking dot products with $\vec{a}$, $\vec{b}$ and $\vec{c}$ successively, we get

$\vec{r}.\vec{a} = x[\vec{b}\ \vec{c}\ \vec{a}]$, $\vec{r}.\vec{b} = y[\vec{c}\ \vec{a}\ \vec{b}]$,

$\vec{r}.\vec{c} = z[\vec{a}\ \vec{b}\ \vec{c}]$

$$\Rightarrow x = \frac{\vec{r}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, y = \frac{\vec{r}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}, z = \frac{\vec{r}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\Rightarrow x = \frac{|\vec{a}|^2}{2[\vec{a}\ \vec{b}\ \vec{c}]}, y = \frac{|\vec{b}|^2}{2[\vec{a}\ \vec{b}\ \vec{c}]}, z = \frac{|\vec{c}|^2}{2[\vec{a}\ \vec{b}\ \vec{c}]}$$

[Using (i)]

Putting the values of x, y, z in (ii), we obtain the position vector of the circumcentre P as

$$\vec{r} = \frac{|\vec{a}|^2(\vec{b}\times\vec{c}) + |\vec{b}|^2(\vec{c}\times\vec{a}) + |\vec{c}|^2(\vec{a}\times\vec{b})}{2[\vec{a}\ \vec{b}\ \vec{c}]}$$

EXERCISE 17.1

1. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that

(i) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

(ii) $\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{a})] = (\vec{a}.\vec{b})(\vec{a}\times\vec{c})$

(iii) $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})] = (\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

2. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

3. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors, prove that

$$\frac{\vec{a}.(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{a}\times\vec{c})}{\vec{c}.(\vec{a}\times\vec{b})} = 0$$

4. If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four coplanar points, prove that

$$[\vec{d}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{d}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

5. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors and $\vec{r}$ is any vector in space, then prove that

$$\vec{r} = \frac{\vec{r}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{r}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{r}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

6. If $\vec{r} = l(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b})$ and $[\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{l+m+n}$, find the value of $\vec{r}.(\vec{a}+\vec{b}+\vec{c})$.

7. Prove that the points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

8. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four vectors, then prove that

(i) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) = 0$

(ii) $\vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = [\vec{b}.\vec{d}][\vec{a}\ \vec{c}\ \vec{d}]$

9. The three concurrent edges of a parallelopiped of volume V represent vectors $\vec{a}, \vec{b}, \vec{c}$. Find the volume of the parallelopiped whose threee concurrent edges are three concurrent diagonals of three faces of the parallelopiped.

10. Prove that:

$$[\vec{a}\times\vec{b}\ \vec{c}\times\vec{d}\ \vec{e}\times\vec{f}] = [\vec{a}\ \vec{b}\ \vec{d}][\vec{c}\ \vec{e}\ \vec{f}] - [\vec{a}\ \vec{b}\ \vec{c}][\vec{d}\ \vec{e}\ \vec{f}]$$
$$= [\vec{a}\ \vec{b}\ \vec{e}][\vec{f}\ \vec{c}\ \vec{d}] - [\vec{a}\ \vec{b}\ \vec{d}][\vec{e}\ \vec{c}\ \vec{d}]$$
$$= [\vec{c}\ \vec{d}\ \vec{a}][\vec{b}\ \vec{e}\ \vec{f}] - [\vec{c}\ \vec{d}\ \vec{b}][\vec{a}\ \vec{e}\ \vec{f}]$$

11. If the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are coplanar, show that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$$

12. Prove that:

(i) $\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})] = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{a}.\vec{b})(\vec{c}\times\vec{d})$

(ii) $\vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = (\vec{b}.\vec{d})[\vec{a}\ \vec{c}\ \vec{d}]$

(iii) $\vec{d}.[(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})] = (\vec{a}.\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$

13. Prove that:

$$\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})] = (\vec{b}.\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\vec{c})(\vec{a}\times\vec{d})$$

14. Prove that:

(i) $(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$
$= -2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$
$= 2[\vec{b}\ \vec{d}\ \vec{c}]\vec{a}$

15. Solve the following vector equations:

(i) $\vec{r}\times\vec{a} = \vec{a}\times\vec{b}$

(ii) $\vec{r}\times\vec{a} = \vec{b}$, $\vec{r}.\vec{a} = 0$

(iii) $\vec{r} + \vec{r}\times\vec{a} = \vec{b}$

16. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a'}$, $\vec{b'}$, $\vec{c'}$ are reciprocal system of vectors, prove that

$$\vec{b'}\times\vec{c'} + \vec{c'}\times\vec{a'} + \vec{a'}\times\vec{b'} = \frac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

17. Find the set of vectors reciprocal to the set $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$.

18. Let $\vec{a}$ and $\vec{b}$ be two non-zero and non-collinear vectors. Show that $\vec{b}$ can be resolved into two components in the plane of $\vec{a}$ and $\vec{b}$, one parallel to $\vec{a}$ and the other perpendicular to it given by

$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{|\vec{a}|^2}$ respectively.

19. If $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system of vectors, then prove that any vector $\vec{r}$ in space can be expressed as

$$\vec{r} = (\vec{r}\cdot\vec{a}')\,\vec{a} + (\vec{r}\cdot\vec{b}')\,\vec{b} + (\vec{r}\cdot\vec{c}')\,\vec{c}$$

20. If $\vec{a} = \vec{b} + \vec{c}$, show that $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector of magnitude $|\vec{a}|\,|\vec{b}|\,|\vec{c}|\sin(\alpha+\beta)$, where α is the angle between $\vec{a}$ and $\vec{c}$ and β is the angle between $\vec{a}$ and $\vec{b}$. What is its direction.

21. A pyramid with vertex at the point P has a regular hexagonal base $ABCDEF$. Position vectors of points A and B are $\hat{i}$ and $\hat{i}+2\hat{j}$ respectively. Centre of the base has the position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Altitude drawn from P on the base meets the diagonal AD at point G. Find all possible position vectors of G. It is given that the volume of the pyramid is $6\sqrt{3}$ cubic units and $AP = 5$ units.

22. If $\vec{AB} = \vec{a}\times(\vec{a}\times\vec{b})$ and $\vec{AC} = \vec{a}\times\vec{b}$ where $|\vec{a}| = \sqrt{3}$, then find the angles of the triangle ABC.

23. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{b}| = |\vec{c}|$ prove that

$$\{(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})\}\times(\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c}) = 0$$

24. For any three vectors, prove that

$$\{(\vec{a}-\vec{b})\times(\vec{a}-\vec{b}-\vec{c})\}\cdot(\vec{a}+2\vec{b}-\vec{c}) = 3\,[\vec{a}\,\vec{b}\,\vec{c}]$$

25. Prove that the ratio of the volume of a tetrahedron with that of the tetrahedron formed by the centroid of its faces is 27 : 1.

26. Lines are drawn in a given direction through the vertices O, A, B, C of a tetrahedron $OABC$ to meet the opposite faces in O', A', B', C'. Prove that

Volume of tetrahedron $O'A'B'C' = 3$ Volume of tetrahedron $OABC$.

27. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and l, m, n are distinct scalars such that the vectors $l\vec{a}+m\vec{b}+n\vec{c}$, $l\vec{b}+m\vec{c}+n\vec{a}$ and $l\vec{c}+m\vec{a}+n\vec{b}$ are coplanar, then prove that $l+m+n=0$.

28. If $\vec{a} = \hat{i}-2\hat{j}+\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-2\hat{k}$, solve the vector equations $\vec{a}\cdot\vec{r} = 4$, $\vec{b}\cdot\vec{r} = -3$ and $(\vec{a}\times\vec{b})\cdot\vec{r} = 10$.

29. Let c be a non-zero scallar, and $\vec{a}$ and $\vec{b}$ be two non-zero vectors such that $\vec{a}\perp\vec{b}$. Find the vector $\vec{r}$ which satisfies the equations

$$\vec{a}\cdot\vec{r} = c \text{ and } \vec{a}\times\vec{r} = \vec{b}$$ [IIT 1987]

30. Given that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, find vector $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations $\vec{r}\cdot\vec{a} = 0$, $\vec{r}\cdot\vec{b} = 1$ and $[\vec{r}\,\vec{a}\,\vec{b}] = 1$.

ANSWERS

9. $2V$

17. (i) $\vec{r} = \lambda\vec{a} - \vec{b}$ (ii) $\vec{r} = \frac{\vec{a}\times\vec{b}}{|\vec{a}|^2}$

(iii) $\vec{r} = \frac{1}{|\vec{a}|^2+1}\,[(\vec{a}\cdot\vec{b})\,\vec{a} + \vec{b} + (\vec{a}\times\vec{b})]$

19. $\hat{i}-\hat{j}-\hat{k}$, $\frac{1}{5}(\hat{i}+k)$, $-\frac{1}{5}(2\hat{i}-5\hat{j}-3\hat{k})$

22. Perpendicular to $\vec{a}$

23. $\hat{i}+2\hat{j}+2\sqrt{3}\,\hat{k}$, $\hat{i}-2\hat{j}-2\sqrt{3}\,\hat{k}$

24. $\angle A = \frac{\pi}{2}, \angle B = \frac{\pi}{3}, \angle C = \frac{\pi}{6}$

28. $\vec{r} = \frac{1}{5}(4\hat{i}-3\hat{j}+10\hat{k})$

29. $\vec{r} = \frac{c\vec{a}-(\vec{a}\times\vec{b})}{|\vec{a}|^2}$

30. $\vec{r} = \frac{1}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{a}\times\vec{b}|^2}(\vec{a}\times\vec{b})$.

HINTS TO SELECTED PROBLEMS

6. We have, $\vec{r}\cdot\vec{a} = l$, $\vec{r}\cdot\vec{b} = m$, $\vec{r}\cdot\vec{n} = c$

$$\therefore\ \vec{r}\cdot(\vec{a}+\vec{b}+\vec{c}) = l+m+n = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

7. Let the given points be P, Q, R and S respectively. Then,

$\vec{PQ} = -\vec{a}-5\vec{b}+4\vec{c}$, $\vec{PR} = \vec{a}+\vec{b}-\vec{c}$ and $\vec{PS} = -\vec{a}-9\vec{b}+7\vec{c}$.

$$\therefore\ [\vec{PQ}\ \vec{PR}\ \vec{PS}] = \begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 7 \end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}] = 0.$$

CHAPTER

18

APPLICATIONS OF VECTORS TO GEOMETRY

18.1 INTRODUCTION

Vectors are very helpful in handling certain type of problems in geometry. In this chapter, we shall study some applications of vectors in geometry.

18.2 VECTOR EQUATIONS OF A LINE

PARAMETRIC FORM *The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is*

$$\vec{r} = \vec{a} + \lambda \vec{b}, \text{ where } \lambda \text{ is a scalar.}$$

PROOF Let O be the origin and let A be the point with position vector $\vec{a}$. Then, $\vec{OA} = \vec{a}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line through A and parallel to $\vec{b}$ as shown in Fig. 18.1. Then,

$$\vec{OP} = \vec{r}$$

Clearly,

$$\vec{AP} \parallel \vec{b}$$

$$\therefore \quad \vec{AP} = \lambda \vec{b} \text{ for some scalar } \lambda.$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda \vec{b}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda \vec{b}$$

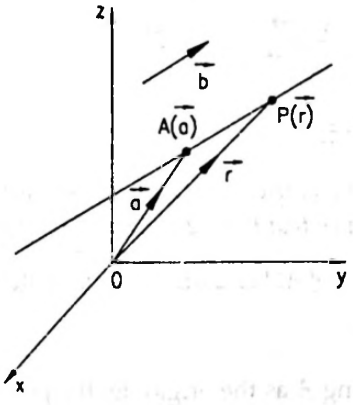


Fig. 18.1

Since every point on the line passing through A and parallel to $\vec{b}$ satisfies the above equation and for each value of λ the above equation gives the position vector of a point P on the line. Hence, the vector equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$.

NON-PARAMETRIC FORM *The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is*

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0} \text{ or, } \vec{r} \times \vec{b} = \vec{a} \times \vec{b}.$$

PROOF In Fig. 18.1, we have

$$\vec{AP} \parallel \vec{b}$$

$$\Rightarrow \quad \vec{AP} \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad (\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

Hence, the vector equation of the line in non-parametric form is

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0} \text{ or, } \vec{r} \times \vec{b} = \vec{a} \times \vec{b}.$$

18.3 VECTOR EQUATION OF A LINE PASSING THROUGH TWO POINTS

PARAMETRIC FORM *The vector equation of a line passing through two given points having position vectors $\vec{a}$ and $\vec{b}$ is*

$$\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a}) \text{ or, } \vec{r} = \vec{b} + \lambda (\vec{b} - \vec{a})$$

PROOF Let O be the origin and let A and B be the points with position vectors $\vec{a}$ and $\vec{b}$ respectively. Then,

$$\vec{OA} = \vec{a} \text{ and } \vec{OB} = \vec{b}.$$

Let $\vec{r}$ be the position vector of any point P on the line through A and B as shown in Fig. 18.2. Then,

$$\vec{OP} = \vec{r}.$$

Clearly, $\vec{AP}$ and $\vec{AB}$ are collinear vectors.

$$\therefore \quad \vec{AP} = \lambda \vec{AB} \quad \text{for some scalar } \lambda$$

$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Since every point on the line satisfies the above equation for each value of λ and the above equation gives the position vector of a point P on the line. Hence, the vector equation of the line is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

NON-PARAMETRIC FORM *The vector equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$ is*

$$(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = \vec{0} \text{ or } \vec{r} \times (\vec{b} - \vec{a}) = \vec{a} \times \vec{b}.$$

PROOF In Fig. 18.2, we have

$\vec{AP} \parallel \vec{AB}$

$\Rightarrow \vec{AP} \times \vec{AB} = \vec{0}$

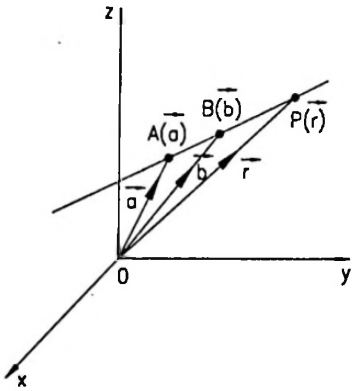


Fig. 18.2

$\Rightarrow (\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = 0$

$\Rightarrow \vec{r} \times (\vec{b} - \vec{a}) = \vec{a} \times \vec{b}.$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In a ΔOAB, E is the mid-point of OB and D is a point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, determine the ratio OP : OD using vector methods.*

[IIT 1989]

SOLUTION Taking O as the origin, let the position vectors of

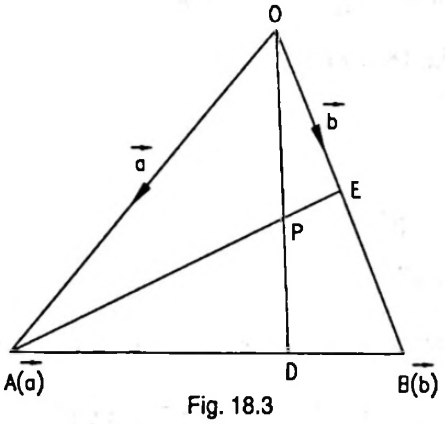


Fig. 18.3

A and B be $\vec{a}$ and $\vec{b}$ respectively.

Since D divides AB in the ratio 2 : 1. So, the position vector of D is $\dfrac{\vec{a} + 2\vec{b}}{3}$.

E is the mid-point of OB.

$\therefore$ P.V. of $E = \dfrac{\vec{b}}{2}$

Now, vector equation of OD is

$$\vec{r} = \vec{0} + \lambda\left(\frac{\vec{a} + 2\vec{b}}{3}\right) \quad \ldots(i)$$

Equation of AE is

$$\vec{r} = \vec{a} + \mu\left(\frac{\vec{b}}{2} - \vec{a}\right) \quad \ldots(ii)$$

Line OD and AE intersect at P. Therefore, for point P, we must have

$$\lambda\left(\frac{\vec{a} + 2\vec{b}}{3}\right) = \vec{a} + \mu\left(\frac{\vec{b}}{2} - \vec{a}\right)$$

$$\Rightarrow \left(\frac{\lambda}{3} - 1 + \mu\right)\vec{a} + \left(\frac{2\lambda}{3} - \frac{\mu}{2}\right)\vec{b} = 0$$

$$\Rightarrow \frac{\lambda}{3} - 1 + \mu = 0 \text{ and } \frac{2\lambda}{3} - \frac{\mu}{2} = 0 \quad \left[\because \vec{a} \text{ and } \vec{b} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \lambda = 3(-\mu + 1) \text{ and } \lambda = \frac{3\mu}{4}$$

$$\Rightarrow 3(-\mu + 1) = \frac{3\mu}{4} \Rightarrow -4\mu + 4 = \mu \Rightarrow \mu = \frac{4}{5}$$

$$\therefore \lambda = \frac{3\mu}{4} \Rightarrow \lambda = \frac{3}{5}.$$

Putting the value of λ in (i), we have

$$\text{P.V. of } P = \frac{\vec{a} + 2\vec{b}}{5}$$

$$\Rightarrow \vec{OP} = \frac{\vec{a} + 2\vec{b}}{5}$$

$$\therefore OP : OD = |\vec{OP}| : |\vec{OD}|$$

$$= \frac{|\vec{a} + 2\vec{b}|}{5} : \frac{|\vec{a} + 2\vec{b}|}{3}$$

$$= 3 : 5.$$

EXAMPLE 2 *In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2 DC and AE = 3 EC. Let P be the point of intersection of AD and BE. Find $\dfrac{BP}{PE}$ using vector methods.*

[IIT 1993]

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

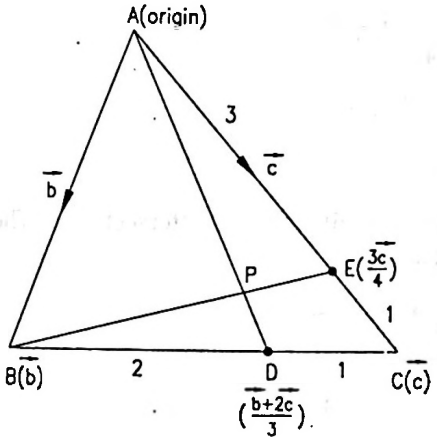


Fig. 18.4

We have,

$BD = 2\,DC$ and $AE = 3\,EC$

$\Rightarrow$ $BD : DC = 2 : 1$ and $AE : EC = 3 : 1$

$\Rightarrow$ P.V. of $D = \dfrac{\vec{b}+2\vec{c}}{3}$

and,

P.V. of $E = \dfrac{3\vec{c}}{4}$.

Equation of AD is

$$\vec{r} = \vec{0} + \lambda\left(\frac{\vec{b}+2\vec{c}}{3}\right) \qquad ...(i)$$

Equation of BE is

$$\vec{r} = \vec{b} + \mu\left(\frac{3\vec{c}}{4} - \vec{b}\right) \qquad ...(ii)$$

Since AD and BE intersect at P. Therefore, for point P, we must have,

$$\lambda\left(\frac{\vec{b}+2\vec{c}}{3}\right) = \vec{b} + \mu\left(\frac{3\vec{c}}{4} - \vec{b}\right)$$

$$\Rightarrow \left(\frac{\lambda}{3} - 1 + \mu\right)\vec{b} + \left(\frac{2\lambda}{3} - \frac{3\mu}{4}\right)\vec{c} = \vec{0}$$

$$\Rightarrow \frac{\lambda}{3} - 1 + \mu = 0 \text{ and } \frac{2\lambda}{3} - \frac{3\mu}{4} = 0 \quad \left[\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \lambda = \frac{9}{11}, \mu = \frac{8}{11}$$

$$\therefore \text{ P.V. of } P = \frac{3}{11}(\vec{b}+2\vec{c}) \qquad [\text{Putting } \lambda = \frac{9}{11} \text{ in (i)}]$$

Now,

$\vec{BP}$ = P.V. of P − P.V. of B

$$= \frac{3}{11}(\vec{b}+2\vec{c}) - \vec{b} = \frac{1}{11}(-8\vec{b}+6\vec{c}) = \frac{2}{11}(-4\vec{b}+3\vec{c})$$

and,

$\vec{PE}$ = P.V. of E − P.V. of P

$$= \frac{3\vec{c}}{4} - \frac{3}{11}(\vec{b}+2\vec{c}) = \frac{3}{44}(-4\vec{b}+3\vec{c})$$

$$\therefore \frac{BP}{PE} = \frac{|\vec{BP}|}{|\vec{PE}|} = \frac{\frac{2}{11}\left|-4\vec{b}+3\vec{c}\right|}{\frac{3}{44}\left|-4\vec{b}+3\vec{c}\right|} = \frac{8}{3}.$$

EXAMPLE 3 *Prove by vector methods or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides.*

[IIT 1998]

SOLUTION Let $ABCD$ be a trapezium whose diagonals intersect at R. Let P and Q be the mid-points of parallel sides AB and DC respectively. Taking A as the origin, let the position vectors of B, C and D be $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Then, the position vectors of P and Q are $\dfrac{\vec{b}}{2}$ and $\dfrac{\vec{c}+\vec{d}}{2}$ respectively.

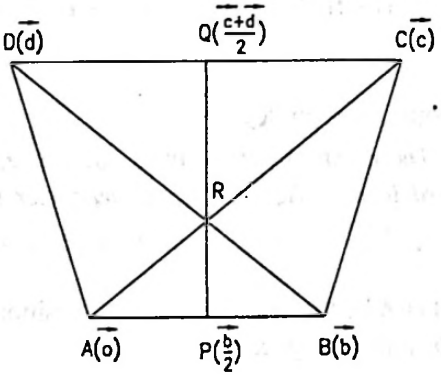


Fig. 18.5

Equation of diagonal AC is

$$\vec{r} = \lambda\vec{c} \qquad ...(i)$$

Equation of diagonal BD is

$$\vec{r} = \vec{b} + \mu(\vec{d} - \vec{b}) \qquad ...(ii)$$

For the point of intersection of AC and BD, we must have

$$\lambda\vec{c} = \vec{b} + \mu(\vec{d} - \vec{b})$$

$$\Rightarrow \lambda\vec{c} = t(\vec{c} - \vec{d}) + \mu[\vec{d} - t(\vec{c} - \vec{d})]$$

$$\Rightarrow (\lambda - t + \mu t)\vec{c} + (t - \mu - \mu t)\vec{d} = \vec{0} \quad \left[\because \vec{AB} \parallel \vec{DC} \therefore \vec{AB} = t\vec{DC} \Rightarrow \vec{b} = t(\vec{c} - \vec{d})\right]$$

$$\Rightarrow \lambda - t + \mu t = 0 \text{ and } t - \mu - \mu t = 0 \quad \left[\because \vec{c} \text{ and } \vec{D} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \lambda - t + \mu t + t - \mu - \mu t = 0$$

$$\Rightarrow \lambda = \mu$$

$$\therefore \lambda = \mu = \frac{t}{1+t} \qquad [\text{Putting } \lambda = \mu \text{ in } \lambda - t + \mu t = 0]$$

Putting this value of λ in (i), we have

Position vector of $R = \left(\frac{t}{1+t}\right)\vec{c}$

The equation of PQ in non-parametric form is

$$\left(\vec{r} - \frac{\vec{b}}{2}\right) \times \left(\frac{\vec{c}+\vec{d}}{2} - \frac{\vec{b}}{2}\right) = \vec{0}$$

or, $$\left\{\vec{r} - \frac{t}{2}(\vec{c}-\vec{d})\right\} \times \left\{\frac{\vec{c}+\vec{d}}{2} - t\left(\frac{\vec{c}-\vec{d}}{2}\right)\right\} = \vec{0} \quad [\because \vec{b} = t(\vec{c}-\vec{d})]$$

or, $$\left\{\vec{r} - \frac{t}{2}(\vec{c}-\vec{d})\right\} \times \left|(1-t)\vec{c} + (1+t)\vec{d}\right| = \vec{0}$$

In order to prove that R lies on it, we must have

$$\left\{\left(\frac{t}{1+t}\right)\vec{c} - \frac{t}{2}(\vec{c}-\vec{d})\right\} \times \left|(1-t)\vec{c} + (1+t)\vec{d}\right| = \vec{0}$$

Now,

$$\left\{\left(\frac{t}{1+t}\right)\vec{c} - \frac{t}{2}(\vec{c}-\vec{d})\right\} \times \left|(1-t)\vec{c} + (1+t)\vec{d}\right|$$

$$= \frac{t}{2(1+t)}\left|(1-t)\vec{c} + (1+t)\vec{d}\right\} \times \left|(1-t)\vec{c} + (1+t)\vec{d}\right|$$

$$= \vec{0}$$

Hence, point R lies on PQ.

EXAMPLE 4 *The median AD of a triangle ABC is bisected at E and BE is produced to meet AC in F. Prove by vector method that* $EF = \frac{1}{4}BF$.

SOLUTION Let A be the origin and let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

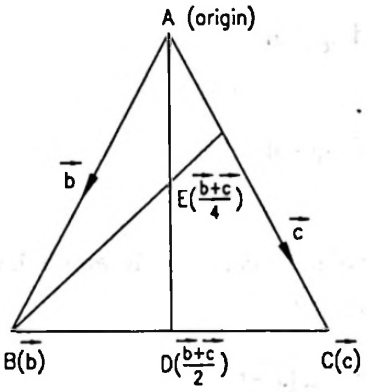


Fig. 18.6

Then,

$$\text{P.V. of } D = \frac{\vec{b}+\vec{c}}{2}$$

Since E is the mid-point of AD.

$$\therefore \quad \text{P.V. of } E = \frac{\vec{b}+\vec{c}}{4}$$

Equation of BE is

$$\vec{r} = \vec{b} + \lambda\left(\frac{\vec{b}+\vec{c}}{4} - \vec{b}\right)$$

or, $$\vec{r} = \vec{b} + \frac{\lambda}{4}(\vec{c} - 3\vec{b}) \qquad ...(i)$$

Equation of AC is

$$\vec{r} = \vec{0} + \mu\vec{c} \qquad ...(ii)$$

It is given that AC and BE intersect at F. Therefore, for point F, we have

$$\mu\vec{c} = \vec{b} + \frac{\lambda}{4}(\vec{c} - 3\vec{b})$$

$$\Rightarrow \quad (4\mu - \lambda)\vec{c} + (3\lambda - 4)\vec{b} = \vec{0}$$

$$\Rightarrow \quad 4\mu - \lambda = 0 \text{ and } 3\lambda - 4 = 0 \qquad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow \quad \lambda = \frac{4}{3} \text{ and } \mu = \frac{1}{3}.$$

Putting $\mu = \frac{1}{3}$ in (ii), we get

$$\text{P.V. of } F = \frac{\vec{c}}{3}$$

$$\therefore \quad \vec{EF} = \frac{\vec{c}}{3} - \frac{\vec{b}+\vec{c}}{4} = \frac{\vec{c} - 3\vec{b}}{12}$$

$$\Rightarrow \quad |\vec{EF}| = \frac{|\vec{c} - 3\vec{b}|}{12}$$

and,

$$\vec{BF} = \frac{\vec{c}}{3} - \vec{b} = \frac{\vec{c} - 3\vec{b}}{3}$$

$$\Rightarrow \quad |\vec{BF}| = \frac{|\vec{c} - 3\vec{b}|}{3}$$

$$\therefore \quad 4|\vec{EF}| = |\vec{BF}| \text{ or, } EF = \frac{1}{4}BF.$$

EXAMPLE 5 *Points P, Q, R divide the sides BC, CA and AB of a triangle ABC in the ratio $1 : 2$. The three lines AP, BQ and CR enclose a triangle XYZ. Prove that the triangles ABC and XYZ have the same centroid.*

SOLUTION Let the position vectors of A, B, C be $\vec{a}, \vec{b}, \vec{c}$ respectively.

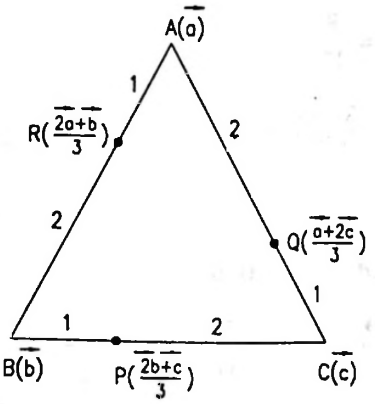


Fig. 18.7

Then, the position vectors of P, Q, R are

$\frac{2\vec{b}+\vec{c}}{3}, \frac{\vec{a}+2\vec{c}}{3}$ and $\frac{2\vec{a}+\vec{b}}{3}$ respectively.

The vector equations of lines AP, BQ and CR are

$$\vec{r} = \vec{a} + \lambda_1\left(\frac{2\vec{b}+\vec{c}}{3} - \vec{a}\right) \quad ...(i)$$

$$\vec{r} = \vec{b} + \lambda_2\left(\frac{\vec{a}+2\vec{c}}{3} - \vec{b}\right) \quad ...(ii)$$

$$\vec{r} = \vec{c} + \lambda_3\left(\frac{2\vec{a}+\vec{b}}{3} - \vec{c}\right) \quad ...(iii)$$

respectively.

Lines AP and BQ intersect at X. Therefore, for point X, we have

$$\vec{a} + \lambda_1\left(\frac{2\vec{b}+\vec{c}}{3} - \vec{a}\right) = \vec{b} + \lambda_2\left(\frac{\vec{a}+2\vec{c}}{3} - \vec{b}\right)$$

$$\Rightarrow (1-\lambda_1)\vec{a} + \left(\frac{2}{3}\lambda_1\right)\vec{b} + \left(\frac{\lambda_1}{3}\right)\vec{c} = \left(\frac{\lambda_2}{3}\right)\vec{a} + (1-\lambda_2)\vec{b}_2 + \frac{2}{3}\lambda_2\vec{c}$$

$$\Rightarrow 1-\lambda_1 = \frac{\lambda_2}{3}, \frac{2\lambda_1}{3} = 1-\lambda_2 \text{ and } \frac{\lambda_1}{3} = \frac{2\lambda_2}{3}$$

$$\Rightarrow 3\lambda_1+\lambda_2 = 3, 2\lambda_1+3\lambda_2 = 3 \text{ and } \lambda_1 = 2\lambda_2$$

$$\Rightarrow \lambda_2 = \frac{3}{7}, \lambda_1 = \frac{6}{7}$$

$\therefore$ Position vector of X

$$= \vec{a} + \frac{6}{7}\left(\frac{2\vec{b}+\vec{c}}{3} - \vec{a}\right) \qquad [\text{Putting } \lambda_1 = \tfrac{6}{7} \text{ in (i)}]$$

$$= \frac{\vec{a}+4\vec{b}+2\vec{c}}{7}$$

Similarly, lines BQ and CR intersect at Y.

$\therefore$ Position vector of $Y = \frac{2\vec{a}+\vec{b}+4\vec{c}}{7}$

Lines AP and CR intersect at Z.

$\therefore$ Position vector of $Z = \frac{4\vec{a}+2\vec{b}+\vec{c}}{7}$

$\therefore$ Position vector of the centroid of ΔXYZ

$$= \frac{\frac{\vec{a}+4\vec{b}+2\vec{c}}{7} + \frac{2\vec{a}+\vec{b}+4\vec{c}}{7} + \frac{4\vec{a}+2\vec{b}+\vec{c}}{7}}{3}$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

= Position vector of the centroid of Δ ABC.

Hence, the triangles ABC and XYZ have the same centroid.

EXAMPLE 6 *Point X and Y are taken on the sides QR and RS, respectively of a parallelogram $PQRS$, so that $QX = 4\,XR$ and $RY = 4YS$. The line XY cuts the line PR at Z. Find the ratio $PZ : ZR$.*

SOLUTION Let P be the origin of reference and the position vectors of Q and S be $\vec{a}$ and $\vec{b}$ respectively. Then, $\vec{PQ} = \vec{a}$ and $\vec{PS} = \vec{b}$.

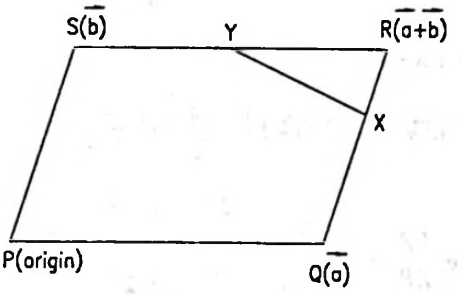


Fig. 18.8

In ΔPQR, we have

$$\vec{PQ} + \vec{QR} = \vec{PR}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{PR}$$

$\Rightarrow$ Position vector of R is $(\vec{a}+\vec{b})$.

Since X and Y divide QR and RS in the ratio 4 : 1 and 1 : 4 respectively. Therefore,

$$\text{Position vector of } X = \frac{4(\vec{a}+\vec{b})+1\,\vec{a}}{4+1} = \frac{5\vec{a}+4\vec{b}}{5}$$

$$\text{Position vector of } Y = \frac{1(\vec{a}+\vec{b})+4\,\vec{b}}{4+1} = \frac{\vec{a}+5\vec{b}}{5}$$

Vector equation of the line PR is

$$\vec{r} = \vec{0} + \lambda(\vec{a}+\vec{b}) \quad ...(i)$$

Vector equation of the line XY is

$$\vec{r} = \frac{5\vec{a}+4\vec{b}}{5} + \lambda\left(\frac{\vec{a}+5\vec{b}}{5} - \frac{5\vec{a}+4\vec{b}}{5}\right)$$

or, $$\vec{r} = \frac{5\vec{a}+4\vec{b}}{5} + \frac{\mu}{5}(\vec{b}-4\vec{a}) \quad ...(ii)$$

Lines (i) and (ii) intersect at Z. Therefore, for point Z, we have

$$\lambda(\vec{a}+\vec{b}) = \frac{5\vec{a}+4\vec{b}}{5} + \frac{\mu}{5}(\vec{b}-4\vec{a})$$

$$\Rightarrow \left\{\lambda - 1 + \frac{4\mu}{5}\right\}\vec{a} + \left\{\lambda - \frac{4}{5} - \frac{\mu}{5}\right\}\vec{b} = \vec{0}$$

$$\Rightarrow \lambda - 1 + \frac{4\mu}{5} = 0 \text{ and } \lambda - \frac{4}{5} - \frac{\mu}{5} = 0 \quad \left[\because \vec{a} \text{ and } \vec{b} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \mu = \frac{1}{5}, \lambda = \frac{21}{25}$$

Putting $\lambda = \frac{9}{10}$ in (i), we get

Position vector of $Z = \frac{21}{25}(\vec{a}+\vec{b})$

$\Rightarrow \vec{PZ} = \frac{21}{25}(\vec{a}+\vec{b})$

Now,

$\vec{ZR} = \vec{PR} - \vec{PZ}$

$\Rightarrow \vec{ZR} = (\vec{a}+\vec{b}) - \frac{21}{25}(\vec{a}+\vec{b}) = \frac{4}{25}(\vec{a}+\vec{b})$

Hence, $\frac{PZ}{ZR} = \frac{|\vec{PZ}|}{|\vec{ZR}|} = \frac{\frac{21}{25}|\vec{a}+\vec{b}|}{\frac{4}{25}|\vec{a}+\vec{b}|} = \frac{21}{4}$

EXAMPLE 7 *Points D, E, F divide the sides BC, CA and AB of a triangle respectively in the ratio 2 : 3, 1 : 2 and 3 : 1. Prove that the lines AD, BE and CF are concurrent and find the position vector of the point of concurrency.*

SOLUTION Taking the vertex A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

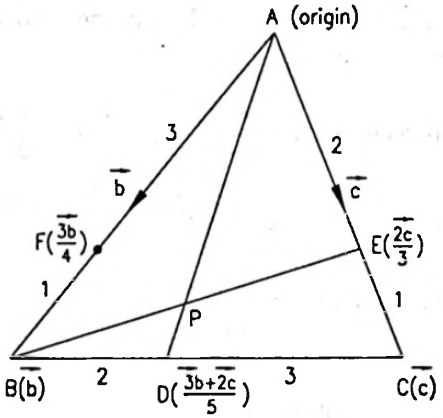


Fig. 18.9

The position vectors of D, E and F are $\frac{3\vec{b}+2\vec{c}}{5}$, $\frac{2\vec{c}}{3}$ and $\frac{3\vec{b}}{4}$ respectively.

The vector equations of the lines AD, BE and CF are

$$\vec{r} = \frac{\lambda_1}{5}(3\vec{b}+2\vec{c}) \quad \ldots(i)$$

$$\vec{r} = \vec{b} + \lambda_2\left(\frac{2\vec{c}}{3} - \vec{b}\right) \quad \ldots(ii)$$

$$\vec{r} = \vec{c} + \lambda_3\left(\frac{3\vec{b}}{4} - \vec{c}\right) \quad \ldots(iii)$$

respectively.

Suppose lines AD and BE intersect at P. For point P, we have

$$\frac{\lambda_1}{5}(3\vec{b}+2\vec{c}) = \vec{b} + \lambda_2\left(\frac{2\vec{c}}{3} - \vec{b}\right)$$

$$\Rightarrow \left(\frac{3\lambda_1}{5} - 1 + \lambda_2\right)\vec{b} + \left(\frac{2\lambda_1}{5} - \frac{2\lambda_2}{3}\right)\vec{c} = \vec{0}$$

$$\Rightarrow \frac{3\lambda_1}{5} - 1 + \lambda_2 = 0 \text{ and } \frac{2\lambda_1}{5} - \frac{2\lambda_2}{3} = 0 \quad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow \lambda_2 = \frac{1}{2}, \lambda_1 = \frac{5}{6}$$

Putting $\lambda_1 = \frac{5}{6}$ in (i), we obtain the position vector of P as $\frac{3\vec{b}+2\vec{c}}{6}$.

Suppose lines BE and CF intersect at Q. For point Q, we have

$$\vec{b} + \lambda_2\left(\frac{2\vec{c}}{3} - \vec{b}\right) = \vec{c} + \lambda_3\left(\frac{3\vec{b}}{4} - \vec{c}\right)$$

$$\Rightarrow \left(1 - \lambda_2 - \frac{3\lambda_3}{4}\right)\vec{b} + \left(\frac{2\lambda_2}{3} - 1 + \lambda_3\right)\vec{c} = \vec{0}$$

$$\Rightarrow 1 - \lambda_2 - \frac{3\lambda_3}{4} = 0 \text{ and } \frac{2\lambda_2}{3} - 1 + \lambda_3 = 0 \quad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow \lambda_2 = \frac{1}{2}, \lambda_3 = \frac{2}{3}.$$

Putting $\lambda_2 = \frac{1}{2}$ in (ii), we obtain the position vector of $\vec{Q}$ as $\vec{b} + \frac{1}{2}\left(\frac{2\vec{c}}{3} - \vec{b}\right) = \frac{3\vec{b}+2\vec{c}}{6}$, which is same as the position vector of P.

Hence, AD, BE and CF are concurrent. The position vector of the point of concurrency is $\frac{3\vec{b}+2\vec{c}}{6}$.

EXAMPLE 8 *In a triangle PQR, S and T are the points on QR and PR respectively, such that QS = 3 SR and PT = 4TR. Let M be the point of intersection of PS and QT. Determine the ratio QM : MT, using vector methods.*

SOLUTION Taking P as the origin, let the position vectors of Q and R be $\vec{b}$ and $\vec{c}$ respectively. The position vectors of S and T are $\frac{\vec{b}+3\vec{c}}{4}$ and $\frac{4\vec{c}}{5}$ respectively.

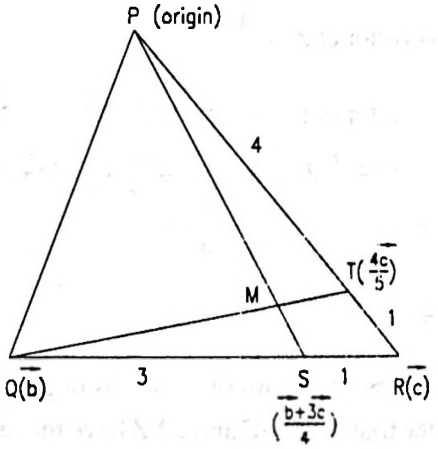


Fig. 18.10

The vector equations of lines PS and QT are

$$\vec{r} = \lambda\left(\frac{\vec{b}+3\vec{c}}{4}\right) \quad ...(i)$$

and,

$$\vec{r} = \vec{b} + \mu\left(\frac{4\vec{c}}{5} - \vec{b}\right) \quad ...(ii)$$

These two lines intersect at point M.

$$\therefore \lambda\left(\frac{\vec{b}+3\vec{c}}{4}\right) = \vec{b} + \mu\left(\frac{4\vec{c}}{5} - \vec{b}\right)$$

$$\Rightarrow \left(\frac{\lambda}{4} - 1 + \mu\right)\vec{b} + \left(\frac{3\lambda}{4} - \frac{4\mu}{5}\right)\vec{c} = \vec{0}$$

$$\Rightarrow \frac{\lambda}{4} - 1 + \mu = 0 \text{ and } \frac{3\lambda}{4} - \frac{4\mu}{5} = 0 \quad \left[\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}\right]$$

$$\Rightarrow \lambda + 4\mu = 4 \text{ and } 15\lambda - 16\mu = 0$$

$$\Rightarrow \lambda = \frac{16}{19} \quad \text{and} \quad \mu = \frac{15}{19}$$

$\therefore$ Position vector of M is $\frac{4}{19}(\vec{b}+3\vec{c})$ [Putting $\lambda = \frac{16}{19}$ in (i)]

Now,

$$\vec{QM} = \text{P.V. of } M - \text{P.V. of } Q$$

$$= \frac{4}{19}(\vec{b}+3\vec{c}) - \vec{b} = \frac{12\vec{c} - 15\vec{b}}{19} = \frac{3}{19}(4\vec{c} - 5\vec{b})$$

$$\vec{MT} = \text{P.V. of } T - \text{P.V. of } M$$

$$= \frac{4\vec{c}}{5} - \frac{4}{19}(\vec{b}+3\vec{c}) = \frac{4}{95}(4\vec{c} - 5\vec{b})$$

$$\therefore QM : MT = |\vec{QM}| : |\vec{MT}|$$

$$= \frac{3}{19}|4\vec{c} - 5\vec{b}| : \frac{4}{95}|4\vec{c} - 5\vec{b}| = 15 : 4$$

EXAMPLE 9 *The position vectors of the points P and Q are $5\hat{i}+7\hat{j}-2\hat{k}$ and $-3\hat{i}+3\hat{j}+6\hat{k}$ respectively. The vector $\vec{A} = 3\hat{i}-\hat{j}+\hat{k}$ passes through the point P and the vector $\vec{B} = -3\hat{i}+2\hat{j}+4\hat{k}$ passes through the point Q. A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vectors $\vec{A}$ and $\vec{B}$. Find the position vectors of the points of intersection.*

SOLUTION Clearly, line L_1 passes through the point P and is in the direction of $\vec{A}$. So, its vector equation is

$$\vec{r} = (5\hat{i}+7\hat{j}-2\hat{k}) + \lambda(3\hat{i}-\hat{j}+\hat{k}) \quad ...(i)$$

Similarly, line L_2 passes through Q and is in the direction of $\vec{B}$. Therefore, vector equation of L_2 is

$$\vec{r} = (-3\hat{i}+3\hat{j}+6\hat{k}) + \mu(-3\hat{i}+2\hat{j}+4\hat{k}) \quad ...(ii)$$

Suppose lines L_1 and L_2 intersect the support of vector $2\hat{i}+7\hat{j}-5\hat{k}$ at R and S respectively.

Let the position vectors of R and S be

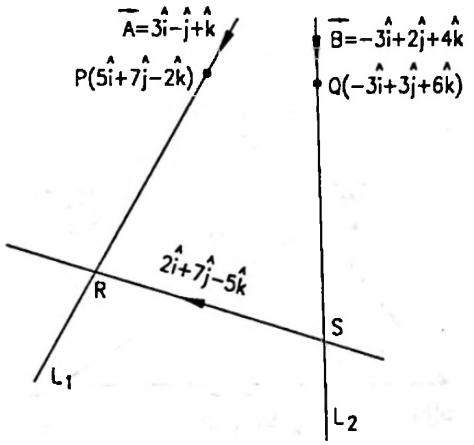


Fig. 18.11

$$(5\hat{i}+7\hat{j}-2\hat{k}) + \lambda(3\hat{i}-\hat{j}+\hat{k})$$

and, $(-3\hat{i}+3\hat{j}+6\hat{k}) + \mu(-3\hat{i}+2\hat{j}+4\hat{k})$

respectively.

$$\therefore \vec{RS} = (-3\mu - 3 - 3\lambda - 5)\hat{i} + (2\mu + 3 + \lambda - 7)\hat{j} + (4\mu + 6 - \lambda + 2)\hat{k}$$

$$= (-3\mu - 3\lambda - 8)\hat{i} + (\lambda + 2\mu - 4)\hat{j} + (-\lambda + 4\mu + 8)\hat{k}$$

Since $\vec{RS}$ is parallel to $2\hat{i}+7\hat{j}-5\hat{k}$.

$$\therefore \frac{-3\mu - 3\lambda - 8}{2} = \frac{\lambda + 2\mu - 4}{7} = \frac{-\lambda + 4\mu + 8}{-5} = t \text{ (say)}$$

$$\Rightarrow -3\mu - 3\lambda - 8 = 2t$$

$$\lambda + 2\mu - 4 = 7t$$

$$-\lambda + 4\mu + 8 = -5t$$

Solving last two equations for λ and μ in terms of t and substituting them in (i), we get

$$\lambda = \mu = t = -1.$$

$\therefore$ Position vector of R

$$= (5\hat{i}+7\hat{j}-2\hat{k}) - (3\hat{i}-\hat{j}+\hat{k})$$

$$= 2\hat{i}+8\hat{j}-3\hat{k} \quad \text{[Putting } \lambda = -1 \text{ in (i)]}$$

Position vector of S

$$= (-3\hat{i}+3\hat{j}+6\hat{k}) - (-3\hat{i}+2\hat{j}+4\hat{k})$$

$$= \hat{j}+2\hat{k} \quad \text{[Putting } \mu = -1 \text{ in (ii)]}$$

EXAMPLE 10 *Prove that the medians of a triangle meet at a point which trisects them.*

SOLUTION Let ABC be a triangle and let AD, BE and CF be its medians. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of the vertices A, B, C respectively. Then, the position vectors of the mid-points D, E, F are respectively $\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}, \frac{\vec{a}+\vec{b}}{2}$. Therefore, the equation of the medians AD, BE and CF are respectively

$$\vec{r} = \vec{a} + \lambda_1\left(\frac{\vec{b}+\vec{c}}{2} - \vec{a}\right) \quad ...(i)$$

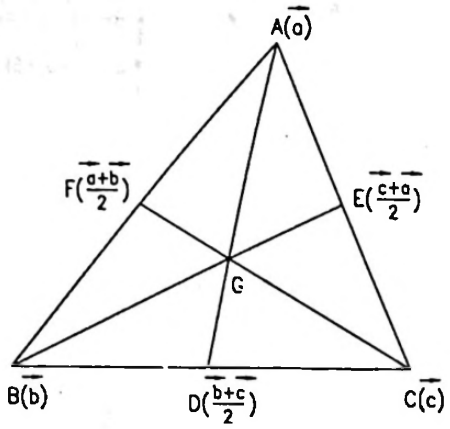


Fig. 18.12

$$\vec{r} = \vec{b} + \lambda_2\left(\frac{\vec{c}+\vec{a}}{2} - \vec{b}\right) \quad ...(ii)$$

$$\vec{r} = \vec{c} + \lambda_3\left(\frac{\vec{a}+\vec{b}}{2} - \vec{c}\right) \quad ...(iii)$$

Suppose the medians AD and BE meet at G. Then, for point G, we have

$$\vec{a} + \lambda_1\left(\frac{\vec{b}+\vec{c}}{2} - \vec{a}\right) = \vec{b} + \lambda_2\left(\frac{\vec{c}+\vec{a}}{2} - \vec{b}\right)$$

$$\Rightarrow \quad 1-\lambda_1 = \frac{\lambda_2}{2}, 1-\lambda_2 = \frac{\lambda_1}{2} \text{ and } \lambda_1 = \lambda_2$$

[On equating the coefficients of $\vec{a}, \vec{b}, \vec{c}$]

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \frac{2}{3}$$

Putting $\lambda_1 = \frac{2}{3}$ in (i), we obtain the position vector of G as

$$\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Now,

$$\vec{AG} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} - \vec{a} = \frac{\vec{b}+\vec{c}-2\vec{a}}{3}$$

and,

$$\vec{GD} = \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}+\vec{c}}{3} = \frac{\vec{b}+\vec{c}-2\vec{a}}{6}$$

Clearly, $\vec{AG} = 2\vec{GD} \Rightarrow AG : GD = 2 : 1.$

Also, $\vec{r} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$ satisfies the equation of CF for $\lambda_3 = \frac{2}{3}$.

Therefore, CF also passes through G.

Also, $\vec{CG} = 2\vec{GF} \Rightarrow CG : GF = 2 : 1$

Hence, G is the point of trisection of the medians of ΔABC.

EXAMPLE 11 *Show that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent at a point of quadrisection.*

SOLUTION Let $OABC$ be a tetrahedron with base as ΔABC. Let G_1, G_2, G_3, G_4 be the centroids of its faces ABC, BOC, COA and AOB respectively. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C referred to O as origin, then the position vectors of G_1, G_2, G_3, and G_4 are $\frac{\vec{a}+\vec{b}+\vec{c}}{3}, \frac{\vec{b}+\vec{c}}{3}, \frac{\vec{c}+\vec{a}}{3}$ and $\frac{\vec{a}+\vec{b}}{3}$ respectively. Thus, the vector equations of OG_1, AG_2, BG_3 and CG_4 are

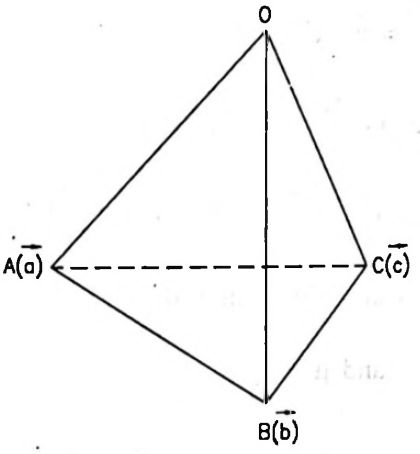


Fig. 18.13

$$\vec{r} = \lambda_1\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) \quad ...(i)$$

$$\vec{r} = \vec{a} + \lambda_2\left(\frac{\vec{b}+\vec{c}}{3} - \vec{a}\right) \quad ...(ii)$$

$$\vec{r} = \vec{b} + \lambda_3\left(\frac{\vec{c}+\vec{a}}{3} - \vec{b}\right) \quad ...(iii)$$

$$\vec{r} = \vec{c} + \lambda_4\left(\frac{\vec{a}+\vec{b}}{3} - \vec{c}\right) \quad ...(iv)$$

For the point of intersection of OG_1 and AG_2, we have

$$\lambda_1\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) = \vec{a} + \lambda_2\left(\frac{\vec{b}+\vec{c}}{3} - \vec{a}\right)$$

$$\Rightarrow \quad \frac{\lambda_1}{3} = 1-\lambda_2, \frac{\lambda_1}{3} = \frac{\lambda_2}{3} \qquad \left[\text{On equating the coefficients of } \vec{a}, \vec{b}, \vec{c}\right]$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \frac{3}{4}$$

Substituting $\lambda_1 = \frac{3}{4}$ in (i), we obtain the position vector of the point of intersection of OG_1 and AG_2 as

$$\frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

Let G be the point of intersection of OG_1 and AG_2. Then,

$$\vec{OG} = \frac{\vec{a}+\vec{b}+\vec{c}}{4} \text{ and } \vec{GG_1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} - \frac{\vec{a}+\vec{b}+\vec{c}}{4} = \frac{\vec{a}+\vec{b}+\vec{c}}{12}$$

$$\Rightarrow \quad \vec{OG} = 3\vec{GG_1}$$

$$\Rightarrow \quad OG : GG_1 = 3 : 1$$

Thus, G divides OG, in the ratio $3 : 1$.

Clearly, $\vec{r} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$ satisfies (iii) and (iv) for

$$\lambda_3 = \frac{3}{4} = \lambda_4.$$

Thus, BG_3 and CG_4 pass through G and G divides them in the same ratio 3 : 1. Hence, G is the point of quadrisection of the line segments OG_1, AG_2, BG_3 and CG_4.

EXAMPLE 12 *Three concurrent straight lines OA, OB, OC are produced to D, E, F respectively. Prove that the points of intersection of AB and DE, BC and EF, CA and FD are collinear.*

SOLUTION Taking O as the origin, let the position vectors of points A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. It is given that OA, OB, OC are produced to D, E, F respectively.

$\therefore \quad \vec{OD} = t_1 \vec{OA} = t_1 \vec{a},$

$\vec{OE} = t_2 \vec{OB} = t_2 \vec{b},$

and, $\vec{OF} = t_3 \vec{OC} = t_3 \vec{c}$ for some scalars t_1, t_2, t_3.

Thus, the position vectors of D, E, F are $t_1\vec{a}, t_2\vec{b}$ and $t_3\vec{c}$ respectively.

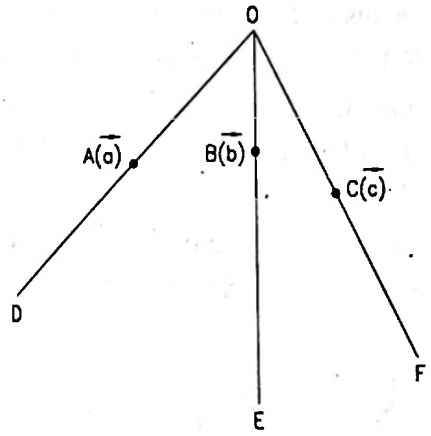


Fig. 18.14

The vector equations of AB and DE are

$$\vec{r} = \vec{a} + m(\vec{b} - \vec{a}) \quad \text{...(i)}$$

and, $$\vec{r} = t_1\vec{a} + n(t_2\vec{b} - t_1\vec{a}) \quad \text{...(ii)}$$

Suppose AB and DE intersect at point P. Then, for the point P, we must have

$$\vec{a} + m(\vec{b} - \vec{a}) = t_1\vec{a} + n(t_2\vec{b} - t_1\vec{a})$$

$\Rightarrow 1 - m - t_1 + nt_1 = 0$ and $m - nt_2 = 0$

[On equating coefficients]

$\Rightarrow 1 - nt_2 - t_1 + nt_1 = 0$ [Putting $m = nt_2$ in first eqn]

$\Rightarrow n = \frac{t_1 - 1}{t_1 - t_2}$

$\therefore \quad m = nt_2 \Rightarrow m = \frac{t_2(t_1 - 1)}{t_1 - t_2}$

Putting the value of m in (i) or that of n in (ii), we get the position vector $\vec{r_1}$ of the point P as

$$\vec{r_1} = \vec{a} + t_2\frac{(t_1 - 1)}{(t_1 - t_2)}(\vec{b} - \vec{a})$$

$$\Rightarrow \vec{r_1} = \frac{t_1(1 - t_2)}{t_1 - t_2}\vec{a} - \frac{t_2(1 - t_1)}{t_1 - t_2}\vec{b}$$

Suppose BC and EF intersect at Q.

Proceeding exactly in the similar manner, we obtain the position vector $\vec{r_2}$ of point Q as

$$\vec{r_2} = \frac{t_2(1 - t_3)\vec{b}}{t_2 - t_3} - \frac{t_3(1 - t_2)\vec{c}}{t_2 - t_3}$$

Now, let CA and FD intersect at R. Then, the position vector $\vec{r_3}$ of point R is given by

$$\vec{r_3} = \frac{t_3(1 - t_1)\vec{c}}{t_3 - t_1} - \frac{t_1(1 - t_3)\vec{a}}{t_3 - t_1}$$

We have to prove that the points P, Q and R are collinear. For this, it is sufficient to show that there exist scalars x, y, z such that

$$x\vec{r_1} + y\vec{r_2} + z\vec{r_3} = \vec{0}, \quad \text{where } x + y + z = 0$$

If we take

$$x = \frac{t_1 - t_2}{(1 - t_2)(1 - t_1)}, y = \frac{t_2 - t_3}{(1 - t_2)(1 - t_3)}$$

and, $$z = \frac{t_3 - t_1}{(1 - t_1)(1 - t_3)},$$

then, we find that

$$x\vec{r_1} + y\vec{r_2} + z\vec{r_3} = \vec{0}$$

and,

$$x + y + z = \frac{t_1 - t_2}{(1 - t_1)(1 - t_2)} + \frac{t_2 - t_3}{(1 - t_2)(1 - t_3)} + \frac{t_3 - t_1}{(1 - t_1)(1 - t_3)} = 0$$

Hence, the points P, Q, R are collinear.

EXAMPLE 13 *If A_1, A_2, A_3 and B_1, B_2, B_3 are two sets of collinear points, then prove that the points of intersection of the pair of lines A_1B_2, A_2B_1; A_2B_3, A_3B_2; A_3B_1, A_1B_3 are collinear.*

SOLUTION Suppose the points A_1, A_2, A_3 lie on the line OA and the points B_1, B_2, B_3 lie on the line OB. Since any two vectors are always coplanar. So, let OA and OB be two coplanar lines.

Taking O, the point of intersection of the given lines, as the origin of reference, let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $\lambda_1 \vec{a}, \lambda_2 \vec{a}, \lambda_3 \vec{a}$ be the position vectors of points A_1, A_2, A_3 respectively. Also, let the position vectors of points B_1, B_2, B_3 be $\mu_1 \vec{b}, \mu_2 \vec{b}$ and $\mu_3 \vec{b}$ respectively.

The vector equations of lines $A_1 B_2$ and $A_2 B_1$ are

$$\vec{r} = \lambda_1 \vec{a} + l(\mu_2 \vec{b} - \lambda_1 \vec{a}) \quad ...(i)$$

and,

$$\vec{r} = \lambda_2 \vec{a} + m(\mu_1 \vec{b} - \lambda_2 \vec{a}) \quad ...(ii)$$

Suppose $A_1 B_2$ and $A_2 B_1$ intersect at P_3. At their point of intersection, we have

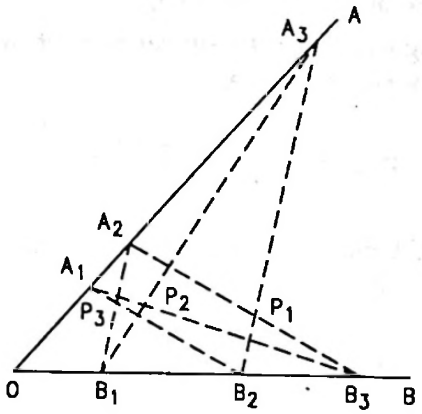


Fig. 18.15

$$\lambda_1 \vec{a} + l(\mu_2 \vec{b} - \lambda_1 \vec{a}) = \lambda_2 \vec{a} + m(\mu_1 \vec{b} - \lambda_2 \vec{a})$$

$$\Rightarrow (\lambda_1 - l\lambda_1 - \lambda_2 + m\lambda_2)\vec{a} + (l\mu_2 - m\mu_1)\vec{b} = \vec{0}$$

$$\Rightarrow (\lambda_1 - \lambda_2) - l\lambda_1 + m\lambda_2 = 0$$

and, $l\mu_2 - m\mu_1 = 0$ [$\because \vec{a}$ and $\vec{b}$ are non-collinear vectors]

$$\Rightarrow (\lambda_1 - \lambda_2) - \frac{m\mu_1}{\mu_2}\lambda_1 + m\lambda_2 = 0$$

$$\Rightarrow m - (\mu_2\lambda_2 - \mu_1\lambda_1) + \mu_2(\lambda_1 - \lambda_2) = 0$$

$$\Rightarrow m = \frac{\mu_2(\lambda_2 - \lambda_1)}{\mu_2\lambda_2 - \mu_1\lambda_1}$$

Substituting the value of m in (ii), we see that the position vector $\vec{r_3}$ of the point of intersection of $A_1 B_2$ and $A_2 B_1$ is given by

$$\vec{r_3} = \lambda_2 \vec{a} + \frac{\mu_2(\lambda_2 - \lambda_1)}{(\mu_2\lambda_2 - \mu_1\lambda_1)}(\mu_1 \vec{b} - \lambda_2 \vec{a})$$

$$\text{or, } \vec{r_3} = \frac{\mu_1\mu_2(\lambda_2 - \lambda_1)}{\mu_2\lambda_2 - \mu_1\lambda_1}.\vec{b} + \frac{\lambda_1\lambda_2(\mu_2 - \mu_1)}{\mu_2\lambda_2 - \mu_1\lambda_1}.\vec{a}$$

$$\text{or, } \vec{r_3} = \frac{1}{\mu_1\lambda_1 - \mu_2\lambda_2}\left\{\lambda_1\lambda_2(\mu_1 - \mu_2)\vec{a} + \mu_1\mu_2(\lambda_1 - \lambda_2)\vec{b}\right\}$$

Similarly, the position vector $\vec{r_2}$ of point P_2, the point of intersection of $A_1 B_3$ and $A_3 B_1$, is given by

$$\vec{r_2} = \frac{1}{\mu_3\lambda_3 - \mu_1\lambda_1}\left\{\lambda_1\lambda_3(\mu_3 - \mu_1)\vec{a} + \mu_1\mu_3(\lambda_3 - \lambda_1)\vec{b}\right\}$$

The position vector $\vec{r_1}$ of the point P_1, the point of intersection of $A_2 B_3$ and $A_3 B_2$, is given by

$$\vec{r_1} = \frac{1}{\mu_2\lambda_2 - \mu_3\lambda_3}\left\{\lambda_2\lambda_3(\mu_2 - \mu_3)\vec{a} + \mu_2\mu_3(\lambda_2 - \lambda_3)\vec{b}\right\}$$

In order to prove that the points P_1, P_2, P_3 are collinear, it is sufficient to show that there exist scalars x_1, x_2, x_3 such that $x_1\vec{r_1} + x_2\vec{r_2} + x_3\vec{r_3} = \vec{0}$, where $x_1 + x_2 + x_3 = 0$.

We observer that, if we take

$$x_1 = (\mu_2\lambda_2 - \mu_3\lambda_3)\lambda_1\mu_1,$$

$$x_2 = (\mu_3\lambda_3 - \mu_1\lambda_1)\lambda_2\mu_2$$

and, $x_3 = (\mu_1\lambda_1 - \mu_2\lambda_2)\lambda_3\mu_3$,

then $x_1\vec{r_1} + x_2\vec{r_2} + x_3\vec{r_3} = \vec{0}$

Also, $x_1 + x_2 + x_3 = \sum \mu_1\lambda_1(\mu_2\lambda_2 - \mu_3\lambda_3) = 0$

Hence, the points P_1, P_2, P_3 are collinear points.

EXAMPLE 14 *If ABC, $A_1 B_1 C_1$ are two triangles such that three lines AA_1, BB_1 and CC_1 are concurrent, then prove that the points of intersection of the three pairs of sides BC, B_1C_1; CA, C_1A_1; AB, A_1B_1 are collinear.*

SOLUTION Suppose AA_1, BB_1 and CC_1 intersect at O. Taking O as the origin, let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

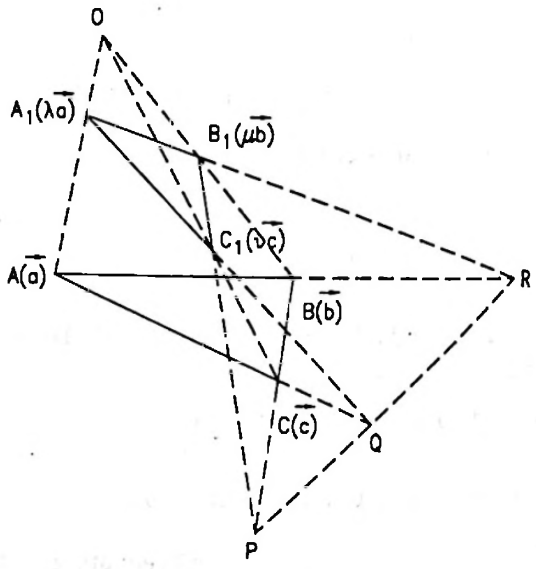


Fig. 18.16

$\therefore \vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{OC} = \vec{c}$

Since $\vec{OA_1}$, $\vec{OB_1}$ and $\vec{OC_1}$ are collinear to vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ respectively. Therefore,

$\vec{OA_1} = \lambda \vec{OA} = \lambda \vec{a}$

$\vec{OB_1} = \mu \vec{OB} = \mu \vec{b}$

and, $\vec{OC_1} = \nu \vec{OC} = \nu \vec{c}$, where λ, μ, ν are scalars.

The vector equations of BC and B_1C_1 are

$$\vec{r} = \vec{b} + t_1(\vec{b} - \vec{c}) \quad ...(i)$$

and, $$\vec{r} = \mu\vec{b} + t_2(\mu\vec{b} - \nu\vec{c}) \quad ...(ii)$$

respectively.

Suppose BC and B_1C_1 intersect at point P. At point P, we must have

$$\vec{b} + t_1(\vec{b} - \vec{c}) = \mu\vec{b} + t_2(\mu\vec{b} - \nu\vec{c})$$

$$\Rightarrow (1 + t_1 - \mu - \mu t_2)\vec{b} + (-t_1 + \nu t_2)\vec{c} = \vec{0}$$

$$\Rightarrow 1 + t_1 - \mu - \mu t_2 = 0 \text{ and } -t_1 + \nu t_2 = 0$$

[$\because \vec{b}$ and $\vec{c}$ are non-collinear]

$$\Rightarrow t_2 = \frac{1-\mu}{\mu-\nu}$$

Substituting the value of t_2 in (ii), we obtain the position vector $\vec{r_1}$ of point P as

$$\vec{r_1} = \mu\vec{b} + \frac{1-\mu}{\mu-\nu}(\mu\vec{b} - \nu\vec{c})$$

$$= \left(\frac{1-\nu}{\mu-\nu}\right)\mu\vec{b} - \left(\frac{1-\mu}{\mu-\nu}\right)\nu\vec{c}$$

Similarly, the position vectors of Q and R are given by

$$\vec{r_2} = \left(\frac{1-\nu}{\lambda-\nu}\right)\lambda\vec{a} - \left(\frac{1-\lambda}{\lambda-\nu}\right)\nu\vec{c}$$

$$= \left(\frac{1-\lambda}{\nu-\lambda}\right)\nu\vec{c} - \left(\frac{1-\nu}{\nu-\lambda}\right)\nu\vec{b}$$

$$\vec{r_3} = \left(\frac{1-\lambda}{\mu-\lambda}\right)\mu\vec{b} - \left(\frac{1-\mu}{\mu-\lambda}\right)\lambda\vec{a}$$

$$= \left(\frac{1-\mu}{\lambda-\mu}\right)\lambda\vec{a} - \left(\frac{1-\lambda}{\lambda-\mu}\right)\mu\vec{b}$$

Now, we observe that if we take

$x_1 = (1-\lambda)(\mu-\nu)$,

$x_2 = (1-\mu)(\nu-\lambda)$

and, $x_3 = (1-\nu)(\lambda-\mu)$, then

$x_1\vec{r_1} + x_2\vec{r_2} + x_3\vec{r_3} = \vec{0}$ and $x_1 + x_2 + x_3 = 0$

Hence, P, Q, R are collinear points.

EXAMPLE 15 *Find the equation of the straight line through the point $\vec{\alpha}$, equally inclined to the vectors $\vec{a}, \vec{b}, \vec{c}$. Hence or otherwise show that if a straight line is equally inclined to three coplanar straight lines, it is perpendicular to their plane.*

SOLUTION Suppose the required line makes angle θ with each of the vectors $\vec{a}, \vec{b}, \vec{c}$. Let $\hat{n}$ be the unit vector parallel to the required line. Then,

$$\hat{n}.\hat{a} = \hat{n}.\hat{b} = \hat{n}.\hat{c} = \cos\theta$$

$$\Rightarrow \hat{n}.\hat{b} - \hat{n}.\vec{a} = 0 \text{ and } \hat{n}.\hat{c} - \hat{n}.\hat{a} = 0$$

$$\Rightarrow \hat{n}.(\hat{b} - \hat{a}) = 0 \text{ and } n.(\hat{c} - \hat{a}) = 0$$

$$\Rightarrow \hat{n} \perp (\hat{b} - \hat{a}) \text{ and } \hat{n} \perp (\hat{c} - \hat{a})$$

$\Rightarrow \hat{n}$ is in the direction of $(\hat{b} - \hat{a}) \times (\hat{c} - \hat{a})$

Now,

$$(\hat{b} - \hat{a}) \times (\hat{c} - \hat{a})$$

$$= \hat{b} \times \hat{c} + \hat{c} \times \hat{a} + \vec{a} \times \vec{b}$$

$$= \frac{\vec{b} \times \vec{c}}{bc} + \frac{\vec{c} \times \vec{a}}{ca} + \frac{\hat{a} \times \hat{b}}{ab}, \text{ where } a = |\vec{a}|, b = |\vec{b}|, c = |\vec{c}|$$

$$= \frac{1}{abc}\left[a(\vec{b} \times \vec{c}) + b(\vec{c} \times \vec{a}) + c(\vec{a} \times \vec{b})\right]$$

$\therefore$ $\hat{n}$ is in the direction of $\frac{1}{abc}\left\{a(\vec{b} \times \vec{c}) + b(\vec{c} \times \vec{a}) + c(\vec{a} \times \vec{b})\right\}$

Hence, the equation of the required line is

$$\vec{r} = \vec{\alpha} + \lambda\hat{n}$$

or, $$\vec{r} = \vec{\alpha} + \frac{\lambda}{abc}\left\{a(\vec{b} \times \vec{c}) + b(\vec{c} \times \vec{a}) + c(\vec{a} \times \vec{b})\right\}$$

or, $$\vec{r} = \alpha + \mu\left\{a(\vec{b} \times \vec{c}) + b(\vec{c} \times \vec{a}) + c(\vec{a} \times \vec{b})\right\}$$

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Now, we have to show that the line is perpendicular to the plane of $\vec{a}, \vec{b}, \vec{c}$.

For this, it is sufficient to show that $\hat{n}$ is perpendicular to $\vec{a}, \vec{b}, \vec{c}$.

We have,

$$\hat{n}.\vec{a} = \frac{1}{abc}\left\{a(\vec{b} \times \vec{c}) + b(\vec{c} \times \vec{a}) + c(\vec{a} \times \vec{b})\right\}.\vec{a}$$

$$= \frac{1}{abc}\left\{a[\vec{b}\ \vec{c}\ \vec{a}] + b[\vec{c}\ \vec{a}\ \vec{a}] + c[\vec{a}\ \vec{b}\ \vec{a}]\right\}$$

$$= \frac{1}{abc}\left\{a[\vec{b}\ \vec{c}\ \vec{a}]\right\} = 0 \qquad [\because [\vec{a}\ \vec{b}\ \vec{c}] = 0]$$

$\therefore$ $\hat{n} \perp \vec{a}$

Similarly, $\hat{n} \perp \vec{b}$ and $\hat{n} \perp \vec{c}$.

Hence, the line is perpendicular to the plane of $\vec{a}, \vec{b}, \vec{c}$.

EXAMPLE 16 *The vertices of a triangle ABC are $A \equiv (2, 0, 2)$, $B \equiv (-1, 1, 1)$ and $C \equiv (1, -2, 4)$. The points D and E divide the sides AB and CA in the ratio 1 : 2 respectively. Another point F is*

taken in space such that perpendicular drawn from F on Δ ABC, meets the triangle at the point of intersection of the line segments CD and BE. If the distance of F from the plane of the Δ ABC is $\sqrt{2}$ units, using vector methods, Find

(i) the volume of the tetrahedron ABCF

and (ii) the equation of the line AF.

SOLUTION The position vectors of D and E are marked in Fig. 18.17

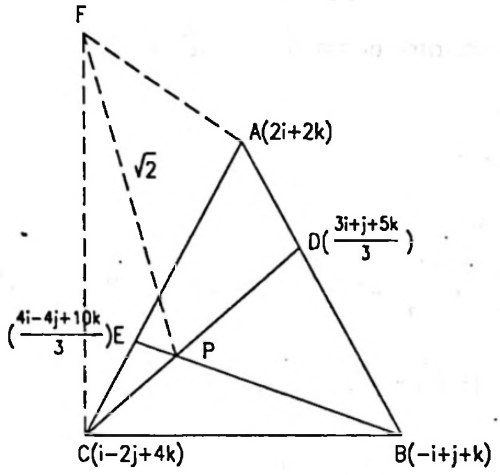


Fig. 18.17

The vector equations of CD and BE are

$$\vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + \frac{\lambda}{3}(7\hat{j} - 7\hat{k}) \quad ...(i)$$

and,

$$\vec{r} = (-\hat{i} + \hat{j} + \hat{k}) + \frac{\mu}{3}(7\hat{i} - 7\hat{j} + 7\hat{k}) \quad ...(ii)$$

respectively.

CD and BE intersect at point P. At their point of intersection, we must have

$$(\hat{i} - 2\hat{j} + 4\hat{k}) + \frac{\lambda}{3}(7\hat{j} - 7\hat{k}) = (-\hat{i} + \hat{j} + \hat{k}) + \frac{\mu}{3}(7\hat{i} - 7\hat{j} + 7\hat{k})$$

$$\Rightarrow \quad 1 = -1 + \frac{7\mu}{3}, -2 + \frac{7\lambda}{3} = 1 - \frac{7\mu}{3}$$

and, $4 - \frac{7\lambda}{3} = 1 + \frac{7\mu}{3}$ [On equating the coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$]

$$\Rightarrow \quad \mu = \frac{6}{7}, \lambda = \frac{3}{7}$$

Substituting the value of λ in (i) or that of μ in (ii), we obtain the position vector $\vec{r_1}$ of point P as

$$\vec{r_1} = \hat{i} - \hat{j} + 3\hat{k}$$

(i) Δ = Area of ΔABC

$$= \frac{1}{2} | \vec{AB} \times \vec{AC} |$$

$$= \frac{1}{2} | (-3\hat{i} + \hat{j} - \hat{k}) \times (-\hat{i} - 2\hat{j} + 2\hat{k}) |$$

$$= \frac{1}{2} | 7\hat{j} + 7\hat{k} | = \frac{7\sqrt{2}}{2} \text{ sq. units}$$

$\therefore$ Volume of the tetrahedron $ABCF$

$$= \frac{1}{3} (\text{Area of the base} \times \text{Height})$$

$$= \frac{1}{3} \times \frac{7\sqrt{2}}{2} \times \sqrt{2} = \frac{7}{3} \text{ cubic units.}$$

(ii) We have,

$$\vec{AB} \times \vec{AC} = 7\hat{j} + 7\hat{k}$$

Since $\vec{PF}$ is parallel to $\vec{AB} \times \vec{AC}$ and $PF = \sqrt{2}$ units. Therefore,

$$\vec{PF} = \sqrt{2} \frac{(7\hat{j} + 7\hat{k})}{\sqrt{49 + 49}} = \hat{j} + \hat{k}.$$

$\Rightarrow$ P.V. of $\vec{F}$ − P.V. of $\vec{P} = \hat{j} + \hat{k}$

$\Rightarrow$ P.V. of $\vec{F} = (\hat{j} + \hat{k}) + (\hat{i} - \hat{j} + 3\hat{k}) = \hat{i} + 4\hat{k}$

$\therefore$ Vector equation of AF is

$$\vec{r} = (2\hat{i} + 2\hat{k}) + \lambda(\hat{i} + 4\hat{k} - 2\hat{i} - 2\hat{k})$$

or, $\vec{r} = (2\hat{i} + 2\hat{k}) + \lambda(-\hat{i} + 2\hat{k})$

EXAMPLE 17 *Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. Show that the point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is $3\hat{i} + \hat{j} - \hat{k}$*

[IIT 1992]

SOLUTION We have,

$$\vec{r} \times \vec{a} = \vec{b} \times \vec{a} \text{ and } \vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

$$\Rightarrow \quad (\vec{r} - \vec{b}) \times \vec{a} = \vec{0} \text{ and } (\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{r} = \vec{b} + \lambda \vec{a} \text{ and } \vec{r} = \vec{a} + \mu \vec{b} \quad ...(i)$$

At the point of intersection of these two lines, we have

$$\vec{b} + \lambda \vec{a} = \vec{a} + \mu \vec{b}$$

$$\Rightarrow \quad (\lambda - 1)\vec{a} + (1 - \mu)\vec{b} = \vec{0}$$

$$\Rightarrow \quad (\lambda - 1)(\hat{i} + \hat{j}) + (1 - \mu)(2\hat{i} - \hat{k}) = \vec{0}$$

$$\Rightarrow \quad (\lambda - 1 + 2 - 2\mu)\hat{i} + (\lambda - 2)\hat{j} - (1 - \mu)\hat{k} = \vec{0}$$

$$\Rightarrow \quad (\lambda + 1 - 2\mu)\hat{i} + (\lambda - 1)\hat{j} + (\mu - 1)\hat{k} = 0$$

$$\Rightarrow \quad \lambda + 1 - 2\mu = 0, \ \lambda - 1 = 0 \text{ and } \mu - 1 = 0$$

$$\Rightarrow \quad \lambda = 1, \mu = 1.$$

Hence, the position vector of the point of intersection is

$$\vec{r} = \vec{b} + \vec{a} \quad \text{[Substituting the value of } \lambda \text{ or } \mu \text{ in (i)]}$$

$$= 2\hat{i} - \hat{k} + \hat{i} + \hat{j}$$

$$= 3\hat{i} + \hat{j} - \hat{k}$$

—❑❑❑—

18.3 BISECTORS OF THE ANGLES BETWEEN TWO LINES

THEOREM *The equations of the bisectors of the angle between the straight lines* $\vec{r} = \vec{a} + \lambda \vec{b}$ *and* $\vec{r} = \vec{a} + \mu \vec{c}$ *are given by*

$$\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right\}, \quad \text{where } t \in R$$

PROOF Let the lines $\vec{r} = \vec{a} + \lambda \vec{b}$ and $\vec{r} = \vec{a} + \mu \vec{c}$ pass through a point $A(\vec{a})$. Let $P(\vec{r})$ be any point on the internal bisector of the angle between the given lines. Draw PN parallel to AM. Therefore,

$$\angle APN = \angle PAM = \angle PAN$$

$$\Rightarrow \quad AN = NP$$

$$\therefore \quad \vec{AN} = t\hat{b} \text{ and } \vec{NP} = t\hat{c} \qquad [\because AN = NP]$$

Now,

$$\vec{AP} = \vec{AN} + \vec{NP}$$

$$\Rightarrow \quad \vec{AP} = t(\hat{b} + \hat{c})$$

$$\Rightarrow \quad \vec{r} - \vec{a} = t\left\{\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right\}$$

$$\Rightarrow \vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right\}$$

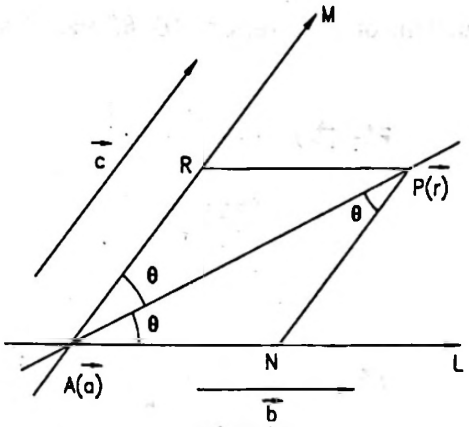


Fig. 18.18

Thus, the equation of the internal bisector of the angle between $\vec{r} = \vec{a} + \lambda \vec{b}$ and $\vec{r} = \vec{a} + \mu \vec{c}$ is

$$\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right\}$$

Similarly, the equation of the external bisector of the angle between the lines $\vec{r} = \vec{a} + \lambda \vec{b}$ and $\vec{r} = a + \mu \vec{c}$ is

$$\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} - \frac{\vec{c}}{|\vec{c}|}\right\}$$

Hence, the two bisectors are given by

$$\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} \pm \frac{\vec{c}}{|\vec{c}|}\right\}$$

18.4 BISECTOR OF THE ANGLE BETWEEN TWO VECTORS

THEOREM *The vector parallel to the bisector of the angle between two non-collinear vectors* $\vec{a}$ *and* $\vec{b}$ *is given by*

$$\lambda(\hat{a} + \hat{b}) \text{ or, } \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$$

PROOF Let $\vec{a}$ and $\vec{b}$ be two given vectors represented by OA and OB respectively as shown in Fig. 18.19

Take points P and Q on OA and OB respectively such that $OP = OQ = 1$ unit. Then, $\vec{OP} = \hat{a}$ and $\vec{OQ} = \hat{b}$. Let R be the mid-point of PQ. Since ΔOPQ is an isosceles triangle, therefore OR is the bisector of $\angle AOB$.

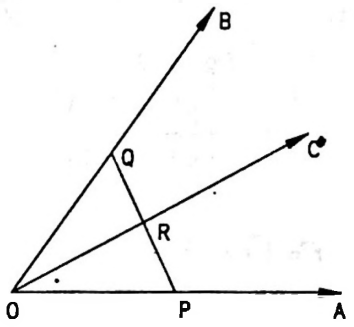


Fig. 18.19

The position vector of R is $\left(\frac{\hat{a} + \hat{b}}{2}\right)$

$$\therefore \quad \vec{OR} = \frac{\hat{a} + \hat{b}}{2}$$

$$\Rightarrow \quad \vec{OC} = \alpha \vec{OR} \text{ for some scalar } \alpha.$$

$$\Rightarrow \quad \vec{OC} = \alpha\left(\frac{\hat{a} + \hat{b}}{2}\right)$$

$$\Rightarrow \quad \vec{OC} = \lambda(\hat{a} + \hat{b}), \text{ where } \lambda = \frac{\alpha}{2} \text{ is a scalar}$$

$$\Rightarrow \quad \vec{OC} = \lambda\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right\}$$

Hence, a vector along the bisector of the angles between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\lambda(\hat{a} + \hat{b}) \text{ or } \lambda\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right\}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Using vectors, show that the internal bisector of angle A of a triangle ABC divides the side BC in the ratio AB : AC.*

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, $\vec{AB} = \vec{b}$ and $\vec{AC} = \vec{c}$. Let AD be the internal bisector of $\angle A$ and it meets BC at D. The vector equations of AD and BC are respectively.

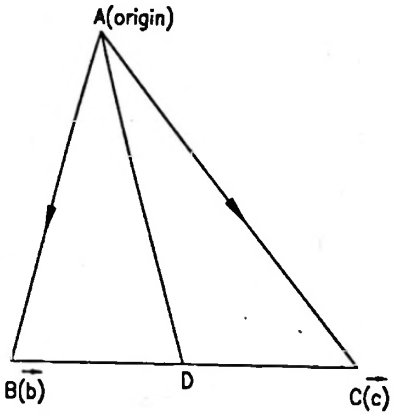


Fig. 18.20

$$\vec{r} = \lambda_1 \left\{ \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{c}}{|\vec{c}|} \right\} \qquad ...(i)$$

and,

$$\vec{r} = \vec{b} + \lambda_2 (\vec{c} - \vec{b}) \qquad ...(ii)$$

Lines AD and BC intersect at D.

For point D, we have

$$\lambda_1 \left\{ \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{c}}{|\vec{c}|} \right\} = \vec{b} + \lambda_2 (\vec{c} - \vec{b})$$

$$\Rightarrow \left\{ \frac{\lambda_1}{|\vec{b}|} - 1 + \lambda_2 \right\} \vec{b} + \left\{ \frac{\lambda_1}{|\vec{c}|} - \lambda_2 \right\} \vec{c} = \vec{0}$$

$$\Rightarrow \frac{\lambda_1}{|\vec{b}|} - 1 + \lambda_2 = 0$$

and, $\dfrac{\lambda_1}{|\vec{c}|} - \lambda_2 = 0$ [$\because \vec{b}$ and $\vec{c}$ are non-collinear vectors]

$$\Rightarrow \lambda_1 = \frac{|\vec{b}|\,|\vec{c}|}{|\vec{b}| + |\vec{c}|} \text{ and } \lambda_2 = \frac{|\vec{b}|}{|\vec{b}| + |\vec{c}|}$$

Putting the value of λ_1 in (i), we obtain the position vector of D as

$$\frac{|\vec{b}|\,|\vec{c}|}{|\vec{b}| + |\vec{c}|} \left\{ \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|} \right\}$$

$$= \frac{|\vec{c}|\,\vec{b} + |\vec{b}|\,\vec{c}}{|\vec{b}| + |\vec{c}|}$$

$$= \frac{(AC)\,\vec{b} + (AB)\,\vec{c}}{AB + AC}$$

$$\therefore \vec{AD} = \frac{(AB)\,\vec{c} + (AC)\,\vec{b}}{AB + AC}$$

$\Rightarrow$ D divides $\vec{BC}$ in the ratio $AB : AC$.

EXAMPLE 2 *Show that the internal bisectors of the angles of a triangle are concurrent.* [IIT 2001]

SOLUTION Let ABC be a triangle and let AD, BE and CF be the three internal bisectors of the angle A, B, and C respectively. Referred to any origin O, let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of the vertices A, B and C respectively. Let a, b, c be the lengths of the sides BC, CA and AB respectively.

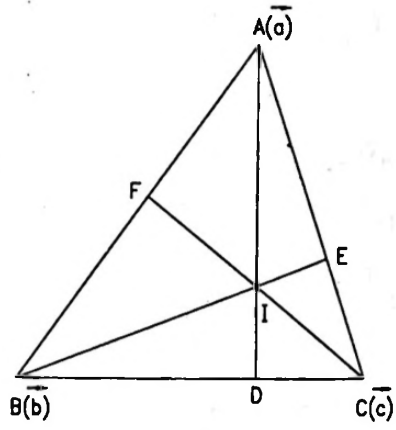


Fig. 18.21

The equations of the bisectors AD, BE and CF are respectively

$$\vec{r} = \vec{a} + \lambda_1 \left\{ \frac{\vec{b} - \vec{a}}{c} + \frac{\vec{c} - \vec{a}}{b} \right\} \qquad ...(i)$$

$$\vec{r} = \vec{b} + \lambda_2 \left\{ \frac{\vec{c} - \vec{b}}{a} + \frac{\vec{a} - \vec{b}}{c} \right\} \qquad ...(ii)$$

and,

$$\vec{r} = \vec{c} + \lambda_3 \left\{ \frac{\vec{a} - \vec{c}}{b} + \frac{\vec{b} - \vec{c}}{a} \right\} \qquad ...(iii)$$

Suppose AD and BE intersect at G. For point I, we have

$$\vec{a} + \lambda_1 \left\{ \frac{\vec{b} - \vec{a}}{c} + \frac{\vec{c} - \vec{a}}{b} \right\} = \vec{b} + \lambda_2 \left\{ \frac{\vec{c} - \vec{b}}{a} + \frac{\vec{a} - \vec{b}}{c} \right\}$$

$$\Rightarrow 1 - \frac{\lambda_1}{c} - \frac{\lambda_1}{b} = \frac{\lambda_2}{c}, \frac{\lambda_1}{c} = 1 - \frac{\lambda_2}{a} - \frac{\lambda_2}{c}$$

and, $\dfrac{\lambda_1}{b} = \dfrac{\lambda_2}{a}$ [On equating the coefficients of $\vec{a}$, $\vec{b}$, $\vec{c}$]

$$\Rightarrow \lambda_1 = \frac{bc}{a+b+c} \text{ and } \lambda_2 = \frac{ac}{a+b+c}$$

On putting the value of λ_1 in (i) or that of λ_2 in (ii), we obtain the position vector $\vec{r_1}$ of I as

$$\vec{r_1} = \vec{a} + \frac{bc}{a+b+c}\left\{\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{c}-\vec{a}}{b}\right\} = \frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$$

Clearly, $\vec{r_1} = \frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$ also satisfies the equation of CF given by (iii) for $\lambda_3 = \frac{ab}{a+b+c}$.

Thus, CF passes through I.

Hence, AD, BE and CF are concurrent. The position vector of the point of concurrency is

$$\frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}.$$

EXAMPLE 3 *Prove that the internal bisector of an angle of a triangle and the external bisector of the other two are concurrent.*

SOLUTION Referred to the vertex C as the origin, let $\vec{a}, \vec{b}$ be the position vectors of the vertices A and B respectively. Let a, b, c be the lengths of the sides BC, CA and AB respectively.

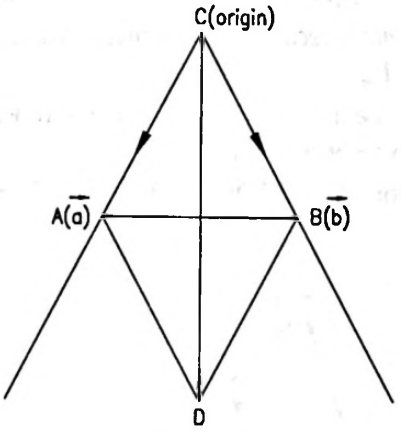


Fig. 18.22

Vector equation of the internal bisector of $\angle C$ is

$$\vec{r} = \lambda_1\left\{\frac{\vec{CA}}{CA} + \frac{\vec{CB}}{CB}\right\}$$

or, $$\vec{r} = \lambda_1\left\{\frac{\vec{a}}{b} + \frac{\vec{b}}{a}\right\}$$

or, $$\vec{r} = \lambda_1\left\{\frac{a\vec{a}+b\vec{b}}{ab}\right\}$$

or, $$\vec{r} = \frac{\lambda_1}{ab}(a\vec{a}+b\vec{b}) \quad \ldots(i)$$

The external bisector of $\angle A$ is the internal bisector of the angle between vectors $\vec{CA}$ and $\vec{AB}$. Therefore, vector equation of the external bisector of $\angle A$ is

$$\vec{r} = \vec{a} + \lambda_2\left\{\frac{\vec{AB}}{AB} + \frac{\vec{CA}}{CA}\right\}$$

or, $$\vec{r} = \vec{a} + \lambda_2\left\{\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{a}}{b}\right\} \quad \ldots(ii)$$

Similarly the external bisector of $\angle B$ is the internal bisector of the angle between the vectors $\vec{CB}$ and $\vec{BA}$. Therefore, the vector equation of the external bisector of $\angle B$ is

$$\vec{r} = \vec{b} + \lambda_3\left\{\frac{\vec{CB}}{CB} + \frac{\vec{BA}}{BA}\right\}$$

or, $$\vec{r} = \vec{b} + \lambda_3\left\{\frac{\vec{b}}{a} + \frac{\vec{a}-\vec{b}}{c}\right\} \quad \ldots(iii)$$

Suppose (ii) and (iii) intersect at D. Then, for the point D, we have

$$\vec{a} + \lambda_2\left\{\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{a}}{b}\right\} = \vec{b} + \lambda_3\left\{\frac{\vec{b}}{a} + \frac{\vec{a}-\vec{b}}{c}\right\}$$

$$\Rightarrow \quad 1 - \frac{\lambda_2}{c} + \frac{\lambda_2}{b} = \frac{\lambda_3}{c}$$

and, $$\frac{\lambda_2}{c} = 1 + \frac{\lambda_3}{a} - \frac{\lambda_3}{c}$$ [On equating the coefficients of $\vec{a}$ and $\vec{b}$]

$$\Rightarrow \quad \frac{\lambda_3}{c} = 1 + \frac{\lambda_2}{bc}(c-b) \text{ and } \frac{\lambda_2}{c} = 1 + \frac{\lambda_3}{ac}(c-a)$$

$$\Rightarrow \quad \frac{\lambda_2}{c} = 1 + \frac{1}{a}(c-a)\left\{1 + \frac{\lambda_2}{bc}(c-b)\right\}$$ [Substituting the value of in second equation]

$$\Rightarrow \quad \frac{\lambda_2}{c} = 1 + \frac{c-a}{a} + \frac{\lambda_2}{abc}(c-a)(c-b)$$

$$\Rightarrow \quad \frac{\lambda_2}{c} = \frac{c}{a} + \frac{\lambda_2}{abc}(c^2 - ac - bc + ab)$$

$$\Rightarrow \quad \frac{\lambda_2}{abc}(ab - c^2 + ac + bc - ab) = \frac{c}{a}$$

$$\Rightarrow \quad \frac{\lambda_2}{ab}(a+b-c) = \frac{c}{a}$$

$$\Rightarrow \quad \lambda_2 = \frac{bc}{a+b-c}$$

Putting this value of λ_2 in $\frac{\lambda_3}{c} = 1 + \frac{\lambda_2}{bc}(c-b)$, we get

$$\frac{\lambda_3}{c} = 1 + \frac{c-b}{a+b-c}$$

$$\Rightarrow \quad \lambda_3 = \frac{ac}{a+b-c}$$

Now, putting the value of λ_2 in (ii) or that of λ_3 in (iii), we get the position vector $\vec{r_1}$ of D as

$$\vec{r_1} = \vec{a} + \frac{bc}{a+b+c}\left\{\frac{\vec{b}-\vec{a}}{c} + \frac{\vec{a}}{b}\right\}$$

$$= \frac{\vec{a}(a+b-c) + b(\vec{b}-\vec{a}) + c\vec{a}}{a+b-c}$$

$$= \frac{a\vec{a} + b\vec{b}}{a+b-c}$$

Clearly, $\vec{r_1} = \frac{a\vec{a} + b\vec{b}}{a+b-c}$ satisfies equation of CD i.e. (i) for $\lambda_1 = \frac{ab}{b+a-c}$

Hence, the internal bisector of $\angle C$ and external bisector of $\angle A$ and $\angle B$ are concurrent.

EXAMPLE 4 *The vector $-\hat{i}+\hat{j}+\hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i}+4\hat{j}$. Determine the unit vector along $\vec{c}$.*

SOLUTION Let the unit vector along $\vec{c}$ be $x\hat{i}+y\hat{j}+3\hat{k}$. Then,

$$x^2+y^2+z^2 = 1$$

Since $-\hat{i}+\hat{j}+\hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i}+4\hat{j}$. Therefore,

Vector parallel to the bisector is $\left(\hat{c} + \frac{3\hat{i}+4\hat{j}}{5}\right)$.

But, $-\hat{i}+\hat{j}+\hat{k}$ is a vector along the bisector. Therefore,

$$\left(\hat{c} + \frac{3\hat{i}+4\hat{j}}{5}\right) = \lambda(-\hat{i}+\hat{j}+\hat{k})$$

$$\Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k} + \frac{3\hat{i}+4\hat{j}}{5}\right) = \lambda(-\hat{i}+\hat{j}+\hat{k})$$

$$\Rightarrow \left(x+\frac{3}{5}\right)\hat{i} + \left(y+\frac{4}{5}\right)\hat{j} + z\hat{k} = \lambda(-\hat{i}+\hat{j}+\hat{k})$$

$$\Rightarrow x+\frac{3}{5} = -\lambda, y+\frac{4}{5} = \lambda, z = \lambda.$$

But, $x^2+y^2+z^2 = 1$

$$\Rightarrow \left(-\lambda-\frac{3}{5}\right)^2 + \left(\lambda-\frac{4}{5}\right)^2 + \lambda^2 = 1$$

$$\Rightarrow (5\lambda+3)^2 + (5\lambda-4)^2 + 25\lambda^2 = 25$$

$$\Rightarrow 75\lambda^2 - 10\lambda = 0$$

$$\Rightarrow \lambda = 0 \text{ or, } \lambda = \frac{2}{15}$$

$$\therefore x = -\frac{3}{5} - \frac{2}{15} = -\frac{11}{15}, y = \frac{2}{15} - \frac{4}{5} = -\frac{10}{15} \text{ and } z = \frac{2}{15}.$$

Hence,

Required vector $= \hat{c} = x\hat{i}+y\hat{j}+z\hat{k}$

$$= \frac{1}{15}(-11\hat{i}-10\hat{j}+2\hat{k})$$

EXAMPLE 5 *The position vectors of the points A and B with respect to origin O are $\vec{a} = \hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b} = 3\hat{i}+\hat{j}-2\hat{k}$ respectively. Determine the vector OP which bisects the angle $\angle AOB$, where P is a point on AB.*

SOLUTION Clearly, $OA = |\vec{a}| = \sqrt{14}$ and $OB = |\vec{b}| = \sqrt{14}$.

Therefore, ΔAOB is an isosceles triangle.

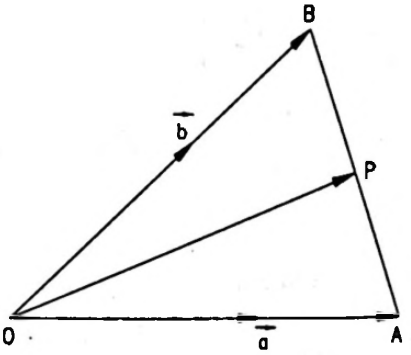


Fig. 18.23

Hence, P is the mid-point of A.

$\therefore$ Position vector of $P = \frac{\vec{a}+\vec{b}}{2} = 2\hat{i}+2\hat{j}-2\hat{k}$

Hence, $\vec{OP} = 2\hat{i}+2\hat{j}-2\hat{k}$

EXAMPLE 6 *The position vectors of two points P and Q with respect to origin O are $\vec{a} = 3\hat{i}-4\hat{j}+12\hat{k}$ and $\vec{b} = 2\hat{i}-6\hat{j}+3\hat{k}$ respectively. Determine the vector OA which bisects the angle $\angle POQ$ where A is a point on PQ.*

SOLUTION We have, $OP = |\vec{a}| = \sqrt{9+16+144} = 13$ and $OQ = |\vec{b}| = \sqrt{4+36+9} = 7$.

The bisector OA of $\angle POQ$ divides PQ in the ratio $OP : OQ$.

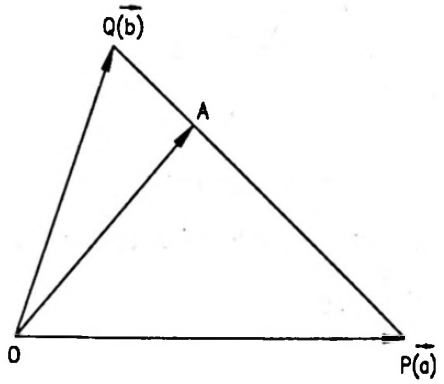


Fig. 18.24

$\therefore$ Position vector of A

$$= \frac{13\vec{b}+7\vec{a}}{13+7}$$

$$= \frac{13(2\hat{i}-6\hat{j}+3\hat{k}) + 7(3\hat{i}-4\hat{j}+12\hat{k})}{20}$$

$$= \frac{1}{20}(47\hat{i}-106\hat{j}+123\hat{k})$$

Hence, $\vec{OA} = \frac{1}{20}(47\hat{i}-106\hat{j}+123\hat{k})$

EXAMPLE 7 *In a rhombus $ABCD$ the interior bisectors of the consecutive angles B and C intersect at P. Use vector methods to find $\angle BPC$.*

SOLUTION Let $\vec{BA} = \vec{CD} = \vec{a}$ and $\vec{BC} = \vec{b}$. Then, $\vec{CB} = -\vec{c}$.

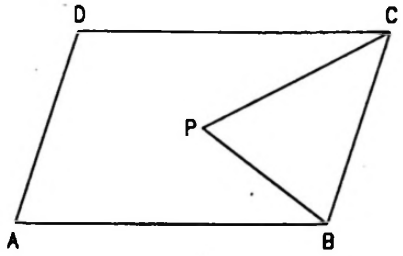


Fig. 18.25

A vector parallel to the internal bisector BP of $\angle B$ is given by

$$\vec{r_1} = \lambda(\hat{a} + \hat{c})$$

Similarly, a vector parallel to the internal bisector CP of $\angle C$ is given by

$$\vec{r_2} = \mu(\hat{a} - \hat{c})$$

Now,

$$\vec{r_1} \cdot \vec{r_2} = \lambda(\hat{a} + \hat{c}) \cdot \mu(\hat{a} - \hat{c})$$

$$= \lambda\mu(|\hat{a}|^2 - |\hat{c}|^2) = 0$$

$\Rightarrow \vec{r_1} \perp \vec{r_2}$

$\Rightarrow \vec{BP} \perp \vec{CP} \Rightarrow \angle BPC = 90°.$

EXAMPLE 8 *Using vector methods, find the ratio in which the bisector of an angle in a triangle divides the opposite side.*

SOLUTION Let ABC be a triangle in which the bisector of $\angle A$ meets side BC at D. Suppose D divides BC in the ratio $\lambda : 1$. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

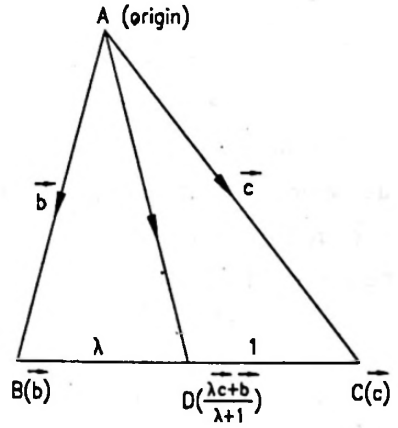


Fig. 18.26

Position vector of $D = \dfrac{\lambda\vec{c} + \vec{b}}{\lambda + 1}$.

Since AD is the bisector of $\angle A$. So, the vector equation of AD is

$$\vec{r} = \vec{0} + \mu(\hat{b} + \hat{c})$$

Let the position vector of D be $\mu(\hat{b} + \hat{c})$

$$\therefore \quad \frac{\lambda\vec{c} + \vec{b}}{\lambda + 1} = \mu(\hat{b} + \hat{c})$$

$$\Rightarrow \frac{\lambda}{\lambda+1}\vec{c} + \frac{1}{\lambda+1}\vec{b} = \mu\left\{\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}\right\}$$

$$\Rightarrow \left\{\frac{\lambda}{\lambda+1} - \frac{\mu}{|\vec{c}|}\right\}\vec{c} + \left(\frac{1}{\lambda+1} - \frac{\mu}{|\vec{b}|}\right)\vec{b} = \vec{0}$$

$$\Rightarrow \frac{\lambda}{\lambda+1} - \frac{\mu}{|\vec{c}|} = 0$$

and, $\dfrac{1}{\lambda+1} - \dfrac{\mu}{|\vec{b}|} = 0$ [$\because \vec{b}$ and $\vec{c}$ are non-collinear vectors]

$$\Rightarrow \frac{\lambda}{\lambda+1} = \frac{\mu}{|\vec{c}|} \text{ and } \frac{1}{\lambda+1} = \frac{\mu}{|\vec{b}|}$$

$$\Rightarrow \lambda = \frac{\mu/|\vec{c}|}{\mu/|\vec{b}|} = \frac{|\vec{b}|}{|\vec{c}|} = \frac{AB}{AC}$$

Hence, AD divides BC in the ratio $AB : AC$.

EXAMPLE 9 *In a parallelogram $OABC$, vectors $\vec{a}, \vec{b}, \vec{c}$ are respectively the position vectors of the vertices A, B, C with reference to O as the origin. A point E is taken on the side BC which divides it in the ratio $2 : 1$. Also, the line segment AE intersects the line bisecting the angle O internally in point P. If CP, when extended meets AB in point F, find :*

(i) *the position vector of point P.*

(ii) *the ratio in which F divides AB.*

SOLUTION Since E divides BC in the ratio $2 : 1$. Therefore,

Position vector of $E = \dfrac{\vec{b} + 2\vec{c}}{3}$

But,

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{a} + \vec{c} = \vec{b}$$

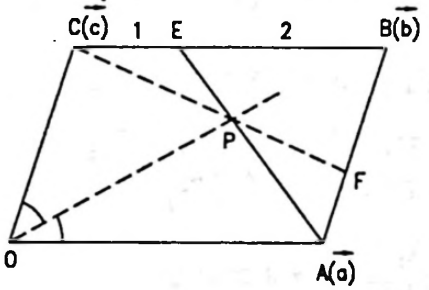


Fig. 18.27

$\therefore$ Position vector of $E = \dfrac{\vec{a} + 3\vec{c}}{3}$

The vector equation of the bisector of $\angle AOC$ is

$$\vec{r} = \lambda\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right\} \quad \ldots(i)$$

The vector equation of AE is

$$\vec{r} = \vec{a} + \mu\left\{\frac{\vec{a}+3\vec{c}}{3} - \vec{a}\right\}$$

or, $$\vec{r} = \vec{a} + \mu\left\{\frac{3\vec{c}-2\vec{a}}{3}\right\} \quad ...(ii)$$

Lines (i) and (ii) intersect at P. Therefore, for point P, we must have

$$\lambda\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right\} = \vec{a} + \mu\left\{\frac{3\vec{c}-2\vec{a}}{3}\right\}$$

$$\Rightarrow \left\{\frac{\lambda}{|\vec{a}|} - 1 + \frac{2\mu}{3}\right\}\vec{a} + \left\{\frac{\lambda}{|\vec{c}|} - \mu\right\}\vec{c} = \vec{0}$$

$$\Rightarrow \frac{\lambda}{|\vec{a}|} - 1 + \frac{2\mu}{3} = 0 \text{ and } \frac{\lambda}{|\vec{c}|} - \mu = 0$$

[$\because \vec{a}$ and $\vec{c}$ are non-collinear]

$$\Rightarrow \lambda = \frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|} \text{ and } \mu = \frac{3|\vec{a}|}{3|\vec{c}| + 2|\vec{a}|}$$

Putting the value of λ in (i) or that of μ in (ii), we obtain the position vector of P as

$$\vec{r_1} = \frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right\}$$

Now,

Vector equation of line CP is

$$\vec{r} = \vec{c} + \lambda_1\left[\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right\} - \vec{c}\right] \quad ...(iii)$$

Vector equation of line AB is

$$\vec{r} = \vec{a} + \lambda_2\vec{c} \quad ...(iv)$$

These two lines intersect at F. For point F, we must have

$$\vec{c} + \lambda_1\left[\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right\} - \vec{c}\right] = \vec{a} + \lambda_2\vec{c}$$

$$\Rightarrow \left[\frac{3|\vec{a}||\vec{c}|\lambda_1}{3|\vec{c}| + 2|\vec{a}|}\cdot\frac{1}{|\vec{a}|} - \right]\vec{a}$$

$$+\left[1 + \frac{3\lambda_1|\vec{a}|}{3|\vec{c}| + 2|\vec{a}|} - \lambda_1 - \lambda_2\right]\vec{c} = \vec{0}$$

$$\Rightarrow \frac{3|\vec{c}|\lambda_1}{3|\vec{c}| + 2|\vec{a}|} - 1 = 0$$

and, $1 + \frac{3\lambda_1|\vec{a}|}{3|\vec{c}| + 2|\vec{a}|} - \lambda_1 - \lambda_2 = 0$ $\left[\because \vec{a} \text{ and } \vec{c} \text{ are non-collinear}\right]$

$$\Rightarrow \lambda_1 = \frac{3|\vec{c}| + 2|\vec{a}|}{3|\vec{c}|} \text{ and } \lambda_2 = \frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}$$

Substituting the value of λ_1 in (iii) or that of λ_2 in (iv), we obtain the position vector of F as

$$\vec{r_4} = \vec{a} + \frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$$

$$\therefore \vec{AF} = \vec{r_4} - \vec{a} = \frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$$

and,

$$\vec{BF} = \vec{r_4} - \vec{b} = \vec{a} + \frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\vec{c} - (\vec{a} + \vec{c}) = \left\{\frac{|\vec{a}| - 3|\vec{c}|}{3|\vec{c}|}\right\}\vec{c}$$

Now,

$$\frac{AF}{BF} = \frac{|\vec{AF}|}{|\vec{BF}|} = \frac{\left\{\frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\right\}|\vec{c}|}{\left\{\frac{|\vec{a}| - 3|\vec{c}|}{3|\vec{c}|}\right\}|\vec{c}|}$$

$$= \frac{\vec{a}}{||\vec{a}| - 3|\vec{c}||}$$

EXAMPLE 10 *If one diagonal of a quadrilateral bisects the other, then using vector methods, prove that it bisects the quadrilateral as well.*

SOLUTION Let $OABC$ be a quadrilateral such that diagonal OB bisects the diagonal AC. We have to prove that the diagonal OB also bisects the quadrilateral $OABC$ i.e.

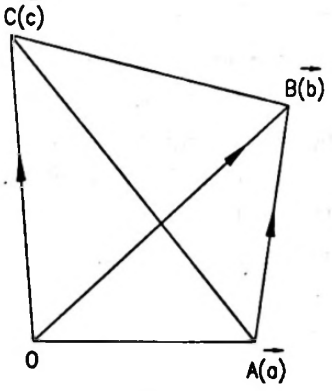


Fig. 18.28

Area of ΔOAB = Area of ΔOBC.

Taking O as the origin, let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

The vector equation of OB is

$$\vec{r} = \lambda\vec{b}$$

Since the mid-point of AC lies on OB. Therefore,

$$\frac{\vec{a}+\vec{c}}{2} = \lambda\vec{b}$$

$$\Rightarrow \vec{a} + \vec{c} = 2\lambda\vec{b}$$

$$\Rightarrow (\vec{a} + \vec{c}) \times \vec{b} = 2\lambda(\vec{b} \times \vec{b}) \quad \text{[Taking cross product with } \vec{b}\text{]}$$

$$\Rightarrow \vec{a} \times \vec{b} + \vec{c} \times \vec{b} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$$

$$\Rightarrow \vec{OA} \times \vec{OB} = \vec{OB} \times \vec{OC}$$

$\Rightarrow \frac{1}{2} | \vec{OA} \times \vec{OB} | = \frac{1}{2} | \vec{OB} \times \vec{OC} |$

$\Rightarrow$ Area of ΔOAB = Area of ΔOBC.

EXAMPLE 11 *The points D, E, F divide the sides BC, CA, AB or triangle ABC in the ratio 1 : 2. The pair of lines AD, BE ; BE, CF ; CF, AD meet at P, Q, R respectively. Prove that :*

$$\text{Area of } \Delta PQR = \frac{1}{7} \text{ Area of } \Delta ABC.$$

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The vector equations of lines AD, BE and CF are respectively.

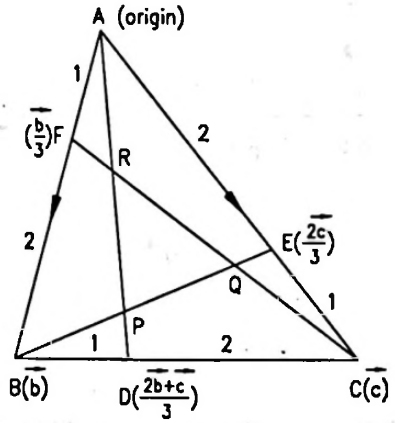


Fig. 18.29

$$\vec{r} = \lambda_1 \left\{\frac{2\vec{b}+\vec{c}}{3}\right\} \quad \ldots(i)$$

$$\vec{r} = \vec{b} + \lambda_2 \left\{\frac{2\vec{c}}{3} - \vec{b}\right\} \quad \ldots(ii)$$

$$\vec{r} = \vec{c} + \lambda_3 \left\{\frac{\vec{b}}{3} - \vec{c}\right\} \quad \ldots(iii)$$

Lines AD and BE intersect at P. At the point of intersection, we have

$$\lambda_1 \left\{\frac{2\vec{b}+\vec{c}}{3}\right\} = \vec{b} + \lambda_2 \left\{\frac{2\vec{c}}{3} - \vec{b}\right\}$$

$$\Rightarrow \left\{\frac{2\lambda_1}{3} - 1 + \lambda_2\right\}\vec{b} + \left\{\frac{\lambda_1}{3} - \frac{2\lambda_2}{3}\right\}\vec{c} = \vec{0}$$

$$\Rightarrow \frac{2\lambda_1}{3} - 1 + \lambda_2 = 0 \text{ and } \frac{\lambda_1}{3} - \frac{2\lambda_2}{3} = 0 \quad \left[\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear}\right]$$

$$\Rightarrow \lambda_1 = \frac{6}{7}, \lambda_2 = \frac{3}{7}$$

Putting $\lambda_1 = \frac{6}{7}$ in (i) or $\lambda_2 = \frac{3}{7}$ in (ii), we obtain the position vector of P as

$$\vec{r_1} = \frac{2}{7}(2\vec{b} + \vec{c})$$

Similarly by solving (i), (ii) and (iii), successively, we get

$\vec{r_2}$ = Position vector of $Q = \frac{\vec{b} + 4\vec{c}}{7}$

$\vec{r_3}$ = Position vector of $R = \frac{2\vec{b} + \vec{c}}{7}$

Now,

$$\text{Area of } \Delta PQR = \frac{1}{2} | \vec{r_1} \times \vec{r_2} + \vec{r_2} \times \vec{r_3} + \vec{r_3} \times \vec{r_1} |$$

$$= \frac{1}{2} \left| \frac{2}{49}(2\vec{b} + \vec{c}) \times (\vec{b} + 4\vec{c}) + \frac{1}{49}(\vec{b} + 4\vec{c}) \times (2\vec{b} + \vec{c}) + \frac{2}{49}(2\vec{b} + \vec{c}) \times (2\vec{b} + \vec{c}) \right|$$

$$= \frac{1}{2} \left| \frac{2}{49}\{8(\vec{b} \times \vec{c}) + (\vec{c} \times \vec{b})\} + \frac{1}{49}\{\vec{b} \times \vec{c} + 8\vec{c} \times \vec{b}\} + 0 \right|$$

$$= \frac{1}{2} \left| \frac{14}{49}(\vec{b} \times \vec{c}) - \frac{7}{49}(\vec{b} \times \vec{c}) \right|$$

$$= \frac{1}{2} \times \frac{1}{7} | \vec{b} \times \vec{c} | = \frac{1}{7} \left\{\frac{1}{2} | \vec{b} \times \vec{c} |\right\}$$

$$= \frac{1}{7} \times \frac{1}{2} | \vec{AB} \times \vec{AC} |$$

$$= \frac{1}{7} (\text{Area of } \Delta ABC)$$

EXAMPLE 12 *ABC is a triangle, E and F are mid-points of AC and AB respectively. CP is drawn parallel to AB to meet BE produced in P. Show that : Area of* ΔFEP = *Area of* $\Delta FCE = \frac{1}{4}$ *(Area of* ΔABC*).*

SOLUTION Let A be the origin with refereene to A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. The position vectors of E and F are $\vec{c}/2$ and $\vec{b}/2$ respectively.

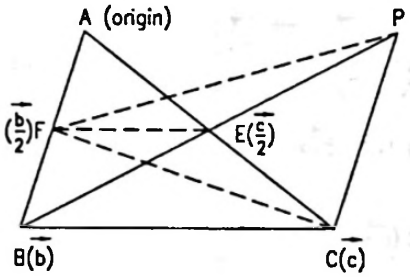


Fig. 18.30

The vector equations of lines CP and BE are respectively

$$\vec{r} = \vec{c} + \lambda\vec{b} \quad \ldots(i)$$

and,

$$\vec{r} = \vec{b} + \mu\left(\frac{\vec{c}}{2} - \vec{b}\right) \quad \ldots(ii)$$

Lines CP and BE intersect at P. At the point of intersection of CP and BE, we have

$$\vec{c}+\lambda\vec{b}=\vec{b}+\mu\left\{\frac{\vec{c}}{2}-\vec{b}\right\}$$

$$\Rightarrow\ (\lambda-1+\mu)\vec{b}+\left\{\frac{\mu}{2}-1\right\}\vec{c}=\vec{0}$$

$$\Rightarrow\ \lambda-1+\mu=0 \text{ and } \frac{\mu}{2}-1=0$$

$$\Rightarrow\ \mu=2 \text{ and } \lambda=-1.$$

Putting the values of λ in (i) or that of μ in (ii), we obtain the position vector of P as $(\vec{c}-\vec{b})$.

Now,

$$\Delta_1 = \text{Area of } \Delta ABC = \frac{1}{2}\,|\,\vec{AB}\times\vec{AC}\,| = \frac{1}{2}\,|\,\vec{b}\times\vec{c}\,|$$

$$\Delta_2 = \text{Area of } \Delta FEP$$

$$= \frac{1}{2}\,|\,\vec{FE}\times\vec{FP}\,|$$

$$= \frac{1}{2}\left|\left(\frac{\vec{c}}{2}-\frac{\vec{b}}{2}\right)\times\left(\vec{c}-\vec{b}-\frac{\vec{b}}{2}\right)\right|$$

$$= \frac{1}{2}\left|\left(\frac{\vec{c}-\vec{b}}{2}\right)\times\left(\frac{2\vec{c}-3\vec{b}}{2}\right)\right|$$

$$= \frac{1}{8}\,|\,(\vec{c}-\vec{b})\times(2\vec{c}-3\vec{b})\,|$$

$$= \frac{1}{8}\,|\,-3(\vec{c}\times\vec{b})-2(\vec{b}\times\vec{c})\,|$$

$$= \frac{1}{8}\,|\,\vec{b}\times\vec{c}\,| = \frac{1}{4}\Delta_1$$

and,

$$\Delta_3 = \text{Area of } \Delta FCE$$

$$= \frac{1}{2}\,|\,\vec{FC}\times\vec{FE}\,|$$

$$= \frac{1}{2}\left|\left(\vec{c}-\frac{\vec{b}}{2}\right)\times\left(\frac{\vec{c}}{2}-\frac{\vec{b}}{2}\right)\right|$$

$$= \frac{1}{8}\,|\,(2\vec{c}-\vec{b})\times(\vec{c}-\vec{b})\,|$$

$$= \frac{1}{8}\,|\,-2(\vec{c}\times\vec{b})-\vec{b}\times\vec{c}\,|$$

$$= \frac{1}{8}\,|\,\vec{b}\times\vec{c}\,| = \frac{1}{4}\Delta_1$$

Hence,

$$\text{Area of } \Delta FEP = \text{Area of } \Delta FCE = \frac{1}{4}(\text{Area of } \Delta ABC)$$

EXAMPLE 13 *In triangle ABC, a point P is taken on the side AB such that AP : BP = 1 : 2 and a point Q is taken on the side BC such that CQ : BQ = 2 : 1. If R be the point of intersection of lines AQ and CP, using vector methods, find the area of triangle ABC, if it is known that area of triangle BRC is one unit.*

SOLUTION Taking A as the origin, let the position vectors o
B and C be $\vec{b}$ and $\vec{c}$ respectively.

The position vectors of P and Q are $\frac{\vec{b}}{3}$ and $\frac{2\vec{b}+\vec{c}}{3}$ respectively.

The vector equations of lines AQ and CP are respectively

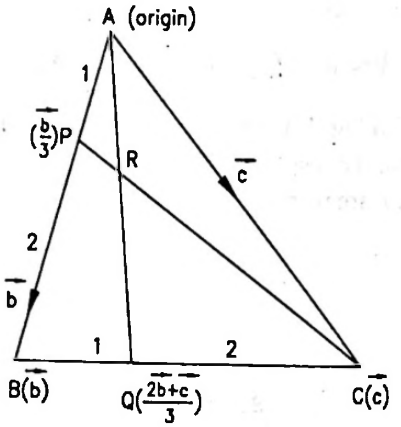


Fig. 18.31

$$\vec{r}=\lambda\left(\frac{2\vec{b}+\vec{c}}{3}\right) \qquad ...(i)$$

$$\vec{r}=\vec{c}+\mu\left(\frac{\vec{b}}{3}-\vec{c}\right) \qquad ...(ii)$$

Lines AQ and CP intersect at R. At point R, we have

$$\lambda\left(\frac{2\vec{b}+\vec{c}}{3}\right)=\vec{c}+\mu\left(\frac{\vec{b}}{3}-\vec{c}\right)$$

$$\Rightarrow\ \left(\frac{2\lambda}{3}-\frac{\mu}{3}\right)\vec{b}+\left(\frac{\lambda}{3}-1+\mu\right)\vec{c}=\vec{0}$$

$$\Rightarrow\ \frac{2\lambda}{3}-\frac{\mu}{3}=0 \text{ and } \frac{\lambda}{3}-1+\mu=0$$

$$\Rightarrow\ \mu=2\lambda \text{ and } \lambda-3+3\mu=0$$

$$\Rightarrow\ \lambda=\frac{3}{7} \text{ and } \mu=\frac{6}{7}$$

Putting the value of λ in (i) or that of μ in (ii), we obtain the position vector of R as $\frac{2\vec{b}+\vec{c}}{7}$

Now,

$$\text{Area of } \Delta BRC = 1$$

$$\Rightarrow\ \frac{1}{2}\,|\,\vec{BR}\times\vec{BC}\,| = 1$$

$$\Rightarrow\ \frac{1}{2}\left|\left(\frac{2\vec{b}+\vec{c}}{7}-\vec{b}\right)\times(\vec{c}-\vec{b})\right| = 1$$

$$\Rightarrow\ \frac{1}{14}\,|\,(\vec{c}-5\vec{b})\times(\vec{c}-\vec{b})\,| = 1$$

$$\Rightarrow\ \frac{1}{14}\,|\,-\vec{c}\times\vec{b}-5(\vec{b}\times\vec{c})\,| = 1$$

$\Rightarrow \frac{4}{14} |\vec{b}\times\vec{c}| = 1$

$\Rightarrow \frac{2}{7} |\vec{b}\times\vec{c}| = 1$

$\Rightarrow \frac{1}{2} |\vec{b}\times\vec{c}| = \frac{7}{4}$

$\Rightarrow \frac{1}{2} |\vec{AB}\times\vec{AC}| = \frac{7}{4}$

$\Rightarrow$ Area of $\Delta ABC = \frac{7}{4}$ square units.

EXAMPLE 14 *The vertices of a ΔABC are $A(1, 0, 2)$, $B(-2, 1, 3)$ and $C(2, -1, 1)$. If D is the foot of the perpendicular drawn from A on BC, using vector methods find :*

(i) *equation of median of ΔABD passing through the vertex A*

(ii) *the equation of the line bisecting angle A in ΔABD.*

SOLUTION Let $\vec{a} = \hat{i} + 2\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \hat{k}$.

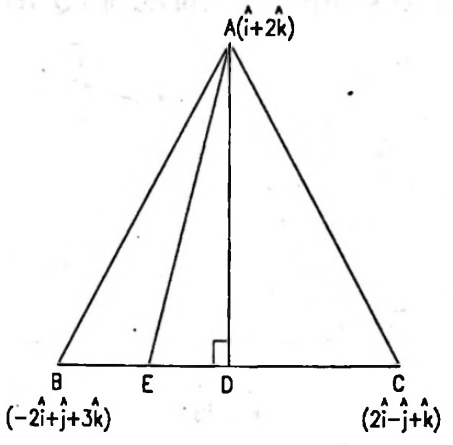


Fig. 18.32

Clearly,

$\vec{BD}$ = Projection vector of $\vec{BA}$ on $\vec{BC}$

$= (\vec{BA} \cdot \hat{BC})\,\hat{BC}$

$= \left\{(\vec{a}-\vec{b}) \cdot \frac{(\vec{c}-\vec{b})}{|\vec{c}-\vec{b}|}\right\} \frac{(\vec{c}-\vec{b})}{|\vec{c}-\vec{b}|}$

$\Rightarrow \vec{BD} = \frac{4}{3}(2\hat{i} - \hat{j} - \hat{k})$

$\Rightarrow$ P.V. of D − P.V. of $B = \frac{4}{3}(2\hat{i} - \hat{j} - \hat{k})$

$\Rightarrow$ P.V. of $D = -2\hat{i} + \hat{j} + 3\hat{k} + \frac{4}{3}(2\hat{i} - \hat{j} - \hat{k}) = \frac{1}{3}(2\hat{i} - \hat{j} + 5\hat{k})$

Since E is the mid-point of BD. Therefore,

$$\text{P.V. of } E = \frac{\frac{1}{3}(2\hat{i} - \hat{j} + 5\hat{k}) + (-2\hat{i} + \hat{j} + 3\hat{k})}{2}$$

$= \frac{1}{6}(-4\hat{i} + 2\hat{j} + 14\hat{k})$

$= \frac{1}{3}(-2\hat{i} + \hat{j} + 7\hat{k})$

Equation of line AE is

$$\vec{r} = (\hat{i} + 2\hat{k}) + \lambda\left\{\frac{1}{3}(-2\hat{i} + \hat{j} + 7\hat{k}) - (\hat{i} + 2\hat{k})\right\}$$

or, $\vec{r} = (\hat{i} + 2\hat{k}) + \frac{\lambda}{3}(-5\hat{i} + \hat{j} + \hat{k})$

We have,

$\vec{AB} = -3\hat{i} + \hat{j} + \hat{k}$ and $\vec{AD} = \frac{1}{3}(-\hat{i} - \hat{j} - \hat{k})$

$\Rightarrow \hat{AB} = \frac{-3\hat{i} + \hat{j} + \hat{k}}{\sqrt{11}}$ and $\hat{AD} = \frac{-\hat{i} - \hat{j} - \hat{k}}{3\sqrt{3}}$

$\therefore$ Vector equation of the bisector of $\angle A$ is given by

$$\vec{r} = (\hat{i} + 2\hat{k}) + \lambda\left(\frac{(-3\hat{i} + \hat{j} + \hat{k})}{\sqrt{11}} + \frac{-\hat{i} - \hat{j} + \hat{k}}{3\sqrt{3}}\right)$$

—❑❑❑—

18.5 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The length of the perpendicular from a point $P(\vec{a})$ on the line $\vec{r} = \vec{b} + \mu\vec{c}$ is given by

$$\sqrt{|\vec{a}-\vec{b}|^2 - \{(\vec{a}-\vec{b}) \cdot \hat{c}\}^2}$$

$$= \sqrt{|\vec{a}-\vec{b}|^2 - \left\{\frac{(\vec{a}-\vec{b}) \cdot \vec{c}}{|\vec{c}|}\right\}^2}$$

PROOF The line $\vec{r} = \vec{b} + \mu\vec{c}$ passes through point $A(\vec{b})$ and is parallel to $\vec{b}$. Let PM be perpendicular from P on the given line.

Clearly,

AM = Projection of $\vec{AP}$ on $\vec{c}$

$= (\vec{AP} \cdot \hat{c})$

$= (\vec{a}-\vec{b}) \cdot \hat{i}$

In right angled triangle AMP, we have

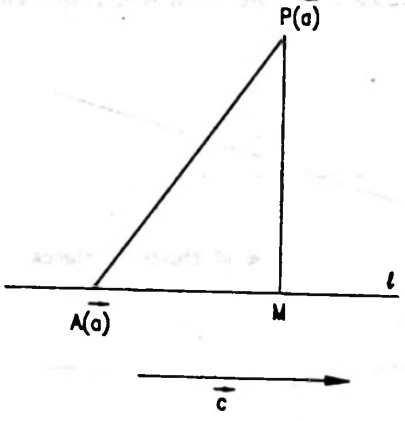


Fig. 18.33

$$AP^2 = AM^2 + PM^2$$

$$\Rightarrow PM = \sqrt{AP^2 - AM^2}$$

$$\Rightarrow PM = \sqrt{|\vec{a}-\vec{b}|^2 - |(\vec{a}.\vec{b}).\hat{c}|^2}$$

$$= \sqrt{|\vec{a}-\vec{b}|^2 - \left|\frac{(\vec{a}.\vec{b}).\vec{c}}{|\vec{c}|}\right|^2}$$

NOTE PM is also equal to $\left|\frac{(\vec{b}-\vec{a})\times\vec{c}}{|\vec{c}|}\right|$

18.6 CONDITION OF COPLANARITY OF TWO LINES

Two lines $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}=\vec{c}+\mu\vec{d}$ are coplanar or intersecting, if

$$(\vec{a}-\vec{c}).(\vec{b}\times\vec{d}) = 0 \Rightarrow [\vec{a}\,\vec{b}\,\vec{d}] = [\vec{c}\,\vec{b}\,\vec{d}]$$

PROOF The line $\vec{r}=\vec{a}+\lambda\vec{b}$, passes through the point $A(\vec{a})$ and is parallel to $\vec{b}$.

The line $\vec{r}=\vec{c}+\mu\vec{d}$ passes through the point $C(\vec{c})$ and is parallel to $\vec{d}$.

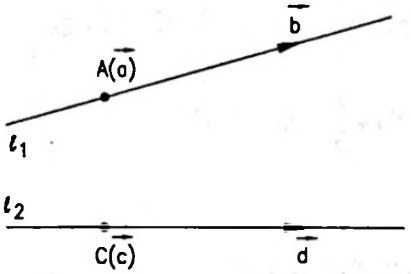


Fig. 18.34

If these two lines lie in the same plane, then $\vec{AC}$ lies in the same plane and $\vec{b}\times\vec{d}$ is normal to the plane. Therefore,

$$\vec{AC}.(\vec{b}\times\vec{d}) = 0$$

$$\Rightarrow (\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = 0$$

$$\Rightarrow [\vec{c}\,\vec{b}\,\vec{d}] = [\vec{a}\,\vec{b}\,\vec{d}]$$

18.7 SHORTEST DISTANCE BETWEEN TWO SKEW LINES

SKEW LINES *Two straight lines in space which are neither parallel nor intersecting are called skew-lines.*

Thus, skew lines are those lines which do not lie in the same plane.

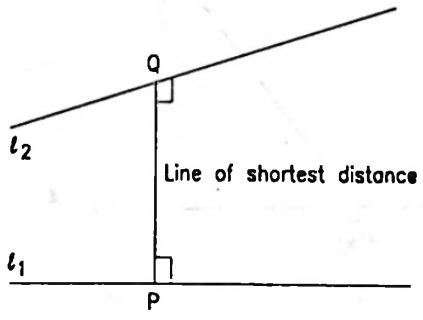


Fig. 18.35

LINE OF SHORTEST DISTANCE *If l_1 and l_2 are two skew lines, then there is one and only one line perpendicular to each of the lines l_1 and l_2 which is known as the line of shortest distance.*

SHORTEST DISTANCE *The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q where the line of shortest distance intersects the two given lines.*

The shortest distance between two skew lines

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

and,

$$\vec{r}=\vec{c}+\mu\vec{d}$$

is given by

$$\text{S.D.} = \left|\frac{(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right|$$

PROOF Let PQ be the shortest distance between the given lines. Then, PQ is perpendicular to both the lines. So, it is parallel to $\vec{b}\times\vec{d}$.

Clearly, PQ is the projection of $\vec{AC}$ on $\vec{PQ}$. Therefore,

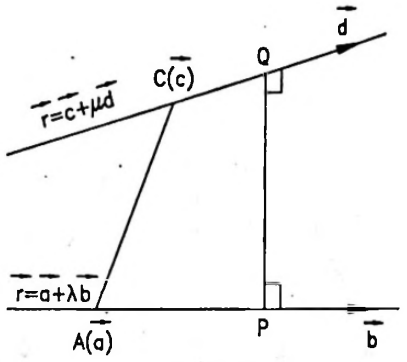


Fig. 18.36

$$PQ = |\vec{AC}.\hat{PQ}|$$

$$= \left|(\vec{c}-\vec{a}).\frac{(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right|$$

$$= \left|\frac{(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right|$$

18.8 EQUATION OF A PLANE

THEOREM *The vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to the vector $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=\vec{0}$ or, $\vec{r}.\vec{n}=\vec{a}.\vec{n}$.*

PROOF Let $P(\vec{r})$ be an arbitrary point on the plane. The plane passes through a point $A(\vec{a})$.

$$\therefore \vec{AP} = \vec{r}-\vec{a}$$

Since the plane is perpendicular to $\vec{n}$. Therefore,

$$\vec{AP} \perp \vec{n}$$

$$\Rightarrow \vec{AP}.\vec{n} = 0$$

$$\Rightarrow (\vec{r}-\vec{a}).\vec{n} = 0$$

$$\Rightarrow \vec{r}.\vec{n} = \vec{a}.\vec{n}$$

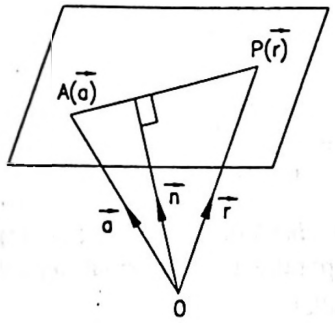


Fig. 18.37

This is the required equation of the plane.

18.9 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN VECTORS

PARAMETRIC FORM *The vector equation of a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is*

$$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c},$$ where λ and μ are scalars.

PROOF Let O be the origin and π be a plane passing through a point A having position vector $\vec{a}$. Let the plane π be parallel to vectors $\vec{b}$ and $\vec{c}$ respectively. Through the point A draw two vectors $\vec{AB}$ and $\vec{AC}$ such that $\vec{AB} = \vec{b}$ and $\vec{AC} = \vec{c}$. Let P be an arbitrary point on the plane. Let the position vector of P be $\vec{r}$. Complete the parallelogram $ALPM$. Since $\vec{AL}$ and $\vec{AM}$ are parallel to $\vec{b}$ and $\vec{c}$ respectively. Therefore,

$\vec{AL} = \lambda\vec{b}$ and $\vec{AM} = \mu\vec{c}$ for some scalars λ and μ.

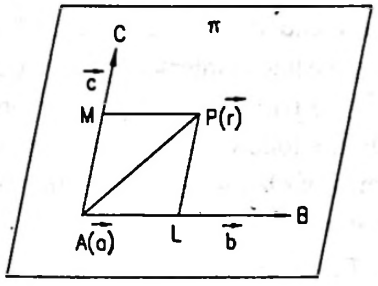


Fig. 18.38

In ΔAPL, we have

$$\vec{AP} = \vec{AL} + \vec{LP}$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b} + \mu\vec{c}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

Since $\vec{r}$ denotes the position vector of any point P on the plane. Therefore, the equation of the plane is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$, where λ and μ are scalars.

NOTE *In the above equation λ and μ are variable scalars, because for different points on the plane the values of λ and μ are different. That is why, it is called the parametric form of the plane passing through a given point and parallel to given vectors.*

NON-PARAMETRIC FORM *The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is*

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0 \text{ or, } \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

or, $[\vec{r}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]$.

PROOF From Fig. 18.38., the plane is parallel to vectors $\vec{AB} = \vec{b}$ and $\vec{AC} = \vec{c}$. Therefore it is perpendicular to the vector $\vec{n} = \vec{AB} \times \vec{AC} = \vec{b} \times \vec{c}$. Since the plane passes through a point A having position vector $\vec{a}$. Therefore, the equation of the plane in scalar product form is

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0 \qquad [\text{Using} : (\vec{r} - \vec{a}) \cdot \vec{n} = 0]$$

$$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c}) - \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow [\vec{r}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]$$

18.10 DISTANCE OF A POINT FROM A PLANE

VECTOR FORM *The length of the perpendicular from a point having position vector $\vec{a}$ to the plane $\vec{r} \cdot \vec{n} = d$ is given by*

$$p = \frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}.$$

PROOF Let π be the given plane and $P(\vec{a})$ be the given point. Let PM be the length of the perpendicular from P on the plane π. Since line PM passes through $P(\vec{a})$ and is parallel to the vector $\vec{n}$ which is normal to the plane π. So, vector equation of line PM is

$$\vec{r} = \vec{a} + \lambda\vec{n}, \qquad \ldots(i)$$

where λ is a scalar.

Point M is the intersection of line (i) and the given plane. Therefore, for point M

$$(\vec{a} + \lambda\vec{n}) \cdot \vec{n} = d \qquad \left[\begin{array}{l}\text{On substituting } \vec{r} = \vec{a} + \lambda\vec{n} \\ \text{in the equation of the plane}\end{array}\right]$$

$$\Rightarrow \quad \vec{a} \cdot \vec{n} + \lambda\vec{n} \cdot \vec{n} = d$$

$$\Rightarrow \lambda = \frac{d - (\vec{a} \cdot \vec{n})}{\vec{n} \cdot \vec{n}}$$

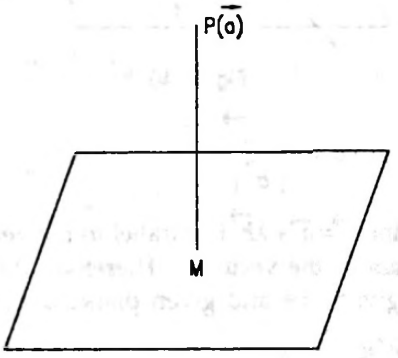


Fig. 18.39

$$\Rightarrow \quad \lambda = \frac{d - (\vec{a} \cdot \vec{n})}{|\vec{n}|^2}$$

Putting this value of λ in (i), we obtain the position vector of M given by

$$\vec{r} = \vec{a} + \left\{ \frac{d - (\vec{a} \cdot \vec{n})}{|\vec{n}|^2} \right\} \vec{n}$$

$$\Rightarrow \quad \vec{PM} = \text{P.V. of } M - \text{P.V. of } P$$

$$\vec{PM} = \vec{a} + \left\{ \frac{d - (\vec{a} \cdot \vec{n})}{|\vec{n}|^2} \right\} \vec{n} - \vec{a}$$

$$\Rightarrow \quad \vec{PM} = \frac{[d - (\vec{a} \cdot \vec{n})]\, \vec{n}}{|\vec{n}|^2}$$

$$\Rightarrow \quad PM = |\vec{PM}| = \left| \frac{[d - (\vec{a} \cdot \vec{n})]\, \vec{n}}{|\vec{n}|^2} \right|$$

$$\Rightarrow \quad PM = \frac{|\vec{a} - (\vec{a} \cdot \vec{n})|\, |\vec{n}|}{|\vec{n}|^2}$$

$$\Rightarrow \quad PM = \frac{|(\vec{a} \cdot \vec{n}) - d|}{|\vec{n}|}.$$

Thus, the length of the perpendicular from a point having position vector $\vec{a}$ on the plane $\vec{r} \cdot \vec{n} = d$ is given by

$$\frac{|(\vec{a} \cdot \vec{n}) - d|}{|\vec{n}|}.$$

18.11 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.

VECTOR FORM *If θ is the angle between a line $\vec{r} = (a + \lambda \vec{b})$ and the plane $\vec{r} \cdot \vec{n} = d$, then*

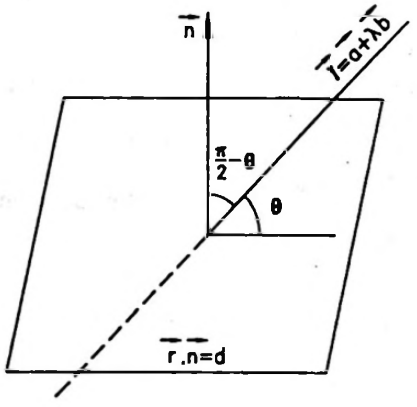


Fig. 18.40

$$\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\, |\vec{n}|}$$

PROOF The line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the vector $\vec{b}$ and the plane is normal to the vector $\vec{n}$. Therefore, if θ is the angle between the given line and given plane, then the angle between $\vec{b}$ and $\vec{n}$ is $\left(\frac{\pi}{2} - \theta\right)$. So,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\, |\vec{n}|}$$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\, |\vec{n}|}.$$

Condition of perpendicularity If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore, $\vec{b}$ and $\vec{n}$ are parallel.

So, $\vec{b} \times \vec{n} = 0$ or, $\vec{b} = \lambda \vec{n}$ for some scalar λ.

Condition of parallelism If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore $\vec{b}$ and $\vec{n}$ are perpendicular. So,

$$\vec{b} \cdot \vec{n} = 0.$$

18.12 ANGLE BETWEEN TWO PLANES

The angle between two planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is defined as the angle between their normals. Thus, if θ is the angle between these two planes, then

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\, |\vec{n_2}|}$$

If the planes are perpendicular, then $\vec{n_1} \cdot \vec{n_2} = 0$. If the planes are parallel, then $\vec{n_1}$ and $\vec{n_2}$ are parallel.

18.13 EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES

The intersection of two planes is always a straight line. For example, xy-plane and xz-plane intersect to form x-axis. The plane containing the line of intersection of two given planes is known as the plane passing through the intersection of two given planes. In the following discussion we will obtain the equation of family of planes passing through the intersection of two given planes.

VECTOR FORM *The equation of a plane passing through the intersection of the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is*

$$(\vec{r} \cdot \vec{n_1} - d_1) + \lambda (\vec{r} \cdot \vec{n_2} - d_2) = 0$$

or, $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

PROOF The equation $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ is of the form $\vec{r} \cdot \vec{n} = d$, so it represents a plane. In order to prove that it represents a plane, it is sufficient to show that every point on the lines of intersection of the plane $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ lies on $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$.

Let $\vec{r_1}$ be the position vector of any point on the line of intersection of $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$. Then,

$$\vec{r_1} \cdot \vec{n_1} = d_1 \text{ and } \vec{r_1} \cdot \vec{n_2} = d_2$$

$\Rightarrow \vec{r_1}\cdot\vec{n_1}-d_1=0$ and $\vec{r_1}\cdot\vec{n_2}-d_2=0$

$\Rightarrow (\vec{r_1}\cdot\vec{n_1}-d_1)+\lambda(\vec{r_1}\cdot\vec{n_2}-d_2)=0$

$\Rightarrow \vec{r_1}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$

$\Rightarrow \vec{r_1}$ lies on the plane $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$.

Thus, $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$ represents a plane passing through the intersection of $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *The position vectors of two points A and C are* $9\hat{i}-\hat{j}+7\hat{k}$ *and* $7\hat{i}-2\hat{j}+7\hat{k}$ *respectively. The point of intersection of vectors* $\vec{AB}=4\hat{i}-\hat{j}+3\hat{k}$ *and* $\vec{CD}=2\hat{i}-\hat{j}+2\hat{k}$ *is P. If vector* $\vec{PQ}$ *is perpendicular to* $\vec{AB}$ *and* $\vec{CD}$ *and* $PQ=15$ *units, find the position vector of Q.*

SOLUTION The vector equations of the lines AB and CD are

$$\vec{r}=(9\hat{i}-\hat{j}+7\hat{k})+\lambda(4\hat{i}-\hat{j}+3\hat{k}) \quad ...(i)$$

and,

$$\vec{r}=(7\hat{i}-\hat{j}+7\hat{k})+\mu(2\hat{i}-\hat{j}+2\hat{k}) \quad ...(ii)$$

respectively.

These two lines intersect at point P. At the intersection point, we must have

$$(9\hat{i}-\hat{j}+7\hat{k})+\lambda(4\hat{i}-\hat{j}+3\hat{k})=(7\hat{i}-\hat{j}+7\hat{k})+\mu(2\hat{i}-\hat{j}+2\hat{k})$$

$\Rightarrow 9+4\lambda=7+2\mu, -1-\lambda=-1-\mu$ and $7+3\lambda=7+2\mu$

$\Rightarrow \lambda=-2, \mu=-3$

Putting the value of λ in (i) or that of μ in (ii), we obtain the position vector $\vec{r_1}$ of point P as

$$\vec{r_1}=\hat{i}+\hat{j}+\hat{k}$$

Now,

$$\vec{AB}\times\vec{CD}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 3\\ 2 & -1 & 2\end{vmatrix}=\hat{i}-2\hat{j}-2\hat{k}$$

Since $\vec{PQ}\perp\vec{AB}$ and $\vec{PQ}\perp\vec{CD}$. Therefore,

$\vec{PQ}\parallel\vec{AB}\times\vec{CD}$

$\Rightarrow \vec{PQ}=m(\vec{AB}\times\vec{CD})$

$\Rightarrow \vec{PQ}=m(\hat{i}-2\hat{j}-2\hat{k})$

Now,

$PQ=15$

$\Rightarrow |m|\sqrt{1+4+4}=15$

$\Rightarrow |m|=5 \Rightarrow m=\pm 5$

When $m=5$, we have

$\vec{PQ}=5(\hat{i}-2\hat{j}-2\hat{k})$

$\Rightarrow$ P.V. of $Q-(\hat{i}+\hat{j}+\hat{k})=5\hat{i}-10\hat{j}-10\hat{k}$

$\Rightarrow$ P.V. of $Q=6\hat{i}-9\hat{j}-9\hat{k}$

When $m=-5$, we have

$\vec{PQ}=-5(\hat{i}-2\hat{j}-2\hat{k})$

$\Rightarrow$ P.V. of Q − P.V. of $P=-5\hat{i}+10\hat{j}+10\hat{k}$

$\Rightarrow$ P.V. of $Q=(\hat{i}+\hat{j}+\hat{k})-5\hat{i}+10\hat{j}+10\hat{k}$

$\Rightarrow$ P.V. of $Q=-4\hat{i}+11\hat{j}+11\hat{k}$.

EXAMPLE 2 *A tetrahedron has vertices at* $O(0,0,0)$, $A(1,2,1)$, $B(2,1,3)$ *and* $C(-1,1,2)$. *Prove that the angle between the faces OAB and ABC will be* $\cos^{-1}\left(\frac{19}{35}\right)$.

SOLUTION We know that the angle between two planes is defined as the angle between their normals.

Let $\vec{n_1}$ and $\vec{n_2}$ be the vectors normals to the faces OAB and ABC respectively. Then,

$$\vec{n_1}=\vec{OA}\times\vec{OB}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ 2 & 1 & 3\end{vmatrix}=5\hat{i}-\hat{j}-3\hat{k}$$

$$\vec{n_2}=\vec{AB}\times\vec{AC}=(\hat{i}-\hat{j}+2\hat{k})\times(-2\hat{i}-\hat{j}+\hat{k})$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ -2 & -1 & 1\end{vmatrix}=\hat{i}-5\hat{j}-3\hat{k}$$

Let θ be the angles between $\vec{n_1}$ and $\vec{n_2}$. Then,

$$\cos\theta=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}=\frac{5+5+9}{\sqrt{35}\sqrt{35}}=\frac{19}{35}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{19}{35}\right).$$

EXAMPLE 3 *If* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar vectors, prove that the lines*

$$\vec{r}=\vec{a}+\lambda(\vec{b}\times\vec{c}) \text{ and } \vec{r}=\vec{b}+\mu(\vec{c}\times\vec{a})$$

will intersect if $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$.

SOLUTION We know that the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ intersect, if

$$(\vec{a_1}-\vec{a_2})\cdot(\vec{b_1}\times\vec{b_2})=0$$

Therefore, the given lines will intersect, if

$$(\vec{a}-\vec{b}).\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} = 0$$

$$\Rightarrow (\vec{a}-\vec{b}).\{(\vec{b}\times\vec{c}).\vec{a})\vec{c}-\{(\vec{b}\times\vec{c}).\vec{c}\}\vec{a}\} = 0$$

$$\Rightarrow (\vec{a}-\vec{b}).\{[\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a}\} = 0$$

$$\Rightarrow (\vec{a}-\vec{b}).[\vec{b}\ \vec{c}\ \vec{a}].\vec{c} = 0$$

$$\Rightarrow [\vec{b}\ \vec{c}\ \vec{a}]\{(\vec{a}-\vec{b}).\vec{c}\} = 0$$

$$\Rightarrow [\vec{b}\ \vec{c}\ \vec{a}](\vec{a}.\vec{c}-\vec{b}.\vec{c}) = 0$$

$$\Rightarrow \vec{a}.\vec{c}-\vec{b}.\vec{c} = 0 \qquad [\because [\vec{a}\ \vec{b}\ \vec{c}] \neq 0]$$

EXAMPLE 4 *Find the perpendicular distance of a corner of a unit cube from a diagonal not passing through it.*

SOLUTION Let the edges OA, OB, OC of the unit cube be along OX, OY and OZ respectively. Since $OA = OB = OC = 1$ unit. Therefore,

$$\vec{OA} = \hat{i},\ \vec{OB} = \hat{j},\ \vec{OC} = \hat{k}.$$

Let CM be perpendicular from the corner C on the diagonal OP.

The vector equation of OP is

$$\vec{r} = \lambda(\hat{i}+\hat{j}+\hat{k})$$

$$\therefore\ OM = \text{Projection of } \vec{OC} \text{ on } \vec{OP}$$

$$= \vec{OC}.\hat{OP}$$

$$= \hat{k}.\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

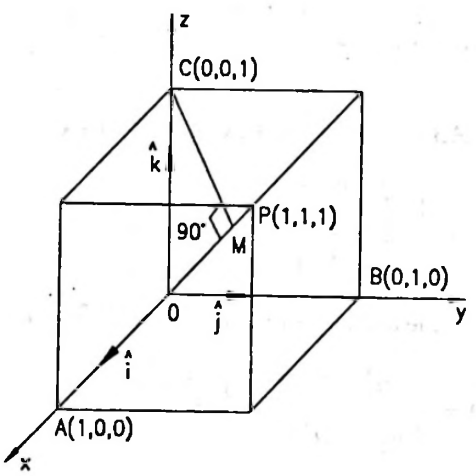


Fig. 18.41

Now, $OC^2 = OM^2 + CM^2$

$$\Rightarrow CM^2 = OC^2 - OM^2 = |\vec{OC}|^2 - OM^2 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow CM = \sqrt{2/3}$$

EXAMPLE 5 *Show that the angle between any edge and a face not containing that edge of a regular tetrahedron is* $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

SOLUTION Let $OABC$ be a regular tetrahedron. Then,

$$OA = OB = OC = AB = BC = CA$$

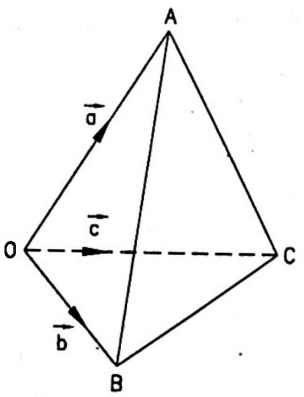


Fig. 18.42

$$\Rightarrow |\vec{OA}|^2 = |\vec{OB}|^2 = |\vec{OC}|^2$$

$$= |\vec{AB}|^2 = |\vec{BC}|^2 = |\vec{CA}|^2$$

$$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2$$

$$= |\vec{b}-\vec{a}|^2 = |\vec{c}-\vec{b}|^2 = |\vec{a}-\vec{c}|^2 = \lambda^2$$

where λ is the length of each edge.

Also, faces of a regular tetrahedron are equilateral triangles of equal size and the angle between any two concurrent edges is equal to 60°.

We know that the angle between a line and a plane is complement of the angle between the line and the normal to the plane. Thus, if θ is the angle between the face OBC and the edge OA, then

$$\sin(90-\theta)$$

$$= \frac{|\vec{OA}\times(\vec{OB}\times\vec{OC})|}{|\vec{OA}|\,|\vec{OB}\times\vec{OC}|} \quad [\because \vec{OB}\times\vec{OC} \text{ is normal to the face } OBC]$$

$$= \frac{|\vec{a}\times(\vec{b}\times\vec{c})|}{|\vec{a}|\,|\vec{b}\times\vec{c}|}$$

Now,

$$|\vec{a}\times(\vec{b}\times\vec{c})|^2 = |(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}|^2$$

$$= (\vec{a}.\vec{c})^2(\vec{b}.\vec{b}) + (\vec{a}.\vec{b})(\vec{c}.\vec{c}) - 2(\vec{a}.\vec{c})(\vec{a}.\vec{b})(\vec{b}.\vec{c})$$

$$= \{|\vec{a}|\,|\vec{c}|\cos 60°\}^2 |\vec{b}|^2 + \{|\vec{a}|\,|\vec{b}|\cos 60°\}^2 |\vec{c}|^2$$

$$-2(|\vec{a}|\,|\vec{c}|\cos 60°)(|\vec{a}|\,|\vec{b}|\cos 60°)(|\vec{b}|\,|\vec{c}|\cos 60°)$$

$$= \frac{\lambda^6}{4} + \frac{\lambda^6}{4} - \frac{2\lambda^6}{8}$$

$$= \frac{\lambda^6}{4} \qquad [\because |\vec{a}|\,|\vec{b}| = |\vec{c}| = \lambda]$$

$$\Rightarrow |\vec{a}\times(\vec{b}\times\vec{c})| = \frac{\lambda^3}{2}$$

and,

$$|\vec{b}\times\vec{c}| = |\vec{b}|\,|\vec{c}|\sin 60^\circ = \frac{\sqrt{3}}{2}\lambda^2$$

$$\therefore \sin(90^\circ-\theta) = \frac{\frac{\lambda^3}{2}}{(\lambda)\left(\frac{\sqrt{3}}{2}\lambda^2\right)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

EXAMPLE 6 *Prove that the shortest distance between the diagonals of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it are*

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$$

SOLUTION Let one vertex of the parallelopiped be at the origin O and three coterminus edges OA, OB, OC be along OX, OY and OZ respectively. The coordinates of the vertices of the parallelopiped are marked in Fig. 18.43.

The edges which do not meet the diagonal OF are AH, AD, BD and their parallels BE, CE and CH respectively.

The vector equation of the diagonal OF is

$$\vec{r} = \vec{0} + \lambda(a\hat{i}+b\hat{j}+c\hat{k}) \quad ...(i)$$

The vector equation of the edge BD is

$$\vec{r} = b\hat{j} + \mu a\hat{i} \quad ...(ii)$$

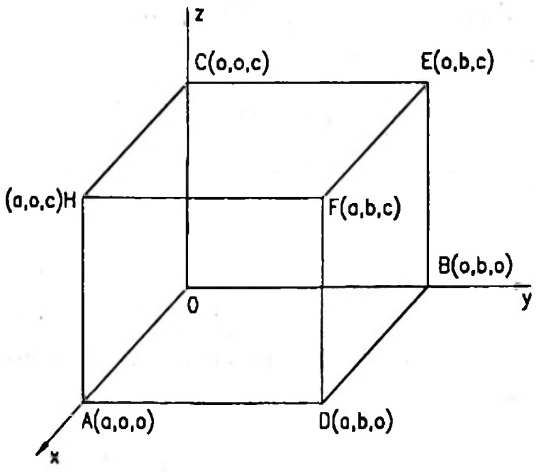


Fig. 18.43

We have,

$$(a\hat{i}+b\hat{j}+c\hat{k})\times a\hat{i} = ba(\hat{j}\times\hat{i}) + ca(\hat{k}\times\hat{i}) = -ba\hat{k} + ca\hat{j}$$

$$\therefore \left|(a\hat{i}\times b\hat{j}+c\hat{k})\times a\hat{i}\right| = \sqrt{b^2a^2+c^2a^2}$$

and, $\left|(a\hat{i}+b\hat{j}+c\hat{k})\times a\hat{i}\right| . (b\hat{j}-\vec{0}) = (-bak+caj).b\hat{j} = abc$

Thus, the shortest distance between (i) and (ii) is given by

$$S.D. = \frac{\left|\{(a\hat{i}+b\hat{j}+c\hat{k})\times a\hat{i}\}.(b\hat{j}-\vec{0})\right|}{\left|(a\hat{i}+b\hat{j}+c\hat{k})\times a\hat{i}\right|}$$

$$= \frac{abc}{\sqrt{b^2a^2+c^2a^2}}$$

$$= \frac{bc}{\sqrt{b^2+c^2}}$$

Similarly, it can be shown that the shortest distance between OF and AD is $\frac{ca}{\sqrt{a^2+c^2}}$ and that between OF and AH is $\frac{ab}{\sqrt{a^2+b^2}}$.

EXAMPLE 7 *Find the position vector of the point of intersection of three planes*

$$\vec{r}.\vec{n_1} = d_1, \vec{r}.\vec{n_2} = d_2, \vec{r}.\vec{n_3} = d_3$$

where $\vec{n_1}, \vec{n_2}$ and $\vec{n_3}$ are non-coplanar vectors.

SOLUTION Since $\vec{n_1}, \vec{n_2}, \vec{n_3}$ are non-coplanar vectors. Therefore, vectors $\vec{n_1}\times\vec{n_2}$, $\vec{n_2}\times\vec{n_3}$ and $\vec{n_3}\times\vec{n_1}$ are also non-coplanar.

Let $\vec{\alpha}$ be the position vector of the point of intersection of the given planes. Then,

$$\vec{\alpha}.\vec{n_1} = d_1, \vec{\alpha}.\vec{n_2} = d_2 \text{ and } \alpha.\vec{n_3} = d_3$$

We know that any vector in space can be written as the linear combination of three non-coplanar vectors. So, let

$$\vec{\alpha} = x(\vec{n_1}\times\vec{n_2}) + y(\vec{n_2}\times\vec{n_3}) + z(\vec{n_3}\times\vec{n_1}) \quad ...(i)$$

Now, $\alpha.\vec{n_1} = d_1$

$$\Rightarrow \{x(\vec{n_1}\times\vec{n_2}) + y(\vec{n_2}\times\vec{n_3}) + z(\vec{n_3}\times\vec{n_1})\}.\vec{n_1} = d_1$$

$$\Rightarrow y\{(\vec{n_2}\times\vec{n_3}).\vec{n_1}\} = d_1$$

$$\Rightarrow y = \frac{d_1}{[\vec{n_2}\ \vec{n_3}\ \vec{n_1}]} = \frac{d_1}{[\vec{n_1}\ \vec{n_2}\ \vec{n_3}]}$$

Similarly, we have

$$\alpha.\vec{n_2} = d_2 \text{ and } \vec{\alpha}.\vec{n_3} = d_3$$

$$\Rightarrow z = \frac{d_2}{[\vec{n_1}\ \vec{n_2}\ \vec{n_3}]} \text{ and } x = \frac{d_3}{[\vec{n_1}\ \vec{n_2}\ \vec{n_3}]}$$

Substituting the value of x, y, z in (i), we get

$$\vec{\alpha} = \frac{1}{[\vec{n_1}\ \vec{n_2}\ \vec{n_3}]}[d_3(\vec{n_1}\times\vec{n_2}) + d_1(\vec{n_2}\times\vec{n_3}) + d_2(\vec{n_1}\times\vec{n_3})]$$

as the required position vector of the point of intersection of the given planes.

EXAMPLE 8 *If the planes $\vec{r}.\vec{n_1} = d_1, \vec{r}.\vec{n_2} = d_2$ and $\vec{r}.n_3 = d_3$ have a common line of intersection, prove that*

$$d_1(\vec{n_2}\times\vec{n_3}) + d_2(\vec{n_3}\times\vec{n_1}) + d_3(\vec{n_1}\times\vec{n_2}) = \vec{0}$$

SOLUTION The equation of a plane passing through the line of intersectin of the planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is

$$\vec{r}.(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2, \text{ where } \lambda \text{ is a parameter.}$$

Since three planes have a common line of intersection, therefore, the abvoe equation should be identical to $\vec{r}.\vec{n_3} = d_3$ for some value of λ.

Thus, for some value of λ, we have

$$\vec{n_1} + \lambda\vec{n_2} = \mu\vec{n_3} \quad ...(i)$$

and, $$d_1 + \lambda d_2 = \mu d_3 \quad ...(ii)$$

Now,

$$\vec{n_1} + \lambda\vec{n_2} = \mu\vec{n_3}$$

$$\Rightarrow (\vec{n_1} + \lambda\vec{n_2}) \times \vec{n_3} = \mu(\vec{n_3} \times \vec{n_3}) \text{ [Taking cross-product with } \vec{n_3}]$$

$$\Rightarrow \vec{n_1} \times \vec{n_3} + \lambda(\vec{n_2} \times \vec{n_3}) = \vec{0} \quad ...(iii)$$

$$\Rightarrow \lambda(\vec{n_2} \times \vec{n_3}) = -(\vec{n_1} \times \vec{n_3})$$

Again,

$$\vec{n_1} + \lambda\vec{n_2} = \mu\vec{n_3}$$

$$\Rightarrow (\vec{n_1} + \lambda\vec{n_2}) \times \vec{n_2} = \mu(\vec{n_3} \times \vec{n_2})$$

$$\Rightarrow \vec{n_1} \times \vec{n_2} = \mu(\vec{n_3} \times \vec{n_2})$$

$$\Rightarrow \mu(\vec{n_2} \times \vec{n_3}) = -(\vec{n_1} \times \vec{n_2}) \quad ...(iv)$$

Now,

$$d_1 + \lambda d_2 = \mu d_3$$

$$\Rightarrow (d_1 + \lambda d_2)(\vec{n_2} \times \vec{n_3}) = \mu d_3(\vec{n_2} \times \vec{n_3})$$

$$\Rightarrow d_1(\vec{n_2} \times \vec{n_3}) + d_2\{\lambda(\vec{n_2} \times \vec{n_3})\} = d_3\{\mu(\vec{n_2} \times \vec{n_3})\}$$

$$\Rightarrow d_1(\vec{n_2} \times \vec{n_3}) - d_2(\vec{n_1} \times \vec{n_3}) = -d_3(\vec{n_1} \times \vec{n_2}) \quad \left[\begin{matrix}\text{Using (iii)} \\ \text{and (iv)}\end{matrix}\right]$$

$$\Rightarrow d_1(\vec{n_2} \times \vec{n_3}) + d_2(\vec{n_3} \times \vec{n_1}) + d_3(\vec{n_1} \times \vec{n_2}) = \vec{0}$$

EXAMPLE 9 *Obtain the vector equation of the line of intersection of the planes*

$$\vec{r} = \vec{b} + \lambda_1(\vec{b} - \vec{a}) + \mu_1(\vec{a} + \vec{c})$$

and, $$\vec{r} = \vec{c} + \lambda_2(\vec{b} - \vec{c}) + \mu_2(\vec{a} + \vec{b})$$

$\vec{a}$, $\vec{b}$, $\vec{c}$ being non-coplanar vectors.

SOLUTION At the point of intersection of the two given planes, we have

$$\vec{b} + \lambda_1(\vec{b} - \vec{a}) + \mu_1(\vec{a} + \vec{c}) = \vec{c} + \lambda_2(\vec{b} - \vec{c}) + \mu_2(\vec{a} + \vec{b})$$

$$\Rightarrow (-\lambda_1 + \mu_1 - \mu_2)\vec{a} + (1 + \lambda_1 - \lambda_2 - \mu_2)\vec{b} + (\mu_1 - 1 + \lambda_2)\vec{c} = \vec{0}$$

$$\Rightarrow -\lambda_1 + \mu_1 - \mu_2 = 0, 1 + \lambda_1 - \lambda_2 - \mu_2 = 0,$$

and,

$$\mu_1 - 1 + \lambda_2 = 0 \quad [\because \vec{a}, \vec{b}, \vec{c} \text{ are non-coplanar vecto}$$

From the last two equations, we get

$$\lambda_1 + \mu_1 - \mu_2 = 0$$

Solving this equation with $-\lambda_1 + \mu_1 - \mu_2 = 0$, we get

$$\mu_1 = \mu_2 \text{ and } \lambda_1 = 0.$$

$$\therefore \lambda_1 = 0, \mu_1 = \mu_2 \text{ and } \lambda_2 = 1 - \mu_1$$

Substituting these values in either of the given equation we obtain

$$\vec{r} = \vec{b} + \mu_1(\vec{a} + \vec{c})$$

as the required line of intersection of the given planes.

EXAMPLE 10 *Find the equation of the plane passing through the lin of intersection of the planes $\vec{r}.\vec{n_1} = d_1$, $\vec{r}.\vec{n_2} = d_2$ and parallel to th line of intersection of the planes $\vec{r}.\vec{n_3} = d_3$ and $\vec{r}.\vec{n_4} = d_4$.*

SOLUTION The equation of the plane passing through the Lin of intersection of the planes

$\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is

$$(\vec{r}.\vec{n_1} - d_1) + \lambda(\vec{r}.\vec{n_2} - d_2) = 0$$

or, $$\vec{r}.(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2 \quad ...(i$$

This plane is parallel to the line of intersection of the planes $\vec{r}.\vec{n_3} = d_3$ and $\vec{r}.\vec{n_4} = d_4$. The line of intersection of these two planes is parallel to $\vec{n_3} \times \vec{n_4}$. So, the plane in (i) is parallel to a vector $\vec{n_3} \times \vec{n_4}$. Therefore,

$$(\vec{n_1} + \lambda\vec{n_2}) \cdot (\vec{n_3} \times \vec{n_4}) = 0$$

$$\Rightarrow [\vec{n_1}\ \vec{n_3}\ \vec{n_4}] + \lambda[\vec{n_2}\ \vec{n_3}\ \vec{n_4}] = 0$$

$$\Rightarrow \lambda = -\frac{[\vec{n_1}\ \vec{n_3}\ \vec{n_4}]}{[\vec{n_2}\ \vec{n_3}\ \vec{n_4}]}$$

Putting the value of λ in (i), we obtain the equation of the required plane as

$$\vec{r}.\left\{\vec{n_1} - \frac{[\vec{n_1}\ \vec{n_3}\ \vec{n_4}]}{[\vec{n_2}\ \vec{n_3}\ \vec{n_4}]}\vec{n_2}\right\} = \left\{d_1 - \frac{[\vec{n_1}\ \vec{n_3}\ \vec{n_4}]}{[\vec{n_2}\ \vec{n_3}\ \vec{n_4}]}d_2\right\}$$

EXAMPLE 11 *ABC is a triangle whose vertices have the positio vectors $\hat{i}, \hat{j}$ and $\hat{k}$ respectively. A line parallel to the vectors $\hat{i} + \hat{j}$ and passing through the point $\hat{i} - 2\hat{j} + 3\hat{k}$ cuts the plane of the triangle at P.*

(i) Find the position vector of point P.

(ii) Find the equations of the plane (s) which is (are) parallel to the plane of triangle ABC and whose perpendicular distance from the origin is $\sqrt{3}$ units.

SOLUTION A vector normal to the plane of ΔABC is given by

$$\vec{n} = \vec{AB} \times \vec{AC}$$
$$= (\hat{j} - \hat{i}) \times (\hat{k} - \hat{i})$$
$$= \hat{i} + \hat{k} + \hat{j} = \hat{i} + \hat{j} + \hat{k}$$

The vector equation of the plane of ΔABC is

$$(\vec{r} - \hat{i}) . (\hat{i} + \hat{j} + \hat{k}) = 0$$
$$\Rightarrow \quad \vec{r} . (\hat{i} + \hat{j} + \hat{k}) = 1 \qquad ...(i)$$

Equation of the line parallel to the vector $\hat{i} + \hat{j}$ and passing through the point $\hat{i} - 2\hat{j} + 3\hat{k}$ is

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda (\hat{i} + \hat{j}) \qquad ...(ii)$$

(i) At the point of intersection of (i) and (ii), we have

$$\{(\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda (\hat{i} + \hat{j})\} . (\hat{i} + \hat{j} + \hat{k}) = 1$$
$$\Rightarrow (1 - 2 + 3) + \lambda (1 + 1) = 1$$
$$\Rightarrow 2\lambda = -1 \Rightarrow \lambda = -\frac{1}{2}$$

Putting $\lambda = -\frac{1}{2}$ in (ii), we obtain the position vector of point P as

$$\vec{r_1} = (\hat{i} - 2\hat{j} + 3\hat{k}) - \frac{1}{2}(\hat{i} + \hat{j}) = \frac{1}{2}(\hat{i} - 3\hat{j} + 6\hat{k})$$

(ii) The equations of the planes parallel to (i) are given by

$$\vec{r} . (\hat{i} + \hat{j} + \hat{k}) = \mu$$

If it is at a distance of $\sqrt{3}$ units from the origin, then

$$\left| \frac{\vec{0} . (\hat{i} + \hat{j} + \hat{k}) - \mu}{|\hat{i} + \hat{j} + \hat{k}|} \right| = \sqrt{3}$$
$$\Rightarrow \quad \frac{|\mu|}{\sqrt{3}} = \sqrt{3}$$
$$\Rightarrow \quad |\mu| = 3$$
$$\Rightarrow \quad \mu = \pm 3$$

So, the equations of the required planes are

$$\vec{r} . (\hat{i} + \hat{j} + \hat{k}) = \pm 3$$

EXERCISE 18.1

1. In a triangle ABC, D is a point on BC such that $\frac{BD}{DC} = \frac{2}{3}$. Point E lies on AB such that $\frac{BE}{EA} = \frac{2}{1}$. AD and CE intersect at F. Using vector methods, find the ratio $\frac{AF}{FD}$.

2. In a triangle ABC, D divides BC in the ratio 3 : 2 and E divides CA in the ratio 1 : 3. The lines AD and BE meet in H and CH meets AB in F. Find the ratio in which F divides AB.

3. In a ΔABC, D is on AC and divides AC in the ratio 2 : 1. BD is produced at F such that $DF = 2\,BD$. Prove by vector method that

(i) $\vec{AF} = 2\vec{BC}$

(ii) $\vec{CF}$ is parallel to the median and is twice the median.

4. Let $OABC$ be a parallelogram with O as the origin and OC a diagonal. Let D be the mid-point of OA. Using vector methods, prove that BD and CO intersect in the same ratio. Find this ratio.

5. If $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$, determine a vector $\vec{r}$ of magnitude $5\sqrt{6}$ which is along the bisector of the angle between $\vec{a}$ and $\vec{b}$.

6. If a straight line is drawn parallel to the base of a triangle, the line joining the vertex to the intersection of the diagonals of the trapezium so formed bisects the base of the triangle.

7. If M, N are the mid-points of the sides AB, CD of a parallelogram $ABCD$, prove that DM and BN cut the diagonal AC at its point of trisection, which are also points of trisection of DM and BN respectively.

8. Show that the external bisector of the angle A of a triangle ABC divides BC externally in the ratio $AB : AC$.

9. If a straight line is drawn parallel to the base of a triangle, prove that the line joining the vertex to the intersection of the diagonals of the trapezoid so formed bisects the base of the triangle.

10. $ABCD$ is a parallelogram, AC a diagonal and a point P is taken on the side AD such that $AP = \frac{AD}{n}$. Show that the line BP cuts the diagonal in a point Q distant $\frac{AC}{n+1}$ from A.

11. D, E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.

12. Find the equation of the plane containing the two parallel lines

$$\vec{r} = \vec{a} + \lambda \vec{b} \text{ and } \vec{r} = \vec{\alpha} + \mu \vec{b}$$

13. Find the equation of the plane containing the lines $\vec{r} = \vec{a} + \lambda \vec{a}$ and $\vec{r} = \vec{a} + \mu \vec{a}$.

14. Prove that the lines $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + \lambda (\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + 2\hat{j} + 2\hat{k} + \mu (\hat{i} - \hat{j} + 2\hat{k})$ are intersecting and hence find their point of intersection.

15. The vertices of a triangle ABC are $A(1, 0, 2)$, $B(-2, 1, 3)$ and $C(2, -1, 1)$. Find the equation of the line BC, the foot of the perpendicular from A on BC and the length of the perpendicular.

16. From the point $A(1, 2, 0)$, perpendicular is drawn to the plane $\vec{r} . (3\hat{i} - \hat{j} + \hat{k}) = 2$ meeting it at the point P. Find the coordinates of point P and the distance AP.

17. Find the equation of the plane passing through the points $A(2, 1, 3)$, $B(-1, 2, 4)$ and $C(0, 2, 1)$. Also, find the coordinates of the point of intersection of the plane ABC and the line $\vec{r} = \hat{i} - \hat{j} + \hat{k} + \lambda (2\hat{i} + \hat{k})$.

18. Find a point R in which the line AB cuts the plane CDE where $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = -4\hat{j} + 4\hat{k}$, $\vec{d} = 2\hat{i} - 2\hat{k}$ and $\vec{e} = 4\hat{i} + \hat{j} + 2\hat{k}$ are position vectors of point A, B, C, D, E respectively.

19. A straight line L passes through a point $A(\vec{a})$ and is parallel to $\vec{b}$. A second line L' passes through $A'(\vec{a}')$ and is parallel to $\vec{b}'$. The vector $\vec{p}$ runs from L to L' along the common perpendicular, show that

$$\vec{p} = \left\{\frac{(\vec{a}' - \vec{a}).\vec{c}}{|\vec{c}|^2}\right\}\vec{c}, \quad \text{where } \vec{c} = \vec{b} \times \vec{b}'.$$

20. In a ΔABC, E and F are the mid-points of AC and A respectively. CP is drawn parallel to AB to meet BE produce at P. Show that $\Delta FEP = \Delta FCE = \frac{1}{4}\Delta ABC$.

21. Let $OACB$ be a parallelogram with O at the origin an OC a diagonal. Let D be the mid-point of OA. Using vect method prove that BD and CO intersect in the same ratio.

22. Find the perpendicular distance of the point $A(1, 4, -2$ from the segment BC, where $B(2, 1, -2)$ and $C(0, -5, 1)$.

ANSWERS

1. 5 : 6 **2.** 2 : 1 **4.** 2 : 1

5. $\frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$ **11.** 15 : 4 and 10 : 9

12. $\vec{r} \cdot \{(\vec{\alpha} - \vec{a}) \times \vec{b}\} = \vec{a} \cdot \{(\vec{\alpha} - \vec{a}) \times \vec{b}\}$

13. $[\vec{r}\ \vec{a}\ \vec{a}'] = 0$

14. (4, 0, 2)

15. $\vec{r} = (-2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(4\hat{i} - 2\hat{j} - 2\hat{k}), \frac{1}{\sqrt{3}}$

16. $\left(\frac{14}{11}, \frac{21}{11}, \frac{1}{11}\right), \frac{1}{\sqrt{11}}$

17. $\vec{r} \cdot (3\hat{i} + 8\hat{j} + \hat{k}) = 17, (7, -11, 4)$

18. $3\hat{i} + 3\hat{k}$ **22.** $\frac{\sqrt{234}}{7}$

—❑❑❑—

CHAPTER

19

COMPLEX NUMBERS

19.1 INTRODUCTION

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b; a, b \in N$ is solvable. But equations of the form $xa = b$, where $a, b \in I, a \neq 0$ are not solvable in I also. Therefore the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b, a \neq 0, a, b \in I$ is uniquely solvable. The equations of the form $x^2 = 2, x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which includes both rational and irrational numbers. This set is known as the set of real numbers. The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 with the property $ci^2 = -1$. He also called this symbol as the imaginary unit.

19.2 INTEGRAL POWERS OF IOTA (i)

Positive integral powers of i :

We have,

$$i = \sqrt{-1}$$

$$\therefore \quad i^2 = -1$$

$$i^3 = i^2 \times 1 = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

In order to compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then,

$$n = 4m + r, \text{ where } 0 \leq r < 4$$

$$\Rightarrow i^n = i^{4m+r} = (i^4)^m i^r = i^r$$

Thus, the value of i^n for $n > 4$ is i^r, where r is the remainder when n is divided by 4.

Negative integral powers of i :

By the law of indices, we have

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then

$i^{-n} = \dfrac{1}{i^n} = \dfrac{1}{i^r}$, where r is the remainder when n is divided by 4

NOTE i^0 *is defined as 1.*

ILLUSTRATION 1 *Evaluate the following:*

(i) i^{135} (ii) i^{19} (iii) i^{-999} (iv) $(-\sqrt{-1})^{4n+3}, n \in N$

SOLUTION (i) 135 leaves remainder as 3 when it is divided by 4. Therefore,

$$i^{135} = i^3 = -i.$$

(ii) The remainder is 3 when 19 is divided by 4. Therefore,

$$i^{19} = i^3 = -i.$$

(iii) We have,

$$i^{-999} = 1/i^{999}.$$

On dividing 999 by 4, we obtain 3 as the remainder. Therefore,

$$i^{999} = i^3.$$

So, $i^{-999} = \frac{1}{i^{999}} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i.$

(iv) We have,

$$(-\sqrt{-1})^{4n+3} = (-i)^{4n+3}$$
$$= (-i)^{4n}(-i)^3$$
$$= \{(-i)^4\}^n(-i^3) = 1^n(-(-i)) = i$$

ILLUSTRATION 2 *Show that*

(i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = -4$

(ii) $\left[i^{17} - \left(\frac{1}{i}\right)^{34}\right]^2 = 2i$

(iii) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, for all $n \in N$.

SOLUTION (i) We have,

$$\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{19} + \frac{1}{i^{25}}\right]^2 = \left[i^3 + \frac{1}{i}\right]^2 = \left[-i + \frac{i^3}{i^4}\right]^2$$
$$= [-i + i^3]^2 = (-i - i)^2 = 4i^2 = -4.$$

(ii) We have,

$$\left[i^{17} - \left(\frac{1}{i}\right)^{34}\right]^2 = \left[i^{17} - \frac{1}{i^{34}}\right]^2 = \left[i - \frac{1}{i^2}\right]^2 = \left[i - \frac{1}{(-1)}\right]^2$$
$$= (i+1)^2$$
$$= i^2 + 2i + 1 = -1 + 2i + 1 = 2i$$

(iii) We have,

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n + i^n \times i + i^n \times i^2 + i^n \times i^3$$
$$= i^n[1 + i + i^2 + i^3]$$
$$= i^n[1 + i - 1 - i] = i^n(0) = 0$$

19.3 IMAGINARY QUANTITIES

DEFINITION *The square root of a negative real number is called an imaginary quantity or an imaginary number.*

For example, $\sqrt{-3}$, $\sqrt{-4}$, $\sqrt{-9/4}$ etc. are imaginary quantities.

A USEFUL RESULT *If a, b are positive real numbers, then*

$$\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}.$$

PROOF We have, $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$

and, $\sqrt{-b} = \sqrt{-1 \times b} = \sqrt{-1} \times \sqrt{b} = i\sqrt{b}$.

Therefore,

$$\sqrt{-a} \times \sqrt{-b}$$
$$= (i\sqrt{a})(i\sqrt{b})$$
$$= i^2(\sqrt{a} \times \sqrt{b}) = -1(\sqrt{ab}) = -\sqrt{ab}$$

NOTE 1 *For any two real numbers $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either positive or zero.*

In other words, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is not valid if a and b both are negative.

NOTE 2. *For any positive real number a, we have*

$$\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a}.$$

ILLUSTRATION 2 *Compute the following:*

(i) $\sqrt{-144}$

(ii) $\sqrt{-4} \times \sqrt{\frac{-9}{4}}$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$.

SOLUTION (i) We have,

$$\sqrt{-144} = \sqrt{-1 \times 144} = \sqrt{-1} \times \sqrt{144} = 12i$$

(ii) We have,

$$\sqrt{-4} \times \sqrt{\frac{-9}{4}} = (2i)\left(\frac{3i}{2}\right) = 3i^2 = -3$$

(iii) We have,

$$\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$$

ILLUSTRATION 3 *A student writes the formula $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Then he substitutes $a = -1$ and $b = -1$ and finds $1 = -1$. Explain where is he wrong?*

SOLUTION Since a and b both are negative. Therefore, $\sqrt{ab}$ cannot be written as $\sqrt{a}\sqrt{b}$. In fact for a and b both negative, we have

$$\sqrt{a}\sqrt{b} = -\sqrt{ab}.$$

ILLUSTRATION 4 *Is the following computation correct? If not give the correct computation:*

$$\text{"}[\sqrt{(-2)}\sqrt{(-3)}] = \sqrt{(-2)(-3)} = \sqrt{6}\text{"}$$

SOLUTION The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is as given below :

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}.$$

19.4 COMPLEX NUMBERS

COMPLEX NUMBER *If a, b are two real numbers, then a number of the form a + ib is called a complex number.*

For example $7 + 2i, -1 + i, 3 - 2i, 0 + 2i, 1 + 0i$ etc. are complex numbers.

REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by Re (z) and the imaginary part by Im (z).

Fo example, if $z = 3 - 4i$, then Re (z) = 3 and Im (z) = -4.

PURELY REAL AND PURELY IMAGINARY COMPLEX NUMBERS A complex number z is purely real if its imaginary part is zero i.e. Im (z) = 0 and purely imaginary if its real part is zero i.e. Re (z) = 0.

SET OF COMPLEX NUMBERS The set of all complex numbers is denoted by C

i.e. $C = \{a + ib \mid a, b \in R\}$.

Since a real number 'a' can be written as $a + 0\,i$, therefore every real number is a complex number. Hence,

$R \subset C$, where R is the set of all real numbers.

19.5 EQUALITY OF COMPLEX NUMBERS

DEFINITION *Two complex numbers $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$ i.e. Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2).*

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}\,(z_1) = \text{Re}\,(z_2)$ and $\text{Im}\,(z_1) = \text{Im}\,(z_2)$.

ILLUSTRATION 1 *If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y.*

SOLUTION We have,

$z_1 = z_2$

$\Rightarrow 2 - iy = x + 3i$

$\Rightarrow 2 = x$ and $-y = 3$

$\Rightarrow x = 2$ and $y = -3$.

ILLUSTRATION 2 *If $(a + b) - i\,(3a + 2b) = 5 + 2\,i$, find a and b.*

SOLUTION We have,

$(a + b) - i\,(3a + 2b) = 5 + 2\,i$

$\Rightarrow \; a + b = 5$ and $-(3a + 2b) = 2$

$\Rightarrow a = -12,\ b = 17$

19.6 ADDITION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.*

It follows from this definition that the sum $z_1 + z_2$ is a complex number such that

$$\text{Re}\,(z_1 + z_2) = \text{Re}\,(z_1) + \text{Re}\,(z_2)$$

and,
$$\text{Im}\,(z_1 + z_2) = \text{Im}\,(z_1) + \text{Im}\,(z_2)$$

For example, if $z_1 = 2 + 3\,i$ and $z_2 = 3 - 2\,i$, then

$z_1 + z_2 = (2 + 3) + (3 - 2)\,i = 5 + i$

PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) *Addition is commutative : For any two complex numbers z_1 and z_2, we have*

$$z_1 + z_2 = z_2 + z_1.$$

(ii) *Addition is associative : For any three complex numbers z_1, z_2, z_3, we have*

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(iii) *Existence of additive identity : The complex number $O = 0 + i\,0$ is the identity element for addition i.e.*

$$z + 0 = z = 0 + z \text{ for all } z \in C.$$

(iv) *Existence of additive identity : The complex number $0 = 0 + i0$ is the identity element for addition i.e.*

$$z + 0 = z = 0 + z, \text{ for all } z \in C.$$

19.7 SUBTRACTION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.*

$$\begin{aligned} \text{Thus,} \quad z_1 - z_2 &= z_1 + (-z_2) \\ &= (a_1 + ib_1) + (-a_2 - i\,b_2) \\ &= (a_1 - a_2) + i\,(b_1 - b_2) \end{aligned}$$

For example, if $z_1 = -2 + 3i$ and $z_2 = 4 + 5i$, then

$$\begin{aligned} z_1 - z_2 &= (-2 + 3i) + (-4 - 5i) \\ &= (-2 - 4) + i\,(3 - 5) \\ &= -6 - 2i \end{aligned}$$

19.8 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by $z_1\,z_2$ and is defined as the complex number $(a_1\,a_2 - b_1\,b_2) + i\,(a_1\,b_2 + a_2\,b_1)$.

$$\begin{aligned} \text{Thus,} \quad z_1\,z_2 &= (a_1 + i\,b_1)\,(a_2 + i\,b_2) \\ &= (a_1\,a_2 - b_1\,b_2) + i\,(a_1\,b_2 + a_2\,b_1) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad z_1\,z_2 &= [\text{Re}\,(z_1)\,\text{Re}\,(z_2) - \text{Im}\,(z_1)\,\text{Im}\,(z_2)] \\ &\quad + i\,[\text{Re}\,(z_1)\,\text{Im}\,(z_2) + \text{Re}\,(z_2)\,\text{Im}\,(z_1)] \end{aligned}$$

For example, if $z_1 = 3 + 2i$ and $z_2 = 2 - 3i$, then

$$\begin{aligned} z_1\,z_2 &= (3 + 2i)\,(2 - 3\,i) \\ &= (3 \times 2 - 2 \times (-3)) + i(3 \times -3 + 2 \times 2) \\ &= 12 - 5i \end{aligned}$$

NOTE *The product $z_1\,z_2$ can also be obtained if we actually carry out the multiplication $(a_1 + i\,b_1)\,(a_2 + i\,b_2)$ as given below :*

$$(a_1 + i\,b_1)\,(a_2 + i\,b_2) = a_1\,a_2 + i\,a_1\,b_2 + i\,b_1\,a_2 + i^2\,b_1\,b_2$$

$$= (a_1\,a_2 - b_1\,b_2) + i(a_1\,b_2 + a_2\,b_1) \; [\because i^2 = -1]$$

PROPERTIES OF MULTIPLICATION

(i) *Multiplication is commutative : For any two complex numbers z_1 and z_2, we have*

$$z_1\,z_2 = z_2\,z_1.$$

(ii) *Multiplication is associative : For any three complex numbers z_1, z_2, z_3, we have*

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iii) *Existence of identity element for multiplication. The complex number $1 = 1 + i0$ is the identity element for multiplication i.e. for every complex number z, we have*

$$z \cdot 1 = z = 1 \cdot z.$$

(iv) *Existence of multiplicative inverse : Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that*

$$z \cdot z_1 = 1 = z_1 \cdot z.$$

PROOF We have,

$$z \cdot z_1 = 1$$

$$\Rightarrow (a + ib)(x + iy) = 1 + i0$$

$$\Rightarrow (ax - by) + i(ay + bx) = 1 + i0$$

$$\Rightarrow ax - by = 1 \text{ and } ay + bx = 0.$$

Solving these two equations, we get

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2} \qquad [\because a \neq 0, b \neq 0]$$

Thus, every non-zero complex number $z = a + i\,b$ possesses multiplicative inverse given by

$$\left(\frac{a}{a^2 + b^2}\right) + i\left(\frac{-b}{a^2 + b^2}\right)$$

NOTE *The multiplicative inverse of z is denoted by z^{-1} or, $\frac{1}{z}$.*

(v) Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3, we have

(i) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ *(Left distributivity)*

(ii) $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$ *(Right distributivity)*

19.9 DIVISION OF COMPLEX NUMBERS

DEFINITION *The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.*

Thus, $$\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \left(\frac{1}{z_2}\right).$$

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$. Then,

$$\frac{z_1}{z_2} = (a_1 + i\,b_1)\left|\frac{a_2}{a_2^2 + b_2^2} + i\frac{(-b_2)}{a_2^2 + b_2^2}\right| \quad \left[\because z = a + i\,b \Rightarrow \frac{1}{z} = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}\right]$$

$$= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) \qquad [\text{By def. of multiplication}]$$

For example, if $z_1 = 2 + 3\,i$ and $z_2 = 1 + 2\,i$, then

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = (2 + 3\,i)\left(\frac{1}{1 + 2\,i}\right)$$

$$= (2 + 3\,i)\left(\frac{1}{5} - \frac{2}{5}i\right)$$

$$= \left(\frac{2}{5} + \frac{6}{5}\right) + i\left(-\frac{4}{5} + \frac{3}{5}\right)$$

$$= \frac{8}{5} - \frac{1}{5}i$$

19.10 CONJUGATE OF A COMPLEX NUMBER

Let $z = a + i\,b$ be a complex number. Then, the conjugate of z is denoted by $\bar{z}$ and is equal to $a - i\,b$.

Thus, $z = a + i\,b$

$\Rightarrow$ $\bar{z} = a - i\,b$

It follows from this definition that the conjugate of a complex number is obtained by replacing i by $-i$.

For examle, $z = 3 + 4\,i$, then $\bar{z} = 3 - 4\,i$.

PROPERTIES OF CONJUGATE

If z, z_1, z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ *is purely real*

(v) $z + \bar{z} = 0 \Rightarrow z$ *is purely imaginary*

(vi) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(vii) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

(viii) $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$

(ix) $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}, \; z_2 \neq 0.$

PROOF (i) Let $z = x + iy$. Then,

$$\bar{z} = x - iy \text{ and } \overline{(\bar{z})} = x + i\,y$$

Clearly, $\overline{(\bar{z})} = z$

(ii) Let $z = x + iy$ Then, $\bar{z} = x - i\,y$.

$$\therefore \quad z + \bar{z} = (x + iy) + (x - iy)$$

$$= 2x = 2\,\text{Re}(z)$$

(iii) Let $z = x + iy$. Then, $\bar{z} = x - iy$.

$\therefore \quad z - \bar{z} = (x + iy) - (x - iy)$

$= 2iy = 2i \operatorname{Im}(z)$

(iv) Let $z = x + iy$. Then, $\bar{z} = x - iy$.

$\therefore \; z = \bar{z} \Leftrightarrow x + iy = x - iy$

$\Leftrightarrow 2iy = 0$

$\Leftrightarrow y = 0$

$\Leftrightarrow \operatorname{Im}(z) = 0$

$\Leftrightarrow z$ is purely real

(v) Let $z = x + iy$. Then, $\bar{z} = x - iy$.

$\therefore \; z + \bar{z} = 0 \Leftrightarrow (x + iy) + (x - iy) = 0$

$\Leftrightarrow 2x = 0$

$\Leftrightarrow x = 0$

$\Leftrightarrow \operatorname{Re}(z) = 0$

$\Leftrightarrow z$ is purely imaginary.

Let $z = x + iy$. Then, $\bar{z} = x - iy$

$\therefore \quad z\bar{z} = (x + iy)(x - iy)$

$= x^2 + y^2$

$= (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$

(vii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$\Rightarrow \overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$

$= (x_1 - iy_1) + (x_2 - iy_2)$

$= \bar{z}_1 + \bar{z}_2$

(viii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\Rightarrow \overline{z_1 - z_2} = (x_1 - x_2) - i(y_1 - y_2)$

$\Rightarrow \overline{z_1 - z_2} = (x_1 - iy_1) - (x_2 - iy_2)$

$= \bar{z}_1 - \bar{z}_2$

(ix) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$\Rightarrow z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

$\Rightarrow \overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1)$

Also, $\bar{z}_1 \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1)$

$\therefore \; \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

(x) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$z_1 = \overline{x_1} - iy_1$ and $\bar{z}_2 = x_2 - iy_2$

$\therefore \quad \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 x_2 + y_1 y_2) - i(x_1 y_2 - x_2 y_1)}{x_2^2 + y_2^2}$

$\Rightarrow \left(\dfrac{z_1}{z_2}\right) = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)}{x_2^2 + y_2^2}$

Also,

$\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{x_1 - iy_1}{x_2 - iy_2}$

$\Rightarrow \dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)}{x_2^2 + y_2^2}$

$\therefore \; \left(\dfrac{z_1}{z_2}\right) = \dfrac{\bar{z}_1}{\bar{z}_2}$

19.11 MODULUS OF A COMPLEX NUMBER

DEFINITION *The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{[\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2}.$$

Clearly, $|z| \geq 0$ for all $z \in C$.

For example,

if $z_1 = 3 - 4i$, $z_2 = -5 + 2i$, and $z_3 = 1 + \sqrt{-3}$, then

$|z_1| = \sqrt{3^2 + (-4)^2} = 5,$

$|z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$

and, $|z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$

REMARK *In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or, $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or, $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.*

PROPERTIES OF MODULUS

THEOREM *If $z, z_1, z_2 \in C$, then*

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\operatorname{Re}(z) = \operatorname{Im}(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \operatorname{Re}(z) \leq |z|$; $-|z| \leq \operatorname{Im}(z) \leq |z|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(vii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}\,(z_1\bar{z}_2)$

(viii) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\,\text{Re}\,(z_1\bar{z}_2)$

(ix) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

(x) $|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b^2)(|z_1|^2+|z_2|^2)$, where $a, b \in R$.

(xi) $|z_1+z_2| \le |z_1|+|z_2|$

(xii) $|z_1-z_2| \le |z_1|+|z_2|$

(xiii) $|z_1+z_2| \ge |z_1|-|z_2|$

(xiv) $|z_1-z_2| \ge |z_1|-|z_2|$

(xv) $|z_1|-|z_2| \le |z_1+z_2|$

and,

$$\big||z_1|-|z_2|\big| \le |z_1-z_2|$$

PROOF (i) Let $z=a+ib$. Then,

$|z| = 0$

$\Leftrightarrow \sqrt{a^2+b^2}=0$

$\Leftrightarrow a^2+b^2=0$

$\Leftrightarrow a=0$ and $b=0$

$\Leftrightarrow \text{Re}(z)=\text{Im}(z)=0.$

(ii) Let $z=a+ib$. Then, $\bar{z}=a-ib$ and $-z=-a-ib$.

$\therefore \quad |z| = \sqrt{a^2+b^2},$

$|\bar{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

and, $|-z| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$

Clearly, $|z| = |\bar{z}| = |-z|.$

(iii) Let $z=a+ib$. Then, $|z| = \sqrt{a^2+b^2}$.

Since, $-\sqrt{a^2+b^2} \le a \le \sqrt{a^2+b^2}$

and, $-\sqrt{a^2+b^2} \le b \le \sqrt{a^2+b^2}$.

$\therefore \; -|z| \le \text{Re}(z) \le |z|$ and, $-|z| \le \text{Im}(z) \le |z|$.

(iv) Let $z=a+ib$. Then, $\bar{z}=a-ib$.

$\therefore \; z\bar{z} = (a+ib)(a-ib)$

$= a^2-i^2b^2$

$= a^2+b^2$

$= \{\sqrt{a^2+b^2}\}^2$

$= |z|^2$

(v) Let $z_1 = a_1+ib_1$ and $z_2=a_2+ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$z_1z_2 = (a_1+ib_1)(a_2+ib_2)$

$\Rightarrow \; z_1z_2 = (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$

$\Rightarrow \; |z_1z_2| = \sqrt{(a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2}$

$\Rightarrow \; |z_1z_2| = \sqrt{a_1^2a_2^2+b_1^2b_2^2+a_1^2b_2^2+a_2^2b_1^2}$

$\Rightarrow \; |z_1z_2| = \sqrt{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)}$

$\Rightarrow \; |z_1z_2| = \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}$

$\Rightarrow \; |z_1z_2| = \sqrt{a_1^2+b_1^2}\,.\,\sqrt{a_2^2+b_2^2}$

$\Rightarrow \; |z_1z_2| = |z_1|\,|z_2|.$

(vi) Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\frac{z_1}{z_2} = \frac{a_1+ib_1}{a_2+ib_2}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = (a_1+ib_1)\left(\frac{1}{a_2+ib_2}\right)$$

$$\Rightarrow \quad \frac{z_1}{z_2} = (a_1+ib_1)\left(\frac{a_2}{a_2^2+b_2^2}+i\frac{(-b_2)}{a_2^2+b_2^2}\right)$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+i\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)^2+\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)^2}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1a_2+b_1b_2)^2+(a_2b_1-a_1b_2)^2}{(a_2^2+b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{a_1^2a_2^2+b_1^2b_2^2+a_2^2b_1^2+a_1^2b_2^2}{(a_2^2+b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1^2+b_1^2)(a_2^2+b_2^2)}{(a_2^2+b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{a_1^2+b_1^2}{a_2^2+b_2^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}} = \frac{|z_1|}{|z_2|}$$

(vii) We have,

$|z_1+z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)}$ $\quad [\because z\bar{z}=|z|^2]$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$ $[\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}]$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + (\overline{z_1\overline{z_2}})$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2})$ $[\because z + \overline{z} = 2\,\mathrm{Re}(z)]$

(viii) We have,

$|z_1 - z_2|^2$

$= (z_1 - z_2)(\overline{z_1 - z_2})$ $[\because z\overline{z} = |z|^2]$

$= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$ $[\because \overline{z_1 + z_2} = \overline{z_1} - \overline{z_2}]$

$= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1}$

$= |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - (\overline{z_1\overline{z_2}})$

$= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z_2})$

(ix) Adding (vii) and (viii), we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(x) We have,

$|az_1 - bz_2|^2$

$= (az_1 - bz_2)(\overline{az_1 - bz_2})$

$= (az_1 - bz_2)(a\overline{z_1} - b\overline{z_2})$

$= a^2 z_1\overline{z_1} - az_1 b\overline{z_2} - bz_2 a\overline{z_1} + b^2 z_2\overline{z_2}$

$= a^2|z_1|^2 - ab\{z_1\overline{z_2} + \overline{z_1}z_2\} + b^2|z_2|^2$

$= a^2|z_1|^2 - ab\left\{z_1\overline{z_2} + (\overline{z_1\overline{z_2}})\right\} + b^2|z_2|^2$

$= a^2|z_1|^2 - ab\left\{2\,\mathrm{Re}(z_1\overline{z_2})\right\} + b^2|z_2|^2$

$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\overline{z_2}) + b^2|z_2|^2$

Similarly, we have

$|bz_1 + az_2|^2$

$= b^2|z_1|^2 + a^2|z_2|^2 + 2ab(z_1\overline{z_2})$

$\therefore |az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\overline{z_2}) + b^2|z_2|^2$

$+ b^2|z_1|^2 + 2ab\,\mathrm{Re}(z_1\overline{z_2}) + a^2|z_2|^2$

$= |z_1|^2(a^2 + b^2) + |z_2|^2(b^2 + a^2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(xi) We have,

$|z_1 + z_2|^2$

$= (z_1 + z_2)(\overline{z_1 + z_2})$ $[\because z\overline{z} = |z|^2]$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$ $[\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}]$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + (\overline{z_1\overline{z_2}})$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2})$

$\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}|$ $\left[\because \mathrm{Re}(z_1\overline{z_2}) \le |z_1\overline{z_2}|\right]$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\therefore |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|.$

(xii) We have,

$|z_1 - z_2|^2$

$= (z_1 - z_2)(\overline{z_1 - z_2})$

$= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - \overline{z_1}z_2$

$= |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - (\overline{z_1\overline{z_2}})$

$= |z_1|^2 + |z_2|^2 - 2|z_1\overline{z_2}|$

$\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}|$ $[\because -|z| \le \mathrm{Re}(z) \le |z|]$

$\therefore |z_1 - z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z_2}|$

$\Rightarrow |z_1 - z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|.$

$\Rightarrow |z_1 - z_2|^2 \le (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 - z_2| \le |z_1| + |z_2|$

(xiii) We have,

$|z_1 + z_2|^2$

$= (z_1 + z_2)(\overline{z_1 + z_2})$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + (\overline{z_1\overline{z_2}})$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2})$

$\ge |z_1|^2 + |z_2|^2 - 2|z_1\overline{z_2}|$ $[\because \mathrm{Re}(z) \ge -|z|]$

$\therefore |z_1 + z_2|^2 \ge |z_1|^2 + |z_2|^2 - 2|z_1||\overline{z_2}|$

$\Rightarrow |z_1 + z_2|^2 \ge |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$\Rightarrow |z_1 + z_2|^2 \ge (|z_1| - |z_2|)^2$

$\Rightarrow \quad |z_1+z_2| \geq |z_1|-|z_2|.$

(xiv) Proceed as in (xiii).

(xv) Proceed as earlier.

19.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z}=\frac{1}{a+ib}$$

$$=\frac{1}{a+ib}\times\frac{a-ib}{a-ib}$$

$$=\frac{a-ib}{a^2-i^2b^2}$$

$$=\frac{a-ib}{a^2+b^2}$$

$$=\frac{a}{a^2+b^2}+\frac{i(-b)}{a^2+b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z. Also,

$$\frac{1}{z}=\frac{a-ib}{a^2+b^2}=\frac{\bar{z}}{|z|^2}.$$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\text{Re}(z)}{|z|^2}+i\frac{(-\text{Im}(z))}{|z|^2}=\frac{\bar{z}}{|z|^2}.$$

19.13 SQUARE ROOTS OF A COMPLEX NUMBER

Let $a + ib$ be a complex number such that $\sqrt{a+ib}=x+iy$, where x and y are real numbers. Then,

$$\sqrt{a+ib}=x+iy$$

$$\Rightarrow \quad (a+ib) = (x+iy)^2$$

$$\Rightarrow \quad a+ib = (x^2-y^2)+2ixy$$

$$\Rightarrow \quad x^2-y^2 = a \qquad \text{...(i)}$$

$$\text{and,} \quad 2xy = b \qquad \text{...(ii)}$$

$$\text{Now,} \quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = a^2+b^2$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{a^2+b^2} \qquad \text{...(iii)}$$

Solving the equations (i) and (ii), we get

$$x^2=\left\{\frac{1}{2}\right\}\left\{\sqrt{a^2+b^2}+a\right\} \text{ and, } y^2=\left\{\frac{1}{2}\right\}\left\{\sqrt{a^2+b^2}-a\right\}$$

$$\Rightarrow \quad x=\pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} \text{ and, } y=\pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}$$

If b is positive, then by equation (ii), x and y are of the same sign. Hence,

$$\sqrt{a+ib}=\pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}}+i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

If b is negative, then by equation (ii), x and y are of different signs. Hence,

$$\sqrt{a+ib}=\pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}}-i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right\}$$

REMARK *To find the square root of $a - ib$, replace i by $-i$ in the above results.*

ILLUSTRATION *Find the square root of $7 - 24i$*

SOLUTION Let $\sqrt{7-24i}=x+iy$. Then,

$$\sqrt{7-24i} = x+iy$$

$$\Rightarrow \quad 7-24i = (x+iy)^2$$

$$\Rightarrow \quad 7-24i = (x^2-y^2)+2ixy$$

$$\Rightarrow \quad x^2-y^2 = 7 \qquad \text{...(i)}$$

$$\text{and,} \quad 2xy = -24 \qquad \text{...(ii)}$$

Now,

$$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = 49+576 = 625$$

$$\Rightarrow \quad x^2+y^2=25 \qquad \text{...(iii)}$$

On solving (i) and (iii), we get

$$x^2 = 16 \text{ and } y^2 = 9$$

$$\Rightarrow \quad x = \pm 4 \text{ and } y = \pm 3$$

From (ii), we find that $2xy$ is negative. So, x and y are of opposite signs.

$$\therefore \quad (x = 4 \text{ and } y = -3) \text{ or, } (x = -4 \text{ and } y = 3)$$

Hence, $\sqrt{7-24i} = \pm(4-3i)$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *The sum and product of two complex numbers are real if and only if they are conjugate of each other.*

SOLUTION First, let the two complex numbers be conjugate of each other. Let them be $z_1 = a + ib$ and $z_2 = a - ib$. Then,

$$z_1+z_2 = (a+ib)+(a-ib) = 2a, \text{ which is real.}$$

Also,

$$z_1z_2 = (a+ib)(a-ib) = a^2-i^2b^2 = a^2+b^2, \text{ which is real.}$$

Conversely, let z_1 and z_2 be two complex numbers such that their sum $z_1 + z_2$ and product $z_1 z_2$ both are real. Then, we have to prove that z_1 and z_2 are conjugate of each other.

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$. Then,

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

Now,

$z_1 + z_2$ and $z_1 z_2$ are real

$\Rightarrow (a_1 + a_2) + i(b_1 + b_2)$

and,

$(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ are real

$\Rightarrow b_1 + b_2 = 0$ and $a_1 b_2 + a_2 b_1 = 0$ $[\because z \text{ is real} \Leftrightarrow \text{Im}(z) = 0]$

$\Rightarrow b_2 = -b_1$ and $a_1 b_2 + a_2 b_1 = 0$

$\Rightarrow b_2 = -b_1$ and $-a_1 b_1 + a_2 b_1 = 0$

$\Rightarrow b_2 = -b_1$ and $(a_2 - a_1) b_1 = 0$

$\Rightarrow b_2 = -b_1$ and $a_2 - a_1 = 0$

$\Rightarrow b_2 = -b_1$ and $a_2 = a_1$

$\Rightarrow z_2 = a_2 + i b_2 = a_1 - i b_1$

$\Rightarrow z_2 = \bar{z}_1$

$\Rightarrow$ z_1 and z_2 are conjugate of each other

EXAMPLE 2 *Find the value of* $x^3 + 7x^2 - x + 16$, *when* $x = 1 + 2i$.

SOLUTION We have,

$x = 1 + 2i$

$\Rightarrow x - 1 = 2i$

$\Rightarrow (x-1)^2 = 4i^2$

$\Rightarrow x^2 - 2x + 1 = -4$

$\Rightarrow x^2 - 2x + 5 = 0.$

Now,

$x^3 + 7x^2 - x + 16$

$= x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$

$= x(0) + 9(0) + 12x - 29$ $[\because x^2 - 2x + 5 = 0]$

$= 12(1 + 2i) - 29$ $[\because x = 1 + 2i]$

$= -17 + 24i$

Hence, the value of the given polynomial when $x = 1 + 2i$ is $-17 + 24i$.

EXAMPLE 3 *Prove that*

$x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

SOLUTION We have,

$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$

$= \{(x+1)^2 - i^2\}\{(x-1)^2 - i^2\}$

$= \{(x+1)^2 + 1\}\{(x-1)^2 + 1\}$

$= \{x^2 + 2x + 2\}(\{x^2 - 2x + 2\}$

$= \{x^2 + 2 + 2x\}\{x^2 + 2 - 2x\}$

$= (x^2 + 2)^2 - (2x)^2$

$= x^4 + 4x^2 + 4 - 4x^2$

$= x^4 + 4$

EXAMPLE 4 *If* $f(z)$ *is a polynomial in* z *such that when it is divided by* $z - i$ *and* $z + i$, *the remainders are respectively* i *and* $1 + i$. *Determine the remainder when* $f(z)$ *is divided by* $z^2 + 1$.

SOLUTION Recall that *if* a polynomial $\phi(x)$ is divided by $(x - a)$, then the remainder is $\phi(a)$. Also, *if* $\phi(x)$ is divided by a quadratic, then the remainder is a linear expression.

It is given that i and $1 + i$ are remainders when $f(z)$ is divided by $z - i$ and $z + i$ respectively.

$\therefore$ $f(i) = i$ and $f(-i) = 1 + i$

Let $(az + b)$ be the remainder when $f(z)$ is divided by $z^2 + 1$ and let the quotient be $Q(z)$. Then,

$f(z) = (z^2 + 1) Q(z) + az + b$

$\Rightarrow$ $f(i) = 0 + ai + b$

and,

$f(-i) = 0 + a(-i) + b$

$\Rightarrow$ $i = ai + b$ and $1 + i = -ai + b$ $[\because f(i) = i \; f(-i) = 1 + i]$

Solving these two equations, we get :

$$a = \frac{1}{2} i \text{ and } b = \frac{1}{2} + i.$$

Hence, remainder $= az + b$

$$= \frac{1}{2} iz + \frac{1}{2} + i$$

EXAMPLE 5 *Find the least positive value of* n, *if* $\left(\frac{1+i}{1-i}\right)^n = 1$.

SOLUTION We have,

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1-i^2}$$

$$= \frac{1 + 2i + i^2}{1 - i^2}$$

$$= \frac{1 + 2i - 1}{1 + 1} = i$$

$\therefore$ $$\left(\frac{1+i}{1-i}\right)^n = 1$$

$\Rightarrow \quad i^n = 1$

$\Rightarrow$ n is a multiple of 4

$\Rightarrow$ The smallest positive value of n is 4

EXAMPLE 6 *If* $z = x + iy$ *and* $w = \dfrac{1 - iz}{z - i}$, *show that* $|w| = 1 \Rightarrow z$ *is purely real.*

SOLUTION We have,

$$|w| = 1$$

$$\Rightarrow \quad \left|\frac{1 - iz}{z - i}\right| = 1$$

$$\Rightarrow \quad \frac{|1 - iz|}{|z - i|} = 1$$

$$\Rightarrow |1 - iz| = |z - i|$$

$$\Rightarrow \quad |1 - i(x + iy)| = |x + iy - i)|, \text{ where } z = x + iy$$

$$\Rightarrow \quad |1 + y - ix| = |x + i(y - 1)|$$

$$\Rightarrow \quad \sqrt{(1 + y)^2 + (-x)^2} = \sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow \quad (1 + y)^2 + x^2 = x^2 + (y - 1)^2$$

$$\Rightarrow \quad y = 0$$

$$\Rightarrow \quad z = x + i0 = x, \text{which is purely real}$$

EXAMPLE 7 *If* $(x + iy)^{1/3} = a + ib$, *prove that*

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2).$$

SOLUTION We have,

$$(x + iy)^{1/3} = a + ib$$

$$\Rightarrow \quad x + iy = (a + ib)^3$$

$$\Rightarrow \quad x + iy = a^3 + i^3 b^3 + 3a^2 ib + 3a i^2 b^2$$

$$\Rightarrow \quad x + iy = (a^3 - 3ab^2) + i(3a^2 b - b^3)$$

$$\Rightarrow \quad x = a^3 - 3ab^2 \text{ and } y = 3a^2 b - b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2)$$

EXAMPLE 8 *Prove the following identities :*

(i) $(x^2 + a^2)^4 = (x^4 - 6x^2 a^2 + a^4)^2 + (4x^3 a - 4xa^3)^2$

(ii) $(x^2 + a^2)^7 = (x^7 - 21x^5 a^2 + 35x^3 a^4 - 7x a^6) + (7x^6 a - 35x^4 a^3 + 21x^2 a^5 - a^7)^2$

SOLUTION (i) We have,

$$(x^2 + a^2)^4$$

$$= \{(x + ai)(x - ai)\}^4$$

$$= (x + ai)^4 (x - ai)^4$$

$$= \{{}^4C_0 x^4 + {}^4C_1 x^3 (ai) + {}^4C_2 x^2 (ai)^2 + {}^4C_3 x (ai)^3 + {}^4C_4 (ai)^4\}$$
$$\times \{{}^4C_0 x^4 - {}^4C_1 x^3 (ai) + {}^4C_2 x^2 (ai)^2 - {}^4C_3 x (ai)^3 + {}^4C_4 (ai)^4\}$$

$$= \{(x^4 - 6x^2 a^2 + a^4) + i(4x^3 a - 4xa^3)\}$$
$$\times \{(x^4 - 6x^2 a^2 + a^4) - i(4x^3 a - 4xa^3)\}$$

$$= (x^4 - 6x^2 a^2 + a^4)^2 + (4x^3 a - 4xa^3)^2$$

(ii) We have,

$$(x^2 + a^2)^7$$

$$= \{(x + ai)^7 (x - ai)^7\}$$

$$= (x + ai)^7 (x - ai)^7$$

$$= \{{}^7C_0 x^7 + {}^7C_1 x^6 (ai) + {}^7C_2 x^5 (ai)^2 + {}^7C_3 x^4 (ai)^3 + {}^7C_4 x^3 (ai)^4$$
$$+ {}^7C_5 x^2 (ai)^5 + {}^7C_6 x (ai)^6 + {}^7C_7 (ai)^7\}$$
$$\times \{{}^7C_0 x^7 - {}^7C_1 x^6 ai + {}^7C_2 x^5 (ai)^2 - {}^7C_3 x^4 (ai)^3 + {}^7C_4 x^3 (ai)^4$$
$$- {}^7C_5 x^2 (ai)^5 + {}^7C_6 x (ai)^6 - {}^7C_7 (ai)^7\}$$

$$= \{(x^7 - 25x^5 a^2 + 35x^3 a^4 - 7x a^6) + i(7x^6 a - 35x^4 a^3 + 21x^2 a^5 - a^7)$$

$$\times \{(x^7 - 25x^5 a^2 + 35x^3 a^4 - 7xa^6) - i(7x^6 a - 35x^4 a^3 + 21x^2 a^5 - a^7)\}$$

$$= \left(x^7 - 21x^5 a^2 + 35x^3 a^4 - 7xa^6\right)^2$$
$$+ \left(7x^6 a - 35x^4 a^3 + 21x^2 a^5 - a^7\right)^2$$

EXAMPLE 9 *For any two complex numbers* z_1 and z_2, *prove that*

(i) $\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right| = |z_1 + z_2| + |z_1 - z_2|$

(ii) $|z_1| + |z_2| = \left|\frac{1}{2}(z_1 + z_2) + \sqrt{z_1 z_2}\right| + \left|\frac{1}{2}(z_1 + z_2) - \sqrt{z_1 z_2}\right|$

(iii) $\left|1 - \bar{z}_1 z_2\right|^2 - \left|z_1 - z_2\right|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$

SOLUTION (i) We have,

$$\left\{\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|\right\}^2$$

$$= \left|z_1 + \sqrt{z_1^2 - z_2^2}\right|^2 + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|^2$$
$$+ 2\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|$$

$$= 2\left\{|z_1|^2 + \left|\sqrt{z_1^2 - z_2^2}\right|^2\right\}$$
$$+ 2\left|(z_1 + \sqrt{z_1^2 - z_2^2})(z_1 - \sqrt{z_1^2 - z_2^2})\right|$$

$$\left[\because \left|\,|z_1+z_2|\,\right|^2+\left|\,|z_1-z_2|\,\right|^2=2\left(|z_1|^2+|z_2|^2\right)\right]$$

$$=2\left\{|z_1|^2+|z_1^2-z_2^2|\right\}+2\left|z_1^2-(z_1^2-z_2^2)\right|$$

$$=2\left\{\left|z_1\right|^2+\left|(z_1+z_2)(z_1-z_2)\right|\right\}+2\left|z_2^2\right|$$

$$=2\left\{|z_1|^2+|z_1+z_2|\,|z_1-z_2|\right\}+2|z_2^2|$$

$$=2\left\{|z_1|^2+|z_2|^2\right\}+2|z_1+z_2|\,|z_1-z_2|$$

$$=\left\{|z_1+z_2|^2+|z_1-z_2|^2\right\}+2|z_1+z_2|\,|z_1-z_2|$$

$$=\left\{|z_1+z_2|+|z_1-z_2|\right\}^2$$

$$\therefore \left|z_1+\sqrt{z_1^2-z_2^2}\right|+\left|\,|z_1-\sqrt{z_1^2-z_2^2}\,\right| = \left|z_1+z_2\right|+\left|z_1-z_2\right|$$

(ii) We have,

$$\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|$$

$$=\frac{1}{2}\left|z_1+z_2+2\sqrt{z_1z_2}\right|+\frac{1}{2}\left|z_1+z_2-2\sqrt{z_1z_2}\right|$$

$$=\frac{1}{2}\left|(\sqrt{z_1}+\sqrt{z_2})^2\right|+\frac{1}{2}\left|(\sqrt{z_1}-\sqrt{z_2})^2\right|$$

$$=\frac{1}{2}\left|\sqrt{z_1}+\sqrt{z_2}\right|^2+\frac{1}{2}\left|\sqrt{z_1}-\sqrt{z_2}\right|^2$$

$$=\frac{1}{2}\left\{\left|\sqrt{z_1}+\sqrt{z_2}\right|^2+\left|\sqrt{z_1}-\sqrt{z_2}\right|^2\right\}$$

$$=\frac{1}{2}\left\{2\left(|\sqrt{z_1}|^2+|\sqrt{z_2}|^2\right)\right\}$$

$$=|\sqrt{z_1}|^2+|\sqrt{z_2}|^2$$

$$=|(\sqrt{z_1})^2|+|(\sqrt{z_2})^2|$$

$$=|z_1|+|z_2|$$

Hence,

$$|z_1|+|z_2|=\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|$$

(iii) We have,

$$|1-\bar{z}_1z_2|^2-|z_1-z_2|^2$$

$$=\left\{1+|\bar{z}_1z_2|^2-2\,\mathrm{Re}\,(\bar{z}_1z_2)\right\}-\left\{|z_1|^2+|z_2|^2-2\,\mathrm{Re}\,(\bar{z}_1z_2)\right\}$$

$$=1+(|\bar{z}_1|\,|z_2|)^2-|z_1|^2-|z_2|^2$$

$$=1+|z_1|^2|z_2|^2-|z_1|^2-|z_2|^2$$

$$=(1-|z_1|^2(1-|z_2|^2)$$

Hence,

$$|1-\bar{z}_1z_2|^2-|z_1-z_2|^2=(1-|z_1|^2)(1-|z_2|^2).$$

EXAMPLE 10 *Let z_1, z_2, z_3 be three complex numbers such that $|z_1|=1, |z_2|=2, |z_3|=3$ and $|z_1+z_2+z_3|=1$. Find $|9z_1z_2+4z_1z_3+z_2z_3|$.*

SOLUTION We have,

$$|z_1+z_2+z_3| = 1$$

$$\Rightarrow |z_1+z_2+z_3|^2 = 1$$

$$\Rightarrow (z_1+z_2+z_3)\overline{(z_1+z_2+z_3)} = 1$$

$$\Rightarrow (z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3) = 1$$

$$\Rightarrow |z_1|^2+|z_2|^2+|z_3|^2+(z_1\bar{z}_2+z_2\bar{z}_1+z_1\bar{z}_3+\bar{z}_1z_3+z_2\bar{z}_3+\bar{z}_2z_3)=1$$

$$\Rightarrow z_1\bar{z}_2+\bar{z}_1z_2+z_1\bar{z}_3+\bar{z}_1z_3+z_2\bar{z}_3+\bar{z}_2z_3=1-1-4-9$$

$$\Rightarrow z_1\bar{z}_2+\bar{z}_1z_2+z_1\bar{z}_3+\bar{z}_1z_3+z_2\bar{z}_3+\bar{z}_2z_3=-13 \quad ...(i)$$

Now,

$$|9z_1z_2+4z_1z_3+z_2z_3|^2$$

$$=(9z_1z_2+4z_1z_3+z_2z_3)\overline{(9z_1z_2+4z_1z_3+z_2z_3)}$$

$$=(9z_1z_2+4z_1z_3+z_2z_3)(9\bar{z}_1\bar{z}_2+4\bar{z}_1\bar{z}_3+\bar{z}_2\bar{z}_3)$$

$$=81|z_1|^2|z_2|^2+16|z_1|^2|z_3|^2+|z_2|^2|z_3|^2$$

$$+36|z_1|^2z_2\bar{z}_3+9|z_2|^2z_1\bar{z}_3+36|z_1|^2\bar{z}_2z_3$$

$$+4z_1\bar{z}_2|z_3|^2+9z_3\bar{z}_1|z_2|^2+4z_2\bar{z}_1|z_3|^2$$

$$=81\times4+16\times9+4\times9+36(z_2\bar{z}_3+z_1\bar{z}_3+\bar{z}_2z_3+z_1\bar{z}_2+\bar{z}_1z_3+z_2\bar{z}_1)$$

$$=324+144+36+36\times-13 \quad \text{[Using (i)]}$$

$$=36$$

Hence,

$$|9z_1z_2+4z_1z_3+z_2z_3| = 6.$$

EXAMPLE 11 *For any two complex numbers z_1, z_2, prove that*

(i) $\left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\le 2\left(|z_1|+|z_2|\right)$, $z_2\neq 0$

(ii) $\left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right|<1$ if $|z_1|<1$ and $|z_2|>1$

(iii) $|z_1+z_2|^2 \le (1+\lambda)|z_1|^2+\left(1+\frac{1}{\lambda}\right)|z_2|^2$

SOLUTION Using Triangle inequality, we have,

$$\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le \left|\frac{z_1}{|z_1|}\right|+\left|\frac{z_2}{|z_2|}\right|$$

$$\Rightarrow \left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le \frac{|z_1|}{|z_1|}+\frac{|z_2|}{|z_2|}$$

$$\Rightarrow \left|\frac{z_1}{|z_1|}+\frac{z_2}{z_2}\right| \le 2$$

$$\therefore (|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \le 2(|z_1|+|z_2|)$$

(ii) We have,

$$\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| < 1$$

$$\Rightarrow \frac{|z_1-z_2|}{|1-\overline{z_1}z_2|} < 1$$

$$\Rightarrow |z_1-z_2| < |1-\overline{z_1}z_2|$$

$$\Rightarrow |z_1-z_2|^2 < |1-\overline{z_1}z_2|^2$$

$$\Rightarrow |z_1|^2+|z_2|^2-2\text{Re}(\overline{z_1}z_2) < 1^2+|\overline{z_1}z_2|^2-2\,\text{Re}(\overline{z_1}z_2)$$

$$\Rightarrow |z_1|^2+|z_2|^2 < 1+|\overline{z_1}|^2|z_2|^2$$

$$\Rightarrow |z_1|^2+|z_2|^2-|z_1|^2|z_2|^2-1<0$$

$$\Rightarrow (|z_1|^2-1)(|z_2|^2-1)<0$$

$$\Rightarrow |z_1|^2<1 \text{ and } |z_2|^2>1 \text{ or, } |z_1|^2>1 \text{ and } |z_2|^2<1$$

$$\Rightarrow |z_1|<1 \text{ and } |z_2|>1 \text{ or, } |z_1|>1 \text{ and } |z_2|<1$$

Hence,

$$\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|<1$$

if $|z_1|<1$ and $|z_2|>1$ or, $|z_1|>1$ and $|z_2|<1$

(iii) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,\text{Re}(z_1\overline{z_2})$$

$$\le |z_1|^2+|z_2|^2+2|z_1\overline{z_2}| \qquad [\because \text{Re}(z) \le |z|]$$

$$= |z_1|^2+|z_2|^2+2|z_1||z_2| \qquad \ldots(i)$$

Now,

A.M $\ge$ G.M

$$\Rightarrow \frac{\lambda|z_1|^2+1/\lambda|z_2|^2}{2} \ge \sqrt{\lambda|z_1|^2.1/\lambda|z_2|^2}$$

$$\Rightarrow \lambda|z_1|^2+\frac{1}{\lambda}|z_2|^2 \ge 2|z_1||z_2|$$

$$\Rightarrow 2|z_1||z_2| \le \lambda|z_1|^2+\frac{1}{\lambda}|z_2|^2$$

$$\Rightarrow |z_1|^2+|z_2|^2+2|z_1||z_2| \le |z_1|^2+|z_2|^2+\lambda|z_1|^2+\frac{1}{\lambda}|z_2|^2$$

$$\Rightarrow |z_1+z_2|^2 \le (1+\lambda)|z_1|^2+\left(1+\frac{1}{\lambda}\right)|z_2|^2 \quad \text{[Using (i)]}$$

EXAMPLE 12 *If $z_1, z_2, \ldots, z_n$ are complex numbers such that $|z_1| = |z_2| = \ldots = |z_n| = 1$, prove that*

$$|z_1+z_2+\ldots+z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right|$$

SOLUTION We have,

$$|z_1| = |z_2| = \ldots = |z_n| = 1$$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \ldots = |z_n|^2 = 1$$

$$\Rightarrow z_1\overline{z_1} = z_2\overline{z_2} = \ldots = z_n\overline{z_n} = 1$$

$$\Rightarrow \overline{z_1} = \frac{1}{z_1}, \overline{z_2} = \frac{1}{z_2}, \ldots \overline{z_n} = \frac{1}{z_n} \qquad \ldots(i)$$

Now,

$$|z_1+z_2+\ldots+z_n|$$

$$= |\overline{z_1+z_2+\ldots+z_n}|$$

$$= |\overline{z_1}+\overline{z_2}+\ldots+\overline{z_n}|$$

$$= \left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right| \qquad \text{[Using (i)]}$$

EXAMPLE 13 *Let $a, b \in R$ and z_1, z_2 be complex numbers such that*

$$\left|\frac{az_1-bz_2}{ab-z_1\overline{z_2}}\right| = 1, \textit{find } z_2, \textit{ if } |z_1| \ne b.$$

SOLUTION We have,

$$\left|\frac{az_1-bz_2}{ab-z_1\overline{z_2}}\right| = 1$$

$$\Rightarrow \left|\frac{az_1-bz_2}{ab-z_1\overline{z_2}}\right|^2 = 1$$

$$\Rightarrow \left(\frac{az_1-bz_2}{ab-z_1\overline{z_2}}\right)\left(\overline{\frac{az_1-bz_2}{ab-z_1\overline{z_2}}}\right) = 1$$

$$\Rightarrow \left(\frac{az_1-bz_2}{ab-z_1\overline{z_2}}\right)\left(\frac{a\overline{z_1}-b\overline{z_2}}{ab-\overline{z_1}z_2}\right) = 1$$

$$\Rightarrow (az_1-bz_2)(a\overline{z_1}-b\overline{z_2}) = (ab-z_1\overline{z_2})(ab-\overline{z_1}z_2)$$

$\Rightarrow a^2 z_1\bar{z}_1 - ab\,z_1\bar{z}_2 - ab\,z_2\bar{z}_1 + b^2 z_2\bar{z}_2$

$= a^2 b^2 - ab(\bar{z}_1 z_2 + z_1\bar{z}_2) + z_1\bar{z}_1 z_2\bar{z}_2$

$\Rightarrow a^2 |z_1|^2 + b^2 |z_2|^2 = a^2 b^2 + |z_1|^2 |z_2|^2$

$\Rightarrow a^2 b^2 - a^2 |z_1|^2 - b^2 |z_2|^2 + |z_1|^2 |z_2|^2 = 0$

$\Rightarrow (a^2 - |z_2|^2)(b^2 - |z_1|^2) = 0$

$\Rightarrow a^2 - |z_2|^2 = 0$ $[\because |z_1| \neq b]$

$\Rightarrow |z_2| = a$

Hence, z_2 is any complex number of modulus a.

EXAMPLE 14 *Let z be a complex number for which $|2z\cos\theta + z^2| > 1$, if $|z| < k$, find k.*

SOLUTION We have,

$|2z\cos\theta + z^2|$

$< |2z\cos\theta| + |z^2|$ $\left[\because |z_1 + z_2| \leq |z_1| + |z_2|\right]$

$< 2|z|\,|\cos\theta| + |z|^2$

$< 2|z| + |z|^2$

$< 2k + k^2$ $[\because |z| < k \text{ (given)}]$

But, $|2z\cos\theta + z^2| > 1$. Therefore,

$2k + k^2 > 1$

$\Rightarrow k^2 + 2k - 1 > 0$

$\Rightarrow k > \sqrt{2} - 1$

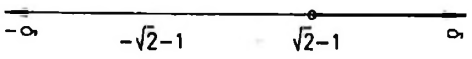


Fig. 19.1

EXAMPLE 15 *Let z_1, z_2 be complex numbers such that $\left|\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$. Find $|z_1|$.*

SOLUTION We have,

$\left|\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow \left|\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right|^2 = 1$

$\Rightarrow \left(\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right)\left(\overline{\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}}\right) = 1$

$\Rightarrow \left(\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right)\left(\frac{\bar{z}_1 - 3\bar{z}_2}{3 - \bar{z}_1 z_2}\right) = 1$

$\Rightarrow (z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1\bar{z}_2)(3 - \bar{z}_1 z_2)$

$\Rightarrow z_1\bar{z}_1 + 9z_2\bar{z}_2 - 3z_1\bar{z}_2 - 3\bar{z}_1 z_2$

$= 9 - 3z_1\bar{z}_2 - 3\bar{z}_1 z_2 + z_1\bar{z}_2 \cdot \bar{z}_1 z_2$

$\Rightarrow |z_1|^2 + 9|z_2|^2 = 9 + |z_1|^2 |z_2|^2$

$\Rightarrow 9 - |z_1|^2 - 9|z_2|^2 - |z_1|^2 |z_2|^2 = 0$

$\Rightarrow (1 - |z_2|^2)(9 - |z_1|^2 = 0$

$\Rightarrow 9 - |z_1|^2 = 0$ $[\because |z_2| \neq 1]$

$\Rightarrow |z_1| = 3$

EXAMPLE 16 *For complex numbers z and w, prove that*

$|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or, $z\bar{w} = 1$.

[IIT 1999]

SOLUTION We have,

$|z|^2 w - |w|^2 z = z - w$

$\Leftrightarrow |z|^2 w - |w|^2 z - z + w = 0$

$\Leftrightarrow (|z|^2 + 1)w - (|w|^2 + 1)z = 0$

$\Leftrightarrow (|z|^2 + 1)w = (|w|^2 + 1)z$

$\Leftrightarrow \frac{z}{w} = \frac{|z|^2 + 1}{|w|^2 + 1}$

$\Leftrightarrow \frac{z}{w}$ is purely real $\left[\because \frac{|z|^2 + 1}{|w|^2 + 1} \text{ is real}\right]$

$\Leftrightarrow \frac{z}{w} = \left(\frac{z}{w}\right)$

$\Leftrightarrow \frac{z}{w} = \frac{\bar{z}}{\bar{w}}$

$\Leftrightarrow z\bar{w} = \bar{z}w$ ...(i)

Again,

$|z|^2 w - |w|^2 z = z - w$

$\Leftrightarrow z\bar{z}w - w\bar{w}z = z - w$

$\Leftrightarrow z\bar{z}w - w\bar{w}z - z + w = 0$

$\Leftrightarrow z(\bar{z}w - 1) - w(z\bar{w} - 1) = 0$

$\Leftrightarrow z(z\bar{w} - 1) - w(z\bar{w} - 1) = 0$ [Using (i)]

$\Leftrightarrow (z - w)(z\bar{w} - 1) = 0$

$\Leftrightarrow z = w$ or, $z\bar{w} = 1$

Hence,

$|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$.

EXAMPLE 17 *If $a \geq 1$, find all complex numbers z which satisfy the equation $z + a|z + 1| + i = 0$*

SOLUTION Let $z = x + iy$. Then,

$$z + a\left|z+1\right| + i = .0$$

$$\Rightarrow (x+iy) + a\left|(x+1)+iy\right| + i = 0$$

$$\Rightarrow (x+iy) + a\sqrt{(x+1)^2+y^2} + i = 0$$

$$\Rightarrow \{x + a\sqrt{(x+1)^2+y^2}\} + i\{y+1\} = 0$$

$$\Rightarrow x + a\sqrt{(x+1)^2+y^2} = 0 \text{ and } y+1 = 0$$

$$\Rightarrow y = -1 \text{ and } x + a\sqrt{(x+1)^2+y^2} = 0$$

$$\Rightarrow x + a\sqrt{(x+1)^2+1^2} = 0 \qquad \text{...(i)}$$

Now, $a \geq 1$ (given) and $\sqrt{(x+1)^2+1^2} > 0$ for all real x. Therefore, equation (i) is true only when $x < 0$.

Now,

$$x + a\sqrt{(x+1)^2+1} = 0$$

$$\Rightarrow a\sqrt{(x+1)^2+1} = -x$$

$$\Rightarrow a\sqrt{x^2+2x+2} = -x$$

$$\Rightarrow a^2(x^2+2x+2) = x^2$$

$$\Rightarrow (a^2-1)x^2 + 2a^2x + 2a^2 = 0$$

If $a = 1$, then

$$(a^2-1)x^2 + 2a^2x + 2a^2 = 0$$

$$\Rightarrow 2x + 2 = 0$$

$$\Rightarrow x = -1.$$

Now, let $a > 1$.

In this case, we have

$$(a^2-1)x^2 + 2a^2x + 2a^2 = 0$$

$$\Rightarrow x = \frac{-2a^2 \pm \sqrt{4a^4 - 8a^2(a^2-1)}}{2(a^2-1)}$$

$$\Rightarrow x = \frac{-a^2 \pm a\sqrt{2-a^2}}{a^2-1}$$

For x to be real, we must have

$$2 - a^2 > 0 \Rightarrow a^2 - 2 < 0$$

$$\Rightarrow -\sqrt{2} \leq a \leq \sqrt{2}$$

$$\Rightarrow 1 < a \leq \sqrt{2} \qquad [\because a > 1]$$

Thus, for $1 < a \leq \sqrt{2}$, we have

$$x = -\frac{a^2 \pm a\sqrt{2-a^2}}{a^2-1} \text{ and } y = -1$$

$$\Rightarrow z = x + iy = \left\{\frac{-a^2 \pm a\sqrt{2-a^2}}{a^2-1}\right\} - iy$$

For $a = 1$, we have

$$x = -1 \text{ and } y = -1$$

$$\therefore z = x + iy = -1 - i$$

For $a > \sqrt{2}$, x is not real. So, the equation has no solution.

EXAMPLE 18 *If $a \geq 0$, find all the complex numbers z which satisfy the equation $2\,|\,z\,| - 4az + 1 + ia = 0$.*

SOLUTION If $a = 0$, the given equation reduces to $2\,|\,z\,| + 1 = 0$, which is not possible. So, let $a > 0$. Let $z = x + iy$. Then,

$$2\,|\,z\,| - 4az + 1 + ia = 0$$

$$\Rightarrow 2\sqrt{x^2+y^2} - 4a(x+iy) + 1 + ia = 0$$

$$\Rightarrow \{2\sqrt{x^2+y^2} - 4ax + 1\} + i\{-4ay + a\} = 0$$

$$\Rightarrow 2\sqrt{x^2+y^2} - 4ax + 1 = 0 \qquad \text{...(i)}$$

and, $-4ay + a = 0$ ...(ii)

Now,

$$-4ay + a = 0$$

$$\Rightarrow a(1-4y) = 0$$

$$\Rightarrow 1 - 4y = 0$$

$$\Rightarrow y = \frac{1}{4} \qquad [\because a \neq 0]$$

Putting $y = \frac{1}{4}$ in (i), we get

$$2\sqrt{x^2 + \frac{1}{16}} - 4ax + 1 = 0 \qquad \text{...(iii)}$$

$$\Rightarrow 4(1-4a^2)x^2 + 8ax - \frac{3}{4} = 0 \qquad \text{...(iv)}$$

$$\Rightarrow x = \frac{4a \pm \sqrt{4a^2+3}}{4(4a^2-1)},$$

provided that $4a^2 - 1 \neq 0$ i.e. $a \neq \frac{1}{2}$ $[\because a > 0]$

When $a = \frac{1}{2}$, we have

$$4x - \frac{3}{4} = 0 \qquad \text{[Putting } a = 1/2 \text{ in (iv)]}$$

$$\Rightarrow x = \frac{3}{16}$$

But, $x = \frac{3}{16}$ and $a = \frac{1}{2}$ do not satisfy (iii). So, the given equation has no solution for $a = \frac{1}{2}$.

When $a \neq \frac{1}{2}$, we have

$$x = \frac{4a \pm \sqrt{4a^2+3}}{4(4a^2-1)} \text{ and } y = \frac{1}{4}$$

From (iii), we have

$$2\sqrt{x^2+\frac{1}{16}} = 4ax - 1$$

$\Rightarrow 4ax - 1 > 0 \qquad [\because 2\sqrt{x^2+\frac{1}{16}} > 0]$

$\Rightarrow 4ax > 1$

$\Rightarrow x > 0 \qquad [\because a > 0]$

Thus, we have to find real values of x which satisfy the following conditions for all $a \neq \frac{1}{2}$.

(i) $4ax - 1 > 0$ (ii) $x > 0$.

We have, $x = \frac{4a \pm \sqrt{4a^2+3}}{4(4a^2-1)}$. So, two cases arise.

CASE I *When* $4a^2 - 1 > 0$ *i.e. when* $a > \frac{1}{2}$

In this case, x will be positive if $4a \pm \sqrt{4a^2+3} > 0$.

Clearly, $4a + \sqrt{4a^2+3} > 0$ and $4a - \sqrt{4a^2+3}$ will be positive, if

$4a - \sqrt{4a^2+3} > 0$

$\Rightarrow 4a > \sqrt{4a^2+3}$

$\Rightarrow 16a^2 > 4a^2 + 3$

$\Rightarrow 12a^2 - 3 > 0$

$\Rightarrow 4a^2 - 1 > 0$, which is true for all $a > \frac{1}{2}$.

Thus, $x = \frac{4a \pm \sqrt{4a^2+3}}{4(4a^2-1)} > 0$

Now,

$$x = \frac{4a + \sqrt{4a^2+3}}{4(4a^2-1)}$$

$$\Rightarrow \quad 4ax = \frac{4a^2 + a\sqrt{4a^2+3}}{4a^2-1}$$

$$\Rightarrow 4ax - 1 = \frac{4a^2 + a\sqrt{4a^2+3}}{4a^2-1} - 1$$

$$\Rightarrow 4ax - 1 = \frac{a\sqrt{4a^2+3}+1}{4a^2-1} > 0$$

$$\text{and, } x = \frac{4a - \sqrt{4a^2+3}}{4(4a^2-1)}$$

$$\Rightarrow \quad 4ax = \frac{4a^2 - a\sqrt{4a^2+3}}{4a^2-1}$$

$$\Rightarrow 4ax - 1 = \frac{-a\sqrt{4a^2+3}+1}{4a^2-1} < 0$$

Thus, when $a > \frac{1}{2}$, then $x = \frac{4a + \sqrt{4a^2+3}}{4(4a^2-1)}$ satisfies both the conditions $x > 0$ and $4ax - 1 > 0$.

CASE II *When* $4a^2 - 1 < 0$ *i.e.* $0 < a < \frac{1}{2}$ $\qquad [\because a > 0]$

In this case, x will be positive if

$$4a \pm \sqrt{4a^2+3} < 0$$

Clearly, $4a + \sqrt{4a^2+3} \not< 0$ for any $a \in (0, 1/2)$

Now,

$4a - \sqrt{4a^2+3} < 0$

$\Rightarrow 4a < \sqrt{4a^2+3}$

$\Rightarrow 16a^2 < 4a^2 + 3$

$\Rightarrow 12a^2 - 3 < 0$

$\Rightarrow 4a^2 - 1 < 0$, which is given to us.

Thus, $x = \frac{4a - \sqrt{4a^2+3}}{4(4a^2-1)} > 0$

But, $x = \frac{4a - \sqrt{4a^2+3}}{4(4a^2-1)}$

$$\Rightarrow \quad 4ax = \frac{4a^2 - a\sqrt{4a^2+3}}{4a^2-1}$$

$$\Rightarrow 4ax - 1 = \frac{-a\sqrt{4a^2+3}+1}{4a^2-1} < 0$$

$\therefore x = \frac{4a - \sqrt{4a^2+3}}{4(4a^2-1)} > 0$ but $4ax - 1 < 0$

Thus, for $0 < a < \frac{1}{2}$, there is no value of x which satisfy both the conditions $x > 0$ and $4ax - 1 > 0$.

Hence, $z = x + iy = \frac{4a + \sqrt{4a^2+3}}{4(4a^2-1)} + \frac{1}{4}i$, where $a > \frac{1}{2}$ is the solution of the given equation.

EXAMPLE 19 *For every real number* $a \geq 0$, *find all the complex numbers* z *which satisfy the equation*

$$|z|^2 - 2iz + 2a(1+i) = 0.$$

SOLUTION Let $z = x + iy$. Then,

$|z|^2 - 2iz + 2a(1+i) = 0$

$\Rightarrow x^2 + y^2 - 2i(x+iy) + 2a(1+i) = 0$

$\Rightarrow (x^2 + y^2 + 2y + 2a) + 2i(a - x) = 0$

$\Rightarrow x^2 + y^2 + 2y + 2a = 0 \qquad \ldots(i)$

and,

$a - x = 0$ ...(ii)

Now,

$a - x = 0 \Rightarrow x = a$

Putting $x = a$ in (i), we get

$y^2 + 2y + a^2 + 2a = 0$

$\Rightarrow y = \dfrac{-2 \pm \sqrt{4 - 4(a^2 + 2a)}}{2}$

$\Rightarrow y = -1 \pm \sqrt{1 - a^2 - 2a}$

Since $y \in R$. Therefore,

$1 - a^2 - 2a \geq 0$

$\Rightarrow a^2 + 2a - 1 \leq 0$

$\Rightarrow -1 - \sqrt{2} \leq a \leq -1 + \sqrt{2}$ $[\because a^2 + 2a - 1 = 0 \Rightarrow a = -1 \pm \sqrt{2}]$

$\Rightarrow 0 \leq a \leq \sqrt{2} - 1$ $[\because a \geq 0]$

Thus,

$z = x + iy = a + i\left\{-1 \pm \sqrt{1 - a^2 - 2a}\right\}$, where $0 \leq a \leq \sqrt{2} - 1$.

EXAMPLE 20 *For every real number $a > 0$, find all complex numbers z, satisfying the equation $z \mid z \mid + az + i = 0$.*

SOLUTION Let $z = x + iy$. Then,

$z \mid z \mid + az + i = 0$

$\Rightarrow (x + iy)\sqrt{x^2 + y^2} + a(x + iy) + i = 0$

$\Rightarrow (x\sqrt{x^2 + y^2} + ax) + i(y\sqrt{x^2 + y^2} + ay + 1) = 0$

$\Rightarrow x\sqrt{x^2 + y^2} + ax = 0$ ...(i)

and,

$y\sqrt{x^2 + y^2} + ay + 1 = 0$ ...(ii)

From (i), we have

$x\sqrt{x^2 + y^2} + ax = 0$

$\Rightarrow x(\sqrt{x^2 + y^2} + a) = 0$

$\Rightarrow x = 0$ $[\because a > 0 \therefore \sqrt{x^2 + y^2} + a > 0]$

Putting $x = 0$ in (ii), we get

$y \mid y \mid + ay + 1 = 0$ ...(iii)

CASE I *When $y \geq 0$.*

If $y \geq 0$, then

$y \mid y \mid + ay + 1 = 0$

$\Rightarrow y^2 + ay + 1 = 0$

$\Rightarrow y = \dfrac{-a \pm \sqrt{a^2 - 4}}{2}$

But, y is a real number. Therefore,

$a^2 - 4 \geq 0$

$\Rightarrow a \leq -2$ or, $a \geq 2$

$\Rightarrow a \geq 2$ $[\because a > 0]$

Thus,

$z = x + iy = 0 + i\left\{\dfrac{-a \pm \sqrt{a^2 - 4}}{2}\right\}$, where $a \geq 2$.

CASE II *When $y < 0$:*

If $y < 0$, then

$y \mid y \mid + ay + 1 = 0$

$\Rightarrow -y^2 + ay + 1 = 0$

$\Rightarrow y^2 - ay - 1 = 0$

$\Rightarrow y = \dfrac{a \pm \sqrt{a^2 + 4}}{2}$

Clearly, $y \in R$ for all $a > 0$. Therefore,

$z = x + iy = 0 + i\left\{\dfrac{a \pm \sqrt{a^2 + 4}}{2}\right\}$

EXAMPLE 21 *Let z be a complex number. Find the range of real values of a for which the equation*

$$z + a \mid z - 1 \mid + 2i = 0$$

has a solution. Also, find the solution.

SOLUTION Let $z = x + iy$. Then,

$z + a \mid z - 1 \mid + 2i = 0$

$\Rightarrow (x + iy) + a \mid (x - 1) + iy \mid + 2i = 0$

$\Rightarrow x + iy + a\sqrt{(x - 1)^2 + y^2} + 2i = 0$

$\Rightarrow x + a\sqrt{(x - 1)^2 + y^2} + (y + 2)i = 0$

$\Rightarrow x + a\sqrt{(x - 1)^2 + y^2} = 0$ ...(i)

and, $y + 2 = 0$ ...(ii)

From (ii), we get $y = -2$.

Putting $y = -2$ in (i), we get

$x + a\sqrt{(x - 1)^2 + 4} = 0$

$\Rightarrow a\sqrt{(x - 1)^2 + 4} = -x$

$\Rightarrow a^2\{(x - 1)^2 + 4\} = x^2$

$\Rightarrow x^2(a^2 - 1) - 2a^2x + 5a^2 = 0$

$\Rightarrow x = \dfrac{2a^2 \pm \sqrt{4a^4 - 20a^2(a^2 - 1)}}{2(a^2 - 1)}$

$\Rightarrow x = \dfrac{a^2 \pm \sqrt{5a^2 - 4a^4}}{(a^2 - 1)}$

For x to be real, we must have

$a^2 - 1 \neq 0$ and $5a^2 - 4a^4 \geq 0$

$\Rightarrow a \neq \pm 1$ and $-a^2(4a^2 - 5) \geq 0$

$\Rightarrow a \neq \pm 1$ and $4a^2 - 5 \leq 0$

$\Rightarrow a \neq \pm 1$ and $-\frac{\sqrt{5}}{2} \le a \le \frac{\sqrt{5}}{2}$

$\Rightarrow a \in \left[\frac{-\sqrt{5}}{2}, -1\right) \cup (-1, 1) \cup \left(1, \frac{\sqrt{5}}{52}\right]$

Thus, the given equation has a solution given by

$$z = \left(\frac{a^2 \pm \sqrt{5a^2 - 4a^4}}{a^2 - 1}\right) - 2i, \text{ if } a \in \left[\frac{-\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right] - \{-1, 1\}.$$

EXAMPLE 22 *Solve the equation $z^2 + |z| = 0$, where z is a complex number.*

SOLUTION Let $z = x + iy$. Then,

$z^2 + |z| = 0$

$\Rightarrow (x+iy)^2 + \sqrt{x^2+y^2} = 0$

$\Rightarrow (x^2 - y^2) + \sqrt{x^2+y^2} + 2ixy = 0$

$\Rightarrow x^2 - y^2 + \sqrt{x^2+y^2} = 0$ ...(i)

and,

$2xy = 0$ ...(ii)

Now,

$2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0$ or, $y = 0$

CASE I *When* $y = 0$

Putting $y = 0$ in (i), we get

$x^2 + \sqrt{x^2} = 0$

$\Rightarrow x^2 + |x| = 0$

Clearly, $x^2 + |x| > 0$ for all $x > 0$.

So, let $x \le 0$.

In this case, we have

$x^2 + |x| = 0$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ $[\because x \le 0]$

$\therefore z = 0$.

CASE II *When* $x = 0$

Putting $x = 0$ in (i), we get

$-y^2 + \sqrt{y^2} = 0$

$\Rightarrow -y^2 + |y| = 0$

If $y > 0$, then

$-y^2 + |y| = 0$

$\Rightarrow -y^2 + y = 0$

$\Rightarrow y = 0, y = 1$

$\Rightarrow y = 1$ $[\because y > 0]$

If $y < 0$, then

$-y^2 + |y| = 0$

$\Rightarrow -y^2 - y = 0$

$\Rightarrow y = 0, -1$

$\Rightarrow y = -1$ $[\because y < 0]$

$\therefore z = 0 + i$ or, $z = 0 - i$

Hence, $z = 0, i$ and $-i$ are solutions of $z^2 + |z| = 0$.

EXAMPLE 23 *Find all complex numbers z satisfying the equation*

$$2|z|^2 + z^2 - 5 + i\sqrt{3} = 0.$$

SOLUTION Let $z = x + iy$. Then,

$2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

$\Rightarrow 2(x^2+y^2) + (x^2 - y^2 + 2ixy) = 5 - i\sqrt{3}$

$\Rightarrow (3x^2 + y^2) + 2ixy = 5 - i\sqrt{3}$

$\Rightarrow 3x^2 + y^2 = 5$ and $2xy = -\sqrt{3}$

$\Rightarrow 3x^2 + y^2 = 5$ and $xy = -\frac{\sqrt{3}}{2}$

$\Rightarrow 3x^2 + y^2 = 5$ and $y = \frac{-\sqrt{3}}{2x}$

Putting $y = -\frac{\sqrt{3}}{2x}$ in $3x^2 + y^2 = 5$, we get

$3x^2 + \frac{3}{4x^2} = 5$

$\Rightarrow 12x^4 - 20x^2 + 3 = 0$

$\Rightarrow (6x^2 - 1)(2x^2 - 3) = 0$

$\Rightarrow x = \pm\frac{1}{\sqrt{6}}, \pm\frac{\sqrt{3}}{2}$

$\because y = \frac{-\sqrt{3}}{2x}$.

$\therefore x = \frac{1}{\sqrt{6}} \Rightarrow y = \frac{-3}{\sqrt{2}}$,

$x = -\frac{1}{\sqrt{6}} \Rightarrow y = \frac{3}{\sqrt{2}}$

$x = \frac{\sqrt{3}}{2} \Rightarrow y = -\frac{1}{\sqrt{2}}$

and, $x = -\frac{\sqrt{3}}{2} \Rightarrow y = \frac{1}{\sqrt{2}}$

Hence, $z = \frac{1}{\sqrt{6}} - \frac{3}{\sqrt{2}}i, -\frac{1}{\sqrt{6}} + \frac{3}{\sqrt{2}}i, \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

EXAMPLE 24 *Find all non-zero complex numbers z satisfying*

$$\bar{z} = iz^2.$$ [IIT 1996]

SOLUTION Let $z = x + iy$. Then,

$\bar{z} = iz^2$

$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow (x + 2xy) + i(x^2 - y^2 + y) = 0$

$\Rightarrow x + 2xy = 0$ ...(i)

and,

$x^2 - y^2 + y = 0$ ...(ii)

Now,

$x + 2xy = 0$

$\Rightarrow x(1 + 2y) = 0$

$\Rightarrow x = 0$ or, $1 + 2y = 0$

$\Rightarrow x = 0$ or, $y = -\frac{1}{2}$.

When $x = 0$, we have

$x^2 - y^2 + y = 0$

$\Rightarrow -y^2 + y = 0$

$\Rightarrow y(y - 1) = 0$

$\Rightarrow y = 0, y = 1$

$\therefore z = 0 + i0 = 0, 0 + 1i = i$

When $y = -\frac{1}{2}$, we have

$x^2 - y^2 + y = 0$

$\Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0$

$\Rightarrow x^2 - \frac{3}{4} = 0$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2}$

$\therefore z = \frac{\sqrt{3}}{2} - \frac{1}{2}i, \frac{-\sqrt{3}}{2} - \frac{1}{2}i$

Hence, $z = 0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

EXAMPLE 25 *Find all complex numbers z satisfying the equation*

$$z^2 + z|z| + |z|^2 = 0$$

SOLUTION Let $z = x + iy$. Then,

$z^2 + z|z| + |z|^2 = 0$

$\Rightarrow (x^2 - y^2 + 2ixy) + (x + iy)\sqrt{x^2 + y^2} + x^2 + y^2 = 0$

$\Rightarrow (x^2 - y^2 + x\sqrt{x^2 + y^2} + x^2 + y^2) + i(2xy + y\sqrt{x^2 + y^2}) = 0$

$\Rightarrow (2x^2 + x\sqrt{x^2 + y^2}) + i(2xy + y\sqrt{x^2 + y^2}) = 0$

$\Rightarrow 2x^2 + x\sqrt{x^2 + y^2} = 0$ ...(i)

and,

$2xy + y\sqrt{x^2 + y^2} = 0$ ...(ii)

Now,

$2xy + y\sqrt{x^2 + y^2} = 0$

$\Rightarrow y(2x + \sqrt{x^2 + y^2}) = 0$

$\Rightarrow y = 0$ or, $2x + \sqrt{x^2 + y^2} = 0$

When $y = 0$, we have

$2x^2 + x\sqrt{x^2} = 0$ [Putting $y = 0$ in (i)]

$\Rightarrow 2x^2 + x|x| = 0$

$\Rightarrow x(2x + |x|) = 0$

$\Rightarrow x = 0$ or, $2x + |x| = 0$

$\Rightarrow x = 0$ or, $2x \pm x = 0$

$\Rightarrow x = 0$

$\therefore z = 0 + i0 = 0$

When $2x + \sqrt{x^2 + y^2} = 0$, equation (i) is true for all x.

Now,

$2x + \sqrt{x^2 + y^2} = 0$

$\Rightarrow 2x = -\sqrt{x^2 + y^2}$

$\Rightarrow 4x^2 = x^2 + y^2$

$\Rightarrow y^2 = 3x^2$

$\Rightarrow y = \pm\sqrt{3}x$.

Also,

$2x = -\sqrt{x^2 + y^2} \Rightarrow x < 0$.

Therefore, $y = \pm\sqrt{3}x$, where $x < 0$.

Hence,

$z = x + iy = x \pm i\sqrt{3}x = (1 \pm i\sqrt{3})x$, where $x < 0$.

$\Rightarrow z = -2x\left(-\frac{1}{2} \mp i\frac{\sqrt{3}}{2}\right) = \lambda w, \lambda w^2$, where $\lambda = -2x > 0$.

ALITER We have,

$z^2 + z|z| + |z|^2 = 0$

$\Rightarrow \frac{z^2}{|z|^2} + \frac{z}{|z|} + 1 = 0$

$\Rightarrow \left(\frac{z}{|z|}\right)^2 + \frac{z}{|z|} + 1 = 0$

$\Rightarrow \frac{z}{|z|} = w, w^2$ [$\because x^2 + x + 1 = 0$ has its roots as w and w^2]

$\Rightarrow z = |z|w, |z|w^2$

$\Rightarrow z = \lambda w, \lambda w^2$, where λ is any positive real number.

EXAMPLE 26 *Show that the equation* $z^2 + p\bar{z} - q^2 = 0, p, q \in R$ *has no solution, with its imaginary part not equal to zero, if* p/q *lies in* $[-2/\sqrt{3}, 2/\sqrt{3}]$. *Find the solution when* p/q *lies outside the interval.*

SOLUTION Let $z = x + iy$. Then,

$z^2 + p\bar{z} - q^2 = 0$

$\Rightarrow (x + iy)^2 + p(x - iy) - q^2 = 0$

$\Rightarrow (x^2 - y^2 + px - q^2) + i(2xy - py) = 0$

$\Rightarrow x^2 - y^2 + px - q^2 = 0$ ...(i)

and,

$$2xy - py = 0 \qquad ...(ii)$$

Now,

$$2xy - py = 0 \qquad \text{[From (ii)]}$$

$$\Rightarrow y(2x - p) = 0$$

$$\Rightarrow 2x - p = 0 \qquad [\because y \neq 0]$$

$$\Rightarrow x = \frac{p}{2}.$$

Putting $x = \frac{p}{2}$ in (i), we get

$$\frac{p^2}{4} - y^2 + \frac{p^2}{2} - q^2 = 0$$

$$\Rightarrow y^2 = \frac{3p^2}{4} - q^2$$

$$\Rightarrow y = \pm\sqrt{\frac{3p^2}{4} - q^2}$$

But, y is a non-zero real number. Therefore,

$$\frac{3p^2}{4} - q^2 > 0$$

$$\Rightarrow \frac{p^2}{q^2} - \frac{4}{3} > 0$$

$$\Rightarrow \frac{p}{q} < -\frac{2}{\sqrt{3}} \text{ or, } \frac{p}{q} > \frac{2}{\sqrt{3}}$$

Thus, the given equation has a solution given by $z = x + iy = \frac{p}{2} \pm i\sqrt{\frac{3p^2}{4} - q^2}$ with its imaginary part not equal to zero, if $\frac{p}{q}$ does not lie in $\left[\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$.

Hence, the given equation has no solution with its imaginary part not equal to zero, if $\frac{p}{q}$ lies in $\left[\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$.

EXAMPLE 27 *If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.* [IIT 1995]

SOLUTION We have,

$$iz^3 + z^2 - z + i = 0$$

$$\Rightarrow z^3 - iz^2 + iz + 1 = 0 \qquad \text{[Dividing both sides by } i\text{]}$$

$$\Rightarrow z^2(z - i) + i(z - i) = 0$$

$$\Rightarrow (z - i)(z^2 + i) = 0$$

$$\Rightarrow z = i \text{ or, } z^2 = -i$$

Now,

$$z = i \Rightarrow |z| = |i| = 1$$

and, $z^2 = -i$

$$\Rightarrow |z^2| = |-i| = 1$$

$$\Rightarrow |z|^2 = 1$$

$$\Rightarrow |z| = 1.$$

Hence, in either case, we have $|z| = 1$.

EXAMPLE 28 *If $|z - 3i| < \sqrt{5}$, then prove that the complex number z also satisfies the inequality*

$$\left|i(z+1) + 1\right| < 2\sqrt{5}.$$

SOLUTION We have,

$$|i(z+1) + 1|$$

$$= \left|i\left(z + 1 + \frac{1}{i}\right)\right|$$

$$= |i|\,|z + 1 - i|$$

$$= |z + 1 - i|$$

$$= |z - 3i + 1 + 2i|$$

$$\leq |z - 3i| + |1 + 2i|$$

$$= |z - 3i| + \sqrt{5}$$

$$< \sqrt{5} + \sqrt{5} = 2\sqrt{5}$$

EXAMPLE 29 *Find the condition for one of the roots of the equation $z^2 + (a_1 + ia_2)z + (b_1 + ib_2) = 0$ to be real. Also, find the condition for both roots to be equal, where a_1, b_1, a_2, b_2 are real.*

SOLUTION Let c be a real root of the equation

$$z^2 + (a_1 + i\,a_2)z + (b_1 + i\,b_2) = 0$$

Then,

$$c^2 + (a_1 + i\,a_2)c + (b_1 + i\,b_2) = 0$$

$$\Rightarrow (c^2 + c\,a_1 + b_1) + i(c\,a_2 + b_2) = 0$$

$$\Rightarrow c^2 + c\,a_1 + b_1 = 0 \qquad ...(i)$$

and,

$$ca_2 + b_2 = 0 \qquad ...(ii)$$

From (ii), we get

$$c = \frac{-b_2}{a_2}$$

Putting the value of c in (i), we get

$$\frac{{b_2}^2}{{a_2}^2} - \frac{b_2}{a_2}a_1 + b_1 = 0$$

$$\Rightarrow {b_2}^2 - a_1 a_2 b_2 + b_1 {a_2}^2 = 0$$

This is the condition for the given equation to have one real root.

To find the condition for both roots of the given equation to be equal, let each root be $\alpha + i\,\beta$. Then,

$$z^2 + (a_1 + i\,a_2)z + (b_1 + i\,b_2) = \{z - (\alpha + i\,\beta)\}\{z - (\alpha + i\,\beta)\}$$

$\Rightarrow z^2 + (a_1 + i a_2) z + (b_1 + i b_2) = z^2 - 2(\alpha + i\beta) z + (\alpha + i\beta)^2$

$\Rightarrow a_1 + i a_2 = -2(\alpha + i\beta)$ ...(iii)

and, $b_1 + i b_2 = (\alpha + i\beta)^2$ ...(iv)

From (i), we get

$a_1 = -2\alpha$ and $a_2 = -2\beta$ ...(v)

From (ii), we get

$b_1 = \alpha^2 - \beta^2$ and $b_2 = 2\alpha\beta$ ...(vi)

Eliminating α and β from (v) and (vi), we get

$$4 b_1 = a_1^2 - a_2^2 \text{ and } 2b_2 = a_1 a_2$$

as the required conditions.

EXAMPLE 30 *Find the condition that the equation $az^2 + bz + c = 0$ has a real root given that a, b, c are complex numbers.*

SOLUTION Let α be a real root of $az^2 + bz + c = 0$. Then,

$$a\alpha^2 + b\alpha + c = 0 \quad ...(i)$$

Taking conjugate on both sides of (i), we get

$$\bar{a}\alpha^2 + \bar{b}\alpha + \bar{c} = 0 \quad ...(ii)$$

From (i) and (ii), we get

$$\frac{\alpha^2}{b\bar{c} - \bar{b}c} = \frac{\alpha}{c\bar{a} - a\bar{c}} = \frac{1}{a\bar{b} - \bar{a}b}$$

$$\Rightarrow \alpha^2 = \frac{b\bar{c} - \bar{b}c}{a\bar{b} - \bar{a}b} \text{ and } \alpha = \frac{c\bar{a} - \bar{c}a}{a\bar{b} - \bar{a}b}$$

$$\Rightarrow \left(\frac{c\bar{a} - \bar{c}a}{a\bar{b} - \bar{a}b}\right)^2 = \left(\frac{b\bar{c} - \bar{b}c}{a\bar{b} - \bar{a}b}\right)$$

$$\Rightarrow (c\bar{a} - \bar{c}a)^2 = (b\bar{c} - \bar{b}c)(a\bar{b} - \bar{a}b)$$

This is the required condition for the given equation to have a real root.

EXAMPLE 31 *If 'a' is a complex number such that $|z| = 1$. Find the value of a, so that the equation $az^2 + z + 1 = 0$ has one purely imaginary root.*

SOLUTION Let α be a purely imaginary root of $az^2 + z + 1 = 0$. Then,

$$a\alpha^2 + \alpha + 1 = 0 \quad ...(i)$$

Taking conjugate on both sides, we get

$$\bar{a}(\bar{\alpha})^2 + \bar{\alpha} + 1 = 0$$

$\Rightarrow \bar{a}\alpha^2 - \alpha + 1 = 0$ ...(ii) $[\because \bar{\alpha} = -\alpha]$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{\alpha^2}{1+1} = \frac{\alpha}{\bar{a} - a} = \frac{1}{-a - \bar{a}}$$

$$\Rightarrow \alpha^2 = \frac{-2}{a + \bar{a}} \text{ and } \alpha = \frac{\bar{a} - a}{-a - \bar{a}}$$

Now,

$\alpha^2 = (\alpha)^2$

$$\Rightarrow \frac{-2}{a + \bar{a}} = \left(\frac{\bar{a} - a}{-a - \bar{a}}\right)^2$$

$$\Rightarrow \frac{-2}{a + \bar{a}} = \frac{(\bar{a} - a)^2}{(a + \bar{a})^2}$$

$\Rightarrow -2(a + \bar{a}) = (\bar{a} - a)^2$

$\Rightarrow (\bar{a} - a)^2 + 2(\bar{a} + a) = 0$ ...(iii)

Now,

$|a| = 1 \Rightarrow a = \cos\theta + i \sin\theta$

Substituting the value of a in (iii), we get

$(-2i \sin\theta)^2 + 4\cos\theta = 0$

$\Rightarrow -4\sin^2\theta + 4\cos\theta = 0$

$\Rightarrow \cos^2\theta + \cos\theta - 1 = 0$

$$\Rightarrow \cos\theta = \frac{\sqrt{5} - 1}{2}, -\frac{\sqrt{5} + 1}{2}$$

Hence, $a = \cos\theta + i \sin\theta$, where $\theta = \cos^{-1}\left\{-\frac{\sqrt{5} + 1}{4}\right\}$

EXAMPLE 32 *Show that the equation $z^4 + 2z^3 + 3z^2 + 4z + 5 = 0$ (z is complex) has no purely real or purely imaginary root.*

SOLUTION If possible, let $z = \alpha$ be a purely real root of the given equation. Then,

$\alpha^4 + 2\alpha^3 + 3\alpha^2 + 4\alpha + 5$ should be zero.

But,

$\alpha^4 + 2\alpha^3 + 3\alpha^2 + 4\alpha + 5$

$= \alpha^2(\alpha + 1)^2 + \alpha^2 + (\alpha + 2)^2 + 1 > 0$ for all $\alpha \in R$.

Therefore the given equation has no real root.

Now, let $z = i\alpha, \alpha \in R$ be a purely imaginary root of the given equation. Then,

$(i\alpha)^4 + 2(i\alpha)^3 + 3(i\alpha)^2 + 4(i\alpha) + 5$ must be zero.

But,

$(i\alpha)^4 + 2(i\alpha)^3 + 3(i\alpha) + 4(i\alpha) + 5 = 0$

$\Rightarrow (\alpha^4 - 3\alpha^2 + 5) + i(2\alpha - \alpha^3) = 0$

$\Rightarrow \alpha^4 - 3\alpha^2 + 5 = 0$ and $2\alpha - \alpha^3 = 0$

Clearly, these two equations are not consistent i.e. the values of α obtained from either of these two equations do not satisfy the other.

Hence, the given equation cannot have a purely real or purely imaginary root.

EXAMPLE 33 *Show that the equation $az^3 + bz^2 + \bar{b}z + \bar{a} = 0$ has a root α, such that $|\alpha| = 1$, a, b, z and α belong to the set of complex numbers.*

SOLUTION Let α be a root of the given equation. Then,

$$a\alpha^3 + b\alpha^2 + \bar{b}\alpha + \bar{a} = 0 \quad ...(i)$$

$\Rightarrow \bar{a}(\bar{\alpha})^3 + \bar{b}(\bar{\alpha})^2 + b\bar{\alpha} + a = 0$

[Taking conjugate on both sides]

$\Rightarrow \bar{a}\alpha^3(\bar{\alpha})^3 + \bar{b}\alpha^3(\bar{\alpha})^2 + b\alpha^3\bar{\alpha} + a\alpha^3 = 0$

[Multiplying on both sides by α^3]

$\Rightarrow \bar{a}(\alpha\bar{\alpha})^3 + \bar{b}\alpha(\alpha\bar{\alpha})^2 + b\alpha^2(\alpha\bar{\alpha}) + a\alpha^3 = 0$

$\Rightarrow \bar{a}\,|\alpha|^6 + \bar{b}\alpha\,|\alpha|^4 + b\alpha^2\,|\alpha|^2 + a\alpha^3 = 0$ ...(ii)

Subtracting (i) from (ii), we get

$\bar{a}(|\alpha|^6 - 1) + \bar{b}\alpha(|\alpha|^4 - 1) + b\alpha^2(|\alpha|^2 - 1) = 0$

Clearly, $|\alpha| = 1$ satisfies this equation.

Hence, the given equation has a root α such that $|\alpha| = 1$.

EXAMPLE 34 *z is the root of $az^2 + bz + c = 0$, where $|z| = 1$ and a, b, c are complex numbers. Prove that :*

$$(\bar{a}b - \bar{b}\bar{c})(a\bar{b} - b\bar{c}) = (a\bar{a} - c\bar{c})^2$$

SOLUTION We have, z as the root of $az^2 + bz + c = 0$ such that $|z| = 1$. Therefore,

$$az^2 + bz + c = 0 \quad ...(i)$$

and, $|z| = 1$

Now, $|z| = 1$

$\Rightarrow z\bar{z} = 1$ ...(ii)

Taking conjugate on both sides of (i), we get

$\bar{a}(\bar{z})^2 + \bar{b}\bar{z} + \bar{c} = 0$

$\Rightarrow \frac{\bar{a}}{z^2} + \frac{\bar{b}}{z} + \bar{c} = 0$ [From (ii) $(\bar{z})^2 = \frac{1}{z^2}$, $\bar{z} = \frac{1}{z}$]

$\Rightarrow \bar{c}z^2 + \bar{b}z + \bar{a} = 0$ ...(iii)

From (i) and (ii), we have

$$\frac{z^2}{b\bar{a} - \bar{b}\bar{c}} = \frac{z}{c\bar{c} - a\bar{a}} = \frac{1}{a\bar{b} - b\bar{c}}$$

$\Rightarrow z = \frac{b\bar{a} - \bar{b}\bar{c}}{c\bar{c} - a\bar{a}}$ and $z = \frac{c\bar{c} - a\bar{a}}{a\bar{b} - b\bar{c}}$

$\Rightarrow \frac{b\bar{a} - \bar{b}\bar{c}}{c\bar{c} - a\bar{a}} = \frac{c\bar{c} - a\bar{a}}{a\bar{b} - b\bar{c}}$

$\Rightarrow (b\bar{a} - \bar{b}\bar{c})(a\bar{b} - b\bar{c}) = (c\bar{c} - a\bar{a})^2$

$\Rightarrow (\bar{a}b - \bar{b}\bar{c})(a\bar{b} - b\bar{c}) = (a\bar{a} - c\bar{c})^2$

EXAMPLE 35 *If n is a positive integer, prove that*

$$\left|\text{Im}(z^n)\right| \le n\left|\text{Im}(z)\right|\,|z|^{n-1}$$

SOLUTION We have,

$$\left|\frac{\text{Im}(z^n)}{\text{Im}(z)}\right|$$

$$= \left|\frac{\frac{z^n - \overline{(z^n)}}{2i}}{\frac{z - \bar{z}}{2i}}\right|$$

$$= \left|\frac{z^n - (\bar{z})^n}{z - \bar{z}}\right|$$

$$= \left|z^{n-1} + z^{n-2}\bar{z} + z^{n-3}(\bar{z})^2 + \ldots + (\bar{z})^{n-1}\right|$$

$$\le |z^{n-1}| + |z^{n-2}\bar{z}| + |z^{n-3}(\bar{z})^2| + \ldots + |(\bar{z})^{n-1}|$$

$$= |z|^{n-1} + |z^{n-2}|\,|\bar{z}| + |z^{n-3}|\,|(\bar{z})^2| + \ldots + |(\bar{z})^{n-1}|$$

$$= |z|^{n-1} + |z|^{n-2}|z| + |z|^{n-3}|z|^2 + \ldots + |z|^{n-1}$$

$$= |z|^{n-1} + |z|^{n-1} + |z|^{n-1} + \ldots + |z|^{n-1}$$

$$= n|z|^{n-1}$$

$\therefore \left|\frac{\text{Im}(z^n)}{\text{Im}(z)}\right| \le n|z|^{n-1}$

$\Rightarrow \frac{|\text{Im}(z^n)|}{|\text{Im}(z)|} \le n|z|^{n-1}$

$\Rightarrow |\text{Im}(z^n)| \le n\,|\text{Im}(z)|\,|z|^{n-1}$.

EXAMPLE 36 *Find all the values of z that satisfy the equation*

$$\exp\left|\frac{|z|^2 - |z| + 4}{|z|^2 + 1}\log 2\right| = \log_{\sqrt{2}}\left|3\sqrt{15} + 11i\right|$$

SOLUTION We have,

$$\exp\left|\frac{|z|^2 - |z| + 4}{|z|^2 + 1}\log 2\right| = \log_{\sqrt{2}}\left|3\sqrt{15} + 11i\right|$$

$$\Rightarrow \exp\left|\frac{|z|^2 - |z| + 4}{|z|^2 + 1}\log 2\right| = \log_{\sqrt{2}} 16$$

$$\Rightarrow e^{\left|\frac{|z|^2 - |z| + 4}{|z|^2 + 1}\right|(\log 2)} = e^{3\log 2} \quad [\because \log_{\sqrt{2}} 16 = 8 = e^{\log 2^3}]$$

$$\Rightarrow \frac{|z|^2 - |z| + 4}{|z|^2 + 1} = 3$$

$\Rightarrow 2|z|^2 + |z| - 1 = 0$

$\Rightarrow (2|z| - 1)(|z| + 1) = 0$

$\Rightarrow (2|z| - 1) = 0$

$\Rightarrow |z| = \frac{1}{2}$

$\Rightarrow z = \frac{1}{2}(\cos\theta + i\sin\theta), 0 \le \theta \le 2\pi$

EXAMPLE 37 *Let z be a complex number such that $\left|2z + \frac{1}{2z}\right| = a$, find the greatest and least values of $|z|$.*

SOLUTION We know that

$$\left|\,|z_1| - |z_2|\,\right| \le |z_1 + z_2|$$

$$\therefore \left|\,|2z| - \left|\frac{1}{2z}\right|\,\right| \le \left|2z + \frac{1}{2z}\right|$$

$$\Rightarrow \left|2|z| - \frac{1}{2|z|}\right| \le a$$

$$\Rightarrow -a \le 2|z| - \frac{1}{2|z|} \le a$$

Now, $-a \le 2|z| - \frac{1}{2|z|}$

$\Rightarrow -a \le \frac{4|z|^2-1}{2|z|}$

$\Rightarrow -2a|z| \le 4|z|^2-1$

$\Rightarrow 4|z|^2+2a|z|-1 \ge 0$

$\Rightarrow$ $|z|$ lies outside the roots of $4|z|^2+2a|z|-1=0$

i.e. $|z| \le \frac{-a-\sqrt{a^2+4}}{4}$ or, $|z| \ge \frac{-a+\sqrt{a^2+4}}{4}$

$\Rightarrow |z| \ge \frac{-a+\sqrt{a^2+4}}{4}$ $[\because |z| \ge 0]$

So, the least value of $|z|$ is $\frac{-a+\sqrt{a^2+4}}{4}$

and,

$2|z|-\frac{1}{2|z|} \le a$

$\Rightarrow 4|z|^2-1 \le 2a|z|$

$\Rightarrow 4|z|^2-2a|z|-1 \le 0$

$\Rightarrow$ $|z|$ lies between the roots of $4|z|^2-2a|z|-1=0$

i.e. $\frac{a-\sqrt{a^2+4}}{2} \le |z| \le \frac{a+\sqrt{a^2+4}}{4}$

$\Rightarrow 0 \le |z| \le \frac{a+\sqrt{a^2+4}}{4}$ $\left[\because \frac{a-\sqrt{a^2+4}}{4}<0 \text{ and } |z| \ge 0\right]$

So, the greatest value of $|z|$ is $\frac{a+\sqrt{a^2+4}}{4}$.

EXAMPLE 38 *Find the greatest and least values of* $|z|$, *if* z *satisfies the equation*

$$\left|z-\frac{4}{z}\right| = 2$$

SOLUTION We know that: $\left||z_1|-|z_2|\right| \le |z_1-z_2|$

$\therefore \quad \left||z|-\left|\frac{4}{z}\right|\right| \le \left|z-\frac{4}{z}\right|$

$\Rightarrow \quad \left||z|-\frac{4}{|z|}\right| \le 2$

$\Rightarrow -2 \le |z|-\frac{4}{|z|} \le 2$

Now,

$-2 \le |z|-\frac{4}{|z|}$

$\Rightarrow -2|z| \le |z|^2-4$

$\Rightarrow |z|^2+2|z|-4 \ge 0$

$\Rightarrow$ $|z|$ lies outside the roots of the equation

$|z|^2+2|z|-4=0$

i.e. $|z| \le -1-\sqrt{5}$ or, $|z| \ge -1+\sqrt{5}$

$\Rightarrow |z| \ge \sqrt{5}-1$ $[\because |z| \ge 0]$

So, the least value of $|z|$ is $\sqrt{5}-1$.

and,

$|z|-\frac{4}{|z|} \le 2$

$\Rightarrow |z|^2-2|z|-4 \le 0$

$\Rightarrow$ $|z|$ lies between the roots of the equation

$|z|^2-2|z|-4=0$

i.e. $1-\sqrt{5} < |z| \le 1+\sqrt{5} \Rightarrow |z| \le \sqrt{5}+1$

So, the greatest value of $|z|$ is $\sqrt{5}+1$.

EXAMPLE 39 *If* z_1 *and* z_2 *are two complex numbers such that* $|z|<1<|z_2|$, *then prove that*

$$\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right| < 1$$

[IIT 2003]

SOLUTION We have,

$\left|1-z_1\bar{z}_2\right|^2-\left|z_1-z_2\right|^2$

$= \left\{1+|z_1\bar{z}_2|^2-2\,\mathrm{Re}\,(z_1\bar{z}_2)\right\}-\left\{|z_1|^2+|z_2|^2-2\,\mathrm{Re}\,(z_1\bar{z}_2)\right\}$

$= 1+|z_1|^2|z_2|^2-|z_1|^2-|z_2|^2$

$= \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)<0$ $[\because |z_1|<1<|z_2|]$

$\therefore \quad |1-z_1\bar{z}_2|^2 < |z_1-z_2|^2$

$\Rightarrow |1-z_1\bar{z}_2| < |z_1-z_2|$

$\Rightarrow \frac{|1-z_1\bar{z}_2|}{|z_1-z_2|} < 1$

$\Rightarrow \left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right| < 1$

EXAMPLEN 40 *Prove that there is no complex number* z *such that*

$|z|<\frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r|<2$. [IIT 2003]

SOLUTION We have,

$\sum_{r=1}^{n} a_r z^r = 1$

$\Rightarrow \left|\sum_{r=1}^{n} a_r z^r\right| = 1$

$\Rightarrow |a_1 z+a_2 z^2+\ldots+a_n z^n| = 1$

$\Rightarrow |a_1 z|+|a_2 z^2|+\ldots+|a_n z^n| \ge 1$ [Using triangle in equality]

$\Rightarrow 1 \le |a_1|\,|z| + |a_2|\,|z|^2 + \ldots + |a_n|\,|z|^n$

$\Rightarrow 1 < 2(|z| + |z|^2 + \ldots + |z|^n)$ $[\because |a_r| < 2, r = 1, 2, \ldots, n]$

$\Rightarrow 1 < 2\,\dfrac{|z|\,(1 - |z|^n)}{1 - |z|}$

$\Rightarrow 1 < \dfrac{2\left(|z| - |z|^{n+1}\right)}{1 - |z|}$

$\Rightarrow 1 < \dfrac{2|z|}{1 - |z|}$

$\Rightarrow 1 - \dfrac{2|z|}{1 - |z|} < 0$

$\Rightarrow \dfrac{1 - 3|z|}{1 - |z|} < 0$

$\Rightarrow \dfrac{|z| - 1}{3|z| - 1} < 0$

$\Rightarrow \dfrac{1}{3} < |z| < 1$

$\Rightarrow |z| > \dfrac{1}{3}$, which is a contradiction.

Hence, there exists no such complex number.

EXERCISE 19.1

1. Find real values of x and y satisfying the following equations:

(i) $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$

(ii) $(x^4 + 2xi) - (3x^2 + y\,i) = (3 - 5i) + (1 + 2iy)$

2. Find real values of θ lying between $-\pi$ and π so that

$$\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ is}$$

(i) purely real (ii) purely imaginary

3. If $f(x) = 2x^3 + 2x^2 - 7x + 72$, evaluate $f\left(\dfrac{3 - 5i}{2}\right)$.

4. If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, evaluate $f(3 + 2i)$

5. If z_1, z_2 and z_3 are three complex numbers, then prove that

$$z_1\,\text{Im}(\bar{z}_2 z_3) + z_2\,\text{Im}(\bar{z}_3 z_1) + z_3\,\text{Im}(\bar{z}_1 z_2) = 0$$

6. If a, b, c are real numbers and z is a complex number such that $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$. Prove that

$$\frac{1 + iz}{1 - iz} = \frac{a + ib}{1 + c}$$

7. Let z_1, z_2 be complex numbers such that

$$\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1 \text{ and } |z_2| \neq 1. \text{ Find } |z_1|.$$

8. Find all complex numbers z for which $\dfrac{z-2}{z+2}$ is purely imaginary.

9. If $\dfrac{1 - \bar{z}_1 z_2}{z_1 - z_2}$ is a unimodular complex number, prove that at least one of the two complex numbers z_1, z_2 must be unimodular.

10. Find all complex numbers z which satisfy the following equtions:

(i) $z = \bar{z}$ (ii) $z = -\bar{z}$ (iii) $z + \bar{z} = 3$ (iv) $z^2 + \bar{z} = 0$

(v) $z^2 = (i\bar{z})^2$ (vi) $z^2 = \bar{z}$ (vii) $\bar{z} = i z^2$

11. If $a, b \in R$, then prove that the equation $z^2 + az + b = 0$ will not have any purely real solution if $a^2 < 4b$.

12. Find the range of values of $a \in R$ for which the equation $z + a\,|\,z - 1\,| + 2i = 0$ has a solution. Also find the solution.

13. Find the complex number z which simultaneously satisfy the equation $\left|\dfrac{z - 12}{z - 8i}\right| = \dfrac{5}{3}$ and $\left|\dfrac{z-4}{z-8}\right| = 1$.

14. Find all complex numbers satisfying

$$2\,|\,z\,|^2 + z^2 - 5 + i\sqrt{3} = 0$$

15. For every real $a > 0$, determine the complex numbers which will satisfy the equation

$$|\,z\,|^2 - 2iz + 2a(1 + i) = 0$$

16. Find the maximum and minimum values of $|\,z\,|$ satisfying $\left|z + \dfrac{1}{z}\right| = 2$.

17. If $|\,z\,| \ge 3$, determine the least value of $\left|z + \dfrac{1}{z}\right|$.

18. If $\left|z + \dfrac{1}{z}\right| = 1$, prove that Re $(z) = 0$ when $|\,z\,|$ is the maximum.

19. If $\left|z + \dfrac{2}{z}\right| = 2$, then find the maximum value of $|\,z\,|$.

20. If $\left|z + \dfrac{1}{z}\right| = a$, where z is a complex number. What are the greatest and least possible values of $|\,z\,|$?

21. Find all the roots of the equation $z^3 - iz^2 - 2iz - 2 = 0$.

ANSWERS

1. (i) $x = \dfrac{5}{13}, y = \dfrac{14}{13}$

(ii) $x = 2, y = 3$ or $x = -2, y = \dfrac{1}{3}$

2. (i) $\theta = 0$ (ii) $\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

3. 4 4. 5 8. $2(\cos\theta + i\sin\theta), \theta \neq n\pi$

10. (i) $z = x + 0i, x \in R$ (ii) $z = iy, y \in R$

(iii) $z = \frac{3}{2} + iy, y \in R$ (iv) $z = 0, z = -1, z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

(v) $z = x(1+i), x \in R$ (vi) $z = 0, 1, \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$

(vii) $i, \pm\frac{\sqrt{3}}{2} - \frac{i}{2}$

12. $a \in \left[\frac{-\sqrt{5}}{2}, 1\right)$, $z = \frac{\alpha^2 \pm |\alpha|\sqrt{5-4\alpha^2}}{\alpha^2 - 1} - 2i$,

where $\alpha \in \left[\frac{-\sqrt{5}}{2}, 1\right)$

13. $z = 6 + 8i, z = 6 + 17i$ 14. $\pm\frac{1-3\sqrt{3}i}{\sqrt{6}}, \pm\frac{\sqrt{3}-i}{\sqrt{2}}$

15. $z = a + i\{-1 \pm \sqrt{1-a^2-2a}\}$, where $0 < a \leq \sqrt{2} - 1$.

16. Min $|z| = \sqrt{2} - 1$, Max. $|z| = \sqrt{2} + 1$

17. 8/3 19. $\sqrt{3} + 1$

20. Least value $= \frac{1}{2}[-a + \sqrt{a^2+4}]$

Greatest value $= \frac{1}{2}[a + \sqrt{a^2+4}]$

19.14 REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following forms:

(i) Geometrical form

(ii) Vectorial form and

(iii) Trigonometrical form or Polar form.

(iv) Eulerian Form

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane. To represent $z = x + iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and imaginary parts of z. This point $P(x, y)$ represents the complex number $z = x + iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the *real axis* and y-axis, as the *imaginary axis*.

Conversely, *if* $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the *affix* of the point P.

Thus, there exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers, *i.e.*, for every complex number $z = x + iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or the **Gaussian plane**. The point P, plotted on the Argand plane, is called the Argand diagram.

The length of the line segment OP is called the *modulus* of z and is denoted by $|z|$.

From Fig. 19.2, we have

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow OP^2 = x^2 + y^2$$

$$\Rightarrow OP = \sqrt{x^2 + y^2}$$

Thus,

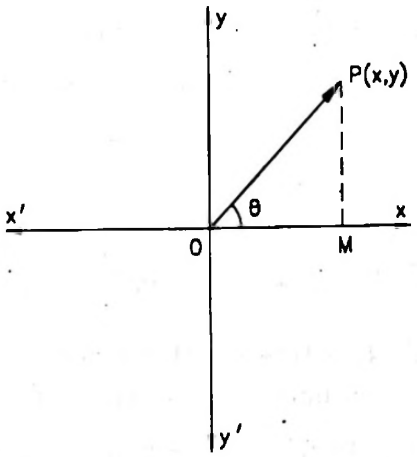


Fig. 19.2

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

The angle θ which OP makes with positive direction of x-axis in anticlockwise sense is called the **argument** or **amplitude** of z and is denoted by arg (z) or amp (z).

From Fig. 19.2, we have,

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left\{\frac{\text{Im}(z)}{\text{Re}(z)}\right\}$$

This angle θ has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi < \theta \leq \pi$ is called the principal value of the amplitude or principal argument. This formula for determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3}$

$= \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Since the above system of equations has infinitely many solutions, therefore there can be infinitely many arguments of $z = x + iy$. The argument θ which satisfies the inequality $-\pi < \theta \leq \pi$ is usually known as the **principal argument** of z. The argument of z depends upon the quadrant in which the point P lies as discussed below.

VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane because a complex number depends on two things viz. (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.

In Fig. 19.3, the complex number $z = x + iy$ is represented by the vector $\overrightarrow{OP}$ and in such a case $|z|$ is the length OP and arg (z) is the angle which the directed line OP makes with the positive direction of x-axis.

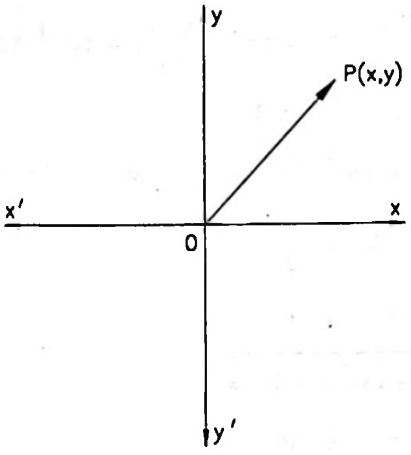


Fig. 19.3

19.15 ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER

In this section, we shall discuss the argument of a complex number for different signs of its real and imaginary parts.

(I) ARGUMENT OF $z = x + iy$ WHEN $x > 0$ AND $y > 0$

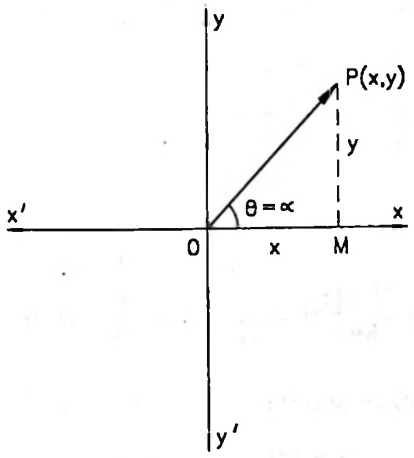


Fig. 19.4

Since x and y both are positive, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the first quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = \left|\frac{y}{x}\right|$. It is evident from Fig. 19.4 that $\theta = \alpha$.

Thus, if x and y both are positive, then the argument of $z = x + iy$ is the acute angle given by $\tan^{-1}\left|\frac{y}{x}\right|$.

(II) ARGUMENT OF $z = x + iy$ WHEN $x < 0$ AND $y > 0$

Since $x < 0$ and $y > 0$, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the second quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$.

It is evident from Fig. 19.5 that $\theta = \pi - \alpha$.

Thus, if $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan^{-1}|y/x|$.

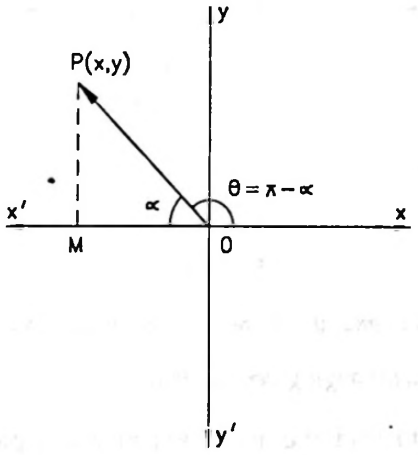


Fig. 19.5

(III) ARGUMENT OF $z = x + iy$ WHEN $x < 0$ AND $y < 0$

Since $x < 0$ and $y < 0$, therefore the point $P(x, y)$ representing $z = x + iy$ lies in the third quadrant. Let θ be the argument of z and α be the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$. It is evident from 19.6, that $\theta = -(\pi - \alpha) = \alpha - \pi$.

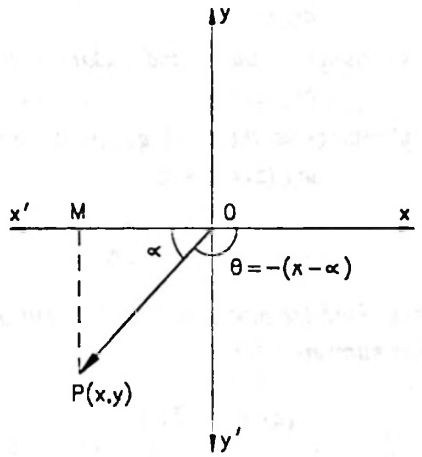


Fig. 19.6

Thus, if $x<0$ and $y<0$ then the argument of $z=x+iy$ is $\alpha-\pi$ where α is the acute angle given by $\tan\alpha=\left|\frac{y}{x}\right|$.

(IV) ARGUMENT OF $z=x+iy$ WHEN $x>0$ AND $y<0$

Since $x>0$ and $y<0$, therefore the point $P(x, y)$ representing $z=x+iy$ lies in the fourth quadrant. Let θ be the argument of z and let α be the acute angle given by $\tan\alpha=\left|\frac{y}{x}\right|$. Then from Fig. 19.7, we have

$$\theta=-\alpha.$$

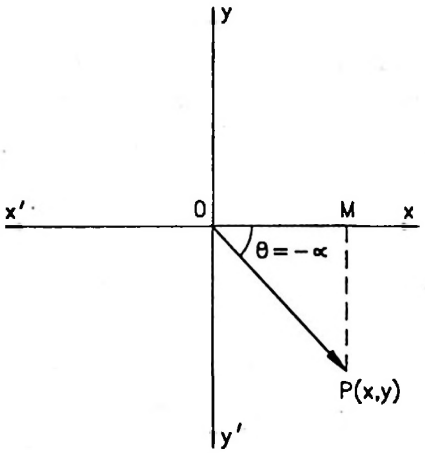


Fig. 19.7

Thus, if $x>0$ and $y<0$, then the argument of $z=x+iy$ is $-\alpha$ where α is the acute angle given by $\tan\alpha=\left|\frac{y}{x}\right|$.

In order to find the argument of a complex number $z=x+iy$, we may use the following algorithm.

ALGORITHM

STEP I *Find the value of* $\tan^{-1}\left|\frac{y}{x}\right|$ *lying between 0 and $\pi/2$. Let it be α.*

STEP II *Determine in which quadrant the point $P(x, y)$ belongs. If $P(x, y)$ belongs to the first quadrant, then*

$$arg\ (z)=\alpha.$$

If $P(x, y)$ belongs to the second quadrant, then

$$arg\ (z)=\pi-\alpha$$

If $P(x, y)$ belongs to the third quadrant, then

$$arg\ (z)=\pi+\alpha$$

If $P(x, y)$ belongs to the fourth quadrant, then

$$arg\ (z)=-\alpha\ or,\ 2\pi-\alpha.$$

ILLUSTRATION 1 *Find the modulus and argument of each of the following complex numbers:*

(i) $1+i\sqrt{3}$ (ii) $-2+2i\sqrt{3}$

(iii) $-\sqrt{3}-i$ (iv) $2\sqrt{3}-2i$

SOLUTION (i) Let $z=1+i\sqrt{3}$ and let α be the acute angle given by

$$\alpha=\tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\tan^{-1}\left|\frac{\sqrt{3}}{1}\right|.$$

$$\tan\alpha=\sqrt{3}$$

$$\Rightarrow\quad \alpha=\frac{\pi}{3}.$$

Since Re (z) > 0 and Im (z) > 0. Therefore,

$$arg\ (z)=\alpha=\frac{\pi}{3}.$$

We have, $|z|=\sqrt{1^2+(\sqrt{3})^2}=2.$

(ii) Let $z=-2+2\sqrt{3}\,i$. Then,

$$|z|=\sqrt{(-2)^2+(2\sqrt{3})^2}=4.$$

Let α be the angle given by

$$\alpha=\tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\tan^{-1}\left|\frac{2\sqrt{3}}{-2}\right|=\tan^{-1}(\sqrt{3}).$$

$\Rightarrow \tan\alpha=\sqrt{3}\Rightarrow\alpha=\pi/3.$

Since Re (z) < 0 and Im (z) > 0. Therefore,

$$arg\ (z)=\pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}.$$

(iii) Let $z=-\sqrt{3}-i$. Then

$$|z|=\sqrt{(-\sqrt{3})^2+(-1)^2}=2.$$

Let α be the acute angle given by

$$\alpha=\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{-1}{-\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\quad \alpha=\frac{\pi}{6}$$

Since Re (z) < 0 and Im (z) < 0. Therefore,

$$arg\ (z)=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$$

(iv) Let $z=2\sqrt{3}-2i$. Then,

$$|z|=\sqrt{(2\sqrt{3})^2+(-2)^2}=4.$$

Let α be the acute angle given by $\tan^{-1}\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{-2}{2\sqrt{3}}\right|=\frac{1}{\sqrt{3}}.\ \Rightarrow\ \alpha=\frac{\pi}{6}$$

Since Re (z) > 0 and Im (z) < 0. Therefore,

$$arg\ (z)=-\alpha=-\pi/6.$$

19.16 POLAR OR TRIGONOMETRICAL FORM OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then, by the geometrical representation of $z = x + iy$, we have

$OP = |z|$ and $\angle POX = \theta = \arg(z)$

In ΔPOM, we have

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|}$$

$$\Rightarrow \quad x = |z|\cos\theta$$

$$\text{and, } \sin\theta = \frac{PM}{OP} = \frac{y}{|z|}$$

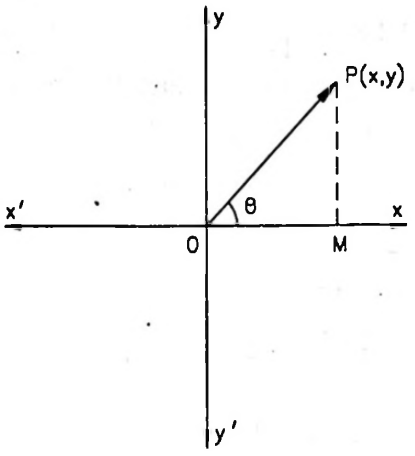


Fig. 19.8

$$\Rightarrow \quad y = |z|\sin\theta$$

$$\therefore \quad z = x + iy$$

$$\Rightarrow \quad z = |z|\cos\theta + i|z|\sin\theta$$

$$\Rightarrow \quad z = |z|(\cos\theta + i\sin\theta)$$

$$\Rightarrow \quad z = r(\cos\theta + i\sin\theta), \text{ where } r = |z| \text{ and } \theta = \arg(z)$$

This form of z is called a polar form of z. If we use the general value of the argument of θ, then the polar form of z is given by

$$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)],$$

where $r = |z|$, $\theta = \arg(z)$ and n is an integer.

POLAR FORM OF $z = x + iy$ FOR DIFFERENT SIGNS OF x AND y.

Let $|z| = r$ and α be the acute angle given by $\tan^{-1}\left(\frac{y}{x}\right)$.

Let θ be the argument of z.

CASE I *Polar form of $z = x + iy$ when $x > 0$ and $y > 0$:*

In this case, we have $\theta = \alpha$. So, the polar form of $z = x + iy$ is

$$r(\cos\alpha + i\sin\alpha).$$

CASE II *Polar form of $z = x\,iy$ when $x < 0$ and $y > 0$:*

In this case, we have $\theta = \pi - \alpha$. So, the polar form of $z = x + iy$ is $r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)$ *or,* $r(-\cos\alpha + i\sin\alpha)$

CASE III *Polar form of $z = x + iy$ when $x < 0$ and $y < 0$:*

In this case, we have $\theta = -(\pi - \alpha)$. So, the polar form of z is

$r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))]$ or $r(-\cos\alpha - i\sin\alpha)$

CASE IV Polar form of $z = x + iy$ when $x > 0$ and $y < 0$:

In this case, we have $\theta = -\alpha$. So, the polar form of z is

$r[\cos(-\alpha) + i\sin(-\alpha)]$ or, $r[\cos\alpha - i\sin\alpha]$

ILLUSTRATION1 *Write the following complex numbers in the polar form:*

(i) $-3\sqrt{2} + 3\sqrt{2}\,i$ (ii) $1 + i$

(iii) $-1 - i$ (iv) $1 - i$

SOLUTION (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then,

$$r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6.$$

$$\text{Let} \quad \tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = 1 \Rightarrow \alpha = \pi/4$$

Since the point representing z lies in the second quadrant. Therefore, the argument of z is given by

$$\theta = \pi - \alpha = \pi - (\pi/4) = (3\pi/4).$$

Hence, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

(ii) Let $z = 1 + i$. Then,

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Let $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point (1, 1) representing z lies in first quadrant. Therefore, the argument of z is given by

$$\theta = \alpha = \frac{\pi}{4}.$$

So, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

(iii) Let $z = -1 - i$. Then,

$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}.$$

Let $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point $(-1, -1)$ representing z lies in the third quadrant, therefore the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}.$$

So, the polar form of $z = -1 - i$ is

$r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\}$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$.

(iv) Let $z = 1 - i$. Then,

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Let $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{1}\right| = 1$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4}$$

Since the point $(1, -1)$ lies in the fourth quadrant. Therefore, the argument of z is given by

$$\theta = -\alpha = -\frac{\pi}{4}.$$

So, the polar form of $z = 1 - i$ is

$r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}$

$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

ILLUSTRATION 2 *Find the modulus and principal argument of* $(1 + i)$ *and hence express it in the polar form.*

SOLUTION Let $z = 1 + i$. Then,

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Since the point $(1, 1)$ representing $z = 1 + i$ lies in first quadrant. Therefore,

$$\theta = \arg(z) = \frac{\pi}{4}.$$

Thus, we have

$r = |z| = \sqrt{2}$ and $\theta = \arg(z) = \frac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

ILLUSTRATION 3 *Find the modulus and principal argument of* $-2i$.

SOLUTION Let $z = -2i = 0 + (-2)i$. Then,

$$|z| = \sqrt{0 + (-2)^2} = 2.$$

Clearly, the point $(0, -2)$ representing $z = -2i$ lies on the negative side of imaginary axis. Therefore, principal argument of z is $-\pi/2$.

ILLUSTRATION 4 *Find the modulus and principal argument of* -4.

SOLUTION Let $z = -4 + 0i$. Then,

$$|z| = \sqrt{(-4)^2 + 0} = 4.$$

Since the point $(-4, 0)$ representing $z = -4 + 0i$ lies on the negative side of real axis. Therefore, principal argument of z is π.

ILLUSTRATION 5 *Express the following complex numbers in the polar form:*

(i) $\frac{1+i}{1-i}$ *(ii)* $\frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

SOLUTION (i) Let $z = \frac{1+i}{1-i}$. Let $z = r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Now, $z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)}$

$$= \frac{1 - 2i + i^2}{1 - i^2}$$

$$= \frac{1 + 2i - 1}{1 + 1} = i = 0 + 1i.$$

$\therefore\ r = |z| = \sqrt{0 + 1} = 1.$

Since the point $(0, 1)$ representing $z = 0 + i$ lies on positive direction of imaginary axis. Therefore,

$\arg(z) = \pi/2$.

Hence, $z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

(ii) Let $z = \frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z, where $r = |z|$ and $\theta = \arg(z)$.

Now, $z = \frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

$$\Rightarrow \quad z = \frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} \cdot \frac{(5-\sqrt{3}\,i)}{(5-\sqrt{3}\,i)}$$

$$\Rightarrow \quad z = \frac{28 + 28\sqrt{3}\,i}{28} = 1 + i\sqrt{3}$$

$\therefore\ r = |z| = \sqrt{1+3} = 2.$

Let α be the smallest positive angle given by

$\tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right) = \tan^{-1}(\sqrt{3}/1) = \tan^{-1}(\sqrt{3})$. Then, $\alpha = \pi/3$.

Since the point $(1, \sqrt{3})$ lies in first quadrant. Therefore, $\theta = \alpha = \pi/3$.

Hence, the polar form of z is $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

19.17 EULERIAN FORM OF A COMPLEX NUMBER

We have,

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta$$

These two are called Euler's notations.

Let z be any complex number such that $|z| = r$ and $\arg(z) = \theta$. Then, in polar from z can be written as

$$z = r(\cos\theta + i\sin\theta)$$

Using Euler's notations, we have

$$z = re^{i\theta}$$

This form of z is known as the *Eulerian form.*

ILLUSTRATION *Express the following complex numbers in Eulerian form :*

(i) $1+i$ *(ii)* $-2+2i$ *(iii)* $-1-i\sqrt{3}$

SOLUTION (i) Let $z = 1 + i$. Then,

$$r = |z| = \sqrt{1^2+1^2} = \sqrt{2} \text{ and } \theta = \arg(z) = \pi/4$$

So, Eulerian form of z is $\sqrt{2}\,e^{i\pi/4}$

(ii) Let $z = -2 + 2i$. Then,

$$r = |z| = \sqrt{(-2)^2+2^2} = 2\sqrt{2}$$

Let θ be the argument of z. Then,

$$\tan\theta = \frac{2}{-2} = -1 \Rightarrow \theta = 3\pi/4$$

So, Eulerian form of z is $2\sqrt{2}\,e^{i3\pi/4}$

(iii) Let $z = -1 - i\sqrt{3}$. Then,

$$r = |z| = \sqrt{(-1)^2+(-\sqrt{3})^2} = 2$$

Let θ be the argument of z. Then,

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} \Rightarrow \theta = 4\pi/3$$

So, Eulerian form of z is $2e^{i4\pi/3}$.

19.18 GEOMETRICAL REPRESENTATIONS OF FUNDAMENTAL OPERATIONS

GEOMETRICAL REPRESENTATION OF ADDITION

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by points $P(a_1, b_1)$ and $Q(a_2, b_2)$ in the Argand plane. Join the origin O with P and Q and complete the parallelogram $OPRQ$ by taking OP and OQ as two adjacent sides. Draw perpendiculars PK, QL and RM from P, Q and R respectively on x-axis. Also draw PN to RM. Since the diagonals of a parallelogram bisect each other, therefore coordinates of the mid point of PQ and also that of OR are $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$. So, the coordinates of R are (a_1+a_2, b_1+b_2). Hence, R represents the complex number

$$(a_1+a_2)+i(b_1+b_2) = (a_1+ib_1)+(a_2+ib_2) = z_1+z_2$$

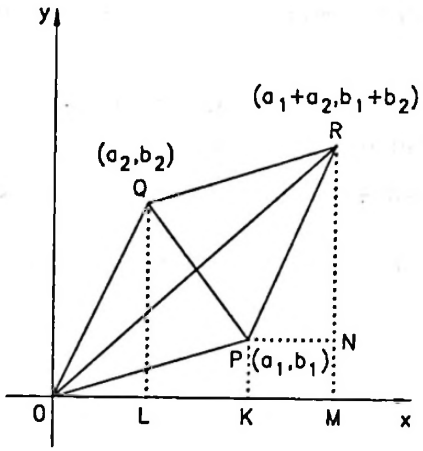


Fig. 19.9

Thus, if two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ$ having OP and OQ as two adjacent sides.

GEOMETRICAL REPRESENTATION OF SUBTRACTION

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by points $P(a_1, b_1)$ and $Q(a_2, b_2)$ in the Argand plane. Produce QO to Q' such that $OQ = OQ'$. Then Q' represents the complex number $(-z_2)$. Complete the parallelogram $OPRQ'$ by taking OP and OQ' as two adjacent sides. As discused above, the sum of z_1 and $-z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ'$.

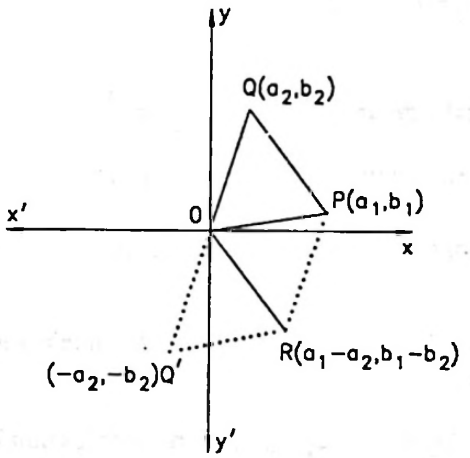


Fig. 19.10

Hence, R represents the complex number $z_1 - z_2$.

19.19 MODULUS AND ARGUMENT OF MULTIPLICATION OF TWO COMPLEX NUMBERS

THEOREM *For any two complex numbers z_1, z_2, we have*

(i) $|z_1z_2| = |z_1||z_2|$,

(ii) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

PROOF Let $z_1 = a_1 + i\,b_1$ .and $z_2 = a_2 + i\,b_2$ be two complex numbers.

Let the polar forms of z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and, $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively i.e.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$r_1 = |z_1|, r_2 = |z_2|, \theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

Now,

$z_1 z_2$

$= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\}$

$\Rightarrow \quad z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\Rightarrow \quad |z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$

and,

$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$.

NOTE *The right hand side of* $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ *provides the general value of the argument of the product* $z_1 z_2$. *This result is not true for principal argument. For, if* $z_1 = -\sqrt{3} + i$ *and* $z_2 = -1 + i\sqrt{3}$, *then*

pr. arg $(z_1) = 5\pi/6$, *pr. arg* $(z_2) = 2\pi/3$, $z_1 z_2 = -4i$.

So, *pr. arg* $(z_1 z_2) = -\pi/2$

and,

$$pr.\ arg(z_1) + pr.\ arg(z_2) = \frac{5\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{2}$$

$\therefore$ pr. arg $(z_1 z_2) \neq$ pr. arg (z_1) + pr. arg (z_2).

In view of the above note the above result can be stated as follows:

The sum of the arguments of two complex numbers is an argument of their product.

GENERALISATION If $z_1, z_2, z_3, \ldots, z_n$ are complex numbers, then

(i) $|z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$

(ii) $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

19.20 GEOMETRICAL REPRESENTATION OF MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers represented by points P and Q in the Argand plane. Let $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$. Then as discussed above we have $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$. So, to find the point R representing the complex number $z_1 z_2$, we have to construct the point with polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. For this take a point L on real axis such that $OL = 1$ and draw a triangle OQR similar to the triangle OLP. Since triangles OQR and OLP are similar. Therefore,

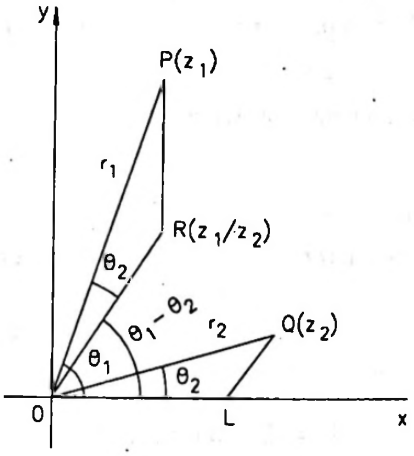


Fig. 19.11

$$\frac{OR}{OQ} = \frac{OP}{OL}$$

$$\Rightarrow \quad \frac{OR}{r_2} = \frac{r_1}{1}$$

$$\Rightarrow \quad OR = r_1 r_2$$

and, $\angle QOR = \angle LOP = \theta_1$

so that $\angle XOR = \angle XOQ + \angle QOR = \theta_2 + \theta_1$.

Hence, R has the polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. Consequently it represents the complex number $z_1 z_2$.

19.21 MODULUS AND ARGUMENT OF DIVISION OF TWO COMPLEX NUMBERS

THEOREM *If* z_1 *and* $z_2 (\neq 0)$ *are two complex numbers, then*

$$(i)\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (ii)\ \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

PROOF Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers. Let the polar forms of z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively. Then,

$$|z_1| = r_1,\ |z_2| = r_2,\ \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2.$$

$$\therefore \quad \frac{z_1}{z_2}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|},$$

and,

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2).$$

NOTE *The right hand side of* $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$ *provides the general value of the argument of* $\frac{z_1}{z_2}$. *This result is not true for principal argument. For, if*

$z_1 = -\sqrt{3} - i$ and $z_2 = -\sqrt{3} + i$, then pr. $\arg(z_1) = -5\pi/6$, pr. $\arg(z_2) = 5\pi/6$, and $\frac{z_1}{z_2} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

$$\therefore \text{ pr. } arg\left(\frac{z_1}{z_2}\right) = \pi/3 \neq \text{pr. } arg(z_1) - \text{pr. } arg(z_2).$$

19.22 GEOMETRICAL REPRESENTATION OF THE DIVISION OF COMPLEX NUMBER

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by points P and Q in the Argand plane. Let $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$. Then as discussed above we have $|z_1/z_2| = r_1/r_2$ and $\arg(z_1/z_2) = \theta_1 - \theta_2$. So, to find a point R representing the complex number z_1/z_2, we have to construct the point with polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$. For this take a point L on real axis OX such that $OL = 1$ and draw a triangle OPR similar to OQL. Since triangles OPR and OQL are similar, therefore

$$\frac{OP}{OQ} = \frac{OR}{OL}$$

$$\Rightarrow \quad \frac{OR}{1} = \frac{r_1}{r_2}$$

$$\Rightarrow \quad OR = \frac{r_1}{r_2}$$

and, $\angle XOR = \angle XOP - \angle ROP = \theta_1 - \theta_2$

Hence, R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$.

Consequently it represents the complex number z_1/z_2.

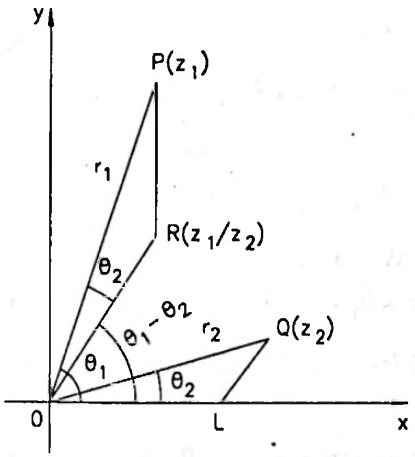


Fig. 19.12

19.23 GEOMETRICAL REPRESENTATION OF CONJUGATE OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex number. Clearly, $z = x + iy$ is represented by a point $P(x, y)$ in the argand plane.

Now, $z = x + iy \Rightarrow \bar{z} = x - iy = x + i(-y)$. So, $\bar{z}$ is represented by a point $Q(x, -y)$ in the argand plane.

Clearly, Q is the image of point P in the real axis.

Thus, if a point P represents a complex number z, then its conjugate $\bar{z}$ *is represented by the image of P in the real axis.*

It is evident from Fig. 19.13 that

$$|z| = |\bar{z}| \text{ and } \arg(\bar{z}) = -\arg(z)$$

The general value of $\arg(\bar{z})$ is $2n\pi + \arg(z)$.

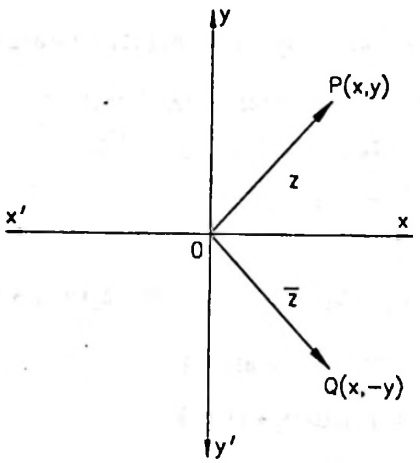


Fig. 19.13

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If z, z_1 and z_2 are complex numbers, prove that :*

(i) $\arg(\bar{z}) = -\arg(z)$.

In general, $\arg(\bar{z}) = 2n\pi - \arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg(z_1 \bar{z_2}) = \arg(z_1) - \arg(z_2)$

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

SOLUTION (i) Let $z = r(\cos\theta + i\sin\theta)$ be the polar form of z. Then,

$|z| = r$ and $\arg(\bar{z}) = \theta$.

Now,

$z = r(\cos\theta\, i + i\sin\theta)$

$\Rightarrow \quad \bar{z} = r(\cos\theta - i\sin\theta)$

$\Rightarrow \quad \bar{z} = r\{\cos(-\theta) + i\sin(-\theta)\}$

$\Rightarrow |\bar{z}| = r$ and $\arg(\bar{z}) = -\theta$.

Since $\cos\theta$ and $\sin\theta$ are periodic functions with period 2π. Therefore, in general

$$\arg(\bar{z}) = 2n\pi - \arg(z).$$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in their polar forms. Then,

$|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

Now,

$z_1 z_2$

$= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\Rightarrow |z_1 z_2| = r_1 r_2$ and, $\arg(z_1 z_2) = \theta_1 + \theta_2$

$\Rightarrow |z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

REMARK *It follows from the above result that*

(i) $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$

$\Rightarrow \quad |z^n| = |z|^n$

and,

(ii) $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

$\Rightarrow \quad \arg(z^n) = n\arg(z)$.

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then,

$\bar{z_2} = r_2(\cos\theta_2 - i\sin\theta_2) = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$\therefore \quad z_1\bar{z_2} = r_1 r_2\,[\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]$ [Using (ii

$\Rightarrow \quad z_1\bar{z_2} = r_1 r_2\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\Rightarrow \arg(z_1\bar{z_2}) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$.

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then, $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Now,

$$\frac{z_1}{z_2}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

$$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2).$$

EXAMPLE 2 *Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then, prove that*

(i) $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2)$

OR

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$

(ii) $|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_1 - \theta_2)$

OR

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$

SOLUTION We have,

$z_1 = r_1\cos\theta_1 + i\sin\theta_1$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow |z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

(i) We have,

$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore \quad |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$\Rightarrow \quad |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$

(ii) We have,

$z_1-z_2 = (r_1\cos\theta_1 - r_2\cos\theta_2)+i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$\therefore |z_1-z_2|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2+(r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$\Rightarrow |z_1-z_2|^2 = r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$\Rightarrow |z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

EXAMPLE 3 *For any two complex numbers z_1 and z_2, prove that :*

(i) $|z_1+z_2| = |z_1-z_2| \Leftrightarrow \arg(z_1)-\arg(z_2) = \frac{\pi}{2}$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(ii) $|z_1+z_2| = |z_1||z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely real.

(iii) $|z_1+z_2|^2 = |z_1|^2+|z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary

SOLUTION Let

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$

Then,

$|z_1| = r_1,\ |z_2| = r_2,\ \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

(i) We have,

$|z_1+z_2| = |z_1-z_2|$

$\Leftrightarrow |z_1+z_2|^2 = |z_1-z_2|^2$

$\Leftrightarrow r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2) = r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$\Leftrightarrow 4r_1r_2\cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow \cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow \theta_1-\theta_2 = \frac{\pi}{2}$ i.e. $\arg(z_1)-\arg(z_2) = \frac{\pi}{2}$

$\Leftrightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ $\left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)\right]$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(ii) We have,

$|z_1+z_2| = |z_1|+|z_2|$

$\Leftrightarrow |z_1+z_2|^2 = (r_1+r_2)^2$ $[\because |z_1| = r_1$ and $|z_2| = r_2]$

$\Leftrightarrow r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2) = r_1^2+r_2^2+2r_1r_2$

$\Leftrightarrow \cos(\theta_1-\theta_2) = 1$

$\Leftrightarrow \theta_1-\theta_2 = 0$ i.e. $\arg(z_1)-\arg(z_2) = 0$ or, $\arg(z_1) = \arg(z_2)$

$\Leftrightarrow \arg\left(\frac{z_1}{z_2}\right) = 0$ $\left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)\right]$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely real

(iii) We have,

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2$

$\Leftrightarrow r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2) = r_1^2+r_2^2$

$\Leftrightarrow 2r_1r_2\cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow \cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow \theta_1-\theta_2 = \frac{\pi}{2}$

$\Leftrightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ $\left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2) = \theta_1-\theta_2\right]$

$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

EXAMPLE 4 *If z and w are two complex numbers such that $|z| \le 1, |w| \le 1$, then prove that*

(i) $|z-w|^2 \le (|z|-|w|)^2+\{\arg(z)-\arg(w)\}^2$

(ii) $|z+w|^2 \ge (|z|+|w|)^2-\{\arg(z)-\arg(w)\}^2$

[IIT 1995]

SOLUTION Let

$z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then,

$|z| = r_1,\ |w| = r_2,\ \arg(z) = \theta_1$ and $\arg(w) = \theta_2$.

(i) We have,

$$
\begin{aligned}
&|z-w|^2\\
&= r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\\
&= r_1^2+r_2^2-2r_1r_2+2r_1r_2-2r_1r_2\cos(\theta_1-\theta_2)\\
&= (r_1-r_2)^2+2r_1r_2(1-\cos(\theta_1-\theta_2))\\
&= (r_1-r_2)^2+4r_1r_2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\\
&= (|z|-|w|)^2+4|z||w|\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\\
&\le (|z|-|w|)^2+4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\\
&\qquad [\because |z|<1, |w| \le 1]\\
&\le (|z|-|w|)^2+4\left(\frac{\theta_1-\theta_2}{2}\right)^2 \qquad \left[\because \sin\theta < \theta \text{ for } \theta > 0,\ \therefore \sin^2\theta < \theta^2 \text{ for all } \theta\right]\\
&= (|z|-|w|)^2+(\theta_1-\theta_2)^2
\end{aligned}
$$

$= (|z| - |w|)^2 + (\arg(z) - \arg(w))^2$

Hence,

$|z-w|^2 \le (|z| - |w|)^2 + (\arg(z) - \arg(w))^2$

(ii) We have,

$|z+w|^2$

$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 + 2r_1 r_2 - 2r_1 r_2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$= (r_1 + r_2)^2 - 2r_1 r_2 (1 - \cos(\theta_1 - \theta_2))$

$= (r_1 + r_2)^2 - 4r_1 r_2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$= (|z| + |w|)^2 - 4|z||w| \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$\ge (|z| + |w|)^2 - 4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$ $[\because |z| \le 1, |w| \le 1]$

$\ge (|z| + |w|)^2 - 4\left(\frac{\theta_1 - \theta_2}{2}\right)^2$ $\left[\because \sin^2\theta < \theta^2 \text{ for all } \theta \in R,\ \therefore \sin^2\frac{\theta_1-\theta_2}{2} < \left(\frac{\theta_1-\theta_2}{2}\right)^2\right]$

$= (|z| + |w|)^2 - (\theta_1 - \theta_2)^2$

$= (|z| + |w|)^2 - (\arg(z) - \arg(w))^2$

Hence,

$|z+w|^2 \ge (|z| + |w|)^2 - (\arg(z) - \arg(w))^2$.

EXAMPLE 5 *For any complex number z, prove that*

$$|\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,|z|$$

SOLUTION Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$|\text{Re}(z)| + \text{Im}(z)| = |r\cos\theta| + |r\sin\theta|$

$= r(|\cos\theta| + |\sin\theta|)$ $[\because r = |z| > 0]$

$\Rightarrow (|\text{Re}(z)| + |\text{Im}(z)|)^2 = r^2(|\cos\theta| + |\sin\theta|)^2$

$= r^2(1 + |2\sin\theta\cos\theta|)$

$= r^2(1 + |\sin 2\theta|)$

$\le r^2(1+1)$ $[\because |\sin 2\theta| \le 1]$

$\Rightarrow |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,r$

$\Rightarrow |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,|z|$

EXAMPLE 6 *For any complex number z, prove that*

(i) $\left|\frac{z}{|z|} - 1\right| \le \arg(z)$

(ii) $|z-1| \le \big||z|-1\big| + |z|\,|\arg(z)|$

SOLUTION (i) Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$\left|\frac{z}{|z|} - 1\right| = |\cos\theta + i\sin\theta - 1|$

$= |(\cos\theta - 1) + i\sin\theta|$

$= [(\cos\theta - 1)^2 + \sin^2\theta]^{1/2}$

$= (2 - 2\cos\theta)^{1/2}$

$= (4\sin^2\theta/2)^{1/2}$

$= 2\left|\sin\frac{\theta}{2}\right|$

$\therefore \left|\frac{z}{|z|} - 1\right| = 2\left|\sin\frac{\theta}{2}\right|$

$\le 2(\theta/2) = \theta$ $[\because \sin\theta \le \theta \text{ for } \theta \ge 0]$

Hence, $\left|\frac{z}{|z|} - 1\right| \le \arg(z)$.

(ii) We have,

$|z-1| = \big|z - |z| + |z| - 1\big|$

$\le \big|z - |z|\big| + \big||z| - 1\big|$ $[\because |z_1 + z_2| \le |z_1| + |z_2|]$

$= |z|\left|\frac{z}{|z|} - 1\right| + \big||z| - 1\big|$

$\le |z|\arg(z) + \big||z| - 1\big|$ [Using (i)]

Hence,

$|z-1| \le |z|\arg(z) + \big||z| - 1\big|$.

EXAMPLE 7 *Let z_1 and z_2 be two complex numbers and a, b be real numbers such that $a^2 + b^2 \ne 0$. Prove that*

$$|\ |^2 + |z_2|^2 - |z_1^2 + z_2^2| \le 2\frac{|az_1 + bz_2|^2}{a^2 + b^2} \le |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$$

SOLUTION Let

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then, $|z_1| = r_1$ and $|z_2| = r_2$.

Also, let $a = r\cos\alpha$ and $b = r\sin\alpha$. Then,

$a^2 + b^2 = r^2$.

Now,

$z_1^2 + z_2^2 = r_1^2(\cos 2\theta_1 + i\sin 2\theta_1) + r_2^2(\cos 2\theta_2 + i\sin 2\theta_2)$

$= (r_1^2\cos 2\theta_1 + r_2^2\cos 2\theta_2) + i(r_1^2\sin 2\theta_1 + r_2^2\sin 2\theta_2)$

$\therefore |z_1^2 + z_2^2| = \sqrt{r_1^4 + r_2^4 + 2r_1^2 r_2^2\cos 2(\theta_1 - \theta_2)}$ ...(i)

and,

$az_1 + bz_2$

$= r\cos\alpha \cdot r_1(\cos\theta_1 + i\sin\theta_1) + r\sin\alpha \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$$= \{r r_1 \cos\alpha \cos\theta_1 + r r_2 \sin\alpha \cos\theta_2\} + i\{r r_1 \cos\alpha \sin\theta_1 + r r_2 \sin\alpha \sin\theta_2\}$$

$$\therefore |az_1 + bz_2|^2 = (r r_1 \cos\alpha \cos\theta_1 + r r_2 \sin\alpha \cos\theta_2)^2 + (r r_1 \cos\alpha \sin\theta_1 + r r_2 \sin\alpha \sin\theta_2)^2$$

$$= r^2 r_1^2 \cos^2\alpha + r^2 r_2^2 \sin^2\alpha + 2r^2 r_1 r_2 \cos\alpha \sin\alpha \cos(\theta_1 - \theta_2)$$

$$= \frac{r^2}{2}[r_1^2(1+\cos 2\alpha) + r_2^2(1-\cos 2\alpha) + 2 r_1 r_2 \sin 2\alpha \cos(\theta_1 - \theta_2)]$$

$$\Rightarrow \frac{2|az_1 + bz_2|^2}{r^2} = r_1^2 + r_2^2 + (r_1^2 - r_2^2)\cos 2\alpha + 2r_1 r_2 \sin 2\alpha \cos(\theta_1 - \theta_2)$$

$$\Rightarrow \frac{2|az_1 + bz_2|^2}{a^2+b^2} - r_1^2 - r_2^2 = (r_1^2 - r_2^2)\cos 2\alpha + 2r_1 r_2 \cos(\theta_1 - \theta_2).\sin 2\alpha \quad ...(ii)$$

We know that, $|p\sin\theta + q\cos\theta| \le \sqrt{p^2+q^2}$ for all values of θ.

$$\therefore |(r_1^2 - r_2^2)\cos 2\alpha + 2r_1 r_2 \cos(\theta_1 - \theta_2)\sin 2\alpha| \le \sqrt{(r_1^2 - r_2^2)^2 + \{2r_1 r_2 \cos(\theta_1 - \theta_2)\}^2}$$

$$= \sqrt{(r_1^2 - r_2^2)^2 + 4 r_1^2 r_2^2 \cos^2(\theta_1 - \theta_2)}$$

$$= \sqrt{(r_1^2 - r_2^2)^2 + 2 r_1^2 r_2^2 \{1 + \cos(2\theta_1 - 2\theta_2)\}}$$

$$= \sqrt{(r_1^4 + r_2^4) + 2 r_1^2 r_2^2 \cos 2(\theta_1 - \theta_2)}$$

$$= |z_1^2 + z_2^2| \quad \text{[Using (i)]}$$

Thus,

$$|(r_1^2 - r_2^2)\cos 2\alpha + 2r_1 r_2 \cos(\theta_1 - \theta_2)\sin 2\alpha| \le |z_1^2 + z_2^2| \quad ...(iii)$$

From (ii) and (iii), we get

$$\left|\frac{2|az_1 + bz_2|^2}{a^2+b^2} - r_1^2 - r_2^2\right| \le |z_1^2 + z_2^2|$$

$$\Leftrightarrow -|z_1^2 + z_2^2| \le \left|\frac{2|az_1 + bz_2|^2}{a^2+b^2} - |z_1|^2 - |z_2|^2\right| \le |z_1^2 + z_2^2|$$

$$\Leftrightarrow |z_1|^2 + |z_2|^2 - |z_1^2 + z_2^2| \le \frac{2|az_1 + bz_2|^2}{a^2+b^2} \le |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$$

EXAMPLE 8 *If* $\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}\, z^{1/3})$, *then find the value of* $|z|$.

SOLUTION We have,

$$\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}\, z^{1/3})$$

$$\Rightarrow 2\arg(z^{1/3}) = \arg(z^2 + \bar{z}\, z^{1/3})$$

$$\Rightarrow \arg(z^{2/3}) = \arg(z^2 + \bar{z}\, z^{1/3}) \quad [\because n\arg(z) = \arg(z^n)]$$

$$\Rightarrow \arg(z^2 + \bar{z}\, z^{1/3}) - \arg(z^{2/3}) = 0$$

$$\Rightarrow \arg\left(\frac{z^2 + \bar{z}\, z^{1/3}}{z^{2/3}}\right) = 0 \quad \left[\because \arg(z_1) - \arg(z_2) = \arg\left(\frac{z_1}{z_2}\right)\right]$$

$$\Rightarrow \arg(z^{4/3} + \bar{z}\, z^{-1/3}) = 0$$

$$\Rightarrow z^{4/3} + \bar{z}\, z^{-1/3} \text{ is purely real}$$

$$\Rightarrow (z^{4/3} + \bar{z}\, z^{-1/3}) = (z^{4/3} + \bar{z}\, z^{1/3})$$

$$\Rightarrow z^{4/3} + \bar{z}\, z^{-1/3} = \bar{(z)}^{4/3} + z\,\bar{(z)}^{-1/3} \quad [\because \bar{(z^n)} = \bar{(z)}^n]$$

$$\Rightarrow z^{4/3} + \frac{\bar{z}}{z^{1/3}} = \bar{(z)}^{4/3} + \frac{z}{\bar{(z)}^{1/3}}$$

$$\Rightarrow z^{4/3} + \frac{\bar{z}.\bar{(z)}^{1/3}}{z^{1/3}\bar{(z)}^{1/3}} = \bar{(z)}^{4/3} + \frac{z.z^{1/3}}{z^{1/3}\bar{(z)}^{1/3}}$$

$$\Rightarrow z^{4/3} + \frac{\bar{(z)}^{4/3}}{(z\bar{z})^{1/3}} = \bar{(z)}^{4/3} + \frac{z^{4/3}}{(z\bar{z})^{1/3}}$$

$$\Rightarrow z^{4/3} + \frac{\bar{(z)}^{4/3}}{|z|^{2/3}} = \bar{(z)}^{4/3} + \frac{z^{4/3}}{|z|^{2/3}}$$

$$\Rightarrow z^{4/3} - \bar{(z)}^{4/3} = \frac{z^{4/3} - \bar{(z)}^{4/3}}{|z|^{2/3}}$$

$$\Rightarrow 1 = \frac{1}{|z|^{2/3}} \quad [\because z \ne \bar{z}]$$

$$\Rightarrow |z|^{2/3} = 1$$

$$\Rightarrow |z| = 1$$

EXAMPLE 9 *If* z_1 *and* z_2 *are two complex numbers such that both of them satisfy the relation* $z + \bar{z} = 2|z-1|$ *and* $\arg(z_1 - z_2) = \frac{\pi}{4}$, *then find the imaginary part of* $(z_1 + z_2)$.

SOLUTION Let $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$. Since z_1 and z_2 satisfy the relation $z + \bar{z} = 2|z-1|$. Therefore,

$$z_1 + \bar{z_1} = 2|\bar{z_1} - 1| \text{ and } z_2 + \bar{z_2} = 2|\bar{z_2} - 1|$$

Now,

$$z_1 + \bar{z_1} = 2|\bar{z_1} - 1|$$

$$\Rightarrow 2x_1 = 2|(x_1 - 1) - i y_1|$$

$$\Rightarrow x_1 = \sqrt{(x_1 - 1)^2 + y_1^2}$$

$$\Rightarrow x_1^2 = (x_1 - 1)^2 + y_1^2$$

$$\Rightarrow 2x_1 = 1 + y_1^2 \quad ...(i)$$

Similarly,

$z_2 + \overline{z_2} = 2 \mid \overline{z_2} - 1 \mid$

$\Rightarrow \quad 2x_2 = 1 + y_2^2$ ...(ii)

Subtracting (i) from (ii), we obtain

$2(x_2 - x_1) = (y_2^2 - y_1^2)$

$\Rightarrow 2(x_2 - x_1) = (y_2 + y_1)(y_2 - y_1)$

$\Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2}{y_2 + y_1}$

$\Rightarrow \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{2}{y_1 + y_2}$ ...(iii)

Now,

$\arg(z_1 - z_2) = \dfrac{\pi}{4}$

$\Rightarrow \arg\{(x_1 - x_2) + i(y_1 - y_2)\} = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{y_1 - y_2}{x_1 - x_2} = \tan\dfrac{\pi}{4}$

$\Rightarrow \dfrac{y_1 - y_2}{x_1 - x_2} = 1$ ...(iv)

From (iii) and (iv), we get

$1 = \dfrac{2}{y_1 + y_2}$

$\Rightarrow y_1 + y_2 = 2$

$\Rightarrow \text{Im}(z_1 + z_2) = 2.$

EXAMPLE 10 *If z is a complex number satisfying $\mid z - 1 \mid = 1$, then prove that*

$$\frac{z-2}{z} = i\tan\{\arg(z)\}.$$

SOLUTION We have,

$\mid z - 1 \mid = 1$

$\Rightarrow (z - 1) = \cos\theta + i\sin\theta$ [Using polar form of $(z-1)$]

$\Rightarrow z - 2 = \cos\theta + i\sin\theta - 1$ and $z = 1 + \cos\theta + i\sin\theta$

$\Rightarrow z - 2 = -(1 - \cos\theta) + i\sin\theta$

and, $z = 2\cos^2\dfrac{\theta}{2} + 2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$\Rightarrow z - 2 = -2\sin^2\dfrac{\theta}{2} + 2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

and, $z = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$

$\Rightarrow z - 2 = 2i\sin\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$

and, $z = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$

$\Rightarrow \dfrac{z-2}{z} = i\tan\dfrac{\theta}{2}$

$\Rightarrow \dfrac{z-2}{z} = i\tan(\arg(z))$ $\left[\because z = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right) \Rightarrow \mid z \mid = 2\cos\dfrac{\theta}{2} \text{ and } \arg(z) = \dfrac{\theta}{2}\right]$

EXAMPLE 11 *If z is a non-real complex number satisfying $\mid z \mid = 1$, then prove that*

$$z = \frac{1 + i\tan\left(\dfrac{\arg(z)}{2}\right)}{1 - i\tan\left(\dfrac{\arg(z)}{2}\right)}$$

SOLUTION We have, $\mid z \mid = 1$. So, z can be written as

$z = \cos\theta + i\sin\theta$, where $\theta = \arg(z)$.

$\Rightarrow z = \dfrac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} + \dfrac{2i\tan\theta/2}{1 + \tan^2\theta/2}$

$\Rightarrow z = \dfrac{1 - \tan^2\theta/2 + 2i\tan\theta/2}{1 + \tan^2\theta/2}$

$\Rightarrow z = \dfrac{(1 + i\tan\theta/2)^2}{(1 + i\tan\theta/2)(1 - i\tan\theta/2)}$

$\Rightarrow z = \dfrac{1 + i\tan\theta/2}{1 - i\tan\theta/2}$

$\Rightarrow z = \dfrac{1 + i\tan\left(\dfrac{\arg(z)}{2}\right)}{1 - i\tan\left(\dfrac{\arg(z)}{2}\right)}$

EXAMPLE 12 *Find all complex numbers z for which*

$$\arg\left(\frac{3z - 6 - 3i}{2z - 8 - 6i}\right) = \frac{\pi}{4} \text{ and } \mid z - 3 + 3i \mid = 3$$

SOLUTION Let $z = x + iy$. Then,

$\arg\left(\dfrac{3z - 6 - 3i}{2z - 8 - 6i}\right) = \dfrac{\pi}{4}$

$\Rightarrow \arg(3z - 6 - 3i) - \arg(2z - 8 - 6i) = \dfrac{\pi}{4}$

$\left[\because \arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$

$\Rightarrow \arg\{3(x-2) + 3(y-1)i\} - \arg\{2(x-4) + 2(y-3)i\} = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left\{\dfrac{3(y-1)}{3(x-2)}\right\} - \tan^{-1}\left\{\dfrac{2(y-3)}{2(x-4)}\right\} = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left\{\dfrac{y-1}{x-2}\right\} - \tan^{-1}\left\{\dfrac{y-3}{x-4}\right\} = \dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{y-1}{x-2}-\frac{y-3}{x-4}}{1+\frac{y-1}{x-2}\times\frac{y-3}{x-4}}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \frac{(y-1)(x-4)-(y-3)(x-2)}{(x-2)(x-4)+(y-1)(y-3)}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2x-2y-2}{x^2+y^2-6x-4y+11}=1$$

$$\Rightarrow x^2+y^2-8x-2y+13=0 \quad ...(i)$$

and,

$$|z-3+i|=3$$

$$\Rightarrow |(x-3)+i(y+1)|=3$$

$$\Rightarrow (x-3)^2+(y+1)^2=9$$

$$\Rightarrow x^2+y^2-6x+2y+1=0 \quad ...(ii)$$

Subtracting (ii) from (i), we get

$$x+2y=6 \Rightarrow x=6-2y \quad ...(iii)$$

Putting $x=6-2y$ in (i), we get

$$5y^2-10y+1=0 \Rightarrow y=1\pm\frac{2}{\sqrt{5}}$$

Putting the values of y in (iii), we get

$$x=4\mp\frac{4}{\sqrt{5}}$$

$$\therefore z=x+iy=\left(4\mp\frac{4}{\sqrt{5}}\right)+i\left(1\pm\frac{2}{\sqrt{5}}\right)$$

EXAMPLE 13 *If z is a complex number such that*

$$\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4},$$ *then prove that* $|z-2i|=2\sqrt{2}$.

SOLUTION Let $z=x+iy$. Then,

$$\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \arg(z-2)-\arg(z+2)=\frac{\pi}{4}\left[\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)\right]$$

$$\Rightarrow \arg\{(x-2)+iy\}-\arg\{(x+2)+iy\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{y}{x-2}-\tan^{-1}\frac{y}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{y}{x-2}-\frac{y}{x+2}}{1+\frac{y^2}{x^2-4}}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \frac{y(x+2)-y(x-2)}{x^2-4+y^2}=\tan\frac{\pi}{4}$$

$$\Rightarrow 4y=x^2+y^2-4$$

$$\Rightarrow x^2+y^2-4y=4$$

$$\Rightarrow (x-0)^2+(y-2)^2=8$$

$$\Rightarrow |(x-0)+i(y-2)|^2=8$$

$$\Rightarrow |z-2i|=2\sqrt{2}$$

EXAMPLE 14 *Let $z_1=10+6i$ and $z_2=4+6i$. If z is a complex number such that* $\arg\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$, *then prove that*

$$|z-7-9i|=3\sqrt{2}$$

[IIT 1990]

SOLUTION Let $z=x+iy$. Then,

$$\arg\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \arg(z-z_1)-\arg(z-z_2)=\frac{\pi}{4}$$

$$\Rightarrow \arg\{(x-10)+i(y-6)\}-\arg\{(x-4)+i(y-6)\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{y-6}{x-10}\right\}-\tan^{-1}\left\{\frac{y-6}{x-4}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{y-6}{x-10}-\frac{y-6}{x-4}}{1+\frac{y-6}{x-10}\times\frac{y-6}{x-4}}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \frac{6(y-6)}{(x-10)(x-4)+(y-6)^2}=1$$

$$\Rightarrow x^2+y^2-14x-12y+76=6y-36$$

$$\Rightarrow x^2+y^2-14x-18y+112=0$$

$$\Rightarrow (x^2-14x+49)+(y^2-18y+81)=18$$

$$\Rightarrow (x-7)^2+(y-9)^2=18$$

$$\Rightarrow |(x-7)+i(y-9)|^2=(3\sqrt{2})^2$$

$$\Rightarrow |z-(7+9i)|=3\sqrt{2}$$

EXAMPLE 15 *Let z_1, z_2 and z_3 be three non-zero complex numbers such that $z_1\neq z_2$. If $|z_1|=a$, $|z_2|=b$ and $|z_3|=c$ such that*

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

show that

$$\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)^2=\arg\left(\frac{z_3}{z_2}\right).$$

SOLUTION We have,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a+b+c & b+c+a & c+a+b \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ [Using } R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0 \quad \text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

$$\Rightarrow (a+b+c)\{-(b-c)^2 - (a-b)(a-c)\} = 0$$

$$\Rightarrow (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) = 0$$

$$\Rightarrow (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow (a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = 0$$

$$\Rightarrow (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow a - b = 0, b - c = 0, c - a = 0$$

$$\Rightarrow a = b = c.$$

$$\Rightarrow |z_1| = |z_2| = |z_3| = r \text{ (say).}$$

Let $z_1 = r(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r(\cos\theta_2 + i\sin\theta_2)$ and $z_3 = r(\cos\theta_3 + i\sin\theta_3)$.

Then, $\arg(z_1) = \theta_1$, $\arg(z_2) = \theta_2$ and $\arg(z_3) = \theta_3$.

Now,

$$\frac{z_3 - z_1}{z_2 - z_1}$$

$$= \frac{r(\cos\theta_3 + i\sin\theta_3) - r(\cos\theta_1 + i\sin\theta_1)}{r(\cos\theta_2 + i\sin\theta_2) - r(\cos\theta_1 + i\sin\theta_1)}$$

$$= \frac{(\cos\theta_3 - \cos\theta_1) + i(\sin\theta_3 - \sin\theta_1)}{(\cos\theta_2 - \cos\theta_1) + i(\sin\theta_2 - \sin\theta_1)}$$

$$= \frac{-2\sin\frac{\theta_3+\theta_1}{2}\sin\frac{\theta_3-\theta_1}{2} + 2i\sin\frac{\theta_3-\theta_1}{2}\cos\frac{\theta_3+\theta_1}{2}}{-2\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2} + 2i\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_2+\theta_1}{2}}$$

$$= \frac{2i\sin\frac{\theta_3-\theta_1}{2}\left\{\cos\frac{\theta_3+\theta_1}{2} + i\sin\frac{\theta_3+\theta_1}{2}\right\}}{2i\sin\frac{\theta_2-\theta_1}{2}\left\{\cos\frac{\theta_2+\theta_1}{2} + i\sin\frac{\theta_2+\theta_1}{2}\right\}}$$

$$= \frac{\sin\left(\frac{\theta_3-\theta_1}{2}\right)}{\sin\left(\frac{\theta_2-\theta_1}{2}\right)}\left\{\cos\left(\frac{\theta_3+\theta_1}{2} - \frac{\theta_2+\theta_1}{2}\right) + i\sin\left(\frac{\theta_3+\theta_1}{2} - \frac{\theta_2+\theta_1}{2}\right)\right\}$$

$$= \frac{\sin\left(\frac{\theta_3-\theta_1}{2}\right)}{\sin\left(\frac{\theta_2-\theta_1}{2}\right)}\left\{\cos\left(\frac{\theta_3-\theta_2}{2} + i\sin\left(\frac{\theta_3-\theta_2}{2}\right)\right\}\right.$$

Since, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$$\therefore \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 = \frac{\sin^2\left(\frac{\theta_3-\theta_1}{2}\right)}{\sin^2\left(\frac{\theta_2-\theta_1}{2}\right)}\left\{\cos(\theta_3 - \theta_2) + i\sin(\theta_3 - \theta_2)\right\}$$

$$\Rightarrow \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 = \theta_3 - \theta_2 = \arg(z_3) - \arg(z_2) = \arg\left(\frac{z_3}{z_2}\right).$$

EXAMPLE 16 *If z be any complex number and if $z^2 + az + b = 0$ has two roots both of which has unit modulus, then prove that $|a| \le 2$, $|b| = 1$ and $\arg(b) = 2\arg(a)$.*

SOLUTION Let $z_1 = \cos\alpha + i\sin\alpha$ and $z_2 = \cos\beta + i\sin\beta$ be two roots of $z^2 + az + b = 0$. Then,

$z_1 + z_2 = -a$ and $z_1 z_2 = b$

Now,

$-a = z_1 + z_2$

$\Rightarrow |-a| = |z_1 + z_2|$

$\Rightarrow |-a| \le |z_1| + |z_2|$

$\Rightarrow |a| \le 1 + 1 \qquad [\because |z_1| = |z_2| = 1]$

$\Rightarrow |a| \le 2.$

and,

$z_1 z_2 = b$

$\Rightarrow |z_1 z_2| = |b|$

$\Rightarrow |b| = |z_1||z_2|$

$\Rightarrow |b| = 1 \qquad [\because |z_1| = |z_2| = 1]$

Again,

$-a = z_1 + z_2$

$\Rightarrow -a = (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)$

$\Rightarrow -a = (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)$

$$\Rightarrow -a = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\Rightarrow -a = 2\cos\left(\frac{\alpha-\beta}{2}\right)\left\{\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right\}$$

$\Rightarrow \arg(-a) = \dfrac{\alpha+\beta}{2}$

$\Rightarrow \pi + \arg(a) = \dfrac{\alpha+\beta}{2}$ $[\because \arg(-z) = \pi + \arg(z)]$

$\Rightarrow \alpha+\beta = 2\pi + 2\arg(a)$

$\Rightarrow \alpha+\beta = 2\arg(a)$ ...(i) $[\because -\pi \le \arg(z) < \pi]$

and,

$b = z_1 z_2$

$\Rightarrow \arg(b) = \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \alpha+\beta$ ...(ii)

From (i) and (ii), we get

$$\arg(b) = 2\arg(a).$$

EXAMPLE 17 *If* $f_n(\alpha) = \left(\cos\dfrac{\alpha}{n^3} + i\sin\dfrac{\alpha}{n^3}\right)\left(\cos\dfrac{4\alpha}{n^3} + i\sin\dfrac{4\alpha}{n^3}\right)\left(\cos\dfrac{9\alpha}{n^3} + i\sin\dfrac{9\alpha}{n^3}\right)\left(\cos\dfrac{16\alpha}{n^3} + i\sin\dfrac{16\alpha}{n^3}\right)$ ... *upto n-terms, then find* $\lim\limits_{n\to\infty} f_n(\pi)$.

SOLUTION We have,

$$f_n(\alpha) = \left(\cos\frac{\alpha}{n^3} + i\sin\frac{\alpha}{n^3}\right)\left(\cos\frac{4\alpha}{n^3} + i\sin\frac{4\alpha}{n^3}\right)\left(\cos\frac{9\alpha}{n^3} + i\sin\frac{9\alpha}{n^3}\right)\left(\cos\frac{16\alpha}{n^3} + i\sin\frac{16\alpha}{n^3}\right) \text{... upto } n\text{-terms}$$

$$= e^{\frac{i\alpha}{n^3}} \cdot e^{\frac{i4\alpha}{n^3}} \cdot e^{\frac{i9\alpha}{n^3}} \cdot e^{\frac{i16\alpha}{n^3}} \ldots e^{\frac{in^2\alpha}{n^3}}$$

$$= e^{\frac{i\alpha}{n^3}(1+4+9+16+\ldots+n^2)}$$

$$= e^{\frac{i\alpha}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}}$$

$$= e^{\frac{i\alpha}{6}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\therefore \lim_{n\to\infty} f_n(\pi) = e^{\frac{i\pi}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

—❑❑❑—

19.24 GEOMETRY OF COMPLEX NUMBERS

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then, we say that the affix of P is z. The use of the word affix is similar to the position vector in vectors. As discussed in earlier sections that the length OP is the modulus of z, denoted by $|z|$, and $\angle XOP = \theta$ is known as the argument of z. The polar and Eulerian form of z are $|z|(\cos\theta + i\sin\theta)$ and $|z|\,e^{i\theta}$ respectively.

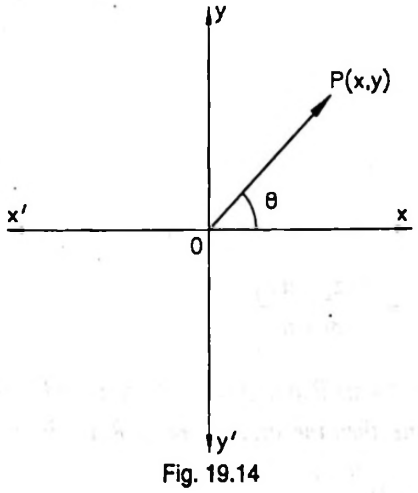


Fig. 19.14

In this section, we shall discuss the geometry of complex numbers.

DISTANCE BETWEEN TWO POINTS

Let P and Q be two points in the *Argand* plane having affixes z_1 and z_2 respectively. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. Using distance formula, we have

$$\therefore \quad PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \qquad \text{...(i)}$$

But, $z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1)$

$$= (x_2 - x_1) + i(y_2 - y_1)$$

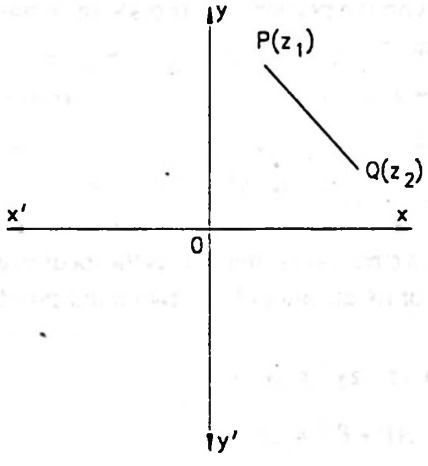


Fig. 19.15

$$\therefore \quad |z_2 - z_1| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$PQ = |z_2 - z_1|$$

Thus, if two points P and Q have affixes z_1 and z_2 respectively. Then,

$PQ = |z_2 - z_1| = |$ affix of Q − affix of $P|$.

REMARK *For any complex number z, we have*

$$|z| = |z - 0| = |z - (0 + i0)|$$

Thus, modulus of a complex number z represented by a point in the Argand plane is its distance from the origin.

ILLUSTRATION 1 *If z_1 and z_2 are two fixed points in the Argand plane, then find the locus of a point z in each of the following :*

(i) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(ii) $|z - z_1| = |z - z_2|$

(iii) $|z - z_1| = k|z - z_2|, k \in R^+, k \neq 1$

(iv) $|z - z_1| + |z - z_2| =$ constant $(\neq |z_1 - z_2|)$

(v) $|z - z_1| - |z - z_2| =$ constant $(|\neq z_1 - z_2|)$

(vi) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

SOLUTION Let A and B be two fixed points having affixes z_1 and z_2 respectively in the Argand plane and let P be a variable point having affix z. Then,

$AP = |z - z_1|$, $BP = |z - z_2|$ and $AB = |z_1 - z_2|$

(i) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow AP + BP = AB$

$\Rightarrow$ P lies on the line segment joining A (z_1) and B (z_2)

(ii) $|z - z_1| = |z - z_2|$

$\Rightarrow AP = BP$

$\Rightarrow$ P is equidistant from A and B

$\Rightarrow$ P lies on the perpendicular bisector of the line segment AB.

(iii) $|z - z_1| = k|z - z_2|$

$\Rightarrow AP = k\,BA$

$\Rightarrow$ P lies on a circle.

Recall that a circle is also defined as the locus of a point such that the ratio of its distances from two fixed points is always constant.

(iv) $|z - z_1| + |z - z_2| =$ constant

$\Rightarrow AP + BP =$ constant

$\Rightarrow$ P moves in such a way that the sum of its distances from two fixed points A and B is always same.

$\Rightarrow$ P lies on an ellipse having its foci at A and B.

(v) $|z - z_1| - |z - z_2| =$ costant $(\neq |z_1 - z_2|)$

$\Rightarrow AP - BP =$ costant $(\neq AB)$

$\Rightarrow$ P lies on a hyperbola having its foci at A and B respectively.

(vi) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

$\Rightarrow AP - BP = AB$

$\Rightarrow$ P lies on the line joining A and B but P does not lie between A and B.

19.25 AFFIX OF A POINT DIVIDING THE LINE SEGMENT JOINING POINTS HAVING AFFIXES z_1 AND z_2

Let z_1 and z_2 be the affixes of points P and Q respectively in the argand plane. Let $z_1 = x_1 + i\,y_1$, and $z_2 = x_2 + i\,y_2$. Then, the coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. Let R be a point on PQ dividing it internally in the ratio $m : n$. Then, the coordinates of R are

$$\left(\frac{m\,x_2 + n\,x_1}{m+n}, \frac{m\,y_2 + n\,y_1}{m+n}\right)$$

$$\Rightarrow \text{Affix of } R = \left(\frac{m\,x_2 + n\,x_1}{m+n}\right) + \left(\frac{m\,y_2 + n\,y_1}{m+n}\right) i$$

$$= \frac{m\,(x_2 + i\,y_2) + n(x_1 + i\,y_1)}{m+n}$$

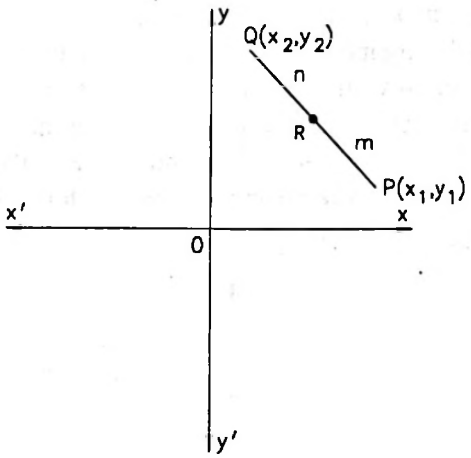


Fig. 19.16

$$= \frac{m\,z_2 + n\,z_1}{m+n}$$

Thus, if two points P and Q have affixes z_1 and z_2 respectively in the argand plane, then the affix of a point R dividing PQ internally in the ratio $m : n$ is $\frac{m\,z_2 + n\,z_1}{m+n}$.

REMARK 1. *If R is the mid-point of PQ, then affix of R is $\frac{z_1 + z_2}{2}$.*

REMARK 2. *If R divides PQ externally in the ratio $m : n$, then affix of R is*

$$\frac{m z_2 - n z_1}{m - n}$$

ILLUSTRATION 1 *If z_1, z_2, z_3 are affixes of the vertices of a triangle, then prove that the affix of its centroid is* $\frac{z_1 + z_2 + z_3}{3}$.

SOLUTION Let ABC be a triangle such that the affixes of its vertices A, B and C are z_1, z_2 and z_3 respectively. Let D be the mid-point of BC. Then,

$$\text{Affix of } D = \frac{z_2 + z_3}{2}$$

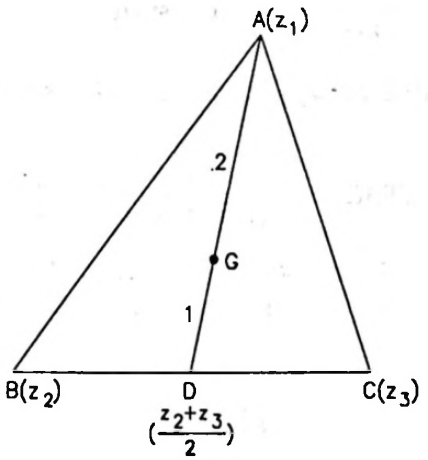


Fig. 19.17

Since G divides AD in the ratio 2 : 1, therefore, affix of G is

$$\frac{1.z_1 + 2.\left(\frac{z_2 + z_3}{2}\right)}{1+2} = \frac{z_1 + z_2 + z_3}{3}$$

ILLUSTRATION 2 *If z_1, z_2, z_3 be the affixes of the vertices A, B and C respectively of a triangle ABC having centroid at G. Such that $z = 0$ is the mid-point of AG, then prove that $4z_1 + z_2 + z_3 = 0$.*

SOLUTION The affix of G is $\frac{z_1 + z_2 + z_3}{3}$. Since $z = 0$ is the mid-point of AG. Therefore,

Affix of the mid-point of $AG = 0$

$$\Rightarrow \frac{\frac{z_1 + z_2 + z_3}{3} + z_1}{2} = 0$$

$$\Rightarrow \frac{4z_1 + z_2 + z_3}{6} = 0$$

$$\Rightarrow 4z_1 + z_2 + z_3 = 0$$

ILLUSTRATION 3 *If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C and D respectively. Then, $ABCD$ is a parallelogram iff $z_1 + z_3 = z_2 + z_4$.*

SOLUTION First, let $ABCD$ be a parallelogram. Since the diagonals of a parallelogram bisect each other.

$\therefore$ Affix of the mid-point of AC = affix of the mid-point of BD.

$$\Rightarrow \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\Rightarrow z_1 + z_3 = z_2 + z_4$$

Conversely, let z_1, z_2, z_3 and z_4 be the affixes of the vertices of a quadrilateral $ABCD$ such that $z_1 + z_3 = z_2 + z_4$. Then, we have to prove that it is a parallelogram.

Now,

$z_1 + z_3 = z_2 + z_4$

$\Rightarrow z_2 - z_1 = z_3 - z_4$ and $z_2 - z_3 = z_1 - z_4$

$\Rightarrow |z_2 - z_1| = |z_3 - z_4|$ and $|z_2 - z_3| = |z_1 - z_4|$

$\Rightarrow AB = DC$ and $CB = DA$

$\Rightarrow ABCD$ is a parallelogram.

ILLUSTRATION 4 *If z_1, z_2, z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin. If z is the affix of its orthocentre, then prove that $z_1 + z_2 + z_3 - z = 0$.*

SOLUTION We know that the centroid G_1, circumcentre O and orthocentre O' of a triangle are collinear such that G divides Oo' in the ratio 1 : 2. Since, affix of G is $\frac{z_1 + z_2 + z_3}{3}$ and O is the origin. Therefore, by section formula, we get



Fig. 19.18

$$\frac{z_1 + z_2 + z_3}{3} = \frac{1.z + 2 \times 0}{1+2}$$

$$\Rightarrow z = z_1 + z_2 + z_3$$

$$\Rightarrow z_1 + z_2 + z_3 - z = 0$$

19.26 EQUATION OF THE PERPENDICULAR BISECTOR

THEOREM *The equation of the perpendicular bisector of the line segment joining points having affixes z_1 and z_2 is*

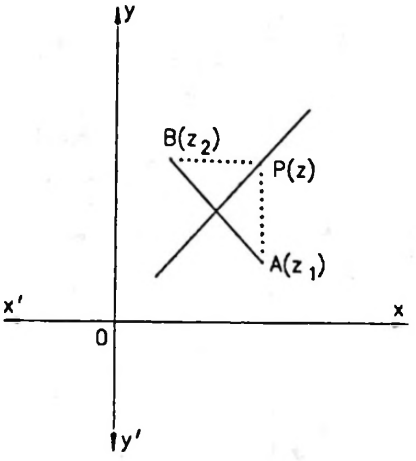


Fig. 19.19

$$z(\overline{z_1}-\overline{z_2})+\bar{z}(z_1-z_2) = |z_1|^2-|z_2|^2$$

PROOF Let $P(z)$ be any point on the perpendicular bisector of AB. Then,

$PA = PB$

$\Rightarrow |z-z_1| = |z-z_2|$

$\Rightarrow |z-z_1|^2 = |z-z_2|^2$

$\Rightarrow (z-z_1)(\overline{z-z_1}) = (z-z_2)(\overline{z-z_2})$

$\Rightarrow (z-z_1)(\overline{z}-\overline{z_1}) = (z-z_2)(\overline{z}-\overline{z_2})$

$\Rightarrow z(\overline{z_1}-\overline{z_2})+\bar{z}(\overline{z_1}-\overline{z_2})-(z_1\overline{z_1}-z_2\overline{z_2}) = 0$

$\Rightarrow z(\overline{z_1}-\overline{z_2})+\bar{z}(z_1-z_2) = |z_1|^2-|z_2|^2$

This is the required equation of the perpendicular bisecter of the line segment joining $A(z_1)$ and $B(z_2)$.

19.27 SOME IMPORTANT RESULTS

THEOREM 1 *If z_1, z_2, z_3 are the affixes of the points A, B, and C in the Argand plane, then*

(i) $\angle BAC = arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$

(ii) $\dfrac{z_3-z_1}{z_2-z_1} = \dfrac{|z_3-z_1|}{|z_2-z_1|}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$.

PROOF Let P and Q be two points in the Argand plane representing z_2-z_1 and z_3-z_1 respectively. Then,

$$\Delta OPQ \cong \Delta ABC$$

$\Rightarrow OP = AB$, $OQ = AC$ and $\angle BAC = \angle POQ$.

(i) $\angle BAC = \angle POQ$

$= \angle XOQ - \angle XOP$

$= \arg(z_3-z_1) - \arg(z_2-z_1)$

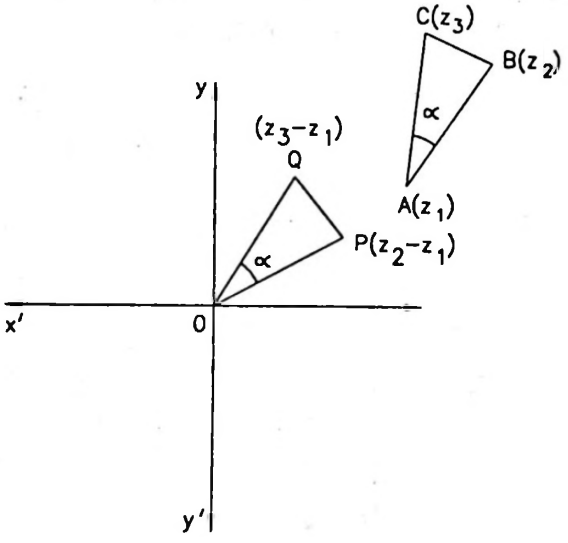


Fig. 19.20

$= \arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$

(ii) Clearly, if $\overrightarrow{OP}$ is rotated through angle α in anticlockwise sense it coincides with $\overrightarrow{OQ}$

$\therefore \overrightarrow{OQ} = \dfrac{OQ}{OP}(\overrightarrow{OP}\,e^{i\alpha})$

$\Rightarrow z_3-z_1 = \dfrac{|z_3-z_1|}{|z_2-z_1|}(z_2-z_1)(\cos\alpha + i\sin\alpha)$

$\Rightarrow \dfrac{z_3-z_1}{z_2-z_1} = \dfrac{|z_3-z_1|}{|z_2-z_1|}(\cos\alpha + i\sin\alpha)$

REMARK *$\angle BAC$ can also be remembered by using the following technique :*

$$\angle BAC = arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$$

THEOREM 2 *If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D respectively in the Argand plane. Then, AB is inclined to CD at the angle.*

$$\arg\left(\frac{z_2-z_1}{z_4-z_3}\right)$$

The proof of this theorem is similar to the proof of the first result.

If CD is inclined at 90° to AB, then

$$\arg\left(\frac{z_2-z_1}{z_4-z_3}\right) = \pm\frac{\pi}{2}$$

$\Rightarrow \dfrac{z_2-z_1}{z_4-z_3} = \dfrac{|z_2-z_1|}{|z_4-z_3|}[\cos(\pm\pi/2) + i\sin(\pm\pi/2)]$

$\Rightarrow \dfrac{z_2-z_1}{z_4-z_3} = \pm\lambda i$, where $\lambda = \dfrac{|z_2-z_1|}{|z_4-z_3|}$

$\Rightarrow z_2-z_1 = \pm\lambda i(z_4-z_3)$

ILLUSTRATION 1 *Show that the two triangles ABC and DEF whose vertices are respectively the complex numbers z_1, z_2, z_3 and a_1, a_2, a_3 are similar if*

$$\begin{vmatrix} z_1 & a_1 & 1 \\ z_2 & a_2 & 1 \\ z_3 & a_3 & 1 \end{vmatrix} = 0$$

SOLUTION Since triangles ABC and DEF are similar. Therefore,

$\angle CBA = \angle FED = \theta$ (*say*) and, $\dfrac{AB}{BC} = \dfrac{DE}{EF}$

Now,

$$\frac{z_1-z_2}{z_3-z_2} = \frac{AB}{BC}(\cos\theta + i\sin\theta)$$

and,

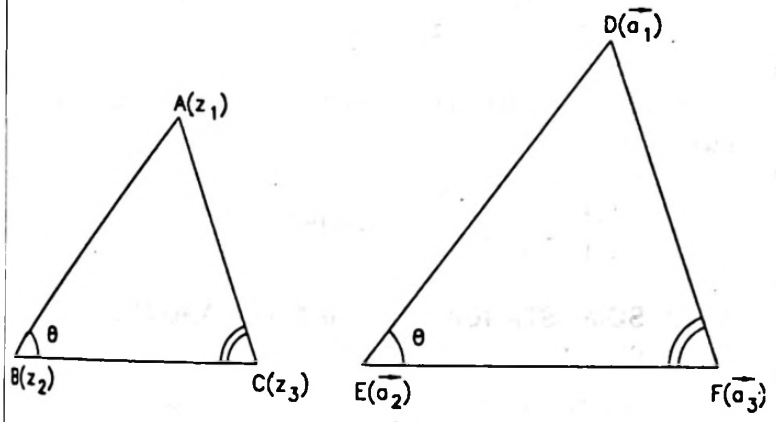


Fig. 19.21

$$\frac{a_1-a_2}{a_3-a_2}=\frac{DE}{EF}(\cos\theta+i\sin\theta)$$

$$\Rightarrow \frac{z_1-z_2}{z_3-z_2}=\frac{a_1-a_2}{a_3-a_2} \qquad \left[\because \frac{AB}{BC}=\frac{DE}{EF}\right]$$

$$\Rightarrow \begin{vmatrix} z_1-z_2 & a_1-a_2 \\ z_3-z_2 & a_3-a_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} z_1-z_2 & a_1-a_2 & 0 \\ z_3-z_2 & a_3-a_2 & 0 \\ z_2 & a_2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} z_1 & a_1 & 1 \\ z_2 & a_3 & 1 \\ z_3 & a_2 & 1 \end{vmatrix}=0 \quad [\text{Applying } R_1\to R_1+R_3,\ R_2\to R_2+R_3]$$

$$\Rightarrow \begin{vmatrix} z_1 & a_1 & 1 \\ z_2 & a_2 & 1 \\ z_3 & a_3 & 1 \end{vmatrix}=0$$

ILLUSTRATION 2 *Let A, B, C, D and E be points on the complex plane representing the complex numbers z_1, z_2, z_3, z_4 and z_5 respectively. If $(z_3-z_2)z_4=(z_1-z_2)z_5$, prove that the triangles ABC and DOE are similar respectively.*

SOLUTION We have,

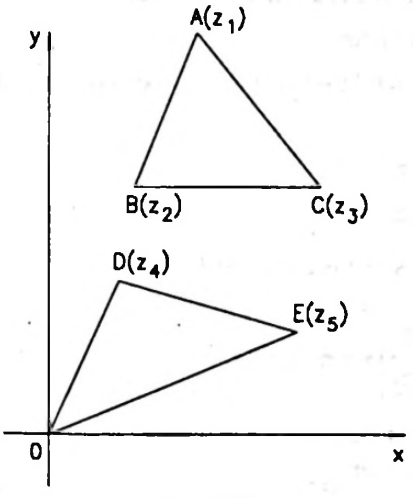


Fig. 19.22

$$(z_3-z_2)z_4=(z_1-z_2)z_5$$

$$\Rightarrow \frac{z_1-z_2}{z_3-z_2}=\frac{z_4}{z_5}$$

$$\Rightarrow \frac{|z_1-z_2|}{|z_3-z_2|}=\frac{|z_4|}{|z_5|} \text{ and, } \arg\left(\frac{z_1-z_2}{z_3-z_2}\right)=\arg\left(\frac{z_4}{z_5}\right)$$

$$\Rightarrow \frac{AB}{BC}=\frac{OD}{OE} \text{ and } \angle CBA=\angle EOD$$

$$\Rightarrow \frac{AB}{OD}=\frac{BC}{OE} \text{ and } \angle CBA=\angle EOD$$

$$\Rightarrow \Delta ABC \sim \Delta DOE.$$

ILLUSTRATION 3 *The equation of the circle having z_1 and z_2 as the end points of a diameter is $(z-z_1)(\bar{z}-\bar{z_2})+(\bar{z}-\bar{z_1})(z-z_2)=0$*

SOLUTION Let $P(z)$ be any point on the circle. Then,

$$\angle APB=\pm\pi/2$$

$$\Rightarrow \arg\left(\frac{z_2-z}{z_1-z}\right)=\pm\pi/2$$

$$\Rightarrow \arg\left(\frac{z-z_2}{z-z_1}\right)=\pm\pi/2$$

$$\Rightarrow \frac{z-z_2}{z-z_1} \text{ is purely imaginary.}$$

$$\Rightarrow \frac{z-z_2}{z-z_1}+\left(\frac{z-z_2}{z-z_1}\right)=0$$

$$\Rightarrow (z-z_2)(\bar{z}-\bar{z_1})+(z-z_1)(\bar{z}-\bar{z_2})=0$$

$$\Rightarrow (z-z_1)(\bar{z}-\bar{z_2})+(\bar{z}-\bar{z_1})(z-z_2)=0$$

REMARK *If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying*

$$\arg\left(\frac{z-z_1}{z-z_2}\right)=\pm\pi/2$$

is a circle having z_1 and z_2 as the end points of a diameter.

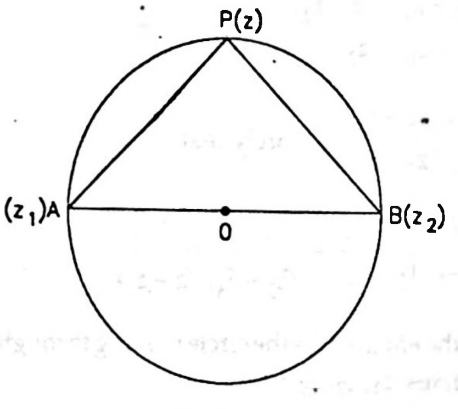


Fig. 19.23

ILLUSTRATION 4 *Find the equation of a circle passing through three points having affixes z_1, z_2 and z_3. Also, deduce the condition for four points to be concyclic.*

SOLUTION Let z_1, z_2, z_3 respectively be the affixes of three points A, B, C on the Argand plane and let z be the affix of any point P on the circle passing through A, B, C. If the relative positions of A, B, C, P be as in Fig. 19.24, the angle of rotation from AP to AB is equal but opposite in sign to the angle of rotation from CB to CP. The sum of these angles is, therefore zero. If the relative positions of A, B, C, P be as in Fig. 19.25, then the angle of rotation from CB to CP is supplementary to the angle of rotation from AP to AB. The sum of these two angles is therefore π. It can be easily seen that in all other cases the sum is either 0 or π.

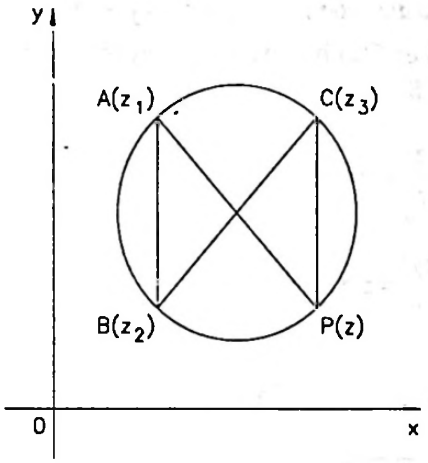


Fig. 19.24

The angle of rotation from AP to AB is $\arg\left(\frac{z_2-z_1}{z-z_1}\right)$

and,

the angle of rotation from CB to CP is $\arg\left(\frac{z-z_3}{z_2-z_3}\right)$

Hence,

$$\arg\left(\frac{z_2-z_1}{z-z_1}\right)+\arg\left(\frac{z-z_3}{z_2-z_3}\right)=0 \text{ or, } \pi$$

$$\Rightarrow \arg\left(\frac{z_2-z_1}{z-z_1}\cdot\frac{z-z_3}{z_2-z_3}\right)=0 \text{ or, } \pi$$

$$\Rightarrow \frac{(z_2-z_1)(z-z_3)}{(z-z_1)(z_2-z_3)} \text{ is purely real}$$

$$\Rightarrow \frac{(z_2-z_1)(z-z_3)}{(z_2-z_3)(z-z_1)}=\frac{(\bar{z}_2-\bar{z}_1)(\bar{z}-\bar{z}_3)}{(\bar{z}_2-\bar{z}_3)(\bar{z}-\bar{z}_1)}$$

Thus, the equation of the circle passing through three points having affixes z_1, z_2, z_3 is

$$\left(\frac{z-z_3}{z-z_1}\right)\left(\frac{z_2-z_1}{z_2-z_3}\right)=\left(\frac{\bar{z}-\bar{z}_3}{\bar{z}-\bar{z}_1}\right)\left(\frac{\bar{z}_2-\bar{z}_1}{\bar{z}_2-\bar{z}_3}\right)$$

If four points having affixes z_1, z_2, z_3, z_4 are concyclic, then

$$\left(\frac{z_1-z_2}{z_1-z_4}\right)\left(\frac{z_3-z_4}{z_3-z_2}\right) \text{ must be purely real.}$$

19.28 SOME STANDARD LOCII IN THE ARGAND PLANE

(I) If z is a variable point in the Argand plane such that $\arg(z)=\theta$, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x-axis.

(II) If z is a variable point and z_1 is a fixed point in the Argand plane such that $\arg(z-z_1)=\theta$, then locus of z is a straight line passing through the point representing z_1 and inclined at an angle θ with x-axis. Note that the point z_1 is excluded from the locus.

(III) If z is a variable point and z_1, z_2 are two fixed points in the Argand plane, then

(a) $|z-z_1|=|z-z_2|$

$\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

(b) $|z-z_1|+|z-z_2|=$ constant $(\neq |z_1-z_2|)$

$\Rightarrow$ Locus of z is an ellipse

(c) $|z-z_1|+|z-z_2|=|z_1-z_2|$

$\Rightarrow$ Locus of z is the line segment joining z_1 and z_2.

(d) $|z-z_1|-|z-z_2|=|z_1-z_2|$

$\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(e) $|z-z_1|-|z-z_2|=$ constant $(\neq |z_1-z_2|)$

$\Rightarrow$ locus of z is a hyperbola.

(f) $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$

$\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

(g) $|z-z_1|=k|z-z_2|, k\neq 1 \Rightarrow$ locus of z is a circle.

(h) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\alpha$ (fixed)

$\Rightarrow$ Locus of z is a segment of circle.

(i) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\pm\pi/2$

$\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

(j) $\arg\left(\frac{z-z_1}{z-z_2}\right)=0$ or, π

$\Rightarrow$ Locus z is a straight line passing through z_1 and z_2

ILLUSTRATION 1 *Find the locus of point z in each of the following:*

(i) $\pi/6 < arg\,(z) < \pi/4$ (ii) $arg\,(z-1)=\pi/4$

(iii) $\pi/6 < arg\,(z+1) < \pi/3$ (iv) $arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

SOLUTION

(i) We observe that $\pi/6 < \arg(z) < \pi/4$ represents the set of all points, except the origin, lying between the lines passing through the origin and making 30° and 45° with the positive direction of x-axis.

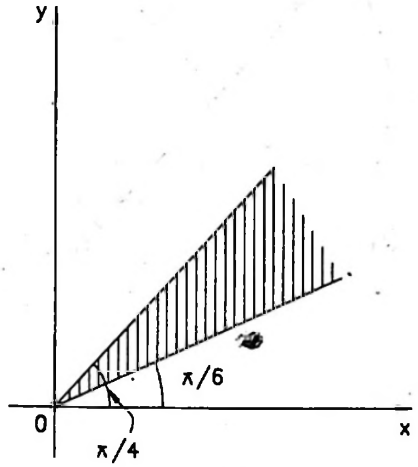


Fig. 19.25

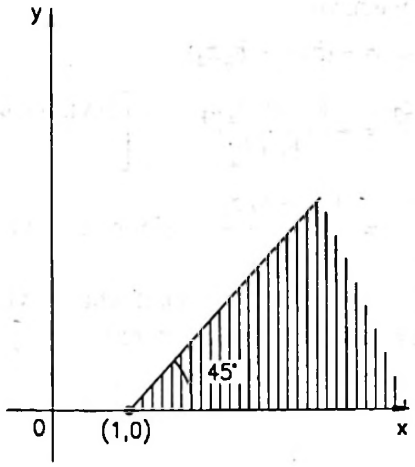


Fig. 19.26

(ii) We observe that $\arg(z-1)=\pi/4$ represents the set of all points, except point (1, 0), lying on the line passing through (1, 0) and making an angle of 45° with x-axis.

(iii) The region represented by $\pi/6 < \arg(z+1) < \pi/3$ is shown in Fig. 19.27. The point (–1, 0) is not in the shaded region.

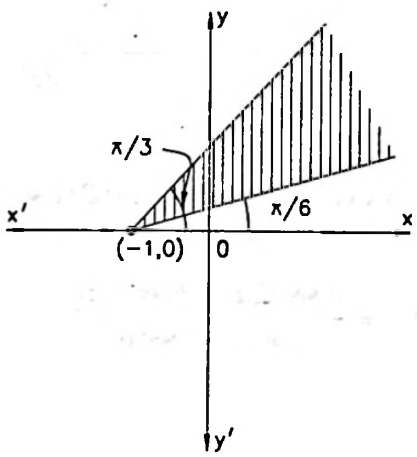


Fig. 19.27

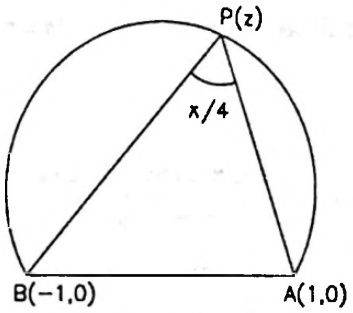


Fig. 19.28

(iv) We observe that $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$ represents the sector APB of angle $\pi/4$ of a circle.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If z_1, z_2, z_3 be the affixes of the vertices, A, B, C respectiely of a triangle ABC, show that the affix of its orthocentre is*

$$\frac{(a\sec A)\,z_1+(b\sec B)\,z_2+(c\sec C)\,z_3}{a\sec A+b\sec B+c\sec C}$$

SOLUTION Let AD and BE be perpendiculars from A and B on BC and CA respectively. Suppose AD and BE intersect at O. Then, O is the orthocentre of ΔABC. We know that the distances of O from the vertices A, B and C are $2R\cos A$, $2R\cos B$ and $2R\cos C$ respectively. The distances of O from the sides BC, CA and AB are respectively $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$

Clearly, $BD = c\cos B$ and $CD = b\cos C$.

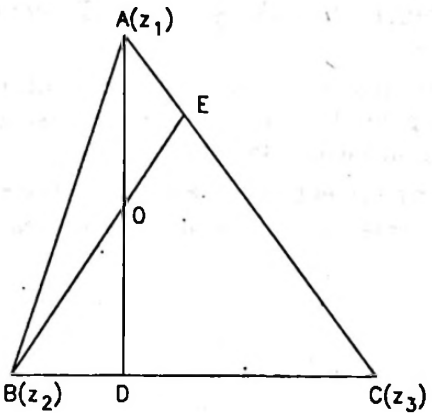


Fig. 19.29

$\therefore \quad BD : DC = c\cos B : b\cos C = c\sec C : b\sec B.$

Thus, D divides BC in the ratio $c\sec C : b\sec B$.

$$\text{So, affix of } D = \frac{(c\sec C)\,z_3 + (b\sec B)\,z_2}{c\sec C + b\sec B}$$

Now,

$OA : OD$

$= 2R\cos A : 2R\cos B\cos C$

$= \cos A : \cos B\cos C$

$= a\cos A : a\cos B\cos C$

$= a\sec B\sec C : a\sec A$

$= (b\cos C + c\cos B)\sec B\sec C : a\sec A \quad [a = b\cos C + c\cos B]$

$= (b\sec B + c\sec C) : a\sec A$

$\therefore$ Affix of O

$$= \frac{(b\sec B + c\sec C)\left(\dfrac{(b\sec B)\,z_2 + c\,(\sec C)\,z_3}{b\sec B + c\sec C}\right) + (a\sec A)\,z_1}{(b\sec B + c\sec C) + a\sec A}$$

$$= \frac{(b\sec B)\;z_2 + (c\sec C)\,z_3 + (a\sec A)\,z_1}{b\sec B + c\sec C + a\sec A}$$

$$= \frac{(a\sec A)\,z_1 + (b\sec B)\,z_2 + (c\sec C)\,z_3}{a\sec A + b\sec B + c\sec C}$$

EXAMPLE 2 *If three points having their affixes z_1, z_2, z_3 are connected by the relation $az_1 + bz_2 + cz_3 = 0$, where $a, b, c \in R$ such that $a + b + c = 0$. Prove that the points are collinear.*

SOLUTION We have,

$az_1 + bz_2 + cz_3 = 0$, where $a + b + c = 0$

$\Rightarrow -cz_3 = az_1 + bz_2$

$\Rightarrow (a + b)\,z_3 = az_1 + bz_2$

$$\Rightarrow \qquad z_3 = \frac{az_1 + bz_2}{a + b}$$

$\Rightarrow z_3$ divides the segment joining z_1 and z_2 in the ratio $b : a$

$\Rightarrow$ Points having affixes z_1, z_2, z_3 are collinear.

EXAMPLE 3 *If $b_1 + b_2 + b_3 + b_4 = 0$, where b_1, b_2, b_3, b_4 are non-zero real numbers, sum of no two being zero, and $b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 = 0$ where no three of the four points z_1, z_2, z_3, z_4 are collinear, then prove that the four points will be concyclic if $b_1b_2 \mid z_1 - z_2 \mid^2 = b_3b_4 \mid z_3 - z_4 \mid^2$.*

SOLUTION Let $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ be the given four points such that no three of these four points are collinear.

We have,

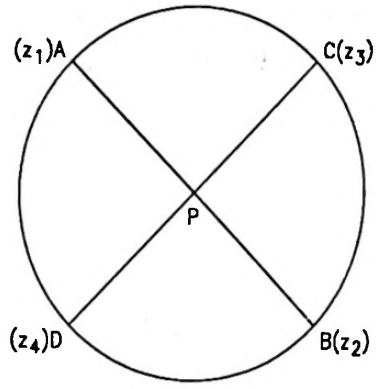


Fig. 19.30

$$b_1 + b_2 + b_3 + b_4 = 0 \qquad \ldots(i)$$

and,

$$b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 = 0 \qquad \ldots(ii)$$

From (ii), we have

$$b_1z_1 + b_2z_2 = -(b_3z_3 + b_4z_4)$$

$$\Rightarrow \frac{b_1z_1 + b_2z_2}{b_1 + b_2} = -\frac{b_3z_3 + b_4z_4}{b_1 + b_2} \qquad \left[\text{Dividing both sides by } b_1 + b_2\right]$$

$$\Rightarrow \frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4} \qquad [\text{From (i) } b_1 + b_2 = -(b_3 + b_4)]$$

$\Rightarrow AB$ and CD have a common point which divides AB and CD in the ratios $b_2 : b_1$ and $b_4 : b_3$ respectively.

$$\Rightarrow P \text{ represents the complex number } \frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4}$$

$$\therefore \quad PA = \left|z_1 - \frac{b_1z_1 + b_2z_2}{b_1 + b_2}\right| = \frac{b_2}{b_1 + b_2} \mid z_1 - z_2 \mid$$

$$PB = \left|z_2 - \frac{b_1z_1 + b_2z_2}{b_1 + b_2}\right| = \frac{b_1}{b_1 + b_2} \mid z_1 - z_2 \mid,$$

$$PC = \left|z_3 - \frac{b_3z_3 + b_4z_4}{b_3 + b_4}\right| = \frac{b_4}{b_3 + b_4} \mid z_3 - z_4 \mid$$

and,

$$PD = \left|z_4 - \frac{b_3z_3 + b_4z_4}{b_3 + b_4}\right| = \frac{b_3}{b_3 + b_4} \mid z_3 - z_4 \mid$$

Now, four points A, B, C, D will be concyclic, if

$$PA \cdot PB = PC \cdot PD$$

$$\Rightarrow \frac{b_1 b_2}{(b_1+b_2)^2} \mid z_1 - z_2 \mid^2 = \frac{b_3 b_4}{(b_3+b_4)^2} \mid z_3 - z_4 \mid^2$$

$$\Rightarrow b_1 b_2 \mid z_1 - z_2 \mid^2 = b_3 b_4 \mid z_3 - z_4 \mid^2$$

$$\left[\because b_1 + b_2 = -(b_3 + b_4) \text{ from (i)} \right.$$
$$\left. \therefore (b_1+b_2)^2 = (b_3+b_4)^2 \right]$$

EXAMPLE 4 *If z_1, z_2, z_3 are the affixes of the vertices of a triangle, then show that the affix of the orthocentre of the triangle is given by*

$$\frac{\Sigma z_1^2 (\bar{z}_2 - \bar{z}_3) + \Sigma \mid z_1 \mid^2 (z_2 - z_3)}{\Sigma (z_1 \bar{z}_2 - \bar{z}_1 z_2)}$$

SOLUTION Let ABC be a triangle such that the affixes of the vertices A, B and C are z_1, z_2 and z_3 respectively. Let the affix of the orthocentre O be z.

Now,

$$AD \perp BC$$

$$\Rightarrow \arg\left(\frac{z_1 - z}{z_3 - z_2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{z_1 - z}{z_3 - z_2} \text{ is purely imaginary}$$

$$\Rightarrow \left(\frac{z_1 - z}{z_3 - z_2}\right) + \overline{\left(\frac{z_1 - z}{z_3 - z_2}\right)} = 0$$

$$\Rightarrow \frac{z_1 - z}{z_3 - z_2} + \frac{\bar{z}_1 - \bar{z}}{\bar{z}_3 - \bar{z}_2} = 0 \qquad \ldots(i)$$

Similarly,

$$BE \perp CA$$

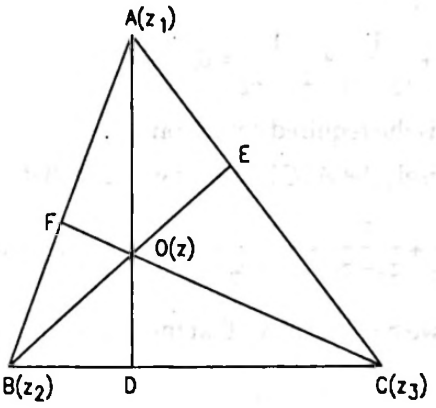


Fig. 19.31

$$\Rightarrow \arg\left(\frac{z_2 - z}{z_3 - z_1}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{z_2 - z}{z_3 - z_1} \text{ is purely imaginary}$$

$$\Rightarrow \left(\frac{z_2 - z}{z_3 - z_1}\right) + \left(\frac{\overline{z_2 - z}}{z_3 - z_1}\right) = 0$$

$$\Rightarrow \frac{z_2 - z}{z_3 - z_1} + \frac{\bar{z}_2 - \bar{z}}{\bar{z}_3 - \bar{z}_1} = 0 \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\bar{z}_1 - \bar{z} = \left(\frac{z - z_1}{z_3 - z_2}\right)(\bar{z}_3 - \bar{z}_2)$$

and,

$$\bar{z}_2 - \bar{z} = \left(\frac{z - z_2}{z_1 - z_3}\right)(\bar{z}_1 - \bar{z}_3)$$

$$\Rightarrow (\bar{z}_1 - \bar{z}) - (\bar{z}_2 - \bar{z}) = \frac{(z - z_1)(\bar{z}_3 - \bar{z}_2)}{z_3 - z_2} - \frac{(z - z_2)(\bar{z}_1 - \bar{z}_3)}{z_1 - z_3}$$

$$\Rightarrow \bar{z}_1 - \bar{z}_2 = \frac{(z - z_1)(\bar{z}_3 - \bar{z}_2)(z_1 - z_3) - (z - z_2)(\bar{z}_1 - \bar{z}_3)(z_3 - z_2)}{(z_3 - z_2)(z_1 - z_3)}$$

$$\Rightarrow (z_3 - z_2)(z_1 - z_3)(\bar{z}_1 - \bar{z}_2)$$
$$= (z - z_1)(z_1 - z_3)(\bar{z}_3 - \bar{z}_2) - (z - z_2)(z_3 - z_2)(\bar{z}_1 - \bar{z}_3)$$

$$\Rightarrow z[(z_1 - z_3)(\bar{z}_3 - \bar{z}_2) - (z_3 - z_2)(\bar{z}_1 - \bar{z}_3)]$$
$$= (z_1 - z_3)(z_3 - z_2)(\bar{z}_1 - \bar{z}_2) + z_1(z_1 - z_3)(\bar{z}_3 - \bar{z}_2) - z_2(z_3 - z_2)(\bar{z}_1 - \bar{z}_3)$$

$$\Rightarrow z[z_1 \bar{z}_3 - \bar{z}_1 z_3 - z_1 \bar{z}_2 + \bar{z}_1 z_2 + \bar{z}_2 z_3 - z_2 \bar{z}_3]$$
$$= z_1^2(\bar{z}_3 - \bar{z}_2) + z_2^2(\bar{z}_1 - \bar{z}_3) + z_3^2(\bar{z}_2 - \bar{z}_1) + \mid z_1 \mid^2 (z_3 - z_2) + \mid z_2 \mid^2 (z_1 - z_3) + \mid z_3 \mid^2 (z_2 - z_1)$$

$$\Rightarrow z[(z_1 \bar{z}_2 - \bar{z}_1 z_2) + (z_2 \bar{z}_3 - \bar{z}_2 z_3) + (z_3 \bar{z}_1 - \bar{z}_3 z_1)]$$
$$= z_1^2(\bar{z}_2 - \bar{z}_3) + z_2^2(\bar{z}_3 - \bar{z}_1) + z_3^2(\bar{z}_1 - \bar{z}_2) + \mid z_1 \mid^2 (z_2 - z_3) + \mid z_2 \mid^2 (z_3 - z_1) + \mid z_3 \mid^2 (z_1 - z_2)$$

$$\Rightarrow z\{\Sigma(z_1 \bar{z}_2 - \bar{z}_1 z_2)\} = \Sigma z_1^2 (\bar{z}_2 - \bar{z}_3) + \Sigma \mid z_1 \mid^2 (z_2 - z_3)$$

$$\Rightarrow z = \frac{\Sigma(z_1^2(\bar{z}_2 - \bar{z}_3) + \Sigma \mid z_1 \mid^2 (z_2 - z_3)}{\Sigma(z_1 \bar{z}_2 - \bar{z}_1 z_2)}$$

EXAMPLE 5 *Show that the circumcentre of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by*

$$z = \frac{\Sigma z_1 \bar{z}_1 (z_2 - z_3)}{\Sigma \bar{z}_1 (z_2 - z_3)}$$

SOLUTION Let ABC be a triangle such that the affixes of its vertices A, B and C are z_1, z_2 and z_3 respecitvely. Let z be the affix of the circumcentre O of ΔABC. Then,

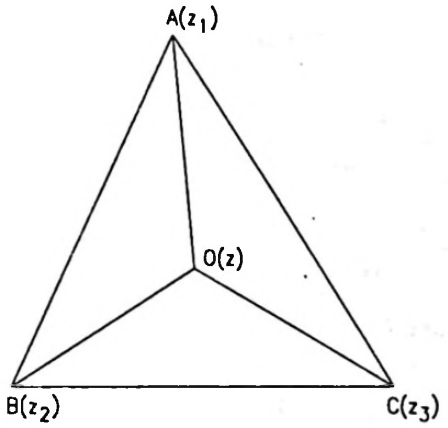


Fig. 19.32

$OA = OB = OC$

$\Rightarrow |z-z_1| = |z-z_2| = |z-z_3|$

Now,

$|z-z_1| = |z-z_2|$

$\Rightarrow |z-z_1|^2 = |z-z_2|^2$

$\Rightarrow |z|^2+|z_1|^2-z\overline{z_1}-\overline{z}z_1 = |z|^2+|z_2|^2-\overline{z}z_2-z\overline{z_2}$

$\Rightarrow z(\overline{z_1}-\overline{z_2})+\overline{z}(z_1-z_2) = |z_1|^2-|z_2|^2$ ...(i)

and,

$|z-z_2| = |z-z_3|$

$\Rightarrow |z-z_2|^2 = |z-z_3|^2$

$\Rightarrow |z|^2| + |z_2|^2-z\overline{z_2}-\overline{z}z_2 = |z|^2+|z_3|^2-z\overline{z_3}-\overline{z}z_3$

$\Rightarrow z(\overline{z_2}-\overline{z_3})+\overline{z}(z_2-z_3) = |z_2|^2-|z_3|^2$ ...(ii)

From (i) and (ii), we have to eliminate $\overline{z}$ to get the value of z.

Multiplying (i) by (z_2-z_3) and (ii) by (z_1-z_2) and subtracting, we get

$z\{(\overline{z_1}-\overline{z_2})(z_2-z_3)-(\overline{z_2}-\overline{z_3})(z_1-z_2)\}$

$= (|z_1|^2-|z_2|^2)(z_2-z_3)$

$\Rightarrow z\{\overline{z_1}(z_2-z_3)+\overline{z_2}(z_3-z_1)+\overline{z_3}(z_1-z_2)\}$

$= |z_1|^2(z_2-z_3)+|z_2|^2(z_3-z_1)+|z_3|^2(z_1-z_2)$

$\Rightarrow z\left\{\Sigma\overline{z_1}(z_2-z_3)\right\} = \Sigma|z_1|^2(z_2-z_3)$

$\Rightarrow z = \dfrac{\Sigma|z_1|^2(z_2-z_3)}{\Sigma\overline{z_1}(z_2-z_3)} = \dfrac{\Sigma z_1\overline{z_1}(z_2-z_3)}{\Sigma\overline{z_1}(z_2-z_3)}$

EXAMPLE 6 *Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle if and only if*

$$\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2} = 0$$

SOLUTION Let ABC be a triangle such that the affixes of its vertices A, B and C are respectively z_1, z_2 and z_3. Further, let $\alpha = z_2-z_3$, $\beta = z_3-z_1$ and $\gamma = z_1-z_2$. Then,

$$\alpha+\beta+\gamma = 0 \quad ...(i)$$

First, let ΔABC be an equilateral triangle. Then,

$BC = CA = AB$

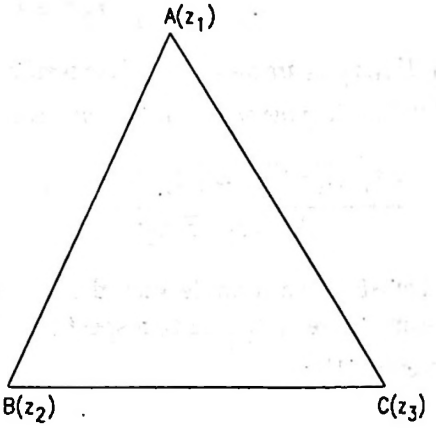


Fig. 19.33

$\Rightarrow |z_2-z_3| = |z_3-z_1| = |z_1-z_2|$

$\Rightarrow |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |\alpha|^2 = |\beta|^2 = |\gamma|^2 = \lambda$ (say)

$\Rightarrow \alpha\overline{\alpha} = \beta\overline{\beta} = \gamma\overline{\gamma} = \lambda$

$\Rightarrow \overline{\alpha} = \dfrac{\lambda}{\alpha}, \overline{\beta} = \dfrac{\lambda}{\beta}, \overline{\gamma} = \dfrac{\lambda}{\gamma}$ ...(ii)

Now,

$\alpha+\beta+\gamma = 0$ [From (i)]

$\Rightarrow \overline{\alpha+\beta+\gamma} = 0$

$\Rightarrow \overline{\alpha}+\overline{\beta}+\overline{\gamma} = 0$

$\Rightarrow \dfrac{\lambda}{\alpha}+\dfrac{\lambda}{\beta}+\dfrac{\lambda}{\gamma} = 0$ [Using (ii)]

$\Rightarrow \dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = 0$

$\Rightarrow \dfrac{1}{z_2-z_3}+\dfrac{1}{z_3-z_1}+\dfrac{1}{z_1-z_2} = 0,$

which is the required condition.

Conversely, let ABC be a triangle such that

$$\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2} = 0 \text{ i.e. } \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = 0$$

Then, we have to prove that the triangle is equilateral.

We have,

$\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = 0$

$\Rightarrow \dfrac{1}{\alpha} = -\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma}\right)$

$$\Rightarrow \frac{1}{\alpha} = -\left(\frac{\beta+\gamma}{\beta\gamma}\right)$$

$$\Rightarrow \frac{1}{\alpha} = \frac{\alpha}{\beta\gamma} \qquad \text{[Using (i)]}$$

$$\Rightarrow \alpha^2 = \beta\gamma$$

$$\Rightarrow |\alpha|^2 = |\beta\gamma|$$

$$\Rightarrow |\alpha|^2 = |\beta|\,|\gamma|$$

$$\Rightarrow |\alpha|^3 = |\alpha|\,|\beta|\,|\gamma|$$

Similarly,

$|\beta|^3 = |\alpha|\,|\beta|\,|\gamma|$ and $|\gamma|^3 = |\alpha|\,|\beta|\,|\gamma|$

$\therefore |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2|$

$\Rightarrow BC = CA = AB$

Hence, the given triangle is an equilateral triangle.

REMARK *The above result can also be restated as : Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle iff*

$$z_1^2 + z_2^2 + z_3^3 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

EXAMPLE 7 *Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$. Let A, B and C be the points in the Argand plane representing z_1, z_2 and z_3 respectively. Prove that $z_1 + z_2 + z_3 = 3$ if and only if ΔABC is an equilateral triangle.*

SOLUTION Let P be a point in the Argand plane whose affix is 1. Then,

$$|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$$

$$\Rightarrow PA = PB = PC$$

$\Rightarrow P$ is the circumcentre of triangle ABC.

Let G be the centroid of ΔABC.

First, let $z_1 + z_2 + z_3 = 3$. Then, we have to prove that the triangle ABC is equilateral.

We have,

$$z_1 + z_2 + z_3 = 3$$

$$\Rightarrow \frac{z_1 + z_2 + z_3}{3} = 1$$

$\Rightarrow$ Affix of the centroid G of ΔABC is 1.

$\Rightarrow$ Circumcentre P and centroid G coincide with each other.

$\Rightarrow \Delta ABC$ is equilateral.

Conversely, let ΔABC be an equilateral triangle. Then, we have to show that $z_1 + z_2 + z_3 = 3$.

The affix of the centroid G of ΔABC is $\frac{z_1 + z_2 + z_3}{3}$.

Since the triangle is equilateral. Therefore, its circumcentre, ortheocentre and centroid coincide. But, the affix of the circumcentre is given as 1 and that of the centroid is $\frac{z_1 + z_2 + z_3}{3}$.

Therefore,

$$\frac{z_1 + z_2 + z_3}{3} = 1 \Rightarrow z_1 + z_2 + z_3 = 3.$$

EXAMPLE 8 *Let $a, b \in R$ such that $0 < a < 1$ and $0 < b < 1$. Find the values of a and b such that the complex numbers $z_1 = -a + i, z_2 = -1 + bi$ and $z_3 = 0$ form an equilateral triangle.*

SOLUTION Let ABC be the equilateral triangle such that the affixes of its vertices A, B, C are z_1, z_2 and z_3 respectively. Then,

$$AB = BC = CA$$

$$\Rightarrow |z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$$

Now,

$$|z_2 - z_1| = |z_3 - z_2|$$

$$\Rightarrow |(a-1) + i(b-1)| = |1 - bi|$$

$$\Rightarrow |(a-1) + i(b-1)|^2 = |1 - bi|^2$$

$$\Rightarrow (a-1)^2 + (b-1)^2 = 1 + b^2$$

$$\Rightarrow 1 + a^2 - 2a - 2b = 0 \qquad \text{...(i)}$$

and,

$$|z_3 - z_2| = |z_1 - z_3|$$

$$\Rightarrow |1 - bi| = |-a + i|$$

$$\Rightarrow |1 - bi|^2 = |-a + i|^2$$

$$\Rightarrow 1 + b^2 = a^2 + 1$$

$$\Rightarrow a^2 = b^2$$

$$\Rightarrow a = b \qquad [\because 0 < a, b < 1]$$

Putting $b = a$ in (i), we get

$$a^2 - 4a + 1 = 0$$

$$\Rightarrow a = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

$$\Rightarrow a = 2 - \sqrt{3} \qquad [\because 0 < a < 1]$$

Hence,

$$a = b = 2 - \sqrt{3}.$$

EXAMPLE 9 *A cubic equation $f(x) = 0$ has one real root α and two complex roots $\beta \pm i\gamma$. Points A, B, C represent roots $\alpha, \beta + i\gamma$ and $\beta - i\gamma$ respectively on the Argand diagram. Show that the roots of the derived equation $f'(x) = 0$ are complex if A falls inside one of the two equilateral triangles described on base BC.*

SOLUTION Since $\alpha, \beta + i\gamma$ and $\beta - i\gamma$ are the roots of $f(x) = 0$. So, let

$$f(x) = k(x - \alpha)(x - \beta - i\gamma)(x - \beta + i\gamma)$$

$$\Rightarrow f(x) = k(x - \alpha)\{(x - \beta)^2 + \gamma^2\}$$

$$\Rightarrow f'(x) = k[(x - \beta)^2 + \gamma^2 + 2(x - \alpha)(x - \beta)]$$

$$\Rightarrow f'(x) = k[3x^2 - 2x(\alpha + 2\beta) + \beta^2 + \gamma^2 + 2\alpha\beta]$$

Let D be the discriminant of $f'(x) = 0$. Then,

$$D = 4(\alpha + 2\beta)^2 - 12(\beta^2 + \gamma^2 + 2\alpha\beta)$$

$$= 4[\alpha^2 + \beta^2 - 3\gamma^2 - 2\alpha\beta] \quad \ldots(i)$$

Now, $BC = |(\beta + i\gamma) - (\beta - i\gamma)| = 2|\gamma|$

In Fig. 19.34 ΔPBC and ΔQBC are equilateral triangles with BC as the base. Therefore,

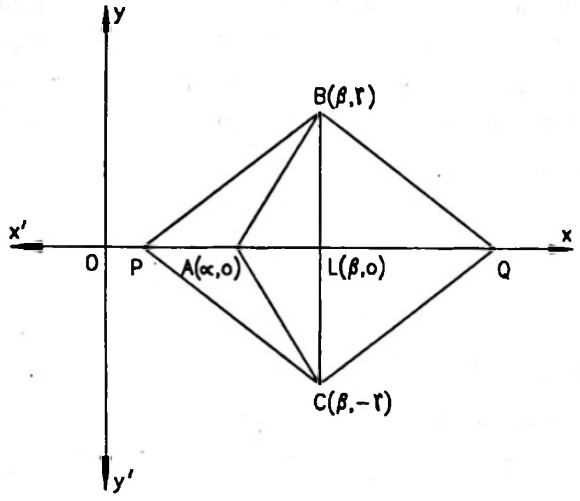


Fig. 19.34

$PB = PC = 2|\gamma|$ and $BQ = CQ = 2|\gamma|$.

In right triangle PLB, we have

$BL = |\gamma|$ and $PB = 2|\gamma|$

$$\therefore PL = \sqrt{PB^2 - BL^2}$$

$$= \sqrt{4\gamma^2 - \gamma^2}$$

$$= \sqrt{3}|\gamma|$$

Similarly, $QL = \sqrt{3}|\gamma|$.

Clearly, $A(\alpha, 0)$ lies inside the equilateral triangles PBC and QBC, if

$AL < PL$

$\Rightarrow |\beta - \alpha| < \sqrt{3}|\gamma|$

$\Rightarrow (\beta - \alpha)^2 < 3\gamma^2$

$\Rightarrow \alpha^2 + \beta^2 - 3\gamma^2 - 2\beta < 0$

$\Rightarrow D < 0$ [Using (i)]

$\Rightarrow f'(x) = 0$ has imaginary roots.

EXAMPLE 10 *Show that the complex numbers whose real and imaginary parts are integers and satisfy the equation* $z(\bar{z})^3 + z^3\bar{z} = 350$ *form a rectangle in the Argand plane with the length of its diagonal having an integral number of units.*

SOLUTION Let $z = x + iy$. Then,

$$z(\bar{z})^3 + z^3\bar{z} = 350$$

$$\Rightarrow z\bar{z}\{(\bar{z})^2 + z^2\} = 350$$

$$\Rightarrow 2(x^2 + y^2)(x^2 - y^2) = 350$$

$$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$$

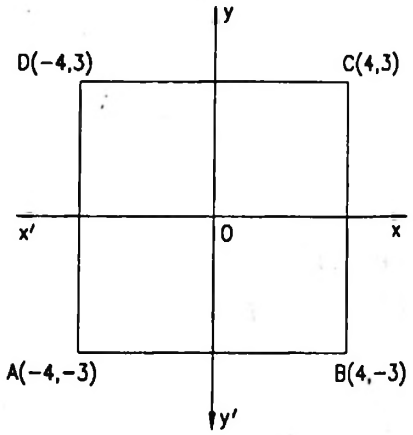


Fig. 19.35

$$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 25 \times 7 \text{ or, } (x^2 + y^2)(x^2 - y^2) = 35 \times 5$$

$$\Rightarrow x^2 + y^2 = 25 \text{ and } x^2 - y^2 = 7$$

or, $x^2 + y^2 = 35$ and $x^2 - y^2 = 5$

Clearly, $x^2 + y^2 = 35$ and $x^2 - y^2 = 5$ do not give integral values for x and y. Therefore,

$$x^2 + y^2 = 25 \text{ and } x^2 - y^2 = 7$$

$$\Rightarrow x = \pm 4, y = \pm 3$$

$$\Rightarrow z = x + iy = 4 + 3i, -4 + 3i, 4 - 3i, -4 - 3i$$

Clearly, these four points form a rectangle whose diagonal AC is of length given by

$$AC = |(4 + 3i) - (-4 - 3i)| = |8 + 6i| = \sqrt{64 + 36} = 10.$$

EXAMPLE 11 *If z is a complex number satisfying the equation $(z-1)^n = z^n$, where $n \in N$. Show that the locus of z is a straight line parallel to the imaginery axis.*

SOLUTION We have,

$$(z-1)^n = z^n$$

$$\Rightarrow \left(\frac{z-1}{z}\right)^n = 1$$

$$\Rightarrow \frac{z-1}{z} = 1^{1/n}$$

$$\Rightarrow \frac{z-1}{z} = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n} \quad \text{[Using De'Moivere's theorem]}$$

$$\Rightarrow \left|\frac{z-1}{z}\right| = 1$$

$$\Rightarrow \frac{|z-1|}{|z|} = 1$$

$$\Rightarrow |z-1| = |z|$$

$$\Rightarrow |z - (1 + 0i)| = |z - 0|$$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $(0, 0)$ and $(1, 0)$.

Hence, the locus of z is a straight line parallel to the imaginary axis.

EXAMPLE 12 *Find the point (s) in the Argand plane at which the curves* $arg\,(z-3i)=\frac{3\pi}{4}$ *and* $arg\,(2z+1-2i)=\frac{\pi}{4}$ *intersect.*

SOLUTION Clearly, $arg\,(z-3i)=\frac{3\pi}{4}$ represents a ray originating from (0, 3) and making an angle of 135° with the real axis as shown in Fig. 19.36.

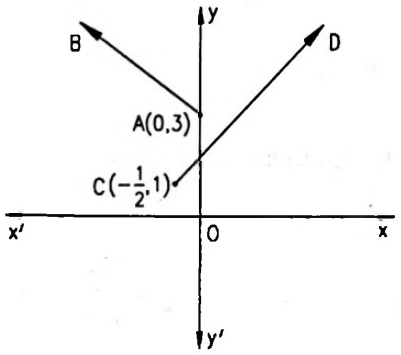


Fig. 19.36

We have,

$$arg\,(2z+1-2i)=\frac{\pi}{4}$$

$$\Rightarrow \arg\left\{2\left(z+\frac{1}{2}-i\right)\right\}=\frac{\pi}{4}$$

$$\Rightarrow \arg(2)+\arg\left(z+\frac{1}{2}-i\right)=\frac{\pi}{4}$$

$$\Rightarrow \arg\left(z+\frac{1}{2}-i\right)=\frac{\pi}{4} \qquad [\because \arg(2)=0]$$

$$\Rightarrow \arg\left\{z-\left(-\frac{1}{2}+i\right)\right\}=\frac{\pi}{4}$$

Clearly, it represents a ray originating from (– 1/2, 1) and making an angle of 45° with the real axis as shown in Fig. 19.36

Clearly, these two rays do not intersect. So, there is no point in the Argand plane where the two curves intersect.

EXAMPLE 13 *Let the complex numbers* z_1, z_2, z_3 *be the vertices of an equilateral triangle. Let* z_0 *be the circumcentre of the triangle. Prove that* $z_1^2+z_2^2+z_3^2=3z_0^2$.

SOLUTION Since the given triangle is equilateral. So, its circumcentre coincides with the centroid. Thus,

$$z_0=\frac{1}{3}(z_1+z_2+z_3)$$

$$\Rightarrow 9z_0^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_1) \qquad ...(i)$$

Again z_1, z_2, z_3 are the affixes of the vertices of an equilateral triangle. Therefore,

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1 \qquad ...(ii)$$

From (i) and (ii), we get

$$9z_0^2=z_1^2+z_2^2+z_3^2+2(z_1^2+z_2^2+z_3^2)$$

$$\Rightarrow z_1^2+z_2^2+z_3^2=3z_0^2.$$

EXAMPLE 14 *If A, B, C and D represent the complex numbers* z_1, z_2, z_3 *and* z_4 *satisfying the identity*

$$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)=0$$

Prove that $AD\,.\,BC\le BD\,.\,CA+CD\,.\,AB$.

SOLUTION We have,

$$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)=0$$

$$\Rightarrow (z_1-z_4)(z_2-z_3)=(z_4-z_2)(z_3-z_1)+(z_4-z_3)(z_1-z_2)$$

$$\Rightarrow |(z_1-z_4)(z_2-z_3)|=|(z_4-z_2)(z_3-z_1)+(z_4-z_3)(z_1-z_2)|$$

$$\Rightarrow |z_1-z_4|\,|z_2-z_3|\le |z_4-z_2|\,|z_3-z_1| + |z_4-z_3|\,|z_1-z_2|$$

$$\Rightarrow DA\,.\,CB\le BD\,.\,CA+CD\,.\,AB$$

EXAMPLE 15 *Find the complex number having least positive argument and satisfying* $|z-5i|\le 3$.

SOLUTION The region represented by $|z-5i|\le 3$ is the region containing all points lying inside or on the circle having centre at (0, 5) and radius 3. Clearly, the complex number having the least positive argument in this region is the complex number represented by the point of contact of tangent drawn from the origin to the circle.

Clearly, $OP=\sqrt{OC^2-CP^2}=\sqrt{25-9}=4$

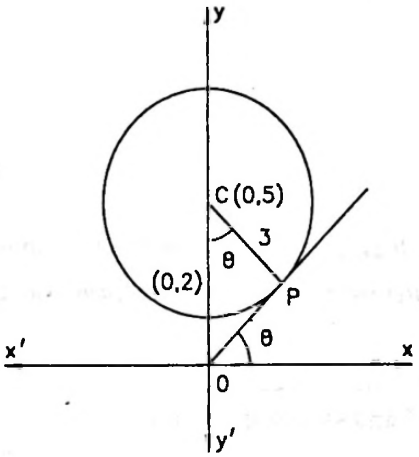


Fig. 19.37

So, the coordinates of P are $(4\cos\theta, 4\sin\theta)$.

From triangle OCP, we have $\cos\theta=3/5$.

So, the coordinates of P are (12/5, 16/5).

Hence, required complex number is $\frac{12}{5}+\frac{16}{5}i$

19.29 CONCEPT OF ROTATION

In this section, we shall learn about the effect of multiplication of a complex number by $e^{i\alpha}$ which will also be interpreted geometrically.

COMPLEX NUMBER AS A ROTATING ARROW IN THE ARGAND PLANE

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ be a complex number represented by a point P in the Argan plane. Then,

$$OP = r \text{ and } \angle XOP = \theta.$$

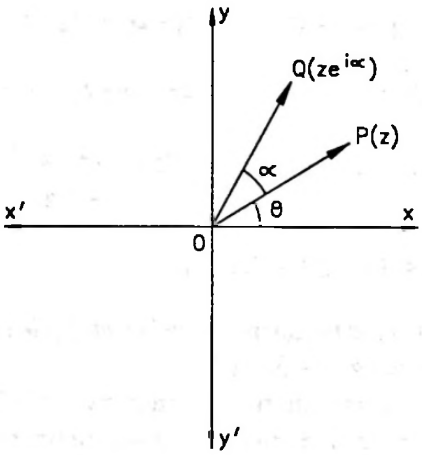


Fig. 19.38

Now, $ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha} = re^{i(\theta+\alpha)}$

This shows that $ze^{i\alpha}$ is the complex number whose modulus is r and argument $\theta + \alpha$. Clearly, $ze^{i\alpha}$ is represented by a point Q in the argand plane such that $OQ = r$ and $\angle XOQ = \theta + \alpha$

In other words, to obtain the point representing $ze^{i\alpha}$ we rotate $\vec{OP}$ through angle α in anticlock wise sense. Thus, multiplication by $e^{i\alpha}$ to z rotates the vector $\vec{OP}$ in anticlockwise sense through an angle α.

Similarly, multiplication by $e^{-i\alpha}$ to z rotates the vector $\vec{OP}$ in clockwise sense through an angle α.

REMARK *Let z_1 and z_2 be two complex numbers represented by points P and Q in the argand plane such that $\angle POQ = \theta$. Then, $z_1 e^{i\theta}$ is a vector of magnitude $|z_1| = OP$ along $\vec{OQ}$ and $\frac{z_1 e^{i\theta}}{|z_1|}$ is a unit vector along $\vec{OQ}$. Consequently, $|z_2| \cdot \frac{z_1 e^{i\theta}}{|z_1|}$* is a vector of magnitude $|z_2| = OQ$ along OQ, i.e.

$$z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta} \Rightarrow z_2 = \left|\frac{z_2}{z_1}\right| \cdot z_1 e^{i\theta}$$

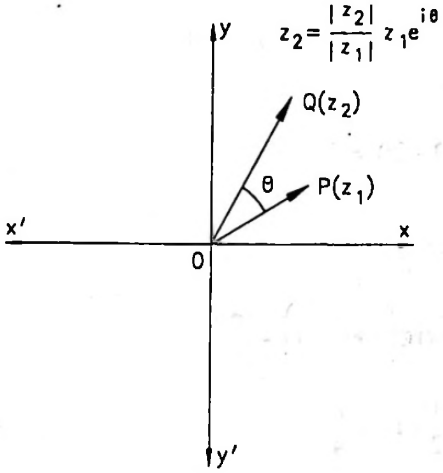


Fig. 19.39

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If z_1, z_2, z_3 are the vertices of an equilateral triangle having its circumcentre at the origin such that $z_1 = 1 + i$, find z_2 and z_3.*

SOLUTION Clearly, $\vec{OB}$ and $\vec{OC}$ are obtained by rotating $\vec{OA}$ through $2\pi/3$ and $4\pi/3$ respectively.

$\therefore \quad \vec{OB} = \vec{OA}\, e^{i\,2\pi/3}$ and, $\vec{OC} = \vec{OA}\, e^{i\,4\pi/3}$

$\Rightarrow z_2 = z_1 e^{i\,2\pi/3}$ and, $z_3 = z_1 e^{i\,4\pi/3}$

$\Rightarrow z_2 = (1+i)(\cos 2\pi/3 + i\sin 2\pi/3)$

and,

$z_3 = (1+i)(\cos 4\pi/3 + i\sin 4\pi/3)$

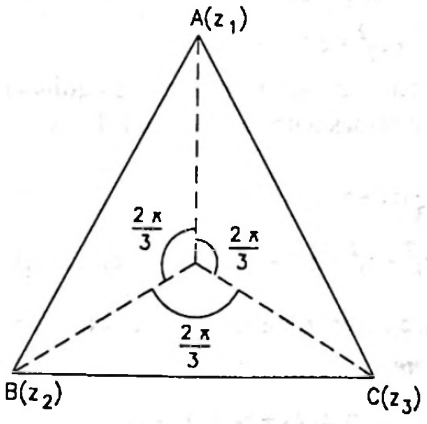


Fig. 19.40

$\Rightarrow z_2 = (1+i)(-1/2 + i\sqrt{3}/2)$ and $z_3 = (1+i)(-1/2 - i\sqrt{3}/2)$

$$\Rightarrow z_2 = -\left(\frac{\sqrt{3}+1}{2}\right) + i\left(\frac{\sqrt{3}-1}{2}\right) \text{ and } z_3 = -\left(\frac{1-\sqrt{3}}{2}\right) - i\left(\frac{1+\sqrt{3}}{2}\right)$$

EXAMPLE 2 *z_1, z_2, z_3 are the vertices of an equilateral triangle having its circumcentre at $(1-2i)$. If $z_1 = 2+i$, find z_2 and z_3.*

SOLUTION We have :

$\vec{OA}$ = affix of A – affix of O = $(2+i) - (1-2i) = 1+3i$

We observe that $\vec{OB}$ is obtained by rotating $\vec{OA}$ in anticlockwise sense through an angle $2\pi/3$. Therefore,

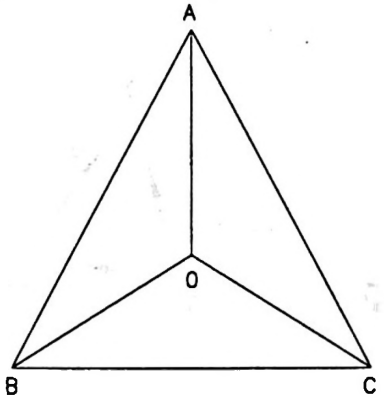


Fig. 19.41

$$\vec{OB} = \vec{OA}\, e^{i\,2\pi/3}$$

$\Rightarrow$ (Affix of B – Affix of O) = $(1+3i)\, e^{i\,2\pi/3}$

$\Rightarrow$ Affix of $B - (1-2i) = (1+3i)(\cos 2\pi/3 + i \sin 2\pi/3)$

$\Rightarrow$ Affix of $B - (1-2i) = (1+3i)(-1/2 + i\sqrt{3}/2)$

$\Rightarrow$ Affix of $B = 1 - 2i + (-1/2 - 3\sqrt{3}/2) + i(-3/2 + \sqrt{3}/2)$

$$\Rightarrow \text{Affix of } B = \frac{1-3\sqrt{3}}{2} + i\left(\frac{\sqrt{3}-7}{2}\right)$$

Similarly,

Affix of C

$= (1-2i) + (1+3i)\, e^{i\,4\pi/3}$

$= (1-2i) + (1+3i)(-1/2 - i\sqrt{3}/2)$

$$= (1-2i) + \left(-\frac{1}{2} + \frac{3\sqrt{3}}{2}\right) + i\left(-\frac{3}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{1+3\sqrt{3}}{2}\right) + i\left(-\frac{7}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{1+3\sqrt{3}}{2}\right) - i\left(\frac{7+\sqrt{3}}{2}\right)$$

EXAMPLE 3 *If the points A, B, C represent the complex numbers z_1, z_2, z_3 respectively and the angles of the ΔABC at B and C are both $\left(\frac{\pi-\alpha}{2}\right)$. Prove that*

$$(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2(\alpha/2).$$

SOLUTION We have,

$$\angle B = \angle C = \frac{\pi-\alpha}{2}.$$

$$\therefore \quad A = \pi - \left(\frac{\pi-\alpha}{2} + \frac{\pi-\alpha}{2}\right) = \alpha$$

Since $\vec{AC}$ is obtained by rotating $\vec{AB}$ through angle α.

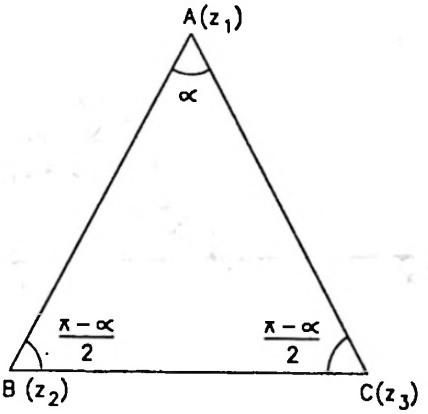


Fig. 19.42

$$\therefore \quad \vec{AC} = \vec{AB}\, e^{i\alpha}$$

$$\Rightarrow (z_3 - z_1) = (z_2 - z_1)\, e^{i\alpha} \qquad \ldots(i)$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = -1 + \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = 2i\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = 2i\sin\frac{\alpha}{2}\, e^{i\alpha/2}$$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_2 - z_1}\right)^2 = 4i^2\sin^2\frac{\alpha}{2}\,(e^{i\alpha/2})^2$$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_2 - z_1}\right)^2 = -4\sin^2\frac{\alpha}{2}\, e^{i\alpha} \qquad \ldots(ii)$$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_2 - z_1}\right)^2 = -4\sin^2\frac{\alpha}{2}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \qquad \text{[Using (i)]}$$

$$\Rightarrow (z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$$

EXAMPLE 4 *Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and OA = OB, where O is the origin, prove that $p^2 = 4q\cos^2\alpha/2$.*

[IIT 1997]

SOLUTION Since z_1 and z_2 are roots of the equation $z^2 + pz + q = 0$

$$\therefore\ z_1 + z_2 = -p \text{ and } z_1 z_2 = q \qquad ...(i)$$

Since $OA = OB$. So, $\vec{OB}$ is obtained by rotating $\vec{OA}$ in

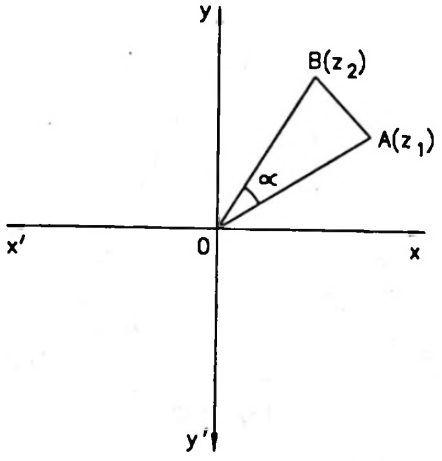


Fig. 19.43

anticlockwise sense through angle α.

$$\therefore\ \vec{OB} = \vec{OA}\,e^{i\alpha}$$

$$\Rightarrow\ z_2 = z_1 e^{i\alpha} \qquad ...(ii)$$

$$\Rightarrow \frac{z_2}{z_1} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_2}{z_1} + 1 = 1 + \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_2 + z_1}{z_1} = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right) = 2\cos\frac{\alpha}{2}e^{i\alpha/2}$$

$$\Rightarrow \frac{z_2 + z_1}{z_1} = 2\cos\frac{\alpha}{2}e^{i\alpha/2}$$

$$\Rightarrow \left(\frac{z_2 + z_1}{z_1}\right)^2 = 4\cos^2\frac{\alpha}{2}e^{i\alpha}$$

$$\Rightarrow \left(\frac{z_2 + z_1}{z_1}\right)^2 = 4\cos^2\frac{\alpha}{2}\left(\frac{z_2}{z_1}\right) \qquad \text{[Using (ii)]}$$

$$\Rightarrow (z_2 + z_1)^2 = 4z_1 z_2\cos^2\frac{\alpha}{2}$$

$$\Rightarrow (-p)^2 = 4q\cos^2\frac{\alpha}{2} \qquad \text{[Using (i)]}$$

$$\Rightarrow p^2 = 4q\cos^2\frac{\alpha}{2}$$

EXAMPLE 5 *Show that the area of the triangle on the Argand plane formed by the complex numbers z, iz and z + iz is $\frac{1}{2}|z|^2$.*

SOLUTION We have,

$$iz = ze^{i\pi/2}$$

This implies that iz is the vector obtained by rotating vector

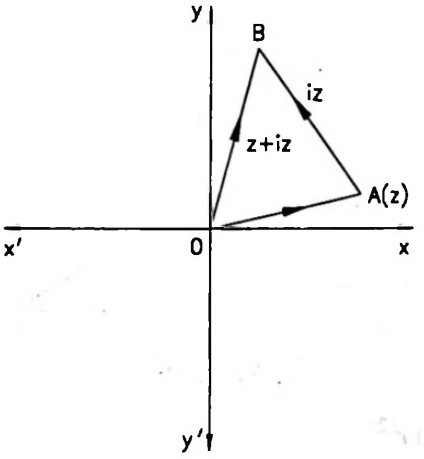


Fig. 19.44

z in anticlockwise direction through 90°. Therefore, $OA \perp AB$. So,

$$\text{Area of } \Delta OAB = \frac{1}{2}OA \times OB$$

$$= \frac{1}{2}|z|\,|iz| = \frac{1}{2}|z|^2.$$

EXAMPLE 6 *A is a point in the Argand plane. On the circle with OA as diameter are taken two points B and C so that $\angle AOB = \angle BOC = \theta$. If O is the origin and A, B, C represent the complex numbers z_1, z_2, z_3 respectively, show that*

$$z_2^2\cos 2\theta = z_1 z_3\cos^2\theta$$

SOLUTION Clearly, OB and OC are obtained by rotating OA in anticlock wise sense through angles θ and 2θ respectively. So,

$$z_3 = \left|\frac{z_3}{z_1}\right| z_1 e^{i2\theta} \text{ and } z_2 = \left|\frac{z_2}{z_1}\right| z_1 e^{i\theta} \text{ Also, } z_3 = \left|\frac{z_3}{z_2}\right| z_2 e^{i\theta}$$

$$\therefore\ \frac{z_2^2}{z_1 z_3} = \frac{\left\{\left|\frac{z_2}{z_1}\right| z_1 e^{i\theta}\right\}^2}{z_1 \cdot \left|\frac{z_3}{z_1}\right| z_1 e^{i2\theta}}$$

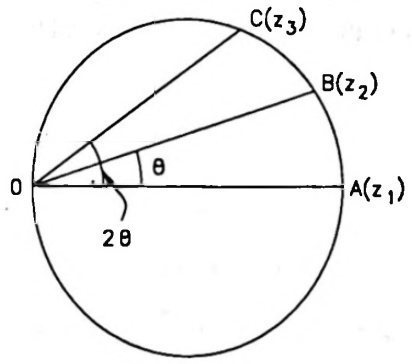


Fig. 19.45

$$= \frac{\left|\frac{z_2}{z_1}\right|^2}{\left|\frac{z_3}{z_1}\right|}$$

$$= \frac{|z_2|^2}{|z_1|^2} \times \frac{|z_1|}{|z_3|}$$

$$= \frac{|z_2|^2}{|z_1|\,|z_3|}$$

$$= \frac{OB^2}{OA \cdot OC}$$

$$= \left(\frac{OB}{OA}\right)^2 + \left(\frac{OC}{OA}\right)$$

$$= \frac{\cos^2\theta}{\cos 2\theta}$$

Hence, $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

EXAMPLE 7 *P is any point, O being the origin. The circle on OP as diameter is drawn. Points Q and R are taken on the circle to lie on the same side of OP such that* $\angle POQ = \theta$ *and* $\angle QOR = 2\theta$. *If P, Q, R are* z_1, z_2, z_3 *such that* $8 z_2^3 = (3\sqrt{3} + 5) z_1^2 z_3$, *find* θ.

SOLUTION We have,

$$z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}, \quad z_3 = \frac{|z_3|}{|z_1|} z_1 e^{i3\theta}$$

and,

$$z_3 = \frac{|z_3|}{|z_2|} z_2 e^{i2\theta}$$

$$\therefore \frac{z_2^3}{z_1^2 z_3} = \frac{\left\{\frac{|z_2|}{|z_1|} z_1 e^{i\theta}\right\}^3}{\left\{z_1^2 \frac{|z_3|}{|z_1|} z_1 e^{i3\theta}\right\}}$$

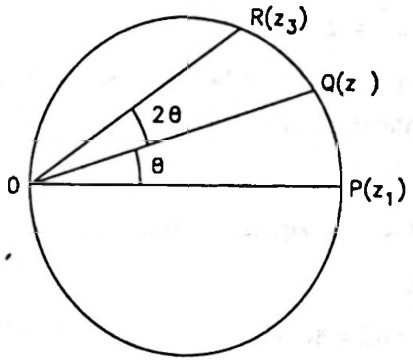


Fig. 19.46

$$\Rightarrow \frac{z_2^3}{z_1^2 z_3} = \frac{|z_2|^3}{|z_1|^2\,|z_3|}$$

$$\Rightarrow \frac{z_2^3}{z_1^2 z_3} = \frac{OQ^3}{OP^2 \cdot OR} = \frac{(OQ/OP)^3}{(OR/OP)}.$$

$$\Rightarrow \frac{z_2^3}{z_1^2 z_3} = \frac{\cos^3\theta}{\cos 3\theta}$$

$$\Rightarrow \frac{z_2^3}{z_1^2 z_3} = \frac{\cos^3\theta}{4\cos^3\theta - 3\cos\theta}$$

$$\Rightarrow \frac{3\sqrt{3}+5}{8} = \frac{\cos^2\theta}{4\cos^2\theta - 3} \quad \left[\because 8 z_2^3 = (3\sqrt{3}+5) z_1^2 z_3 \;\therefore\; z_2^3/z_1^2 z_3 = (3\sqrt{3}+5)/8\right]$$

$$\Rightarrow \frac{8}{3\sqrt{3}+5} = \frac{4\cos^2\theta - 3}{\cos^2\theta}$$

$$\Rightarrow \frac{8}{3\sqrt{3}+5} = 4 - 3\sec^2\theta$$

$$\Rightarrow 3\sec^2\theta = 4 - \frac{8}{3\sqrt{3}+5}$$

$$\Rightarrow 3\sec^2\theta = \frac{12\sqrt{3}+12}{3\sqrt{3}+5}$$

$$\Rightarrow \cos^2\theta = \frac{3\sqrt{3}+5}{4\sqrt{3}+4}$$

$$\Rightarrow \cos^2\theta = \frac{3\sqrt{3}+5}{4(\sqrt{3}+1)}$$

$$\Rightarrow \cos^2\theta = \frac{3\sqrt{3}+5}{4(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$$

$$\Rightarrow \cos^2\theta = \frac{(\sqrt{3}+1)^2}{8}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\Rightarrow \cos\theta = \cos 15^\circ \Rightarrow \theta = 15^\circ$$

EXAMPLE 8 *Let z_1, z_2, z_3 be the affixes of the vertices A, B and C respectively of a ΔABC. Prove that the triangle is equilateral iff*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

SOLUTION First, let z_1, z_2, z_3 be the affixes of the vertices A, B, C of an equilateral triangle ABC. Then, we have to prove that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

Since ΔABC is an equilateral triangle.

Therefore,

$$AB = BC = AC \text{ and } \angle A = \angle B = \angle C = \pi/3.$$

Clearly, $\vec{AC}$ can be obtained by rotating $\vec{AB}$ in anticlockwise sense through 60°.

$$\therefore \quad z_3 - z_1 = (z_2 - z_1)\, e^{i\pi/3} \qquad \ldots(i)$$

$$\Rightarrow \quad z_2 - z_1 = (z_3 - z_1)\, e^{-i\pi/3} \quad [\text{Dividing both sides by } e^{i\pi/3}]$$

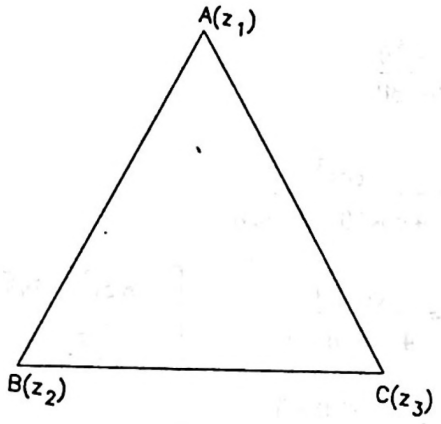


Fig. 19.47

$$\Rightarrow e^{i\pi}(z_2 - z_1) = (z_3 - z_1)\, e^{i\pi} \cdot e^{-i\pi/3}$$

[Multiplying both sides by $e^{i\pi}$]

$$\Rightarrow \quad -(z_2 - z_1) = (z_3 - z_1)\, e^{i2\pi/3}$$

$$\Rightarrow \quad z_1 - z_2 = (z_3 - z_1)\,\omega \qquad [\because \omega = e^{i2\pi/3}]$$

$$\Rightarrow z_1(1 + \omega) - z_2 - z_3\omega = 0$$

$$\Rightarrow z_1\omega^2 + z_2 + z_3\omega = 0 \qquad [\because 1 + \omega = -\omega^2]$$

$$\Rightarrow z_1\omega^3 + z_2\omega + z_3\omega^2 = 0\omega \qquad [\text{Multiplying both sides by } \omega]$$

$$\Rightarrow z_1 + z_2\omega + z_3\omega^2 = 0$$

$$\Rightarrow (z_1 + z_2\omega + z_3\omega^2)(z_1 + z_2\omega^2 + z_3\omega) = 0$$

[Multiplying both sides by $(z_1 + z_2\omega^2 + z_3\omega)$]

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Conversely, let z_1, z_2, z_3 be the affixes of the vertices of a triangle ABC such that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$. Then, we have to prove that ΔABC is an equilateral triangle.

Now,

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

$$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$$

$$\Rightarrow z_1 + \omega z_2 + \omega^2 z_3 = 0 \text{ or, } z_1 + \omega^2 z_2 + \omega z_3 = 0$$

If $z_1 + \omega z_2 + \omega^2 z_3 = 0$, then

$$z_1 + \omega z_2 + (-1 - \omega) z_3 = 0$$

$$\Rightarrow (z_1 - z_3) = \omega(z_3 - z_2)$$

$$\Rightarrow (z_1 - z_3) = -\omega(z_2 - z_3)$$

$$\Rightarrow (z_1 - z_3) = (z_2 - z_3)\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \qquad \left[\because \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow (z_1 - z_3) = (z_2 - z_3)\, e^{-i\pi/3}$$

$$\Rightarrow \quad z_2 - z_3 = (z_1 - z_3)\, e^{i\pi/3}$$

$$\Rightarrow \vec{CB} = \vec{CA}\, e^{i\pi/3}$$

$\Rightarrow$ $\vec{CB}$ is obtained by rotating $\vec{CA}$ in anticlockwise sense through $\pi/3$.

$$\Rightarrow |\vec{CB}| = |\vec{CA}| \text{ and } \angle BCA = 60°.$$

$$\Rightarrow CB = CA \text{ and } \angle BCA = 60°.$$

$\Rightarrow$ ΔABC is an equilateral triangle.

Similarly, $z_1 + \omega^2 z_2 + \omega z_3 = 0$ implies that ΔABC is an equilateral triangle.

Hence, ΔABC is an equilateral triangle.

EXAMPLE 9 *If z_1, z_2, z_3 are the vertices of an equilateral triangle having its circumcentre at z_0. Prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.*

SOLUTION In an equilateral triangle circumcentre, centroid and orthocentre are coincident.

$$\therefore \quad z_0 = \frac{z_1 + z_2 + z_3}{3}$$

$$\Rightarrow \quad z_1 + z_2 + z_3 = 3z_0.$$

Now, z_1, z_2, z_3 are the affixes of the vertices of an equilateral triangle.

$$\therefore \quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Rightarrow \quad (z_1 + z_2 + z_3)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$\Rightarrow (3z_0)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$\Rightarrow z_1 z_2 + z_2 z_3 + z_3 z_1 = 3 z_0^2$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = 3 z_0^2 \quad [\because z_1^2 + z_2^2 z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1]$

EXAMPLE 10 *Prove that the complex number z_1, z_2 and the origin form an isosceles triangle with vertical angle $2\pi/3$, if $z_1^2 + z_2^2 + z_1 z_2 = 0$.*

SOLUTION Let A and B be the points having affixes z_1 and z_2 respectively. Then,

$z_1^2 + z_2^2 + z_1 z_2 = 0.$

$\Rightarrow (z_1 - \omega z_2)(z_1 - \omega^2 z_2) = 0$

$\Rightarrow z_1 = \omega z_2$ or, $z_1 = \omega^2 z_2$

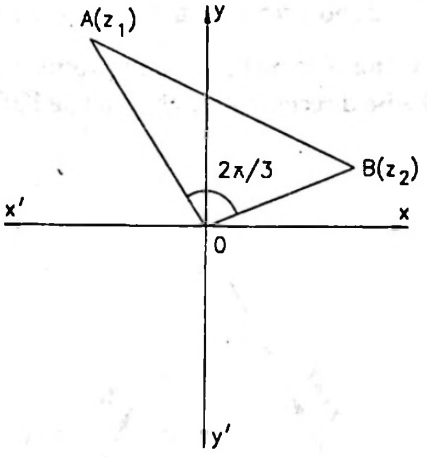


Fig. 19.48

$\Rightarrow z_1 = e^{i2\pi/3} z_2$ or, $z_1 = e^{4i\pi/3} z_2$

$\Rightarrow z_1 = e^{i2\pi/3} z_2$ or, $z_1 = e^{-i2\pi/3} z_2 \quad [\because e^{i4\pi/3} = e^{-i2\pi/3}]$

Now,

$z_1 = e^{i2\pi/3} z_2$

$\Rightarrow |z_1| = |e^{i2\pi/3}|\,|z_2| = |z_2|$

$\therefore z_1 = e^{i2\pi/3} z_2$

$\Rightarrow \vec{OA}$ is obtained by rotating $\vec{OB}$ through $2\pi/3$ in anticlock wise sense.

$\Rightarrow OA = OB$ and $\angle BOA = 2\pi/3$

$\Rightarrow \Delta OAB$ is an isosceles triangle

Similarly, $z_1 = z_2 e^{-i2\pi/3}$ implies that ΔOAB is an isosceles triangle.

EXAMPLE 11 *Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle, if $z_1^2 + z_2^2 - z_1 z_2 = 0$*

SOLUTION Let A and B be the points having affixes z_1 and z_2 respectively. Then,

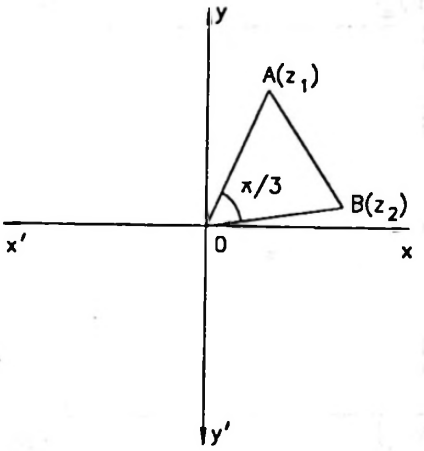


Fig. 19.49

$z_1^2 + z_2^2 - z_1 z_2 = 0$

$\Rightarrow (z_1 + \omega z_2)(z_1 + \omega^2 z_2) = 0$

$\Rightarrow z_1 + \omega z_2 = 0$ or, $z_1 + \omega^2 z_2 = 0$

$\Rightarrow z_1 = -\omega z_2$ or, $z_1 = -\omega^2 z_2$

$\Rightarrow z_1 = \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z_2$ or, $z_1 = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z_2$

$\Rightarrow z_1 = e^{-i\pi/3} z_2$ or, $z_1 = e^{i\pi/3} z_2$

Now,

$z_1 = e^{-i\pi/3} z_2$

$\Rightarrow |z_1| = |e^{-i\pi/3}|\,|z_2| = |z_2|$

$\therefore z_1 = e^{-i\pi/3} z_2$

$\Rightarrow \vec{OA}$ is obtained by rotating $\vec{OB}$ through $\pi/3$ in clockwise sense

$\Rightarrow OA = OB$ and $\angle BOA = \pi/3$

$\Rightarrow \Delta OAB$ is an equilateral triangle.

Similarly, $z_1 = e^{i\pi/3} z_2$ implies that ΔOAB is an equilateral triangle.

EXAMPLE 12 *If the vertices of a square are z_1, z_2, z_3 and z_4 taken in the anticlockwise order, prove that $z_3 = iz_1 + (1+i) z_2$ and $z_4 = (1-i) z_1 + i z_2$.*

SOLUTION Let $ABCD$ be the given square such that the affixes of the vertices A, B, C and D are z_1, z_2, z_3 and z_4 respectively. Since diagonals AC and BD bisect each other. Therefore,

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2} \Rightarrow z_1 + z_3 = z_2 + z_4 \qquad \ldots(i)$$

Clearly, $\vec{AD}$ can be obtained by rotating $\vec{AB}$ through $\pi/2$.

$\therefore \quad \vec{AD} = \vec{AB}\, e^{i\pi/2}$

$\Rightarrow z_4 - z_1 = (z_2 - z_1) e^{i\pi/2}$

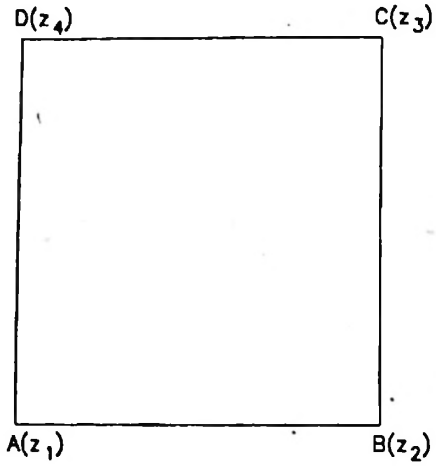


Fig. 19.50

$$\Rightarrow \quad z_4 - z_1 = (z_2 - z_1)\, i$$

$$\Rightarrow \quad z_4 = i\, z_2 + (1 - i)\, z_1$$

Putting $z_4 = i\, z_2 + (1 - i)\, z_1$ in (i), we get $z_3 = i\, z_1 + (1 + i)\, z_2$.

EXAMPLE 13 *ABCD is a rhombus in the Argand plane. if the affixes of the vertices are z_1, z_2, z_3 and z_4 respectively, and $\angle CBA = \pi/3$, show that*

$$2\, z_2 = z_1\,(1 + i\sqrt{3}) + z_3\,(1 - i\sqrt{3})$$

and, $$2z_4 = z_1\,(1 - i\sqrt{3}) + z_3\,(1 + i\sqrt{3}).$$

SOLUTION Since the diagonals of a rhombus bisect each other. Therefore,

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\Rightarrow z_1 + z_3 = z_2 + z_4 \qquad \ldots(i)$$

Since $BC = BA$ and $\vec{BA}$ is obtained by rotating $\vec{BC}$ through $\pi/3$.

$$\therefore \quad \vec{BA} = \vec{BC}\, e^{i\pi/3}$$

$$\Rightarrow (z_1 - z_2) = (z_3 - z_2)(\cos \pi/3 + i \sin \pi/3)$$

$$\Rightarrow (z_1 - z_2) = (z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right)$$

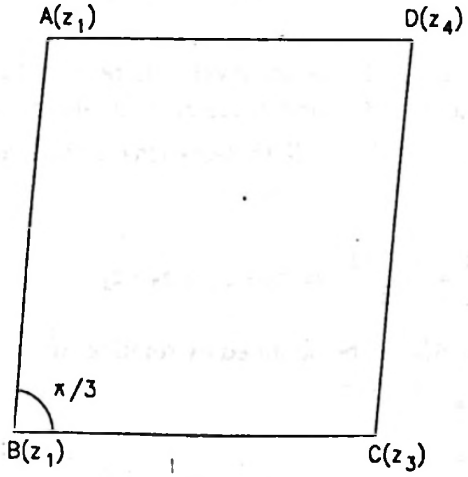


Fig. 19.51

$$\Rightarrow \quad 2\,(z_1 - z_2) = (z_3 - z_2)\,(1 + i\sqrt{3})$$

$$\Rightarrow \quad z_2\,(2 - 1 - i\sqrt{3}) = 2\,z_1 - z_3\,(1 + i\sqrt{3})$$

$$\Rightarrow \quad z_2\,(1 - i\sqrt{3}) = 2\,z_1 - z_3\,(1 + i\sqrt{3})$$

$$\Rightarrow \quad z_2\,(1 - i\sqrt{3})\,(1 + i\sqrt{3}) = 2\,z_1\,(1 + i\sqrt{3}) - z_3\,(1 + i\sqrt{3})^2$$

$$\Rightarrow \quad 4\,z_2 = 2\,z_1\,(1 + i\sqrt{3}) - z_3\,(1 - 3 + 2i\sqrt{3})$$

$$\Rightarrow \quad 2\,z_2 = z_1\,(1 + i\sqrt{3}) + z_3\,(1 - i\sqrt{3}) \qquad \ldots(ii)$$

From (i) and (ii), we get

$$2z_4 = z_1\,(1 - i\sqrt{3}) + z_3\,(1 + i\sqrt{3}).$$

EXAMPLE 14 *If $|z|^2 = 4$, find the area of the triangle formed by the complex numbers z, wz and z + wz as its sides.*

SOLUTION We know that,

$$w = e^{i\,2\pi/3} \text{ and } |w| = 1.$$

$$\Rightarrow \quad wz = e^{i\,2\pi/3}\, z \text{ and } |wz| = |w|\,|z| = |z|$$

$\Rightarrow$ wz is the vector obtained by rotating vector z in anticlockwise direction through an angle 120°.

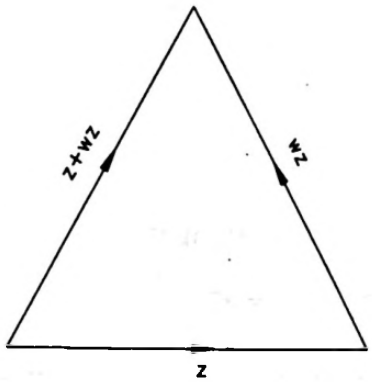


Fig. 19.52

From Fig. 19.52, we find that the third side of the triangle is $z + wz = z\,(1 + w) = -w^2\, z$.

Clearly, $|z + wz| = |-w^2\, z| = |w^2|\ |z| = |z|$.

Thus, the triangle formed by the complex numbers, z, wz and $z + wz$ is an equilateral triangle. Hence,

$$\text{Area} = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}\,|z|^2 = \frac{\sqrt{3}}{4} \times 4 = \sqrt{3} \text{ sq. units.}$$

EXAMPLE 15 *If $z_1, z_2, z_3, z_1', z_2'$ and z_3' are complex numbers such that*

$$(z_1 - z_2)\,(z_1' - z_2') = (z_2 - z_3)\,(z_2' - z_3') = (z_3 - z_1)\,(z_3' - z_1')$$

then show that the triangles whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' are equilateral.

SOLUTION We have,

$$(z_1 - z_2)(z_1' - z_2') = (z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1')$$

$$\Rightarrow (z_1 - z_2)(z_1' - z_2') = (z_2 - z_3)(z_2' - z_3')$$

$$\Rightarrow \frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_2 - z_3}{z_1' - z_2'}$$

$$\Rightarrow \frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_2 - z_3}{z_1' - z_2'} = \frac{(z_1 - z_2) + (z_2 - z_3)}{(z_2' - z_3') + (z_1' - z_2')}$$

$$\Rightarrow \frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_2 - z_3}{z_1' - z_2'} = \frac{z_1 - z_3}{z_1' - z_3'}$$

$$\Rightarrow \frac{z_1 - z_2}{z_2' - z_3'} = \frac{z_1 - z_3}{z_1' - z_3'}$$

$$\Rightarrow \frac{z_2' - z_3'}{z_1' - z_3'} = \frac{z_1 - z_2}{z_1 - z_3} \quad \ldots(i)$$

Again,

$$(z_1 - z_2)(z_1' - z_2') = (z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1')$$

$$\Rightarrow (z_2 - z_3)(z_2' - z_3') = (z_3 - z_1)(z_3' - z_1')$$

$$\Rightarrow \frac{z_2' - z_3'}{z_1' - z_3'} = \frac{z_1 - z_3}{z_2 - z_3} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{z_1 - z_2}{z_1 - z_3} = \frac{z_1 - z_3}{z_2 - z_3}$$

$$\Rightarrow (z_1 - z_3)^2 = (z_1 - z_2)(z_2 - z_3)$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Hence, the triangle whose vertices are z_1, z_2, z_3 is equilateral. Similarly, it can be shown that the triangle whose vertices are $z_1'\ z_2', z_3'$ is equilateral.

EXAMPLE 16 *ABC is a triangle formed of the points represented by the complex numbers z_1, z_2 and z_3. Triangles BCP, CAQ, ABR are drawn respectively on BC, CA and AB directly similar to one another. Show that the centroids of triangles ABC and PQR are at the same point.*

SOLUTION Let z_1', z_2', z_3' be the affixes of the vertices P, Q, R of ΔPQR. The triangles ABC and PQR are so constructed that the triangles BCP, CAQ and ABZ two by two and in that order are similar.

Since ΔBCP and ΔCAQ are similar. Therefore,

$$\frac{BC}{CP} = \frac{AC}{AQ} \quad \ldots(i)$$

and, $\angle BCP = \angle CAQ = \theta$ (say)

Now,

$$\vec{CP} = \frac{|\vec{CP}|}{|\vec{CB}|}\vec{CB}\, e^{i\theta}$$

$$\Rightarrow z_1' - z_3 = \frac{CP}{CB}(z_2 - z_3)\, e^{i\theta} \quad \ldots(ii)$$

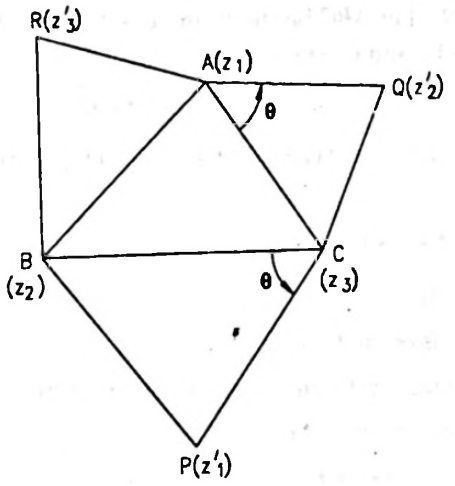


Fig. 19.53

and,

$$\vec{AQ} = \frac{AQ}{AC}\vec{AC}\, e^{i\theta}$$

$$\Rightarrow z_2' - z_1 = \frac{AQ}{AC}(z_3 - z_1)\, e^{i\theta}$$

From (ii) and (iii), we get

$$\left(\frac{z_1' - z_3}{z_2' - z_1}\right) = \frac{CP}{CB}\cdot\frac{AC}{AQ}\left(\frac{z_2 - z_3}{z_3 - z_1}\right)$$

$$\Rightarrow \frac{z_1' - z_3}{z_2' - z_1} = \frac{z_2 - z_3}{z_3 - z_1} \quad \text{[Using (i)]}$$

$$\Rightarrow \frac{z_1' - z_3}{z_2 - z_3} = \frac{z_2' - z_1}{z_3 - z_1} \quad \ldots(iv)$$

Similarly, we can obtain that

$$\frac{z_2' - z_1}{z_3' - z_2} = \frac{z_3 - z_1}{z_1 - z_2}$$

$$\Rightarrow \frac{z_2' - z_1}{z_3 - z_1} = \frac{z_3' - z_2}{z_1 - z_2} \quad \ldots(v)$$

From (iv) and (v), we get

$$\frac{z_1' - z_3}{z_2 - z_3} = \frac{z_2' - z_1}{z_3 - z_1} = \frac{z_3' - z_2}{z_1 - z_2}$$

$$\Rightarrow \frac{z_1' - z_3}{z_2 - z_3} = \frac{z_2' - z_1}{z_3 - z_1} = \frac{z_3' - z_2}{z_1 - z_2} = \frac{(z_1' - z_3) + (z_2' - z_1) + (z_3' - z_2)}{(z_2 - z_3) + (z_3 - z_1) + (z_1 - z_2)}$$

$$= \frac{(z_1' + z_2' + z_3') - (z_1 + z_2 + z_3)}{0}$$

$$\Rightarrow z_1' + z_2' + z_3' = z_1 + z_2 + z_3$$

$$\Rightarrow \frac{z_1' + z_2' + z_3'}{3} = \frac{z_1 + z_2 + z_3}{3}$$

Hence, the centroids of triangles ABC and PQR coincide.

EXAMPLE 17 *A triangle with vertices at $3 + i$, $i(1 + \sqrt{3})$ and $i(1 - \sqrt{3})$ is rotated about its circumcentre through an angle $\pi/3$ in the counter clockwise direction. Find the new position of its vertices.*

SOLUTION Let ABC be the triangle with its vertices $A(3+i)$, $B(i(\sqrt{3}+1))$ and $C(i(1-\sqrt{3}))$. Then,

$AB = |i(1+\sqrt{3})-(3+i)| = |-3+i\sqrt{3}| = \sqrt{9+3} = 2\sqrt{3}$,

$BC = |i(1-\sqrt{3})-i(1+\sqrt{3})| = |-2i\sqrt{3}| = \sqrt{0+12} = 2\sqrt{3}$ and,

$CA = |(3+i)-i(1-\sqrt{3})| = |3+i\sqrt{3}| = \sqrt{9+3} = 2\sqrt{3}$.

$\therefore\ AB = BC = CA$

$\Rightarrow\ \Delta ABC$ is equilateral

$\Rightarrow$ Circumcentre P coincides with the incentre.

Thus, the circumcentre is at

$$\frac{3+i+i(1+\sqrt{3})+i(1-\sqrt{3})}{3} = 1+i$$

Let A', B', C' be the new positions of vertices A, B and C respectively. Let the affixes of A', B', C' be z_1, z_2 and z_3 respectively.

If PA is rotated through 60°, then it coincides with PA'. Therefore,

$$z_1-(1+i) = \{(3+i)-(1+i)\}e^{i\pi/3}$$

$$\Rightarrow z_1-(1+i) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow z_1-(1+i) = 1+i\sqrt{3}$$

$$\Rightarrow z_1 = 2+i(\sqrt{3}+1)$$

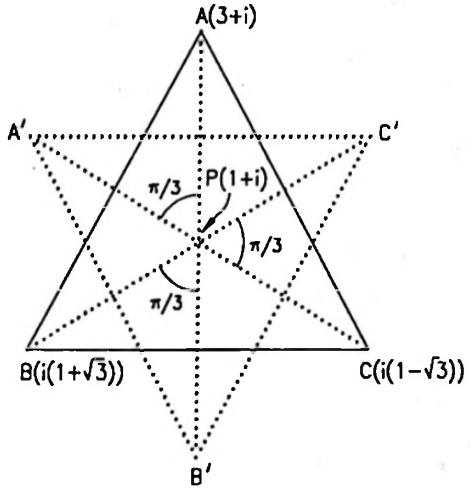


Fig. 19.54

Similarly, if PB and PC are rotated through $\frac{\pi}{3}$, then they coincide with PB' and PC' respectively.

$$\therefore\ z_2-(1+i) = \{i(1+\sqrt{3})-(1+i)\}e^{i\pi/3}$$

$$\Rightarrow z_2-(1+i) = (-1+i\sqrt{3})\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow z_2-(1+i) = (-1+i\sqrt{3})\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow z_2 = 1+i+\left(-\frac{1}{2}-\frac{3}{2}\right) = -1+i$$

and,

$$z_3-(1+i) = \{i(1-\sqrt{3})-(1+i)\}e^{i\pi/3}$$

$$\Rightarrow z_3-(1+i) = (-1-i\sqrt{3})\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow z_3-(1+i) = (-1-i\sqrt{3})\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow z_3-(1+i) = -\frac{1}{2}(1+2i\sqrt{3}-3) = 1-i\sqrt{3}$$

$$\Rightarrow z_3 = 2+i(1-\sqrt{3}).$$

EXAMPLE 18 *Equilateral triangles are described externally on the sides BC, CA and AB of a given triangle ABC. Prove using complex numbers that their centroids form an equilateral triangle.*

SOLUTION Let the complex numbers z_1, z_2, z_3 represent the vertices A, B and C respectively of ΔABC. Equilateral triangles BCD, CAE and ABF are constructed on the sides BC, CA and AB respectively.

Let P, Q and R be the centroids of triangles BCD, CAE and ABF respectively.

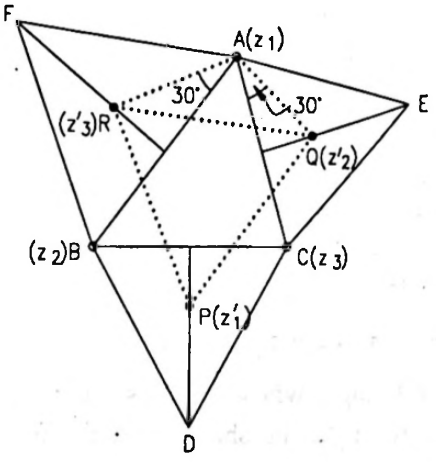


Fig. 19.55

We have,

$$CP = \frac{2}{3}(\text{Median of equilateral triangle } BCD)$$

$$= \frac{2}{3}\left(\frac{\sqrt{3}}{2}BC\right)$$

$$= \frac{BC}{\sqrt{3}}$$

Similarly, $AQ = \frac{AC}{\sqrt{3}}$ and $AR = BR = \frac{AB}{\sqrt{3}}$

Let P, Q, R represent the complex numbers z_1', z_2' and z_3' respectively. We have,

$$\angle CAQ = 30°$$

$$\therefore\ z_2'-z_1 = \frac{AQ}{AC}(z_3-z_1)e^{i\pi/6}$$

$$\Rightarrow z_2' - z_1 = \frac{1}{\sqrt{3}}(z_3 - z_1)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow z_2' - z_1 = \frac{1}{\sqrt{3}}(z_3 - z_1)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\Rightarrow z_2' - z_1 = \frac{1}{2\sqrt{3}}(z_3 - z_1)(\sqrt{3} + i) \quad ...(i)$$

$\angle BAR = 30°$

$$\therefore z_3' - z_1 = \frac{AR}{AB}(z_2 - z_1)e^{-i\pi/6}$$

$$\Rightarrow z_3' - z_1 = \frac{1}{\sqrt{3}}(z_2 - z_1)\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow z_3' - z_1 = \frac{1}{2\sqrt{3}}(z_2 - z_1)(\sqrt{3} - i) \quad ...(ii)$$

Now,

$$z_2' - z_3'$$

$$= (z_2' - z_1) - (z_3' - z_1)$$

$$= \frac{1}{2\sqrt{3}}(z_3 - z_1)(\sqrt{3} + i) - \frac{1}{2\sqrt{3}}(z_2 - z_1)(\sqrt{3} - i) \quad [\text{Using (i) and (ii)}]$$

$$= \frac{1}{2}\{(z_3 - z_1) - (z_2 - z_1)\} + \frac{1}{2\sqrt{3}}i\{(z_3 - z_1) + (z_2 - z_1)\}$$

$$= \frac{1}{2}(z_3 - z_2) + \frac{1}{2\sqrt{3}}i\{(z_3 - z_1) + (z_2 - z_1)\} \quad ...(iii)$$

Similarly,

$$z_3' - z_1' = \frac{1}{2}(z_1 - z_3) + \frac{i}{2\sqrt{3}}\{(z_3 - z_2) + (z_1 - z_2)\} \quad ...(iv)$$

Now,

$$(z_1' - z_3')e^{i\pi/3}$$

$$= (z_1' - z_3')\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= (z_1' - z_3')\left(\frac{1 + i\sqrt{3}}{2}\right)$$

$$= \left[\frac{1}{2}(z_3 - z_1) + \frac{i}{2\sqrt{3}}\{(z_2 - z_3) + (z_2 - z_1)\}\right]\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \left[\frac{1}{4}(z_3 - z_1) - \frac{1}{4}\{(z_2 - z_3) + (z_2 - z_1)\}\right]$$

$$+ i\left[\left\{\frac{\sqrt{3}}{4}(z_3 - z_1) + \frac{1}{4\sqrt{3}}\{(z_2 - z_3) + (z_2 - z_1)\}\right\}\right]$$

$$= \frac{1}{4}[(z_3 - z_1) - z_2 + z_3 - z_2 + z_1]$$

$$+ \frac{i}{4\sqrt{3}}[3(z_3 - z_1) + z_2 - z_3 + z_2 - z_1]$$

$$= \frac{1}{4}[2(z_3 - z_2)] + \frac{i}{2\sqrt{3}}[(z_3 - z_1) + (z_2 - z_1)]$$

$$= \frac{1}{2}(z_3 - z_2) + \frac{i}{2\sqrt{3}}[(z_3 - z_1) + (z_2 - z_1)]$$

$$= (z_2' - z_3') \quad [\text{From (iii)}]$$

Thus,

$$z_2' - z_3' = (z_1' - z_3')e^{i\pi/3}$$

$\Rightarrow RQ = RP$ and $\angle PRQ = \frac{\pi}{3}$

$\Rightarrow \Delta PQR$ is an equilateral triangle.

EXAMPLE 19 *In the Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with AC = BC and equal angle θ. If z_4 is the incentre of the triangle, prove that :*

(i) $(z_2 - z_1)(z_3 - z_1) = \sec\theta(1 + \cos\theta)(z_4 - z_1)^2$

(ii) $(z_2 - z_1)^2 = -\cot\theta\cot\frac{\theta}{2}(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$

SOLUTION (i) Let I be the incentre of ΔABC. Then,

$$\angle IAB = \frac{\theta}{2} \text{ and } \angle IAC = \frac{\theta}{2}.$$

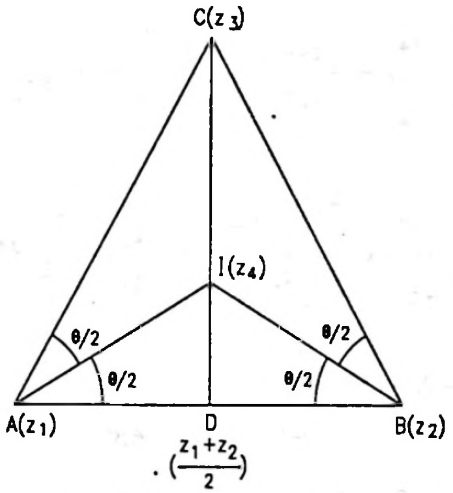


Fig. 19.56

$$\Rightarrow \frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{-i\theta/2} \quad [\text{Rotating } IA \text{ in clockwise direction}]$$

and,

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad [\text{Rotating } IA \text{ in anti-clockwise direction}]$$

$$\Rightarrow \frac{(z_2 - z_1)(z_3 - z_1)}{|z_2 - z_1||z_3 - z_1|} = \frac{(z_4 - z_1)^2}{|z_4 - z_1|^2}$$

$$\Rightarrow \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{AB \cdot AC}{(IA)^2}$$

$$\Rightarrow \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \left(\frac{AB}{IA}\right)^2 \cdot \left(\frac{AC}{AB}\right)$$

$$\Rightarrow \frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2} = 2\left(\frac{AD}{IA}\right)^2\left(\frac{AC}{AD}\right) \qquad [\because AB = 2AD]$$

$$\Rightarrow \frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2} = 2\left(\cos^2\frac{\theta}{2}\right)\left(\frac{1}{\cos\theta}\right)$$

$$\Rightarrow \frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2} = (1+\cos\theta)(\sec\theta)$$

$$\Rightarrow (z_2-z_1)(z_3-z_1) = (1+\cos\theta)\sec\theta\,(z_4-z_1)^2$$

(ii) We have, $AB \perp CD$. Therefore,

$$\frac{\frac{z_1+z_2}{2}-z_3}{\left|\frac{z_1+z_2}{2}-z_3\right|} = \frac{z_2-z_1}{|z_2-z_1|}\,e^{i\pi/2} \left[\begin{array}{l}\text{Rotating } AB \text{ through } \pi/2\\ \text{in anticlockwise direction}\end{array}\right]$$

Also,

$$\angle IDB = \frac{\pi}{2}$$

$$\therefore \frac{\frac{z_1+z_2}{2}-z_4}{\left|\frac{z_1+z_2}{2}-z_4\right|} = \frac{z_2-z_1}{|z_2-z_1|}\,e^{i\pi/2} \left[\text{Rotating } AB \text{ through } \frac{\pi}{2}\right]$$

$$\Rightarrow \frac{\frac{z_1+z_2}{2}-z_3}{\left|\frac{z_1+z_2}{2}-z_3\right|} \times \frac{\frac{z_1+z_2}{2}-z_4}{\left|\frac{z_1+z_2}{2}-z_4\right|} = \frac{(z_2-z_1)^2}{|z_2-z_1|^2}\,e^{i\pi}$$

$$\Rightarrow \frac{z_1+z_2-2z_3}{2\,CD} \times \frac{z_1+z_2-2z_4}{2\,ID} = -\frac{(z_2-z_1)^2}{AB^2}$$

$$\Rightarrow -\frac{(z_1+z_2-2z_3)(z_1+z_2-2z_4)}{(z_2-z_1)^2} = \frac{4\,(CD\times ID)}{4\,AD^2} \qquad [\because AB = 2AD]$$

$$= \frac{CD}{AD}\times\frac{ID}{AD}$$

$$= \tan\theta \times \tan\frac{\theta}{2}$$

$$\Rightarrow -(z_1+z_2-2z_3)(z_1+z_2-2z_4)\cot\theta\cot\frac{\theta}{2} = (z_2-z_1)^2$$

EXAMPLE 20 *If z_1, z_2, z_3 and z_1', z_2', z_3' represent the vertices of two similar triangles ABC and PQR respectively in the Argand plane, then prove that*

$$\left|\frac{\overline{z_1'}}{\overline{z_2}-\overline{z_1}}\right| \cdot \left|\frac{z_2-z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2-z_1}\right| \cdot \left|\frac{\overline{z}_3-\overline{z}_1}{\overline{z}_3'}\right| \geq 1.$$

SOLUTION Since ΔABC and ΔPQR are similar. Therefore,

$$\angle BAC = \angle QPR = \theta \text{ (say) and } \frac{AB}{AC} = \frac{PQ}{PR}$$

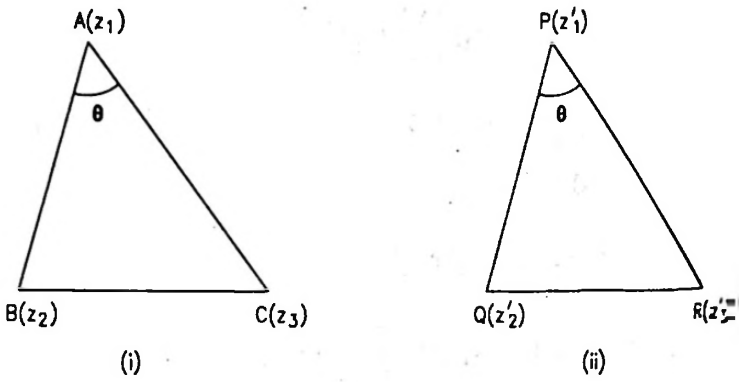


Fig. 19.57

Now,

$$\angle BAC = \angle QPR = \theta$$

$$\Rightarrow z_3-z_1 = \frac{AC}{AB}(z_2-z_1)e^{i\theta} \text{ and } z_3'-z_1' = \frac{PR}{PQ}(z_2'-z_1')e^{i\theta}$$

$$\Rightarrow \frac{z_3-z_1}{z_2-z_1} = \frac{AC}{AB}e^{i\theta} \text{ and } \frac{z_3'-z_1'}{z_2'-z_1'} = \frac{PR}{PQ}e^{i\theta}$$

$$\Rightarrow \frac{z_3-z_1}{z_2-z_1} = \frac{z_3'-z_1'}{z_2'-z_1'} \qquad \left[\because \frac{AC}{AB} = \frac{PR}{PQ}\right]$$

$$\Rightarrow (z_3-z_1)(z_2'-z_1') = (z_3'-z_1')(z_2-z_1)$$

$$\Rightarrow z_2'(z_3-z_1) - z_1'(z_3-z_1) = z_3'(z_2-z_1) - z_1'(z_2-z_1)$$

$$\Rightarrow z_1'(z_2-z_3) + z_2'(z_3-z_1) = z_3'(z_2-z_1)$$

$$\Rightarrow \frac{z_1'}{z_2-z_1}\cdot\frac{z_2-z_3}{z_3'} + \frac{z_2'}{z_2-z_1}\cdot\frac{z_3-z_1}{z_3'} = 1$$

$$\Rightarrow \left|\frac{z_1'}{z_2-z_1}\cdot\frac{z_2-z_3}{z_3'} + \frac{z_2'}{z_2-z_1}\cdot\frac{z_3-z_1}{z_3'}\right| = 1$$

$$\Rightarrow \left|\frac{z_1'}{z_2-z_1}\cdot\frac{z_2-z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2-z_1}\cdot\frac{z_3-z_1}{z_3'}\right| \geq 1$$

$$\left[\because |z_1+z_2| \leq |z_1|+|z_2|\right]$$

$$\Rightarrow \left|\frac{z_1'}{z_2-z_1}\right|\cdot\left|\frac{z_2-z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2-z_1}\right|\cdot\left|\frac{z_3-z_1}{z_3'}\right| \geq 1$$

$$\Rightarrow \left|\overline{\left(\frac{z_1'}{z_2-z_1}\right)}\right|\left|\frac{z_2-z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2-z_1}\right|\left|\overline{\left(\frac{z_3-z_1}{z_3'}\right)}\right| \geq 1$$

$$[\because |z| = |\bar{z}|]$$

$$\Rightarrow \left|\frac{\overline{z_1'}}{\overline{z}_2-\overline{z}_1}\right|\left|\frac{z_2-z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2-z_1}\right|\left|\frac{\overline{z}_3-\overline{z}_1}{\overline{z}_3'}\right| \geq 1$$

EXAMPLE 21 *Let A, B, C, D and E be points on the complex plane representing the complex numbers z_1, z_2, z_3, z_4 and z_5 respectively. If $(z_3-z_2)z_4 = (z_1-z_2)z_5$, prove that the triangles ABC and DOE are similar.*

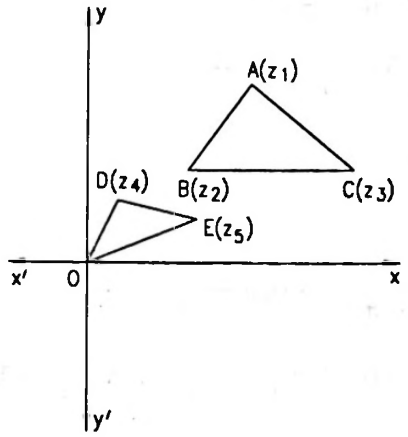


Fig. 19.58

SOLUTION We have,

$$(z_3 - z_2)\, z_4 = (z_1 - z_2)\, z_5$$

$$\Rightarrow \frac{z_3 - z_2}{z_1 - z_2} = \frac{z_5 - 0}{z_4 - 0}$$

$$\Rightarrow \left|\frac{z_3 - z_2}{z_1 - z_2}\right| = \left|\frac{z_5 - 0}{z_4 - 0}\right| \text{ and, } \arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \arg\left(\frac{z_5 - 0}{z_4 - 0}\right)$$

$$\Rightarrow \frac{|z_3 - z_2|}{|z_1 - z_2|} = \frac{|z_5 - 0|}{|z_4 - 0|} \text{ and } \angle ABC = \angle DOE$$

$$\Rightarrow \frac{BC}{AB} = \frac{OE}{OD} \text{ and } \angle ABC = \angle DOE.$$

Hence, triangles ABC and DOE are similar.

EXAMPLE 22 *Find the vertices of a regular polygon of n sides if its centre is located at z = 0 and one of its vertices z_1 is known.*

SOLUTION Let O be the origin and let the point A_1 represent the vertex z_1. Let A_2 be the vertex adjacent to A_1.

Clearly,

$$OA_1 = OA_2 = OA_3 = \ldots = OA_n$$

and, $\angle A_1 O A_2 = \angle A_2 OA_3 = \ldots = \angle A_n OA_1 = \frac{2\pi}{n}$.

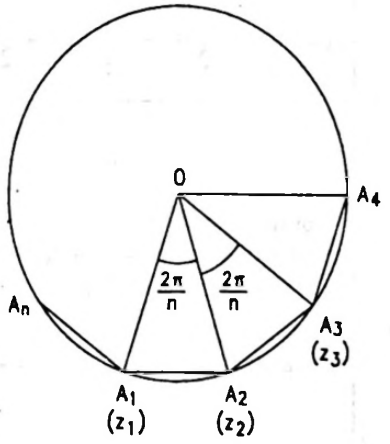


Fig. 19.59

$\therefore\ z_2 = z_1\, e^{i2\pi/n}$, $z_3 = z_1\, e^{i4\pi/n}$, $z_4 = e^{i6\pi/n}$ and so on.

If we take $\alpha = e^{i2\pi/n}$. Then,

$$z_2 = z_1\,\alpha,\ z_3 = z_1\,\alpha^2,\ z_4 = z_1\,\alpha^3,\ \ldots,\ z_n = z_1\,\alpha^{n-1}.$$

In general,

$$z_k = z_1\,\alpha^{k-1},\ \alpha = e^{i2\pi/n},\ k = 1, 2, \ldots, n.$$

EXAMPLE 23 *If points z_1 and z_2 are the adjacent vertices of a regular polygon of n sides, find the vertex z_3 ($\neq z_1$) adjacent to z_2.*

SOLUTION Let C (z_0) be the centre of the regular polygon and let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of the polygon representing complex numbers $z_1, z_2, \ldots, z_n$ respectively. Then,

$$CA_1 = CA_2 = \ldots = CA_n$$

and, $\angle A_1 CA_2 = \angle A_2 CA_3 = \ldots = \angle A_n CA_1 = \frac{2\pi}{n}$

Now,

$$CA_1 = CA_2 \text{ and } \angle A_1 CA_2 = \frac{2\pi}{n}$$

$$\Rightarrow \angle CA_2 A_1 = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) = \frac{\pi}{2} - \frac{\pi}{n}$$

$$\Rightarrow \angle A_1 A_2 A_3 = 2\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \pi - \frac{2\pi}{n}.$$

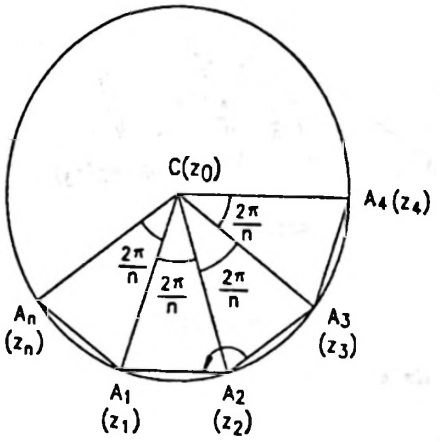


Fig. 19.60

Thus, we have,

$$A_1 A_2 = A_2 A_3 \text{ and } \angle A_1 A_2 A_3 = \pi - \frac{2\pi}{n}$$

$$\therefore\ z_1 - z_2 = (z_3 - z_2)\, e^{i(\pi - 2\pi/n)}$$

$$\Rightarrow z_1 - z_2 = (z_3 - z_2)\, e^{i\pi}.\, e^{\frac{-i2\pi}{n}}$$

$$\Rightarrow z_1 - z_2 = -(z_3 - z_2)\, e^{-i2\pi/n} \quad [\because e^{i\pi} = \cos\pi + i\sin\pi = -1]$$

$$\Rightarrow z_2 - z_3 = (z_1 - z_2)\, e^{i2\pi/n}$$

$$\Rightarrow z_3 = z_2 + (z_2 - z_1)\, e^{i2\pi/n}$$

If points $A_2, A_3, A_4, \ldots$ are marked on the left side of A_1 in clockwise sense, then by proceeding in the same manner, we obtain

$$z_3 = z_2 + (z_2 - z_1)\, e^{i\,2\pi/n}.$$

Hence, $z_3 = z_2 + (z_2 - z_1)\, e^{\pm i\,2\pi/n}$.

EXAMPLE 24 *Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-sided regular polygon inscribed in a circle of radius unity, then show that*

(i) $|A_1A_2|^2 + |A_1A_3|^2 + \ldots + |A_1A_n|^2 = 2n$

(ii) $|A_1A_2| \cdot |A_1A_3| \ldots |A_1A_n| = n$.

SOLUTION Let the centre of the circle of radius unity be at the origin and let the complex numbers $z_1, z_2, \ldots, z_n$ represent the vertices $A_1, A_2, \ldots A_n$ respectively.

Then, $A_1A_2 = A_2A_3 = \ldots = A_nA_1$

and,

$$\angle A_1OA_2 = \angle A_2OA_3 = \angle A_3OA_4 = \ldots = \angle A_nOA_1 = \frac{2\pi}{n}.$$

Since $OA_2 = OA_1$ and $\angle A_1OA_2 = \frac{2\pi}{n}$.

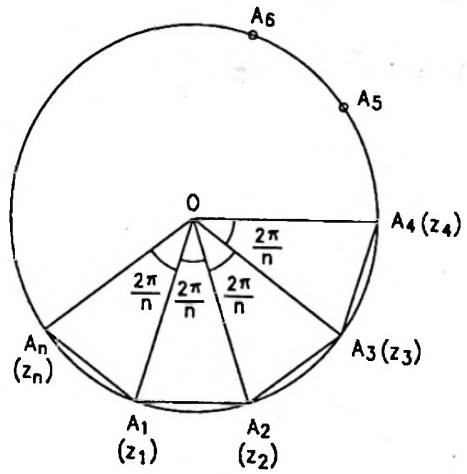


Fig. 19.61

$\therefore\ z_2 = z_1\, e^{i2\pi/n}$

Similarly,

$$z_3 = z_1\, e^{i4\pi/n},\ z_4 = z_1 = e^{i\,6\pi/i},\ \ldots,\ z_n = e^{i\,2(n-1)\pi/n}.$$

Let $\alpha = e^{i\,2\pi/n}$. Then,

$$z_2 = z_1\alpha,\ z_3 = z_1\alpha^2,\ z_4 = z_1\alpha^3,\ \ldots,\ z_n = z_1\alpha^{n-1}.$$

In general, $z_r = z_1\alpha^{r-1}$, $r = 2, 3, \ldots, n$

Now,

$$|A_1A_r|^2$$

$$= |z_r - z_1|^2$$

$$= |z_1\alpha^{r-1} - z_1|^2$$

$$= |z_1|^2\,|\alpha^{r-1} - 1|^2$$

$$= |z_1|^2 \left|e^{\frac{i\,2(r-1)\pi}{n}} - 1\right|^2$$

$$= \left|e^{\frac{i\,2(r-1)\pi}{n}} - 1\right|^2 \qquad [\because\ |z_1| = OA_1 = 1]$$

$$= \left|\cos\frac{2(r-1)\pi}{n} + i\sin\frac{2(r-1)\pi}{n} - 1\right|^2$$

$$= \left|-2\sin^2\frac{(r-1)\pi}{n} + 2i\sin(r-1)\frac{\pi}{n}\cos\frac{(r-1)\pi}{n}\right|^2$$

$$= \left|2i\sin(r-1)\frac{\pi}{n}\left\{\cos(r-1)\frac{\pi}{n} + i\sin(r-1)\frac{\pi}{n}\right\}\right|^2$$

$$= 4\sin^2\frac{(r-1)\pi}{n}\left|\cos(r-1)\frac{\pi}{4} + i\sin(r-1)\frac{\pi}{n}\right|^2$$

$$= 4\sin^2\frac{(r-1)\pi}{n}$$

$$= 2\left[1 - \cos\frac{2(r-1)\pi}{n}\right]$$

$$= 2 - 2\cos\frac{2(r-1)\pi}{n} \qquad \ldots(i)$$

(i) $|A_1A_2|^2 + |A_1A_3|^2 + \ldots + |A_1A_n|^2$

$$= \sum_{r=2}^{n} |A_1A_r|^2$$

$$= \sum_{r=2}^{n}\left\{2 - 2\cos\frac{2(r-1)\pi}{n}\right\} \qquad [\text{Using (i)}]$$

$$= \sum_{r=2}^{n} 2 - 2\sum_{r=2}^{n}\cos\frac{2(r-1)\pi}{n}$$

$$= 2(n-1) - 2\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{2(n-1)\pi}{n}\right]$$

$$= 2(n-1) - 2\left[\frac{\cos\left(\frac{\frac{2\pi}{n} + \frac{2(n-1)\pi}{n}}{2}\right)\sin\left(\frac{(n-1)2\pi}{2n}\right)}{\sin\frac{\pi}{n}}\right]$$

$$\left[\text{Using : } \cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha+(n-1)\beta) = \frac{\cos\left(\frac{\alpha+(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= 2(n-1) - 2\frac{\cos\pi \sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

$$= 2(n-1) + 2\frac{\sin\left(\pi - \frac{\pi}{n}\right)}{\sin\frac{\pi}{n}}$$

$$= 2(n-1) + 2\frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 2n.$$

(ii) We have,

$$|A_1A_r|$$

$$= |z_r - z_1|$$

$$= |z_1\alpha^{r-1} - z_1|$$

$$= |z_1||1-\alpha^{r-1}|, r = 2, 3, ..., n$$

$$= |1-\alpha^{r-1}|, r = 2, 3, ..., n \qquad [\because |z_1| = 1]$$

$$\therefore |A_1A_2|\,|A_1A_3|\,|A_1A_4|\,...\,|A_1A_n|$$

$$= |(1-\alpha)|\,|1-\alpha^2|\,|1-\alpha^3|\,....\,|1-\alpha^{n-1}|$$

$$= |(1-\alpha)(1-\alpha^2)(1-\alpha^3)...(1-\alpha^{n-1})| \qquad ...(ii)$$

Since $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ are nth roots of unity. Therefore,

$$z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2)....(z-\alpha^{n-1})$$

$$\Rightarrow \frac{z^n-1}{z-1} = (z-\alpha)(z-\alpha^2)...(z-\alpha^{n-1})$$

$$\Rightarrow 1 + z + z^2 + ... + z^{n-1} = (z-\alpha)(z-\alpha^2)...(z-\alpha^{n-1})$$

$$\Rightarrow n = (1-\alpha)(1-\alpha)(1-\alpha^2)....(1-\alpha^{n-1}) \qquad \text{[Putting } z = 1\text{]}$$

Substituting this value in (ii), we get

$$|A_1A_2|\,|A_1A_3|\,....\,|A_1A_n| = n.$$

EXAMPLE 25 *Let $A_1, A_2, ..., A_n$ be the vertices of an n-sided regular polygon such that*

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

Find the value of n.

[IIT 1994]

SOLUTION Let $C(z_0)$ be the centre of the polygon (See Fig.19.60 in Ex. 23) and let the vertices $A_1, A_2, ..., A_n$ represent the complex numbers $z_1, z_2, ... , z_n$ respectively. Then,

$$\angle A_1CA_2 = \angle A_2CA_3 = ... = \angle A_{n-1}CA_n = \frac{2\pi}{n}.$$

Also,

$$CA_1 = CA_2 = ... = CA_n.$$

$$\therefore z_2 - z_1 = (z_1 - z_0)e^{i2\pi/n}, z_3 - z_0 = (z_1 - z_0)e^{i4\pi/n},$$

$$z_4 - z_0 = (z_1 - z_0)e^{i6\pi/n} \text{ and so on.}$$

In general,

$$z_r - z_0 = (z_1 - z_0)e^{i(r-1)2\pi/n}, r = 2, 3, ..., n$$

$$\Rightarrow z_r = z_0 + (z_1 - z_0)e^{i2(r-1)\pi/n}, r = 2, 3, ... n$$

Now,

$$A_1A_r = |z_r - z_1|$$

$$= \left|z_0 + (z_1 - z_0)e^{i2(r-1)\pi/n} - z_1\right|$$

$$= |(z_0 - z_1) + (z_1 - z_0)e^{i2(r-1)\pi/n}|$$

$$= |z_1 - z_0|\left|1 - e^{i2(r-1)\pi/n}\right|$$

$$= |z_1 - z_0|\left|1 - \cos\frac{2(r-1)\pi}{n} - i\sin\frac{2(r-1)\pi}{n}\right|$$

$$= |z_1 - z_0|\left|2\sin^2\frac{(r-1)\pi}{n} - 2i\sin\frac{(r-1)\pi}{n}\frac{\cos(r-1)\pi}{n}\right|$$

$$= |z_1 - z_0|\,2\sin\frac{(r-1)\pi}{n}\left|\cos(r-1)\frac{\pi}{n} + i\sin\frac{(r-1)\pi}{n}\right|$$

$$= 2|z_1 - z_0|\sin(r-1)\frac{\pi}{n}, r = 2, 3, ..., n$$

$$\therefore A_1A_2 = 2|z_1 - z_0|\sin\frac{\pi}{n}, A_1A_3 = 2|z_1 - z_0|\sin\frac{2\pi}{n},$$

$$A_1A_4 = 2|z_1 - z_0|\sin\frac{3\pi}{n}$$

Now,

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

$$\Rightarrow \frac{1}{2|z_1 - z_0|\sin\frac{\pi}{4}} = \frac{1}{2|z_1 - z_0|\sin\frac{2\pi}{n}} + \frac{1}{2|z_1 - z_0|\sin\frac{3\pi}{2}}$$

$$\Rightarrow \frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$$

$$\Rightarrow \sin\frac{2\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{3\pi}{n}\sin\frac{\pi}{n} + \sin\frac{\pi}{n}\sin\frac{2\pi}{n}$$

$$\Rightarrow 2\cos\frac{\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{3\pi}{n} + \sin\frac{2\pi}{n}$$

$\Rightarrow \sin\frac{4\pi}{n} + \sin\frac{2\pi}{n} = \sin\frac{3\pi}{n} + \sin\frac{2\pi}{n}$

$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n}$

$\Rightarrow \frac{4\pi}{n} = \pi - \frac{3\pi}{n}$ [$\because \sin\theta = \sin\alpha \Rightarrow \theta = \pi - \alpha$]

$\Rightarrow \frac{7\pi}{n} = \pi$

$\Rightarrow n = 7.$

—❑❑❑—

19.30 STRAIGHT LINE

PARAMETRIC FORM : *Equation of a straight line joining the points having affixes z_1 and z_2 is*

$z = tz_1 + (1-t)z_2$, where $t \in R$ [*Parametric form*]

PROOF Let P be any point on the line joining the points A and B whose affixes are z_1 and z_2 respectively. Suppose P divides the join of A and B in the ratio $\lambda : 1$ ($\lambda \neq -1$). Then, the affix of P is given by

$$z = \frac{\lambda z_2 + 1 \cdot z_1}{\lambda + 1}$$

$\Rightarrow$ $$z = t z_1 + (1-t) z_2 \quad \text{...(i)}$$

,where $t = 1/(\lambda + 1)$

As t takes different real values, we get affixes of different points on the line AB. For $t = 0$, we get $z = z_2$ and for $t = 1$, we get $z = z_1$ and if t takes any value different from 0 and 1, we get the affix of a point P that divides the join of A and B in the ratio $1 - t : t$.

Conversely, every point on the line joining the points with affixes z_1 and z_2 has affix of the form given by (i).

Hence, the equation of the line joining the points with affixes z_1 and z_2 is

$$z = t z_1 + (1-t) z_2, \text{ where } t \in R$$

ILLUSTRATION *Let z_1 and z_2 be any two given complex numbers on arg and plane, find the locus of z in complex form if z varies as $z = \alpha z_1 + \beta z_2$ and where α and β are varying real numbers satisfying $\alpha + \beta = 1$.*

SOLUTION We have,

$z = \alpha z_1 + \beta z_2$, where $\alpha, \beta \in R$ such that $\alpha + \beta = 1$.

$\Rightarrow z = \alpha z_1 + (1-\alpha) z_2, \ \alpha \in R$

$\Rightarrow$ z lies on the line joining z_1 and z_2.

Hence, the locus of z is the line joining z_1 and z_2.

To find the locus in the complex form we proceed as follows :

$z = \alpha z_1 + (1-\alpha) z_2$

$\Rightarrow z = z_2 + \alpha(z_1 - z_2)$

$\Rightarrow z - z_2 = \alpha(z_1 - z_2)$

$\Rightarrow \bar{z} - \bar{z_2} = \alpha(\bar{z_1} - \bar{z_2})$ [Taking Conjugate]

Eliminating α from these two, we get

$$\frac{z - z_2}{\bar{z} - \bar{z_2}} = \frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}}$$

$$\Rightarrow \frac{z - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z_2}}{\bar{z_1} - \bar{z_2}}$$

$$\Rightarrow \begin{vmatrix} z - z_2 & \bar{z} - \bar{z_2} \\ z_1 - z_2 & \bar{z_1} - \bar{z_2} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z - z_2 & \bar{z} - \bar{z_2} & 0 \\ z_1 - z_2 & \bar{z_1} - \bar{z_2} & 0 \\ z_2 & \bar{z_2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z - z_2 & \bar{z} - \bar{z_2} & 0 \\ z_1 - z_2 & \bar{z_1} - \bar{z_2} & 0 \\ z_2 & \bar{z_2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z_1} & 1 \\ z_2 & \bar{z_2} & 1 \end{vmatrix} = 0 \quad [\text{Applying } R_1 \to R_1 + R_3, R_2 \to R_2 + R_3]$$

This is the required locus of z.

NON-PARAMETRIC FORM *Equation of a straight line joining the points having affixes z_1 and z_2 is*

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z_1} & 1 \\ z_2 & \bar{z_2} & 1 \end{vmatrix} = 0$$

OR

$$z(\bar{z_1} - \bar{z_2}) - \bar{z}(z_1 - z_2) + z_1\bar{z_2} - \bar{z_1}z_2 = 0$$

PROOF Equation of a straight line joining points having affixes z_1 and z_2 is

$z = t z_1 + (1-t) z_2$, where $t \in R$.

$\Rightarrow z - z_2 = t(z_1 - z_2)$ ...(i)

$\Rightarrow \overline{z - z_2} = t(\overline{z_1 - z_2})$

$\Rightarrow \bar{z} - \bar{z_2} = t(\bar{z_1} - \bar{z_2})$ ...(ii)

From (i) and (ii), we get

$$\frac{z - z_2}{\bar{z} - \bar{z_2}} = \frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}}$$

$$\Rightarrow \frac{z-z_2}{z_1-z_2} = \frac{\bar{z}-\bar{z}_2}{\bar{z}_1-\bar{z}_2}$$

$$\Rightarrow \begin{vmatrix} z-z_2 & \bar{z}-\bar{z}_2 & 0 \\ z_1-z_2 & \bar{z}_1-\bar{z}_2 & 0 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \quad [\text{Applying } R_1 \to R_1 + R_3, R_2 \to R_2 + R_3]$$

$$\Rightarrow z(\bar{z}_1-\bar{z}_2) - \bar{z}(z_1-z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

EQUATION OF A STRAIGHT LINE BY USING THE CONCEPT OF ROTATION

Let there be a line passing through two points $A(z_1)$ and $B(z_2)$.

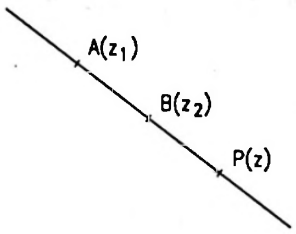


Fig. 19.62

Let $P(z)$ be an arbitrary point on the line. Then,

$\angle BAP = 0$ or π

$$\Rightarrow \arg\left(\frac{z-z_1}{z_2-z_1}\right) = 0 \text{ or } \pi$$

$$\Rightarrow \frac{z-z_1}{z_2-z_1} \text{ is purely real}$$

$$\Rightarrow \frac{z-z_1}{z_2-z_1} = \left(\frac{\overline{z-z_1}}{z_2-z_1}\right)$$

$$\Rightarrow \frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

$$\Rightarrow z(\bar{z}_2-\bar{z}_1) - \bar{z}(z_2-z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

GENERAL EQUATION OF A STRAIGHT LINE

THEOREM 1 *The general equation of a straight line is of the form*

$$a\bar{z} + \bar{a}z + b = 0$$

where a is a complex number and b is a real number.

PROOF The equation of a straight line passing through points having affixes z_1 and z_2 is given by

$$z(\bar{z}_1-\bar{z}_2) - \bar{z}(z_1-z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0 \qquad ...(i)$$

Multiplying (i) by i, we get

$$zi(\bar{z}_1-\bar{z}_2) - \bar{z}i(z_1-z_2) + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow \bar{z}\{-i(z_1-z_2)\} + z\{i(\bar{z}_1-\bar{z}_2)\} + i(z_1\bar{z}_2 - (z_1\bar{z}_2)) = 0$$

$$\Rightarrow \bar{z}\{-i(z_1-z_2)\} + z\{\overline{-i(\bar{z}_1-z_2)}\} + \{i(2i\,\text{Re}(z_1\bar{z}_2)\}$$

$$\Rightarrow \bar{z}\{-i(z_1-z_2)\} + z\{\overline{-i(z_1-z_2)}\} + \{-2\,\text{Re}(z_1\bar{z}_2)\} = 0$$

$$\Rightarrow \bar{z}a + z\bar{a} + b = 0, \text{ where } a = -i(z_1-z_2),\ b = -2\,\text{Re}(z_1\bar{z}_2)$$

Hence, the general equation of a straight line is of the form $a\bar{z} + \bar{a}z + b = 0$, where $b \in R$.

SLOPE OF THE LINE SEGMENT JOINING TWO POINTS

If A and B represent complex number z_1 and z_2 in the Argand plane, then the complex slope of AB is defined to be $\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$.

SLOPE OF THE LINE $a\bar{z} + \bar{a}z + b = 0$

Let A and B be two points on the line $a\bar{z} + \bar{a}z + b = 0$ such that affixes of A and B are z_1 and z_2. Then,

$$a\bar{z}_1 + \bar{a}z_1 + b = 0 \text{ and, } a\bar{z}_2 + \bar{a}z_2 = b = 0$$

$$\therefore\ a(\bar{z}_1-\bar{z}_2) + \bar{a}(z_1-z_2) = 0$$

$$\Rightarrow \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} = \frac{-a}{\bar{a}}$$

$$\Rightarrow \text{Slope of } AB = \frac{-a}{\bar{a}} = -\frac{\text{Coeff. of } \bar{z}}{\text{Coeff of } z}$$

Thus, the slope of the line $a\bar{z} + \bar{a}z + b = 0$ is $\frac{-a}{\bar{a}}$.

THEOREM 2 *If w_1 and w_2 are the complex slope of two lines on the Argand plane, then prove that the lines are :*

(i) *perpendicular, if $w_1 + w_2 = 0$*

(ii) *parallel, if $w_1 = w_2$*

PROOF Let z_1 and z_2 be the affixes of two points on the line of slope w_1 and z_3 and z_4 be two points on the line of slope w_2.

$$\text{Then, } w_1 = \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} \text{ and } w_2 = \frac{z_3-z_4}{\bar{z}_3-\bar{z}_4}$$

(i) If the lines are perpendicular, then

$$\frac{z_1-z_2}{|z_1-z_2|} = \frac{z_3-z_4}{|z_3-z_4|} e^{i\pi/2}$$

$$\Rightarrow \frac{(z_1-z_2)^2}{|z_1-z_2|^2} = \frac{(z_3-z_4)^2}{|z_3-z_4|^2} e^{i\pi}$$

$$\Rightarrow \frac{(z_1-z_2)^2}{(z_1-z_2)(\bar{z}_1-\bar{z}_2)} = \frac{(z_3-z_4)^2}{(z_3-z_4)(\bar{z}_3-\bar{z}_4)} \times -1$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = -\frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4}$$

$$\Rightarrow w_1 + w_2 = 0.$$

(ii) If the lines are parallel, then

$$\frac{z_1 - z_2}{|z_1 - z_2|} = \frac{z_3 - z_4}{|z_3 - z_4|} e^{i0}$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{|z_1 - z_2|^2} = \frac{(z_3 - z_4)^2}{|z_3 - z_4|^2}$$

$$\Rightarrow \frac{(z_1 - z_2)^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(z_3 - z_4)^2}{(z_3 - z_4)(\bar{z}_3 - \bar{z}_4)}$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4}$$

$$\Rightarrow w_1 = w_2.$$

REMARK *The equation of a line passing through two points having their affixes as z_1 and z_2 is*

$$z(\bar{z}_2 - \bar{z}_1) - \bar{z}(z_2 - z_1) + z_1(\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\text{or, } z(\bar{z}_2 - \bar{z}_1) - \bar{z}(z_2 - z_1) + \bar{z}_1 \bar{z}_2 - \overline{(z_1 \bar{z}_2)} = 0$$

$$\text{or, } z(\bar{z}_2 - \bar{z}_1) - \bar{z}(z_2 - z_1) + 2i \operatorname{Im}(z_1 \bar{z}_2) = 0$$

$$\text{or, } z(\bar{z}_1 - \bar{z}_2) i + \bar{z}(z_2 - z_1) i + 2 \operatorname{Im} z_1 \bar{z}_2 = 0$$

Compairing this equation with $a\bar{z} + \bar{a}z + b = 0$, we get

$$\bar{a} = (\bar{z}_1 - \bar{z}_2) i, a = (z_2 - z_1) i \text{ and } b = 2 \operatorname{Im}(z_1 \bar{z}_2)$$

$$\Rightarrow ai = z_1 - z_2$$

Since $z_1 - z_2$ is a vector parallel to the line passing through points z_1 and z_2. It follows from this that ai is a vector parallel to the line $a\bar{z} + \bar{a}z + b = 0$

EQUATION OF A LINE PARALLEL TO A GIVEN LINE

THEOREM 3 *The equation of a line parallel to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} + \bar{a}z + \lambda = 0$, where $\lambda \in R$.*

PROOF Let $P(z)$ be an arbitrary point on the line parallel to the given line. Suppose it passes through $A(z_1)$. Then,

$\frac{z - z_1}{|z - z_1|}$ is a unit vector parallel to the given line and $\frac{z - z_1}{\bar{z} - \bar{z}_1}$ is its slope. Also, slope of the line $a\bar{z} + \bar{a}z + b = 0$ is $\frac{-a}{\bar{a}}$. Since the lines are parallel. Therefore, their slopes are same i.e.

$$\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{-a}{\bar{a}}$$

$$\Rightarrow \bar{a}z - \bar{a}z_1 = -a\bar{z} + a\bar{z}_1$$

$$\Rightarrow \bar{a}z + a\bar{z} - (\bar{a}z_1 + a\bar{z}_1) = 0$$

$$\Rightarrow \bar{a}z + a\bar{z} - (\bar{a}z_1 + \overline{(\bar{a}z_1)}) = 0$$

$$\Rightarrow \bar{a}z + a\bar{z} - 2 \operatorname{Re}(\bar{a}z_1) = 0$$

$$\Rightarrow \bar{a}z + a\bar{z} + \lambda = 0, \text{ where } \lambda = -2 \operatorname{Re}(\bar{a}z_1)$$

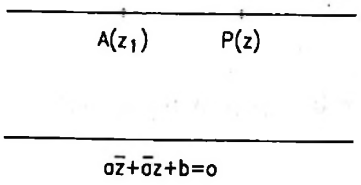


Fig. 19.63

EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE

THEOREM 4 *The equation of a line perpendicular to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} - \bar{a}z + i\lambda = 0$, , where $\lambda \in R$.*

PROOF Let $P(z)$ be an arbitrary point on the line perpendicular to the given line and suppose it passes through $A(z_1)$. Then, its slope is $\frac{z - z_1}{\bar{z} - \bar{z}_1}$. The slope of the given line is $\frac{-a}{\bar{a}}$. These two lines are perpendicular.

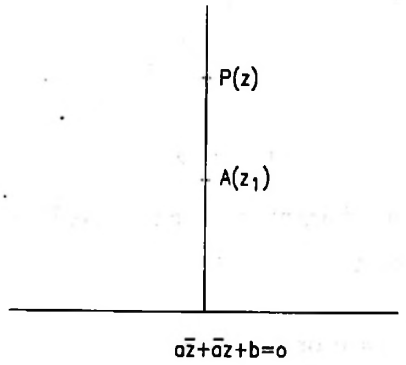


Fig. 19.64

Therefore,

$$\frac{z - z_1}{\bar{z} - \bar{z}_1} + \frac{-a}{\bar{a}} = 0$$

$$\Rightarrow \bar{a}z - \bar{a}z_1 - a\bar{z} + a\bar{z}_1 = 0$$

$$\Rightarrow \bar{a}z - a\bar{z} + (a\bar{z}_1 - \bar{a}z_1) = 0$$

$$\Rightarrow \bar{a}z - a\bar{z} + (a\bar{z}_1 - \overline{(a\bar{z}_1)}) = 0$$

$$\Rightarrow \bar{a}z - a\bar{z} + 2i \operatorname{Im}(a\bar{z}_1) = 0$$

$$\Rightarrow \bar{a}z - a\bar{z} + i\lambda = 0, \text{ where } \lambda = 2 \operatorname{Im}(a\bar{z}_1).$$

LENGTH OF THE PERPENDICULAR FROM A GIVEN POINT ON A GIVEN LINE

THEOREM 5 *The length of the perpendicualr from a point $P(z_0)$ to the line $a\bar{z} + \bar{a}z + b = 0$ is given by*

$$\left| \frac{a\bar{z}_0 + \bar{a}z_0 + b}{2|a|} \right|$$

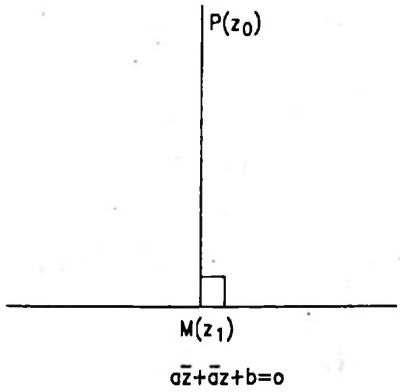


Fig. 19.65

PROOF Let PM be perpendicular from P on the line $a\bar{z} + \bar{a}z + b = 0$, and let the affix of M be z_1. Then,

$PM = |z_1 - z_0|$.

Since M lies on the line $a\bar{z} + \bar{a}z + b = 0$.

Therefore,

$a\bar{z}_1 + \bar{a}z_1 + b = 0$ ...(i)

Since PM is perpendicular to the given line. Therefore,

$$\frac{z_0 - z_1}{\bar{z}_0 - \bar{z}_1} + \left(\frac{-a}{\bar{a}}\right) = 0 \qquad \text{[Sum of the slopes = 0]}$$

$\Rightarrow \bar{a}z_0 - \bar{a}z_1 - a\bar{z}_0 + a\bar{z}_1 = 0$

$\Rightarrow a\bar{z}_0 + \bar{a}z_0 + b = 2\bar{a}z_0 + a\bar{z}_1 + b - \bar{a}z_1$ [From (i) $a\bar{z}_1 + b = \bar{a}z_1$]

$\Rightarrow a\bar{z}_0 + \bar{a}z_0 + b = 2\bar{a}z_0 - 2\bar{a}z_1$

$\Rightarrow 2\bar{a}(z_0 - z_1) = a\bar{z}_0 + \bar{a}z_0 + b$

$\Rightarrow |2\bar{a}(z_0 - z_1)| = |a\bar{z}_0 + \bar{a}z_0 + b|$

$$\Rightarrow |z_0 - z_1| = \frac{|a\bar{z}_0 + \bar{a}z_0 + b|}{2|\bar{a}|}$$

$$\Rightarrow PM = \frac{|a\bar{z}_0 + \bar{a}z_0 + b|}{2|a|} \qquad [\because |a| = |\bar{a}|]$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Let α and β be two fixed non-zero complex numbers and 'z' a variable complex number. If the lines $\alpha\bar{z} + \bar{\alpha}z + 1 = 0$ and $\beta\bar{z} + \bar{\beta}z + 1 = 0$ are mutually perpendicular, then prove that $\alpha\bar{\beta} + \bar{\alpha}\beta = 0$.*

SOLUTION The slopes of the two lines are $-\frac{\alpha}{\bar{\alpha}}$ and $\frac{-\beta}{\bar{\beta}}$ respectively. The lines will be perpendicular if

$$\frac{-\alpha}{\bar{\alpha}} + \left(\frac{-\beta}{\bar{\beta}}\right) = 0$$

$\Rightarrow \quad \alpha\bar{\beta} + \bar{\alpha}\beta = 0.$

EXAMPLE 2 *Let $\bar{b}z + b\bar{z} = c$, $b \neq 1$ be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2 \bar{b}$.*

[IIT 1997]

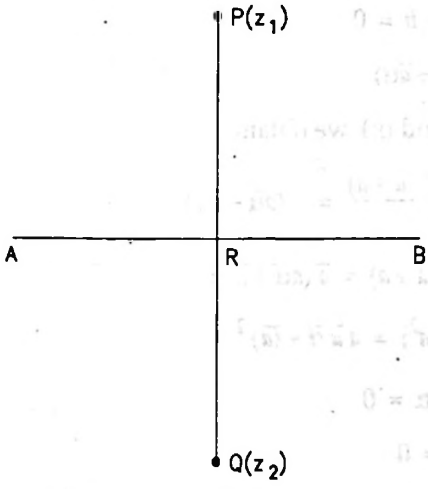


Fig. 19.66

SOLUTION Since z_1 is the reflection of point z_2 through the line. So, the mid-point of PQ lies on the line i.e. the point with affix $\frac{z_1 + z_2}{2}$ lies on $\bar{b}z + b\bar{z} = c$.

$$\Rightarrow \bar{b}\left(\frac{z_1 + z_2}{2}\right) + b\left(\frac{\bar{z_1} + \bar{z_2}}{2}\right) = c$$

$\Rightarrow \bar{b}(z_1 + z_2) + b(\bar{z_1} + \bar{z_2}) = 2c$ ...(i)

Since $PQ \perp AB$. Therefore,

Slope of PQ + slope of AB = 0

$$\Rightarrow \frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}} + \left(\frac{-b}{\bar{b}}\right) = 0$$

$\Rightarrow (\bar{z_1} - \bar{z_2})b - (z_1 - z_2)\bar{b} = 0$ ...(i)

Adding (i) and (ii), we get

$$\bar{z_1}b + z_2\bar{b} = c$$

EXAMPLE 3 *Let $A(z_1)$, $B(z_2)$ and (z_3) be the vertices of a triangle ABC and P be the foot of the altitude from A on BC. Find the complex number associated with the point P.*

SOLUTION Let the affix of point P be z.

Now,

$AP \perp BC$

$\Rightarrow \angle APC = \frac{\pi}{2}$

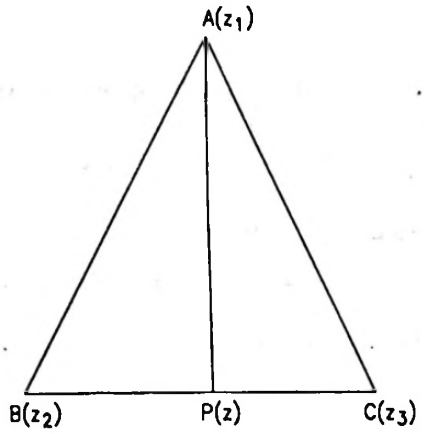


Fig. 19.67

$\Rightarrow$ Slope of AP + Slope of BC = 0

$$\Rightarrow \frac{z-z_1}{\bar{z}-\bar{z}_1}+\frac{z_2-z_3}{\bar{z}_2-\bar{z}_3}=0$$

$$\Rightarrow (z-z_1)(\bar{z}_2-\bar{z}_3)+(\bar{z}-\bar{z}_1)(z_2-z_3)=0$$

$$\Rightarrow z(\bar{z}_2-\bar{z}_3)-z_1(\bar{z}_2-\bar{z}_3)-\bar{z}_1(z_2-z_3)=-\bar{z}(z_2-z_3) \quad ...(i)$$

Equation of line BC is

$$z(\bar{z}_2-\bar{z}_3)-\bar{z}(z_2-z_3)+z_2\bar{z}_3-\bar{z}_2z_3=0$$

Since $P(z)$ lies on BC. Therefore,

$$z(\bar{z}_2-\bar{z}_3)-\bar{z}(z_2-z_3)+z_2\bar{z}_3-\bar{z}_2z_3=0$$

$$\Rightarrow -\bar{z}(z_2-z_3)=-z(\bar{z}_2-\bar{z}_3)+\bar{z}_2z_3-z_2\bar{z}_3 \quad ...(ii)$$

Eliminating $\bar{z}$ from (i) and (ii), we get

$$z(\bar{z}_2-\bar{z}_3)-z_1(\bar{z}_2-\bar{z}_3)-\bar{z}_1(z_2-z_3)$$

$$=-z(\bar{z}_2-\bar{z}_3)+\bar{z}_2z_3-z_2\bar{z}_3$$

$$\Rightarrow 2z(\bar{z}_2-\bar{z}_3)=z_1(\bar{z}_2-\bar{z}_3)+\bar{z}_1(z_2-z_3)+\bar{z}_2z_3-z_2\bar{z}_3$$

$$\Rightarrow 2z(\bar{z}_2-\bar{z}_3)=\bar{z}_1(z_2-z_3)-\bar{z}_2(z_2-z_3)+(z_1+z_2)(\bar{z}_2-\bar{z}_3)$$

$$z=\frac{1}{2}\left[\frac{(\bar{z}_1-\bar{z}_2)(z_2-z_3)}{\bar{z}_2-\bar{z}_3}+(z_1+z_2)\right]$$

EXAMPLE 4 *Find the locus of mid-point of line segment intercepted between real and imaginary axes, by the line $a\bar{z}+\bar{a}z+b=0$, where b is a real parameter and a is a fixed complex number with non-zero real and imaginary part.*

SOLUTION Suppose the line $a\bar{z}+\bar{a}z+b=0$ meets the real and imaginary axes at P and Q respectively. Let $R(\alpha)$ be the mid-point of segment PQ.

At point P on the real axis, we have $z=\bar{z}$. Therefore, putting $\bar{z}=z$ in $a\bar{z}+\bar{a}z+b=0$, we get

$$az+\bar{a}z+b=0 \Rightarrow z=-\frac{b}{a+\bar{a}}$$

So, the affix of P is given by

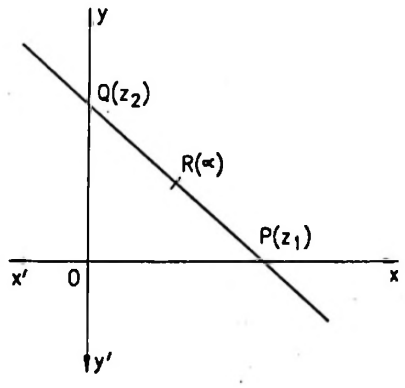


Fig. 19.68

$$z_1=-\frac{b}{a+\bar{a}}$$

At point Q on the imaginary axis, we have $\bar{z}=-z$. Therefore, putting $\bar{z}=-z$ in $a\bar{z}+\bar{a}z+b=0$, we get

$$\therefore -az+\bar{a}z+b=0 \Rightarrow z=\frac{-b}{\bar{a}-a}$$

So, the affix of Q is given by

$$z_2=-\frac{b}{\bar{a}-a}$$

Since R is the mid-point of PQ. Therefore,

$$\alpha=\frac{z_1+z_2}{2}$$

$$\Rightarrow 2\alpha=\frac{-b}{a+\bar{a}}-\frac{b}{\bar{a}-a}$$

$$\Rightarrow 2\alpha=\frac{-2\bar{a}b}{(\bar{a}-a)(\bar{a}+a)}$$

$$\Rightarrow \alpha=\frac{-\bar{a}b}{(\bar{a}-a)(\bar{a}+a)}$$

$$\Rightarrow b=\frac{-\alpha(\bar{a}-a)(\bar{a}+a)}{\bar{a}} \quad ...(i)$$

Since R lies on $a\bar{z}+\bar{a}z+b=0$. Therefore,

$$a\bar{\alpha}+\bar{a}\alpha+b=0$$

$$\Rightarrow b=-(a\bar{\alpha}+\bar{a}\alpha) \quad ...(ii)$$

From (i) and (ii), we obtain

$$\frac{-\alpha(\bar{a}-a)(\bar{a}+a)}{\bar{a}}=-(a\bar{\alpha}+\bar{a}\alpha)$$

$$\Rightarrow \alpha(\bar{a}-a)(\bar{a}+a)=\bar{a}(a\bar{\alpha}+\bar{a}\alpha)$$

$$\Rightarrow \alpha\{(\bar{a})^2-a^2\}=a\bar{a}\bar{\alpha}+(\bar{a})^2\alpha$$

$$\Rightarrow a\bar{a}\bar{\alpha}+a^2\alpha=0$$

$$\Rightarrow \bar{a}\bar{\alpha}+a\alpha=0$$

Hence, the locus of point R is $\bar{a}\bar{z}+az=0$

EXAMPLE 5 *Prove that the equations of two lines making an angle of 45° with the line $a\bar{z}+\bar{a}z+b=0$, where $a \in C$ and $b \in R$ and passing through the point c (c is a complex number) are*

$$\frac{z-c}{a} \pm i\,\frac{\bar{z}-\bar{c}}{\bar{a}} = 0$$

SOLUTION Let $P(z)$ be any point on the required line. Then,

$\dfrac{\vec{CP}}{|\vec{CP}|}$ i.e. $\dfrac{z-c}{|z-c|}$ is a unit vector parallel to it.

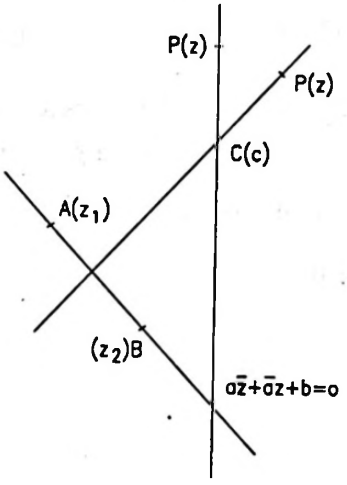


Fig. 19.69

Also, let $A(z_1)$ and $B(z_2)$ be two points on $a\bar{z}+\bar{a}z+b=0$.

Then, $\dfrac{z_2-z_1}{|z_2-z_1|}$ is a unit vector parallel to the line $a\bar{z}+\bar{a}z+b=0$.

Since the two lines are making an angle of 45° with each other. Therefore,

$$\frac{z-c}{|z-c|} = \frac{z_2-z_1}{|z_2-z_1|}e^{\pm i\pi/4}$$

$$\Rightarrow \frac{(z-c)^2}{|z-c|^2} = \frac{(z_2-z_1)^2}{|z_2-z_1|^2}e^{\pm i\pi/2}$$

$$\Rightarrow \frac{z-c}{\bar{z}-\bar{c}} = \frac{z_2-z_1}{\bar{z}_2-\bar{z}_1}(\pm i)$$

$$\Rightarrow \frac{z-c}{\bar{z}-\bar{c}} = \pm i\left(\frac{z_2-z_1}{\bar{z}_2-\bar{z}_1}\right) \quad ...(i)$$

Since $A(z_1)$ and $B(z_2)$ are on the line $a\bar{z}+\bar{a}z+b=0$. Therefore,

$$a\bar{z}_1+\bar{a}z_1+b = 0 \text{ and } a\bar{z}_2+\bar{a}z_2+b = 0$$

$$\Rightarrow a(\bar{z}_2-\bar{z}_1)+\bar{a}(z_2-z_1) = 0$$

$$\Rightarrow \frac{z_2-z_1}{\bar{z}_2-\bar{z}_1} = -\frac{a}{\bar{a}} \quad ...(ii)$$

From (i) and (ii), we get

$$\frac{z-c}{\bar{z}-\bar{c}} = \pm i\left(\frac{-a}{\bar{a}}\right)$$

$$\Rightarrow \frac{z-c}{a} \pm i\,\frac{\bar{z}-\bar{c}}{\bar{a}} = 0$$

These are the equations of the required lines.

EXAMPLE 6 *Find the equations of two lines making an angle of 45° with the line $(2-i)z+(2+i)\bar{z}+3=0$ and passing through $(-1, 4)$.*

SOLUTION Let $P(z)$ be any point on the required line. Then,

$\dfrac{z-(-1+4i)}{|z-(-1+4i)|}$ is a unit vector along it.

Also, let $A(z_1)$ and $B(z_2)$ be two points on the line $(2-i)z+(2+i)\bar{z}+3=0$. Then, a unit vector along it is

$$\frac{z_2-z_1}{|z_2-z_1|}$$

Since these two lines make an angle of 45° with each other. Therefore,

$$\frac{z-(-1+4i)}{|z-(-1+4i)|} = \frac{z_2-z_1}{|z_2-z_1|}e^{\pm i\pi/4}$$

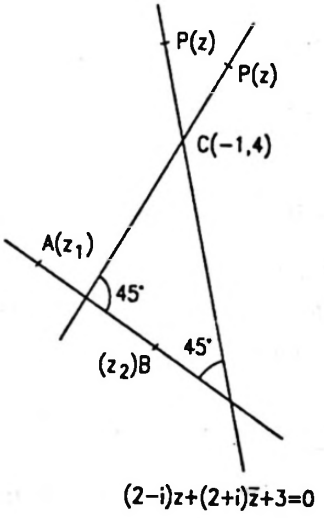


Fig. 19.70

$$\Rightarrow \frac{\{z-(-1+4i)\}^2}{|z-(-1+4i)|^2} = \frac{(z_2-z_1)^2}{|z_2-z_1|^2}e^{\pm i\pi/2}$$

$$\Rightarrow \frac{z-(-1+4i)}{\{z-(-1+4i)\}} = \frac{z_2-z_1}{(z_2-z_1)}(\pm i)$$

$$\Rightarrow \frac{z-(-1+4i)}{\bar{z}-(-1-4i)} = \frac{\pm i(z_2-z_1)}{(\bar{z}_2-\bar{z}_1)} \quad ...(i)$$

Since $A(z_1)$ and $B(z_2)$ lie on the line

$$(2-i)z+(2+i)\bar{z}+3=0.$$

Therefore,

$$(2-i)z_1+(2+i)\bar{z}_1+3=0 \text{ and } (2-i)z_2+(2+i)\bar{z}_2+3=0$$

$$\Rightarrow (2-i)(z_2-z_1)+(2+i)(\bar{z}_2-\bar{z}_1) = 0$$

$$\Rightarrow \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} = -\left(\frac{2+i}{2-i}\right) \quad \ldots(ii)$$

From (i) and (ii), we obtain the equations of the required lines as

$$\frac{z-(-1+4i)}{\bar{z}-(-1-4i)} = \mp\, i\left(\frac{2+i}{2-i}\right)$$

$$\Rightarrow \frac{z+(1-4i)}{\bar{z}+(1+4i)} = i\left(\frac{2+i}{2-i}\right) \text{ and } \frac{z+1-4i}{\bar{z}+1+4i} = -i\left(\frac{2+i}{2-i}\right)$$

$$\Rightarrow (z+1-4i)(2-i) = i(\bar{z}+1+4i)(2+i)$$

and,

$$(z+1-4i)(2-i) = -i(\bar{z}+1+4i)(2+i)$$

$$\Rightarrow z(2-i) - 2 - 9i = \bar{z}(2i-1) + (-9-2i)$$

and,

$$z(2-i) - 2 - 9i = -\bar{z}(2i-1) + 9 + 2i$$

$$\Rightarrow z(2-i) + \bar{z}(1-2i) + 7 - 7i = 0$$

and,

$$z(2-i) + \bar{z}(2i-1) - 11 - 11i = 0$$

$$\Rightarrow z(2-i)(1+i) + \bar{z}(1-2i)(1+i) + 7(1-i)(1+i) = 0$$

and,

$$z(2-i)(1-i) + \bar{z}(2i-1)(1-i) - 11(1+i)(1-i) = 0$$

$$\Rightarrow z(3+i) + \bar{z}(3-i) + 14 = 0$$

and,

$$z(1-3i) + \bar{z}(1+3i) - 22 = 0.$$

These are the equations of the required lines.

EXAMPLE 7 *Show that the equation of the line joining the complex numbers a and ib, where $a, b \in R$ and $a, b \neq 0$, is*

$$z\left(\frac{1}{2a} - \frac{i}{2b}\right) + \bar{z}\left(\frac{1}{2a} + \frac{i}{2b}\right) = 1$$

SOLUTION We know that the equation of the line passing through two points having affixes z_1 and z_2 is

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

So, the equation of the line passing through $z_1 = a + 0i$ and $z_2 = 0 + ib$ is

$$z(a+ib) - \bar{z}(a-ib) + a(-ib) - a(ib) = 0$$

or, $z(a+ib) - \bar{z}(a-ib) - 2aib = 0$

or, $z\left(\frac{a+ib}{2iab}\right) - \bar{z}\left(\frac{a-ib}{2iab}\right) = 1$

or, $z\left(\frac{1}{2a} - \frac{i}{2b}\right) + \bar{z}\left(\frac{1}{2a} + \frac{i}{2b}\right) = 1.$

EXAMPLE 8 *If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, prove that $\frac{iz_1}{z_2} = k$, where $k \in R$. Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of k.*

SOLUTION We have,

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$$

$$\Rightarrow |z_1 - z_2| = |z_1 + z_2|$$

$$\Rightarrow |z_1 - z_2|^2 = |z_1 + z_2|^2$$

$$\Rightarrow |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2),$$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

$$\Rightarrow 4|z_1||z_2|\cos(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary.}$$

$$\Rightarrow \frac{iz_1}{z_2} \text{ is purely real}$$

$$\Rightarrow \frac{iz_1}{z_2} = k, \text{ for some real number } k.$$

Let θ be the angle between the lines OP and OQ. Then,

$$\theta = \arg\left(\frac{z_1 + z_2}{z_1 - z_2}\right)$$

$$= \arg\left(\frac{\frac{z_1}{z_2} + 1}{\frac{z_1}{z_2} - 1}\right)$$

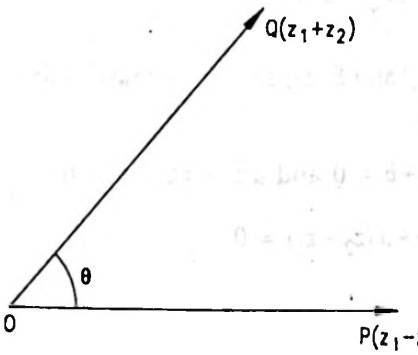


Fig. 19.71

$$= \arg\left(\frac{-ik+1}{-ik-1}\right)$$

$$= \arg\left(\frac{ik-1}{ik+1}\right)$$

$$= \arg\left(\frac{k^2-1+2ik}{k^2+1}\right)$$

$$= \tan^{-1}\left(\frac{2k}{k^2-1}\right)$$

EXAMPLE 9 *If a, b, c are three collinear points in the Argand plane such that $\left|(c-a)(c-b)\right| = R^2$, then prove that $b = c + \frac{R^2}{\bar{a}-\bar{c}}$*

SOLUTION Since a, b and c are collinear points. Therefore,

$$\frac{c-a}{|c-a|} = \frac{b-c}{|b-c|}e^{i\alpha} \text{ or, } \frac{c-a}{|c-a|} = \frac{b-c}{|b-c|}e^{i\pi}$$

$$\Rightarrow \frac{(c-a)^2}{|c-a|^2} = \frac{(b-c)^2}{|b-c|^2}$$

$$\Rightarrow \frac{c-a}{\bar{c}-\bar{a}} = \frac{b-c}{\bar{b}-\bar{c}} \qquad \left[\because \frac{z^2}{|z|^2} = \frac{z^2}{z\bar{z}} = \frac{z}{\bar{z}}\right]$$

$$\Rightarrow (c-a)(\bar{b}-\bar{c}) = (b-c)(\bar{c}-\bar{a})$$

$$\Rightarrow (c-a) = \frac{(b-c)(\bar{c}-\bar{a})}{(\bar{b}-\bar{c})} \qquad \ldots(i)$$

We have,

$$|(c-a)(c-b)| = R^2$$

$$\Rightarrow |(c-a)(c-b)|^2 = R^2$$

$$\Rightarrow \{(c-a)(c-b)\}\{(\bar{c}-\bar{a})(\bar{c}-\bar{b})\} = R^4$$

$$\Rightarrow \left\{\frac{(b-c)(\bar{c}-\bar{a})}{(\bar{b}-\bar{c})}(c-b)\right\} \times \{(\bar{c}-\bar{a})(\bar{c}-\bar{b})\} = R^4$$

$$\Rightarrow (b-c)^2(\bar{a}-\bar{c})^2 = R^4$$

$$\Rightarrow (b-c)(\bar{a}-\bar{c}) = R^2$$

$$\Rightarrow b-c = \frac{R^2}{\bar{a}-\bar{c}}$$

$$\Rightarrow b = c + \frac{R^2}{\bar{a}-\bar{c}}$$

ALITER It is given that a, b, c are collinear points in the Argand plane. So, let b divide the segment joining a and c in the ratio $\lambda : 1$ (say).

$$\therefore b = \frac{a+\lambda c}{\lambda+1}$$

$$\Rightarrow c-b = c - \frac{a+\lambda c}{\lambda+1}$$

$$\Rightarrow c-b = \frac{c-a}{\lambda+1} \qquad \ldots(i)$$

$$\Rightarrow (c-b)(c-a) = \frac{(c-a)^2}{\lambda+1}$$

$$\Rightarrow |(c-a)(c-b)| = \frac{|(c-a)^2|}{\lambda+1}$$

$$\Rightarrow |(c-a)(c-b)| = \frac{|c-a|^2}{\lambda+1}$$

$$\Rightarrow R^2 = \frac{|c-a|^2}{\lambda+1}$$

$$\Rightarrow R^2 = \frac{(c-a)(\bar{c}-\bar{a})}{\lambda+1}$$

$$\Rightarrow \frac{R^2}{\bar{c}-\bar{a}} = \frac{c-a}{\lambda+1}$$

$$\Rightarrow \frac{R^2}{\bar{c}-\bar{a}} = c-b \qquad \text{[Using (i)]}$$

$$\Rightarrow \frac{R^2}{\bar{a}-\bar{c}} = b-c$$

$$\Rightarrow b = c + \frac{R^2}{\bar{a}-\bar{c}}$$

EXAMPLE 10 *Find the equation of the parabola (in complex form) with focus at $(1+i)$ and directrix $(1-i)z + (1+i)\bar{z} + 2 = 0$.*

SOLUTION We know that a parabola is the locus of a point which moves in such a way that its distance from a fixed point (focus) is always equal to the distance from a fixed line (directrix).

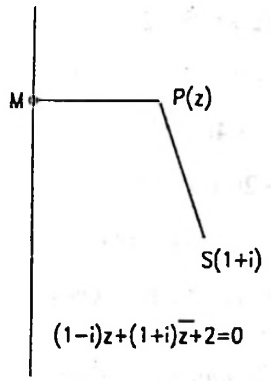


Fig. 19.72

So, let $P(z)$ be any point on the parabola. Then,

$$SP = PM$$

$$\frac{|(1-i)z+(1+i)\bar{z}+2|}{2|1+i|} = |z-(1+i)|$$

$$\Rightarrow \frac{|(1-i)z+(1+i)\bar{z}+2|}{2\sqrt{2}} = |z-(1+i)|$$

$$\Rightarrow |(1-i)z+(1+i)\bar{z}+2|^2 = 8|z-1-i|^2$$

$$\Rightarrow \{(1-i)z+(1+i)\bar{z}+2\}\{(1+i)\bar{z}+(1-i)z+2\}$$

$$= 8(z-1-i)(\bar{z}-1+i) \qquad [\because |z|^2 = z\bar{z}]$$

$$\Rightarrow 2z\bar{z}+(1-i)^2z^2+2(1-i)z+(1+i)^2(\bar{z})^2$$
$$+2z\bar{z}+2(1+i)\bar{z}+2(1+i)\bar{z}+2(1-i)z+4$$
$$=8(z\bar{z}-z+iz-\bar{z}+1-i-i\bar{z}+i+1)$$

$$\Rightarrow 4z\bar{z}-2iz^2+2i(\bar{z})^2+4(1-i)z+4(1+i)\bar{z}+4$$
$$=8z\bar{z}-8(1-i)z-8(1+i)\bar{z}+16$$

$$\Rightarrow 4z\bar{z}+2i\left(z^2-(\bar{z})^2\right)-12(1-i)z-12(1+i)\bar{z}+12=0$$

$$\Rightarrow z\bar{z}-3(1-i)z-3(1+i)\bar{z}-2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)+3=0$$

$$\Rightarrow z\bar{z}-6\left(\frac{z+\bar{z}}{2}\right)-6\left(\frac{z-\bar{z}}{2i}\right)-2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)+3=0$$

EXAMPLE 11 *The transformation* $w=\dfrac{z+2}{z+i}$, *where* $z\neq i$, $w\neq 1$ *maps the complex number* $z=x+iy$ *onto the complex number* $w=u+iv$. *If the point representing* w *lies on the real axis, find the locus of* z.

SOLUTION Since the point representing $w=\dfrac{z+2}{z+i}$ lies on the real axis. Therefore, w is purely real. Thus,

$$w=\bar{w}$$

$$\Rightarrow \frac{z+2}{z+i}=\overline{\left(\frac{z+2}{z+i}\right)}$$

$$\Rightarrow \frac{z+2}{z+i}=\frac{\bar{z}+2}{\bar{z}-i}$$

$$\Rightarrow (z+2)(\bar{z}-i)=(z+i)(\bar{z}+2)$$

$$\Rightarrow z\bar{z}-iz+2\bar{z}-2i=z\bar{z}+2z+i\bar{z}+2i$$

$$\Rightarrow (2+i)z+\bar{z}(i-2)+4i=0$$

$$\Rightarrow (1-2i)z+\bar{z}(1+2i)+4=0$$

Clearly, it represents a straight line.

EXAMPLE 12 *If* z_1 and z_2 *are two complex numbers representing consecutive vertices of a regular hexagon in anticlockwise direction, then (i) find the complex number* z_3 *representing the vertex adjacent to* z_2. *(ii) If* z_0 *is the circumcentre of the triangle formed by* z_1, z_2 *and* z_3, *then find* z_0 *in terms of* z_1 and z_2.

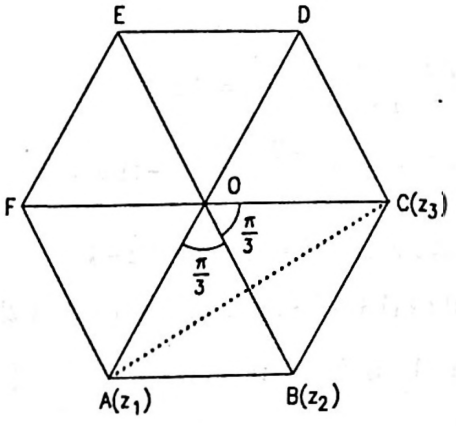


Fig. 19.73

SOLUTION (i) Let the circumcentre of the hexagon be at the origin. Then,

$$z_2=z_1e^{i\pi/3} \text{ and } z_3=z_1e^{i2\pi/3}.$$

(ii) Clearly, ΔABC is the triangle formed by z_1, z_2 and z_3. It is an isosceles triangle. The circumcentre of ΔABC is at the origin. Therefore, $z_0=0$.

Clearly, $z_0=z_2-z_1e^{\pi/3}$

EXAMPLE 13 *If* $z_1+z_2+\ldots+z_n=0$, *then prove that any straight line passing through the origin separates the points representing* $z_1, z_2, \ldots z_n$, *provided that they do not lie on this line.*

SOLUTION If possible, let there be a line passing through the origin, say $a\bar{z}+\bar{a}z=0$ such that the points representing $z_1, z_2, \ldots, z_n$ lie on one side of it. Then,

$$a\bar{z}_i+\bar{a}z_i>0 \text{ or, } a\bar{z}_i+\bar{a}z_i<0 \text{ for all } i=1,2,\ldots,n$$

$$\Rightarrow a\left(\sum_{i=1}^{n}\bar{z}_i\right)+\bar{a}\left(\sum_{i=1}^{n}z_i\right)>0 \text{ or, } a\left(\sum_{i=1}^{n}\bar{z}_i\right)+\bar{a}\left(\sum_{i=1}^{n}z_i\right)<0$$

$$\Rightarrow a\left(\sum_{i=1}^{n}z_i\right)+\bar{a}\left(\sum_{i=1}^{n}z_i\right)>0 \text{ or, } a\left(\sum_{i=1}^{n}z_i\right)+\bar{a}\left(\sum_{i=1}^{n}z_i\right)<0$$

$$\Rightarrow a\times 0+\bar{a}\times 0>0 \text{ or, } a\times 0+\bar{a}\times 0<0$$

$$\Rightarrow 0>0 \text{ or, } 0<0$$

This is a contradiction. So, our supposition is wrong.

Hence, any straight line passing through the origin separates points $z_1, z_2, \ldots, z_n$.

EXAMPLE 14 *If the points representing complex numbers* $z_1, z_2, \ldots, z_n$ *in the Argand plane lie on one side of a line drawn through the origin, then prove that the same is true for the points representing complex numbers* $\dfrac{1}{z_1}, \dfrac{1}{z_2}, \ldots \dfrac{1}{z_n}$. *Also, show that*

$$z_1+z_2+\ldots+z_n\neq 0 \text{ and } \frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\neq 0.$$

SOLUTION The equation of any line passing through the origin is

$$\bar{a}z+a\bar{z}=0 \qquad \ldots(i)$$

Suppose the points representing $z_1, z_2, \ldots, z_n$ lie in the half plane $a\bar{z}+\bar{a}z>0$. Then,

$$a\bar{z}_i+\bar{a}z_i>0, i=1,2,\ldots,n \qquad \ldots(ii)$$

$$\Rightarrow a\left(\frac{\bar{z}_iz_i}{z_i}\right)+\bar{a}\left(\frac{z_i\bar{z}_i}{\bar{z}_i}\right)>0, i=1,2,\ldots,n$$

$$\Rightarrow |z_i|^2\left(\frac{a}{z_i}+\frac{\bar{a}}{\bar{z}_i}\right)>0, i=1,2,\ldots,n$$

$$\Rightarrow \quad \frac{a}{z_i}+\frac{\bar{a}}{\bar{z}_i} > 0, i = 1, 2, ..., n \qquad [\because \ | z_1 |^2 > 0]$$

$$\Rightarrow a\left(\frac{1}{\bar{z}_i}\right)+\bar{a}\left(\frac{1}{z_i}\right) > 0 \ ; i = 1, 2, .., n \qquad ...(iii)$$

This shows that the points $\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}, ... , \frac{1}{\bar{z}_n}$ lie on one side of line $a\bar{z}+\bar{a}z = 0$.

From (i), we have

$$a\bar{z}_i+\bar{a}\,z_i > 0, i = 1, 2, ..., n$$

$$\Rightarrow \quad a\left(\sum_{i=1}^{n}\bar{z}_i\right)+\bar{a}\left(\sum_{i=1}^{n} z_i\right) > 0$$

$$\Rightarrow \quad \bar{a}\left(\sum_{i=1}^{n} z_i\right)+a\left(\overline{\sum_{i=1}^{n} z_i}\right) > 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} z_i \neq 0.$$

From (iii), we have

$$a\left(\frac{1}{z_i}\right)+\bar{a}\left(\frac{1}{\bar{z}_i}\right) > 0, i = 1, 2, \ldots n$$

$$\Rightarrow \quad a\left(\sum_{i=1}^{n}\frac{1}{z_i}\right)+\bar{a}\left(\sum_{i=1}^{n}\frac{1}{\bar{z}_i}\right) > 0$$

$$\Rightarrow \quad \sum_{i=1}^{n}\left(\frac{1}{z_i}\right) \neq 0.$$

EXAMPLE 15 *Let z_1, z_0 be two complex numbers. It is given that $| z | = 1$ and numbers $z, z_0, z\bar{z}_0, 1$ and 0 are represented in an Argand plane by the points P, P_0, Q, A and the origin respectively. Show that the triangles $PO\,P_0$ and AOQ are congruent. Hence, or otherwise, prove that $| z - z_0 | = | z\bar{z}_0 - 1 |$.*

SOLUTION We have,

$$| OP | = | z | = 1 = OA$$

$$| OQ | = | z z_0 | = | z | \, | z_0 | = | z_0 | \qquad [\because \ | z | = 1]$$

$$= | OP_0 |$$

and, $\angle AOQ = \arg\left(\frac{1-0}{z\bar{z}_0-0}\right)$

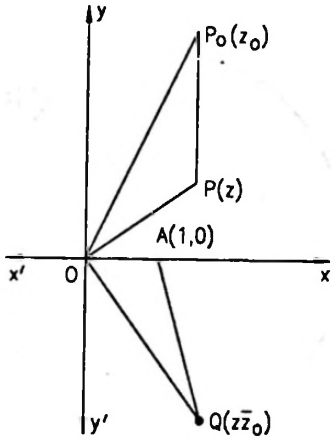


Fig. 19.74

$$= \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{| z |^2}{z\bar{z}_0}\right) \qquad [\because \ | z | = 1]$$

$$= \arg\left(\frac{z\bar{z}}{z\bar{z}_0}\right) \qquad [\because \ | z |^2 = z\bar{z}]$$

$$= \arg\left(\frac{\bar{z}}{\bar{z}_0}\right)$$

$$= \arg\left(\frac{z}{z_0}\right)$$

$$= -\arg\left(\frac{z}{z_0}\right)$$

$$= \arg\left(\frac{z_0}{z}\right)$$

$$= \arg\left(\frac{z_0-0}{z-0}\right)$$

$$= \angle PO\,P_0.$$

Thus, in triangles POP_0 and AOQ, we have

$$| OP | = | OA |, | OQ | = | OP_O | \text{ and } \angle AOQ = \angle POP_0$$

Therefore,

$$\Delta\, POP_O \cong \Delta\, AOQ$$

$$\Rightarrow \ | PP_O | = | AQ |$$

$$\Rightarrow \ | z - z_0 | = | z\bar{z}_0 - 1 |$$

—❑❑❑—

19.31 CIRCLE

The equation of a circle whose centre is at point having affix z_0 and radius R is

$$|z - z_0| = R$$

PROOF Let C be the centre of the circle such that the affix of C is z_0 and let $P(z)$ be any point on the circle. Then,

$$CP = R$$

$$\Rightarrow \ |z - z_0| = R$$

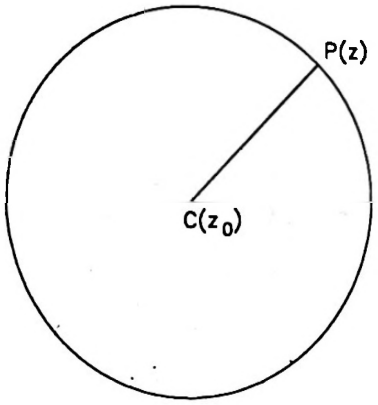


Fig. 19.75

Since P is an arbitrary point on the circle such that $|z-z_0| = R$. Hence, the equation of the circle is $|z-z_0| = R$.

REMARK 1 *If the centre of the circle is at the origin and radius R, then its equation is* $|z| = R$.

REMARK 2 $|z-z_0| < R$ *represents the interior of the circle and* $|z-z_0| \le R$ *is the set of all points lying inside and on the circle* $|z\ z_0| = R$. *Similarly,* $|z-z_0| > R$ *is the set of all points lying outside the circle and* $|z-z_0| \ge R$ *is the set of all points lying outside and on the circle* $|z-z_0| = R$.

The equation of a circle having centre at z_0 and radius R is

$$|z-z_0| = R$$

$$\Rightarrow \quad |z-z_0|^2 = R^2$$

$$\Rightarrow (z-z_0)(\bar{z}-\bar{z_0}) = R^2$$

$$\Rightarrow (z-z_0)(\bar{z}-\bar{z_0}) = R^2$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - z\bar{z_0} + z_0\bar{z_0} = R^2$$

$$\Rightarrow z\bar{z} + (-z_0)\bar{z} + (-\bar{z_0})z + |z_0|^2 - R^2 = 0$$

$$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0, \text{ where } a = -z_0 \text{ and } b = |z_0|^2 - R^2$$

$$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0, \text{ where } b \in R.$$

represents a circle having centre at $-a$ and radius $= \sqrt{|a|^2 - b}$.

ILLUSTRATION 1 *Find the radius and centre of the circle*

$$z\bar{z} + (1-i)z + (1+i)\bar{z} - 7 = 0.$$

SOLUTION The given equation can be written as

$$z\bar{z} + \overline{(1+i)}z + (1+i)\bar{z} - 7 = 0$$

So, it represents a circle with centre at $(-1-i)$ and

$$\text{Radius} = \sqrt{|1+i|^2 - (-7)} = \sqrt{2+7} = 3$$

ILLUSTRATION 2 *Determine the values of k for which the equation* $z\bar{z} + (-3+4i)\bar{z} - (3+4i)z + k = 0$ *represents a circle.*

SOLUTION We have,

$$z\bar{z} + (-3+4i)\bar{z} + (-3-4i)z + k = 0.$$

This equation represents a circle with centre at $(3-4i)$ and,

$$\text{Radius} = \sqrt{|-3+4i|^2 - k} = \sqrt{25-k}$$

For the circle to exist, we must have

$$25-k \ge 0 \Rightarrow k \le 25.$$

Hence, the given equation will represent a circle, if $k \le 25$.

ILLUSTRATION 3 *Find the centre and radius of the circles represented by the following equations:*

(i) $|z-(2+3i)| = 5$

(ii) $|z+3-2i| = 4$

(iii) $|z-4+3i| = 2$

(iv) $|z| = 5$

(v) $|2z-3+4i| = 4$

SOLUTION Recall that the equation $|z-z_0| = R$ represents a circle having centre at a point whose affix is z_0 and radius R.

(i) $|z-(2+3i)| = 5$

Clearly, this equation represents a circle having centre at $(2+3i)$ and radius 5.

(ii) We have, $|z+3-2i| = 4$

or, $|z-(-3+2i)| = 4$,

Clearly, this represents a circle of radius 4 having its centre at $(-3, 2)$.

(iii) We have, $|z-4+3i| = 2$

$$\Rightarrow \quad |z-(4-3i)| = 2$$

Clearly, it is the equation of a circle having centre at $(4,-3)$, and radius 2.

(iv) We have, $|z| = 5$

or, $|z-(0+i0)| = 5$

It is the equation of a circle of radius 5 centred at the origin.

(v) We have, $|2z-3+4i| = 4$

$$\Rightarrow \quad 2|z-(3/2-2i)| = 4$$

$$\Rightarrow \quad |z-(3/2-2i)| = 2.$$

Clearly, it represents a circle of radius 2 having centre at $(3/2, -2)$.

ILLUSTRATION 4 *If the complex numbers* z_1, z_2, z_3 *represent the vertices, of an equilateral triangle such that* $|z_1| = |z_2| = |z_3|$, *then show that* $z_1 + z_2 + z_3 = 0$.

SOLUTION Let ABC be a triangle such that the complex numbers z_1, z_2 and z_3 represent the vertices A, B and C respectively.

Now,

$$|z_1| = |z_2| = |z_3|$$

$$\Rightarrow |z_1 - 0| = |z_2 - 0| = |z_3 - 0|$$

$\Rightarrow OA = OB = OC$, where O is the origin.

$\Rightarrow$ Origin is the circumcentre of tirangle ABC.

But, the circumcentre of an equilateral triangle is same as its cenroid. So, the centroid of triangle ABC is at the origin.

$$\Rightarrow \frac{z_1+z_2+z_3}{3} = 0 \qquad \left[\because \text{Affix of the centroid} = \frac{z_1+z_2+z_3}{3}\right]$$

$$\Rightarrow z_1+z_2+z_3 = 0$$

ILLUSTRATION 5 *If z_1, z_2, z_3 are the affixes of three points A, B and C respectively and z is the affix of a point P such that $|z_1-z| = |z_2-z| = |z_3-z|$, show that A, B, C lie on a circle having centre at P.*

SOLUTION We have,

$$|z_1-z| = |z_2-z| = |z_3-z|$$

$$\Rightarrow \quad PA = PB = PC$$

$\Rightarrow$ *A, B, C* lie on a circle whose centre is *P* and radius = *PA*

ILLUSTRATION 6 *What is the region represented by the inequality $3 < |z-2-3i| < 4$ in the Argand plane?*

SOLUTION We know that $|z-2-3i| = 3$ represents a circle of radius 3 having centre at a point represented by $2+3i$. Therefore, $3 < |z-(2+3i)|$ represents the exterior of the circle $|z-(2+3i)| = 3$. Similarly $|z-(2+3i)| < 4$ represents the interior of the circle of radius 4 having centre at a point represented by $2+3i$.

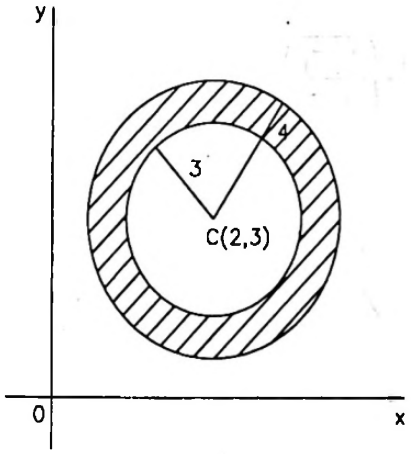


Fig. 19.76

Hence, $3 < |z-(2+3i)| < 4$ represents the circular annulus lying between the circles $|z-(2+3i)| = 3$ and $|z-(2+3i)| = 4$ as shown in Fig. 19.76.

ILLUSTRATION 7 *Locate the points representing the complex number z on the Argand diagram for which $|i-1-2z| > 9$.*

SOLUTION We have,

$$|i-1-2z| > 9$$

$$\Rightarrow \left|-2\left\{z+\frac{1}{2}(1-i)\right\}\right| > 9$$

$$\Rightarrow \quad 2\left|z-\frac{1}{2}(-1+i)\right| > 9$$

$$\Rightarrow \left|z-\frac{1}{2}(-1+i)\right| > \frac{9}{2}$$

Since $\left|z-\frac{1}{2}(-1+i)\right| = \frac{9}{2}$ represents a circle having centre at a point represented by $\frac{1}{2}(-1+i)$ and radius 9/2. Therefore $\left|z-\frac{1}{2}(-1+i)\right| > 9/2$ represents the exterior of the circle.

ILLUSTRATION 8 *Show that the equation of a circle on a line segment joining points having affixes z_1 and z_2 as diameter is*

$$|z-z_1|^2+|z-z_2|^2 = |z_1-z_2|^2$$

SOLUTION Let *AB* be a diameter of the circle such that the affixex of *A* and *B* are z_1 and z_2 respectively. Let *P* be any point on the circle and let the affix of *P* be z. Then, triangle *APB* is a right angled triangle.

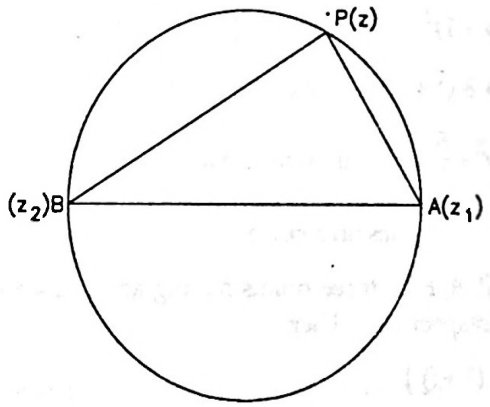


Fig. 19.77

$$\therefore \qquad AP^2 + BP^2 = AB^2$$

$$\Rightarrow |z-z_1|^2+|z-z_2|^2 = |z_1-z_2|^2$$

Since z is the affix of an arbitrary point on the circle. Therefore, the equation

$$|z-z_1|^2+|z-z_2|^2 = |z_1-z_2|^2$$

represents the required circle.

ILLUSTRATION 9 *Locate the points in the Argand plane representing the complex numbers $z = x+iy$ for which*

(i) $|z-1|+|z+1| = 3$

(ii) $|z-1|^2+|z+1|^2 = 4$

(iii) $\left|\frac{z-i}{z+i}\right| = 3$

(iv) $|z-(1+i)|-|z-(1-i)| = 5$

SOLUTION (i) Let *P, A, B* be three points having affixes $z = x+iy$, $1+i0$ and $-1+i0$ respectively. Then,

$$|z-1|+|z+1| = 3$$

$\Rightarrow \quad AP + BP = 3$

This shows that P is a variable point which moves in such a way that the sum of its distances from A and B is always constant equal to 3 units. Hence, by definition of an ellipse P lies on an ellipse having its foci at $(1, 0)$ and $(-1, 0)$.

(ii) We have, $|z-1|^2+|z+1|^2=4$

$\Rightarrow PA^2+PB^2=4$

$\Rightarrow$ P lies on a circle having $A(1, 0)$ and $B(-1, 0)$ as the end points of a diameter.

(iii) We have, $\left|\frac{z-i}{z+i}\right|=3$

$\Rightarrow \left|\frac{x+iy-i}{x+iy+i}\right|=3$

$\Rightarrow |x+i(y-1)|=3|x+i(y+1)|$

$\Rightarrow \sqrt{x^2+(y-1)^2}=3\sqrt{x^2+(y+1)^2}$

$\Rightarrow x^2+(y-1)^2=9[x^2+(y+1)^2]$

$\Rightarrow 8x^2+8y^2+20y+8=0$

$\Rightarrow x^2+y^2+\frac{5}{2}y+1=0$, which is a circle.

Hence, $z=x+iy$ lies on a circle.

(iv) Let P, A, B be three points having affixes $z=x+iy$, $1+i$ and $1-i$ respectively. Then,

$|z-(1+i)|-|z-(1-i)|=5$

$\Rightarrow \quad PA - PB = 5$

$\Rightarrow$ P lies on a hyperbola having its foci at $A(1, 1)$ and $B(1, -1)$

EQUATION OF A CIRCLE IN DIAMETER FORM

THEOREM *The equation of a circle described on a line segment joining $A(z_1)$ and $B(z_2)$ as diameter is*

$$(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$$

PROOF Let $P(z)$ be an arbitrary point on the circle. Then,

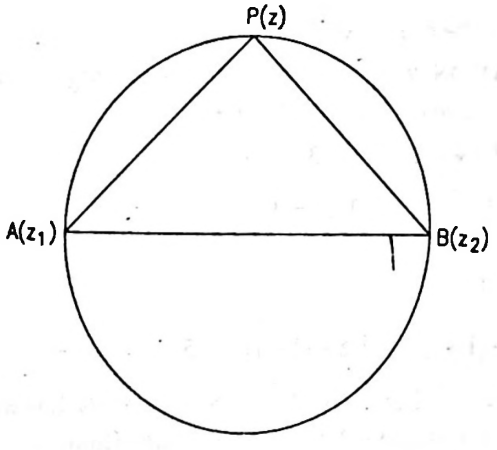


Fig. 19.78

$$\angle APB = \frac{\pi}{2}$$

$\Rightarrow \arg\left(\frac{z_1-z}{z_2-z}\right)=\pm\frac{\pi}{2}$

$\Rightarrow \frac{z_1-z}{z_2-z}$ is purely imaginary.

$\Rightarrow \frac{z_1-z}{z_2-z}+\left(\frac{z_1-z}{z_2-z}\right)=0$

$\Rightarrow \frac{(z-z_1)}{(z-z_2)}+\left(\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right)=0$

$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_2)+(\bar{z}-\bar{z}_1)(z-z_2)=0$, which is the equation of the required circle.

EQUATION OF A TANGENT TO A GIVEN CIRCLE AT A GIVEN POINT

Let $|z-z_0|=r$ be the given circle and let $A(z_1)$ be a point on the circle. Clearly,

$$\angle PAC=\frac{\pi}{2}.$$

$\therefore \arg\left(\frac{z-z_1}{z_0-z_1}\right)=\pm\frac{\pi}{2}$

$\Rightarrow \frac{z-z_1}{z_0-z_1}$ is purely imaginary

$\Rightarrow \left(\frac{z-z_1}{z_0-z_1}\right)+\left(\overline{\frac{z-z_1}{z_0-z_1}}\right)=0$

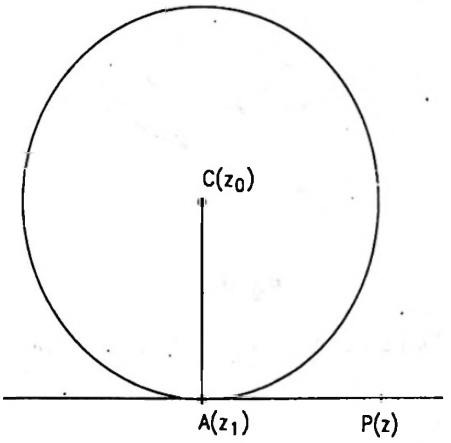


Fig. 19.79

$\Rightarrow \frac{z-z_1}{z_0-z_1}+\frac{\bar{z}-\bar{z}_1}{\bar{z}_0-\bar{z}_1}=0$

$\Rightarrow (z-z_1)(\bar{z}_0-\bar{z}_1)+(\bar{z}-\bar{z}_1)(z_0-z_1)=0$

$\Rightarrow z(\bar{z}_0-\bar{z}_1)+\bar{z}(z_0-z_1)+2(z_1\bar{z}_1)-z_1\bar{z}_0-\bar{z}_1z_0=0$

$\Rightarrow z(\bar{z}_0 - \bar{z}_1) + \bar{z}(z_0 - z_1) + 2(z_1\bar{z}_1 - \text{Re}(z_1\bar{z}_0)) = 0$

This is the equation of the tangent to $|z - z_0| = r$ at point z_1.

PARTICULAR CASE The equation of the tangent to $|z| = r$ at point $z = z_1$ is given by

$$z\bar{z}_1 + z_1\bar{z} = 2|z_1|^2$$

or, $z\bar{z}_1 + z_1\bar{z} = 2r^2$ [$\because z_1$ lies on $|z| = r \therefore |z_1| = r$]

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that all roots of the equation*

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = n,$$

where $|a_i| \le 1, i = 0, 1, 2, ..., n$ *lie outside the circle with centre at the origin and radius* $\frac{n-1}{n}$.

SOLUTION We have,

$$n = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{n-1} z + a_n$$

$$\Rightarrow |n| = |a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{n-1} z + a_n|$$

$$\Rightarrow n \le |a_0 z^n| + |a_1 z^{n-1}| + ... + |a_{n-1} z| + |a_n|$$

$$\Rightarrow n \le |a_0||z|^n + |a_1||z|^{n-1} + ... + |a_{n-1}||z| + |a_n|$$

$$\Rightarrow n \le |z|^n + |z|^{n-1} + ... + |z| + 1$$

$$\Rightarrow n \le \frac{1 - |z|^{n+1}}{1 - |z|}$$

$$\Rightarrow n < \frac{1}{1 - |z|}$$

$$\Rightarrow 1 - |z| < \frac{1}{n}$$

$$\Rightarrow |z| > 1 - \frac{1}{n}$$

$$\Rightarrow |z| > \frac{n-1}{n}$$

$\Rightarrow z$ lies outside the circle $|z - 0| = \frac{n-1}{n}$.

EXAMPLE 2 *Find the complex number z which satisfieds the condition* $|z - 2 + 2i| = 1$ *and has the least absolute value. Also, find the complex number having the greatest absolute value.*

SOLUTION We have,

$$|z - 2 + 2i| = 1$$

or, $$|z - (2 - 2i)| = 1$$

Clearly, it represents a circle with centre $C(2, -2)$ and radius 1 unit.

It is also clear that OC makes an angle of 45° with OX in clockwise sense.

It is evident from the figure that point P represents the complex number having the least absolute value and Q represents the complex number having the greatest absolute value.

We have,

$OC = 2\sqrt{2}$ and $CP = CQ = 1$.

$\therefore$ $OP = (2\sqrt{2} - 1)$ and $OQ = (2\sqrt{2} + 1)$

Since OP makes an angle of $-45°$ with OX. Therefore, P represents the complex number z_1 given by

$$z_1 = (2\sqrt{2} - 1)e^{-i\pi/4}$$

$$= (2\sqrt{2} - 1)(\cos \pi/4 - i \sin \pi/4)$$

$$= \left(\frac{2\sqrt{2} - 1}{\sqrt{2}}\right)(1 - i)$$

$$= \left(2 - \frac{1}{\sqrt{2}}\right) - \left(2 - \frac{1}{\sqrt{2}}\right)i$$

and,

Q represent the complex number z_2 given by

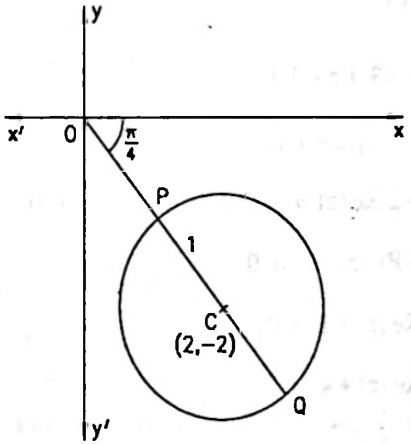


Fig. 19.80

$$z_2 = (2\sqrt{2} + 1)e^{-i\pi/4}$$

$$= (2\sqrt{2} + 1)(\cos \pi/4 - i \sin \pi/4)$$

$$= (2\sqrt{2} + 1)\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \left(2 + \frac{1}{\sqrt{2}}\right) - i\left(2 + \frac{1}{\sqrt{2}}\right)$$

Hence, the complex numbers having the least and the greatest absolute values are

$$z_1 = \left(2-\frac{1}{\sqrt{2}}\right)-\left(2-\frac{1}{\sqrt{2}}\right)i$$

and,

$$z_2 = \left(2+\frac{1}{\sqrt{2}}\right)-\left(2+\frac{1}{\sqrt{2}}\right)i \quad \text{respectively.}$$

EXAMPLE 3 *If $z=(2+t)+i\sqrt{3-t^2}$, where t is real and $t^2<3$, show that the modulus of $\frac{z+1}{z-1}$ is independent of t. Also, show that the locus of the points z for different values of t is a circle and find its centre and radius.*

SOLUTION We have,

$$z = (2+t)+i\sqrt{3-t^2}$$

$$\Rightarrow \frac{z}{1} = (2+t)+i\sqrt{3-t^2}$$

$$\Rightarrow \frac{z+1}{z-1} = \frac{(3+t)+i\sqrt{3-t^2}}{(1+t)+i\sqrt{3-t^2}}$$

$$\Rightarrow \left|\frac{z+1}{z-1}\right|^2 = \left|\frac{(3+t)+i\sqrt{(3-t^2)}}{(1+t)+i\sqrt{3-t^2}}\right|^2$$

$$\Rightarrow \left|\frac{z+1}{z-1}\right|^2 = \frac{|(3+t)+i\sqrt{3-t^2}|^2}{|(1+t)+i\sqrt{3-t^2}|^2}$$

$$\Rightarrow \left|\frac{z+1}{z-1}\right|^2 = \frac{(3+t)^2+(3-t^2)}{(1+t)^2+(3-t^2)} = \frac{12+6t}{4+2t} = 3$$

$$\Rightarrow \left|\frac{z+1}{z-1}\right| = \sqrt{3}$$

$$\Rightarrow |z+1| = \sqrt{3}\,|z-1|$$

$$\Rightarrow |z+1|^2 = 3\,|z-1|^2$$

$$\Rightarrow |z|^2+1+2\,\mathrm{Re}(z) = 3\{|z|^2-2\,\mathrm{Re}(z)+1\}$$

$$\Rightarrow 2|z|^2-8\,\mathrm{Re}(z)+2 = 0$$

$$\Rightarrow |z|^2-4\,\mathrm{Re}(z)+1 = 0$$

$$\Rightarrow |z|^2-4\,\mathrm{Re}(z)+4 = 3$$

$$\Rightarrow |z-2|^2 = 3$$

$$\Rightarrow |z-2| = \sqrt{3}$$

$\Rightarrow$ Locus of z is a circle with centre at (2, 0) and radius $\sqrt{3}$.

EXAMPLE 4 *Let $z=t^2-1+\sqrt{t^4-t^2}$, where $t\in R$ is a parameter. Find the locus of 'z' independent of t, and draw the locus of 'z' in the Argand plane.*

SOLUTION We have,

$$z = (t^2-1)+\sqrt{t^2(t^2-1)}$$

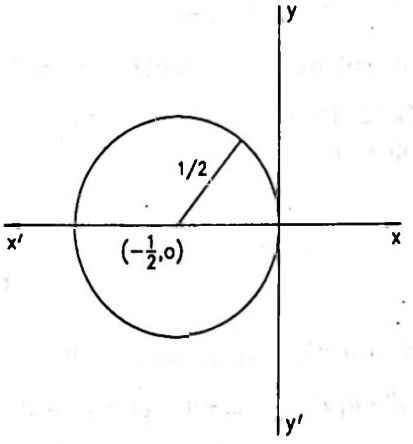


Fig. 19.81

$$\Rightarrow z = (t^2-1)+|t|\sqrt{t^2-1}$$

Now, two cases arise.

CASE I *When $t^2<1$ i.e. $-1<t<1$.*

In this case,

$$z = (t^2-1)+|t|\sqrt{t^2-1}$$

$$\Rightarrow z = (t^2-1)+i\,|t|\sqrt{1-t^2}$$

$$\Rightarrow z = t^2-1+i\sqrt{t^2-t^4}$$

$$\Rightarrow z = t^2-1+i\sqrt{-\left(t^4-t^2+\frac{1}{4}\right)+\frac{1}{4}}$$

$$\Rightarrow z = t^2-1+i\sqrt{\frac{1}{4}-\left(t^2-\frac{1}{2}\right)^2}$$

$$\Rightarrow z+\frac{1}{2} = \left(t^2-\frac{1}{2}\right)+i\sqrt{\frac{1}{4}-\left(t^2-\frac{1}{2}\right)^2}$$

$$\Rightarrow \left|z+\frac{1}{2}\right|^2 = \left(t^2-\frac{1}{2}\right)^2+\frac{1}{4}-\left(t^2-\frac{1}{2}\right)^2$$

$$\Rightarrow \left|z+\frac{1}{2}\right| = \frac{1}{2}$$

$$\Rightarrow \left|z-\left(-\frac{1}{2}+0i\right)\right| = \frac{1}{2}$$

$\Rightarrow$ z lies on a circle with centre at $-\frac{1}{2}+0i$ and radius $\frac{1}{2}$.

CASE II *When $t^2\geq 1$ i.e. $t\geq 1$ or $t\leq -1$*

In this case,

$$z = (t^2-1)+\sqrt{t^4-t^2}$$

Clearly, z is purely real and is an even function of t.

Also, $$\frac{dz}{dt} = 2t+\frac{4t^3-2t}{2\sqrt{t^4-t^2}}$$

$$= t\left|2+\frac{2t^2-1}{\sqrt{t^2(t^2-1)}}\right| > 0 \quad \text{for } t\in(1,\infty)$$

Therefore, z increases with t. At $t = 1$, we have $z = 0$ and $z \to \infty$ as $t \to \infty$.

Thus, z takes all values from 0 to ∞ for $t \in (1, \infty)$. As z is an even function of t. Therefore, z takes all real values from $-\infty$ to ∞. Hence, the locus of z is the real axis in the Argand plane.

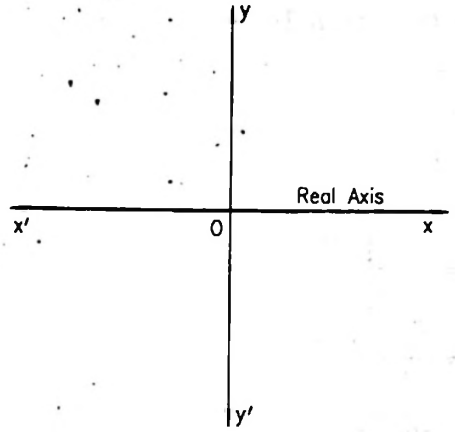


Fig. 19.82

EXAMPLE 5 *If* $\log_{(1/\sqrt{3})}\left(\frac{|z|^2-|z|+1}{2+|z|}\right) > -2$, *then show that z lies in the interior of a circle with centre at the origin and radius 5 units.*

SOLUTION We have,

$$\log_{(1/\sqrt{3})}\left(\frac{|z|^2-|z|+1}{2+|z|}\right) > -2$$

$$\Rightarrow \frac{|z|^2-|z|+1}{2+|z|} < \left(\frac{1}{\sqrt{3}}\right)^{-2}$$

$$\Rightarrow \frac{|z|^2-|z|+1}{|z|+2} < 3$$

$$\Rightarrow |z|^2-|z|+1 < 3(|z|+2)$$

$$\Rightarrow |z|^2-4|z|-5<0$$

$$\Rightarrow (|z|-5)(|z|+1)<0$$

$$\Rightarrow |z|-5<0 \qquad [\because |z|+1>0]$$

$$\Rightarrow |z| < 5$$

Hence, z lies in the interior of the circle $|z-0| = 5$.

EXAMPLE 6 *Show that the equation of a circle passing through the origin and having intercepts a and b on real and imaginary axes respectively on the Argand plane is given by*

$$z\bar{z} = a\,\mathrm{Re}(z) + b\,\mathrm{Im}(z).$$

SOLUTION Let the equation of the required circle be

$$z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0 \qquad ...(i)$$

It passes through the origin. So, $z = 0$ must satisfy (i).

$\therefore \beta = 0$

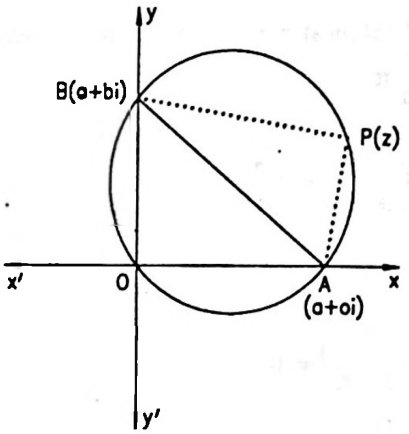


Fig. 19.83

The circle also passes through $z = a + 0i$ and $z = 0 + bi$. Therefore, putting $z = a$ and $z = bi$ respectively in (i), we have

$$a^2 + \alpha a + \bar{\alpha}a + \beta = 0 \qquad ...(ii)$$

and,

$$(bi)(\overline{bi}) + \alpha(\overline{bi}) + \alpha(bi) + \beta = 0 \qquad ...(iii)$$

Putting $\beta = 0$ in (ii), we obtain

$$a^2 + \alpha a + \bar{\alpha}a = 0$$

$$\Rightarrow a = -(\alpha + \bar{\alpha})$$

$$\Rightarrow \alpha + \bar{\alpha} = -a \qquad ...(iv)$$

Putting $\beta = 0$ in (iii), we obtain

$$(bi)(\overline{bi}) + \alpha(\overline{bi}) + \bar{\alpha}(bi) + 0 = 0$$

$$\Rightarrow (bi)(-bi) + \alpha(-bi) + \alpha(bi) = 0$$

$$\Rightarrow b^2 - \alpha bi + \bar{\alpha}bi = 0$$

$$\Rightarrow b - (\alpha - \bar{\alpha})i = 0$$

$$\Rightarrow b = (\alpha - \bar{\alpha})i$$

$$\Rightarrow \alpha - \bar{\alpha} = -bi \qquad ...(v)$$

From (iv) and (v), we get

$$\alpha = -\left(\frac{a+bi}{2}\right) \text{ and } \bar{\alpha} = -\left(\frac{a-bi}{2}\right)$$

Putting the values of α and β in (i), we obtain the equation of the required circle as

$$z\bar{z} - \left(\frac{a+bi}{2}\right)\bar{z} - \left(\frac{a-bi}{2}\right)z = 0$$

$$\Rightarrow z\bar{z} - \frac{a}{2}(\bar{z}+z) - \frac{bi}{2}(\bar{z}-z) = 0$$

$$\Rightarrow z\bar{z} - \frac{a}{2}(2\,\mathrm{Re}(z)) - \frac{bi}{2}(-2i\,\mathrm{Im}(z)) = 0$$

$$\Rightarrow z\bar{z} - a\,\mathrm{Re}(z) - b\,\mathrm{Im}(z) = 0$$

$z\bar{z} = a\,\text{Re}(z) + b\,\text{Im}(z)$.

ALITER Let $P(z)$ be any point on the circle. Then,

$$\angle APB = \pm\frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{z-(a+0i)}{z-(0+bi)}\right) = \pm\frac{\pi}{2}$$

$$\Rightarrow \frac{z-a}{z-bi} \text{ is purely imaginary}$$

$$\Rightarrow \left(\frac{z-a}{z-bi}\right)+\left(\overline{\frac{z-a}{z-bi}}\right) = 0$$

$$\Rightarrow \frac{z-a}{z-bi}+\frac{\bar{z}-a}{\bar{z}+bi} = 0$$

$$\Rightarrow (z-a)(\bar{z}+bi)+(\bar{z}-a)(z-bi) = 0$$

$$\Rightarrow 2z\bar{z}-(a-bi)z-(a+bi)\bar{z}-abi+abi = 0$$

$$\Rightarrow 2z\bar{z}-a(z+\bar{z})+bi(z-\bar{z}) = 0$$

$$\Rightarrow 2z\bar{z}-a(2\,\text{Re}(z))+bi(2i\,\text{Im}(z)) = 0$$

$$\Rightarrow 2z\bar{z}-2a\,\text{Re}(z)-2b\,\text{Im}(z) = 0$$

$$\Rightarrow z\bar{z} = a\,\text{Re}(z)+b\,\text{Im}(z).$$

EXAMPLE 7 *If* $\alpha=\dfrac{z-i}{z+i}$, *show that, when z lies above the real axis,* α *will lie within the unit circle which has centre at the origin. Find the locus of* α *as z travels on the real axis from* $-\infty$ *to* ∞.

SOLUTION CASE I *When z lies above the real axis :*

If z lies above the real axis, we have

$$PB < PA$$

$$\Rightarrow |z-i| < |z+i|$$

$$\Rightarrow \frac{|z-i|}{|z+i|} < 1$$

$$\Rightarrow \left|\frac{z-i}{z+i}\right| < 1$$

$$\Rightarrow |\alpha| < 1 \qquad \left[\because \left|\frac{z-i}{z+i}\right| < 1\right]$$

$\Rightarrow$ α lies within the unit circle which has its centre at the origin.

CASE II *When z travels on real axis from* $-\infty$ *to* ∞ *:*

If z travels on real axis from $-\infty$ to ∞, then $\text{Im}(z) = 0$ and $\text{Re}(z)$ varies from $-\infty$ to ∞.

Let $z = t+0\,i, t \in R$. Then,

$$\alpha = \frac{z-i}{z+i}$$

$$\Rightarrow \alpha = \frac{t-i}{t+i}$$

$$\Rightarrow |\alpha| = \left|\frac{t-i}{t+i}\right|$$

$$\Rightarrow |\alpha| = \frac{|t-i|}{|t+i|}$$

$$\Rightarrow |\alpha| = \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}} = 1$$

$\Rightarrow$ α moves on the unit circle which has its centre at the origin.

EXAMPLE 8 *Locate the region in the Argand plane determined by* $z^2+(\bar{z})^2+2|z|^2 < 8i(\bar{z}-z)$, *where* $z \in C$.

SOLUTION Let $z = x + iy$. Then,

$$z^2+(\bar{z})^2+2|z|^2 < 8i(\bar{z}-z)$$

$$\Rightarrow (x^2-y^2+2ixy)+(x^2-y^2-2ixy)+2(x^2+y^2) < 8i(-2iy)$$

$$\Rightarrow 4x^2 < 16y$$

$$\Rightarrow x^2 < 4y$$

$\Rightarrow$ (x, y) lies inside the parabola having vertex at the origin and axis along OY.

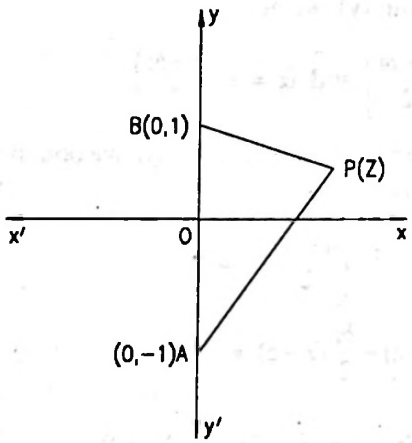


Fig. 19.84

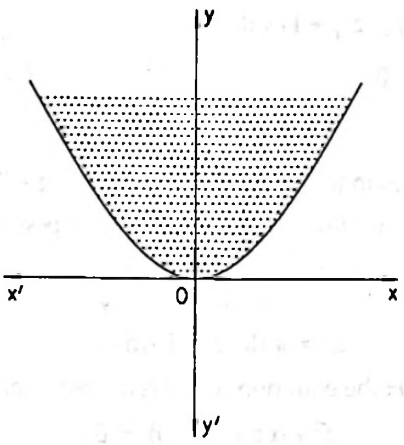


Fig. 19.85

Hence, the given inequality represents the part of the Argand plane lying inside the parabola with vertex at the origin, axis along positive side of imaginary axis and latus rectum 4 units in length.

EXAMPLE 9 *If z_1, z_2, z_3, z_4 are distinct complex numbers lying on the circle $|z| = r$, then shows that $\dfrac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is a real number.*

SOLUTION Let $\angle BAC = \theta$. Then, $\angle DCB = \pi - \theta$

$\therefore \quad \angle BAC + \angle DCB = \pi$

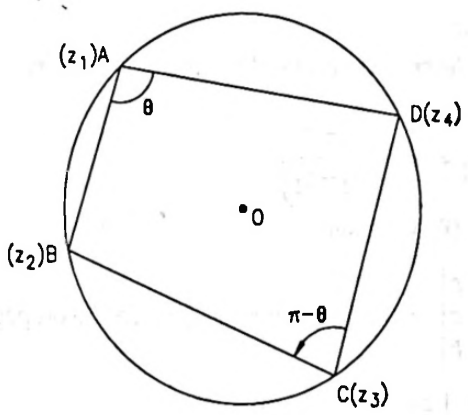


Fig. 19.86

$$\Rightarrow \arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \arg\left(\frac{z_4 - z_1}{z_2 - z_1} \cdot \frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \frac{z_4 - z_1}{z_2 - z_1} \cdot \frac{z_2 - z_3}{z_4 - z_3} \text{ is purely real}$$

$$\Rightarrow \frac{z_2 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_3}{z_2 - z_3} \text{ is purely real}$$

$$\Rightarrow \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)} \text{ is purely real.}$$

EXAMPLE 10 *show that if $z_1 z_2 + z_3 z_4 = 0$ and $z_1 + z_2 = 0$, the points represented by the complex numbers $z_1 z_2, z_3, z_4$ taken in order are ancyclic.*

SOLUTION For points z_1, z_2, z_3 and z_4 to be concyclic, $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ should be real.

We have,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$$= \frac{(z_1 - z_2)(z_3 - z_4)}{z_1 z_2 + z_3 z_4 - z_1 z_3 - z_2 z_4}$$

$$= \frac{(z_1 - z_2)(z_3 - z_4)}{z_2(z_3 - z_4)} \qquad \left[\because z_1 z_2 + z_3 z_4 = 0 \text{ and } z_1 = -z_2\right]$$

$$= \frac{z_1 - z_2}{z_2}$$

$$= \frac{-2z_2}{z_2} = -2, \text{ which is real.}$$

Hence, the given points are concyclic.

EXAMPLE 11 *Find the locus of points z such that $\arg\left(\dfrac{z - i}{z + 1}\right) = \theta$, when (i) $\theta = 0$ (ii) $\theta = \pi$ (iii) $\theta = \alpha$, where $0 < \alpha < \dfrac{\pi}{2}$.*

SOLUTION We have,

$$\arg\left(\frac{z - i}{z + 1}\right) = \theta$$

$$\Rightarrow \quad \angle APB = \theta$$

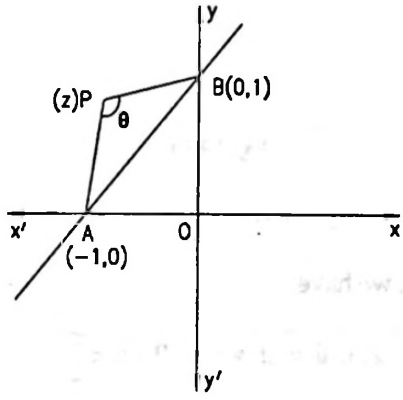


Fig. 19.87

(i) *When* $\theta = 0$:

In this case, we have

$\angle APB = 0$

$\Rightarrow$ *P* lies either above *B* or it lies below *A*

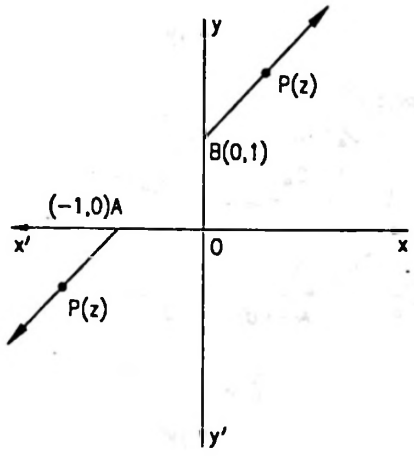


Fig. 19.88

Thus, the locus of P is two rays, one beginning from A and going downwards and other beginning from B and going upwords as shown in Fig. 19.88

(ii) *When* $\theta = \pi$:

In this case, we have

$\angle APB = \pi$

$\Rightarrow$ P lies on the segment AB, excluding its end points A and B.

Thus, the locus of P is the line segment joining A and B excluding end points A and B.

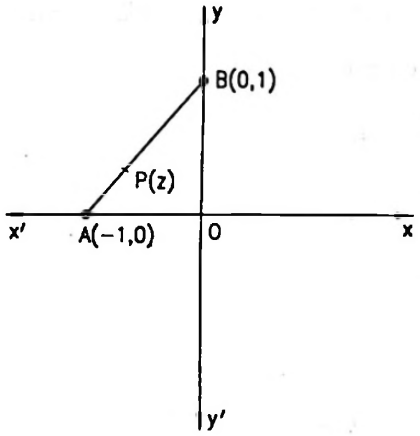


Fig. 19.89

(iii) *When* $\theta = \alpha$, *where* $0 < \alpha < \frac{\pi}{2}$:

In this case, we have

$$\angle APB = \alpha, \text{ where } 0 < \alpha < \frac{\pi}{2}$$

$\Rightarrow$ P moves on the major arc of the circle passing through A and B and lying in the upper half plane formed by the line passing through A and B.

Note that P will not lie on the major arc of the circle lying in the lower half plane determined by the line passing through A and B, because for any point $P(z)$ on the major arc of this circle we have

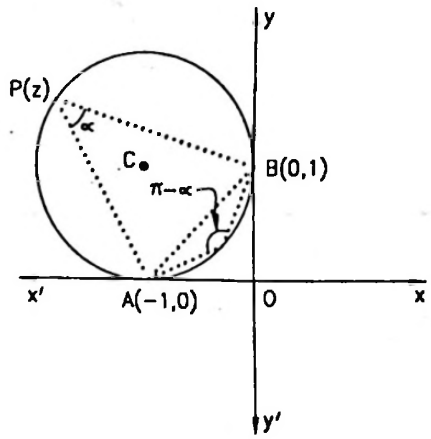


Fig. 19.90

$$\arg\left(\frac{z+1}{z-i}\right) = \alpha$$

and,

$$\arg\left(\frac{z-i}{z+1}\right) = -\alpha.$$

EXAMPLE 12 *Let* z_1, z_2, z_3 *be three complex numbers such that* $|z_1| = a, |z_2| = b, |z_3| = c$ *and* $z_1 \neq z_2$. *If*

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$$

then prove that :

(i) z_1, z_2, z_3 *lie on a circle with the centre at the origin.*

(ii) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$

SOLUTION (i) We have,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \Rightarrow a = b = c \quad \text{[See Ex. 16 on page 19.37]}$$

$\Rightarrow$ $|z_1| = |z_2| = |z_3| = \lambda$ (say)

$\Rightarrow$ z_1, z_2, z_3 lie on the circle $|z| = \lambda$ having centre at the origin.

(ii) Clearly,

$$\angle BOC = 2\angle BAC$$

$$\Rightarrow \arg\left(\frac{z_3 - 0}{z_2 - 0}\right) = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

$$\Rightarrow \arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 \qquad [\because \arg(z^n) = n \arg(z)]$$

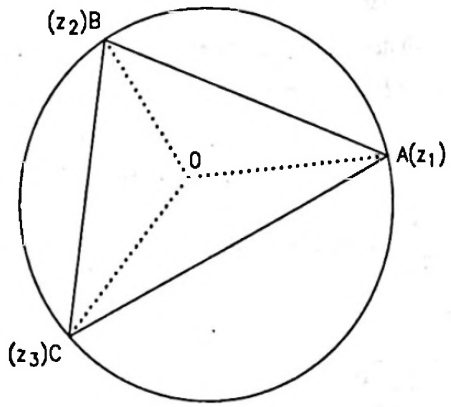


Fig. 19.91

EXAMPLE 13 *If* z_1 and z_2 *are two given complex numbers and* z *is a variable complex number such that*

$$|z-z_1|^2+|z-z_2|^2=k,$$

find the set of values of k for which this equation represents a circle.

SOLUTION We have,

$$|z-z_1|^2+|z-z_2|^2=k$$

$$\Rightarrow |z|^2+|z_1|^2-2\,\text{Re}\,(z\bar{z}_1)+|z|^2+|z_2|^2-2\,\text{Re}\,(z\bar{z}_2)=k$$

$$\Rightarrow |z|^2-2\,\text{Re}\left\{\frac{z(\bar{z}_1+\bar{z}_2)}{2}\right\}=\frac{1}{2}[k-|z_1|^2-|z_2|^2]$$

$$\Rightarrow |z|^2-2\,\text{Re}\left\{\frac{z(\bar{z}_1+\bar{z}_2)}{2}\right\}+\left|\frac{z_1+z_2}{2}\right|^2$$

$$=\frac{1}{2}\left\{k-|z_1|^2-|z_2|^2\right\}+\left|\frac{z_1+z_2}{2}\right|^2$$

$$\Rightarrow \left|z-\frac{z_1+z_2}{2}\right|^2=\frac{1}{4}[2k-2|z_1|^2-2|z_2|^2+|z_1|^2$$

$$+|z_2|^2|+2\,\text{Re}\,(z_1\bar{z}_2)]$$

$$\Rightarrow \left|z-\frac{z_1+z_2}{2}\right|^2=\frac{1}{4}[2k-|z_1-z_2|^2]$$

$$\Rightarrow \left|z-\frac{z_1+z_2}{2}\right|=\frac{1}{2}\sqrt{2k-|z_1-z_2|^2}$$

Clearly, it represents a circle having centre at $\frac{z_1+z_2}{2}$ and

$$\text{Radius}=\frac{1}{2}\sqrt{2k-|z_1-z_2|^2}.$$

For the circle to exist, we must have

$$2k-|z_1-z_2|^2\geq 0 \Rightarrow k\geq\frac{1}{2}|z_1-z_2|^2$$

EXAMPLE 14 *Find the locus of* $\omega=\frac{1}{4+i-z}$, *when z lies on the circle* $|z+1|=3$.

SOLUTION We have,

$$\omega=\frac{1}{4+i-z}$$

$$\Rightarrow 4+i-z=\frac{1}{\omega}$$

$$\Rightarrow z=4+i-\frac{1}{\omega}$$

$$\Rightarrow z+1=5+i-\frac{1}{\omega}$$

$$\Rightarrow |z+1|=\left|5+i-\frac{1}{\omega}\right|$$

$$\Rightarrow 3=\left|\frac{(5+i)\omega-1}{\omega}\right| \qquad [\because |z+1|=3]$$

$$\Rightarrow \frac{|(5+i)\omega-1|}{|\omega|}=3$$

$$\Rightarrow |(5+i)\omega-1|=3|\omega|$$

$$\Rightarrow |(5+i)\omega-1|^2=9|\omega|^2$$

$$\Rightarrow \{(5+i)\omega-1\}\{\overline{(5+i)\omega-1}\}=9|\omega|^2$$

$$\Rightarrow ((5+i)\omega-1)((5-i)\bar{\omega}-1)=9|\omega|^2$$

$$\Rightarrow 26\,\omega\bar{\omega}+1-(5+i)\omega-(5-i)\bar{\omega}=9|\omega|^2$$

$$\Rightarrow 26|\omega|^2-(5+i)\omega-(5-i)\bar{\omega}+1=9|\omega|^2$$

$$\Rightarrow \omega\bar{\omega}-\left(\frac{5}{17}+\frac{1}{17}i\right)\omega-\left(\frac{5}{17}-\frac{1}{17}i\right)\bar{\omega}+\frac{1}{17}=0$$

$$\Rightarrow \omega\bar{\omega}+\left(-\frac{5}{17}-\frac{1}{17}i\right)\omega+\left(-\frac{5}{17}+\frac{1}{17}i\right)\bar{\omega}+\frac{1}{17}=0$$

This shows that ω lies on a circle having its centre at

(– Coefficient of $\bar{\omega}$) i.e. at $\frac{5}{17}-\frac{1}{17}i$

and,

$$\text{Radius}=\sqrt{\left|\frac{-5}{17}+\frac{1}{17}i\right|^2-\frac{1}{17}}=\frac{3}{17}$$

EXAMPLE 15 *Find all complex numbers common to* $|z|=2\sqrt{2}$ *and* $\arg\left\{\frac{z-(5+i)}{z-(1+i)}\right\}=\frac{\pi}{4}$.

SOLUTION Clearly, $|z|=2\sqrt{2}$ represents a circle with centre at (0, 0) and radius $2\sqrt{2}$ and $\arg\left\{\frac{z-(5+i)}{z-(1+i)}\right\}=\frac{\pi}{4}$ is the major segment of the circle described on the join of (5, 1) and (1, 1) as a chord.

We observe that there is only one common point of the two circles.

Now,

$$|z|=2\sqrt{2}$$

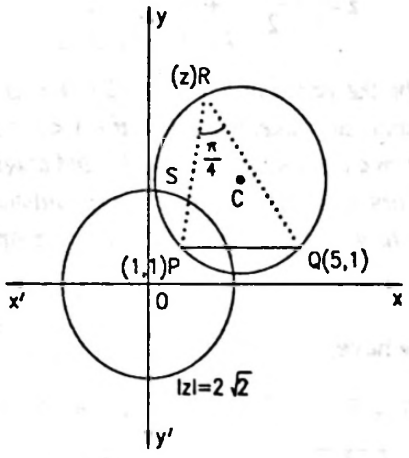


Fig. 19.92

$\Rightarrow |z|^2 = 8$

$\Rightarrow x^2+y^2 = 8$ ...(i)

and,

$$\arg\left\{\frac{z-(5+i)}{z-(1+i)}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \arg\left\{\frac{(x-5)+i(y-1)}{(x-1)+i(y-1)}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \arg\{(x-5)+i(y-1)\} - \arg\{(x-1)+i(y-1)\} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y-1}{x-5}\right) - \tan^{-1}\left(\frac{y-1}{x-1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{y-1}{x-5}-\frac{y-1}{x-1}}{1+\frac{y-1}{x-5}\times\frac{y-1}{x-1}}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \frac{4(y-1)}{(x-5)(x-1)+(y-1)^2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4(y-1)}{x^2+y^2-6x-2y+6} = 1$$

$\Rightarrow x^2+y^2-6x-6y+10 = 0$ ...(ii)

From (i) and (ii), we get

$-6x-6y+18 = 0 \Rightarrow x+y-3 = 0 \Rightarrow y = 3-x$

Putting $y = 3-x$ in (i), we get

$$(3-x)^2+x^2 = 8 \Rightarrow 2x^2-6x+1 = 0 \Rightarrow x = \frac{3\pm\sqrt{7}}{2}$$

Putting $x = \frac{3\pm\sqrt{7}}{2}$ in $x+y-3=0$, we get $y = \frac{3\mp\sqrt{7}}{2}$

Clearly, $\left(\frac{3-\sqrt{7}}{2}, \frac{3+\sqrt{7}}{2}\right)$ is the only common point. Hence, the common point is represented by the complex number

$$z = \left(\frac{3-\sqrt{7}}{2}\right) + i\left(\frac{3+\sqrt{7}}{2}\right)$$

EXAMPLE 16 *In the equation $z^2+2\lambda z+1=0$, λ is a parameter which can take any real value. Show that if $-1<\lambda<1$ the roots of this equation lie on a certain circle in the Argand diagram, but that if $\lambda>1$, one root lies inside the unit circle and one outside. Also, prove that for every large values of λ the roots are approximately $-2\lambda, -\frac{1}{2\lambda}$.*

SOLUTION We have,

$z^2+2\lambda z+1 = 0$

$$\Rightarrow z = \frac{-2\lambda\pm\sqrt{4\lambda^2-4}}{2}$$

$\Rightarrow z = -\lambda\pm\sqrt{\lambda^2-1}$ ...(i)

CASE I *When* $-1<\lambda<1$.

If $-1<\lambda<1$, we have

$\lambda^2 < 1$

$\Rightarrow \lambda^2-1 < 0$

$\Rightarrow \lambda^2-1 = -\mu^2$ (say), where $\mu>0$.

$\Rightarrow \sqrt{\lambda^2-1} = \pm i\mu$

$\therefore z = -\lambda\pm i\mu$ [Putting $\sqrt{\lambda^2-1} = \pm i\mu$ in (i)]

$\Rightarrow z+\lambda = \pm i\mu$

$\Rightarrow |z+\lambda| = |\pm i\mu|$

$\Rightarrow |z+\lambda| = \mu$

$\Rightarrow$ z lies on the circle having centre at $-\lambda+0i$ and radius μ.

CASE II *When* $\lambda>1$

If $\lambda>1$, then

$\lambda^2 > 1 \Rightarrow \lambda^2-1 > 0$

Let $\lambda^2-1 = \mu^2$, where $\mu \in R$ ...(ii)

$\Rightarrow \sqrt{\lambda^2-1} = \pm\mu$

$\therefore z = -\lambda\pm\mu$ [Putting $\sqrt{\lambda^2-1} = \pm\mu$ in (i)]

Let $z_1 = -\lambda+\mu$ and $z_2 = -\lambda-\mu$. Then,

$z_1z_2 = \lambda^2-\mu^2$

$\Rightarrow z_1z_2 = 1$ [Using (ii)]

$\Rightarrow |z_1z_2| = 1$

$\Rightarrow |z_1||z_2| = 1$

$\Rightarrow$ either $|z_1|<1$ and $|z_2|>1$ or $|z_1|>1$ and $|z_2|<1$.

$\Rightarrow$ one root lies inside $|z|<1$ and the other outside $|z|<1$.

CASE III *When λ is very large* :

In this case, roots of the given equation are $z_1 = -\lambda+\mu$ and $z_2 = -\lambda-\mu$. Also, z_1 and z_2 are real numbers.

We have,

$\lambda^2-1 = \mu^2$ [From (ii)]

$\therefore \sqrt{\lambda^2-1} = \mu$

Since λ is large. Therefore,

$\mu = \sqrt{\lambda^2-1}$

$\Rightarrow$ μ is also large and $\mu \simeq \lambda$

Hence, the roots are :

$$z_1 = -\lambda + \mu = \frac{\mu^2 - \lambda^2}{\mu + \lambda} = \frac{-1}{2\lambda}$$

and,

$$z_2 = -\lambda - \mu = -\lambda - \lambda = -2\lambda$$

EXAMPLE 17 *Find the real values of parameter 'a' for which at least one complex number z satisfies both the equality* $|z + \sqrt{2}| = a^2 - 3a + 2$ *and the inequality* $|z + i\sqrt{2}| < a^2$.

SOLUTION The equation $|z + \sqrt{2}| = a^2 - 3a + 2$ will represent a circle with centre at $(-\sqrt{2}, 0)$ and radius $(a^2 - 3a + 2)$, if

$$a^2 - 3a + 2 > 0 \Rightarrow (a-2)(a-1) > 0 \Rightarrow a < 1 \text{ or } a > 2$$

The equation $|z + i\sqrt{2}| < a^2$ represents the interior of the circle having centre at $(0, -\sqrt{2})$ and radius a.

We have to find the values of parameter a for which at least one complex number z satisfies both the equality $|z + \sqrt{2}| = a^2 - 3a + 2$ and the inequality $|z + i\sqrt{2}| < a^2$. This is possible only when the circle $|z + \sqrt{2}| = a^2 - 3a + 2$ intersects with the circle $|z + i\sqrt{2}| = a^2$. For this, we must have

Difference of the radii < Distance between the centres of the two circles < Sum of the radii

$$\Rightarrow a^2 - 3a + 2 - a^2 < 2 < a^2 + a^2 - 3a + 2$$

$$\Rightarrow -3a + 2 < 2 < 2a^2 - 3a + 2$$

$$\Rightarrow -3a + 2 < 2 \text{ and } 2 < 2a^2 - 3a + 2$$

$$\Rightarrow -3a < 0 \text{ and } 2a^2 - 3a > 0$$

$$\Rightarrow a > 0 \text{ and } a(2a - 3) > 0.$$

But, $a < 1$ or $a > 2$. Therefore, we must have $a > 2$.

EXAMPLE 18 *If* $z_1 = 7 + 8i$ *and* $z_2 = 2 + 3i$ *and z be any complex number such that* $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{8}$, *find the locus traced by z in the Argand plane.*

SOLUTION Let A and B be the points representing $z_1 = 7 + 8i$ and $z_2 = 2 + 3i$ in the Argand plane. Let $P(z)$ be any point in the Argand plane such that

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{8}$$

This implies that point P lies on the major segment of the circle drawn on AB and containing the angle $\frac{\pi}{8}$.

Let C be the centre of the circle and $CM \perp AB$. Clearly, M is the mid-point of AB. So, the coordinates of M are $\left(\frac{9}{2}, \frac{11}{2}\right)$.

$$\therefore AM = \sqrt{\left(7 - \frac{9}{2}\right)^2 + \left(8 - \frac{11}{2}\right)^2} = \frac{5}{\sqrt{2}}$$

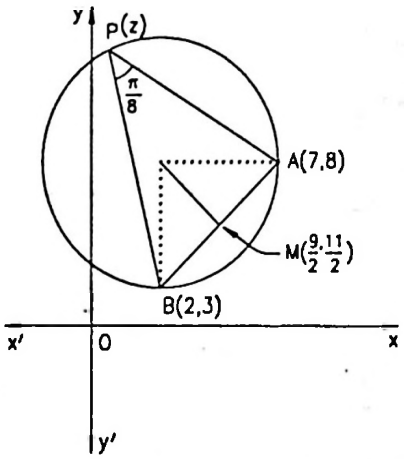


Fig. 19.93

We have,

$$\angle ACB = 2\angle APB = \frac{\pi}{4}$$

$$\Rightarrow \angle ACM = \frac{1}{2}\angle ACB = \frac{\pi}{8}$$

In ΔACM, we have

$$\tan \angle ACM = \frac{AM}{CM}$$

$$\Rightarrow \tan\frac{\pi}{8} = \frac{5}{\sqrt{2}\, CM}$$

$$\Rightarrow CM = \frac{5}{\sqrt{2}(\sqrt{2} - 1)} \qquad [\because \tan\frac{\pi}{8} = \sqrt{2} - 1]$$

$$\Rightarrow CM = \frac{5}{\sqrt{2}}(\sqrt{2} + 1)$$

Now,

$$AC^2 = AM^2 + CM^2$$

$$\Rightarrow AC = \sqrt{\left(\frac{5}{\sqrt{2}}\right)^2 + \left\{\frac{5}{\sqrt{2}}(\sqrt{2} + 1)\right\}^2}$$

$$\Rightarrow AC = \sqrt{\frac{25}{2} + \frac{25}{2}(3 + 2\sqrt{2})}$$

$$\Rightarrow AC = 5\sqrt{2 + \sqrt{2}}$$

Now,

$$\text{Slope of } AB = \frac{8 - 3}{7 - 2} = 1.$$

$$\therefore \text{ Slope of } CM = -1$$

So, the equation of CM is

$$\frac{x-\frac{9}{2}}{\cos\frac{3\pi}{4}}=\frac{y-\frac{11}{2}}{\sin\frac{3\pi}{4}}$$

Since $CM=\frac{5}{\sqrt{2}}(\sqrt{2}+1)$. So, the coordinates of C are given by

$$\frac{x-\frac{9}{2}}{-\frac{1}{\sqrt{2}}}=\frac{y-\frac{11}{2}}{\frac{1}{\sqrt{2}}}=\frac{5}{\sqrt{2}}(\sqrt{2}+1)$$

$$\Rightarrow x=\frac{9}{2}-\frac{5}{2}(\sqrt{2}+1) \text{ and } y=\frac{11}{2}+\frac{5}{2}(\sqrt{2}+1)$$

$$\Rightarrow x=2-\frac{5}{\sqrt{2}} \qquad \text{and } y=8+\frac{5}{\sqrt{2}}$$

$\therefore$ Coordinates of C are $\left(2-\frac{5}{\sqrt{2}}, 8+\frac{5}{\sqrt{2}}\right)$.

Hence, the locus of z is the major segment of the circle

$$\left|z-\left(2-\frac{5}{\sqrt{2}}\right)-i\left(8+\frac{5}{\sqrt{2}}\right)\right|=5\sqrt{2+\sqrt{2}}$$

drawn on AB and containing the angle $\frac{\pi}{8}$.

EXAMPLE 19 *Find $a \in R$, if the complex number z is to satisfy : $|z|=3$, $|z-\{a(1+i)-i\}| \le 3$ and $|z+2a-(a+1)i|>3$ simultaneously for at least one z.*

SOLUTION We have,

$$|z| = 3 \qquad \ldots(i)$$

$$|z-\{a(1+i)-i\}| \le 3 \qquad \ldots(ii)$$

and,

$$|z+2a-(a+1)i| > 3 \qquad \ldots(iii)$$

If the complex number z is to satisfy (i), (ii) and (iii) simultaneously, then z should lie on the circle $|z|=3$ but inside or on the circle $|z-\{a(1+i)-i\}|=3$ and outside the circle $|z+2a-(a+1)i|=3$.

If z is to satisfy (i) and (ii), then the circles $|z|=3$ and $|z-\{a(1+i)-i\}|=3$ should cut or touch each other. Therefore,

Difference of radii < Distance between the centres $\le$ Sum of the radii

$$\Rightarrow 0 < \left|\{a(1+i)-i\}\right| \le 6$$

$$\Rightarrow 0 < \sqrt{a^2+(a-1)^2} \le 6$$

$$\Rightarrow 0 < 2a^2-2a+1 \le 36$$

$$\Rightarrow 2a^2-2a+1 > 0 \text{ and } 2a^2-2a+1 \le 36$$

$$\Rightarrow a \in R \text{ and } 2a^2-2a-35 \le 0$$

$$\Rightarrow a \in R \text{ and } \frac{1-\sqrt{71}}{2} \le a \le \frac{1+\sqrt{71}}{2} \qquad \left[\because 2a^2-2a-35=0 \Rightarrow a=\frac{1\pm\sqrt{71}}{2}\right]$$

$$\Rightarrow a \in \left[\frac{1-\sqrt{71}}{2}, \frac{1+\sqrt{71}}{2}\right] \qquad \ldots(iv)$$

If z is to satisfy (i) and (iii), then the circles $|z|=3$ and $|z-\{-2a+(a+1)i\}|=3$ should not intersect or touch each other.

$\therefore$ Distance between their centres > Sum of their radii

$$\Rightarrow |-2a+(a+1)i| > 3+3$$

$$\Rightarrow \sqrt{(-2a)^2+(a+1)^2} > 6$$

$$\Rightarrow 5a^2+2a+1 > 36$$

$$\Rightarrow 5a^2+2a-35 > 0$$

$$\Rightarrow a < \frac{-1-4\sqrt{11}}{5} \text{ or, } a > \frac{-1+4\sqrt{11}}{5} \qquad \left[\because 5a^2+2a-35=0 \Rightarrow a=\frac{-1\pm4\sqrt{11}}{5}\right]$$

$$\Rightarrow a \in \left(-\infty, \frac{-1-4\sqrt{11}}{4}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \infty\right) \qquad \ldots(v)$$

From (iv) and (v), we get

$$a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$$

EXAMPLE 20 *Find the equation in complex variables of all the circles which are orthogonal to $|z|=1$ and $|z-1|=4$.*

SOLUTION We have,

$$|z| = 1 \qquad \ldots(i)$$

and, $$|z-1| = 4 \qquad \ldots(ii)$$

Let the equation of the circle cutting (i) and (ii) orthogonally be

$$|z-(a+ib)| = r \qquad \ldots(iii)$$

where $a, b, r \in R$ and $r > 0$.

We know that two circles cut each other orthogonally, if the square of the distance between their centres is equal to the sum of the squares of their radii. Therefore, (i) and (iii) will intersect orthogonally, if

(Distance between their centres)^2 = (Sum of the squares of their radii)

$$\Rightarrow |a+bi|^2 = 1+r^2$$

$$\Rightarrow a^2+b^2 = 1+r^2 \qquad \ldots(iv)$$

If the circles (ii) and (iii) intersect orthogonally, then

$$|(a+bi)-1|^2 = 4^2+r^2$$

$$\Rightarrow |(a-1)+ib|^2 = 16+r^2$$

$$\Rightarrow (a-1)^2+b^2 = 16+r^2$$

$$\Rightarrow a^2+b^2-2a = 15+r^2$$

$\Rightarrow 1+r^2-2a = 15+r^2$ [Using (iv)]

$\Rightarrow -2a = 14$

$\Rightarrow a = -7$

Putting $a=-7$ in (iv), we get

$49+b^2 = 1+r^2$

$\Rightarrow r = \sqrt{b^2+48}$

Substituting the values of a and r in (iii), we obtain that the required family of circles is given by

$|z+7-bi| = \sqrt{b^2+48}$, where $b \in R$

EXAMPLE 21 *Find all the points in the complex plane which satisfy the equations*

$\log_5(|z|+3) - \log_{\sqrt{5}}\left||z|-1\right| = 1$ and $\arg(z-1-i) = \frac{\pi}{4}$

SOLUTION We have,

$\log_5(|z|+3) - \log_{\sqrt{5}}\left||z|-1\right| = 1$

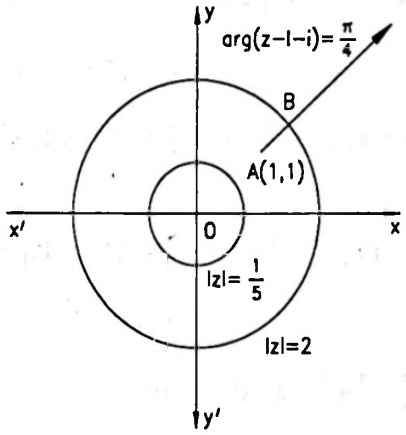


Fig. 19.94

$\Rightarrow \log_5(|z|+3) - 2\log_{\sqrt{5}}\left||z|-1\right| = 1$

$\Rightarrow \log_5\left\{\frac{|z|+3}{\left||z|-1\right|^2}\right\} = 1$

$\Rightarrow \frac{|z|+3}{\left||z|-1\right|^2} = 5$

$\Rightarrow |z|+3 = 5\{|z|^2-2|z|+1\}$

$\Rightarrow 5|z|^2-11|z|+2 = 0$

$\Rightarrow (5|z|-1)(|z|-2) = 0$

$\Rightarrow |z| = \frac{1}{5}$ or, $|z| = 2$

Thus, we have

$\left(|z| = \frac{1}{5} \text{ or, } |z| = 2\right)$ and $\arg\{z-(1+i)\} = \frac{\pi}{4}$.

Clearly, $\arg\{z-(1+i)\} = \frac{\pi}{4}$ it represents a ray originating from $A(1, 1)$ and making an angle of 45° with x-axis i.e. along the ling $y=x$.

It does not intersect the circle $|z| = \frac{1}{5}$ but it intersects $|z| = 2$ at B.

To find the coordinates of B, we solve $|z| = 2$ and $y=x$ simultaneously.

Now,

$|z| = 2 \Rightarrow |z|^2 = 2^2 \Rightarrow x^2+y^2 = 4$

Putting $y=x$ in this, we get

$2x^2 = 4 \Rightarrow x = \pm\sqrt{2} \Rightarrow x = \sqrt{2}$ [$\because x>0$]

$\therefore y = x = \sqrt{2}$.

Hence, required point is $(\sqrt{2}, \sqrt{2})$ which represents the complex number $\sqrt{2}(1+i)$.

EXAMPLE 22 *A point z moves in the Argand plane such that* $\arg(z-2-i) = \frac{\pi}{4}$. *Find the path traced by* $\frac{1}{z}$.

SOLUTION We have,

$\arg(z-2-i) = \frac{\pi}{4}$

$\Rightarrow \arg\{z-(2+i)\} = \frac{\pi}{4}$

$\Rightarrow z$ lies on the ray originating from (2, 1) and making an angle of $\pi/4$ with the real axis. Cartesian equation of this line is

$y-1 = (\tan\pi/4)(x-2)$

$\Rightarrow y-x+1 = 0$ ...(i)

Let $\omega = \frac{1}{z}$. Also, let $\omega = a+ib$. Then,

$\omega = \frac{1}{z}$

$\Rightarrow a+ib = \frac{1}{x+iy}$ [$\because z = x+iy$]

$\Rightarrow x+iy = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$

$\Rightarrow x = \frac{a}{a^2+b^2}, y = \frac{-b}{a^2+b^2}$

Substituting the values of x and y in (i), we get

$\frac{-b}{a^2+b^2} - \frac{a}{a^2+b^2} + 1 \Rightarrow 0$

$\Rightarrow a^2+b^2-a-b = 0$

$$\Rightarrow \omega\bar{\omega} - \left(\frac{\omega+\bar{\omega}}{2}\right) - \left(\frac{\omega-\bar{\omega}}{2i}\right) = 0$$

$$\Rightarrow \omega\bar{\omega} - \frac{1}{2}(1-i)\omega - \frac{1}{2}(1+i)\bar{\omega} = 0,$$

Clearly, this equation represents a circle centred at $\frac{1}{2}(1+i)$

and,

$$\text{Radius} = \sqrt{\frac{1}{4}(1+1)} = \frac{1}{\sqrt{2}}$$

EXAMPLE 23 *Consider an equilateral triangle having vertices at the points* $A\left(\frac{2}{\sqrt{3}}e^{i\pi/2}\right), B\left(\frac{2}{\sqrt{3}}e^{-i\pi/6}\right), C\left(\frac{2}{\sqrt{3}}e^{-i5\pi/6}\right)$. *Let* $P(z)$ *be any point on its incircle. Prove that* $AP^2 + BP^2 + CP^2 = 5$.

SOLUTION Let $z_1 = \frac{2}{\sqrt{3}}e^{i\pi/2}, z_2 = \frac{2}{\sqrt{3}}e^{-i\pi/6}$

and, $z_3 = \frac{2}{\sqrt{3}}e^{-i5\pi/6}$.

Clearly,

$$|z_1| = |z_2| = |z_3| = \frac{2}{\sqrt{3}}$$

$\Rightarrow$ z_1, z_2, z_3 lie on the circle $|z| = \frac{2}{\sqrt{3}}$

$\Rightarrow$ Circumcentre of ΔABC is at the origin.

Since the triangle is equilateral. Therefore, its incentre coincides with the circumcentre. So, the incentre of ΔABC is at the origin. In order to obtain the equation of the incircle, we require its radius (= OD) only.

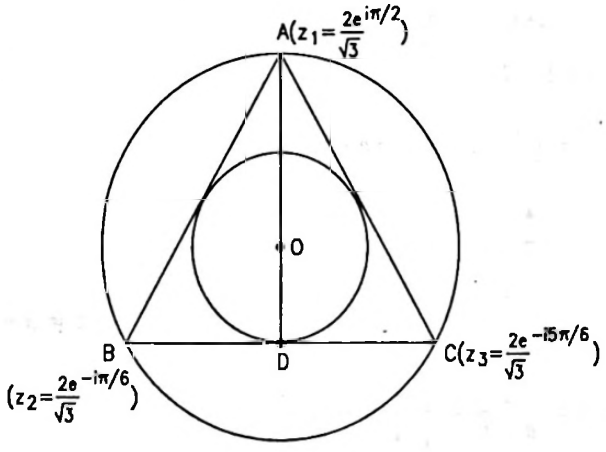


Fig. 19.95

Let l be the length of the side of ΔABC. Then,

$$l = AB = |z_2 - z_1|$$

$$= \left|\frac{2}{\sqrt{3}}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) - \frac{2}{\sqrt{3}}i\right|$$

$$= \frac{2}{\sqrt{3}}\left|\frac{\sqrt{3}}{2} - \frac{3}{2}i\right|$$

$$= \frac{2}{\sqrt{3}} \times \sqrt{\frac{3}{4} + \frac{9}{4}} = 2$$

In ΔABD, we have

$$\sin 60° = \frac{AD}{AB}$$

$$\Rightarrow \quad AD = AB\frac{\sqrt{3}}{2} = \sqrt{3} \qquad [\because AB = 2]$$

$$\therefore \quad OD = \frac{1}{3}AD = \frac{1}{\sqrt{3}}$$

So, the equation of the incircle is $|z| = \frac{1}{\sqrt{3}}$.

Let $P(z)$ be any point on the incircle. Then,

$$z = \frac{1}{\sqrt{3}}e^{i\theta}, \text{ where } 0 \le \theta \le 2\pi$$

Now,

$$AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_1)$$

$$BP^2 = |z - z_2|^2 = |z|^2 + |z_2|^2 - (z\bar{z}_2 + \bar{z}z_2)$$

$$CP^2 = |z - z_3|^2 = |z|^2 + |z_3|^2 - (z\bar{z}_3 + \bar{z}z_3)$$

$$\therefore \quad AP^2 + BP^2 + CP^2$$
$$= 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3)$$

$$= 3 \times \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3} - z \times 0 - \bar{z} \times 0$$

$$\left[\because \text{Origin is also the centroid of } \Delta ABC \;\; \therefore \frac{z_1 + z_2 + z_3}{3} = 0 \Rightarrow z_1 + z_2 + z_3 = 0\right]$$

$$= 1 + 3 \times \frac{4}{3} = 5$$

EXAMPLE 24 *A, B, C are the points representing the complex numbers* z_1, z_2, z_3 *respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude of the triangle through the vertex A meets the circumcircle again at P, then prove that P represents the complex number* $\frac{-z_2 z_3}{z_1}$.

SOLUTION Let AD be the altitude through vertex A of ΔABC. AD produced meets the circumcircle at $P(z)$.

Since O is the circumcentre of ΔABC.

Therefore,

Now,

$$OA = OB = OC = OP$$

$$OA = OP$$

$\Rightarrow |z_1| = |z|$

$\Rightarrow |z_1|^2 = |z|^2$

$\Rightarrow z_1\bar{z}_1 = z\bar{z}$

$\Rightarrow \dfrac{\bar{z}_1}{\bar{z}} = \dfrac{z}{z_1}$ ...(i)

Similarly,

$OB = OP \Rightarrow \dfrac{\bar{z}_2}{\bar{z}} = \dfrac{z}{z_2}$ ...(ii)

and,

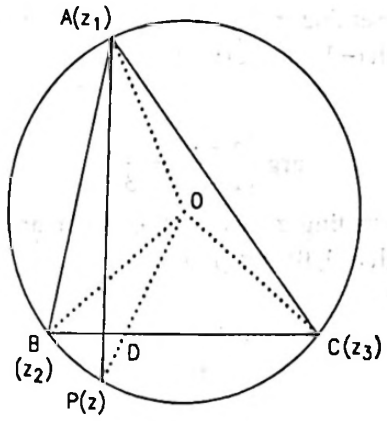


Fig. 19.96

$OC = OP \Rightarrow \dfrac{\bar{z}_3}{\bar{z}} = \dfrac{z}{z_3}$ ...(iii)

Now,

$AD \perp BC$

$\Rightarrow AP \perp BC$

$\Rightarrow \arg\left(\dfrac{z_1 - z}{z_3 - z_2}\right) = \dfrac{\pi}{2}$

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2}$ is purely imaginary

$\Rightarrow \left(\dfrac{z_1 - z}{z_3 - z_2}\right) + \left(\overline{\dfrac{z_1 - z}{z_3 - z_2}}\right) = 0$

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2} + \dfrac{\bar{z}_1 - \bar{z}}{\bar{z}_3 - \bar{z}_2} = 0$

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2} + \dfrac{\dfrac{\bar{z}_1}{\bar{z}} - 1}{\dfrac{\bar{z}_3}{\bar{z}} - \dfrac{\bar{z}_2}{\bar{z}}} = 0$

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2} + \dfrac{\dfrac{z}{z_1} - 1}{\dfrac{z}{z_3} - \dfrac{z}{z_2}} = 0$ [Using (i), (ii) and (iii)]

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2} + \dfrac{(z - z_1)z_2 z_3}{zz_1(z_2 - z_3)} = 0$

$\Rightarrow \dfrac{z_1 - z}{z_3 - z_2}\left(1 + \dfrac{z_2 z_3}{zz_1}\right) = 0$

$\Rightarrow 1 + \dfrac{z_2 z_3}{zz_1} = 0$ $\left[\because \dfrac{z_1 - z}{z_3 - z_2} \neq 0\right]$

$\Rightarrow z = -\dfrac{z_2 z_3}{z_1}$

EXAMPLE 25 *Let $z_1 = 10 + 6i$ and $z_2 = 4 + 2i$ be two complex numbers and z be a complex number such that $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{4}$. Find the centre and radius of the locus of complex number z.*

[IIT 1990]

SOLUTION We have, $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{4}$. This means that z is lying on the major arc of the circle passing through z_1 and z_2.

Let $O(z_0)$ be the centre of the circle.

Then,

$\angle AOB = 2\angle APB = 2 \times \dfrac{\pi}{4} = \dfrac{\pi}{2}$

We have,

$AB = |z_1 - z_2|$

$= |6 + 4i| = \sqrt{36 + 16} = \sqrt{52}$

Let r be the radius of the circle. Then,

$AB^2 = OA^2 + OB^2$

$\Rightarrow r^2 + r^2 = (\sqrt{52})^2$

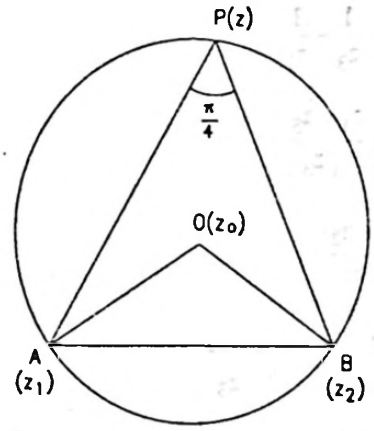


Fig. 19.97

$\Rightarrow 2r^2 = 52$

$\Rightarrow r = \sqrt{26}$

Also,

$$z_2 - z_0 = (z_1 - z_0)\, e^{i\pi/2}$$

$\Rightarrow z_2 - z_0 = (z_1 - z_0)\, i$

$\Rightarrow z_0(1-i) = -z_1 i + z_2$

$$\Rightarrow z_0 = \frac{-(10+6i)\,i + (4+2i)}{1-i} = \frac{10-8i}{1-i}$$

$$\Rightarrow z_0 = \frac{10-8i}{1-i} \times \frac{1+i}{1+i} = \frac{18+2i}{2} = 9+i$$

Hence, the centre of the circle is at $9+i$ and radius is $\sqrt{26}$ units.

EXAMPLE 26 *If z_1, z_2, z_3 are complex numbers such that at least one of them is not real and $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.*

SOLUTION We have,

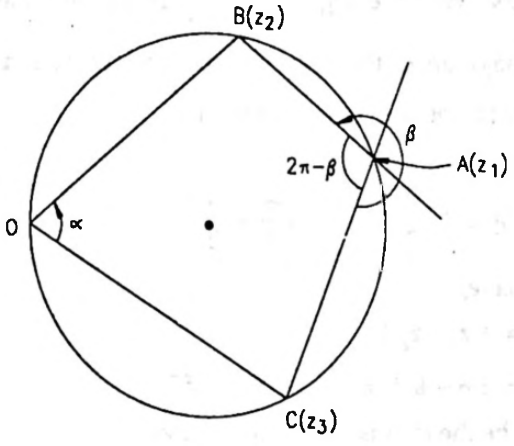


Fig. 19.98

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$$

$$\Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$

$$\Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$$

$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = \frac{-z_2}{z_3}$$

$$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{-z_2}{z_3}\right)$$

$$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right) \qquad [\because \arg(-z) = \pi + \arg(z)]$$

$\Rightarrow \beta = \pi + \alpha$ ...(i)

Now,

$$\angle BOC + \angle BAC = \alpha + 2\pi - \beta$$
$$= \alpha + 2\pi - (\pi + \alpha) \qquad \text{[Using (i)]}$$
$$= \pi$$

Thus, the sum of a pair of opposite angles of a quadrilateral is 180°. Hence, the points $0, z_1, z_2$ and z_3 are the vertices of a cyclic quadrilateral i.e. they lie on a circle.

EXAMPLE 27 *Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.*

SOLUTION We know that

$$\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$$

$\Rightarrow$ Point representing z lies on the minor arc of the circle passing through $(-1, 0)$ and $(1, 0)$

and,

$$\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$$

$\Rightarrow$ Point representing z lies on the major arc of the circle passing through $(-1, 0)$ and $(1, 0)$.

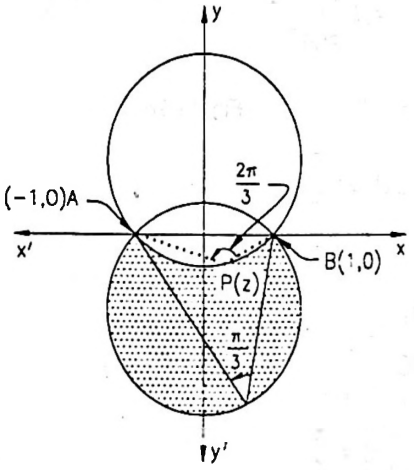


Fig. 19.99

Thus, if $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$, then z lies in the shaded region marked in Fig. 19.99 except points $(1, 0)$ and $(-1, 0)$.

EXAMPLE 28 *Two different non-parallel lines meet the circle $|z| = r$ in the points a, b and c, d respectively. Prove that these lines meet in the point z given by*

$$z = \frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1} b^{-1} - c^{-1} d^{-1}}$$

SOLUTION Since P, A, B are collinear points. Therefore,

Slope of AP = Slope of BA

$$\Rightarrow \frac{z-a}{\bar{z}-\bar{a}} = \frac{a-b}{\bar{a}-\bar{b}}$$

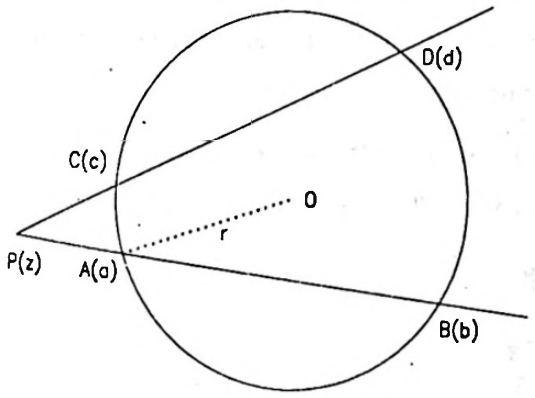


Fig. 19.100

$$\Rightarrow \frac{\bar{z}-\bar{a}}{z-a} = \frac{\bar{a}-\bar{b}}{a-b}$$

$$\Rightarrow \bar{z}-\bar{a} = \left(\frac{\bar{a}-\bar{b}}{a-b}\right)(z-a)$$

$$\Rightarrow \bar{z} = \bar{a}+\left(\frac{\bar{a}-\bar{b}}{a-b}\right)(z-a) \quad ...(i)$$

Similarly, P, C, D are collinear points. Therefore,

Slope of CP = Slope of DC

$$\Rightarrow \frac{z-c}{\bar{z}-\bar{c}} = \frac{c-d}{\bar{c}-\bar{d}}$$

$$\Rightarrow \frac{\bar{z}-\bar{c}}{z-c} = \frac{\bar{c}-\bar{d}}{c-d}$$

$$\Rightarrow \bar{z}-\bar{c} = \left(\frac{\bar{c}-\bar{d}}{c-d}\right)(z-c)$$

$$\Rightarrow \bar{z} = \bar{c}+\left(\frac{\bar{c}-\bar{d}}{c-d}\right)(z-c) \quad ...(ii)$$

From (i) and (ii), we get

$$\bar{a}+\left(\frac{\bar{a}-\bar{b}}{a-b}\right)(z-a) = \bar{c}+\left(\frac{\bar{c}-\bar{d}}{c-d}\right)(z-c)$$

$$\Rightarrow z\left(\frac{\bar{a}-\bar{b}}{a-b}-\frac{\bar{c}-\bar{d}}{c-d}\right) = (\bar{c}-\bar{a})+a\left(\frac{\bar{a}-\bar{b}}{a-b}\right)-c\left(\frac{\bar{c}-\bar{d}}{c-d}\right)$$

$$\Rightarrow z\left(\frac{\frac{r^2}{a}-\frac{r^2}{b}}{a-b}-\frac{\frac{r^2}{c}-\frac{r^2}{d}}{c+d}\right) = \left(\frac{r^2}{c}-\frac{r^2}{a}\right)+a\left(\frac{\frac{r^2}{a}-\frac{r^2}{b}}{a-b}\right)-c\left(\frac{\frac{r^2}{c}-\frac{r^2}{d}}{c-d}\right)$$

$$\left[\begin{array}{l}\because A(a), B(b), C(c) \text{ and } D(d) \text{ lie on } |z| = r \\ \therefore |a| = |b| = |c| = |d| = r \\ \Rightarrow |a|^2 = |b|^2 = |c|^2 = |d|^2 = r^2, \\ \Rightarrow a\bar{a} = r^2, b\bar{b} = r^2, c\bar{c} = r^2, d\bar{d} = r^2\end{array}\right]$$

$$\Rightarrow z\left(-\frac{1}{ab}+\frac{1}{cd}\right) = \left(\frac{1}{c}-\frac{1}{a}\right)+a\left(-\frac{1}{ab}\right)-c\left(-\frac{1}{cd}\right)$$

$$\Rightarrow z(a^{-1}b^{-1}-c^{-1}d^{-1}) = a^{-1}-c^{-1}+b^{-1}-d^{-1}$$

$$\Rightarrow z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

EXAMPLE 29 *Three points represented by complex numbers a, b, c lie on a circle with centre origin and radius r. The tangent at C cuts the chord joining the points a, b at z. Show that*

$$z = \frac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$$

SOLUTION The equation of the circle with centre at the origin and radius r is

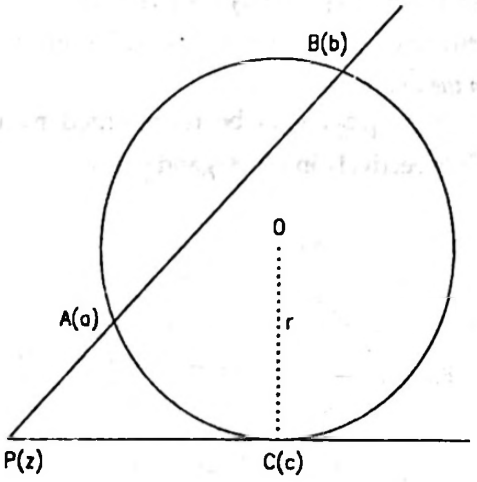


Fig. 19.101

$$|z| = r$$

Since points $A(a), B(b), C(c)$ lie on the circle $|z| = r$. Therefore,

$$|a| = |b| = |c| = r$$

$$\Rightarrow |a|^2 = |b|^2 = |c|^2 = r^2$$

$$\Rightarrow a\bar{a} = b\bar{b} = c\bar{c} = r^2$$

$$\Rightarrow \bar{a} = \frac{r^2}{a}, \bar{b} = \frac{r^2}{b}, \bar{c} = \frac{r^2}{c} \quad ...(i)$$

The equation of the tangent to the circle $|z| = r$ at $z = c$ is given by

$$z\bar{c}+\bar{z}c = 2r^2 \quad ...(ii)$$

The equation of the chord joining $A(a)$ and $B(b)$ is given by

$$z(\bar{a}-\bar{b})-\bar{z}(a-b)+a\bar{b}-\bar{a}b = 0 \quad ...(iii)$$

Lines (ii) and (iii) intersect at $P(z)$. Eliminating $\bar{z}$ from these two equations, we get

$$\frac{z(\bar{a}-\bar{b})+a\bar{b}-\bar{a}b}{a-b} = \frac{2r^2-z\bar{c}}{c}$$

$$\Rightarrow z\left(\frac{\bar{a}-\bar{b}}{a-b}+\frac{\bar{c}}{c}\right) = \frac{2r^2}{c}-\left(\frac{a\bar{b}-\bar{a}b}{a-b}\right)$$

$$\Rightarrow z\left(\frac{r^2}{a}-\frac{r^2}{b}+\frac{r^2}{c^2}\right)=\frac{2r^2}{c}-\left(\frac{ar^2}{b}-\frac{br^2}{a}\right) \qquad \text{[Using (i)]}$$

$$\Rightarrow z\left(\frac{-1}{ab}+\frac{1}{c^2}\right)=\frac{2}{c}-\frac{a+b}{ab}$$

$$\Rightarrow z(a^{-1}b^{-1}-c^{-2})=a^{-1}+b^{-1}-2c^{-1}$$

$$\Rightarrow z\,\frac{(a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$$

EXAMPLE 30 *If z_1, z_2, z_3 are the complex numbers such that $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|$, then prove that the points representing z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the unit circle.*

SOLUTION Let z_1, z_2 and z_3 be represented by the points A, B and C respectively in the Argand plane.

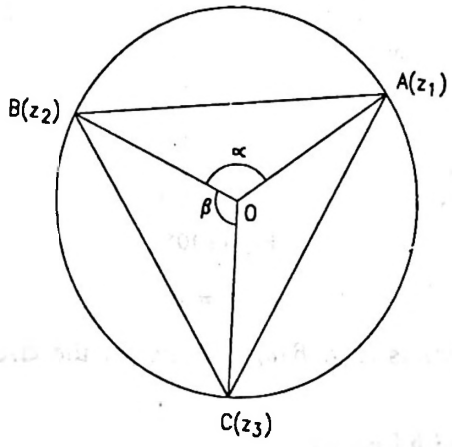


Fig. 19.102

We have,

$$|z_1|=|z_2|=|z_3|=1$$

$$\Rightarrow \quad OA=OB=OC=1$$

$\Rightarrow$ A, B, C lie on the unit circle with centre at the origin.

Let $\angle AOB=\alpha$ and $\angle BOC=\beta$.

Clearly, OB and OC are obtained by rotating OA through α and $\alpha+\beta$ respectively. Therefore,

$$\therefore \quad z_2=z_1e^{i\alpha} \text{ and } z_3=z_1e^{i(\alpha+\beta)}$$

Now,

$$z_1+z_2+z_3=0$$

$$\Rightarrow z_1+z_1e^{i\alpha}+z_1e^{i(\alpha+\beta)}=0$$

$$\Rightarrow z_1(1+e^{i\alpha}+e^{i(\alpha+\beta)}=0$$

$$\Rightarrow z_1\{(1+\cos\alpha+\cos(\alpha+\beta))+i(\sin\alpha+\sin(\alpha+\beta)\}=0$$

$$\Rightarrow 1+\cos\alpha+\cos(\alpha+\beta)=0 \text{ and } \sin\alpha+\sin(\alpha+\beta)=0$$

$$\Rightarrow \cos\alpha+\cos(\alpha+\beta)=-1 \quad \text{and} \quad \sin\alpha+\sin(\alpha+\beta)=0$$

Squaring and adding, we get

$$\{\cos\alpha+\cos(\alpha+\beta)\}^2+\{\sin\alpha+\sin(\alpha+\beta)\}^2=1$$

$$\Rightarrow 2+2\cos(\alpha+\beta-\alpha)=1$$

$$\Rightarrow \cos\beta=\frac{-1}{2}$$

$$\Rightarrow \beta=\frac{2\pi}{3}$$

Now,

$$\sin\alpha+\sin(\alpha+\beta)=0$$

$$\Rightarrow \sin(\alpha+\beta)=-\sin\alpha$$

$$\Rightarrow \sin(\alpha+\beta)=\sin(2\pi-\alpha)$$

$$\Rightarrow \alpha+\beta=2\pi-\alpha$$

$$\Rightarrow 2\alpha=2\pi-\frac{2\pi}{3}$$

$$\Rightarrow \alpha=\frac{2\pi}{3}$$

Now,

$$\alpha=\frac{2\pi}{3} \text{ and } \beta=\frac{2\pi}{3}$$

$$\Rightarrow 2\angle C=\frac{2\pi}{3} \text{ and } 2\angle A=\frac{2\pi}{3}$$

$$\Rightarrow \quad \angle A=\angle C=\frac{\pi}{3}.$$

Hence, triangle ABC is an equilateral triangle.

EXAMPLE 31 *If A and B represent the complex numbers $6i$ and $3+0i$ in the Argand plane and $P(z)$ is a variable point such that $PA=2PB$, then show that the locus of P is a circle. Also, find its centre and radius.*

SOLUTION We have,

$$PA=2PB$$

$$\Rightarrow PA^2=4PB^2$$

$$\Rightarrow |z-6i|^2=4|z-3|^2$$

$$\Rightarrow (z-6i)(\overline{z-6i})=4(z-3)(\overline{z-3})$$

$$\Rightarrow (z-6i)(\bar{z}+6i)=4(z-3)(\bar{z}-3)$$

$$\Rightarrow z\bar{z}+6iz-6i\bar{z}+36=4(z\bar{z}-3\bar{z}-3z+9)$$

$$\Rightarrow 3z\bar{z}-(12-6i)\bar{z}-(12+6i)z=0$$

$$\Rightarrow z\bar{z}-(4-2i)\bar{z}-(4+2i)z=0$$

Clearly, it represents a circle having its centre at $4-2i$

and radius $= \sqrt{|-4+2i|^2 - 0} = \sqrt{20}$.

EXAMPLE 32 *Consider the triangle ABC, where A (z_a), B (z_b), C (z_c) whose circumcircle is $|z| = r$. If the internal bisector of angle 'A' meets the circumcircle again at D. Find the complex number associated with D. Hence or otherwise also prove that for a fixed position of B (z_b) and C (z_c) all the bisectors (internal) of $\angle A$ will pass through a fixed point.*

SOLUTION Let D represent the complex number z.

Since $\angle BAD = \angle ADC = \theta$. Therefore, we have

$$\frac{z - z_a}{|z - z_a|} = \frac{z_b - z_a}{|z_b - z_a|} e^{i\theta}$$

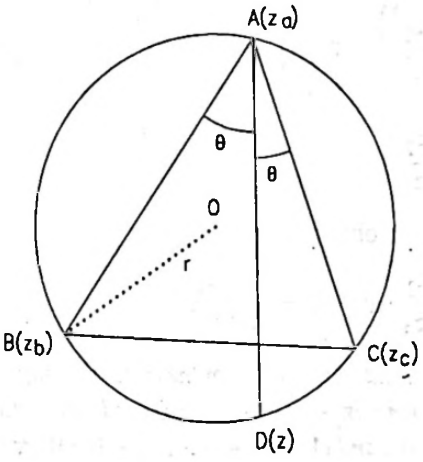


Fig. 19.103

and,

$$\frac{z_c - z_a}{|z_c - z_a|} = \frac{z - z_a}{|z - z_a|} e^{i\theta}$$

$$\Rightarrow \quad \frac{(z - z_a)^2}{|z - z_a|^2} = \frac{(z_b - z_a)}{|z_b - z_a|^2} e^{i2\theta}$$

and, $\dfrac{(z_c - z_a)^2}{|z_c - z_a|^2} = \dfrac{(z - z_a)^2}{|z - z_a|^2} e^{i2\theta}$

$$\Rightarrow \frac{z - z_a}{\bar{z} - \bar{z}_a} = \frac{z_b - z_a}{\bar{z}_b - \bar{z}_a} e^{i2\theta} \text{ and } \frac{z_c - z_a}{\bar{z}_c - \bar{z}_a} = \frac{z - z_a}{\bar{z} - \bar{z}_a} e^{i2\theta}$$

$$\Rightarrow \frac{z - z_a}{\bar{z} - \bar{z}_a} \times \frac{\bar{z}_c - \bar{z}_a}{z_c - z_a} = \frac{z_b - z_a}{\bar{z}_b - \bar{z}_a} \times \frac{\bar{z} - \bar{z}_a}{z - z_a}$$

$$\Rightarrow \left(\frac{z - z_a}{\bar{z} - \bar{z}_a}\right)^2 = \left(\frac{z_b - z_a}{\bar{z}_b - \bar{z}_a}\right) \cdot \left(\frac{z_c - z_a}{\bar{z}_c - \bar{z}_a}\right)$$

$$\Rightarrow \left(\frac{z - z_a}{\frac{r^2}{z} - \frac{r^2}{z_a}}\right)^2 = \left(\frac{z_b - z_a}{\frac{r^2}{z_b} - \frac{r^2}{z_a}}\right)\left(\frac{z_c - z_a}{\frac{r^2}{z_c} - \frac{r^2}{z_a}}\right)$$

$$\left[\begin{array}{l} \because z_a, z_b, z_c \text{ and} \\ z \text{ lie on } |z| = r \\ \therefore |z_a| = |z_b| \\ = |z_c| = |z| = r \\ \Rightarrow z_a \bar{z}_a = z_b \bar{z}_b \\ = z_c \bar{z}_c = z\bar{z} = r^2 \end{array}\right]$$

$$\Rightarrow (zz_a)^2 = (z_a z_b)(z_a z_c)$$

$$\Rightarrow z^2 = z_b z_c$$

Hence, the complex number assoicated with D is given by $z^2 = z_b z_c$. Clearly, if z_b and z_c are fixed, then $z^2 = z_b z_c$ gives a fixed value of z. Hence, all the bisectors of $\angle A$ will pass through a fixed point given by $z^2 = z_b z_c$.

EXAMPLE 33 *If $\alpha \in R$ and z is a complex number, then find the real part u and imaginary part v of the complex number*

$$(z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$$

Also, prove that the locus of the points on the Argand plane representing the complex numbers z such that $v = 0$ is a circle of unit radius with centre at the point (1, 0) or a straight line through (1, 0).

SOLUTION Let $z = x + iy$. Then,

$$u + iv = (z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$$

$$\Rightarrow u + iv$$

$$= \{(x-1) + iy\}(\cos\alpha - i\sin\alpha) + \frac{\cos\alpha + i\sin\alpha}{(x-1) + iy}$$

$$= \{(x-1)\cos\alpha + y\sin\alpha\} + i\{y\cos\alpha - (x-1)\sin\alpha\}$$

$$+ \frac{\{(x-1) - iy\}(\cos\alpha + i\sin\alpha)}{(x-1)^2 + y^2}$$

$$= \{(x-1)\cos\alpha + y\sin\alpha\} + i\{y\cos\alpha - (x-1)\sin\alpha\}$$

$$+ \frac{\{(x-1)\cos\alpha + y\sin\alpha\} + i\{(x-1)\sin\alpha - y\cos\alpha\}}{(x-1)^2 + y^2}$$

$$= \{(x-1)\cos\alpha + y\sin\alpha\}\left\{1 + \frac{1}{(x-1)^2 + y^2}\right\}$$

$$+ i\{y\cos\alpha - (x-1)\sin\alpha\}\left\{1 - \frac{1}{(x-1)^2 + y^2}\right\}$$

$$\Rightarrow u = \{(x-1)\cos\alpha + y\sin\alpha\}\left\{1 + \frac{1}{(x-1)^2 + y^2}\right\}$$

and,

$$v = \{y\cos\alpha - (x-1)\sin\alpha\}\left\{1 - \frac{1}{(x-1)^2 + y^2}\right\}$$

Now,

$$v = 0$$

$$\Rightarrow y\cos\alpha - (x-1)\sin\alpha = 0 \text{ or, } 1 = \frac{1}{(x-1)^2 + y^2}$$

$$\Rightarrow y = (x-1)\tan\alpha \qquad \text{or, } (x-1)^2 + y^2 = 1$$

$$\Rightarrow y = (x-1)\tan\alpha \qquad \text{or, } |z - 1| = 1$$

$\Rightarrow$ Locus of z is either a circle with centre at $1 + 0i$ and radius 1 or a straight line passing through (1, 0)

EXAMPLE 34 *Let $|z| = r$ be the incircle of a ΔABC in the Argand plane. If contact points opposite to the vertices A, B, C are A_1 (z_1), B_1 (z_2) and C_1 (z_3) respectively, obtain the complex numbers associated with the vertices A, B, C in terms of z_1, z_2 and z_3.*

SOLUTION Let z_a, z_b and z_c be the complex numbers represented by the vertices A, B and C respectively. Then,

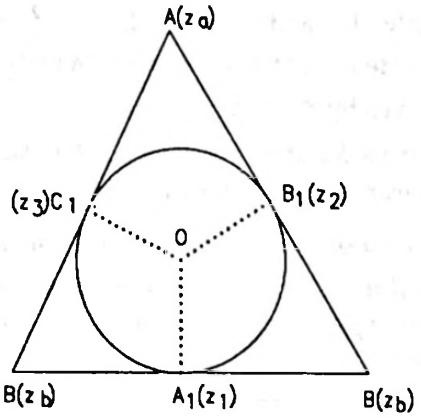


Fig. 19.104

$$\angle OA_1C = \frac{\pi}{2}$$

$$\Rightarrow \frac{z_c - z_1}{|z_c - z_1|} e^{i\pi/2} = \frac{z_1}{|z_1|}$$

$$\Rightarrow \frac{z_1^2}{|z_1|^2} = \frac{(z_c - z_1)^2}{|z_c - z_1|^2} e^{i\pi}$$

$$\Rightarrow \frac{z_1}{\bar{z}_1} = \frac{z_c - z_1}{\bar{z}_c - \bar{z}_1} e^{i\pi} \qquad \left[\because \frac{z^2}{|z|^2} = \frac{z^2}{z\bar{z}} = \frac{z}{\bar{z}}\right]$$

$$\Rightarrow \frac{z_1}{\bar{z}_1} = -\frac{z_c - z_1}{\bar{z}_c - \bar{z}_1}$$

$$\Rightarrow \bar{z}_c - \bar{z}_1 = -(z_c - z_1)\frac{\bar{z}_1}{z_1}$$

$$\Rightarrow \bar{z}_c = \bar{z}_1 - \frac{\bar{z}_1}{z_1}(z_c - z_1) \qquad \ldots(i)$$

Now,

$$\angle OB_1C = \frac{\pi}{2}$$

$$\Rightarrow \frac{z_c - z_2}{|z_c - z_2|} = \frac{z_2 - 0}{|z_2 - 0|} e^{i\pi/2}$$

$$\Rightarrow \frac{(z_c - z_2)^2}{|z_c - z_2|^2} = \frac{z_2^2}{|z_2|^2} e^{i\pi}$$

$$\Rightarrow \frac{z_c - z_2}{\bar{z}_c - \bar{z}_2} = \frac{-z_2}{\bar{z}_2}$$

$$\Rightarrow \bar{z}_c - \bar{z}_2 = -\frac{\bar{z}_2}{z_2}(z_c - z_2)$$

$$\Rightarrow \bar{z}_c = \bar{z}_2 - \frac{\bar{z}_2}{z_2}(z_c - z_2) \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\bar{z}_1 - \frac{\bar{z}_1}{z_1}(z_c - z_1) = \bar{z}_2 - \frac{\bar{z}_2}{z_2}(z_c - z_2)$$

$$\frac{\bar{z}_2}{z_2}(z_c - z_2) - \frac{\bar{z}_1}{z_1}(z_c - z_1) = \bar{z}_2 - \bar{z}_1$$

$$\Rightarrow z_c\left(\frac{\bar{z}_2}{z_2} - \frac{\bar{z}_1}{z_1}\right) = 2(\bar{z}_2 - \bar{z}_1)$$

$$\Rightarrow z_c\left(\frac{r^2}{z_2^2} - \frac{r^2}{z_1^2}\right) = 2\left(\frac{r^2}{z_2} - \frac{r^2}{z_1}\right) \left[\begin{array}{l}\because |z_1| = r, \ |z_2| = r \\ \Rightarrow |z_1|^2 = r^2, \ |z_2|^2 = r^2 \\ \Rightarrow z_1\bar{z}_1 = r^2, \ z_2\bar{z}_2 = r^2\end{array}\right]$$

$$\Rightarrow z_c\left(\frac{z_1^2 - z_2^2}{z_1^2 z_2^2}\right) = 2\left(\frac{z_1 - z_2}{z_1 z_2}\right)$$

$$\Rightarrow z_c = \frac{2z_1z_2}{z_1 + z_2}$$

Similarly, we obtain

$$z_a = \frac{2z_2z_3}{z_2 + z_3} \text{ and } z_b = \frac{2z_1z_3}{z_1 + z_3}.$$

EXAMPLE 35 *Find the equation of the circle in complex form which tourches the line* $iz + \bar{z} + (1+i) = 0$ *and for which the lines* $(1-i)z = (1+i)\bar{z}$ *and* $(1+i)z + (i-1)\bar{z} - 4i = 0$ *are normals.*

SOLUTION Clearly, centre of the circle is the point of intersection of the normals

$$(1-i)z = (1+i)\bar{z} \qquad \ldots(i)$$

and, $$(1+i)z + (i-1)\bar{z} - 4i = 0 \qquad \ldots(ii)$$

From (i), we get

$$\bar{z} = \left(\frac{1-i}{1+i}\right)z$$

$$\Rightarrow \bar{z} = \frac{(1-i)^2}{(1+i)(1-i)}z = -iz$$

Putting the value of $\bar{z}$ in (ii), we obtain

$$(1+i)z + (i-1)(-iz) - 4i = 0$$

$$\Rightarrow (1+i+1+i)z = 4i$$

$$\Rightarrow z = \frac{2i}{1+i} = \frac{2i(1-i)}{(1+i)(1-i)} = 1+i$$

So, the centre of the circle is at $1+i$.

The equation of the tangent to the circle is

$$iz + \bar{z} + (1+i) = 0$$

Multiplying both sides by $(1-i)$ so that the last term becomes real, we get

$$i(1-i)z + \bar{z}(1-i) + (1+i)(1-i) = 0$$

$$\Rightarrow (1+i)z + (1-i)\bar{z} + 2 = 0 \qquad \ldots(iii)$$

$\therefore$ Radius of the required circle

= Length of the perpendicular from $(1+i)$ to (iii)

$$= \frac{|(1+i)(1+i)+(1-i)(1-i)+2|}{2|1-i|}$$

$$= \frac{1}{\sqrt{2}}$$

Thus, the equation of the required circle is

$$|z-(1+i)| = \frac{1}{\sqrt{2}}$$

EXAMPLE 36 *Find the equation of the line in the Argand plane which touches the circle* $|z| = \sqrt{2}a$ *at point b (where b is a complex number). Also, prove that the equations of lines passing thorugh the centre of the circle and making an angle* $\pi/4$ *with the normal at b are* $z = \pm i\frac{b^2}{2a^2}\bar{z}$.

SOLUTION Let $P(z)$ be any point on the tangent at $B(b)$. Then,

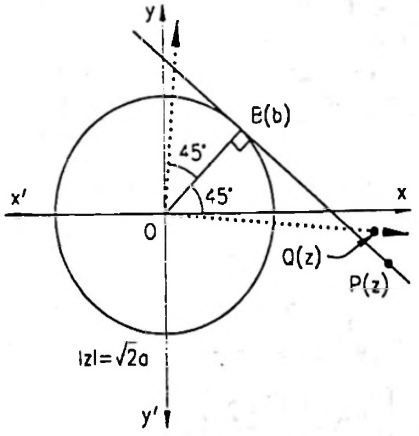


Fig. 19.105

$OB \perp PB$

$$\Rightarrow \frac{z-b}{|z-b|} = \frac{b-0}{|b-0|}e^{i\pi/2}$$

$$\Rightarrow \frac{z-b}{|z-b|} = \frac{b}{|b|}i$$

$$\Rightarrow \frac{(z-b)^2}{|z-b|^2} = \frac{-b^2}{|b|^2}$$

$$\Rightarrow \frac{z-b}{(z-b)(\bar{z}-\bar{b})} = \frac{-b^2}{b\bar{b}}$$

$$\Rightarrow \frac{z-b}{\bar{z}-\bar{b}} = \frac{-b}{\bar{b}}$$

$$\Rightarrow z\bar{b}-b\bar{b} = -\bar{z}b+b\bar{b}$$

$$\Rightarrow z\bar{b}+\bar{z}b = 2b\bar{b}$$

$$\Rightarrow z\bar{b}+\bar{z}b = 2|b|^2$$

$$\Rightarrow z\bar{b}+\bar{z}b = 4a^2 \qquad [\because b \text{ lie on } z = \sqrt{2}a \Rightarrow |b| = \sqrt{2}a]$$

So, the equation of the tangent at b is

$$z\bar{b}+\bar{z}b = 4a^2$$

Let $Q(z)$ be any point on the required lines. Then,

$$\frac{z}{|z|} = \frac{b}{|b|}e^{\pm i\pi/4}$$

$$\Rightarrow \frac{z^2}{|z|^2} = \frac{b^2}{|b|^2}e^{\pm i\pi/2}$$

$$\Rightarrow \frac{z^2}{z\bar{z}} = \frac{b^2}{b\bar{b}}(\pm i)$$

$$\Rightarrow \frac{z}{\bar{z}} = \pm\frac{b}{\bar{b}}i$$

$$\Rightarrow z = \pm\frac{b}{\bar{b}}\bar{z}i$$

$$\Rightarrow z = \pm\frac{ib^2}{|\bar{b}|^2}\bar{z}$$

$$\Rightarrow z = \pm i\frac{b^2}{2a^2}\bar{z} \qquad [\because b \text{ lies on } |z| = \sqrt{2}a \Rightarrow |b| = \sqrt{2}a \Rightarrow b^2 = 2a^2]$$

Hence, the equations of the required lines are

$$z = \pm i\frac{b^2}{2a^2}\bar{z}.$$

EXAMPLE 37 *Find the equation of the line making equal intercepts on the axes and touching the circle* $|z| = \sqrt{2}a$.

SOLUTION Let the equation of the required line be

$$\alpha\bar{z}+\bar{\alpha}z+\beta = 0 \qquad \ldots(i)$$

For intercept on real axis, we have

$$z = \bar{z}$$

Putting $z = \bar{z}$ in (i), we get

$$z = \frac{-\beta}{\alpha+\bar{\alpha}}$$

At the point where line (i) cuts imaginary axis, we have

$$\bar{z} = -z.$$

Putting $\bar{z} = -z$ in (i), we get

$$z = \frac{\beta}{\alpha-\bar{\alpha}}$$

It is given that the line in (i) cuts equal intercepts on the axes. Therefore,

$$\left|\frac{-\beta}{\alpha+\bar{\alpha}}\right| = \left|\frac{\beta}{\alpha-\bar{\alpha}}\right|$$

$\Rightarrow |\alpha-\bar{\alpha}| = |\alpha+\bar{\alpha}|$

$\Rightarrow |2i\mu| = |2\lambda|$ [Taking $\alpha = \lambda + i\mu$]

$\Rightarrow |\mu| = |\lambda|$

$\Rightarrow \mu = \pm\lambda$

$\therefore \alpha = \lambda(1 \pm i)$

Line (i) touches the circle $|z| = \sqrt{2}a$. Therefore,

Length of the perpendicular from the centre = Radius

$$\Rightarrow \frac{|\alpha(0)+\bar{\alpha}(0)+\beta}{2|\alpha|} = \sqrt{2}a$$

$$\Rightarrow \frac{|\beta|}{2\sqrt{2}\lambda} = \sqrt{2}a$$

$\Rightarrow |\beta| = 4\lambda a$

$\Rightarrow \beta = \pm 4\lambda a$

Putting the values of α and β in (i), we obtain the equation of the required line as

$$(1 \pm i)\bar{z} + (1 \mp i)z \pm 4a = 0$$

EXERCISE 19.2

1. If z is a unimodular complex number, prove that
$$\arg(z^2 + \bar{z}) = \frac{1}{2}\arg(z).$$

2. Prove that the product of two unimodular complex numbers is a unimodular complex number whose argument is equal to the sum of their arguments.

3. If $z = \cos\theta + i\sin\theta$, find the modulus and argument of $z_1 = z^2 - z$.

4. If z is a complex number such that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, then prove that
$$\left|z - \frac{1}{\sqrt{3}}i\right| = \frac{2}{\sqrt{3}}.$$

5. Show that all the roots of the equation $a_0 z^3 + a_1 z^2 + a_2 z + a_3 = 3$, where $|a_i| \le 1, i = 0, 1, 2, 3$ lie outside the circle with centre at the origin and radius $\frac{2}{3}$.

6. Show that all roots of the equation
$$z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \ldots\ldots + z\cos\theta_{n-1} + \cos\theta_n = 2$$
lie outside the circle $|z| = 1/2$, where $\theta_0, \theta_1, \ldots, \theta_n \in R$.

7. Given that $|z - a| = a$, where z is a point in the Argand plane and $a \in R$. Show that $\frac{z-2a}{z} = i\tan\{\arg(z)\}$.

8. Find the real part (u) and imaginary part (v) of the complex number $(z-2)(\cos\theta - i\sin\theta) + (z-2)^{-1}(\cos\theta + i\sin\theta)$. Also, determine the locus represented by $v = 0$.

9. Locate the region represented by the complex number z in the Argand plane in each of the following :

(i) $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{|z| + 2}\right) < 2$

(ii) $\log_{\cos\pi/6}\left(\frac{|z-2|+5}{4|z-2|-4}\right) < 2$

(iii) $\log_{\sec\pi/4}\left(\frac{|z|^2 + |z| + 4}{2|z| - 1}\right) > 2$

(iv) $\log_{0.5}|z-2| > \log_{0.5}|z|$

(v) $\log_{1/3}|z+1| > \log_{1/3}|z-1|$

(vi) $\log_{1/2}\left(\frac{|z-1|+4}{|z-1|-2}\right) > 1$

(vii) $\left|\frac{z-5i}{z+5i}\right| = 1$

(viii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$

(ix) $\left|\frac{z-a}{z-\bar{a}}\right| = 1$, where a is a complex number such that $\mathrm{Re}(a) \ne 0$

(x) $\log_{\sin\pi/3}\left\{\frac{|z-2|+5}{4|z-2|-4}\right\} > 2$

(xi) $\log_{(1/3)}\left\{\frac{|z-3|^2+2}{11|z-3|-2}\right\} > 1$

10. If $z_1 = 3 + 3i$, $z_2 = 1 + i$ and $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{6}$, find the locus of the point representing z.

11. A triangle has vertices at the points represented by the complex numbers z_1, z_2, z_3. Show that the complex number associated to the orthocentre is
$$\frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$
or, $z_1\cot B\cot C + z_2\cot C\cot A + z_3\cot A\cot B$

12. If $\left|\log_{(1/\sqrt{3})}\left\{\frac{|z|^2 - |z| + 1}{2 + |z|}\right\}\right| < 2$, show that either $|z|$ lies within the circle $|z| = \frac{1}{3}$ or within the circular annulus bounded by $|z| = 1$ and $|z| = 5$.

13. Find $a \in R$ if at least one complex number z is to satisfy $|z+3| = a^2 - 2a + 6$ and $|z - 3\sqrt{3}i| < a^2$ simultaneously.

14. Locate the complex number $z = x + iy$ such that $|z - i| = 1$ and $\arg\left(\frac{z}{z+i}\right) = \frac{\pi}{2}$.

15. Among the complex number z which satisfy the condition $|z-25i| \le 15$, find the number having the least positive argument and greatest positive argument.

16. Prove that the area of the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argand plane is $\frac{\Sigma (z_2 - z_3) |z_1|^2}{4i z_1}$.

17. If z_1, z_2, z_3 be the complex numbers such that $z_1+z_2+z_3=0$ and $|z_1|+|z_2|+|z_3|=1$, prove that $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=1$

18. If complex numbers z_1, z_2, z_3 represent the vertices A, B, C of a right angled triangle with right angle at C, show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

19. If z_1, z_2, z_3 are non-zero complex numbers such that $z_1+z_2+z_3=0$ and $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=0$, then prove that z_1, z_2, z_3 represent the vertices of an equilateral triangle.

20. $ABCD$ is a rhombus described in the clockwise direction in the Argand plane. If the vertices A, B, C, D represent complex numbers z_1, z_2, z_3, z_4 respectively and $\angle CBA = 2\pi/3$, show that

$$2\sqrt{3}\, z_2 = (\sqrt{3} - i) z_1 + (\sqrt{3} + i) z_3$$

and, $2\sqrt{3}\, z_4 = (\sqrt{3} + i) z_1 + (\sqrt{3} - i) z_3$.

21. If the vertices of a square are z_1, z_2, z_3, z_4 taken in the anti-clockwise order, prove that $z_3 = -i z_1 + (1+i) z_2$ and $z_4 = (1-i) z_1 + i z_2$.

22. $ABCD$ is a rhombus in the Argand plane. If the vertices A, B, C, D represent complex numbers z_1, z_2, z_3, z_4 respectively and $\angle CBA = \frac{\pi}{3}$, show that

$$2z_2 = (1 + i\sqrt{3}) z_1 + (1 - i\sqrt{3}) z_3$$

and,

$$2z_4 = (1 - i\sqrt{3}) z_1 + (1 + i\sqrt{3}) z_3$$

23. If $z = x + iy$ and $w = \frac{1 - zi}{z - i}$, $|w| = 1$, find the locus of z

24. Prove that any complex number z of unit modulus can be expressed in the form $\frac{a+i}{a-i}$, where $a \in R$

25. A regular octagon is inscribed in a circle with centre at $3 + \sqrt{2}i$. If $5 + \sqrt{2}i$ is one of the vertices, on a diagonal of maximum length, find the other vertices and the lengths of the diagonals.

26. In the Argand plane, the curve represented by $z^2 - 2z\bar{z} + (\bar{z})^2 + 8a(z + \bar{z}) = 0$ passes through $1 + 6i$. Find a. Examine whether it passes through the points $2 + 6\sqrt{2}i$ and $-5 + 6\sqrt{5}i$. Identify the curve.

27. If $x + iy = be^{i(\beta+\theta)} + ce^{i(\gamma-\theta)}$, where b, c, β, γ are real constants and x, y, θ are real variables, prove that as θ varies the point (x, y) describes a conic section.

28. If $z_1^2 + z_2^2 - 2z_1 z_2 \cos\theta = 0$, then prove that the origin, z_1, z_2 form the vertices of an isosceles triangles with vertical angle θ.

29. Show that the point a' is the reflectin of the point a in the line $z\bar{b} + \bar{z}b + c = 0$ if $a'\bar{b} + \bar{a}b + c = 0$

30. If $z = w + \frac{1}{w}$ and w is any point on a fixed circle with centre at the origin, then prove that z lies on an ellipse.

31. If two consective vertices of a regular hexagon be z and $\bar{z}$, find the other vertices if the centre is at the origin.

ANSWERS

3. If $\theta = 2n\pi$, $|z_1| = 0$, $\arg(z_1)$ is undefined.

If $4n\pi < \theta < (4n+2)\pi, n \in z$, then $|z_1| = 2\sin\frac{\theta}{2}$, $\arg z_1 = \frac{\pi + 3\theta}{2}$

If $(4n+2)\pi < \theta < (4n+4)\pi, n \in Z$, then $|z_1| = 2\sin\frac{\theta}{2}$, $\arg(z_1) = \frac{3\pi + 3\theta}{2}$.

8. $u = X\left(1 + \frac{1}{X^2 + Y^2}\right), v = \left(1 - \frac{1}{X^2 + Y^2}\right)$,

where $x + iy = (z - 2)(\cos\theta - i\sin\theta)$

Locus represented by $v = 0$ is either straight lines or circles of unit radius.

9. (i) Interior of the circle $|z| = 5$.

(ii) Circularannulus $1 < |z-2| < 4$ i.e. the region lying in side the circle $|z-2| = 4$ and outside the circle $|z-2| = 1$.

(iii) Region outside the circle $|z-0| = \frac{1}{2}$.

(iv) z lies on the right side of $x = 1$.

(v) z lies on the LHS of imaginary axis.

(vi) There is no z for which the inequality exists.

(vii) Real axis

(viii) Major arcs of the circles passing through (1, 0) and (−1, 0).

(ix) y–axis.

(x) Region outside the circle whose centre is at (2, 0) and radius is 4.

(xi) Circular annulus $1 < |z-3| < 8/3$.

10. Major segment of $\left| z - \{(2-\sqrt{3}) + i(2+\sqrt{3})\} \right| = 2\sqrt{2}$ angle $\pi/6$.

13. $(1, \infty)$

25. Length of the diagonals : $2\sqrt{2}, \sqrt{8+2\sqrt{2}}, 4$

Vertices: $5+\sqrt{2}i, 3+\sqrt{2}+2\sqrt{2}\,i, 3+(2+\sqrt{2})\,i, 3-\sqrt{2}+2\sqrt{2}i$, $1+\sqrt{2}i, 3-\sqrt{2}, 3+(\sqrt{2}-2)\,i$ and $3+\sqrt{2}$

26. $a = 9$, Parabola

31. $i\,|z|, -\bar{z}, -z, -|z|$

HINTS TO SELECTED PROBLEMS

9. (iv) $\log_{0.5}|z-2| > \log_{0.5}|z-0|$

$\Rightarrow |z-2| < |z-0|$

$\Rightarrow$ z lies on the right side of the perpendicular bisector of the segment joining (0, 0) and (2, 0).

(vi) Given inequality holds, if $|z-1|-2>0$ i.e. $|z-1|>2$. When $|z-1|>2$ we have

$\frac{|z-1|+4}{|z-1|-2} < \frac{1}{2} \Rightarrow |z-1| < -10$, which is not possible.

13. First find the values of a for which the two circles never cut each other i.e. $C_1C_2 > r_1 + r_2$, Required set of values of a is the complement of the set of values of a satisfying $C_1C_2 > r_1 + r_2$.

17. Take $z_i = \cos\theta_i + i\sin\theta_i$.

23. $|w| = 1$

$\Rightarrow |z-i| = |1-2i|$

$\Rightarrow |z-(\theta+i)| = |z+(0+i)|$

$\Rightarrow$ Locus of z is the perpendicular bisector of the segment $(0,-1)$ and $(0, 1)$ i.e. the real axis.

26. The equation can be written as

$$\left(\frac{z-\bar{z}}{2i}\right)^2 = 4a\left(\frac{z+\bar{z}}{2}\right) \text{ or, } (\text{Im } z)^2 = 4a\,\text{Re}(z)]$$

—❑❑❑—

19.32 De' MOIVERE'S THEOREM

STATEMENT (i) *If $n \in Z$ (the set of integers), then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) *If $n \in Q$ (the set of rational numbers), then*

$\cos n\theta + i\sin n\theta$ *is one of the values of* $(\cos\theta + i\sin\theta)^n$.

PROOF (i) We divide the proof into the following three cases:

CASE I *When $n = 0$.*

In this case, we have

$(\cos\theta + i\sin\theta)^n$

$= (\cos\theta + i\sin\theta)^0$

$= 1 = \cos(0.\theta) + i\sin(0.\theta)$

$= \cos n\theta + i\sin n\theta$ $[\because n = 0]$

So, the theorem holds good for $n = 0$.

CASE II *When $n \in N$ i.e. n is a positive integer*

In this case, we shall prove the theorem by induction on n.

The theorem is true for $n = 1$, because

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos(1.\theta) + i\sin(1.\theta)$

Let us assume that the theorem is true for $n = m$. Then,

$$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta \quad \ldots(a)$$

Now, we shall show that the theorem is true for $n = m + 1$. We have,

$(\cos\theta + i\sin\theta)^{m+1}$

$= (\cos\theta + i\sin\theta)^m (\cos\theta + i\sin\theta)$

$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$ [Using (a)]

$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)$

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta)$

$= \cos(m+1)\theta + i\sin(m+1)\theta$

So, the theorem holds good for $n = m + 1$.

Hence, by induction the theorem holds good for all $n \in N$.

CASE III *When n is a negative integer.*

Let $n = -m$, where $m \in N$. Then,

$(\cos\theta + i\sin\theta)^n$

$= (\cos\theta + i\sin\theta)^{-m}$

$= \frac{1}{(\cos\theta + i\sin\theta)^m}$

$= \frac{1}{\cos m\theta + i\sin m\theta}$ [By case II]

$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$

$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$

$= \cos(-m)\theta + i\sin(-m)\theta$

$= \cos n\theta + i\sin n\theta$ $[\because -m = n]$

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in Z$.

(ii) Let $n = \frac{p}{q}$, where p, q are integers and $q > 0$.

From part (i), we have

$$\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)^q$$
$$=\cos\left\{\left(\frac{p\theta}{q}\right)q\right\}+i\sin\left\{\left(\frac{p\theta}{q}\right)q\right\}$$
$$=\cos p\theta+i\sin p\theta$$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos p\theta+i\sin p\theta)^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $[(\cos\theta+i\sin\theta)^p]^{1/q}$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta+i\sin\theta)^{p/q}$

REMARK 1 *The theorem is also true for* $(\cos\theta-i\sin\theta)$ *i.e.*

$$(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta, \textit{ because}$$
$$(\cos\theta-i\sin\theta)^n=[\cos(-\theta)+i\sin(-\theta)]^n$$
$$=\cos(n(-\theta)+i\sin(n(-\theta))$$
$$=\cos(-n\theta)+i\sin(-n\theta)$$
$$=\cos n\theta-i\sin n\theta$$

REMARK 2 $\dfrac{1}{\cos\theta+i\sin\theta}=(\cos\theta+i\sin\theta)^{-1}=\cos\theta-i\sin\theta$

NOTE 1. $(\sin\theta\pm i\cos\theta)^n\neq\sin n\theta\pm i\cos n\theta$

NOTE 2 $(\sin\theta+i\cos\theta)^n=[\cos(\pi/2-\theta)+i\sin(\pi/2-\theta)]^n$
$$=\cos(n\pi/2-n\theta)+i\sin(n\pi/2-n\theta)$$

NOTE 3. $(\cos\theta+i\sin\phi)^n\neq\cos n\theta+i\sin n\phi$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 Simplify $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$.

SOLUTION

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$$
$$=\left\{\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)}\right\}^n$$
$$=\left\{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right\}^n$$
$$=\left\{\frac{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right\}^n$$
$$=\frac{\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n}{\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n}$$
$$=\frac{\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n}{\left[\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^{-1}\right]^n}$$
$$=\frac{\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n}{\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^{-n}}$$
$$=\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^{2n}$$
$$=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

EXAMPLE 2 *If* $x+(1/x)=2\cos\theta$ and $y+(1/y)=2\cos\phi$ *etc, then prove that*

(i) $xyz\ldots+\dfrac{1}{xyz}\ldots=2\cos(\theta+\phi+\ldots)$

(ii) $\dfrac{x}{y}+\dfrac{y}{x}=2\cos(\theta-\phi)$

(iii) $x^m y^n+\dfrac{1}{x^m y^n}=2\cos(m\theta+n\phi)$

(iv) $\dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}=2\cos(m\theta-n\phi)$

SOLUTION We have, $x+\dfrac{1}{x}=2\cos\theta, y+\dfrac{1}{y}=2\cos\phi$ etc.

Now,

$$x+\frac{1}{x}=2\cos\theta$$
$$\Rightarrow x^2-2x\cos\theta+1=0$$
$$\Rightarrow x=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

$\Rightarrow x = \cos\theta + i\sin\theta$, taking +ve sign only.

Similarly $y = \cos\phi + i\sin\phi$ etc.

(i) $xyz\ldots = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\ldots$

$= \cos(\theta+\phi+\ldots) + i\sin(\theta+\phi+\ldots)$

$$\therefore \quad \frac{1}{xyz} = \frac{1}{\cos(\theta+\phi+\ldots)+i\sin(\theta+\phi+\ldots)}$$

$= \cos(\theta+\phi+\ldots) - i\sin(\theta+\phi+\ldots)$

So, $xyz\ldots + \dfrac{1}{xyz\ldots} = 2\cos(\theta+\phi+\ldots)$

(ii) $\dfrac{x}{y} + \dfrac{y}{x}$

$$= \frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi} + \frac{\cos\phi + i\sin\phi}{\cos\theta + i\sin\theta}$$

$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1}$

$+ (\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta)^{-1}$

$= (\cos\theta + i\sin\theta)\{\cos(-\phi) + i\sin(-\phi)\}$

$+ (\cos\phi + i\sin\phi)\{\cos(-\theta) + i\sin(-\theta)\}$

$= \cos(\theta-\phi) + i\sin(\theta-\phi) + \cos(\phi-\theta) + i\sin(\phi-\theta)$

$= \cos(\theta-\phi) + i\sin(\theta-\phi) + \cos(\theta-\phi) - i\sin(\theta-\phi)$

$= 2\cos(\theta-\phi)$

(iii) We have,

$x^m y^n$

$= (\cos\theta + i\sin\theta)^m (\cos\phi + i\sin\phi)^n$

$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

$$\therefore \quad \frac{1}{x^m y^n}$$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$= \cos(m\theta + n\phi) - i\sin(m\theta + n\theta)$

So, $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

(iv) We have,

$$\frac{x^m}{y^n} = \frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n}$$

$= (\cos\theta + i\sin\theta)^m (\cos\phi + i\sin\phi)^{-n}$

$= (\cos m\theta + i\sin m\theta)[\cos(-n\phi) + i\sin(-n\phi)]$

$= \cos(m\theta - n\phi) + i\sin(m\theta - n\phi)$

$$\therefore \quad \frac{y^n}{x^m} = [\cos(m\theta - n\phi) + i\sin(m\theta - n\phi)]^{-1}$$

$= \cos(m\theta - n\phi) - i\sin(m\theta - n\theta)$.

So, $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

EXAMPLE 3 *Prove that :*

$$(1+i)^n + (1-i)^n = 2^{(n/2)+1}\cos\left(\frac{n\pi}{4}\right)$$

SOLUTION Let $1 + i = r(\cos\theta + i\sin\theta)$. Then,

$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

and, $\tan\theta = \dfrac{1}{1} \Rightarrow \theta = \dfrac{\pi}{4}$

$$\therefore \quad 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \quad (1+i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$\Rightarrow \quad (1+i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \qquad \ldots(i)$$

Let the polar form of $1 - i$ be $r(\cos\theta + i\sin\theta)$. Then,

$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and, $\tan\theta = \left(\dfrac{-1}{1}\right) \Rightarrow \theta = -\dfrac{\pi}{4}$.

$$\therefore \quad 1 - i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \quad (1-i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$$

$$\Rightarrow \quad (1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \qquad \ldots(ii)$$

Adding (i) and (ii), we obtain

$(1+i)^n + (1-i)^n$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= 2^{n/2} \cdot 2\cos\frac{n\pi}{4}$$

$$= 2^{(n/2)+1}\cos\frac{n\pi}{4}.$$

EXAMPLE 4 *If n is a positive integer, prove that*

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$$

SOLUTION Let $r(\cos\theta + i\sin\theta)$ be the polar form of $\sqrt{3} + i$. Then,

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ and $\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \dfrac{\pi}{6}$.

$$\therefore \quad \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow (\sqrt{3}+i)^n = 2^n\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n$$

$$\Rightarrow (\sqrt{3}+i)^n = 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right) \quad ...(i)$$

The polar form of $(\sqrt{3}-i)$ is

$$2\left\{\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right\} = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right).$$

$$\therefore (\sqrt{3}-i)^n = 2^n\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^n$$

$$\Rightarrow (\sqrt{3}-i)^n = 2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right) \quad ...(ii)$$

Adding (i) and (ii), we obtain

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$$

$$= 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right) + 2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right)$$

$$= 2^n\left(2\cos\frac{n\pi}{6}\right) = 2^{n+1}\cos\frac{n\pi}{6}$$

EXAMPLE 5 *Let $r, \theta)$ denote the point $r(\cos\theta + i\sin\theta)$ in the Argand plane. If $a \equiv (1, \alpha), b \equiv (1, \beta), c \equiv (1, \gamma)$ and $a+b+c=0$, show that $a^{-1}+b^{-1}+c^{-1}=0$.*

SOLUTION Since $(r, \theta) \equiv r(\cos\theta + i\sin\theta)$. Therefore,

$$a \equiv (1, \alpha) = 1(\cos\alpha + i\sin\alpha),$$

$$b \equiv (1, \beta) = 1.(\cos\beta + i\sin\beta)$$

and, $c \equiv (1, \gamma) = 1.(\cos\gamma + i\sin\gamma)$.

We have,

$$a+b+c=0$$

$$\Rightarrow (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$$

$$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$$

$$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0 \text{ and } \sin\alpha + \sin\beta + \sin\gamma = 0 \quad ..(i)$$

Now,

$$a^{-1}+b^{-1}+c^{-1}$$

$$= (\cos\alpha + i\sin\alpha)^{-1} + (\cos\gamma + i\sin\gamma)^{-1} + (\cos\beta + i\sin\beta)^{-1}$$

$$= [\cos(-1)\alpha + i\sin(-1)\alpha] + [\cos(-1)\beta + i\sin(-1)\beta] + [\cos(-1)\gamma + i\sin(-1)\gamma]$$

$$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 - i0 = 0 \qquad \text{[Using (i)]}$$

EXAMPLE 6 *If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, prove that*

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(iv) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

SOLUTION Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and, $c = \cos\gamma + i\sin\gamma$. Then,

$$a+b+c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= 0 + i0 = 0.$$

(i) Since $a+b+c=0$. Therefore,

$$a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$$
$$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

$$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

(iii) We have,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 - i0 = 0$$

$$\Rightarrow ab + bc + ca = 0$$

$$\therefore (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$$

$$\Rightarrow 0 = a^2 + b^2 + c^2 + 0$$

$$\Rightarrow a^2 + b^2 + c^2 = 0$$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$

$$\Rightarrow (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0$$

$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i0$$

$$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

and, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

EXAMPLE 7 *If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$, prove that $x_1 . x_2 . x_3 \ldots$ to infinity $= -1$.*

SOLUTION $x_1 . x_2 . x_3 \ldots$ to infinity

$$= \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3}+i\sin\frac{\pi}{2^3}\right)\ldots\infty$$

$$= \cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\ldots\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\ldots\right)$$

$$= \cos\left(\frac{\pi/2}{1-1/2}\right) + i\sin\left(\frac{\pi/2}{1-1/2}\right) \qquad \left[\because a + ar + ar^2 + \ldots = \frac{a}{1-r}\right]$$

$= \sqrt{(u^2 - v^2)^2 + 1 - 2(u^2 - v^2) + 4u^2 v^2}$

EXAMPLE 8 *If $z = x + iy$ is a complex number with rational x and y and $|z| = 1$, show that $|z^{2n} - 1|$ is a rational number for every $n \in N$.*

SOLUTION Let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of z.

We have,

$|z| = 1.$

$\Rightarrow \quad z = \cos\theta + i\sin\theta$

Also, $z = x + iy$, where $x, y \in Q$.

$\therefore \; x = \cos\theta$ and $y = \sin\theta$

Since x and y are rational numbers. Therefore, $\cos\theta$ and $\sin\theta$ are also rational numbers.

Let $z^n = u + iv$.

Using De'Moivere's theorem, we have

$\Rightarrow z^n = (\cos\theta + i\sin\theta)^n$

$\Rightarrow z^n = \cos n\theta + i\sin n\theta$

$\Rightarrow u + iv = \cos n\theta + i\sin n\theta$

$\Rightarrow u = \cos n\theta$ and $v = \sin n\theta$

Now,

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$\Rightarrow \cos n\theta + i\sin n\theta$

$= {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta(i\sin\theta) + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2 + \ldots + {}^nC_n\cos^0\theta(i\sin\theta)^n$

$\Rightarrow \cos n\theta + i\sin n\theta$

$= ({}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta \ldots) + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta \ldots)$

$\Rightarrow \cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots$

and,

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots$

$\Rightarrow \quad u = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots.$

and,

$v = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots.$

$\Rightarrow u$ and v are rationals $\quad [\because \cos\theta$ and $\sin\theta$ are rationals$]$

Now,

$|z^{2n} - 1|$

$= |(z^n)^2 - 1|$

$= |(u + iv)^2 - 1|$

$= |(u^2 - v^2 - 1) + 2iuv|$

$= \sqrt{(u^2 - v^2 - 1)^2 + 4u^2v^2}$

$= \cos\pi + i\sin\pi = -1$

$= \sqrt{(u^2 + v^2)^2 + 1 - 2u^2 + 2v^2}$

$= \sqrt{1 + 1 - 2u^2 + 2v^2} \qquad [\because u^2 + v^2 = \cos^2 n\theta + \sin^2 n\theta = 1]$

$= \sqrt{2 - 2u^2 + 2v^2}$

$= \sqrt{2(u^2 + v^2) - 2u^2 + 2v^2} \qquad [\because u^2 + v^2 = 1]$

$= \sqrt{4v^2}$

$= 2|v|$

Since $v \in Q$. Therefore,

$|z^{2n} - 1| = 2|v| \in Q$

EXAMPLE 9 *If α and β are the roots of*

$$z + \frac{1}{z} = 2(\cos\theta + i\sin\theta),\ 0 < \theta < \pi,\ \textit{show that}$$

$|\alpha - i| = |\beta - i|$.

SOLUTION We have,

$z + \frac{1}{z} = 2(\cos\theta + i\sin\theta)$

$\Rightarrow z + \frac{1}{z} = 2e^{i\theta} \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$

$\Rightarrow z^2 - 2ze^{i\theta} + 1 = 0$

$\Rightarrow z = \frac{2e^{i\theta} \pm \sqrt{4e^{i2\theta} - 4}}{2}$

$\Rightarrow z = e^{i\theta} \pm \sqrt{e^{i2\theta} - 1}$

Let $\alpha = e^{i\theta} + \sqrt{e^{i2\theta} - 1}$ and $\beta = e^{i\theta} - \sqrt{e^{i2\theta} - 1}$

Now,

$e^{i2\theta} - 1 = \cos 2\theta + i\sin 2\theta - 1$

$= (\cos 2\theta - 1) + i\sin 2\theta$

$= -2\sin^2\theta + 2i\sin\theta\cos\theta$

$= 2\sin\theta(-\sin\theta + i\cos\theta)$

$= 2\sin\theta\left\{\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right\}$

$\Rightarrow \sqrt{e^{i2\theta} - 1} = \sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right\}^{1/2}$

$= \sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$

$\therefore \; \alpha = e^{i\theta} + \sqrt{e^{i2\theta} - 1}$

$= (\cos\theta + i\sin\theta) + \sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$

$= \left\{\cos\theta + \sqrt{2\sin\theta}\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\} + i\left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$

$$\Rightarrow \alpha - i = \left\{\cos\theta + \sqrt{2\sin\theta}\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\} + i\left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - 1\right\}$$

$$\Rightarrow |\alpha - i|^2 = \left\{\cos\theta + \sqrt{2\sin\theta}\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}^2 + \left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - 1\right\}^2$$

$$\Rightarrow |\alpha - i|^2$$

$$= (\cos^2\theta + \sin^2\theta) + 2\sin\theta\left\{\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\} + 1 + 2\sqrt{2\sin\theta}\left\{\cos\theta\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \sin\theta\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\} - 2\sin\theta - 2\sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= 2 + 2\sin\theta + 2\sqrt{2\sin\theta}\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) - 2\sin\theta - 2\sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= 2 + 2\sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) - \sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= 2 + 2\sqrt{2\sin\theta}\left\{\sin\left\{\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} - \sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= 2 + 2\sqrt{2\sin\theta}\left\{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= 2.$$

Similarly, we have

$$|\beta - i|^2 = 2$$

Hence, $|\alpha - i|^2 = |\beta - i|^2 = 2$

$$\Rightarrow \quad |\alpha - i| = |\beta - i|$$

EXAMPLE 10 *Form an equation whose roots are*

$$\sin^2\frac{\pi}{2n+1}, \sin^2\frac{2\pi}{2n+1}, \sin^2\frac{3\pi}{2n+1}, \ldots, \sin^2\frac{n\pi}{2n+1}$$

SOLUTION By the binomial theorem, we have

$$(\cos\theta + i\sin\theta)^{2n+1}$$

$$= {}^{2n+1}C_0\cos^{2n+1}\theta + {}^{2n+1}C_1\cos^{2n}\theta\,(i\sin\theta) + {}^{2n+1}C_2\cos^{2n-1}\theta\,(i\sin\theta)^2 + {}^{2n+1}C_3\cos^{2n-2}\theta\,(i\sin\theta)^3 + \ldots + {}^{2n+1}C_{2n+1}(i\sin\theta)^{2n+1}$$

$$\Rightarrow \cos(2n+1)\theta + i\sin(2n+1)\theta$$

$$= \left\{{}^{2n+1}C_0\cos^{2n+1}\theta - {}^{2n+1}C_2\cos^{2n-1}\theta\sin^2\theta + {}^{2n+1}C_4\cos^{2n-3}\theta\sin^4\theta + \ldots\right\} + i\left\{{}^{2n+1}C_1\cos^{2n}\theta\sin\theta - {}^{2n+1}C_3\cos^{2n-2}\theta\sin^3\theta + \ldots\right\}$$

On equating imaginary parts, we get

$$\sin(2n+1)\theta = {}^{2n+1}C_1\cos^{2n}\theta\sin\theta - {}^{2n+1}C_3\cos^{2n-2}\theta\sin^3\theta + {}^{2n+1}C_5\cos^{2n-4}\theta\sin^5\theta + \ldots + (-1)^n\,{}^{2n+1}C_{2n+1}\sin^{2n+1}\theta$$

$$\Rightarrow \sin(2n+1)\theta = {}^{2n+1}C_1(1-\sin^2\theta)^n\sin\theta - {}^{2n+1}C_3(1-\sin^2\theta)^{n-1}\sin^3\theta + \ldots + (-1)^n\,{}^{2n+1}C_{2n+1}\sin^{2n+1}\theta$$

$$\Rightarrow \frac{\sin(2n+1)\theta}{\sin\theta} = {}^{2n+1}C_1(1-\sin^2\theta)^n - {}^{2n+1}C_3(1-\sin^2\theta)^{n-1}\sin^2\theta + \ldots + (-1)^n(\sin^2\theta)^n$$

We observe that LHS = 0 for $\theta = \frac{\pi}{2n+1}, \frac{2\pi}{2n+1}, \ldots, \frac{n\pi}{2n+1}$

Therefore,

$\sin^2\frac{\pi}{2n+1}, \sin^2\frac{2\pi}{2n+1}, \ldots, \sin^2\frac{n\pi}{2n+1}$ are the roots of the equation

$${}^{2n+1}C_1(1-x)^n - {}^{2n+1}C_3(1-x)^{n-1}x + {}^{2n+1}C_5(1-x)^{n-2}x^2 + \ldots + (-1)^n x^n = 0$$

EXAMPLE 11 *Form an equation whose roots are*

$$\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \ldots, \cot^2\frac{n\pi}{2n+1}$$

Hence or otherwise find the values of

(i) $\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \ldots + \cot^2\frac{n\pi}{2n+1}$

(ii) $\operatorname{cosec}^2\frac{\pi}{2n+1} + \operatorname{cosec}^2\frac{2\pi}{2n+1} + \ldots + \operatorname{cosec}^2\frac{n\pi}{2n+1}$

SOLUTION Proceeding as in Ex 10, we obtain

$$\sin(2n+1)\theta = {}^{2n+1}C_1\cos^{2n}\theta\sin\theta - {}^{2n+1}C_3\cos^{2n-2}\theta\sin^3\theta + {}^{2n+1}C_5\cos^{2n-4}\theta\sin^5\theta + \ldots + (-1)^n\,{}^{2n+1}C_{2n+1}\sin^{2n+1}\theta$$

$$\Rightarrow \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = {}^{2n+1}C_1\cot^{2n}\theta - {}^{2n+1}C_3\cot^{2n-2}\theta + {}^{2n+1}C_5\cot^{2n-4}\theta \ldots + (-1)^n\,{}^{2n+1}C_{2n+1}$$

We observe that LHS = 0 for

$$\theta = \frac{\pi}{2n+1}, \frac{2\pi}{2n+1}, \ldots, \frac{n\pi}{2n+1}$$

Therefore,

$\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \cot^2\frac{3\pi}{2n+1} \ldots, \cot^2\frac{n\pi}{2n+1}$ are roots of the equation.

$${}^{2n+1}C_1x^n - {}^{2n+1}C_3x^{n-1} + {}^{2n+1}C_5x^{n-2} + \ldots + (-1)^n\,{}^{2n+1}C_{2n+1} = 0$$

Now,

$$\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \ldots + \cot^2\frac{n\pi}{2n+1}$$

= Sum of the roots

$$= \frac{^{2n+1}C_3}{^{2n+1}C_1}$$

$$= \frac{n(2n-1)}{3}$$

$$\therefore \quad \operatorname{cosec}^2\frac{\pi}{2n+1} + \operatorname{cosec}^2\frac{2\pi}{2n+1} + \operatorname{cosec}^2\frac{3\pi}{2n+1} + \ldots + \operatorname{cosec}^2\frac{n\pi}{2n+1}$$

$$= \left(1+\cot^2\frac{\pi}{2n+1}\right) + \left(1+\cot^2\frac{2\pi}{2n+1}\right) + \left(1+\cot^2\frac{3\pi}{2n+1}\right) + \ldots + \left(1+\cot^2\frac{n\pi}{2n+1}\right)$$

$$= n + \left(\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \ldots + \cot^2\frac{n\pi}{2n+1}\right)$$

$$= n + \frac{n(2n-1)}{3}$$

$$= \frac{2n(n+1)}{3}$$

EXAMPLE 12 *If α, β are the roots of the equation $t^2 - 2t + 2 = 0$ and*

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{(\sin\theta)^n}$$

then prove that $x = \cot\theta - 1$.

SOLUTION We have,

$$t^2 - 2t + 2 = 0$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

So, let $\alpha = 1 + i$ and $\beta = 1 - i$

$$\Rightarrow x + \alpha = (x+1)\,i \text{ and } x + \beta = (x+1) - i$$

Let $x + \alpha = r(\cos\phi + i\sin\phi)$. Then,

$$(x+1) + i = r(\cos\phi + i\sin\phi)$$

$$\Rightarrow r = \sqrt{(x+1)^2 + 1^2} \text{ and } \tan\phi = \frac{1}{x+1}.$$

$$\Rightarrow r = \sqrt{\cot^2\phi + 1} = \operatorname{cosec}\phi, \text{ where } \tan\phi = \frac{1}{x+1}.$$

Clearly, $x + \beta$ is conjugate of $x + \alpha$. Therefore,

$$x + \beta = r(\cos\phi - i\sin\phi)$$

Now,

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow \frac{r^n(\cos\phi + i\sin\phi)^n - r^n(\cos\phi - i\sin\phi)^n}{(1+i) - (1-i)} = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow \frac{r^n(\cos n\phi + i\sin n\phi) - r^n(\cos n\phi - i\sin n\phi)}{2i} = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow \frac{2i\,r^n \sin n\phi}{2i} = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow r^n \sin n\phi = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow (\operatorname{cosec}\phi)^n \sin n\phi = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow \frac{\sin n\phi}{(\sin\phi)^n} = \frac{\sin n\theta}{(\sin\theta)^n}$$

$$\Rightarrow \phi = \theta.$$

$$\therefore \tan\phi = \frac{1}{x+1}$$

$$\Rightarrow \tan\theta = \frac{1}{x+1}$$

$$\Rightarrow x + 1 = \cot\theta$$

$$\Rightarrow x = \cot\theta - 1$$

—❑❑❑—

19.33 ROOTS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then by De Moivere's theorem $r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$.

Here, we shall show that $z^{1/n}$ has n distinct values.

We know that

$$\cos\theta + i\sin\theta = \cos(2m\pi + \theta) + i\sin(2m\pi + \theta),\ m = 0, 1, 2, \ldots$$

So, $z^{1/n} = r^{1/n}[\cos(2m\pi + \theta) + i\sin(2m\pi + \theta)]^{1/n}$

$$= r^{1/n}\left[\cos\frac{2m\pi + \theta}{n} + i\sin\frac{2m\pi + \theta}{n}\right]$$

Now by giving m the values $0, 1, 2, \ldots, (n-1)$; we shall obtain distinct values of $z^{1/n}$.

For the values $m = n, n+1, \ldots$, the values of $z^{1/n}$ will repeat. For example, if $m = n$, then

$$z^{1/n} = r^{1/n}\left[\cos\frac{2n\pi + \theta}{n} + i\sin\frac{2n\pi + \theta}{n}\right]$$

$$= r^{1/n}\left[\cos\left(2\pi + \frac{\theta}{n}\right) + i\sin\left(2\pi + \frac{\theta}{n}\right)\right]$$

$$= r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

which is same as the value obtained by taking $m = 0$. On taking $m = n + 1$, the value comes out to be identical with corresponding to $m = 1$. Hence, $z^{1/n}$ has n distinct values.

In order to find the roots of a complex number by using De' Moivere's theorem, we follow the following algorithm.

ALGORITHM

STEP I *Write the given complex number in polar form.*

STEP II *Add* $2m\pi$ *to the argument.*

STEP III *Apply De Moivere's theorem.*

STEP IV *Put* $m = 0, 1, 2, \ldots, (n-1)$ *i.e. one less than the number in the denominator of the given index in the lowest form.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the value of* $\{1+\sqrt{-3}\}^{3/4}$.

SOLUTION Let $z = 1+\sqrt{-3} = 1+i\sqrt{3}$. Then,

$$|z| = \sqrt{1^2+(\sqrt{3})^2} = 2,$$

and,

$$\theta = \arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

So, polar form of z is $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$$\therefore \quad z = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$z = 2\left[\cos\left(2m\pi+\frac{\pi}{3}\right)+i\sin\left(2m\pi+\frac{\pi}{3}\right)\right]$$

$$\Rightarrow z^{3/4} = 2^{3/4}\left[\cos\left(2m\pi+\frac{\pi}{3}\right)+i\sin\left(2m\pi+\frac{\pi}{3}\right)\right]^{3/4}$$

$$\Rightarrow z^{3/4} = 2^{3/4}\left[\cos(6m+1)\frac{\pi}{4}+i\sin(6m+1)\frac{\pi}{4}\right],$$

$$m = 1, 2, 3.$$

For $m = 0$, we have

$$z^{3/4} = 2^{3/4}\left\{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right\}$$

$$= 2^{3/4}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = 2^{1/4}(1+i)$$

For $m = 1$, we have

$$z^{3/4} = 2^{3/4}\left\{\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right\}$$

$$= 2^{3/4}\left[\cos\left(2\pi-\frac{\pi}{4}\right)+i\sin\left(2\pi-\frac{\pi}{4}\right)\right]$$

$$= 2^{3/4}\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]$$

$$= 2^{1/4}(1-i)$$

For $m = 2$, we have

$$z^{3/4} = 2^{3/4}\left\{\cos\frac{13\pi}{4}+i\sin\frac{13\pi}{4}\right\}$$

$$= 2^{3/4}\left[\cos\left(3\pi+\frac{\pi}{4}\right)+i\sin\left(3\pi+\frac{\pi}{4}\right)\right]$$

$$= 2^{3/4}\left[-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]$$

$$= -2^{1/4}(1+i)$$

For $m = 3$, we have

$$z^{3/4} = 2^{3/4}\left[\cos\frac{19\pi}{4}+i\sin\frac{19\pi}{4}\right]$$

$$= 2^{3/4}\left[\cos\left(5\pi-\frac{\pi}{4}\right)+i\sin\left(5\pi-\frac{\pi}{4}\right)\right]$$

$$= 2^{3/4}\left[-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$$

$$= 2^{1/4}(-1+i)$$

Thus, the values of $(1+i\sqrt{3})^{3/4}$ are

$$\pm 2^{1/4}(1+i),\ \pm 2^{1/4}(1-i)$$

EXAMPLE 2 *Find* $\sqrt[3]{i}$.

SOLUTION Let $z = i$.

We have, $|z| = 1$ and $\arg(z) = \frac{\pi}{2}$.

So, z in polar form is $\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$.

Now,

$$z^{1/3} = \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{1/3}$$

$$= \left[\cos\left(2m\pi+\frac{\pi}{2}\right)+i\sin\left(2m\pi+\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \cos(4m+1)\frac{\pi}{6}+i\sin(4m+1)\frac{\pi}{6},\ m = 0, 1, 2.$$

For $m = 0$, we have

$$z^{1/3} = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = \frac{1}{2}(\sqrt{3}+i).$$

For $m = 1$, we have

$$z^{1/3} = \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}$$

$$= \cos\left(\pi - \frac{\pi}{6}\right) + i \sin\left(\pi - \frac{\pi}{6}\right)$$

$$= \left(-\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

$$= -\frac{1}{2}(\sqrt{3} - i)$$

For $m = 2$, we have

$$z^{1/3} = \cos\frac{9\pi}{6} + i \sin\frac{9\pi}{6} = \cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} = -i$$

Thus, the values of $\sqrt[3]{i}$ are $\frac{1}{2}(\sqrt{3} + i)$, $-\frac{1}{2}(\sqrt{3} - i)$ and $-i$.

EXAMPLE 3 *Find* $\sqrt[3]{-1}$

SOLUTION Let $z = -1$. Then, $|z| = 1$ and $\arg(z) = \pi$.

So, the polar form of z is $(\cos\pi + i \sin\pi)$.

Now,

$$z^{1/3} = (\cos\pi + i \sin\pi)^{1/3}$$

$$= [\cos(2m\pi + \pi) + i \sin(2m\pi + \pi)]^{1/3}$$

$$= \cos(2m + 1)\frac{\pi}{3} + i \sin(2m + 1)\frac{\pi}{3}, \quad m = 0, 1, 2$$

For $m = 0$, we have

$$z^{1/3} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2}(1 + i\sqrt{3}).$$

For $m = 1$, we have

$$z^{1/3} = \cos\pi + i \sin\pi = -1$$

For $m = 2$, we have

$$z^{1/3} = \cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}$$

$$= \cos\left(2\pi - \frac{\pi}{3}\right) + i \sin\left(2\pi - \frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{3} - i \sin\frac{\pi}{3}$$

$$= \frac{1}{2}(1 - i\sqrt{3})$$

Hence, the values of $z^{1/3}$ are $\frac{1}{2}(1 \pm i\sqrt{3})$, -1.

EXAMPLE 4 *Use De Moivere's theorem to solve the equation*

$$2\sqrt{2}\, z^4 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$$

SOLUTION We have,

$$2\sqrt{2}\, z^4 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$$

$$\Rightarrow \quad z^4 = \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

Let $r(\cos\theta + i \sin\theta)$ be the polar form of

$$\left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right).$$

Then,

$$r = \sqrt{\left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)^2} = 1$$

and,

$$\tan\theta = \frac{(\sqrt{3} + 1)/2\sqrt{2}}{(\sqrt{3} - 1)/2\sqrt{2}}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$$

$$= \frac{1 + \tan\pi/6}{1 - \tan\pi/6} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12}$$

$$\Rightarrow \theta = \frac{5\pi}{12}$$

$$\therefore \quad \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) = \cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}$$

$$\Rightarrow \quad z^4 = \cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}$$

$$\Rightarrow \quad z^4 = \cos\left(2r\pi + \frac{5\pi}{12}\right) + i \sin\left(2r\pi + \frac{5\pi}{12}\right), r \in Z$$

$$\Rightarrow \quad z = \left\{\cos(24r + 5)\frac{\pi}{12} + i \sin(24r + 5)\frac{\pi}{12}\right\}^{1/4}, r \in Z$$

$$\Rightarrow \quad z = \cos(24r + 5)\frac{\pi}{48} + i \sin(24r + 5)\frac{\pi}{48}, r = 0, 1, 2, 3.$$

For $r = 0$, we have

$$z = \cos\frac{5\pi}{48} + i \sin\frac{5\pi}{48}$$

For $r = 1$, we have

$$z = \cos\frac{29\pi}{48} + i \sin\frac{29\pi}{48}$$

$$= \cos\left(\pi - \frac{19\pi}{48}\right) + i \sin\left(\pi - \frac{19\pi}{48}\right)$$

$$= -\cos\frac{19\pi}{48} + i \sin\frac{19\pi}{48}$$

$$= -\left(\cos\frac{19\pi}{48} - i \sin\frac{19\pi}{48}\right)$$

For $r = 2$, we have

$$z = \cos\frac{53\pi}{48} + i \sin\frac{53\pi}{48}$$

$$= \cos\left(\pi + \frac{5\pi}{48}\right) + i \sin\left(\pi + \frac{5\pi}{48}\right)$$

$$= -\cos\frac{5\pi}{48} - i\sin\frac{5\pi}{48}$$

$$= -\left(\cos\frac{5\pi}{48} + i\sin\frac{5\pi}{48}\right)$$

For $r = 3$, we have

$$z = \cos\frac{77\pi}{48} + i\sin\frac{77\pi}{48}$$

$$\Rightarrow z = \cos\left(2\pi - \frac{19\pi}{48}\right) + i\sin\left(2\pi - \frac{19\pi}{48}\right)$$

$$\Rightarrow z = \cos\frac{19\pi}{48} - i\sin\frac{19\pi}{48}$$

Hence, the roots of the given equation are

$$\pm\left(\cos\frac{5\pi}{48} + i\sin\frac{5\pi}{48}\right) \text{ and } \pm\left(\cos\frac{19\pi}{48} - i\sin\frac{19\pi}{48}\right)$$

EXAMPLE 5 *Find all the roots of the equation* $(3z-1)^4 + (z-2)^4 = 0$ *in the simplified form of* $a + ib$.

SOLUTION We have,

$$(3z-1)^4 + (z-2)^4 = 0$$

$$\Rightarrow (3z-1)^4 = -(z-2)^4$$

$$\Rightarrow \left(\frac{3z-1}{z-2}\right)^4 = -1$$

$$\Rightarrow \left(\frac{3z-1}{z-2}\right) = (-1)^{1/4} = (\cos\pi + \sin\pi)^{1/4}$$

$$\Rightarrow \frac{3z-1}{z-2}\,[\cos(2r\pi+\pi) + i\sin(2r\pi+\pi)]^{1/4},\ r \in Z$$

$$\Rightarrow \frac{3z-1}{z-2} = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4},\ r = 0, 1, 2, 3$$

$$\Rightarrow \frac{3(z-2)+5}{z-2} = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4}$$

$$\Rightarrow 3 + \frac{5}{z-2} = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4}$$

$$\Rightarrow \frac{5}{z-2} = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4} - 3$$

$$\Rightarrow z - 2 = \frac{5}{\cos(2r+1)\pi/4 + i\sin(2r+1)\pi/4 - 3}$$

$$\Rightarrow z = \frac{5 + 2\cos(2r+1)\pi/4 + 2i\sin(2r+1)\pi/4 - 6}{\cos(2r+1)\pi/4 + i\sin(2r+1)\pi/4 - 3}$$

$$\Rightarrow z = \frac{2\cos(2r+1)\pi/4 + 2i\sin(2r+1)\pi/4 - 1}{\cos(2r+1)\pi/4 + i\sin(2r+1)\pi/4 - 3},\ r = 0, 1, 2, 3.$$

$$\Rightarrow z = \frac{2e^{i\theta} - 1}{e^{i\theta} - 3}, \text{ where } \theta = (2r+1)\pi/4,\ r = 0, 1, 2, 3.$$

$$\Rightarrow z = \frac{(2e^{i\theta} - 1)(e^{-i\theta} - 3)}{(e^{i\theta} - 3)(e^{-i\theta} - 3)}$$

$$\Rightarrow z = \frac{2 - e^{-i\theta} - 6e^{i\theta} + 3}{1 + 9 - 3(e^{i\theta} + e^{-i\theta})}$$

$$\Rightarrow z = \frac{5 - e^{-i\theta} - 6e^{i\theta}}{1 + 9 - 3 \times 2\cos\theta}$$

$$\Rightarrow z = \frac{5 - (\cos\theta - i\sin\theta) - 6(\cos\theta + i\sin\theta)}{10 - 6\cos\theta}$$

$$\Rightarrow z = \frac{5 - 7\cos\theta}{10 - 6\cos\theta} - \frac{5i\sin\theta}{10 - 6\cos\theta}$$

$$\Rightarrow z = \frac{5 - 7\cos(2r+1)\pi/4}{10 - 6\cos(2r+1)\pi/4} - \frac{5i\sin(2r+1)\pi/4}{10 - 6\cos(2r+1)\pi/4}, \text{ where}$$

$r = 0, 1, 2, 3$

EXAMPLE 6 *Show that the roots of* $(x+1)^6 + (x-1)^6 = 0$ *are* $\pm i\cot(2r+1)\pi/12$, $r = 0, 1, 2$, *and deduce that*

$$\sin\frac{\pi}{12}\sin\frac{3\pi}{12}\sin\frac{5\pi}{12} = \frac{1}{4\sqrt{2}}.$$

SOLUTION We have,

$$(x+1)^6 + (x-1)^6 = 0$$

$$\Rightarrow (x-1)^6 = -(x+1)^6$$

$$\Rightarrow \left(\frac{x-1}{x+1}\right)^6 = -1$$

$$\Rightarrow z^6 = -1, \text{ where } z = \frac{x-1}{x+1}$$

$$\Rightarrow z = (-1)^{1/6}$$

$$\Rightarrow z = (\cos\pi + i\sin\pi)^{1/6}$$

$$\Rightarrow z = [\cos(2r\pi+\pi) + i\sin(2r\pi+\pi)]^{1/6},\ r \in Z$$

$$\Rightarrow z = \cos(2r+1)\frac{\pi}{6} + i\sin(2r+1)\frac{\pi}{6},\ r = 0, 1, 2, 3, 4, 5$$

$$\Rightarrow \frac{x-1}{x+1} = \cos(2r+1)\frac{\pi}{6} + i\sin(2r+1)\frac{\pi}{6}$$

$$\Rightarrow \frac{(x-1)+(x+1)}{(x-1)-(x+1)} = \frac{\cos(2r+1)\pi/6 + 1 + i\sin(2r+1)\pi/6}{\cos(2r+1)\pi/6 - 1 + i\sin(2r+1)\pi/6}$$

$$\Rightarrow -x = \frac{2\cos^2(2r+1)\pi/12 + 2i\sin(2r+1)\frac{\pi}{12}\cos(2r+1)\frac{\pi}{12}}{-2\sin^2(2r+1)\frac{\pi}{12} + 2i\sin(2r+1)\frac{\pi}{12}\cos(2r+1)\frac{\pi}{12}}$$

$$\Rightarrow -x = \frac{2\cos(2r+1)\frac{\pi}{12}\left\{\cos(2r+1)\frac{\pi}{12} + i\sin(2r+1)\frac{\pi}{12}\right\}}{2i\sin(2r+1)\frac{\pi}{12}\left\{\cos(2r+1)\frac{\pi}{12} + i\sin(2r+1)\frac{\pi}{12}\right\}}$$

$$\Rightarrow -x = -i\cot(2r+1)\frac{\pi}{12}$$

$\Rightarrow x = i\cot(2r+1)\frac{\pi}{12}, r = 0, 1, 2, 3, 4, 5$

Thus, the roots of the given equation are

$$i\cot\frac{\pi}{12}, i\cot\frac{3\pi}{12}, i\cot\frac{5\pi}{12}, i\cot\frac{7\pi}{12}, i\cot\frac{9\pi}{12}, i\cot\frac{11\pi}{12}$$

But, $i\cot\frac{11\pi}{12} = i\cot\left(\pi-\frac{\pi}{12}\right) = i\cot\frac{\pi}{12}$

$$i\cot\frac{9\pi}{12} = i\cot\left(\pi-\frac{3\pi}{12}\right) = -i\cot\frac{3\pi}{12}$$

and, $i\cot\frac{7\pi}{12} = i\cot\left(\pi-\frac{5\pi}{12}\right) = -i\cot\frac{5\pi}{12}$

Therefore, the roots of the given equation are

$$\pm i\cot\frac{\pi}{12}, \pm i\cot\frac{3\pi}{12}, \pm i\cot\frac{5\pi}{12}$$

i.e. $\pm i\cot(2r+1)\frac{\pi}{12}, r = 0, 1, 2.$

EXAMPLE 7 *Prove that the equation* $(1-z)^5 - z^5 = 0$ *has roots* $z = \frac{1}{2}\left(1 - i\tan\frac{r\pi}{5}\right)$, $r = 0, 1, 2, 3, 4$. *Take out the factor* $(2z-1)$ *corresponding to* $z = \frac{1}{2}$ *and verify that the resulting equation can be factorised as* $\left(z^2 - z + \frac{3+\sqrt{5}}{2}\right)\left(z^2 - z + \frac{3-\sqrt{5}}{2}\right) = 0$

SOLUTION We have,

$$(1-z)^5 - z^5 = 0$$

$$\Rightarrow \left(\frac{1-z}{z}\right)^5 = 1$$

$$\Rightarrow \left(\frac{1-z}{z}\right)^5 = \cos 0° + i\sin 0°$$

$$\Rightarrow \left(\frac{1-z}{z}\right)^5 = \cos 2r\pi + i\sin 2r\pi, r \in Z$$

$$\Rightarrow \frac{1-z}{z} = (\cos 2r\pi + i\sin 2r\pi)^{1/5}, r \in Z$$

$$\Rightarrow \frac{1-z}{z} = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}, r = 0, 1, 2, 3, 4$$

For $r = 0$, we have

$$\frac{1-z}{z} = 1 \Rightarrow z = \frac{1}{2}$$

For any $r \neq 0$, we have

$$\frac{1-z}{z} = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$$

$$\Rightarrow \frac{1}{z} = 1 + \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$$

$$\Rightarrow \frac{1}{z} = 2\cos^2\frac{r\pi}{5} + 2i\sin\frac{r\pi}{5}\cos\frac{r\pi}{5}$$

$$\Rightarrow \frac{1}{z} = 2\cos^2\frac{r\pi}{5}\left(1 + i\tan\frac{r\pi}{5}\right)$$

$$\Rightarrow z = \frac{1}{2\cos^2\frac{r\pi}{2}\left(1 + i\tan\frac{r\pi}{5}\right)}$$

$$\Rightarrow z = \frac{1\left(1 - i\tan\frac{r\pi}{5}\right)}{2\cos^2\frac{r\pi}{5}\left(1 + \tan^2\frac{r\pi}{5}\right)}$$

$$\Rightarrow z = \frac{1}{2}\left(1 - i\tan\frac{r\pi}{5}\right), r = 1, 2, 3, 4.$$

Thus, the roots of the given equation are given by

$$z = \frac{1}{2}\left(1 - i\tan\frac{r\pi}{5}\right), r = 0, 1, 2, 3, 4.$$

Now,

$$r = 1 \Rightarrow z_1 = \frac{1}{2}\left(1 - i\tan\frac{\pi}{5}\right)$$

$$r = 4 \Rightarrow z_4 = \frac{1}{2}\left(1 - \tan\frac{4\pi}{5}\right) = \frac{1}{2}\left(1 + i\tan\frac{\pi}{5}\right)$$

$$r = 2 \Rightarrow z_2 = \frac{1}{2}\left(1 - i\tan\frac{2\pi}{5}\right)$$

$$r = 3 \Rightarrow z_3 = \frac{1}{2}\left(1 - i\tan\frac{3\pi}{5}\right) = \frac{1}{2}\left(1 + i\tan\frac{2\pi}{5}\right)$$

The equation having z_1, z_2, z_3, z_4 as its roots is

$$(z-z_1)(z-z_2)(z-z_3)(z-z_4) = 0$$

$$\Rightarrow \left\{z - \frac{1}{2}\left(1 - i\tan\frac{\pi}{5}\right)\right\}\left\{z - \frac{1}{2}\left(1 + i\tan\frac{\pi}{5}\right)\right\} \times \left\{z - \frac{1}{2}\left(1 - i\tan\frac{2\pi}{5}\right)\right\}\left\{z - \frac{1}{2}\left(1 + i\tan\frac{2\pi}{5}\right)\right\} = 0$$

$$\Rightarrow \left(z - \frac{1}{2} + \frac{1}{2}i\tan\frac{\pi}{5}\right)\left(z - \frac{1}{2} - \frac{1}{2}i\tan\frac{\pi}{5}\right)\left(z - \frac{1}{2} + \frac{1}{2}i\tan\frac{2\pi}{5}\right)\left(z - \frac{1}{2} - \frac{1}{2}i\tan\frac{2\pi}{5}\right) = 0$$

$$\Rightarrow \left\{\left(z - \frac{1}{2}\right)^2 + \frac{1}{4}\tan^2\frac{\pi}{5}\right\}\left\{\left(z - \frac{1}{2}\right)^2 + \frac{1}{4}\tan^2\frac{2\pi}{5}\right\} = 0$$

$$\Rightarrow \left\{z^2 - z + \frac{1}{4} + \frac{1}{4}\tan\frac{2\pi}{5}\right\}\left\{z^2 - z + \frac{1}{4} + \frac{1}{4}\tan^2\frac{2\pi}{5}\right\} = 0$$

$$\Rightarrow \left\{z^2 - z + \frac{1}{4}\sec^2\frac{\pi}{5}\right\}\left\{z^2 - z + \frac{1}{4}\sec^2\frac{2\pi}{5}\right\} = 0$$

$$\Rightarrow \left(z^2 - z + \frac{1}{\cos^2\frac{\pi}{5}}\right)\left(z^2 - z + \frac{1}{4\cos^2\frac{2\pi}{5}}\right) = 0$$

$$\Rightarrow \left(z^2-z+\frac{4}{(\sqrt{5}+1)^2}\right)\left(z^2-z+\frac{4}{(\sqrt{5}-1)^2}\right)=0$$

$$\Rightarrow \left(z^2-z+\frac{3-\sqrt{5}}{2}\right)\left(z^2-z+\frac{3+\sqrt{5}}{2}\right)=0,$$

which is the required equation.

EXAMPLE 8 *Solve:* $(1+x)^8+x^8=0$ *and show that every root of the equation has* $-1/2$ *for its real part.*

SOLUTION We have,

$$(1+x)^8+x^8=0$$

$$\Rightarrow (1+x)^8=-x^8$$

$$\Rightarrow \left(\frac{1+x}{x}\right)^8=-1$$

$$\Rightarrow \left(\frac{1+x}{x}\right)^8=(\cos\pi+i\sin\pi)$$

$$\Rightarrow \left(\frac{1+x}{x}\right)^8=[\cos(2r\pi+\pi)+i\sin(2r\pi+\pi)]$$

$$\Rightarrow \left(\frac{1+x}{x}\right)=\cos(2r+1)\frac{\pi}{8}+i\sin(2r+1)\frac{\pi}{8},$$

$$r=0,1,2,3,4,5,6,7$$

$$\Rightarrow \frac{1}{x}=\cos(2r+1)\frac{\pi}{8}-1+i\sin(2r+1)\frac{\pi}{8}$$

$$\Rightarrow \frac{1}{x}=-2\sin^2(2r+1)\frac{\pi}{16}+2i\sin(2r+1)\frac{\pi}{16}\cos(2r+1)\frac{\pi}{16}$$

$$\Rightarrow \frac{1}{x}=-2\sin(2r+1)\frac{\pi}{16}\left\{\sin(2r+1)\frac{\pi}{16}-i\cos(2r+1)\frac{\pi}{16}\right\}$$

$$\Rightarrow x=\frac{-1}{2\sin(2r+1)\frac{\pi}{16}\left\{\sin(2r+1)\frac{\pi}{16}-i\cos(2r+1)\frac{\pi}{16}\right\}}$$

$$\Rightarrow x=\frac{-1}{2\sin(2r+1)\frac{\pi}{16}}\left\{\sin(2r+1)\frac{\pi}{16}+i\cos(2r+1)\frac{\pi}{16}\right\}$$

$$\Rightarrow x=-\frac{1}{2}\left\{1+i\cot(2r+1)\frac{\pi}{16}\right\},\ r=0,1,2,3,4,5,6,7$$

Clearly, each value of x has $-\frac{1}{2}$ as its real part.

EXAMPLE 9 *Solve the equation* $z^{10}-1=0$ *and deduce that*

$$\sin 5\theta=5\sin\theta\left\{1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{5}}\right\}\left\{1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{5}}\right\}$$

SOLUTION We have,

$$z^{10}-1=0$$

$$\Rightarrow z^{10}=1$$

$$\Rightarrow z^{10}=\cos 2r\pi+i\sin 2r\pi,\ r\in Z$$

$$\Rightarrow z=(\cos 2r\pi+i\sin 2r\pi)^{1/10},\ r\in Z$$

$$\Rightarrow z=\cos\frac{r\pi}{5}+i\sin\frac{r\pi}{5},\ r=0,1,2,3,4,\ldots,9$$

Now,

$r=0\Rightarrow z=1$

$$r=1\Rightarrow z=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$$

$$r=2\Rightarrow z=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$$

$$r=3\Rightarrow z=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$$

$$r=4\Rightarrow z=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$$

$$r=5\Rightarrow z=\cos\pi+i\sin\pi=-1$$

$$r=6\Rightarrow z=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$$

$$=\cos\left(2\pi-\frac{4\pi}{5}\right)+i\sin\left(2\pi-\frac{4\pi}{5}\right)$$

$$=\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}$$

$$r=7\Rightarrow z=\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$$

$$=\cos\left(2\pi-\frac{3\pi}{5}\right)+i\sin\left(2\pi-\frac{3\pi}{5}\right)$$

$$=\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}$$

$$r=8\Rightarrow z=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$$

$$=\cos\left(2\pi-\frac{2\pi}{5}\right)+i\sin\left(2\pi-\frac{2\pi}{5}\right)$$

$$=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$$

and,

$$r=9\Rightarrow z=\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$$

$$\Rightarrow z=\cos\left(2\pi-\frac{\pi}{5}\right)+i\sin\left(2\pi-\frac{\pi}{5}\right)$$

$$\Rightarrow z=\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}$$

Thus, the roots of the equation $z^{10}-1=0$ are

$$z=\pm 1,\ \cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\ \cos\frac{2\pi}{5}\pm i\sin\frac{2\pi}{5}$$

$$\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5},\ \cos\frac{4\pi}{5}\pm i\sin\frac{4\pi}{5}$$

$\therefore\ z^{10}-1$

$$= (z+1)(z-1)\left(z-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}\right)\left(z-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$$
$$\left(z-\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right)\left(z-\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)$$
$$\left(z-\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)\left(z-\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$
$$\left(z-\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}\right)\left(z-\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)$$

$$\Rightarrow z^{10}-1$$
$$= (z^2-1)\left\{\left(z-\cos\frac{\pi}{5}\right)^2+\sin^2\frac{\pi}{5}\right\}\left\{\left(z-\cos\frac{2\pi}{5}\right)^2+\sin^2\frac{2\pi}{5}\right\}$$
$$\left\{\left(z-\cos\frac{3\pi}{5}\right)^2+\sin^2\frac{3\pi}{5}\right\}\left\{\left(z-\cos\frac{4\pi}{5}\right)^2+\sin^2\frac{4\pi}{5}\right\}$$

$$\Rightarrow z^{10}-1$$
$$= (z^2-1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{2\pi}{5}+1\right)$$
$$\left(z^2-2z\cos\frac{3\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right)$$

Dividing both sides by z^5, we get

$$z^5-\frac{1}{z^5}=\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}-2\cos\frac{\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{2\pi}{5}\right)$$
$$\left(z+\frac{1}{z}-2\cos\frac{3\pi}{5}\right)\left(z+\frac{1}{z}-2\cos\frac{4\pi}{5}\right) \quad ...(i)$$

Now, let

$z = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta,\ z^n+\frac{1}{z^n} = 2\cos n\theta$$

and $$z^n-\frac{1}{z^n} = 2i\sin n\theta$$

Substituting these values in (i), we get

$$2i\sin 5\theta = (2i\sin\theta)\left(2\cos\theta-2\cos\frac{\pi}{5}\right)\left(2\cos\theta-2\cos\frac{2\pi}{5}\right)$$
$$\left(2\cos\theta-2\cos\frac{3\pi}{5}\right)\left(2\cos\theta-2\cos\frac{4\pi}{5}\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\left(\cos\theta-\cos\frac{\pi}{5}\right)\left(\cos\theta-\cos\frac{2\pi}{5}\right)$$
$$\left(\cos\theta-\cos\frac{3\pi}{5}\right)\left(\cos\theta-\cos\frac{4\pi}{5}\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\left(\cos\theta-\cos\frac{\pi}{5}\right)\left(\cos\theta-\cos\frac{2\pi}{5}\right)$$
$$\left(\cos\theta+\cos\frac{2\pi}{5}\right)\left(\cos\theta+\cos\frac{\pi}{5}\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\left(\cos^2\theta-\cos^2\frac{\pi}{5}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{5}\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\left(\sin^2\frac{\pi}{5}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{5}-\sin^2\theta\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}$$
$$\times\left(1-\frac{\sin^2\theta}{\sin^2\pi/5}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\pi/5}\right)$$

$$\Rightarrow \sin 5\theta = 2^4\sin\theta\left(\frac{\sqrt{10-2\sqrt5}}{4}\right)^2\left(\frac{\sqrt{10+2\sqrt5}}{4}\right)^2$$
$$\times\left(1-\frac{\sin^2\theta}{\sin^2\pi/5}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\pi/5}\right)$$

$$\Rightarrow \sin 5\theta = 5\sin\theta\left(1-\frac{\sin^2\theta}{\sin^2\pi/5}\right)\left(1-\frac{\sin^2\theta}{\sin^2 2\pi/5}\right)$$

EXAMPLE 10 *Solve the equation* $z^8+1=0$ *and deduce that* $\cos 4\theta = 8(\cos\theta-\cos\pi/8)(\cos\theta-\cos 3\pi/8)(\cos\theta-\cos 5\pi/8)(\cos\theta-\cos 7\pi/8)$

SOLUTION We have,

$$z^8+1 = 0$$
$$\Rightarrow z^8 = -1$$
$$\Rightarrow z^8 = \cos\pi + i\sin\pi$$
$$\Rightarrow z^8 = \cos(2r\pi+\pi)+i\sin(2r\pi+\pi),\ r\in Z$$
$$\Rightarrow z^8 = \cos(2r+1)\pi + i\sin(2r+1)\pi$$
$$\Rightarrow z = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/8}$$
$$\Rightarrow z = \cos(2r+1)\frac{\pi}{8}+i\sin(2r+1)\frac{\pi}{8},\ r=0,1,2,....,7$$

Now,

$$r=0 \Rightarrow z = \cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$$

$$r=1 \Rightarrow z = \cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}$$

$$r=2 \Rightarrow z = \cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}$$

$$r=3 \Rightarrow z = \cos\frac{7\pi}{8}+i\sin\frac{7\pi}{8}$$

$$r=4 \Rightarrow z = \cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}$$
$$= \cos\left(2\pi-\frac{7\pi}{8}\right)+i\sin\left(2\pi-\frac{7\pi}{8}\right)$$
$$= \cos\frac{7\pi}{8}-i\sin\frac{7\pi}{8}$$

$$r=5 \Rightarrow z = \cos\frac{11\pi}{8}+i\sin\frac{11\pi}{8}$$
$$= \cos\left(2\pi-\frac{5\pi}{8}\right)+i\sin\left(2\pi-\frac{5\pi}{8}\right)$$

$$= \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}$$

$$r = 6 \Rightarrow z = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$$

$$= \cos\left(2\pi - \frac{3\pi}{8}\right) + i\sin\left(2\pi - \frac{3\pi}{8}\right)$$

$$= \cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}$$

$$r = 7 \Rightarrow z = \cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}$$

$$= \cos\left(2\pi - \frac{\pi}{8}\right) + i\sin\left(2\pi - \frac{\pi}{8}\right)$$

$$= \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}$$

Thus, the roots of $z^8 = 1 = 0$ are :

$$\cos\frac{\pi}{8} \pm i\sin\frac{\pi}{8}, \cos\frac{3\pi}{8} \pm i\sin\frac{3\pi}{8},$$

$$\cos\frac{5\pi}{8} \pm i\sin\frac{5\pi}{8}, \cos\frac{7\pi}{8} \pm i\sin\frac{7\pi}{8}$$

$$\therefore z^8 + 1 = \left(z - \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)\left(z - \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$$
$$\times\left(z - \cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right)\left(z - \cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right)$$
$$\times\left(z - \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}\right)\left(z - \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$$
$$\times\left(z - \cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8}\right)\left(z - \cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$$

$$= \left\{\left(z - \cos\frac{\pi}{8}\right)^2 + \sin^2\frac{\pi}{8}\right\}\left\{\left(z - \cos\frac{3\pi}{8}\right)^2 + \sin^2\frac{3\pi}{8}\right\}$$
$$\times\left\{\left(z - \cos\frac{5\pi}{8}\right)^2 + \cos^2\frac{5\pi}{8}\right\}\left\{\left(z - \cos\frac{7\pi}{8}\right)^2 + \sin^2\frac{7\pi}{8}\right\}$$

$$= \left(z^2 - 2z\cos\frac{\pi}{8} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{8} + 1\right)$$
$$\times\left(z^2 - 2z\cos\frac{5\pi}{8} + 1\right)\left(z^2 - 2z\cos\frac{7\pi}{8} + 1\right)$$

Dividing both sides by z^4, we get

$$z^4 + \frac{1}{z^4} = \left(z + \frac{1}{z} - 2\cos\frac{\pi}{8}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{8}\right)$$
$$\times\left(z + \frac{1}{z} - 2\cos\frac{5\pi}{8}\right)\left(z + \frac{1}{z} - 2\cos\frac{7\pi}{8}\right)$$

Putting $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$

and,

$z^4 + \frac{1}{z^4} = 2\cos 4\theta$, we get

$$2\cos 4\theta = \left(2\cos\theta - 2\cos\frac{\pi}{8}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{8}\right)$$
$$\times\left(2\cos\theta - 2\cos\frac{5\pi}{8}\right)\left(2\cos\theta - 2\cos\frac{7\pi}{8}\right)$$

$$\Rightarrow \cos 4\theta = 8\left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)$$
$$\times\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right)$$

EXAMPLE 11 *Solve the equation* $z^7 + 1 = 0$ *and deduce that*

(i) $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$

(ii) $\cos\frac{\pi}{4}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

(iii) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$

(iv) $\tan\left(\frac{\pi}{14}\right)\tan\left(\frac{3\pi}{14}\right)\tan\left(\frac{5\pi}{14}\right) = \frac{1}{\sqrt{7}}$

Also show that :

$$(1+y)^7 + (1-y)^7 = 14\left(y^2 + \tan^2\frac{\pi}{14}\right)\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

and deduce that

$$\tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14} = 5$$

SOLUTION We have,

$$z^7 + 1 = 0$$

$$\Rightarrow z^7 = -1$$

$$\Rightarrow z^7 = \cos\pi + i\sin\pi$$

$$\Rightarrow z^7 = \cos(2r\pi + \pi) + i\sin(2r\pi + \pi), r \in Z$$

$$\Rightarrow z = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/7}, r \in Z$$

$$\Rightarrow z = \cos(2r+1)\frac{\pi}{7} + i\sin(2r+1)\frac{\pi}{7}, r = 0, 1, 2, ..., 6$$

Now,

$$r = 0 \Rightarrow z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$$

$$r = 1 \Rightarrow z = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$$

$$r = 2 \Rightarrow z = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$$

$$r = 3 \Rightarrow z = \cos\pi + i\sin\pi = -1$$

$$r = 4 \Rightarrow z = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$$

$$= \cos\left(2\pi - \frac{5\pi}{7}\right) + i\sin\left(2\pi - \frac{5\pi}{7}\right)$$

$$= \cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}$$

$$r = 5 \Rightarrow z = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$$
$$= \cos\left(2\pi - \frac{3\pi}{7}\right) + i\sin\left(2\pi - \frac{3\pi}{7}\right)$$
$$= \cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}$$

$$r = 6 \Rightarrow z = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$$
$$= \cos\left(2\pi - \frac{\pi}{7}\right) + i\sin\left(2\pi - \frac{\pi}{7}\right)$$
$$= \cos\frac{\pi}{7} - i\sin\frac{\pi}{7}$$

Thus, the roots of $z^7 + 1 = 0$ are :

$$z = -1, \cos\frac{\pi}{7} \pm i\sin\frac{\pi}{7}, \cos\frac{3\pi}{7} \pm i\sin\frac{3\pi}{7}, \cos\frac{5\pi}{7} \pm i\sin\frac{5\pi}{7}$$

$$\therefore z^7 + 1 = (z+1)\left(z - \cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right)\left(z - \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$$
$$\times \left(z - \cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}\right)\left(z - \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)$$
$$\times \left(z - \cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}\right)\left(z - \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$$

$$\Rightarrow z^7 + 1$$
$$= (z+1)\left\{\left(z - \cos\frac{\pi}{7}\right)^2 + \sin^2\frac{\pi}{7}\right\}\left\{\left(z - \cos\frac{3\pi}{7}\right)^2 + \sin^2\frac{3\pi}{7}\right\}$$
$$\left\{\left(z - \cos\frac{5\pi}{7}\right)^2 + \sin^2\frac{5\pi}{7}\right\}$$

$$\Rightarrow z^7 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)$$
$$\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right) \ldots(i)$$

$$\Rightarrow \frac{z^7+1}{z+1} = \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)$$
$$\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

$$\Rightarrow z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$$
$$= \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

Dividing both sides by z^3, we get

$$\left(z^3 + \frac{1}{z^3}\right) - \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1$$
$$= \left(z + \frac{1}{z} - 2\cos\frac{\pi}{7}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{7}\right)\left(z + \frac{1}{z} - 2\cos\frac{5\pi}{7}\right) \ldots(ii)$$

Putting $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$ *and*,

$z^n + \frac{1}{z^n} = 2\cos n\theta$, we get

$$2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1$$
$$= \left(2\cos\theta - 2\cos\frac{\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{5\pi}{7}\right) \ldots(iii)$$

Putting $\theta = \frac{\pi}{2}$ on both sides of (iii), we get

$$2 - 1 = \left(-2\cos\frac{\pi}{7}\right)\left(-2\cos\frac{3\pi}{7}\right)\left(-2\cos\frac{5\pi}{7}\right)$$

$$\Rightarrow \cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$$

(ii) Putting $z = -1$ in (ii), we get

$$-7 = \left(-2 - 2\cos\frac{\pi}{7}\right)\left(-2 - 2\cos\frac{3\pi}{7}\right)\left(-2 - 2\cos\frac{5\pi}{7}\right)$$

$$\Rightarrow 7 = 8\left(1 + \cos\frac{\pi}{7}\right)\left(1 + \cos\frac{3\pi}{7}\right)\left(1 + \cos\frac{5\pi}{7}\right)$$

$$\Rightarrow 7 = 8\left(2\cos^2\frac{\pi}{14}\right)\left(2\cos^2\frac{3\pi}{14}\right)\left(2\cos^2\frac{5\pi}{14}\right)$$

$$\Rightarrow \frac{7}{64} = \left(\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14}\right)^2$$

$$\Rightarrow \cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8} \quad \left[\because \cos\frac{\pi}{14}, \cos\frac{3\pi}{14}, \cos\frac{5\pi}{14} > 0\right]$$

(iii) Putting $z = \pm 1$ in (ii), we get

$$1 = \left(2 - 2\cos\frac{\pi}{7}\right)\left(2 - 2\cos\frac{3\pi}{7}\right)\left(2 - 2\cos\frac{5\pi}{7}\right)$$

$$\Rightarrow 1 = 8\left(2\sin^2\frac{\pi}{14}\right)\left(2\sin^2\frac{3\pi}{14}\right)\left(2\sin^2\frac{5\pi}{14}\right)$$

$$\Rightarrow \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$$

(iv) Dividing (iii) by (ii), we get

$$\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14} = \frac{1}{\sqrt{7}}$$

Let $z = \frac{1+y}{1-y}$. Then,

$$z^2 - 2z\cos\alpha + 1$$
$$= \left(\frac{1+y}{1-y}\right)^2 - 2\left(\frac{1+y}{1-y}\right)\cos\alpha + 1$$
$$= \frac{1}{(1-y)^2}\left\{(1+y)^2 - 2(1-y^2)\cos\alpha + (1-y)^2\right\}$$
$$= \frac{1}{(1-y)^2}\left\{2(1+y)^2 - 2(1-y)^2\cos\alpha\right\}$$
$$= \frac{2}{(1-y)^2}\left\{(1-\cos\alpha) + y^2(1+\cos\alpha)\right\}$$

$$= \frac{2}{(1-y)^2}\left\{2\sin^2\frac{\alpha}{2} + 2y^2\cos^2\frac{\alpha}{2}\right\}$$

$$= \frac{2^2}{(1-y)^2}\left\{\sin^2\frac{\alpha}{2} + y^2\cos^2\frac{\alpha}{2}\right\} \quad ...(iv)$$

Putting $z = \frac{1+y}{1-y}$ in (i), we get

$$\left(\frac{1+y}{1-y}\right)^7 + 1 = \left(\frac{1+y}{1-y} + 1\right) \times \frac{2^2}{(1-y)^2}\left\{\sin^2\frac{\pi}{14} + y^2\cos^2\frac{\pi}{14}\right\}$$

$$\times \frac{2^2}{(1-y)^2}\left(\sin^2\frac{3\pi}{14} + y^2\cos^2\frac{3\pi}{14}\right)$$

$$\times \frac{2^2}{(1-y)^2}\left\{\sin^2\frac{5\pi}{14} + y^2\cos^2\frac{5\pi}{14}\right\}$$

$$\Rightarrow (1+y)^7 + (1-y)^7 = 2^7\left(\sin^2\frac{\pi}{14} + y^2\cos^2\frac{\pi}{14}\right)$$

$$\left(\sin^2\frac{3\pi}{14} + y^2\cos^2\frac{3\pi}{14}\right)\left(\sin^2\frac{5\pi}{14} + y^2\cos^2\frac{5\pi}{14}\right)$$

$$\Rightarrow (1+y)^7 + (1-y)^7 = 2^7\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}\left(y^2 + \tan^2\frac{\pi}{14}\right)$$

$$\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

$$\Rightarrow (1+y)^7 + (1-y)^7 = 2^7 \times \left(\frac{\sqrt{7}}{8}\right)^2\left(y^2 + \tan^2\frac{\pi}{14}\right)$$

$$\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right) \text{ [Using (ii)]}$$

$$\Rightarrow (1+y)^7 + (1-y)^7$$

$$= 14\left(y^2 + \tan^2\frac{\pi}{14}\right)\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

Equating coefficients of y^4 on both sides, we get

$${}^7C_4 + {}^7C_4 = 14\left(\tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14}\right)$$

$$\Rightarrow \tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14} = 5$$

EXAMPLE 12 *Solve the equation* $z^{14} - 1 = 0$ *and deduce that*

$$\sin 7\theta = 64\sin\theta\left(\sin^2\frac{\pi}{7} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{7} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{7} - \sin^2\theta\right)$$

SOLUTION We have,

$$z^{14} - 1 = 0$$

$$\Rightarrow z^{14} = 1$$

$$\Rightarrow z^{14} = \cos 0° + i\sin 0°$$

$$\Rightarrow z^{14} = \cos 2r\pi + i\sin 2r\pi, r \in Z$$

$$\Rightarrow z = (\cos 2r\pi + i\sin 2r\pi)^{1/14}$$

$$\Rightarrow z = \cos\frac{2r\pi}{14} + i\sin\frac{2r\pi}{14}, r = 0, 1, 2, 3, ..., 13$$

$$\Rightarrow z = \cos\frac{r\pi}{7} + i\sin\frac{r\pi}{7}, r = 0, 1, 2, ..., 13$$

Now,

$r = 0 \Rightarrow z = 1$

$z = \cos\frac{r\pi}{7} + i\sin\frac{r\pi}{7}$ for $r = 1, 2, 3, ..., 6$

For $r = 7, z = \cos\pi + i\sin\pi = -1$

For $r = 8, 9, ..., 13$, we have

$$z = \cos\frac{r\pi}{7} + i\sin\frac{r\pi}{7}$$

$$= \cos\left\{2\pi - \left(\frac{14-r}{7}\right)\pi\right\} + i\sin\left\{2\pi - \left(\frac{14-r}{7}\right)\pi\right\}$$

$$= \cos\frac{(14-r)\pi}{7} - i\sin\frac{(14-r)\pi}{7}$$

$$\therefore r = 8 \Rightarrow z = \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$$

$$r = 9 \Rightarrow z = \cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}$$

$$r = 10 \Rightarrow z = \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$$

$$r = 11 \Rightarrow z = \cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}$$

$$r = 12 \Rightarrow z = \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$$

$$r = 13 \Rightarrow z = \cos\frac{\pi}{7} - i\sin\frac{\pi}{7}$$

Thus, the roots of $z^{14} - 1 = 0$ are :

$$z = \pm 1, \cos\frac{r\pi}{7} \pm \sin\frac{r\pi}{7}, r = 1, 2, 3, ..., 6$$

$$\therefore z^{14} - 1$$

$$= (z-1)(z+1)\prod_{r=1}^{6}\left(z - \cos\frac{r\pi}{7} - i\sin\frac{r\pi}{7}\right)\left(z - \cos\frac{r\pi}{7} + i\sin\frac{r\pi}{7}\right)$$

$$\Rightarrow z^{14} - 1 = (z^2-1)\prod_{r=1}^{6}\left\{\left(z - \cos\frac{r\pi}{7}\right)^2 + \sin^2\frac{r\pi}{7}\right\}$$

$$\Rightarrow z^{14} - 1 = (z^2-1)\prod_{r=1}^{6}\left(z^2 - 2z\cos\frac{r\pi}{7} + 1\right)$$

Dividing both sides by z^7, we get

$$z^7 - \frac{1}{z^7} = \left(z - \frac{1}{z}\right)\prod_{r=1}^{6}\left(z - 2\cos\frac{r\pi}{7} + \frac{1}{z}\right)$$

$$\Rightarrow z^7 - \frac{1}{z^7} = \left(z - \frac{1}{z}\right)\prod_{r=1}^{6}\left(z + \frac{1}{z} - 2\cos\frac{r\pi}{7}\right)$$

Putting $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$,

$z - \frac{1}{z} = 2i\sin\theta$ and $z^7 - \frac{1}{z^7} = 2i\sin 7\theta$, we get

$$2i\sin 7\theta = (2i\sin\theta)\prod_{r=1}^{6}\left(2\cos\theta - 2\cos\frac{r\pi}{7}\right)$$

$$\Rightarrow \sin 7\theta = (\sin\theta)(2^6)\left(\cos\theta - \cos\frac{\pi}{7}\right)\left(\cos\theta - \cos\frac{2\pi}{7}\right) \times\left(\cos\theta - \cos\frac{3\pi}{7}\right)\left(\cos\theta - \cos\frac{4\pi}{7}\right) \times\left(\cos\theta - \cos\frac{5\pi}{7}\right)\left(\cos\theta - \cos\frac{6\pi}{7}\right)$$

$$\Rightarrow \sin 7\theta = 64\sin\theta\left(\cos\theta - \cos\frac{\pi}{7}\right)\left(\cos\theta + \cos\frac{\pi}{7}\right) \times\left(\cos\theta - \cos\frac{2\pi}{7}\right)\left(\cos\theta + \cos\frac{2\pi}{7}\right) \times\left(\cos\theta - \cos\frac{3\pi}{7}\right)\left(\cos\theta + \cos\frac{2\pi}{7}\right)$$

$$\Rightarrow \sin 7\theta = 64\sin\theta\left(\cos^2\theta - \cos^2\frac{\pi}{7}\right) \times\left(\cos^2\theta - \cos^2\frac{2\pi}{7}\right)\left(\cos^2\theta - \cos^2\frac{3\pi}{7}\right)$$

$$\Rightarrow \sin 7\theta = 64\sin\theta\left(\sin^2\frac{\pi}{7} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{7} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{7} - \sin^2\theta\right)$$

19.34 ROOTS OF UNITY

In this section, we shall obtain various order roots of unity by using De' Moivere's theorem. We shall also learn about some properties of these roots.

nth ROOTS OF UNITY : *Let* $z = 1^{1/n}$. *Then,*

$$z = (\cos 0° + i\sin 0°)^{1/n}$$

$$\Rightarrow z = (\cos 2r\pi + i\sin 2r\pi)^{1/n}, r \in Z$$

$$\Rightarrow z = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}, r = 0, 1, 2, ..., (n-1)$$

[Using De' Moivere's theorem]

$$\Rightarrow z = e^{\frac{i2r\pi}{n}}, r = 0, 1, 2, ..., (n-1)$$

$$\Rightarrow z = \left\{e^{i2\pi/n}\right\}^r, r = 0, 1, 2, ..., (n-1)$$

$$\Rightarrow z = \alpha^r, \alpha = e^{i2\pi/n}, r = 0, 1, 2, ..., (n-1)$$

Thus nth roots of unity are :

$$1, \alpha, \alpha^2, ..., \alpha^{n-1}, \text{where } \alpha = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

PROPERTIES OF nth ROOTS OF UNITY

(I) *nth roots of unity form a G.P. with common ratio* $e^{i2\pi/n}$

PROOF nth roots of unity are $1, \alpha, \alpha^2, ..., \alpha^{n-1}$, where $\alpha = e^{i2\pi/n}$.

Clearly, $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ is a G.P. with common ratio $\alpha = e^{i2\pi/n}$

(II) *Sum of nth roots of unity is always zero.*

PROOF nth roots of unity are : $1, \alpha, \alpha^2, ..., \alpha^{n-1}$, where $\alpha = e^{i2\pi/n}$

Now,

$$\text{Sum} = 1 + \alpha + \alpha^2 + ... + \alpha^{n-1}$$

$$= \frac{1-\alpha^n}{1-\alpha} \quad \left[\text{Using : } a + ar + ar^2 + ... + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)\right]$$

$$= \frac{1-e^{i2\pi}}{1-\alpha} \quad [\because \alpha = e^{i2\pi/n} \therefore \alpha^n = e^{i2\pi}]$$

$$= \frac{1-1}{1-\alpha} = 0 \quad [\because e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1]$$

(III) *Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.*

PROOF nth roots of unity are : $1, \alpha, \alpha^2, ..., \alpha^{n-1}$.

$$\therefore 1 + \alpha^p + (\alpha^2)^p + \alpha^{2p} + (\alpha^3)^p + ... + (\alpha^{n-1})^p$$

$$= 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + ... + \alpha^{(n-1)^p}$$

$$= \frac{1-(\alpha^p)^n}{1-\alpha^p}$$

$$= \frac{1-(e^{i2\pi p/n})^n}{1-e^{i2\pi p/n}} \quad \left[\because \alpha = e^{i2\pi/n} \therefore \alpha^p = e^{i2\pi p/n}\right]$$

$$= \frac{1-e^{i2\pi p}}{1-e^{i2\pi p/n}}$$

$$= \frac{1-1}{1-e^{i2\pi p/n}} = 0 \quad [\because e^{i2\pi p} = \cos 2\pi p + i\sin 2\pi p = 1]$$

(IV) *Sum of pth powers of nth roots of unity is n, if p is a multiple of n.*

PROOF Let $p = \lambda n$, where $\lambda \in N$. Then,

$$\alpha^p = e^{i2\pi p/n} = e^{i2\pi\lambda n/n} = e^{i2\pi\lambda} = \cos 2\pi\lambda + i\sin 2\pi\lambda = 1.$$

$\therefore$ Sum of pth powers of nth roots of unity

$$= 1 + \alpha^p + \alpha^{2p} + ... + \alpha^{(n-1)p} = 1 + 1 + 1 + ... + 1 = n$$

(V) *Product of nth roots of unity is* $(-1)^{n-1}$

PROOF nth roots of unity are $1, \alpha, \alpha^2, ..., \alpha^{n-1}$.

$$\therefore \text{Product} = 1 \cdot \alpha \cdot \alpha^2 ... \alpha^{n-1}$$

$$= \alpha^{1+2+3+...+(n-1)}$$

$$= \alpha^{n(n-1)/2}$$

$= (\alpha^{n/2})^{n-1}$

$= (e^{i\pi})^{n-1}$ $[\because \alpha = e^{i2\pi/n} \Rightarrow \alpha^{n/2} = e^{i\pi}]$

$= (\cos\pi + i\sin\pi)^{n-1}$

$= (-1)^{n-1}$

(VI) *nth roots of unity lie on the unit circle* $|z| = 1$ *and divide its circumference into n equal parts.*

PROOF *n*th roots of unity are : $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

For any $r = 0, 1, 2, \ldots, (n-1)$, we have

$$\alpha^r = e^{i2\pi r/n} = \cos(2\pi r/n) + i\sin(2\pi r/n)$$

$$\Rightarrow |\alpha^r| = \sqrt{\cos^2\left(\frac{2\pi r}{n}\right) + \sin^2\left(\frac{2\pi r}{n}\right)} = 1$$

$\Rightarrow \alpha^r$ lies on $|z| = 1$, where $r = 0, 1, 2, \ldots, (n-1)$.

Thus, $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ lie on the circle $|z| = 1$.

We have, $\arg(1) = 0, \arg(\alpha) = 2\pi/n, \arg(\alpha^2) = 4\pi/n, \ldots, \arg(\alpha^{n-1}) = 2(n-1)\pi/n$.

Thus, difference between the arguments of any two consecutive *n*th roots of unity is $2\pi/n$.

Hence, they divide the circumference of the circle $|z| = 1$ into *n* equal parts.

REMARK 1. *nth roots of 1 are the solutions of the equation* $z = 1^{1/n}$

i.e. $z^n = 1$ *or,* $z^n - 1 = 0$

$\therefore z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1})$

REMARK 2 *nth roots of −1 are the soltuions of the equation* $z = (-1)^{1/n}$ *i.e.* $z^n = -1$ *or,* $z^n + 1 = 0$

SOME PARTICULAR CASES

(I) *Square roots of unity*

We know that the *n*th roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where $\alpha = e^{i2\pi/n}$

For square roots of unity, we have $n = 2$.

$\therefore \alpha = e^{i\pi} = \cos\pi + i\sin\pi = -1$.

Thus, square roots of unity are $1, -1$.

(II) *Cube roots of unity*

For cube roots of unity, we have $n = 3$.

$\therefore \alpha = e^{i2\pi/3}$

So, cube roots of unity are $1, \alpha, \alpha^2$ i.e. $1, e^{i2\pi/3}$ and $e^{i4\pi/3}$

We denote $e^{i2\pi/3}$ by ω i.e.

$$\omega = \cos(2\pi/3 + i)\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Thus, cube roots of unity are $1, \omega, \omega^2$, where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and, $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Note that the cube roots of unity follow all the properties listed for *n*th roots of unity, but due to their importance, we shall list the properties of cube roots of unity separately in the following section.

19.35 CUBE ROOTS OF UNITY

As discussed in the previous section that the cube roots of unity are $1, \omega$ and ω^2, where

$$\omega = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2 \text{ and } \omega^2 = e^{i4\pi/3} = -1/2 - i\sqrt{3}/2.$$

We shall now obtain cube roots of unity by an alternative method.

Let $z = 1^{1/3}$. Then,

$$z^3 = 1 \Rightarrow z^3 - 1 = 0$$

$$\Rightarrow (z-1)(z^2 + z + 1) = 0$$

$$\Rightarrow z = 1,\ z^2 + z + 1 = 0$$

$$\Rightarrow z = 1,\ z = \frac{-1 \pm i\sqrt{3}}{2}$$

Thus, cube roots of unity are $-1, \frac{-1 + i\sqrt{3}}{2}$ and $\frac{-1 \pm i\sqrt{3}}{2}$.

It can be easily seen that $-1/2 + i\sqrt{3}/2$ and $-1/2 - i\sqrt{3}/2$ are complex numbers of unit modulus such that

$\arg(-1/2 + i\sqrt{3}/2) = 2\pi/3$ and $\arg(-1/2 - i\sqrt{3}/2) = 4\pi/3$.

$$\therefore -\frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{i2\pi/3} \text{ and } -\frac{1}{2} - \frac{i\sqrt{3}}{2} = e^{i4\pi/3}$$

Let $\omega = e^{i2\pi/3}$. Then, $\omega^2 = e^{i4\pi/3}$

Thus, cube roots of unity are $1, \omega, \omega^2$, where $\omega = e^{i2\pi/3}$

PROPERTIES OF CUBE ROOTS OF UNITY AND SOME USEFUL RESULTS RELATED TO THEM

(I) Cube roots of unity are $1, \omega, \omega^2$, where

$$\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{i2\pi/3}$$

(II) $\arg(\omega) = 2\pi/3$ and $\arg(\omega^2) = 4\pi/3$

(III) $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$

(IV) ω and ω^2 are roots of the equation $z^2 + z + 1 = 0$

(V) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into three equal parts.

(VI) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{ if } n \text{ is not a multiple of } 3 \\ 3, \text{ if } n \text{ is a multiple of } 3 \end{cases}$

(VII) Cube roots of -1 are $-1, -\omega, -\omega^2$.

(VIII) $z^3 + 1 = (z+1)(z+\omega)(z+\omega^2)$

(IX) $-\omega$ and $-\omega^2$ are roots of $z^2 - z + 1 = 0$

(X) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

$a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

$a^2+b^2+c^2-ab-bc-ca = (a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ $\quad$ $a^3+b^3+c^3-3abc = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

ILLUSTRATIVE EXAMLES

EXAMPLE 1 *If* $1, \alpha_1, \alpha_2, ..., \alpha^{n-1}$ *are the nth roots of unity then prove that*

(i) $(1+\alpha_1)(1+\alpha_2)...(1+\alpha_{n-1}) = 1$, if n is an odd natural number.

(ii) $(1+\alpha_1)(1+\alpha_2)...(1+\alpha_{n-1}) = 0$, if n is an even natural number.

SOLUTION Since $1, \alpha_1, \alpha_2, ..., \alpha_{n-1}$ are the nth roots of unity i.e. these are the roots of $x^n-1=0$. Therefore,

$$x^n-1 = (x-1)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})$$

$$\Rightarrow \frac{x^n-1}{x-1} = (x-\alpha_1)(x-\alpha_2)....(x-\alpha_{n-1})$$

$$\Rightarrow (x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1}) = 1+x+x^2+...+x^{n-1}$$

Putting $x=-1$ on both sides, we get

$$(-1-\alpha_1)(-1-\alpha_2)...(-1-\alpha_{n-1}) = 1+(-1)+(-1)^2+...+(-1)^{n-1}$$

$$\Rightarrow (-1)^{n-1}(1+\alpha_1)(1+\alpha_2)...(1+\alpha_{n-1}) = \begin{cases}0, \text{ if } n \text{ is even}\\ 1, \text{ if } n \text{ is odd}\end{cases}$$

$$\Rightarrow (1+\alpha_1)(1+\alpha_2)....(1+\alpha_{n-1}) = \begin{cases}0, \text{ if } n \text{ is even}\\ 1, \text{ if } n \text{ is odd}\end{cases}$$

EXAMPLE 2 *If* $\alpha^r, r=0, 1, 2, ...,\sim(n-1)$ *are the nth roots of unity, show that*

$$\sum_{r=0}^{n-1}\left|\,|z_1+\alpha^r z_2|\right|^2 = n(|z_1|^2+|z_2|^2)$$

SOLUTION Since $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ are the roots of the equation $x^n-1=0$

Therefore,

$$1+\alpha+\alpha^2+...+\alpha^{n-1}=0$$

and, $|\alpha^r|=1$ for $r=0, 1, 2, ...(n-1)$

Now,

$$|z_1+\alpha^r z_2|^2$$

$$= |z_1|^2+|\alpha^r z_2|^2+z_1\overline{(\alpha^r z_2)}+\bar{z}_1(\alpha^r z_2)$$

$$= |z_1|^2+|\alpha^r|^2|z_2|^2+z_1\bar{z}_2\bar{\alpha}_r+\bar{z}_1 z_2\alpha^r$$

$$= |z_1|^2+|z_2|^2+z_1\bar{z}_2\bar{\alpha}^r+\bar{z}_1 z_2\alpha^r, \quad r=0, 1, 2, ..., (n-1)$$

$$\therefore \sum_{r=0}^{n-1}|z_1+\alpha^r z_2|^2$$

$$= \sum_{r=0}^{n-1}\left\{|z_1|^2+|z_2|^2+z_1\bar{z}_2\bar{\alpha}^r+\bar{z}_1 z_2\alpha^r\right\}$$

$$= \sum_{r=0}^{n-1}\left\{|z_1|^2+|z_2|^2\right\}+z_1\bar{z}_2\left(\sum_{r=0}^{n-1}\bar{\alpha}^r\right)+\bar{z}_1 z_2\left(\sum_{r=0}^{n-1}\alpha^r\right)$$

$$= n\left\{|z_1|^2+|z_2|^2\right\}+z_1\bar{z}_2\left(\overline{\sum_{r=0}^{n-1}\alpha^r}\right)+\bar{z}_1 z_2\left(\sum_{r=0}^{n-1}\alpha^r\right)$$

$$= n\left\{|z_1|^2+|z_2|^2\right\}+z_1\bar{z}_2\times 0+\bar{z}_1 z_2\times 0 \quad \left[\because \sum_{r=0}^{n-1}\alpha^r = 0\right]$$

$$= n\left\{|z_1|^2+|z_2|^2\right\}$$

EXAMPLE 3 *If* $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ *are the roots of the equation* $x^5-1=0$, *then prove that*

$$\frac{\omega-\alpha_1}{\omega^2-\alpha_1}\cdot\frac{\omega-\alpha_2}{\omega^2-\alpha_2}\cdot\frac{\omega-\alpha_3}{\omega^2-\alpha_3}\cdot\frac{\omega-\alpha_4}{\omega^2-\alpha_4}=\omega$$

where ω *is a non-real cube root of unity.*

SOLUTION Since $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the equation

$$x^5-1 = 0$$

$$\therefore x^5-1 = (x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

$$\Rightarrow \frac{x^5-1}{x-1} = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4) \quad ...(i)$$

Putting $x=\omega$ in (i), we get

$$\frac{\omega^5-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)$$

$$\Rightarrow \frac{\omega^2-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4) \quad ...(ii)$$

Putting $x=\omega^2$ in (i), we get

$$\frac{\omega^{10}-1}{\omega^2-1} = (\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)$$

$$\frac{\omega-1}{\omega^2-1} = (\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4) \quad ...(iii)$$

Dividing (ii) by (iii), we get

$$\frac{\omega-\alpha_1}{\omega^2-\alpha_1}\cdot\frac{\omega-\alpha_2}{\omega^2-\alpha_2}\cdot\frac{\omega-\alpha_3}{\omega^2-\alpha_3}\cdot\frac{\omega-\alpha_4}{\omega^2-\alpha_4}$$

$$= \frac{(\omega^2-1)^2}{(\omega-1)^2}$$

$$= \frac{\omega^4-2\omega^2+1}{\omega^2-2\omega+1}$$

$$= \frac{\omega-2\omega^2+1}{\omega^2+1-2\omega}$$

$$= \frac{-3\omega^2}{-3\omega} = \omega.$$

EXAMPLE 4 *If ω is an imaginary cube root of unity, prove that*

$$\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$$

SOLUTION We have,

$$1+\omega+\omega^2=0$$

$$\therefore \frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}$$

$$= \frac{1}{(1+\omega)+\omega}+\frac{1}{1+(1+\omega)}-\frac{1}{1+\omega}$$

$$= \frac{1}{\omega-\omega^2}+\frac{1}{1-\omega^2}-\frac{1}{-\omega^2}$$

$$= \frac{1}{\omega-\omega^2}+\frac{1}{1-\omega^2}+\frac{1}{\omega^2}$$

$$= \frac{\omega(1+\omega)+\omega^2+1-\omega^2}{\omega^2(1-\omega^2)}$$

$$= \frac{1+\omega+\omega^2}{\omega^2(1-\omega^2)}=0$$

EXAMPLE 5 *If ω, ω^2 are complex cube roots of unity and $a, b, c, d \in R$ such that*

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2$$

and, $$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=2\omega$$

then prove that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2$$

SOLUTION We have,

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2$$

and,

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=2\omega$$

$$\Rightarrow \frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=\frac{2}{\omega} \quad ...(i)$$

and,

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=\frac{2}{\omega^2} \quad ...(ii)$$

It is evident from these two results that ω and ω^2 are the roots of the equation

$$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x}=\frac{2}{x}$$

In order to prove the desired result, we have to prove that $x = 1$ is a root of the above equation.

Consider the polynomial

$$f(x) = x(a+x)(b+x)(c+x)(d+x)$$

$$\times\left[\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x}-\frac{2}{x}\right]$$

or, $f(x) = x(a+x)(b+x)(c+x)+x(a+x)(b+x)(d+x)$

$$+x(a+x)(c+x)(d+x)+x(b+x)(c+x)(d+x)$$

$$-2(a+x)(b+x)(c+x)(d+x)$$

$$= 2x^4+x^3(a+b+c+d)-x(abc+abd+acd+bcd)-2abcd$$

We have,

$$f(\omega) = \omega(a+\omega)(b+\omega)(c+\omega)(d+\omega)$$

$$\times\left[\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}-\frac{2}{\omega}\right] \quad \text{[Using (i)]}$$

$$= 0$$

and,

$$f(\omega^2) = \omega^2(a+\omega^2)(b+\omega^2)(c+\omega^2)(d+\omega^2)$$

$$\times\left[\frac{1}{(a+\omega)^2}+\frac{1}{(b+\omega)^2}+\frac{1}{(c+\omega)^2}+\frac{1}{(d+\omega)^2}-\frac{2}{\omega^2}\right]$$

$$= 0 \quad \text{[Using (ii)]}$$

Therefore, ω and ω^2 are two roots of $f(x) = 0$.

Since $f(x) = 0$ is a fourth degree equation in x. So, it has four roots.

Let α, β be two other roots of $f(x) = 0$. Then,

$$\alpha\beta+\omega.\omega^2+(\alpha+\beta)(\omega+\omega^2)=0$$

$$\Rightarrow \alpha\beta+1-\alpha-\beta=0$$

$$\Rightarrow (\alpha-1)(\beta-1)=0$$

$$\Rightarrow \alpha=1 \text{ or, } \beta=1$$

Now, $\alpha = 1$ is a root of $f(x) = 0$

$$\therefore f(1)=0$$

$$\Rightarrow (a+1)(b+1)(c+1)+(a+1)(b+1)(d+1)$$

$$+(a+1)(c+1)(d+1)+(b+1)(c+1)(d+1)$$

$$-2(a+1)(b+1)(c+1)(d+1)=0$$

$$\Rightarrow \frac{1}{d+1}+\frac{1}{c+1}+\frac{1}{b+1}+\frac{1}{a+1}=2$$

$$\Rightarrow \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2$$

EXAMPLE 5 *If 1, $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ be nth roots of unity and ω be any non-real cube root of unity, then prove that*

$$(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha^{n-1})=\begin{cases}0 & \text{, if } n \text{ is a multiple of } 3\\ 1 & \text{, if } n=3m+1, m\in N\\ 1+\omega & \text{, if } n=3m+2, m\in N\end{cases}$$

SOLUTION Since 1, $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ are nth roots of unity. Therefore, 1, $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ are the roots of $x^n-1=0$.

$$\therefore\quad x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})$$

Putting $x=\omega$, we get

$$\omega^n-1=(\omega-1)(\omega-\alpha_1)(w-\alpha_2)\ldots(\omega-\alpha_{n-1})$$

$$\Rightarrow (\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)\ldots(\omega-\alpha^{n-1})=\frac{\omega^n-1}{\omega-1}$$

Now,

$$\frac{\omega^n-1}{\omega-1}=\begin{cases}\dfrac{\omega^{3m}-1}{\omega-1}=\dfrac{1-1}{\omega-1}=0, & \text{if } n=3m\\ \dfrac{\omega^{3m+1}-1}{\omega-1}=\dfrac{\omega-1}{\omega-1}=1, & \text{if } n=3m+1\\ \dfrac{\omega^{3m+2}-1}{\omega-1}=\dfrac{\omega^2-1}{\omega-1}=\omega+1, & \text{if } n=3m+2\end{cases}$$

Hence,

$$(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha_{n-1})=\begin{cases}0 & \text{, if } n=3m\\ 1 & \text{, if } n=3m+1\\ \omega+1 & \text{, if } n=3m+2\end{cases}$$

EXAMPLE 6 *If ω and ω^2 are complex cube roots of unity, prove that*

(i) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^3+b^3+c^3-3abc$

(ii) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$

$=(2a-b-c)(2b-c-a)(2c-a-b)$

$=27abc$, if $a+b+c=0$

SOLUTION (i) We have,

$$(a+b+c)(a+b\omega+c\omega)^2(a+b\omega^2+c\omega)$$

$$=(a+b+c)\{a^2+b^2+c^2+ab(\omega+\omega^2)+bc(\omega^2+\omega^4)+ca(\omega+\omega^2)\}$$

$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$=a^3+b^3+c^3-3abc$$

(ii) We know that,

$$x^3+y^3=(x+y)(x\omega+y\omega^2)(x\omega^2+y\omega) \qquad ...(i)$$

Let $x=a+b\omega+c\omega^2$ and $y=a+b\omega^2+c\omega$. Then,

$$x+y=(a+b\omega+c\omega)^2+(a+b\omega^2+c\omega)$$

$$=2a+b(\omega+\omega^2)+c(\omega^2+\omega)$$

$$=2a-b-c$$

$$x\omega+y\omega^2=(a+b\omega+c\omega^2)\omega+(a+b\omega^2+c\omega)\omega^2$$

$$=a(\omega+\omega^2)+b(\omega^2+\omega^4)+c(\omega^3+\omega^3)$$

$$=(2c-a-b)$$

and,

$$x\omega^2+y\omega=(a+b\omega+c\omega^2)\omega^2+(a+b\omega^2+c\omega)\omega$$

$$=a(\omega^2+\omega)+b(\omega^3+\omega^3)+c(\omega^4+\omega^2)c$$

$$=-a+2b-c$$

$$=(2b-a-c)$$

Putting these values in (i), we get

$$(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$$

$$=(2a-b-c)(2c-a-b)(2b-a-c)$$

EXAMPLE 7 *If it is given that n is an odd integer greater than 3 but n is not a multiple of 3, prove that x^3+x^2+x is a factor of $(x+1)^n-x^n-1$.*

SOLUTION We have,

$$x^3+x^2+x=x(x^2+x+1)=x(x-\omega)(x-\omega^2)$$

Let $f(x)=(x+1)^n-x^n-1$

Since $x=0, \omega, \omega^2$ are roots of x^3+x^2+x. Therefore, to show that $f(x)=(x+1)^n-x^n-1$ is divisible by x^3+x^2+x, it is sufficient to show that

$$f(0)=0, f(\omega)=0 \text{ and } f(\omega^2)=0$$

Now,

$$f(0)=(0+1)^n-0^n-1=1^n-1=0$$

$$f(\omega)=(\omega+1)^n-\omega^n-1$$

$$=(-\omega^2)^n-\omega^n-1$$

$$=(-1)^n\omega^{2n}-\omega^n-1$$

$$=-\omega^{2n}-\omega^n-1 \qquad [\because n \text{ is odd}] \qquad ...(i)$$

$$=-(1+\omega^n+\omega^{2n})$$

$$=0\left[\because 1+\omega^n+\omega^{2n}=\begin{cases}0 & \text{, if } n \text{ is not a multiple of } 3\\ 3 & \text{, if } n \text{ is a multiple of } 3\end{cases}\right]$$

and,

$$f(\omega^2)=(\omega^2+1)^n-(\omega^2)^n-1$$

$$=(-\omega)^n-\omega^{2n}-1$$

$$=-\omega^n-\omega^{2n}-1 \qquad [\because n \text{ is odd}]$$

$$=-(1+\omega^n+\omega^{2n})$$

$$=0$$

Hence, $x(x-\omega)(x-\omega^2) = x^3 + x^2 + x$ is a factor of $f(x)$.

EXAMPLE 8 *If n is an odd integer greater than 3 but not a multiple of 3, prove that*

$$xy(x^2+xy+y^2) \text{ is a factor of } (x+y)^n - x^n - y^n.$$

SOLUTION We have,

$$xy(x^2+xy+y^2) = xy(x-y\omega)(x-y\omega^2)$$

In order to prove that $xy(x^2+xy+y^2)$ is a factor of $(x+y)^n - x^n - y^n$, it is sufficient to show that $(x+y)^n - x^n - y^n$ vanishes for $x=0$, $y=0$, $x=y\omega$ and $x=y\omega^2$.

For $x=0$, we have

$$(x+y)^n - x^n - y^n = y^n - 0 - y^n = 0$$

For $y = 0$, we have

$$(x+y)^n - x^n - y^n = x^n - x^n - 0 = 0$$

For $x = y\omega$, we have

$$(x+y)^n - x^n - y^n$$
$$= (y\omega + y)^n - (y\omega)^n - y^n$$
$$= (-y\omega^2)^n - (y\omega)^n - y^n$$
$$= (-1)^n y^n \omega^{2n} - y^n \omega^n - y^n$$
$$= -y^n \omega^{2n} - y^n \omega^n - y^n \qquad \left[\because n \text{ is odd}, \therefore (-1)^n = -1\right]$$
$$= -y^n(1 + \omega^{2n} + \omega^n + 1)$$
$$= -y^n \times 0$$
$$= 0$$

For $x = y\omega^2$, we have

$$(x+y)^n - x^n - y^n$$
$$= (y\omega^2 + y)^n - (y\omega^2)^n - y^n$$
$$= (-y\omega)^n - (y\omega^2)^n - y^n$$
$$= (-1)^n y^n \omega^n - y^n \omega^{2n} - y^n$$
$$= -y^n \omega^n - y^n \omega^{2n} - y^n \qquad \left[\because n \text{ is odd}, \therefore (-1)^n = -1\right]$$
$$= = -y^n(1 + \omega^n + \omega^{2n})$$
$$= -y^n \times 0$$
$$= 0$$

Hence, $xy(x^2+xy+y^2)$ is a factor of $(x+y)^n - x^n - y^n$

EXAMPLE 9 *If* $z_1 + z_2 + z_3 = A$,

$$z_1 + \omega z_2 + \omega^2 z_3 = B$$

and, $z_1 + \omega^2 z_2 + \omega z_3 = C$

(i) Express z_1, z_2, z_3 in terms of A, B, C.

(ii) Prove that

$$|A|^2 + |B|^2 + |C|^2 = 3\left\{|z_1|^2 + |z_2|^2 + |z_3|^2\right\}$$

SOLUTION We have,

$$z_1 + z_2 + z_3 = A \qquad \ldots(i)$$

$$z_1 + \omega z_2 + \omega^2 z_3 = B \qquad \ldots(ii)$$

$$z_1 + \omega^2 z_2 + \omega z_3 = C \qquad \ldots(iii)$$

$$\therefore A + B + C = 3z_1 + (1 + \omega + \omega^2) z_2 + (1 + \omega^2 + \omega) z_3$$
$$= 3z_1.$$

$$\Rightarrow z_1 = \frac{A+B+C}{3}$$

Multiplying (i) by 1, (ii) by ω^2, (iii) by ω and then adding, we get

$$z_1(1 + \omega^2 + \omega) + 3z_2 + z_3(1 + \omega + \omega^2) = A + B\omega^2 + C\omega$$

$$\Rightarrow z_2 = \frac{A + B\omega^2 + C\omega}{3}$$

Similarly, multiplying (i) by 1, (ii) by ω and (iii) by ω^2 and then adding, we get

$$z_3 = \frac{A + B\omega + C\omega^2}{3}$$

(ii) We have,

$$A = z_1 + z_2 + z_3$$

$$\Rightarrow \overline{A} = \overline{z}_1 + \overline{z}_2 + \overline{z}_3$$

$$\therefore |A|^2$$
$$= A\overline{A}$$
$$= (z_1 + z_2 + z_3)(\overline{z}_1 + \overline{z}_2 + \overline{z}_3)$$
$$= z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3 + z_1(\overline{z}_2 + \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 + \overline{z}_2)$$
$$= |z_1|^2 + |z_2|^2 + |z_3|^2 + z_1(\overline{z}_2 + \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 + \overline{z}_2)$$
$$= |z_1|^2 |z_2|^2 + |z_3|^2 + z_1(\overline{z}_2 + \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 + \overline{z}_2)$$

Similarly,

$$B = z_1 + \omega z_2 + \omega^2 z_3$$

$$\Rightarrow \overline{B} = \overline{z}_1 + \omega^2 \overline{z}_2 + \omega \overline{z}_3 \qquad [\because \overline{\omega} = \omega^2 \text{ and } \overline{\omega^2} = \omega]$$

$$\therefore |B|^2$$

$$= B\overline{B} = (z_1 + \omega z_2 + \omega^2 z_3)(\overline{z}_1 + \omega^2 \overline{z}_2 + \omega \overline{z}_3)$$

$$= z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3 + z_1(\omega^2 \overline{z}_2 + \omega \overline{z}_3) + z_2(\omega \overline{z}_1 + \omega^2 \overline{z}_3)$$
$$+ z_3(\omega^2 \overline{z}_1 + \omega \overline{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + |z_3|^2 + z_1(\omega^2\bar{z}_2 + \omega\bar{z}_3) + z_2(\omega\bar{z}_1 + \omega^2\bar{z}_3) + z_3(\omega^2\bar{z}_1 + \omega\bar{z}_2)$$

and,

$$C = z_1 + \omega^2 z_2 + \omega z_3$$

$$\Rightarrow \bar{C} = \bar{z}_1 + \omega\bar{z}_2 + \omega^2\bar{z}_3$$

$$\therefore |C|^2$$

$$= C\bar{C} = (z_1 + \omega^2 z_2 + \omega z_3)(\bar{z}_1 + \omega\bar{z}_2 + \omega^2\bar{z}_3)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_1(\omega\bar{z}_2 + \omega^2\bar{z}_3) + z_2(\omega^2\bar{z}_1 + \omega\bar{z}_3) + z_3(\omega\bar{z}_1 + \omega^2\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + |z_3|^2 + z_1(\omega\bar{z}_2 + \omega^2\bar{z}_3) + z_2(\omega^2\bar{z}_1 + \omega\bar{z}_3) + z_3(\omega\bar{z}_1 + \omega^2\bar{z}_2)$$

$$\therefore |A|^2 + |B|^2 + |C|^2$$

$$= 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$$

$$+ z_1[(1+\omega+\omega^2)\bar{z}_2 + (1+\omega+\omega^2)\bar{z}_3]$$

$$+ z_2[(1+\omega+\omega^2)\bar{z}_1 + (1+\omega+\omega^2)\bar{z}_3]$$

$$+ z_3[(1+\omega+\omega^2)\bar{z}_2 + (1+\omega+\omega^2)\bar{z}_1]$$

$$= 3(|z_1|^2 + |z_2|^2 + |z_3|^2$$

EXAMPLE 10 *Find the common roots of the equations*

$$x^3 + 2x^2 + 2x + 1 = 0 \text{ and } x^{2000} + x^{2002} + 1 = 0$$

SOLUTION We have,

$$x^3 + 2x^2 + 2x + 1 = 0$$

$$\Rightarrow (x^3 + 1) + 2x(x+1) = 0$$

$$\Rightarrow (x+1)(x^2 - x + 1) + 2x(x+1) = 0$$

$$\Rightarrow (x+1)(x^2 + x + 1) = 0$$

$$\Rightarrow (x-1)(x-\omega)(x-\omega^2) = 0$$

$$\Rightarrow x = -1, +\omega, \omega^2$$

Thus, $x = -1, \omega, \omega^2$ are the roots of the equation $x^3 + 2x^2 + 2x + 1 = 0$. We shall now see which of these values of x satisfy the second equation.

Clearly, $x = 0$ does not satisfy the equation

$$x^{2000} + x^{2002} + 1 = 0$$

For $x = \omega$, we have

$$\omega^{2000} + \omega^{2002} + 1 = \omega^2 + \omega + 1 = 0$$

For $x = \omega^2$, we have

$$(\omega^2)^{2000} + (\omega^2)^{2002} + 1 = \omega^{4000} + \omega^{4004} + 1 = \omega + \omega^2 + 1 = 0$$

Thus, ω and ω^2 are also the roots of $x^{2000} + x^{2002} + 1 = 0$.

Hence, ω and ω^2 are the common roots of the given equations.

EXAMPLE 11 *If* $ax + cy + bz = X$, $cx + by + az = Y$, $bx + ay + cz = Z$, *show that :*

(i) $(a + b^2 + c^2 - ab - bc - ca)(x^2 + y^2 + z^2 - xy - yz - zx)$

$$= X^2 + Y^2 + Z^2 - XY - YZ - ZX$$

(ii) $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz)$

$$= X^3 + Y^3 + Z^3 - 3XYZ.$$

SOLUTION (i) We know that :

$$(a + bw + cw^2)(a + bw^2 + cw) = a^2 + b^2 + c^2 - ab - bc - ca$$

$$\therefore (a^2 + b^2 + c^2 - ab - bc - ca)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= (a + bw + cw^2)(a + bw^2 + cw)(x + yw + zw^2)(x + yw^2 + zw)$$

$$= \{(a + bw + cw^2)(x + yw + zw^2)\}\{(a + bw^2 + cw)(x + yw^2 + zw)\}$$

$$= \{(ax + bz + cy) + (bx + ay + cz)w + (cx + by + az)w^2\}$$

$$\times \{(ax + cy + bz) + (az + by + cx)w + (bx + cz + ay)w^2\}$$

$$= (X + wZ + Yw^2)(X + Yw + Zw^2)$$

$$= (X + Yw^2 + Zw)(X + Yw + Zw^2)$$

$$= X^2 + Y^2 + Z^2 - XY - YZ - ZX$$

(ii) We have,

$$(a + b + c)(x + y + z)$$

$$= (ax + cy + bz) + (cx + by + az) + (bx + ay + cz)$$

$$= X + Y + Z.$$

Now, $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz)$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\times (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= (a + b + c)(x + y + z)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$(x^2 + y^2 - z^2 - xy - yz - zx)$$

$$= (X + Y + Z)(X^2 + Y^2 + Z^2 - XY - YZ - ZX) \quad \text{[Using (i)]}$$

$$= X^3 + Y^3 + Z^3 - XY - YZ - ZX.$$

EXAMPLE 12 *If* $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, *and,*

$$E_1 = a_0 + a_3 + a_6 + \ldots,$$

$$E_2 = a_1 + a_4 + a_7 + \ldots,$$

$$E_3 = a_2 + a_5 + a_8 + \ldots$$

prove that $E_1 = E_2 = E_3 = 3^{n-1}$.

SOLUTION We have,

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$$

$$\Rightarrow (1 + x + x^2)^n = (a_0 + a_3 x^3 + a_6 x^6 + \ldots)$$

$$+ x(a_1 + a_4 x^3 + a_7 x^6 + \ldots)$$

$$+ x^2(a_2 + a_5 x^3 + a_8 x^6 + \ldots)$$

Putting $x = 1, w$ and w^2 respectively, we get

$$3^n = E_1 + E_2 + E_3 \quad ...(i)$$

$$0 = E_1 + wE_2 + w^2 E_3 \quad ...(ii)$$

$$0 = E_1 + w^2 E_2 + w E_3 \quad ...(iii)$$

Adding (i), (ii) and (iii), we get

$$3E_1 = 3^n \Rightarrow E_1 = 3^{n-1}$$

Multiplying (ii) by w^2, (iii) by w and adding to (i), we get

$$3E_2 = 3^n \Rightarrow E_2 = 3^{n-1}$$

Putting the values of E_1 and E_2 in (i), we get

$$E_3 = 3^{n-1}$$

Hence, $E_1 = E_2 = E_3 = 3^{n-1}$.

EXAMPLE 13 *If* $(1+x)^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, *prove that*

(i) $a_0 + a_3 + a_6 + ... = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$

(ii) $a_1 + a_4 + a_7 + ... = \frac{1}{3}\left[2^n + 2\cos(n-2)\frac{\pi}{3}\right]$

(iii) $a_2 + a_5 + a_8 + ... = \frac{1}{3}\left[2^n + 2\cos(n+2)\frac{\pi}{3}\right]$

SOLUTION Let

$E_1 = a_0 + a_3 + a_6 + ...,$

$E_2 = a_1 + a_4 + a_7 + ...,$

$E_3 = a_2 + a_5 + a_8 + ...$

We have,

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

$$\Rightarrow (a_0 + a_3 x^3 + a_6 x^6 + ...) + x(a_1 + a_4 x^3 + a_7 x^6 + ...) + x^2 (a_2 + a_5 x^3 + ...) = (1+x)^n \quad ...(i)$$

Putting $x = 1$ in (i), we get

$$E_1 + E_2 + E_3 = 2^n \quad ...(ii)$$

Putting $x = w$ in (i), we get

$$E_1 + wE_2 + w^2 E_3 = (1+w)^n$$

$$\Rightarrow E_1 + wE_2 + w^2 E_3 = (-w^2)^n$$

$$\Rightarrow E_1 + wE_2 + w^2 E_3 = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n$$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$$

$$= \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$$

$$\Rightarrow E_1 + wE_2 + w^2 E_3 = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} \quad ...(iii)$$

Putting $x = w^2$ in (i), we get

$$E_1 + w^2 E_2 + wE_3 = (1+w^2)^n$$

$$= (-w)^n$$

$$= \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^n$$

$$= \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n$$

$$\Rightarrow E_1 + w^2 E_2 + wE_3 = \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3} \quad ...(iv)$$

Adding (ii), (iii) and (iv), we obtain

$$3E_1 = 2^n + 2\cos\frac{n\pi}{3}$$

$$\Rightarrow E_1 = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

$$\Rightarrow a_0 + a_3 + a_6 + ... = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

Multiplying (iii) by w^2 and (iv) by w and adding to (ii), we get

$$3E_2 = 2^n + \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)w^2 + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)w$$

$$\Rightarrow 3E_2 = 2^n + \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\Rightarrow 3E_2 = 2^n + \cos(n-2)\frac{\pi}{3} + i\sin(n-2)\frac{\pi}{3} + \cos(n-2)\frac{\pi}{3} - i\sin(n-2)\frac{\pi}{3}$$

$$\Rightarrow E_2 = \frac{1}{3}\left[2^n + 2\cos(n-2)\frac{\pi}{3}\right]$$

Multiplying (iii) by w and (iv) by w^2 and adding to (ii), we get

$$E_3 = \frac{1}{3}\left[2^n + 2\cos(n+2)\frac{\pi}{3}\right]$$

EXAMPLE 14 *Find the sum of the series :*

$$2(1+w)(1+w^2) + 3(2w+1)(2w^2+1) + 4(3w+1)(3w^2+1) + ... + (n+1)(nw+1)(nw^2+1)$$

SOLUTION We have,

Required sum

$$= \sum_{r=1}^{n} (r+1)(rw+1)(rw^2+1)$$

$$= \sum_{r=1}^{n} (r+1)(r+w^2)(r+w)$$

$$= \sum_{r=1}^{n} (r^3+1)$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + n$$

EXAMPLE 15 *Evaluate :*

$$\sum_{n=1}^{32} (3n+2) \left| \sum_{r=1}^{10} \left(\sin\frac{2r\pi}{11} - i\cos\frac{2r\pi}{11} \right) \right|^n$$

SOLUTION We have,

$$\sum_{r=1}^{10} \left(\sin\frac{2r\pi}{11} - i\cos\frac{2r\pi}{11} \right)$$

$$= -i \sum_{r=1}^{10} \left(\cos\frac{2r\pi}{11} + i\sin\frac{2r\pi}{11} \right)$$

$$= -i \sum_{r=1}^{10} e^{\frac{i2r\pi}{11}}$$

$$= -i \sum_{r=1}^{10} \left(e^{\frac{i2\pi}{11}} \right)^r$$

$$= -i \sum_{r=1}^{10} \alpha^r, \text{ where } \alpha = e^{\frac{i2\pi}{11}}$$

$$= -i\,\alpha \left(\frac{1-\alpha^{10}}{1-\alpha} \right)$$

$$= \left| \frac{i(\alpha - \alpha^{11})}{\alpha - 1} \right|$$

$$= i \left(\frac{\alpha - 1}{\alpha - 1} \right) \qquad [\because \alpha^{11} = e^{i2\pi} = 1]$$

$$= i$$

$$\therefore \sum_{n=1}^{32} (3n+2) \left| \sum_{r=1}^{10} \left(\sin\frac{2r\pi}{11} - i\cos\frac{2r\pi}{11} \right) \right|^n$$

$$= \sum_{n=1}^{32} (3n+2)\, i^n$$

$$= 3 \sum_{n=1}^{32} n \cdot i^n + 2 \sum_{n=1}^{32} i^n$$

$$= 3S_1 + 2S_2 \qquad \ldots(i)$$

where

$$S_1 = \sum_{n=1}^{32} n \cdot i^n \text{ and } S_2 = \sum_{n=1}^{32} i^n$$

Now,

$$S_1 = \sum_{n=1}^{32} n \cdot i^n$$

$$\Rightarrow S_1 = i + 2 \cdot i^2 + 3 \cdot i^3 + \ldots + 32 \cdot i^{32}$$

$$\Rightarrow i S_1 = i^2 + 2 \cdot i^3 + \ldots + 31\, i^{32} + 32\, i^{33}$$

$$\therefore S_1 - i S_1 = [i + i^2 + i^3 + \ldots + i^{32}] - 32\, i^{33}$$

$$\Rightarrow S_1(1-i) = i\left(\frac{1-i^{32}}{1-i} \right) - 32\, i^{33}$$

$$\Rightarrow S_1(1-i) = \frac{i(1-1)}{1-i} - 32\, i^{33}$$

$$\Rightarrow S_1 = \frac{-32i}{1-i}$$

$$= \frac{-32i(1+i)}{(1-i)(1+i)}$$

$$= -16\, i(1+i)$$

$$= 16(1-i)$$

and,

$$S_2 = \sum_{n=1}^{32} i^n = \frac{i(1-i^{32})}{1-i} = \frac{i(1-1)}{1-i} = 0$$

Putting the values of S_1 and S_2 in (i), we get

$$\sum_{n=1}^{32} (3n+2) \left| \sum_{r=1}^{10} \left(\sin\frac{2r\pi}{11} - i\cos\frac{2r\pi}{11} \right) \right|^n = 48(1-i)$$

EXAMPLE 16 *Find the integral solutions of the equation* $(1-i)^n = 2^n$.

SOLUTION We have,

$$(1-i)^n = 2^n$$

$$\Rightarrow |(1-i)^n| = |2^n|$$

$$\Rightarrow |1-i|^n = 2^n$$

$$\Rightarrow (\sqrt{2})^n = 2^n$$

$$\Rightarrow 2^{n/2} = 2^n$$

$$\Rightarrow 2^{n-n/2} = 1$$

$$\Rightarrow 2^{n/2} = 1$$

$$\Rightarrow \frac{n}{2} = 0$$

$$\Rightarrow n = 0$$

EXAMPLE 17 *Solve the following system of equations in terms of complex numbers :*

$$z^{13} w^{19} = 1, z^5 w^7 = 1, z^2 + w^2 = -2$$

SOLUTION We have, $z^{13} w^{19} = 1$ ...(i)

$$z^5 w^7 = 1 \quad ...(ii)$$

$$z^2 + w^2 = -2 \quad ...(iii)$$

From (ii), we have

$z^{15} w^{21} = 1$ ...(iv) [Cubing both sides]

$\Rightarrow \frac{z^{15} w^{21}}{z^{13} w^{19}} = \frac{1}{1}$ [Dividing (iv) by (i)]

$\Rightarrow z^2 w^2 = 1$

$\Rightarrow z^6 w^6 = 1$ ...(v) [Cubing both sides]

Dividing (ii) by (v), we get

$$\frac{w}{z} = 1 \Rightarrow w = z$$

Putting $z = w$ in (iii), we get

$$2z^2 = -2 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$$

But $z = w$. Therefore, $w = \pm i$

Hence, the solutions are : $z = i, w = i$; $z = -i, w = -i$

EXAMPLE 18 *Find the values of a, b and c if $x^3 - 1$ is a factor of $x^6 + ax^4 + bx^3 + cx^2 + 3x + 2$.*

SOLUTION Let $f(x) = x^6 + ax^4 + bx^3 + cx^2 + 3x + 2$.

We have,

$$x^3 - 1 = (x-1)(x^2 + x + 1) = (x-1)(x-w)(x-w^2)$$

If $x^3 - 1$ is a factor of $f(x)$, then $(x-1)$, $(x-w)$ and $(x-w^2)$ are also its factors.

$\therefore f(1) = 0, f(w) = 0$ and $f(w^2) = 0$

$\Rightarrow 1 + a + b + c + 3 + 2 = 0,$

$$w^6 + aw^4 + bw^3 + cw^2 + 3w + 2 = 0$$

and, $w^{12} + aw^8 + bw^6 + cw^4 + 3w^2 + 2 = 0$

$\Rightarrow a + b + c = -6$ ...(i)

$$aw + b + cw^2 = 3w^2 \quad ...(ii)$$

and, $aw^2 + b + cw = 3w$ ...(iii)

Adding (i), (ii) and (iii), we get

$$3b = -6 + 3w^2 + 3w$$

$\Rightarrow 3b = -6 - 3$

$\Rightarrow b = -3$

Multiplying (ii) by w^2, (iii) by w and adding them to (i), we obtain

$$3a = -6 + 3w^4 + 3w^2$$

$\Rightarrow 3a = -6 - 3$

$\Rightarrow a = -3$

Putting $a = -3, b = -3$ in (i), we get $c = 0$

Hence, $a = b = -3, c = 0$

EXAMPLE 19 *Given* $z = \cos\left(\frac{2\pi}{2n+1}\right) + i \sin\left(\frac{2\pi}{2n+1}\right)$, *n is a positive integer, find the equation whose roots are :*

$$\alpha = z + z^3 + z^5 + ... + z^{2n-1} \text{ and } \beta = z^2 + z^4 + ... + z^{2n}$$

SOLUTION We have,

$$\alpha + \beta = z + z^2 + ... + z^{2n}$$

$$= z\left(\frac{z^{2n} - 1}{z - 1}\right)$$

$$= \frac{z^{2n+1} - z}{z - 1}$$

$$= \frac{1 - z}{z - 1}$$

$$= -1 \qquad [\because z^{2n+1} = \cos 2\pi + i \sin 2\pi = 1]$$

and,

$$\alpha\beta = z(z + z^3 + z^5 + ... z^{2n-1})^2$$

$$= z^3 (1 + z^2 + z^4 + ... + z^{2n-2})^2$$

$$= z^3 \left[\frac{(z^2)^n - 1}{z^2 - 1}\right]^2$$

$$= z^3 \left[\frac{z^{2n} - 1}{z^2 - 1}\right]^2$$

$$= z \left[\frac{z^{2n+1} - z}{z^2 - 1}\right]^2$$

$$= z \left[\frac{1 - z}{z^2 - 1}\right]^2 \qquad [\because z^{2n+1} = \cos 2\pi + i \sin 2\pi = 1]$$

$$= \frac{z}{(z+1)^2}$$

$$= \frac{\cos\frac{2\pi}{2n+1} + i \sin\frac{2\pi}{2n+1}}{\left[\cos\frac{2\pi}{2n+1} + i \sin\frac{2\pi}{2n+1} + 1\right]^2}$$

$$= \frac{\cos\frac{2\pi}{2n+1} + i \sin\frac{2\pi}{2n+1}}{\left[2\cos^2\frac{\pi}{2n+1} + 2i \sin\frac{\pi}{2n+1}\cos\frac{\pi}{2n+1}\right]^2}$$

$$= \frac{\cos\frac{2\pi}{2n+1} + i \sin\frac{2\pi}{2n+1}}{4\cos^2\frac{\pi}{2n+1}\left[\cos\frac{2\pi}{2n+1} + i \sin\frac{2\pi}{2n+1}\right]}$$

$$= \frac{1}{4\cos^2\frac{\pi}{2n+1}}$$

So, the required equation is

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

or, $$x^2 + x + \frac{1}{4\cos^2\frac{\pi}{2n+1}} = 0$$

EXAMPLE 20 *If* $\alpha = e^{\frac{i2\pi}{7}}$ *and* $f(x) = a_0 + \sum_{k=1}^{20} A_k x^k$, *then find the value of* $f(x) + f(\alpha x) + \ldots + f(\alpha^6 x)$, *independent of* α.

SOLUTION We have,

$$f(x) + f(\alpha x) + \ldots + f(\alpha^6 x)$$

$$= \sum_{r=0}^{6} f(\alpha^r x)$$

$$= \sum_{r=0}^{6}\left[A_o + \sum_{k=1}^{20} A_k (\alpha^r x)^k\right]$$

$$= \sum_{r=0}^{6} A_o + \sum_{k=1}^{20} A_k \left|\sum_{r=0}^{6} (\alpha^r x)^k\right|$$

$$= \sum_{r=0}^{6} A_0 + \sum_{k=1}^{20} A_k \{x^k + (\alpha x)^k + (\alpha^2 x)^k + \ldots + (\alpha^6 x)^k\}$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k (1 + \alpha^k + \alpha^{2k} + \ldots + \alpha^{6k})$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left|\frac{1-(\alpha^k)^7}{1-\alpha^k}\right|$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left|\frac{1-\alpha^{7k}}{1-\alpha^k}\right|$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left\{\frac{1-1}{1-\alpha^k}\right\} \qquad [\because \alpha^{7k} = e^{i2k\pi} = 1]$$

$$= 7A_0$$

EXAMPLE 21 *Let a complex number* α, $\alpha \neq 1$, *be a root of the equation*

$$z^{p+q} - z^p - z^q + 1 = 0,$$

where p, q *are distinct primers. Show that either* $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ *or,* $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$, *but not both together.*

[IIT 2002]

SOLUTION Since $\alpha \neq 1$, is a root of the equation

$$z^{p+q} - z^p - z^q + 1 = 0$$

$$\therefore \quad \alpha^{p+q} - \alpha^p - \alpha^q + 1 = 0$$

$$\Rightarrow (\alpha^p - 1)(\alpha^q - 1) = 0$$

$$\Rightarrow \text{either } \alpha^p - 1 = 0 \text{ or } \alpha^q - 1 = 0$$

CASE I *When* $\alpha^p - 1 = 0$:

If $\alpha^p - 1 = 0$, then

$$1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1}$$

$$= \frac{1-\alpha^p}{1-\alpha}$$

$$= 0$$

CASE II When $\alpha^q - 1 = 0$:

If $\alpha^q - 1 = 0$, then

$$1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1}$$

$$= \frac{1-\alpha^q}{1-\alpha}$$

$$= 0$$

Now we have to show that

$1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ and $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$ do not hold simultaneously.

Suppose $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ and $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$ hold together.

Then, $\frac{1-\alpha^p}{1-\alpha} = 0$ and $\frac{1-\alpha^q}{1-\alpha} = 0$

$$\Rightarrow \alpha^p - 1 = 0 \text{ and } \alpha^q - 1 = 0$$

$$\Rightarrow \alpha^p = 1 \text{ and } \alpha^q = 1$$

$$\Rightarrow \alpha = 1^{1/p} \text{ and } \alpha = 1^{1/q}$$

$\Rightarrow \alpha$ is both pth and qth root of unity

$$\Rightarrow \alpha = e^{i2m\pi/p} \text{ and } \alpha = e^{i2n\pi/q}, \text{ where } m = 0, 1, 2, \ldots, p-1$$

and, $n = 0, 1, 2, \ldots, q-1$

$$\Rightarrow e^{i2m\pi/p} = e^{i2n\pi/q} \text{ for some values of } m \text{ and } n \text{ satisfying}$$

$$0 \le m \le p-1 \text{ and } 0 \le n \le q-1.$$

$$\Rightarrow e^{i2\pi\left(\frac{m}{p} - \frac{n}{q}\right)} = 1$$

$$\Rightarrow 2\pi\left(\frac{m}{p} - \frac{n}{q}\right) = 2\pi k \text{ for some integer } k$$

$$\Rightarrow \frac{m}{p} - \frac{n}{q} = k \text{ for some integer } k \qquad \ldots(i)$$

But, $0 \le m \le p-1$ and $0 \le n \le q-1$

$$\Rightarrow 0 \le \frac{m}{p} < 1 \text{ and } 0 \le \frac{n}{q} < 1$$

$$\Rightarrow -1 < \frac{m}{p} - \frac{n}{q} < 1$$

$\Rightarrow \frac{m}{p} - \frac{n}{q} = 0$ $[\because \frac{m}{p} - \frac{n}{q}$ is an integer (see (i))]

$\Rightarrow mq = np$

$\Rightarrow p \mid mq$ $[\because p \mid np$ and $np = mq]$

$\Rightarrow p \mid m$ $[\because p$ and q are distinct primes]

$\Rightarrow m = 0$ $[\because 0 \le m < p-1]$

$\Rightarrow n = 0$ $[\because np = mq]$

$\Rightarrow \alpha = e^{i\phi} = 1$ $[\because \alpha = e^{\frac{i2\pi n}{q}}]$

This is a contradiction to the hypothesis that $\alpha \neq 1$.

Hence, $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ and $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$ do not hold simultaneously.

EXERCISE 19.4

1. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$, $p_1, p_2, \ldots, p_n$ being real, prove that

$$(1+\alpha_1^2)(1+\alpha_2^2)\ldots(1+\alpha_n^2) = (1 - p_2 + p_4 + \ldots)^2 + (p_1 - p_3 + p_5 \ldots)^2$$

2. If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, prove that

$$(a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 \ldots)^2 = 2^n$$

3. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the nth roots of unity, then prove that

$$\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \ldots + \frac{1}{2-\alpha_{n-1}} = \frac{(n-2)2^{n-1}+1}{2^n - 1}$$

4. If α is a non-real complex nth roots of unity, prove that :

(i) $1 + 2\alpha + 3\alpha^2 + \ldots + n\alpha^{n-1} = -\frac{n}{2} - i\frac{n}{2}\cot\frac{r\pi}{n}$,

$r = 1, 2, \ldots, (n-1)$

(ii) $1 + 3\alpha + 5\alpha^2 + \ldots + (2n-1)\alpha^{n-1} = -n - in\cot\frac{r\pi}{n}$,

$r = 1, 2, \ldots, (n-1)$

5. If n is an odd integer but not a multiple of 3, prove that $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y)(x^2+xy+y^2)$

6. Find the common roots of the equation

$$x^3 - 2x^2 + 2x - 1 = 0 \text{ and } x^{2002} + x^{2000} + 1 = 0$$

7. Show that the polynomial $x^{4p} + x^{4p+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$, where $p, q, r, s \in N$.

8. If p, q, r are three consecutive integers greater than or equal to 3, prove that $x^p + x^q + x^r$ is divisible by $x^3 + x^2 + x$.

9. Solve $z^9 - 1 = 0$ and deduce that

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9} = \frac{1}{16}$$

10. If $\frac{1}{a+w} + \frac{1}{b+w} + \frac{1}{c+w} + \frac{1}{d+w} = \frac{2}{w}$, where a, b, c are real and w is a non-real cube root of unity, then prove that :

(i) $\frac{1}{a+w^2} + \frac{1}{b+w^2} + \frac{1}{c+w} + \frac{1}{d+w} = \frac{2}{w^2}$

(ii) $\Sigma abc = 2$

(iii) $a + b + c + d = 2abcd$

(iv) $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 2$

11. If w is a non-real cube root of unity, prove that

$$1 + w^n + w^{2n} = \begin{cases} 3, \text{ if } n \text{ is a multiple of } 3 \\ 0, \text{ if } n \text{ is not a multiple of } 3 \end{cases}$$

12. If w is a complex cube root of unity and $a, b, c \in R$, prove that

$$\frac{a+bw+cw^2}{c+aw+bw^2} + \frac{a+bw+cw^2}{b+cw+aw^2} = 1$$

13. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, prove that

(i) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

(iii) $\sin(2\alpha - \beta - \gamma) + 8\sin(2\beta - \gamma - \alpha) + 27\sin(2\gamma - \alpha - \beta) = 0$

14. If $P(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then show that $g(x)$ and $h(x)$ are divisible by $(x-1)$.

15. Show that for any real numbers p and m,

$$(e^{2mi\cot^{-1}p})\left(\frac{1+ip}{ip-1}\right)^m = 1$$

16. Compute :

$$\left(1+\frac{1+i}{2}\right)\left\{1+\left(\frac{1+i}{2}\right)^2\right\}\left\{1+\left(\frac{1+i}{2}\right)^{2^2}\right\}\left\{1+\left(\frac{1+i}{2}\right)^{2^3}\right\} \ldots \left\{1+\left(\frac{1+i}{2}\right)^{2^n}\right\}$$

17. If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$; and

$E_1 = a_0 + a_3 + a_6 + \ldots$,

$E_2 = a_1 + a_4 + a_7 + \ldots$,

$E_3 = a_2 + a_5 + \ldots$

Show that the values of E_1, E_2 and E_3 are respectively $\frac{1}{3}\left[2^n + 2\cos\frac{r\pi}{3}\right]$ with $r = n$ for E_1, $r = n-2$ for E_2 and $r = n+2$ for E_3.

18. If $\alpha = e^{i2\pi/7}$, $p = \alpha + \alpha^2 + \alpha^4$ and $q = \alpha^3 + \alpha^4 + \alpha^6$, then find the equation whose roots are p and q.

ANSWERS

6. $-w, -w^2$

16. $\frac{2}{1-i}\left\{1-\left(\frac{1}{2}\right)^{2^{n-1}}\right\}$

18. $x^2 + x + 2 = 0$

HINTS TO SELECTED PROBLEM

16. Let $x = \frac{1+i}{2}$. Then,

Given product $= (1+x)(1+x^2)(1+x^{2^2})\ldots(1+x^{2^n})$

$= \frac{1}{1-x}[1-(x^{2^n})^2]$

$= \frac{1}{1-x}[1-(x^2)^{2^n}]$

—❑❑❑—

CHAPTER

20

THEORY OF EQUATIONS

20.1 SOME DEFINITIONS

REAL POLYNOMIAL *Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is called a real polynomial of real variable x with real coefficients.*

For example, $2x^3 - 6x^2 + 11x - 6$, $x^2 - 4x + 3$ etc. are real polynomials.

COMPLEX POLYNOMIAL *If $a_0, a_1, a_2, \ldots, a_n$ be complex numbers and x is a varying complex number, then*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

is called a complex polynomial or a polynomial of complex variable with complex coefficients.

For example, $2x^2 - (3 + 7i)x + (9i - 3)$, $x^3 - 5i\,x^2 + (1 - 2i)x + 9$ etc. are complex polynomials.

DEGREE OF A POLYNOMIAL *A polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.*

For example, the polyonomials $2x^3 - 7x^2 + x + 5$ and $(3 - 2i)x^2 - ix + 5$ are of degree 3 and 2 respectively.

A polynomial of second degree is generally called a quadratic polynomial. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials.

POLYNOMIAL EQUATION *If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.*

If $f(x)$ is a polynomial of second degree, then $f(x) = 0$ is called a *quadratic* equation. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a, b, c \in C$, set of all complex numbers, and $a \neq 0$.

ROOTS OF AN EQUATION *The values of the variable satisfying the given equation are called its roots.*

In other words, $x = \alpha$ is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

The real roots of an equation $f(x) = 0$ are the x-coordinates of the points where the curve $y = f(x)$ crosses x-axis.

20.2 SOME RESULTS ON ROOTS OF AN EQUATION

The following are some results on the roots of a polynomial equation with rational coefficients :

I *An equation of degree n has n roots, real or imaginary.*

II *Surd and imaginary roots always occur in pairs i.e. if $2 - 3i$ is a root of an equation, then $2 + 3i$ is also its root. Similarly, if $2 + \sqrt{3}$ is a root of a given equation, then $2 - \sqrt{3}$ is also its roots.*

III *An odd degree equation has at least one real root, whose sign is opposite to that of its last term, provided that the coefficient of highest degree term is positive.*

IV *Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive, has at least two real roots, one positive and one negative.*

20.3 POSITION OF ROOTS OF A POLYNOMIAL EQUATION

If $f(x) = 0$ is an equation and a, b are two real numbers such that $f(a) f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots between a and b. In case $f(a)$ and $f(b)$ are of the same sign, then either no real root or an even number of real roots of $f(x) = 0$ lie between a and b.

DEDUCTIONS

1. *Every equation of an odd degree has at least one real root, whose sign is opposite to that of its last term, provided that the coefficient of first term is positive..*

2. *Every equation of an even degree whose last term is negative and the coefficient of first term positive, has at least two real roots, one positive and one negative.*

3. *If an equation has only one change of sign, it has one positive. root and no more.*

4. *If all the terms of an equation are positive.and the equation involves no odd powers of x, then all its roots are complex.*

ILLUSTRATION 1 *If $a, b, c, d \in R$ such that $a < b < c < d$, then show that the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct.*

SOLUTION Let $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$.

Then,

$f(a) = 2(a-b)(a-d) > 0$ $[\because a-b<0$ and $a-d<0]$

$f(b) = (b-a)(b-c) < 0$ $[\because b-a>0$ and $b-c<0]$

and, $f(d) = (d-a)(d-b) > 0$ $[\because d-a>0$ and $d-b>0]$

So, a root of $f(x) = 0$ lies between a and b and another root lies between b and d.

Hence, the roots of the given equation are real and distinct.

20.4 DESCARTES RULE OF SIGNS

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs from positive to negative and negative to positive in f(x).

Consider the equation $x^3 + 6x^2 + 11x - 6 = 0$. The signs of the various terms are :

$$+ \quad + \quad + \quad -$$

Clearly, there is only one change of sign in the expression $x^3 + 6x^2 + 11x - 6$. So, the given equation has at most one positive real root.

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of change of signs from positive to negative and negative to positive in $f(-x)$.

Now,

Consider the equation $x^4 + x^3 + x^2 - x - 1 = 0$. Let $f(x) = x^4 + x^3 + x^2 - x - 1$. Then, $f(-x) = x^4 - x^3 + x^2 + x - 1$.

The signs of various terms in $f(-x)$ are :

$$+ \quad - \quad + \quad + \quad -$$

We find that there are three changes of signs. So, the given equation has at most three negative real roots.

Finally, let us consider the polynomial $f(x) = x^4 + x^2 + 1$. We observe that $f(x)$ and $f(-x)$ do not have any change of signs. So, the equation f(x) = 0 has no real roots i.e. its all roots are imaginary.

20.5 ROOTS OF A QUADRATIC EQUATION WITH REAL COEFFICIENTS

An equation of the form

$$ax^2 + bx + c = 0 \qquad \text{...(i)}$$

where $a \neq 0, a, b, c \in R$ is called a quadratic equation with real coefficients.

The quantity $D = b^2 - 4ac$ is known as the discriminant of the quadratic equation in (i) whose roots are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The nature of the roots is as given below :

1. *The roots are real and distinct iff* $D > 0$.
2. *The roots are real and equal iff* $D = 0$.
3. *The roots are complex with non-zero imaginary part iff* $D < 0$.
4. *The roots are rational iff a, b, c are rational and D is a perfect square.*
5. *The roots are of the form* $p + \sqrt{q}$ $(p, q \in Q)$ *iff a, b, c are rational and D is not a perfect square.*
6. *If* $a = 1, b, c \in I$ *and the roots are rational numbers, then these roots must be integers.*
7. *If a quadratic equation in x has more than two roots, then it is an identity in x that is* $a = b = c = 0$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve for real* $x : x(x^2-1)(x+2) + 1 = 0$

SOLUTION We have,

$x(x^2-1)(x+2) + 1 = 0$

$\Leftrightarrow x(x-1)(x+1)(x+2) + 1 = 0$

$\Leftrightarrow \{x(x+1)\}\{(x-1)(x+2)\} + 1 = 0$

$\Leftrightarrow (x^2+x)(x^2+x-2) + 1 = 0$

$\Leftrightarrow y(y-2) - 1 = 0$, where $y = x^2 + x$

$\Leftrightarrow y^2 - 2y + 1 = 0$

$\Leftrightarrow (y-1)^2 = 0$

$\Leftrightarrow y = 1.$

$\Leftrightarrow x^2 + x = 1$

$\Leftrightarrow x^2 + x - 1 = 0$

$\Leftrightarrow x = \frac{-1 \pm \sqrt{5}}{2}$

Hence, the roots of the given equations are

$$\frac{-1+\sqrt{5}}{2} \text{ and } \frac{-1-\sqrt{5}}{2}.$$

EXAMPLE 2 *Solve :* $x^2 + \frac{x^2}{(x+1)^2} = 3.$

SOLUTION We have,

$$x^2 + \frac{x^2}{(x+1)^2} = 3$$

$\Leftrightarrow x^2+\left(\frac{x}{x+1}\right)^2=3$

$\Leftrightarrow \left(x-\frac{x}{x+1}\right)^2+2x\left(\frac{x}{x+1}\right)=3$ [IJsing $a^2+b^2=(a-b)^2+2ab$]

$\Leftrightarrow \left(\frac{x^2+x-x}{x+1}\right)^2+2\left(\frac{x^2}{x+1}\right)=3$

$\Leftrightarrow \left(\frac{x^2}{x+1}\right)^2+2\left(\frac{x^2}{x+1}\right)-3=0$

$\Leftrightarrow y^2+2y-3=0$, where $y=\frac{x^2}{x+1}$.

$\Leftrightarrow (y+3)(y-1)=0$

$\Leftrightarrow y=1$ or, $y=-3$

Now,

$y=1$

$\Rightarrow \frac{x^2}{x+1}=1$

$\Rightarrow x^2-x-1=0$

$\Rightarrow x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$

And,

$y=-3$

$\Rightarrow \frac{x^2}{x+1}=-3$

$\Rightarrow x^2+3x+3=0$

$\Rightarrow x=\frac{-3\pm\sqrt{9-12}}{2}=\frac{-3\pm i\sqrt{3}}{2}$

Hence, the roots of the given equation are

$$\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},\frac{-3+i\sqrt{3}}{2}\text{ and }\frac{-3-i\sqrt{3}}{2}.$$

EXAMPLE 3 *Solve* : $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$.

SOLUTION We have,

$(5+2\sqrt{6})(5-2\sqrt{6})=25-24=1$

$\Rightarrow 5-2\sqrt{6}=\frac{1}{5+2\sqrt{6}}$

$\therefore (5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$

$\Leftrightarrow (5+2\sqrt{6})^{x^2-3}+\frac{1}{(5+2\sqrt{6})^{x^2-3}}=10$

$\Leftrightarrow y+\frac{1}{y}=10$, where $y=(5+2\sqrt{6})^{x^2-3}$

$\Leftrightarrow y^2-10y+1=0$

$\Leftrightarrow y=\frac{10\pm\sqrt{100-4}}{2}$

$\Leftrightarrow y=5\pm2\sqrt{6}$

$\Leftrightarrow (5+2\sqrt{6})^{x^2-3}=5\pm2\sqrt{6}$

$\Leftrightarrow (5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})$ or, $(5+2\sqrt{6})^{x^2-3}=5-2\sqrt{6}$

$\Leftrightarrow (5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})$ or, $(5+2\sqrt{6})^{x^2-3}=\frac{1}{5+2\sqrt{6}}$

$\Leftrightarrow (5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})^1$ or $(5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})^{-1}$

$\Leftrightarrow x^2-3=1$ or, $x^2-3=-1$

Now,

$x^2-3=1 \Rightarrow x^2=4 \Rightarrow x=\pm2$

And,

$x^2-3=-1\Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

Hence, $x=\pm2,\pm\sqrt{2}$.

EXAMPLE 4 *Solve* $x^{(2/3)\{\log_2 x-1\}}=\sqrt{2}$

SOLUTION We have,

$x^{(2/3)\{\log_2 x-1\}}=\sqrt{2}$

$\Leftrightarrow \frac{2}{3}\{\log_2 x-1\}=\log_x\sqrt{2}$ [$\because a^x=N\Leftrightarrow \log_a N=x$]

$\Leftrightarrow \frac{2}{3}\{\log_2 x-1\}=\frac{1}{2}\log_x 2$

$\Leftrightarrow \frac{2}{3}\{\log_2 x-1\}=\frac{1}{2\log_2 x}$ $\left[\because \log_b a=\frac{1}{\log_a b}\right]$

$\Leftrightarrow \frac{2}{3}(y-1)=\frac{1}{2y}$, where $y=\log_2 x$

$\Leftrightarrow 4y^2-4y=3$

$\Leftrightarrow 4y^2-4y-3=0$

$\Leftrightarrow 4y^2-6y+2y-3=0$

$\Leftrightarrow (2y-3)(2y+1)=0$

$\Leftrightarrow y=\frac{3}{2}$ or, $y=-\frac{1}{2}$

Now, $y=\frac{3}{2}\Rightarrow \log_2 x=\frac{3}{2}\Rightarrow x=2^{3/2}=2\sqrt{2}$

And, $y=-\frac{1}{2}\Rightarrow \log_2 x=-\frac{1}{2}\Rightarrow x=2^{-1/2}=\frac{1}{\sqrt{2}}$

Hence, $x=2\sqrt{2}$ or, $x=\frac{1}{\sqrt{2}}$

EXAMPLE 5 *Solve* : $4^x+6^x=9^x$.

SOLUTION We have,

$4^x+6^x=9^x$

$\Rightarrow \frac{4^x}{9^x}+\frac{6^x}{9^x}=1$

$\Leftrightarrow \left(\frac{4}{9}\right)^x+\left(\frac{6}{9}\right)^x=1$

$$\Leftrightarrow \left(\frac{2}{3}\right)^{2x} + \left(\frac{2}{3}\right)^{x} - 1$$

$$\Leftrightarrow y^2 + y - 1 = 0, \text{ where } y = \left(\frac{2}{3}\right)^{x}$$

$$\Leftrightarrow y = \frac{-1 \pm \sqrt{1+4}}{2} \Leftrightarrow y = \frac{-1 \pm \sqrt{5}}{2}$$

Now, $y = \frac{-1+\sqrt{5}}{2} \Rightarrow \left(\frac{2}{3}\right)^{x} = \frac{-1+\sqrt{5}}{2}$

$$\Rightarrow \left(\frac{2}{3}\right)^{x} = \frac{\sqrt{5}-1}{2}$$

$$\Rightarrow \log_{10}\left(\frac{2}{3}\right)^{x} = \log_{10}\left(\frac{\sqrt{5}-1}{2}\right)$$

$$\Rightarrow x(\log_{10} 2 - \log_{10} 3) = \log_{10}(\sqrt{5}-1) - \log_{10} 2$$

$$\Rightarrow x = \frac{\log_{10}(\sqrt{5}-1) - \log_{10} 2}{\log_{10} 2 - \log_{10} 3}$$

And, $y = \frac{-1-\sqrt{5}}{2} \Rightarrow \left(\frac{2}{3}\right)^{x} = \frac{-1-\sqrt{5}}{2}$

This is not possible, because LHS is always positive whereas RHS is negative.

EXAMPLE 6 *Solve the equation* $e^{\sin x} - e^{-\sin x} = 4$.

SOLUTION Putting $e^{\sin x} = y$, the given equation reduces to

$$y - \frac{1}{y} = 4$$

$$\Leftrightarrow y^2 - 4y - 1 = 0$$

$$\Leftrightarrow y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

Now, $y = 2 + \sqrt{5}$

$$\Rightarrow e^{\sin x} = 2 + \sqrt{5}$$

$$\Rightarrow \sin x = \log_e (2+\sqrt{5}) \qquad [\because a^x = N \Leftrightarrow \log_a N = x]$$

This is not possible because LHS of this relation is always less than or equal to 1 whereas RHS is a always greater than one.

$$y = 2 - \sqrt{5} \Rightarrow e^{\sin x} = 2 - \sqrt{5}$$

This is also not possible, because LHS > 0 whereas RHS < 0.

Hence, the given equation has no solution.

EXAMPLE 7 *Find all the roots of the equation :*

$$4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0 \text{ if one root is } 3 + i\sqrt{6}.$$

SOLUTION We know that in an equation with real coefficients imaginary roots always occur in pairs. Therefore, $3 - i\sqrt{6}$ is also a root of the given equation.

Thus, $3 \pm i\sqrt{6}$ are two roots of the equation

$$4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0$$

We have,

$$(x - 3 + i\sqrt{6})(x - 3 - i\sqrt{6})$$

$$= (x-3)^2 + 6$$

$$= x^2 - 6x + 15$$

Now,

$$4x^4 - 24x^3 + 57x^2 + 18x - 45$$

$$= 4x^2(x^2 - 6x + 15) - 3(x^2 - 6x + 15)$$

$$= (x^2 - 6x + 15)(4x^2 - 3)$$

$$\therefore 4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0$$

$$\Rightarrow (x^2 - 6x + 15)(4x^2 - 3) = 0$$

$$\Rightarrow 4x^2 - 3 = 0, \; x^2 - 6x + 15 = 0$$

$$\Rightarrow x = \pm\frac{\sqrt{3}}{2}, 3 \pm i\sqrt{6}$$

Thus, the other roots of the given equation are :

$$3 - i\sqrt{6}, \pm\frac{\sqrt{3}}{2}$$

EXAMPLE 8 *Solve :* $\left|x^2 + 2x - 8\right| + x - 2 = 0$

SOLUTION We have,

$$x^2 + 2x - 8 \geq 0$$

$$\Rightarrow (x+4)(x-2) \geq 0$$

$$\Rightarrow x \leq -4 \text{ or, } x \geq 2.$$

Now, two cases arise.

CASE I *When* $x \leq -4$ *or* $x \geq 2$:

In this case, we have

$$x^2 + 2x - 8 \geq 0$$

$$\Rightarrow \left|x^2 + 2x - 8\right| = x^2 + 2x - 8.$$

Therefore,

$$\left|x^2 + 2x - 8\right| + x - 2 = 0$$

$$\Rightarrow x^2 + 2x - 8 + x - 2 = 0$$

$$\Rightarrow x^2 + 3x - 10 = 0$$

$$\Rightarrow (x+5)(x-2) = 0$$

$$\Rightarrow x = -5, 2.$$

Clearly, these two values of x satisfy the condition $x \leq -4$ or, $x \geq 2$.

Hence, in this case the solution is $x = -5, 2$.

CASE II *When* $-4 < x < 2$:

In this case, we have

$$x^2 + 2x - 8 < 0$$

$$\Rightarrow \left|x^2 + 2x - 8\right| = -(x^2 + 2x - 8)$$

Therefore,

$$\left|x^2 + 2x - 8\right| + x - 2 = 0$$

$$\Rightarrow -(x^2 + 2x - 8) + x - 2 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0$$

$$\Rightarrow x = 2, -3$$

Clearly, $x=-3$ satisfies the condition $-4<x<2$.

Hence, in this case the solution is $x=-3$.

Combining both the cases, the solutions of the given equation are :

$$x = 2, -3, -5.$$

EXAMPLE 9 *Solve the equation :*

$$x^2+2a\left|x-a\right|-3a^2=0, \text{ if } a<0.$$

SOLUTION We have,

$$|x-a| = \begin{cases} x-a, & \text{if } x\geq a \\ -(x-a), & \text{if } x<a \end{cases}$$

So, two cases arise.

CASE I *When* $x\geq a$ and $a<0$:

In this case, the given equation reduces to

$$x^2+2a(x-a)-3a^2 = 0$$

$$\Rightarrow \quad x^2+2ax-5a^2 = 0$$

$$\Rightarrow x = \frac{-2a\pm\sqrt{4a^2+20a^2}}{2} = -a\pm\sqrt{6}a$$

$$\Rightarrow x = -(\sqrt{6}+1)a, (\sqrt{6}-1)a$$

We observe that $(\sqrt{6}-1)a<0$ and $(\sqrt{6}-1)a<a$, because $a<0$.

Therefore, $x=(\sqrt{6}-1)a$ does not satisfy the condition $x\geq a$.

However, $x=-(\sqrt{6}+1)a$ satisfies the condition $x\geq a$.

Hence, in this case $x=-(\sqrt{6}+1)a$ is the solution.

CASE II *When* $x<a$ and $a<0$:

In this case, the given equation reduces to

$$x^2-2a(x-a)-3a^2 = 0$$

$$\Rightarrow \quad x^2-2ax-a^2 = 0$$

$$\Rightarrow x = \frac{2a\pm\sqrt{4a^2+4a^2}}{2} = a\pm\sqrt{2}a = a(1\pm\sqrt{2})$$

We observe that $a(1-\sqrt{2})>0$ $\quad[\because a<0]$

Therefore, $x=a(1-\sqrt{2})$ does not satisfy the condition $x<a$.

But, $x=a(1+\sqrt{2})$ satisfies this condition.

Thus, in this case $x=a(1+\sqrt{2})$ is the solution of the given equation.

Hence, $x=-(\sqrt{6}+1)a$ and $x=(1+\sqrt{2})a$ are the solutions of the given equation.

EXAMPLE 10 *Solve the equation*

$$\left|2x^2-5x+3\right|+(x-1) = 0$$

SOLUTION We have,

$$2x^2-5x+3 = 2x^2-2x-3x+3 = (x-1)(2x-3)$$

$$\therefore \quad 2x^2-5x+3\geq 0 \quad \text{, if } x\leq 1 \text{ or } x\geq\frac{3}{2}$$

and, $2x^2-5x+3<0$, if $1<x<\frac{3}{2}$.

Now two cases arise :

CASE I *When* $x\leq 1$ *or* $x\geq\frac{3}{2}$:

In this case, we have

$$2x^2-5x+3\geq 0$$

$$\therefore \quad |2x^2-5x+3| = 2x^2-5x+3.$$

Now,

$$|2x^2-5x+3|+(x-1) = 0$$

$$\Rightarrow 2x^2-5x+3+x-1 = 0$$

$$\Rightarrow 2x^2-4x+2 = 0$$

$$\Rightarrow x^2-2x+1 = 0$$

$$\Rightarrow x = 1$$

Clearly, $x=1$ satisfies the condition $x\leq 1$ or $x\geq\frac{3}{2}$.

Hence, $x=1$ is the only solution in this case.

CASE II *When* $1<x<\frac{3}{2}$:

In this case, we have

$$2x^2-5x+3<0$$

$$\Rightarrow |2x^2-5x+3| = -(2x^2-5x+3)$$

$$\therefore \quad |2x^2-5x+3|+(x-1) = 0$$

$$\Rightarrow -(2x^2-5x+3)+x-1 = 0$$

$$\Rightarrow -2x^2+6x-4 = 0$$

$$\Rightarrow x^2-3x+2 = 0$$

$$\Rightarrow x = 1, 2$$

Clearly, none of these two values belong to the interval $(1, 3/2)$ i.e. they do not satisfy the condition $1<x<3/2$. So, there is no solution in this case.

Hence, the only solution of the given equation is $x=1$.

EXAMPLE 11 *Solve the equation :*

$$|x^2-3x+2|+12x-24 = 0$$

SOLUTION We have,

$$x^2-3x+2 = (x-1)(x-2)$$

$$\therefore \quad x^2-3x+2\geq 0 \quad \text{, if } x\leq 1 \text{ or } x\geq 2$$

and, $x^2-3x+2<0$, if $1<x<2$

Two cases arise.

CASE I *When* $x\leq 1$ *or* $x\geq 2$:

In this case, we have

$$x^2-3x+2\geq 0.$$

$$\therefore \quad |x^2-3x+2| = x^2-3x+2$$

The given equation is

$$|x^2-3x+2|+12x-24=0$$

$\Rightarrow x^2-3x+2+12x-24=0$

$\Rightarrow x^2+9x-22=0$

$\Rightarrow (x+11)(x-2)=0$ $[\because x\geq 2 \text{ or } x\leq 1]$

$\Rightarrow x=-11, 2$

CASE II *When* $1<x<2$:

In this case, we have

$x^2-3x+2<0$

$\Rightarrow |x^2-3x+2|=-(x^2-3x+2)$

$\therefore |x^2-3x+2|+12x-24=0$

$\Rightarrow -(x^2-3x+2)+12x-24=0$

$\Rightarrow -x^2+15x-26=0$

$\Rightarrow x^2-15x+26=0$

$\Rightarrow (x-13)(x-2)=0$

$\Rightarrow x=2, 13$

But, x must lie between 1 and 2.

So, in this case there is no value of x satisfying the given equation.

Hence, $x=-11, 2$ are the only solutions of the given equation.

EXAMPLE 12 *Find all roots of the equation*

$$2\left|x^2-1\right|-\left|x^2-x-2\right|=0$$

SOLUTION We have, $x^2-1=(x-1)(x+1)$. The signs of x^2-1 for different values of x are as shown below :



Fig. 20.1

$$\therefore |x^2-1|=\begin{cases} x^2-1, & \text{if } x\leq -1 \text{ or } x\geq 1 \\ -(x^2-1), & \text{if } -1<x<1 \end{cases}$$

Also, $x^2-x-2=(x-2)(x+1)$.

So, the signs of x^2-x-2 for different values of x are as shown in Fig. 20.2.

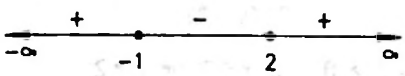


Fig. 20.2

$$\therefore |x^2-x-2|=\begin{cases} x^2-x-2, & \text{if } x\leq -1 \text{ or } x\geq 2 \\ -(x^2-x-2), & \text{if } -1<x<2 \end{cases}$$

Now, we have the following cases :

CASE I *When* $x\leq -1$:

In this case, we have

$|x^2-1|=x^2-1$

and, $|x^2-x-2|=x^2-x-2$.

$\therefore 2|x^2-1|-|x^2-x-2|=0$

$\Rightarrow 2(x^2-1)-(x^2-x-2)=0$

$\Rightarrow x^2+x=0$

$\Rightarrow x=0, -1$

$\Rightarrow x=-1$ $[\because x\leq -1]$

Thus, $x=-1$ is the only solution in this case.

CASE II *When* $-1<x<1$:

In this case, we have

$|x^2-1|=-(x^2-1)$

and, $|x^2-x-2|=-(x^2-x-2)$

$\therefore 2|x^2-1|-|x^2-x-2|=0$

$\Rightarrow -2(x^2-1)+x^2-x-2=0$

$\Rightarrow -x^2-x=0$

$\Rightarrow x=0, -1$

$\Rightarrow x=0$ $[\because -1<x<1]$

Thus, $x=0$ is the only solution in this case.

CASE III *When* $1\leq x<2$:

In this case, we have

$\left|x^2-1\right|=x^2-1$ and $\left|x^2-x-2\right|=-(x^2-x-2)$

$\therefore 2\left|x^2-1\right|-\left|x^2-x-2\right|=0$

$\Rightarrow 2(x^2-1)+x^2-x-2=0$

$\Rightarrow 3x^2-x-4=0$

$\Rightarrow (3x-4)(x+1)=0$

$\Rightarrow x=\frac{4}{3}, -1$

$\Rightarrow x=\frac{4}{3}$ $[\because -1\leq x<2]$

Thus, $x=\frac{4}{3}$ is the only solution in this case.

CASE IV *When* $x\geq 2$:

In this case, we have

$\left|x^2-1\right|=x^2-1$ and $\left|x^2-x-2\right|=x^2-x-2$

$\therefore 2\left|x^2-1\right|-\left|x^2-x-2\right|=0$

$\Rightarrow x^2+x=0$

$\Rightarrow x=0, -1$

But, none of these values satisfy the condition $x\geq 2$.

So, there is no solution in this case.

It follows from all these solutions that $x=-1, 0, \frac{4}{3}$ are the solutions of the given equation.

EXAMPLE 13 *Solve the following equation :*

$$| 4x - x^2 | - 2 | x - 2 | = 1$$

SOLUTION We have, $4x - x^2 = -x(x-4)$

The signs of $4x - x^2$ for different values of x are shown in Fig. 20.3.

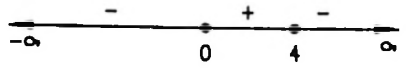


Fig. 20.3

$$\therefore \quad | 4x - x^2 | = \begin{cases} 4x - x^2, & \text{if } 0 \le x \le 4 \\ -(4x - x^2) = x^2 - 4x, & \text{if } x < 0 \text{ or } x > 4 \end{cases}$$

$$\text{Also, } | x - 2 | = \begin{cases} x - 2, & \text{if } x \ge 2 \\ -(x-2), & \text{if } x < 2 \end{cases}$$

Thus, we have the following cases :

CASE I *When* $x < 0$:

In this case, we have

$| 4x - x^2 | = x^2 - 4x$ and $| x - 2 | = -(x-2) = 2 - x$.

$\therefore \quad | 4x - x^2 | - 2 | x - 2 | = 1$

$\Rightarrow \quad x^2 - 4x - 2(2 - x) = 1$

$\Rightarrow \quad x^2 - 2x - 5 = 0$

$\Rightarrow \quad x = \dfrac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6}$

$\Rightarrow \quad x = 1 - \sqrt{6}$ $\quad [\because x < 0]$

So, $x = 1 - \sqrt{6}$ is the only solution in this case.

CASE II *When* $0 \le x < 2$:

In this case, we have

$| 4x - x^2 | = 4x - x^2$ and $| x - 2 | = 2 - x$.

$\therefore \quad | 4x - x^2 | - 2 | x - 2 | = 1$

$\Rightarrow \quad 4x - x^2 - 2(2 - x) = 1$

$\Rightarrow \quad -x^2 + 6x - 5 = 0$

$\Rightarrow \quad x^2 - 6x + 5 = 0$

$\Rightarrow \quad (x-1)(x-5) = 0$

$\Rightarrow \quad x = 1, 5$

$\Rightarrow \quad x = 1$ $\quad [\because 0 \le x < 2]$

So, $x = 1$ is the only solution in this case.

CASE III *When* $2 \le x \le 4$:

In this case, we have

$| 4x - x^2 | = 4x - x^2$ and $| x - 2 | = x - 2$.

$\therefore \quad | 4x - x^2 | - 2 | x - 2 | = 0$

$\Rightarrow \quad 4x - x^2 - 2(x - 2) = 0$

$\Rightarrow \quad -x^2 + 2x + 4 = 0$

$\Rightarrow \quad x^2 - 2x - 4 = 0$

$\Rightarrow \quad x = \dfrac{2 \pm \sqrt{4 + 16}}{2} = 1 \pm \sqrt{5}$

$\Rightarrow \quad x = 1 + \sqrt{5}$ $\quad [\because 2 \le x \le 4]$

Thus, $x = 1 + \sqrt{5}$ is the only solution in this case.

CASE IV *When* $x > 4$:

In this case, we have

$| 4x - x^2 | = -(4x - x^2)$ and $| x - 2 | = x - 2$.

$\therefore \quad | 4x - x^2 | - 2 | x - 2 | = 1$

$\Rightarrow \quad -(4x - x^2) - 2(x - 2) = 1$

$\Rightarrow \quad x^2 - 6x + 3 = 0$

$\Rightarrow \quad x = \dfrac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm \sqrt{6}$

$\Rightarrow \quad x = 3 + \sqrt{6}$ $\quad [\because x > 4]$

Thus, $x = 3 + \sqrt{6}$ is the only solution in this case.

Hence, $x = 1 - \sqrt{6}, 1, 1 + \sqrt{5}$ and $3 + \sqrt{6}$ are the solutions of the given equation.

EXAMPLE 14 *Solve :* $(x+2)^2 - 3 | x + 2 | + 2 = 0$

SOLUTION We have,

$$| (x+2)^2 | = (x+2)^2 = | x + 2 |^2 \qquad [\because (x+2)^2 \ge 0]$$

$\therefore \quad (x+2)^2 - 3 | x + 2 | + 2 = 0$

$\Rightarrow \quad | x + 2 |^2 - 3 | x + 2 | + 2 = 0$

$\Rightarrow \quad (| x + 2 | - 2)(| x + 2 | - 1) = 0$

$\Rightarrow \quad | x + 2 | = 2, | x + 2 | = 1$

$\Rightarrow \quad x + 2 = \pm 2, x + 2 = \pm 1$

$\Rightarrow \quad x = 0, -4, -1, -3$

EXAMPLE 15 *Solve :* $| x^2 - 3x - 4 | = 9 - | x^2 - 1 |$

SOLUTION We have, $x^2 - 3x - 4 = (x-4)(x+1)$

$\therefore \quad x^2 - 3x - 4 \ge 0$ $\quad$ for $x \le -1$ or $x \ge 4$

and,

$x^2 - 3x - 4 < 0$ $\quad$ for $-1 < x < 4$.

Also,

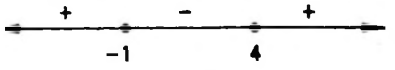
$x^2 - 1 = (x-1)(x+1)$

$\Rightarrow \quad x^2 - 1 \ge 0$ for $x \le -1$ or $x \ge 1$

and,

$x^2 - 1 < 0$ for $-1 < x < 1$.

The sign of $x^2 - 3x - 4$ and $x^2 - 1$ are for different values of x are shown in Fig. 20.4.



Signs of $(x^2 - 3x - 4)$

Fig. 20.4

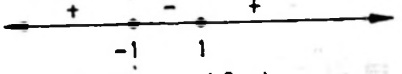


Fig. 20.5 Signs of $(x^2 - 1)$

Thus, we have the following cases :

CASE I *When* $x \le -1$

In this case, we have,

$x^2-1\geq 0$ and $x^2-3x-4\geq 0$.

$\Rightarrow |x^2-1| = x^2-1$ and $|x^2-3x-4| = x^2-3x-4$

$\therefore |x^2-3x-4| = 9-|x^2-1|$

$\Rightarrow x^2-3x-4 = 9-(x^2-1)$

$\Rightarrow 2x^2-3x-14 = 0$

$\Rightarrow (2x-7)(x+2) = 0$

$\Rightarrow x = \frac{7}{2}, -2$

$\Rightarrow x = -2$ $[\because x\leq -1]$

CASE II *When* $-1<x<1$:

In this case, we have

$x^2-1<0$ and $x^2-3x-4<0$

$\Rightarrow |x^2-1| = -(x^2-1) = 1-x^2$

and,

$|x^2-3x-4| = -(x^2-3x-4)$

$\therefore |x^2-3x-4| = 9-|x^2-1|$

$\Rightarrow -(x^2-3x-4) = 9+x^2-1$

$\Rightarrow 2x^2-3x+4 = 0$

The discriminant of this equation is imaginary. So, the equation has imaginary roots.

Thus, there is no solution in this case.

CASE III When $1\leq x<4$:

In this case, we have

$x^2-1\geq 0$ and $x^2-3x-4<0$

$\Rightarrow |x^2-1| = x^2-1$ and $|x^2-3x-4| = -(x^2-3x-4)$

$\therefore |x^2-3x-4| = 9-|x^2-1|$

$\Rightarrow -(x^2-3x-4) = 9-(x^2-1)$

$\Rightarrow 3x-6 = 0$

$\Rightarrow x = 2$

Clearly, $x=2$ satisfies $1\leq x<4$.

So, $x=2$ is the solution in this case.

CASE IV *When* $x\geq 4$.

In this case, we have,

$x^2-3x-4\geq 0$ and $x^2-1\geq 0$

$\Rightarrow |x^2-3x-4| = x^2-3x-4$ and $|x^2-1| = x^2-1$

$\therefore |x^2-3x-4| = 9-|x^2-1|$

$\Rightarrow x^2-3x-4 = 9-(x^2-1)$

$\Rightarrow 2x^2-3x-14 = 0$

$\Rightarrow (2x-7)(x+2) = 0$

$\Rightarrow x = \frac{7}{2}, -2$

But $x\geq 4$; so there is no solution in this case.

Hence, the solutions of the given equation are: $x=-2, 2$.

EXAMPLE 16 *For what values of a, the equation*

$$(a^2-a-2)x^2+(a^2-4)x+a^2-3a+2=0$$

will have more than two solutions ?

Does there exist a value of x for which the above equation will be an identity in a ?

SOLUTION We know that the equation $ax^2+bx+c=0$ is an identity in x (i.e. the equation has more than two roots), if $a=b=c=0$. So, the given quadratic equation will have more than two solutions, if

$a^2-a-2=0, a^2-4=0$ and $a^2-3a+2=0$

Now,

$$a^2-a-2=0 \Rightarrow (a-2)(a+1)=0 \Rightarrow a=-1, 2$$

$$a^2-4=0 \Rightarrow a=-2, 2$$

and, $a^2-3a+2=0 \Rightarrow a=1, 2$

Clearly, $a=2$ is the common solution of three equations in a.

Thus, the given equation will have more than two solutions, if $a=2$.

To find the values of x for which the equation becomes an identity in a, we write it as an equation in a as given below:

$a^2(x^2+x+1)-(x^2+3)a-2x^2-4x+2 = 0$

This will be an identity in a, if

$x^2+x+1=0, -(x^2+3)=0$ and $-2x^2-4x+2=0$

Clearly, there is no real value of x satisfying all the above equations in x simultaneously.

Hence, the given equation cannot be an identity in a.

EXAMPLE 17 *For* $x\in R, f(x)$ *is defined as follows:*

$$f(x) = \begin{cases} x+1 & ,0\leq x<2 \\ |x-4| & ,x\geq 2 \end{cases}$$

then solve the equation $f^2(x)+x = f(x)+x^2$.

SOLUTION We have,

$$f(x) = \begin{cases} x+1 & ,0\leq x<2 \\ -(x-4) & ,2\leq x<4 \\ x-4 & ,x\geq 4 \end{cases}$$

So, we have the following cases:

CASE I *When* $0\leq x<2$:

In this case, we have $f(x)=x+1$.

$\therefore f^2(x)+x = f(x)+x^2$

$\Rightarrow (x+1)^2+x = x+1+x^2$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

CASE II *When* $2\leq x<4$:

In this case, we have

$f(x) = -(x-4) = 4-x$

$\therefore f^2(x)+x = f(x)+x^2$

$\Rightarrow (4-x)^2+x = 4-x+x^2$

$\Rightarrow -6x+12 = 0$

$\Rightarrow x = 2$

CASE III *When* $x \geq 4$:

In this case, we have

$f(x) = x-4$

$\therefore f^2(x)+x = f(x)+x^2$

$\Rightarrow (x-4)^2+x = x-4+x^2$

$\Rightarrow -8x+20 = 0$

$\Rightarrow x = \frac{5}{4}$

But, $x \geq 4$. Therefore, there is no solution in this case.

Hence, $x = 0, 2$.

EXAMPLE 18 *Solve the equation :*

$$x \mid x+1 \mid + a = 0 \text{ for every real number } a.$$

SOLUTION We have,

$$\mid x+1 \mid = \begin{cases} x+1 & , \text{if } x \geq -1 \\ -(x+1) & , \text{if } x < -1 \end{cases}$$

Therefore, we have the following two cases :

CASE I *When* $x < -1$:

In this case, we have

$\mid x+1 \mid = -(x+1)$

$\therefore x \mid x+1 \mid + a = 0$

$\Rightarrow -x(x+1)+a = 0$

$\Rightarrow x^2+x-a = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1+4a}}{2}$

$\Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{1+4a}}{2}$.

Clearly, the values of x are real, if $1+4a \geq 0$ i.e. $a \geq -\frac{1}{4}$.

Out of these values of x, we have to take only those values which satisfy the inequation $x < -1$ i.e. $x+1 < 0$.

We have,

$x = -\frac{1}{2} \pm \frac{\sqrt{1+4a}}{2}$

$\Rightarrow x+1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4a}$

$\Rightarrow x+1 = \frac{1}{2}(1 \pm \sqrt{1+4a})$

Clearly, $x+1 < 0$, if $(1 \pm \sqrt{1+4a}) < 0$.

We observe that $1+\sqrt{1+4a} > 0$ for all $a \geq -\frac{1}{4}$

Also, $1-\sqrt{1+4a} \geq 0$ for all $a \in [-1/4, 0]$.

However, $1-\sqrt{1+4a} < 0$ for all $a > 0$.

Hence, $x = -\frac{1}{2} - \frac{1}{2}\sqrt{1+4a}$ for all $a > 0$ is a solution of the given equation.

CASE II *When* $x \geq -1$

In this case, we have

$\mid x+1 \mid = x+1$.

$\therefore x \mid x+1 \mid + a = 0$

$\Rightarrow x(x+1)+a = 0$

$\Rightarrow x^2+x+a = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1-4a}}{2}$

$\Rightarrow x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$

Clearly, x is real for $1-4a \geq 0$ i.e. $a \leq \frac{1}{4}$.

We have to take the value of x for which $x \geq -1$ i.e. $x+1 \geq 0$.

Now,

$x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$

$\Rightarrow x+1 = \frac{1}{2} \pm \frac{\sqrt{1-4a}}{2} = \frac{1}{2}(1 \pm \sqrt{1-4a})$

Clearly,

$x+1 \geq 0$, if $x = \frac{1}{2}(-1+\sqrt{1-4a})$ and $a \leq \frac{1}{4}$

Also,

$x+1 \geq 0$, if $x = \frac{1}{2}(-1-\sqrt{1-4a})$ and $0 \leq 1-4a < 1$ i.e. $0 \leq a \leq 1/4$

Hence, $x = \begin{cases} -\frac{1}{2} - \frac{\sqrt{1-4a}}{2} \text{ for all } a \leq \frac{1}{4} \\ -\frac{1}{2} - \frac{\sqrt{1-4a}}{2} \text{ for } 0 \leq a \leq \frac{1}{4} \end{cases}$

EXAMPLE 19 *If* $P(x) = ax^2+bx+c$ and $Q(x) = -ax^2+bx+c$, *where* $ac \neq 0$, *then show that the equation* $P(x) \cdot Q(x) = 0$ *has at least two real roots.*

SOLUTION Let D_1 and D_2 be the discriminants of equations $P(x) = 0$ and $Q(x) = 0$ respectively. Then,

$$D_1 = b^2-4ac \text{ and } D_2 = b^2+4ac$$

Since $ac \neq 0$, therefore either $ac > 0$ or $ac < 0$

If $ac > 0$, then $D_2 = b^2+4ac > 0$ but $D_1 = b^2-4ac$ may be positive or negative. Consequently roots of $Q(x) = 0$ are real.

If $ac < 0$, then $D_1 = b^2-4ac > 0$ but $D_2 = b^2+4ac$ may be negative or positive. Consequently roots of $P(x) = 0$ are real.

Hence, at least two roots of $P(x) \cdot Q(x) = 0$ are real.

EXAMPLE 20 *Show that if p, q, r and s are real numbers and $pr = 2(q+s)$, then at least one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.*

SOLUTION Let D_1 and D_2 be the discriminats of $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ respectively. Then,

$$D_1 + D_2 = (p^2 - 4q) + (r^2 - 4s)$$

$$= (p^2 + r^2) - 4(q+s)$$

$$= p^2 + r^2 - 2pr \qquad [\because pr = 2(q+s)]$$

$$= (p-r)^2 \geq 0$$

$\Rightarrow$ At least one of D_1 and D_2 is positive

$\Rightarrow$ At least one of the two equations has real roots.

EXAMPLE 21 *If the roots of the equation*

$$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$$

are equal, show that $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$.

SOLUTION Since the roots of the given equations are equal, therefore its discriminant is zero i.e.

$$b^2(c-a)^2 - 4a(b-c) \cdot c(a-b) = 0$$

$$\Rightarrow b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0$$

$$\Rightarrow a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0$$

$$\Rightarrow (ab + bc - 2ac)^2 = 0$$

$$\Rightarrow ab + bc - 2ac = 0$$

$$\Rightarrow ab + bc = 2ac$$

$$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b} \qquad \text{[Dividing both sides by } abc\text{]}$$

$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

ALITER Clearly, $x = 1$ is a root of the given equation. Since the given equation has equal roots. So, the other root is also 1.

$\therefore$ Product of the roots = 1

$$\Rightarrow \frac{c(a-b)}{a(b-c)} = 1$$

$$\Rightarrow 2ca = ab + bc$$

$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \qquad \text{[Dividing both sides by } abc\text{]}$$

Hence, a, b, c are in H.P.

EXAMPLE 22 *If the roots of the equation*

$$(b-c)x^2 + (c-a)x + (a-b) = 0$$

are equal, then prove that $2b = a + c$.

SOLUTION If the roots of the given equation are equal, then its discriminant is zero i.e.

$$(c-a)^2 - 4(b-c)(a-b) = 0$$

$$\Rightarrow c^2 + a^2 - 2ac + 4b^2 - 4ab + 4ac - 4bc = 0$$

$$\Rightarrow c^2 + a^2 + 4b^2 + 2ac - 4ab - 4bc = 0$$

$$\Rightarrow (c + a - 2b)^2 = 0 \Rightarrow c + a = 2b.$$

ALITER Clearly, $x = 1$ satisfies the given equation. As the given equation has equal roots. So, the other root is also 1.

$\therefore$ Product of the roots = 1

$$\Rightarrow \frac{a-b}{b-c} = 1$$

$$\Rightarrow 2b = a + c$$

$\Rightarrow a, b, c$ are in AP.

EXAMPLE 23 *If c, d are the roots of the equation $(x-a)(x-b) - k = 0$, prove that a, b are the roots of the equation $(x-c)(x-d) + k = 0$.* [IIT 1992]

SOLUTION Since c and d are the roots of the equation $(x-a)(x-b) - k = 0$. Therefore,

$$(x-a)(x-b) - k = (x-c)(x-d)$$

$$\Rightarrow (x-a)(x-b) = (x-c)(x-d) + k$$

$$\Rightarrow (x-c)(x-d) + k = (x-a)(x-b)$$

Clearly, a and b are roots of the equation $(x-a)(x-b) = 0$

$$\Rightarrow a, b \text{ are roots of } (x-c)(x-d) + k = 0 \left[\because \begin{matrix}(x-a)(x-b) \\ = (x-c)(x-d) + k\end{matrix}\right]$$

EXAMPLE 24 *If $a < b < c < d$, then show that the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct.*

SOLUTION We have,

$$(x-a)(x-c) + 2(x-b)(x-d) = 0$$

$$\Rightarrow 3x^2 - (a + c + 2b + 2d)x + (ac + 2bd) = 0$$

Let D be the discriminant of this equation. Then,

$$D = (a + c + 2b + 2d)^2 - 12(ac + 2bd)$$

$$= \{(a + 2d) + (c + 2b)\}^2 - 12(ac + 2bd)$$

$$= \{(a + 2d) - (c + 2b)\}^2 + 4(a + 2d)(c + 2b) - 12(ac + 2bd)$$

$$= \{(a + 2d) - (c + 2b)\}^2 + 8(ab + cd - ac - bd)$$

$$= \{(a + 2d) - (c + 2b)\}^2 + 8(c - b)(d - a)$$

$$> 0 \qquad [\because a < b < c < d \Rightarrow c - b > 0 \text{ and } d - a > 0]$$

Hence, the roots of the given equation are real and distinct.

ALITER Let $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$. Then,

$$f(b) = (b-a)(b-c) < 0 \quad \left[\begin{matrix}\because a < b < c < d \\ \therefore (b-a) > 0 \text{ and } (b-c) < 0\end{matrix}\right]$$

and, $f(d) = (d-a)(d-b) > 0 \quad \left[\begin{matrix}\because a < b < c < d \\ \therefore (d-a) > 0 \text{ and } (d-b) > 0\end{matrix}\right]$

So, a root of $f(x) = 0$ lies between b and d.

Now, $f(a) = 2(a-b)(a-d) > 0 \quad [\because (a-b) < 0 \text{ and } (a-d) < 0]$

So, a root of $f(x) = 0$ lies between a and b.

Hence, the roots of the given equation are real and distinct.

EXAMPLE 25 *If the coefficients of a quadratic equation* $ax^2 + bx + c = 0\ (a \neq 0)$ *are all odd integers, show that the roots cannot be rational.*

SOLUTION Let $a = 2m + 1, b = 2n + 1$ and $c = 2p + 1$, where m, n, p are integers. Further, let D be the discriminant of the given quadratic equation. Then,

$$\begin{aligned} D &= b^2 - 4ac \\ &= (2n+1)^2 - 4(2m+1)(2p+1) \\ &= 4n^2 + 4n + 1 - 4(2m+1)(2p+1) \\ &= 4n^2 + 4n - 4(2m+1)(2p+1) + 1 \end{aligned}$$

For roots to be rational, D must be a perfect square. This is possible only if the discriminant of the quadratic

$$4n^2 + 4n - 4(2m+1)(2p+1) + 1 = 0 \qquad ...(i)$$

is zero.

Let D_1 be the discriminant of (i). Then,

$$\begin{aligned} D_1 &= 16 - 16\{1 - 4(2m+1)(2p+1)\} \\ &= 64(2m+1)(2p+1) \end{aligned}$$

Clearly, $D_1 \neq 0$, because m and p are integers.

Therefore, the roots of the given equation cannot be rational.

EXAMPLE 26 *If the coefficients of the equations* $a_i x^2 + b_i x + c_i = 0, i = 1, 2$ *are real and* $2(a_1 c_1 + a_2 c_2) = b_1 b_2$, *then prove that at least one of the two given equations will have real roots.*

SOLUTION The two given quadratic equations are :

$$a_1 x^2 + b_1 x + c_1 = 0 \qquad ...(i)$$

and,

$$a_2 x^2 + b_2 x + c_2 = 0 \qquad ...(ii)$$

Let D_1 and D_2 be the discriminants of (i) and (ii) respectively. Then,

$D_1 = b_1^2 - 4a_1 c_1$ and $D_2 = b_2^2 - 4a_2 c_2$.

We have,

$$\begin{aligned} D_1 + D_2 &= b_1^2 - 4a_1 c_1 + b_2^2 - 4a_2 c_2 \\ &= b_1^2 + b_2^2 - 4(a_1 c_1 + a_2 c_2) \\ &= b_1^2 + b_2^2 - 2b_1 b_2 \quad [\because 2(a_1 c_1 + a_2 c_2) = b_1 b_2 \text{ (given)}] \\ &= (b_1 - b_2)^2 \geq 0 \end{aligned}$$

$\Rightarrow$ At least one of D_1, D_2 should be positive

Hence, at least one of the given equations has real roots.

EXAMPLE 27 *If a, b, c are positive integers and if the roots of the equation* $ax^2 - bx + c = 0$ *lie in the open interval* $(0, 1)$, *find the minimum values of a, b, c.*

SOLUTION Let α, β be the roots of the equation $ax^2 - bx + c = 0$. Then,

$$\alpha + \beta = \frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

It is given that

$\alpha, \beta \in (0, 1)$

$\Rightarrow 0 < \alpha < 1$ and $0 < \beta < 1$

$\Rightarrow 0 < 1 - \alpha < 1$ and $0 < 1 - \beta < 1$.

Now, A.M $\geq$ G.M.

$$\Rightarrow \frac{\alpha + (1-\alpha)}{2} \geq \{\alpha(1-\alpha)\}^{1/2} \text{ and } \frac{\beta + (1-\beta)}{2} \geq \{\beta(1-\beta)\}^{1/2}$$

$$\Rightarrow \frac{1}{2} \geq \{\alpha(1-\alpha)\}^{1/2} \text{ and } \frac{1}{2} \geq \{\beta(1-\beta)\}^{1/2}$$

$$\Rightarrow 0 < \alpha(1-\alpha) \leq \frac{1}{4} \text{ and } 0 < \beta(1-\beta) \leq \frac{1}{4}$$

$[\because \alpha, \beta, (1-\alpha), (1-\beta) \in (0, 1)]$

$$\Rightarrow 0 < \alpha(1-\alpha)\beta(1-\beta) < \frac{1}{16}$$

and, $0 < \beta(1-\beta) < \frac{1}{4}$ $\quad \left[\because \alpha(1-\alpha) = \frac{1}{4} \text{ and } \beta(1-\beta) = \frac{1}{4} \Rightarrow \alpha = \beta = \frac{1}{2}. \text{ But } \alpha \neq \beta \text{ (given)}\right]$

$$\Rightarrow 0 < \alpha\beta(1 - \alpha - \beta + \alpha\beta] < \frac{1}{16}$$

$$\Rightarrow 0 < \alpha\beta[1 - (\alpha + \beta) + \alpha\beta] < \frac{1}{16}$$

$$\Rightarrow 0 < \frac{c}{a}\left[1 - \frac{b}{a} + \frac{c}{a}\right] < \frac{1}{16}$$

$$\Rightarrow 0 < \frac{c(a-b+c)}{a^2} < \frac{1}{16}$$

$$\Rightarrow 0 < c(a-b+c) < \frac{a^2}{16} \qquad ...(i)$$

Since a, b, c are integers, therefore $c(a - b + c)$ is an integer.

From (i), we have

$$c(a-b+c) > 0$$

$\therefore\ c(a-b+c)$ is a positive integer.

$\Rightarrow c(a-b+c) \geq 1$

$\Rightarrow 1 < \frac{a^2}{16}$ [Using (i)]

$\Rightarrow a^2 > 16$

$\Rightarrow a > 4$

$\Rightarrow a = 5, 6, 7, ...$

Hence, the minimum value of a is 5.

Since the roots of $ax^2 - bx + c = 0$ are real and distinct as they lie in (0, 1). Therefore,

$b^2 - 4ac > 0$ [Discriminant ≥ 0]

$\Rightarrow b^2 > 4ac$ $[\because a \geq 5]$

$\Rightarrow b^2 > 20$ [$\because$ The minimum value of c is 1]

$\Rightarrow b \geq 5$

$\Rightarrow b = 5, 6, 7, \ldots,$

So, the minimum value of b is also 5.

Since there is no other condition and $c \in N$, therefore the minimum value of c is 1.

EXAMPL 28 *Consider the quadratic equation $ax^2 - bx + c = 0$, $a, b, c \in N$. If the given equation has two distinct real roots belonging to the interval $(1, 2)$, show that $a \geq 5$ and $b \geq 11$. Also, find the minimum value of c.*

SOLUTION Let α, β be the roots of the given equation. Then,

$$\alpha + \beta = \frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

It is given that $\alpha, \beta \in (1, 2)$. Therefore,

$$(\alpha - 1), (\beta - 1), (2 - \alpha), (2 - \beta) \in (0, 1).$$

Now,

$$\text{AM} \geq \text{GM}$$

$$\Rightarrow \frac{(\alpha - 1) + (2 - \alpha)}{2} > \{(\alpha - 1)(2 - \alpha)\}^{1/2}$$

$$\text{and, } \frac{(\beta - 1) + (2 - \beta)}{2} > \{(\beta - 1)(2 - \beta)\}^{1/2}$$

$$\Rightarrow (\alpha - 1)(2 - \alpha) < \frac{1}{4} \text{ and } (\beta - 1)(2 - \beta) < \frac{1}{4}$$

$$\Rightarrow (\alpha - 1)(2 - \alpha) < \frac{1}{4} \text{ and } (\beta - 1)(2 - \beta) < \frac{1}{4}$$

$$\Rightarrow (\alpha - 1)(\beta - 1)(2 - \alpha)(2 - \beta) < \frac{1}{16}$$

Also,

$$(\alpha - 1), (\beta - 1), (2 - \alpha), (2 - \beta) \in (0, 1)$$

$$\therefore 0 < (\alpha - 1)(\beta - 1)(2 - \alpha)(2 - \beta) < \frac{1}{16}$$

$$\Rightarrow 0 < \{\alpha\beta - (\alpha + \beta) + 1\}\{4 - 2(\alpha + \beta) + \alpha\beta)\} < \frac{1}{16}$$

$$\Rightarrow 0 < \left(\frac{c}{a} - \frac{b}{a} + 1\right)\left(4 - \frac{2b}{a} + \frac{c}{a}\right) < \frac{1}{16}$$

$$\Rightarrow 0 < \frac{(a - b + c)(4a - 2b + c)}{a^2} < \frac{1}{16}$$

$$\Rightarrow 0 < (a - b + c)(4a - 2b + c) < \frac{a^2}{16} \quad \ldots(i)$$

Since $a, b, c \in N$. Therefore, $(a - b + c)$ and $(4a - 2b + c)$ are integers.

From, (i), we have

$(a - b + c)(4a - 2b + c) > 0$

$\Rightarrow (a - b + c)(4a - 2b + c)$ is a natural number.

$\Rightarrow (a - b + c)(4a - 2b + c) \geq 1$

$\Rightarrow 1 < \frac{a^2}{16}$ [Using (i)]

$\Rightarrow a^2 > 16$

$\Rightarrow a = 5, 6, 7, \ldots..$

$\Rightarrow a \geq 5, \ a \in N.$

We have,

$\alpha\beta = \frac{c}{a}$

$\Rightarrow \frac{c}{a} > 1$ [$\because \alpha, \beta \in (1, 2) \therefore \alpha\beta > 1$]

$\Rightarrow c > a$

$\Rightarrow c > 5$ [$\because a \geq 5$]

$\Rightarrow c = 6, 7, 8, \ldots..$

Since the roots are real and distinct. Therefore,

$b^2 - 4ac > 0$

$\Rightarrow b^2 > 4ac$

$\Rightarrow b^2 > 120$ [$\because a \geq 5$ and $c \geq 6 \therefore 4ac \geq 120$]

$\Rightarrow b \geq 11$

$\Rightarrow b = 11, 12, 13, \ldots.$

EXAMPLE 29 *If $ab + bc + ca = 0$, find the roots of the equation*

$$a(b - 2c)x^2 + b(c - 2a)x + c(a - 2b) = 0.$$

SOLUTION We have,

$a(b - 2c) + b(c - 2a) + c(a - 2b) = -(ab + bc + ca) = 0$

So, $x = 1$ is a root of the given quadratic.

Let the other root be β. Then,

$$1.\beta = \frac{c(a - 2b)}{a(b - 2c)}$$

$$\Rightarrow \beta = \frac{c(a - 2b)}{a(b - 2c)}.$$

Hence, the roots of the given equation are 1 and $\frac{c(a - 2b)}{a(b - 2c)}$.

EXAMPLE 30 *If a and c are prime numbers and b an integer. Given that $ax^2 + bx + c = 0$ has rational roots. Show that one of the roots is independent of the coefficients. Find the two roots.*

SOLUTION It is given that the equation $ax^2 + bx + c = 0$ has rational roots. Therefore,

$b^2 - 4ac$ must be a perfect square

$\Rightarrow b^2 - 4ac = n^2$ for some integer n [$\because a, b, c$ are integers]

$\Rightarrow b^2 - n^2 = 4ac$

$\Rightarrow (b + n)(b - n) = 4ac$

$\Rightarrow (b + n = 4a \text{ and } b - n = c)$ or $(b + n = 4c \text{ and } b - n = a)$

or $(b + n = 2a \text{ and } b - n = 2c)$ or $(b + n = 2c \text{ and } b - n = 2a)$

[$\because a$ and c are primes]

CASE I *When* $b+n=4a$ and $b-n=c$:

In this case, we have

$2b = 4a+c$

$\Rightarrow b = 2a+\frac{c}{2}$, which is not possible as c is prime and b is an integer

$\therefore$ $b+n = 4a$ and $b-n=c$ is not possible.

CASE II *When* $b+n=4c$ *and* $b-n=a$:

In this case, we have

$2b = 4c+a$

$\Rightarrow b = 2c+\frac{a}{2}$, which is not possible as a is prime and b is an integer.

CASE III *When* $b+n=2a$ *and* $b-n=2c$ *or* $b+n=2c$ *and* $b-n=2a$

In this case, we have

$2b = 2a+2c \Rightarrow b = a+c.$

$\therefore$ $ax^2+bx+c = 0$

$\Rightarrow ax^2+(a+c)x+c = 0$

$\Rightarrow (x+1)(ax+c) = 0$

$\Rightarrow x = -1, -\frac{c}{a}$

Clearly, $x=-1$ is independent of the coefficients a, b and c.

EXAMPLE 31 *Find all integral values of 'a' for which the equation* $(x-a)(x-10)+1=0$ *has integral roots.*

SOLUTION We have,

$(x-a)(x-10)+1 = 0$

$\Rightarrow x^2-x(a+10)+10a+1 = 0$

Let D be the discriminant of this equation. Then,

$D = (a+10)^2-4(10a+1)$

$= a^2-20a+96.$

$= (a-10)^2-4$

The given equation will have integeral roots if D is a perfect square.

Let $D = n^2$.

$\Rightarrow (a-10)^2-4 = n^2$

$\Rightarrow (a-10)^2 = n^2+4$

$\Rightarrow n = 0.$

$\therefore$ $(a-10)^2 = 4 \Rightarrow a-10 = \pm 2 \Rightarrow a = 12, 8.$

EXAMPLE 32 *Find all integral values of* p *such that* $(x-p)(x-12)+2$ *is factorizable as* $(x+q)(x+r)$ *where* q *and* r *are integers.*

SOLUTION $(x-p)(x-12)+2$ is factorizable as $(x+q)(x+r)$, where q, r are integers, means that the equation $(x-p)(x-12)+2=0$ has integral roots.

Now,

$(x-p)(x-12)+2 = 0$ ...(i)

$\Rightarrow x^2-x(p+12)+12p+2 = 0$

Let D be the discriminant of this equation. Then,

$D = (p+12)^2-4(12p+2)$

$= p^2-24p+136$

$= (p-12)^2-8$

Equation (i) will have integral roots, if D is a perfect square. Let $D=n^2$. Then,

$(p-12)^2-8 = n^2$

$\Rightarrow (p-12)^2 = n^2+8$

$\Rightarrow n^2 = 1.$

$\therefore$ $(p-12)^2 = 1+8$

$\Rightarrow p-12 = \pm 3 \Rightarrow p = 15, 9.$

EXAMPLE 33 *Show that the roots of the equation*

$$(a^4+b^4)x^2+4abcd\,x+c^4+d^4=0$$

cannot be distinct, if real.

SOLUTION Let D be the discriminant of the given equation. Then,

$D = 16a^2b^2c^2d^2-4(a^4+b^4)(c^4+d^4)$

$= -4[(a^4+b^4)(c^4+d^4)-4a^2b^2c^2d^2]$

$= -4[(a^4c^4+b^4d^4-2a^2b^2c^2d^2)+a^4d^4+b^4c^4-2a^2b^2c^2d^2]$

$= -4[(a^2c^2-b^2d^2)^2+(a^2d^2-b^2c^2)^2]$

$\Rightarrow D \le 0$ ...(i)

If the roots of the given equation are real, then

$D \ge 0$ ...(ii)

From (i) and (ii), we obtain

$D = 0.$

Hence, the roots of the given equation are equal.

EXAMPLE 34 *If the equation* $ax^3+3bx^2+3cx+d=0$ *has two equal roots, show that each of them is equal to* $\frac{bc-ad}{2(ac-b^2)}$.

SOLUTION Let $f(x) = ax^3+3bx^2+3cx+d$ and let $x=\alpha$ be a repeated root of $f(x)=0$. Then, $x=\alpha$ is also a root of $f'(x)=0$.

Now,

$f(x) = ax^3+3bx^2+3cx+d$

$\Rightarrow f'(x) = 3ax^2+6bx+3c.$

Since α is a root of $f(x)=0$ and $f'(x)=0$. Therefore,

$f(\alpha) = 0$ and $f'(\alpha) = 0$

$\Rightarrow a\alpha^3+3b\alpha^2+3c\alpha+d = 0$ ...(i)

and, $3a\alpha^2+6b\alpha+3c = 0$

or $a\alpha^2 + 2b\alpha + c = 0$ ...(ii)

Multiplying (ii) by α and subtracting from (i), we get

$$b\alpha^2 + 2c\alpha + d = 0 \quad \text{...(iii)}$$

From (ii) and (iii), we get

$$\frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)}$$

$$\Rightarrow \alpha = \frac{bc - ad}{2(ac - b^2)}.$$

EXAMPLE 35 *For what real values of p, the equation*

$$x(x+1)(x+p)(x+p+1) = p^2$$

has four distinct real roots ?

SOLUTION We have,

$$x(x+1)(x+p)(x+p+1) = p^2$$

$$\Rightarrow \{x(x+p+1)\}\{(x+1)(x+p)\} = p^2$$

$$\Rightarrow \{x^2 + x(p+1)\}\{x^2 + x(p+1) + p\} = p^2$$

$$\Rightarrow y(y+p) = p^2, \text{ where } y = x^2 + x(p+1)$$

$$\Rightarrow y^2 + py - p^2 = 0$$

Clearly, discriminant of this equation is positive, so its roots are real and are given by

$$y = \frac{-p \pm \sqrt{5}p}{2}$$

$$\Rightarrow x^2 + x(p+1) = \frac{-p \pm \sqrt{5}p}{2}$$

$$\Rightarrow \quad x^2 + x(p+1) + \left(\frac{1-\sqrt{5}}{2}\right)p = 0 \quad \text{...(i)}$$

$$\text{and,} \quad x^2 + x(p+1) + \left(\frac{1+\sqrt{5}}{2}\right)p = 0 \quad \text{...(ii)}$$

Thus, the given equation will have four distinct real roots, if equation (i) and (ii) have distinct real roots.

Let D_1 and D_2 be the discriminants of (i) and (ii) respectively. Then,

$$D_1 = (p+1)^2 - 2(1-\sqrt{5})p = p^2 + 2\sqrt{5}p + 1$$

$$\text{and,} \quad D_2 = (p+1)^2 - 2(1+\sqrt{5})p = p^2 - 2\sqrt{5}p + 1.$$

Now, the roots of (i) will be real and distinct if

$D_1 > 0$

$$\Rightarrow p^2 + 2\sqrt{5}p + 1 > 0$$

$$\Rightarrow (p + \sqrt{5})^2 - 4 > 0$$

$$\Rightarrow |p + \sqrt{5}| > 2 \quad \text{...(iii)} \quad [\because x^2 - a^2 > 0 \Rightarrow |x| > a]$$

Roots of (ii) will be real and distinct, if

$D_2 > 0$

$$\Rightarrow p^2 - 2\sqrt{5}p + 1 > 0$$

$$\Rightarrow (p - \sqrt{5})^2 - 4 > 0$$

$$\Rightarrow |p - \sqrt{5}| > 2 \quad \text{...(iv)}$$

Now,

$D_1 > 0$ and $D_2 > 0$

$$\Rightarrow |p + \sqrt{5}| > 2 \text{ and } |p - \sqrt{5}| > 2$$

$$\Rightarrow p < -\sqrt{5} - 2 \quad \text{or} \quad -\sqrt{5} + 2 < p < \sqrt{5} - 2 \text{ or } p > \sqrt{5} + 2$$

$$\Rightarrow |p| > \sqrt{5} + 2 \text{ or } |p| < \sqrt{5} - 2.$$

EXAMPLE 36 *Show that, if a, b, c are real, the roots of the equation* $\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} = \frac{3}{x}$, $x \neq 0$ *are all real.*

SOLUTION We have,

$$\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} = \frac{3}{x}$$

$$\Rightarrow \frac{1}{x+a} - \frac{1}{x} + \frac{1}{x+b} - \frac{1}{x} + \frac{1}{x+c} - \frac{1}{x} = 0$$

$$\Rightarrow \frac{-a}{x(x+a)} - \frac{b}{x(x+b)} - \frac{c}{x(x+c)} = 0$$

$$\Rightarrow -\{a(x+b)(x+c) + b(x+c)(x+a) + c(x+a)(x+b)\} = 0$$

$$\Rightarrow x^2(a+b+c) + 2(ab+bc+ca)x + 3abc = 0$$

Le D be the discriminant of this equation. Then,

$$D = 4(ab+bc+ca)^2 - 12abc(a+b+c)$$

$$= 4[a^2b^2 + b^2c^2 + c^2a^2 - ab^2c - bc^2a - ca^2b]$$

$$= 4[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha],$$

where $\alpha = ab$, $\beta = bc$ and $\gamma = ac$

$$= 2[(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2] > 0.$$

Hence, the roots of the given equation are real.

EXAMPLE 37 *Find the integral values of x for which* $x^2 + 19x + 92$ *is a perfect square.*

SOLUTION Let $x^2 + 19x + 92 = k^2$ for some integer k.

$$\Rightarrow x^2 + 19x + 92 - k^2 = 0 \quad \text{...(i)}$$

$$\Rightarrow x = \frac{-19 \pm \sqrt{361 - 4(92 - k^2)}}{2}$$

$\Rightarrow x$ will attain integral values, if

$361 - 4(92 - k^2)$ is a perfect square of an odd integer

i.e. if $361 - 4(92 - k^2) = (2n+1)^2$ for some integer n

$$\Rightarrow 4k^2 - 7 = (2n+1)^2$$

$$\Rightarrow 4k^2 - (2n+1)^2 = 7$$

$$\Rightarrow (2k - 2n - 1)(2k + 2n + 1) = 7$$

$$\Rightarrow 2k + 2n + 1 = 7 \text{ and } 2k - 2n - 1 = 1$$

$$\Rightarrow k = 2, n = 1$$

Putting $k = 2$ in (i), we get

$$x^2 + 19x + 88 = 0$$

$\Rightarrow (x+11)(x+8) = 0$

$\Rightarrow x = -8, -11.$

Hence, $x^2 + 19x + 92$ is a perfect square for $x = -8$ and for $x = -11$.

EXAMPLE 38 *Let [a] denote the greatest integer less than or equal to a.. Given that the quadratic equation* $x^2 + [a^2 - 5a + b + 4]x + b = 0$ *has roots* -5 *and* 1, *find all the values of a.*

SOLUTION Since -5 and 1 are the roots of the equation

$$x^2 + [a^2 - 5a + b + 4]x + b = 0$$

$\therefore -5 + 1 = -[a^2 - 5a + b + 4]$ and $-5 \times 1 = b$

$\Rightarrow [a^2 - 5a + b + 4] = 4$ and $b = -5$

$\Rightarrow [a^2 - 5a - 5 + 4] = 4$

$\Rightarrow [a^2 - 5a - 1] = 4$

$\Rightarrow 4 \le a^2 - 5a - 1 < 5$

$\Rightarrow a^2 - 5a - 1 \ge 4$ and $a^2 - 5a - 1 < 5$

$\Rightarrow a^2 - 5a - 5 \ge 0$ and $a^2 - 5a - 6 < 0$

$$\Rightarrow \left(a \le \frac{5-3\sqrt{5}}{2} \text{ or } a \ge \frac{5+3\sqrt{5}}{2}\right) \text{and } -1 < a < 6$$

$$\Rightarrow a \in \left(-1, \frac{5-3\sqrt{5}}{2}\right] \cup \left[\frac{5+3\sqrt{5}}{2}, 6\right)$$

EXAMPLE 39 *The quadratic equation* $x^2 + px + q = 0$ *where p and q are integers has rational roots. Prove that the roots are all integers.*

SOLUTION We have,

$$x^2 + px + q = 0$$

$$\Rightarrow x = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad \ldots(i)$$

The given equation has rational roots. Therefore, $p^2 - 4q$ must be a perfect square. Since p and q are integers. Therefore, $p^2 - 4q$ must be a perfect square of some integer.

Let $p^2 - 4q = n^2$, where n is an integer.

Now, two cases arise :

CASE I *When p is an even integer :*

Let $p = 2k$, where k is an integer.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow p^2$ is a multiple of 4

$\Rightarrow p^2 - 4q$ is a multiple of 4

$\Rightarrow p^2 - 4q$ is an even integer.

$\Rightarrow n^2$ is an even integer

$\Rightarrow n$ is an even integer

$\Rightarrow n = 2l$ for some integer.

Substituting the values of p and n in (i), we get

$$x = \frac{-2k \pm 2l}{2} = -k \pm l$$

Clearly, the values of x are integers.

CASE II *When p is an odd integer :*

Let $p = 2m + 1$, where m is an integer.

$\Rightarrow p^2 - 4q$ is an odd integer

$\Rightarrow n^2$ is an odd integer

$\Rightarrow n$ is an odd integer

$\Rightarrow n = 2l + 1$ for some integer l.

Substituting the values of p and n in (i), we get

$$x = \frac{-(2m+1) \pm (2l+1)}{2}$$

$\Rightarrow x = -m + l, -m - l - 1$

Clearly, the values of x are integers.

EXAMPLE 40 *Show that the equation*

$$\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + \ldots + \frac{A_n^2}{x - a_n} = x + l, \text{ has no imaginary roots,}$$

where $A_1, A_2, \ldots, A_n, a_1, a_2, \ldots, a_n$ *and are real.*

SOLUTION Let $\alpha + i\beta$ be a root of the given equation. In order to prove that the given equation has no imaginary roots, it is sufficient to show that $\beta = 0$.

Since imaginary roots always occur in pairs. Therefore, $\alpha - i\beta$ is also a root of the given equation.

Putting $x = \alpha + i\beta$ and $x = \alpha - i\beta$ in the given equation, we get

$$\frac{A_1^2}{(\alpha - a_1) + i\beta} + \frac{A_2^2}{(\alpha - a_2) + i\beta} + \ldots + \frac{A_n^2}{(\alpha - a_n) + i\beta} = \alpha + i\beta + l \quad \ldots(i)$$

$$\frac{A_1^2}{(\alpha - a_1) - i\beta} + \frac{A_2^2}{(\alpha - a_2) - i\beta} + \ldots + \frac{A_n^2}{(\alpha - a_n) - i\beta} = \alpha - i\beta + l \quad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$2i\beta\left[\frac{A_1^2}{(\alpha - a_1)^2 + \beta^2} + \frac{A_2^2}{(\alpha - a_2)^2 + \beta^2} + \ldots + \frac{A_n^2}{(\alpha - a_n)^2 + \beta^2}\right] = -2i\beta$$

$$\Rightarrow 2i\beta\left[\frac{A_1^2}{(\alpha - a_1)^2 + \beta^2} + \frac{A_2^2}{(\alpha - a_2)^2 + \beta^2} + \ldots + \frac{A_n^2}{(\alpha - a_n)^2 + \beta^2} + 1\right] = 0$$

$\Rightarrow 2i\beta = 0$ [$\because$ The expression inside the bracket is positive]

$\Rightarrow \beta = 0$

Hence, all the roots of the given equation are real.

EXAMPLE 41 *Find the range of values of a for which two of the roots of the equation* $(a-1)(1 + x + x^2)^2 = (a+1)(1 + x^2 + x^4)$ *are real and distinct.*

SOLUTION We have,

$$(a-1)(1 + x + x^2)^2 = (a+1)(1 + x^2 + x^4)$$

$\Rightarrow (a-1)(1+x+x^2)^2 = (a+1)(1+x+x^2)(1-x+x^2)$

$\Rightarrow (a-1)(1+x+x^2) = (a+1)(1-x+x^2)$

$[\because 1+x+x^2 \neq 0 \text{ for any } x \in R]$

$\Rightarrow x^2 - ax + 1 = 0$

This equation will have real and distinct roots, if

$a^2 - 4 > 0 \Rightarrow |a| > 2$.

EXAMPLE 42 *Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. Find the possible values of p such that A, B, C are the angles of a triangle.* [IIT 1997]

SOLUTION For A, B, C to be the angles of a triangle, we must have

$A + B + C = \pi$

$\Rightarrow B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow \tan(B+C) = \tan\frac{3\pi}{4}$

$\Rightarrow \frac{\tan B + \tan C}{1 - \tan B \tan C} = -1$

$\Rightarrow \frac{\tan B + \frac{p}{\tan B}}{1-p} = -1$

$\Rightarrow \tan^2 B + p = -\tan B \,.\, (1-p)$

$\Rightarrow \tan^2 B + (1-p)\tan B + p = 0$

Since $\tan B$ is real. Therefore,

Discriminant ≥ 0

$\Rightarrow (1-p)^2 - 4p \geq 0$

$\Rightarrow p^2 - 6p + 1 \geq 0$

$\Rightarrow p \leq 3 - 2\sqrt{2}$ or $p \geq 3 + 2\sqrt{2}$.

$\Rightarrow p \in (-\infty, 3 - 2\sqrt{2}] \cup [3 + 2\sqrt{2}, \infty)$.

EXAMPLE 43 *Given two quadratic equations $x^2 - x + a = 0$ and $x^2 - x + 3a = 0$. Find the value of a for which one of the roots of the second equation is equal to the double of one of the root of the first equation.*

SOLUTION Let α be a root of the first equation. Then 2α is a root of the second equation.

$\therefore \quad \alpha^2 - \alpha + a = 0$

and, $4\alpha^2 - 2\alpha + 3a = 0$

$\Rightarrow \frac{\alpha^2}{-3a+2a} = \frac{\alpha}{4a-3a} = \frac{1}{-2+4}$

$\Rightarrow \frac{\alpha^2}{-a} = \frac{\alpha}{a} = \frac{1}{2}$

$\Rightarrow \alpha^2 = -\frac{a}{2}$ and $\alpha = \frac{a}{2}$

Now, $\alpha^2 = (\alpha)^2 \Rightarrow -\frac{a}{2} = \frac{a^2}{4}$

$\Rightarrow a^2 = -2a \Rightarrow a = 0, -2$.

But, for $a = 0$, the two equations, have same roots. Therefore, $a = -2$.

EXAMPLE 44 *If the quadratic equation $4^{\sec^2\alpha}x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$ has real roots, then find all possible values of $\cos\alpha + \cos^{-1}\beta$.*

SOLUTION The equation

$4^{\sec^2\alpha}x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$ has real roots. Therefore,

Discriminant ≥ 0

$\Rightarrow 4 - 4 \times 4^{\sec^2\alpha}\left(\beta^2 - \beta + \frac{1}{2}\right) \geq 0$

$\Rightarrow \quad 4^{\sec^2\alpha}\left(\beta^2 - \beta + \frac{1}{2}\right) \leq 1 \qquad \ldots(i)$

But,

$4^{\sec^2\alpha} \geq 4$ and $\beta^2 - \beta + \frac{1}{2} = \left(\beta - \frac{1}{2}\right)^2 + \frac{1}{4} \geq \frac{1}{4}$.

Therefore, inequation (i) will be true, if

$4^{\sec^2\alpha} = 4$ and $\beta^2 - \beta + \frac{1}{2} = \frac{1}{4}$

$\Rightarrow \sec^2\alpha = 1$ and $\left(\beta - \frac{1}{2}\right)^2 = 0$

$\Rightarrow \cos^2\alpha = 1$ and $\beta = \frac{1}{2} \Rightarrow \alpha = n\pi$ and $\beta = \frac{1}{2}$

$\therefore \cos\alpha + \cos^{-1}\beta$

$= \cos n\pi + \cos^{-1}\left(\frac{1}{2}\right) = \begin{cases} 1 + \frac{\pi}{3}, & \text{if } n \text{ is even} \\ -1 + \frac{\pi}{3}, & \text{if } n \text{ is odd} \end{cases}$

Thus, the values of $\cos\alpha + \cos^{-1}\beta$ are $\frac{\pi}{3} \pm 1$.

EXAMPLE 45 *Find the condition that $x^4 + ax^3 + bx^2 + cx + d$ is a perfect square.*

SOLUTION If $x^4 + ax^3 + bx^2 + cx + d$ is a perfect square, then

$x^4 + ax^3 + bx^2 + cx + d = (x^2 + px + q)^2$ for some $p \,.\, q$.

Comparing the coefficients of corresponding terms, we get

$d = q^2, 2pq = c, p^2 + 2q = b$ and $2p = a$.

Now,

$(2pq)^2 = 4p^2q^2$

$\Rightarrow c^2 = 4\left(\frac{a^2}{4}\right)d$

$\Rightarrow c^2 = a^2 d$, which is the required condition.

EXAMPLE 46 *If $x^2+(a-b)x+(1-a-b)=0$ where $a, b \in R$, then find the values of a for which the equation has unequal real roots for all values of b.* [IIT 2003]

SOLUTION The given equation will have unequal real roots, if

$$(a-b)^2-4(1-a-b)>0$$

$$\Rightarrow b^2+b(4-2a)+a^2+4a-4>0.$$

For the above quadratic inequation to be true for all $b \in R$, we must have

$$(4-2a)^2-4(a^2+4a-4)<0$$

$$\Rightarrow -32a+32<0 \Rightarrow a>1. \Rightarrow a \in (1,\infty).$$

EXERCISE 20.1

1. Solve :

(i) $(2+\sqrt{3})^{x^2-2x+1}+(2-\sqrt{3})^{x^2-2x-1}=\dfrac{101}{10(2-\sqrt{3})}$

(ii) $(a+\sqrt{a^2-1})^{x^2-2x}+(a-\sqrt{a^2-1})^{x^2-2x}=a, a>1.$

2. If the roots of $lx^2+nx+n=0$ are in the ratio $p:q$, show that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$$

3. Prove that the equation $x^2+2bx+2c=0$ cannot have rational roots if b and c are odd integers.

4. Solve : $|x^2+4x+3|+2x+5=0$

5. If the equation $x^2-2px+q^2=0$ has real roots, prove that p cannot lie between $-2q$ and $2q$.

6. Determine the limits within which n must lie in order that the equation $2ax(ax+nc)+(n^2-2)c^2=0$ may have real roots.

7. If $ax^2+(b-c)x+a-b-c=0$ has unequal real roots for all $c \in R$, then prove that either $b<a<0$ or $b>a>0$.

8. If $a<b<c<d$, prove that the equation $(x-a)(x-c)+\lambda(x-b)(x-d)=0$ has real roots for all $\lambda \in R$.

9. If a, b, c are positive integers and if the roots of the equation $ax^2-bx+c=0$ lie in the open interval $(0, 1)$, show that the minimum value of $a+b+c$ is 11. Also, prove that

$$\log_5(abc) \geq 2.$$

10. Show that the equation

$$\left(1-q+\frac{p^2}{2}\right)x^2+p(1+q)x+q(q-1)+\frac{p^2}{2}=0$$

has equal roots iff $4, p, q$ are in GP.

11. If a, b, c are positive real numbers, prove that both the roots of $ax^2+bx+c=0$ have negative real parts.

12. Show that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and k are real numbers such that at least one of $a_i \neq 0$, then the roots of the equation

$$\frac{a_1^2}{x-b_1}+\frac{a_2^2}{x-b_2}+\ldots+\frac{a_n^2}{x-b_n}-k=0 \text{ are real.}$$

13. If the roots of the equation

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

are equal, then prove that

(i) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

(ii) $a+bw+cw^2=0$ or $a+bw^2+cw=0$ or $a+b+c=0$

14. Show that the equation

$$\frac{a}{x-1}+\frac{b}{x-2}+\frac{c}{x-3}+\frac{d}{x-4}=1$$

cannot have imaginary roots if a, b, c, d are any four real numbers of the same sign.

15. If the roots of the equation

$$\left(1-b+\frac{a^2}{2}\right)x^2+a(1+b)x+b(b-1)+\frac{a^2}{2}=0$$

are equal, show that $4, a, b$ are in GP.

16. Find all the integral values of a for which the quadratic equation $(x+a)(x+1991)+1=0$ has integral roots.

17. Prove that the roots of $(a-b)^2x^2+2(a+b-2c)x+1=0$ are real or imaginary according as c does not or does lie between a and $b, a<b$.

18. Find the set of values of 'a' for which the equation $(a-2)x^2-2ax+2a-3=0$ has two distinct roots.

19. Find the values of 'a' for which the equation $(a^2+4a+3)x^2+(a^2-a-2)x+a(a+1)=0$ has more than two roots.

20. Solve for x : $|x^2+2x-8|+|x|(x-2)=0.$

21. Solve for x :

(i) $|x^2+4x+3|+2x+5=0$ (ii) $|x^2-x-6|=x+2$

(iii) $\left|\dfrac{x^2-5x+4}{x^2-4}\right|=1$ (iv) $|6x^2-5x-1|=5x-6x^2-1$

(v) $(x+1)(|x|-1)=-\dfrac{1}{2}$

(vi) $|x^2-9|+|x^2-4|=5$

22. Find the sum of all the real roots of the equation

$$|x-2|^2+|x-2|-2=0$$

23. Solve each of the following equations for real x :

(i) $x^2-2|x|-3=0$ (ii) $|x|^2-3|x|+2=0$

(iii) $x^2+5|x|+4=0$ (iv) $|x|^2-7|x|+12=0$

24. Solve each of the following equations for real x :

(i) $|x^2+x|-5=0$

(ii) $|x^2-2x-8|=2x$

(iii) $|x^2+x+1|=3$

(iv) $|x^2-x|+|x+1|=5$

(v) $|x^2+x+1|+|2x-3| = 4x$

(vi) $2|x+1|^2-|x+1| = 3$

ANSWERS

1. (i) $1 \pm \sqrt{1+\log_{2+\sqrt{3}} 10}$ (ii) $x = 1, 1 \pm \sqrt{2}$
4. $-4, -1-\sqrt{3}$
6. -2 and 2.
16. 1989, 1993
19. $a = -1$
20. $x = \pm 2$
21. (i) $-4, -1-\sqrt{3}$ (ii) $-2, 2, 4$ (iii) $0, \frac{8}{5}, \frac{5}{2}$
(iv) $\frac{-1 \pm \sqrt{5}}{2}$ (v) $\frac{1}{\sqrt{2}}, -1 \pm \frac{1}{\sqrt{2}}$
(vi) $x \in [2, 3] \cup [-3, -2]$
22. 4
23. (i) ± 3
(ii) $\pm 1, \pm 2$
(iii) No solution
(iv) $\pm 3, \pm 4$
24. (i) $\frac{-1 \pm \sqrt{21}}{2}$
(ii) $2\sqrt{2}, 2(\sqrt{3}+1)$
(iii) $1, -2$
(iv) $1-\sqrt{7}, 2$
(v) $2, 1$
(vi) $\frac{1}{2}, -\frac{5}{2}$

—❑❑❑—

20.6 RELATIONS BETWEEN ROOTS AND COEFFICIENTS

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are roots of the equation

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$, then

$f(x) = a_0 (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n)$

$\therefore\ a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$

$= a_0 (x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_n)$

Comparing the coefficients of like powers of x on both sides, we get

$$S_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

or, $$\Sigma \alpha_i = -\frac{a_1}{a_0}$$

or, $$S_1 = -\frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n}$$

$$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots$$

or, $$\sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

or, $$S_2 = \frac{(-1)^2 \text{ coeff of } x^{n-2}}{\text{coeff of } x^n}$$

$$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots$$

or, $$\sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

or, $$S_3 = \frac{(-1)^3 \text{ coeff of } x^{n-3}}{\text{coeff of } x^n}$$

....

....

$$S_n = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{const term}}{\text{coeff of } x^n}$$

Here, S_k denotes the sum of the products of the roots taken k at a time.

PARTICULAR CASES

Quadratic equation : If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Cubic equation : If α, β, γ are roots of a cubic equation

$ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and, $$\alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

Biquadratic equation : If $\alpha, \beta, \gamma, \delta$ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta$$

$$= -\frac{b}{a}$$

$$S_2 = \alpha\beta + \beta\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$= (-1)^2 \frac{c}{a} = \frac{c}{a}$$

or, $$S_2 = (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta$$

$$= \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta$$

$$= (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

or, $$S_3 = \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta)$$

$$= -\frac{d}{a}$$

and, $S_4 = \alpha\beta\gamma\delta$

$$= (-1)^4 \frac{e}{a} = \frac{e}{a}$$

20.7 FORMATION OF A POLYNOMIAL EQUATION FROM GIVEN ROOTS

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of an nth degree equation, then the equation is

$$x^n - S_1 x^{n-1} + S_2 x^{n-2} - S_3 x^{n-3} + \ldots + (-1)^n S_n = 0$$

where S_k denotes the sum of the products of roots taken k at a time.

PARTICULAR CASES :

Quadratic equation : If α, β are the roots of a quadratic equation, then the equation is

$$x^2 - S_1 x + S_2 = 0$$

i.e. $$x^2 - (\alpha + \beta) x + \alpha\beta = 0$$

Cubic equation : If α, β, γ are the roots of a cubic equation, then the equation is

$$x^3 - S_1 x^2 + S_2 x - S_3 = 0$$

i.e. $$x^3 - (\alpha + \beta + \gamma) x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha) x - \alpha\beta\gamma = 0$$

Biquadratic equation : If $\alpha, \beta, \gamma, \delta$ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1 x^3 + S_2 x^2 - S_3 x + S_4 = 0$$

i.e. $x^4 - (\alpha + \beta + \gamma + \delta) x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma) x^2$

$$- (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha) x + \alpha\beta\gamma\delta = 0$$

20.8 TRANSFORMATION OF EQUATIONS

Sometimes to solve a given equation $f(x) = 0$ we transform it into another equation whose roots bear a certain assigned relation with those of the given equation. This is generally done when the solution of the transformed equation is more convenient than that of the given equation. Having solved the transformed equation we can find the roots of the given equation from the given relation between the roots of the two equations.

In this section we shall study some simple transformations.

T1 : *Transformation of an equation into another equation whose roots are the reciprocals of the roots of the given equation.*

Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0 \quad \ldots(i)$$

be the given equation. Let x and y be respectively the roots of the given equation and that of the transformed equation. Then,

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

Putting $x = \frac{1}{y}$ in (i), we get

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \ldots + \frac{a_{n-1}}{y} + a_n = 0$$

$$\Rightarrow \quad a_n y^n + a_{n-1} y^{n-1} + \ldots + a_1 y + a_n = 0$$

This is the required equation.

Thus, to obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation.

ILLUSTRATION 1 *Find the condition so that the roots of the equation* $x^3 - px^2 + qx - r = 0$ *be in H.P.*

SOLUTION The equation whose roots are reciprocals of the roots of the given equation is given by

$$\frac{1}{x^3} - \frac{p}{x^2} + \frac{q}{x} - r = 0$$

or, $$rx^3 - qx^2 + px - 1 = 0 \quad \ldots(i)$$

Since the roots of the given equation are in H.P. So, the roots of this equation are in A.P. Let its roots be $a - d$, a and $a + d$. Then,

$$(a - d) + a + (a + d) = -\left(-\frac{q}{r}\right)$$

$$\Rightarrow \quad 3a = \frac{q}{r}$$

$$\Rightarrow \quad a = \frac{q}{3r}$$

Since a is a root of (i). Therefore,

$$ra^3 - qa^2 + pa - 1 = 0$$

$$\Rightarrow \quad r\left(\frac{q}{3r}\right)^3 - q\left(\frac{q}{3r}\right)^2 + p\left(\frac{q}{3r}\right) - 1 = 0$$

$$\Rightarrow \quad \frac{q^3}{27r^2} - \frac{q^3}{9r^2} + \frac{pq}{3r} - 1 = 0$$

$$\Rightarrow \quad q^3 - 3q^3 + 9pqr - 27r^2 = 0$$

$$\Rightarrow \quad 27r^2 - 9pqr + 2q^3 = 0$$

T2 : *Transformation of an equation to another equation whose roots are negative of the roots of a given equation.*

Let the given equation be

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

Let x be a root of the given equation and y be a root of the transformed equation. Then, $y = -x$ or $x = -y$. Thus, the transformed equation is obtained by putting $x = -y$ in $f(x) = 0$ and is therefore

$$f(-y) = 0$$

or, $a_0 y^n - a_1 y^{n-1} + a_1 y^{n-2} + \ldots + (-1)^n a_n = 0$

For example, the equation whose roots are negative of the roots of the equation $x^3 - 5x^2 - 7x - 3 = 0$ is

$$(-x)^3 - 5(-x)^2 - 7(-x) - 3 = 0$$

or, $-x^3 - 5x^2 + 7x - 3 = 0$

or, $x^3 + 5x^2 - 7x + 3 = 0.$

T3 : *Transformation of an equation to another equation whose roots are square of the roots of a given equation.*

Let x be a root of the given equation and y be that of the transformed equation. Then,

$$y = x^2 \Rightarrow x = \sqrt{y}.$$

Thus, an equation whose roots are squares of the roots of a given equation is obtained by replacing x *by* $\sqrt{x}$ in the given equation.

ILLUSTRATION 2 *Form an equation whose roots are squares of the roots of the equation*

$$x^3 - 6x^2 + 11x - 6 = 0.$$

SOLUTION Replacing x by $\sqrt{x}$ in the given equation, we get

$$(\sqrt{x})^3 - 6(\sqrt{x})^2 + 11\sqrt{x} - 6 = 0$$

$$\Rightarrow x^{3/2} + 11\sqrt{x} = 6x + 6$$

$$\Rightarrow \sqrt{x}(x + 11) = 6(x + 1)$$

$$\Rightarrow x(x + 11)^2 = 36(x + 1)^2$$

$$\Rightarrow x^3 - 14x^2 + 45x - 36 = 0$$

T4 : *Transformation of an equation to another equation whose roots are cubes of the roots of a give equation.*

Let x be a root of the given equation and y be that of the transformed equation. Then,

$$y = x^3 \Rightarrow x = y^{1/3}$$

Thus, an equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equation.

ILLUSTRATION 3 *Form an equation whose roots are cubes of the roots of equation*

$$ax^3 + bx^2 + cx + d = 0.$$

SOLUTION Replacing x by $x^{1/3}$ in the given equation, we get

$$a(x^{1/3})^3 + b(x^{1/3})^2 + c(x^{1/3}) + d = 0$$

$$\Rightarrow ax + d = -(bx^{2/3} + cx^{1/3})$$

$$\Rightarrow (ax + d)^3 = -(bx^{2/3} + cx^{1/3})^3$$

$$\Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^2 + c^3x + 3bcx(bx^{2/3} + cx^{1/3})\}$$

$$\Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^2 + c^3x - 3bcx(ax + d)\}$$

$$\Rightarrow a^3x^3 + x^2(3a^2d - 3abc + b^3) + x(3ad^2 - 3bcd + c^3) + d^3 = 0$$

This is the required equation..

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* α, β *are the roots of the equation* $x^2 - px + q = 0$ *and* $\alpha > 0, \beta > 0$, *then find the value of*

(i) $\alpha^5 + \beta^5$ (ii) $\alpha^{1/4} + \beta^{1/4}$

SOLUTION Since α and β are the roots of the given equation. Therefore,

$$\alpha + \beta = p \quad \ldots(i)$$

and, $$\alpha\beta = q \quad \ldots(ii)$$

(i) We have,

$$(\alpha + \beta)^5 = \alpha^5 + 5\alpha^4\beta + 10\alpha^3\beta^2 + 10\alpha^2\beta^3 + 5\alpha\beta^4 + \beta^5$$

[Using binomial theorem]

$$\Rightarrow (\alpha + \beta)^5 = \alpha^5 + \beta^5 + 5\alpha\beta(\alpha^3 + \beta^3) + 10\alpha^2\beta^2(\alpha + \beta)$$

$$\Rightarrow (\alpha + \beta)^5 = \alpha^5 + \beta^5 + 5\alpha\beta\{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)\} + 10\alpha^2\beta^2(\alpha + \beta)$$

$$\Rightarrow \alpha^5 + \beta^5 = (\alpha + \beta)^5 - 5\alpha\beta(\alpha + \beta)^3 + 15(\alpha\beta)^2(\alpha + \beta) - 10(\alpha\beta)^2(\alpha + \beta)$$

$$\Rightarrow \alpha^5 + \beta^5 = p^5 - 5p^3q + 15pq^2 - 10pq^2$$

$$= p^5 - 5p^3q + 5pq^2$$

(ii) We have,

$$(\alpha^{1/4} + \beta^{1/4})^4$$

$$= (\alpha^{1/4})^4 + 4(\alpha^{1/4})^3\beta^{1/4} + 6(\alpha^{1/4})^2(\beta^{1/4})^2 + 4(\alpha^{1/4})(\beta^{1/4})^3 + (\beta^{1/4})^4$$

$$= (\alpha + \beta) + 4\alpha^{1/4}\beta^{1/4}(\alpha^{1/2} + \beta^{1/2}) + 6\alpha^{1/2}\beta^{1/2}$$

$$= \alpha + \beta + 4(\alpha\beta)^{1/4}\sqrt{(\alpha^{1/2} + \beta^{1/2})^2} + 6(\alpha\beta)^{1/2}$$

$$= (\alpha + \beta) + 4(\alpha\beta)^{1/4}\sqrt{\alpha + \beta + 2(\alpha\beta)^{1/2}} + 6(\alpha\beta)^{1/2}$$

$$= p + 4q^{1/4}\sqrt{p + 2q^{1/2}} + 6q^{1/2}$$

$$\therefore \alpha^{1/4} + \beta^{1/4} = \left[p + 4q^{1/4}\sqrt{p + 2q^{1/2}} + 6q^{1/2}\right]^{1/4}$$

ALITER We have,

$$(\alpha^{1/4} + \beta^{1/4})^4$$

$$= \left\{(\alpha^{1/4} + \beta^{1/4})^2\right\}^2$$

$$= \left\{\alpha^{1/2} + \beta^{1/2} + 2(\alpha\beta)^{1/4}\right\}^2$$

$$= \left\{\sqrt{(\alpha^{1/2} + \beta^{1/2})^2} + 2(\alpha\beta)^{1/4}\right\}^2$$

$$= \left\{\sqrt{\alpha + \beta + 2(\alpha\beta)^{1/2}} + 2(\alpha\beta)^{1/4}\right\}^2$$

$$= \left\{\sqrt{p + 2q^{1/2}} + 2q^{1/4}\right\}^2$$

$$= \left\{p + 2q^{1/2} + 4q^{1/2} + 2 \times 2q^{1/4}\sqrt{p + 2q^{1/2}}\right\}$$

$$= \left\{p + 6q^{1/2} + 4q^{1/4}\sqrt{p + 2q^{1/2}}\right\}$$

$$\therefore (\alpha^{1/4} + \beta^{1/4}) = \left\{p + 6q^{1/2} + 4q^{1/4}\sqrt{p + 2q^{1/2}}\right\}^{1/4}$$

EXAMPLE 2 *If* α and β *are the roots of the equation* $x^2 - p(x+1) - c = 0$, *show that* $(\alpha+1)(\beta+1) = 1 - c$

Hence, prove that

$$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c} = 1$$

SOLUTION We have,

$$x^2 - p(x+1) - c = 0$$

$$\Rightarrow x^2 - px - (p + c) = 0 \quad \ldots(i)$$

Since α and β are the roots of the equation (i). Therefore,

$\alpha + \beta = p$ and $\alpha\beta = -(p + c)$

Now, $(\alpha+1)(\beta+1)$

$$= \alpha\beta + (\alpha + \beta) + 1$$

$$= -(p + c) + p + 1 = 1 - c \quad \ldots(ii)$$

This proves the first part.

Now,

$$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (\alpha+1)(\beta+1)}$$

[From (ii), $(1-c) = (\alpha+1)(\beta+1)$]

$$= \frac{\alpha+1}{(\alpha+1) - (\beta+1)} + \frac{\beta+1}{(\beta+1) - (\alpha+1)}$$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha}$$

$$= \frac{\alpha+1-\beta-1}{\alpha-\beta}$$

$$= \frac{\alpha-\beta}{\alpha-\beta} = 1.$$

EXAMPLE 3 *If* p and q *are the roots of the equation* $x^2 - 2x + A = 0$ *and* r *and* s *are the roots of the equation* $x^2 - 18x + B = 0$ *such that* $p < q < r < s$ *and* p, q, r, s *are in A.P., then find the values of* A and B.

[IIT 1997]

SOLUTION Since p, q are the roots of $x^2 - 2x + A = 0$ and r, s are the roots of $x^2 - 18x + B = 0$. Therefore,

$$p + q = 2 \quad \ldots(i)$$

$$pq = A \quad \ldots(ii)$$

$$r + s = 18 \quad \ldots(iii)$$

and, $$rs = B \quad \ldots(iv)$$

It is given that p, q, r, s are in A.P. Let $p = a - 3d$, $q = a - d$, $r = a + d$ and $s = a + 3d$. Then,

$$p + q + r + s = 4a$$

$\Rightarrow 2 + 18 = 4a$ [Using (i) and (iii)]

$\Rightarrow a = 5.$

Now,

$$p + q = 2$$

$\Rightarrow a - 3d + a - d = 2$

$\Rightarrow 2a - 4d = 2$

$\Rightarrow 10 - 4d = 2$

$\Rightarrow d = 2.$

$\therefore \quad p = a - 3d = 5 - 6 = -1,$

$q = a - d = 5 - 2 = 3,$

$r = a + d = 5 + 2 = 7,$

and, $s = a + 3d = 5 + 6 = 11.$

Putting these values in (ii) and (iv), we get

$A = pq = -3$ and $B = rs = 77.$

EXAMPLE 4 *If the ratio of the roots of the equation* $x^2 + px + q = 0$ *be equal to the ratio of the roots of the equation* $x^2 + bx + c = 0$, *then prove that* $p^2 c = b^2 q$.

SOLUTION Let α, β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of the equation $x^2 + bx + c = 0$. Then,

$$\alpha + \beta = -p, \alpha\beta = q \quad \ldots(i)$$

and, $$\gamma + \delta = -b, \gamma\delta = c \quad \ldots(ii)$$

We have, $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$

$\Rightarrow \frac{\alpha+\beta}{\alpha-\beta} = \frac{\gamma+\delta}{\gamma-\delta}$ [Applying componendo and dividendo]

$\Rightarrow \frac{(\alpha+\beta)^2}{(\alpha-\beta)^2} = \frac{(\gamma+\delta)^2}{(\gamma-\delta)^2}$

$\Rightarrow \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} = \frac{(\gamma-\delta)^2}{(\gamma+\delta)^2}$

$$\Rightarrow \frac{(\alpha+\beta)^2-4\alpha\beta}{(\alpha+\beta)^2}=\frac{(\gamma+\delta)^2-4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow 1-\frac{4\alpha\beta}{(\alpha+\beta)^2}=1-\frac{4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \frac{\alpha\beta}{(\alpha+\beta)^2}=\frac{\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \frac{q}{p^2}=\frac{c}{b^2}$$

$$\Rightarrow p^2c=b^2q.$$

EXAMPLE 5 *If α, β be the non-zero roots of $ax^2+bx+c=0$ and γ, δ be the non-zero roots of $px^2+qx+r=0$ such that the system of equations*

$$\alpha y+\beta z=0$$
$$\gamma y+\delta z=0$$

have a non-trivial solution, then prove that $\frac{a}{p}, \frac{b}{q}$ and $\frac{c}{r}$ are in G.P.

SOLUTION We have, $ax^2+bx+c=0$ ...(i)

and, $px^2+qx+r=0$ ...(ii)

It is given that α, β are the roots of (i) and γ, δ are the roots of (ii). Therefore,

$$\left.\begin{aligned}\alpha+\beta&=-\frac{b}{a}\\ \alpha\beta&=\frac{c}{a}\end{aligned}\right\} \quad ...(iii)$$

and,

$$\left.\begin{aligned}r+\delta&=-\frac{q}{p}\\ \gamma\delta&=\frac{r}{p}\end{aligned}\right\} \quad ...(iv)$$

It is given that the system of equations

$$\alpha y+\beta z=0$$
$$\gamma y+\delta z=0$$

have a non-trivial solution.

$$\therefore \begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}=0$$

$$\Rightarrow \alpha\delta-\beta\gamma=0$$

$$\Rightarrow \frac{\alpha}{\beta}=\frac{\gamma}{\delta}$$

$$\Rightarrow \frac{\alpha+\beta}{\alpha-\beta}=\frac{\gamma+\delta}{\gamma-\delta}$$ [Applying componendo and dividendo]

$$\Rightarrow \frac{\alpha-\beta}{\alpha+\beta}=\frac{\gamma-\delta}{\gamma+\delta}$$

$$\Rightarrow \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2=\left(\frac{\gamma-\delta}{\gamma+\delta}\right)^2$$

$$\Rightarrow \frac{(\alpha+\beta)^2-4\alpha\beta}{(\alpha+\beta)^2}=\frac{(\gamma+\delta)^2-4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow 1-\frac{4\alpha\beta}{(\alpha+\beta)^2}=1-\frac{4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \frac{\alpha\beta}{(\alpha+\beta)^2}=\frac{\gamma\delta}{(r+\delta)^2}$$

$$\Rightarrow \frac{c/a}{(-b/a)^2}=\frac{r/q}{(-q/p)^2}$$

$$\Rightarrow \frac{ac}{b^2}=\frac{pr}{q^2}$$

$$\Rightarrow \frac{b^2}{q^2}=\frac{ac}{pr}$$

$$\Rightarrow \left(\frac{b}{q}\right)^2=\left(\frac{a}{p}\right)\left(\frac{c}{r}\right)$$

$\Rightarrow \frac{a}{p}, \frac{b}{q}$ and $\frac{c}{r}$ are in G.P.

EXAMPLE 6 *Let a, b be the roots of $x^2-3x+A=0$ and c, d be the roots of $x^2-12x+B=0$. If a, b, c, d form an increasing GP, find A and B.*

SOLUTION Let r be the common ratio of the GP. Then,

$b=ar$, $c=ar^2$ and $d=ar^3$. Also, $r>0$.

Now, a and b are the roots of the equation $x^2-3x+A=0$.

$\therefore \quad a+b=3$ and $ab=A$

$\Rightarrow \quad a+ar=3$ and $ar^2=A$

$\Rightarrow a(1+r)=3$ and $a^2r=A$ ...(i)

Since c and d are the roots of the equation $x^2-12x+B=0$. Therefore,

$c+d=12$ and $cd=B$

$\Rightarrow ar^2+ar^3=12$ and $a^2r^5=B$

$\Rightarrow ar^2(1+r)=12$ and $a^2r^5=B$ ...(ii)

From (i) and (ii), we have

$a(1+r)=3$ and $ar^2(1+r)=12$

$$\Rightarrow \frac{ar^2(1+r)}{a(1+r)}=\frac{12}{3}$$

$\Rightarrow r^2=4$

$\Rightarrow r=2$ [$\because r>0$]

Putting the value of r in $a(1+r)=3$, we get $a=1$.

$\therefore b=ar=2, c=ar^2=4$ and $d=ar^3=8$.

Now,

$A=ab \Rightarrow A=2$

and, $B=cd \Rightarrow B=32$.

EXAMPLE 7 *If α, β be the roots of $ax^2+2bx+c=0$ and $\alpha+\delta, \beta+\delta$ be those of $Ax^2+2Bx+C=0$, then prove that*

$$\frac{b^2-ac}{B^2-AC}=\left(\frac{a}{A}\right)^2 \quad \text{[IIT 2000]}$$

SOLUTION Since, α, β are roots of $ax^2+2bx+c=0$.

$$\therefore \quad \alpha+\beta=-\frac{2b}{a} \text{ and } \alpha\beta=\frac{c}{a} \quad ...(i)$$

Let $\alpha' = \alpha+\delta$ and $\beta' = \beta+\delta$

It is given that $\alpha' = \alpha+\delta, \beta' = \beta+\delta$ are the roots of the equation

$Ax^2+2Bx+C = 0$

$$\therefore \alpha'+\beta' = -\frac{2B}{A} \text{ and } \alpha'\beta' = \frac{C}{A} \quad ...(ii)$$

Now,

$\alpha' = \alpha+\delta$ and $\beta' = \beta+\delta$

$\Rightarrow \alpha'-\beta' = (\alpha+\delta)-(\beta+\delta) = \alpha-\beta$

$\Rightarrow (\alpha'-\beta')^2 = (\alpha-\beta)^2$

$\Rightarrow (\alpha'+\beta')^2-4\alpha'\beta' = (\alpha+\beta)^2-4\alpha\beta$

$$\Rightarrow \frac{4B^2}{A^2}-\frac{4C}{A}=\frac{4b^2}{a^2}-\frac{4c}{a}$$

$$\Rightarrow \frac{B^2-AC}{A^2}=\frac{b^2-ac}{a^2}$$

$$\Rightarrow \frac{b^2-ac}{B^2-AC}=\left(\frac{a}{A}\right)^2.$$

EXAMPLE 8 *If $\alpha, \beta, \gamma, \delta$ be in AP of which the first two terms are roots of $x^2-2px+q=0$ and the last two terms are roots of $x^2-2rx+s=0$, then prove that $s-q=r^2-p^2$.*

SOLUTION Since α, β are the roots of $x^2-2px+q=0$ and γ, δ are the roots of $x^2-2rx+s=0$. Therefore,

$$\alpha+\beta = 2p, \alpha\beta = q$$

and, $$\gamma+\delta = 2r, \quad \gamma\delta = s.$$

Now,

$\alpha, \beta, \gamma, \delta$ are in A.P.

$\Rightarrow \beta-\alpha = \delta-\gamma$

$\Rightarrow (\beta-\alpha)^2 = (\delta-\gamma)^2$

$\Rightarrow (\beta+\alpha)^2-4\alpha\beta = (\delta+\gamma)^2-4\gamma\delta$

$\Rightarrow 4p^2-4q = 4r^2-4s$

$\Rightarrow s-q = r^2-p^2.$

EXAMPLE 9 *If α is a root of the equation $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root.*

SOLUTION Let the other root of the equation $4x^2+2x-1=0$ be β. Then we have to prove that $\beta = 4\alpha^3-3\alpha$.

Now, $$\alpha+\beta = -\frac{2}{4} \Rightarrow \beta = -\frac{1}{2}-\alpha \quad ...(i)$$

Since α is a root of the given equation.

$\therefore 4\alpha^2+2\alpha-1 = 0$

$$\Rightarrow \alpha^2 = \frac{1-2\alpha}{4} \quad ...(ii)$$

Now, $4\alpha^3-3\alpha = \alpha\cdot 4\alpha^2-3\alpha$

$= \alpha(1-2\alpha)-3\alpha$ [Using (ii)]

$= -2\alpha^2-2\alpha$

$$= -2\left(\frac{1-2\alpha}{4}\right)-2\alpha \quad \text{[Using (ii)]}$$

$$= -\frac{1}{2}+\alpha-2\alpha$$

$$= -\alpha-\frac{1}{2} = \beta \quad \text{[Using (i)]}$$

Hence, $\beta = 4\alpha^3-3\alpha$ is the other root of the given equation.

EXAMPLE 10 *Find the number of quadratic equations which are unchanged by squaring their roots.*

SOLUTION Let α, β be the roots of a quadratic and α^2, β^2 be the roots of another quadratic, since the quadratics remain same, we have

$$\alpha+\beta = \alpha^2+\beta^2 \quad ...(i)$$

and, $$\alpha\beta = \alpha^2\beta^2 \quad ...(ii)$$

Now,

$\alpha\beta = \alpha^2\beta^2$

$\Rightarrow \alpha\beta(1-\alpha\beta) = 0$

$\Rightarrow \alpha = 0 \text{ or } \beta = 0 \text{ or } \alpha\beta = 1.$

CASE I *When* $\alpha = 0$.

From (i), we have

$\alpha+\beta = \alpha^2+\beta^2$

$\Rightarrow \beta = \beta^2$

$\Rightarrow \beta(1-\beta) = 0$

$\Rightarrow \beta = 0 \text{ or, } \beta = 1$

Thus, we get two sets of values of α and β viz. $\alpha = 0$, $\beta = 0$ and $\alpha = 0, \beta = 1$.

CASE II When $\beta = 0$.

From (i), we have

$\alpha+\beta = \alpha^2+\beta^2$

$\Rightarrow \alpha = \alpha^2$

$\Rightarrow \alpha(1-\alpha) = 0$

$\Rightarrow \alpha = 0 \text{ or, } \alpha = 1$

Thus, we get two sets of values of α and β viz. $\alpha = 0, \beta = 0$ and $\alpha = 1, \beta = 0$.

CASE III *When* $\alpha\beta = 1$

From (i), we have

$$\alpha + \beta = \alpha^2 + \beta^2$$

$$\Rightarrow \alpha + \frac{1}{\alpha} = \alpha^2 + \frac{1}{\alpha^2} \qquad \left[\because \alpha\beta = 1 \Rightarrow \beta = \frac{1}{\alpha}\right]$$

$$\Rightarrow \alpha + \frac{1}{\alpha} = \left(\alpha + \frac{1}{\alpha}\right)^2 - 2$$

$$\Rightarrow \left(\alpha + \frac{1}{\alpha}\right)^2 - \left(\alpha + \frac{1}{\alpha}\right) - 2 = 0$$

$$\Rightarrow y^2 - y - 2 = 0, \text{ where } y = \alpha + \frac{1}{\alpha}$$

$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = 2 \text{ or, } y = -1.$$

Now,

$$y = 2 \Rightarrow \alpha + \frac{1}{\alpha} = 2$$

$$\Rightarrow \alpha^2 - 2\alpha + 1 = 0$$

$$\Rightarrow (\alpha - 1)^2 = 0 \Rightarrow \alpha = 1$$

And,

$$y = -1 \Rightarrow \alpha + \frac{1}{\alpha} = -1$$

$$\Rightarrow \alpha^2 + \alpha + 1 = 0$$

$$\Rightarrow \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2$$

If $\alpha = 1$, then $\alpha\beta = 1 \Rightarrow \beta = 1$.

If $\alpha = \omega$, then $\alpha\beta = 1 \Rightarrow \beta = \frac{1}{\omega} = \omega^2$

If $\alpha = \omega^2$, then $\alpha\beta = 1 \Rightarrow \beta = \frac{1}{\omega^2} = \omega$

Thus, when $\alpha\beta = 1$, we get two sets of values of α and β viz $\alpha = 1, \beta = 1$ and $\alpha = \omega, \beta = \omega^2$.

Thus, there are four sets of values of α and β viz. $\alpha = 0, \beta = 0$; $\alpha = 1, \beta = 0$; $\alpha = 1, \beta = 1$ and $\alpha = \omega, \beta = \omega^2$.

Hence, there are four quadratic equations which do not change by squaring their roots.

EXAMPLE 11 *If the roots of* α and β *of the quadratic equation* $ax^2 + bx + c = 0$ *are real and of opposite signs. Prove that the roots of the quadratic equation* $\alpha(x-\beta)^2 + \beta(x-\alpha)^2 = 0$ *are also real and of opposite signs.*

SOLUTION Since α, β are the roots of $ax^2 + bx + c = 0$

$$\therefore \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Since α and β are real and of opposite signs. Therefore,

$$b^2 - 4ac \geq 0 \text{ and } \alpha\beta < 0$$

$$\Rightarrow b^2 - 4ac \geq 0 \text{ and } \frac{c}{a} < 0$$

$$\Rightarrow b^2 - 4ac \geq 0 \text{ and } a, c \text{ are of opposite signs.}$$

$$\Rightarrow b^2 - 4ac \geq 0 \text{ and } ac < 0 \qquad \ldots(i)$$

Now,

$$\alpha(x-\beta)^2 + \beta(x-\alpha)^2 = 0$$

$$\Rightarrow x^2(\alpha+\beta) - 4x\alpha\beta + \alpha\beta(\alpha+\beta) = 0$$

$$\Rightarrow -\frac{bx^2}{a} - \frac{4xc}{a} - \frac{bc}{a^2} = 0 \qquad \left[\because \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\right]$$

$$\Rightarrow abx^2 + 4acx + bc = 0$$

Let D_1 be the discriminant of this equation and α_1, β_1 be its roots. Then,

$$D_1 = 16a^2c^2 - 4ab^2c$$

$$= -4ac(b^2 - 4ac) > 0 \qquad \text{[Using (i)]}$$

and, $\alpha_1\beta_1 = \frac{bc}{ab} = \frac{c}{a} < 0$ [Using (i)]

Hence, the roots of the second equation are also real and of opposite signs.

EXAMPLE 12 *If* α, β *be the non-zero roots of* $ax^2 + bx + c = 0$ *and* α^2, β^2 *the roots of* $a^2x^2 + b^2x + c^2 = 0$, *then find the number of possible sets of values of* α, β. *Also, prove that a, b, c are in G.P.*

SOLUTION We have,

$$\alpha + \beta = -\frac{b}{a} \qquad \ldots(i)$$

$$\alpha\beta = \frac{c}{a} \qquad \ldots(ii)$$

$$\alpha^2 + \beta^2 = -\frac{b^2}{a^2} \qquad \ldots(iii)$$

and,

$$\alpha^2\beta^2 = \frac{c^2}{a^2} \qquad \ldots(iv)$$

From (i) and (iii), we obtain

$$\alpha^2 + \beta^2 = -(\alpha+\beta)^2$$

$$\Rightarrow 2\alpha^2 + 2\beta^2 + 2\alpha\beta = 0$$

$$\Rightarrow \alpha^2 + \alpha\beta + \beta^2 = 0$$

$$\Rightarrow \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right) + 1 = 0$$

$$\Rightarrow \frac{\alpha}{\beta} = \omega, \omega^2 \qquad [\because x^2 + x + 1 = 0 \text{ has roots } \omega \text{ and } \omega^2]$$

$$\Rightarrow \alpha = \beta\omega, \beta\omega^2.$$

CASE I When $\alpha = \beta\omega$:

Putting $\alpha = \beta\omega$ in (ii), we obtain

$$\beta^2\omega = \frac{c}{a}$$

$\Rightarrow \beta^2 = \frac{c}{\omega a}$

$\Rightarrow \beta^2 = \frac{c}{a}\omega^2$

$\Rightarrow \beta = \pm\sqrt{\frac{c}{a}}\,\omega.$

$\therefore \alpha = \beta\omega \Rightarrow \alpha = \pm\sqrt{\frac{c}{a}}\,\omega^2$

Thus, the roots are :

$$\alpha = \sqrt{\frac{c}{a}}\,\omega^2,\ \beta = \sqrt{\frac{c}{a}}\,w$$

and, $$\alpha = -\sqrt{\frac{c}{a}}\,\omega^2, \beta = -\sqrt{\frac{c}{a}}\,\omega.$$

CASE II When $\alpha = \beta\omega^2$:

Putting $\alpha = \beta\omega^2$ in (ii), we obtain

$\beta^2\omega^2 = \frac{c}{a}$

$\Rightarrow \beta^2 = \frac{c}{a\omega^2}$

$\Rightarrow \beta = \pm\sqrt{\frac{c}{a}}\cdot\frac{1}{\omega} = \pm\sqrt{\frac{c}{a}}\,\omega^2$

$\therefore \alpha = \beta\omega^2 \Rightarrow \alpha = \pm\sqrt{\frac{c}{a}}\,\omega^2$

In this case, the roots are :

$\alpha = \pm\sqrt{\frac{c}{a}}\,\omega,\ \beta = \pm\sqrt{\frac{c}{a}}\,\omega^2$

Putting $\alpha = \beta\omega$ in (i) and (ii), we obtain

$$\beta = \beta\omega = -\frac{b}{a} \text{ and } \beta^2\omega = \frac{c}{a}$$

$\Rightarrow -\beta\omega^2 = -\frac{b}{a}$ and $\beta^2\omega = \frac{c}{a}$

$\Rightarrow \beta\omega^2 = \frac{b}{a}$ and $\beta^2\omega = \frac{c}{a}$

$\Rightarrow \beta^2\omega^4 = \frac{b^2}{a^2}$ and $\beta^2\omega = \frac{c}{a}$

$\Rightarrow \beta^2\omega = \frac{b^2}{a^2}$ and $\beta^2\omega = \frac{c}{a}$

$\Rightarrow \frac{b^2}{a^2} = \frac{c}{a}$

$\Rightarrow b^2 = ac$

$\Rightarrow a, b, c$ are in GP.

EXAMPLE 13 *Let a, b and c be integers with $a > 1$, and let p be a prime number. Show that if $ax^2 + bx + c$ is equal to p for two distinct integral values of x, then it cannot be equal to $2p$ forany integral value of x.*

SOLUTION Let x_1 and x_2 be two distinct integral values of x such that

$ax_1^2 + bx_1 + c = p$ and $ax_2^2 + bx_2 + c = p$

$\Rightarrow ax_1^2 + bx_1 + c - p = 0$ and $ax_2^2 + bx_2 + c - p = 0$

$\Rightarrow x_1$ and x_2 are the roots of $ax^2 + bx + c - p = 0$

$\Rightarrow ax^2 + bx + c - p \equiv a(x - x_1)(x - x_2)$ ...(i)

If possible, let there be an integer α such that

$a\alpha^2 + b\alpha + c = 2p$

$\Rightarrow a\alpha^2 + b\alpha + c - p = p$

$\Rightarrow a(\alpha - x_1)(\alpha - x_2) = p$ ...(ii) [Using (i)]

But p is a prime number and $a > 1$. Therefore,

$a(\alpha - x_1)(\alpha - x_2) = p$

$\Rightarrow a = p.$

Putting $a = p$ in (ii), we get

$(\alpha - x_1)(\alpha - x_2) = 1$

$\Rightarrow \alpha - x_1 = 1, \alpha - x_2 = 1$ or, $\alpha - x_1 = -1, \alpha - x_2 = -1$

[$\because \alpha - x_1$ and $\alpha - x_2$ are integers]

$\Rightarrow x_1 = x_2$, which is not possible as $x_1 \neq x_2$.

Hence, there cannot be any integer α for which the expression $ax^2 + bx + c$ is equal to $2p$.

ALITER It is given that $ax^2 + bx + c = p$ for two distinct integral values of x. This means that the equation $ax^2 + bx + c - p = 0$ has integral roots.

$\therefore$ Sum of the roots = an integer

and,

Product of the roots = an integer

$\Rightarrow -\frac{b}{a} = an$ integer and, $\frac{c-p}{a}$ = an integer

$\Rightarrow -\frac{b}{a} = l$ (say) and $\frac{c-p}{a} = m$ (say) ...(i)

We have to show that $ax^2 + bx + c \neq 2p$ for any integral value of x.

In order to prove this, it is sufficient to show that the equation

$$ax^2 + bx + c - 2p = 0 \quad \text{...(ii)}$$

does not have integral roots.

Let α, β be the roots of equation (ii). Then,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c-2p}{a}$$

Now,

$\alpha\beta = \frac{c-2p}{a}$

$\Rightarrow \alpha\beta = \frac{c-p}{a} - \frac{p}{a}$

$\Rightarrow \alpha\beta = m - \frac{p}{a}$ [Using (i)]

$\Rightarrow \alpha\beta$ is not an integer $[\because m \in Z$ but $\frac{p}{a} \notin Z$ as p is prime$]$

$\Rightarrow$ at least one of α, β is not an integer

Also,

$\alpha + \beta = -\frac{b}{a}$

$\Rightarrow \alpha + \beta = l$ [Using (i)]

$\Rightarrow \alpha + \beta$ is an integer

But, one of α and β is not an integer. Therefore, $\alpha + \beta$ can be an integer only when the other one is also not an integer.

Thus, equation (i) does not have integral roots.

EXAMPLE 14 *If one root of the equation $ax^2 + bx + c = 0$ is equal to the nth power of the other root, then show that*

$$(ac^n)^{1/n+1} + (a^n c)^{1/n+1} + b = 0.$$

SOLUTION Let the roots of the given equation be α and α^n. Then,

$$\alpha + \alpha^n = -b/a \quad ...(i)$$

and, $$\alpha \cdot \alpha^n = c/a \quad ...(ii)$$

Now,

$\alpha \cdot \alpha^n = \frac{c}{a}$

$\Rightarrow \alpha^{n+1} = \frac{c}{a}$

$\Rightarrow \alpha = \left(\frac{c}{a}\right)^{1/n+1}$

Substituting this value of α in (i), we get

$\left(\frac{c}{a}\right)^{1/n+1} + \left(\frac{c}{a}\right)^{n/n+1} = -\frac{b}{a}$

$\Rightarrow a \cdot \left(\frac{c}{a}\right)^{1/n+1} + a \cdot \left(\frac{c}{a}\right)^{n/n+1} = -b$

$\Rightarrow a \cdot \frac{c^{1/n+1}}{a^{1/n+1}} + a\frac{c^{n/n+1}}{a^{n/n+1}} + b = 0$

$\Rightarrow c^{1/n+1} a^{n/n+1} + a^{1/n+1} c^{n/n+1} + b = 0$

$\Rightarrow (ca^n)^{1/n+1} + (ac^n)^{1/n+1} + b = 0$

$\Rightarrow (ac^n)^{1/n+1} + (a^n c)^{1/n+1} + b = 0.$

EXAMPLE 15 *Given that α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 are the two values of λ for which the roots α, β are related by : $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, find the value of $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$.*

SOLUTION Since α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$ or, $\lambda x^2 + x(1 - \lambda) + 5 = 0$.

$\therefore \quad \alpha + \beta = \frac{\lambda - 1}{\lambda}$ ...(i)

and, $\alpha\beta = \frac{5}{\lambda}$ ...(ii)

Now,

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$

$\Rightarrow 5(\alpha^2 + \beta^2) = 4\alpha\beta$

$\Rightarrow 5[(\alpha + \beta)^2 - 2\alpha\beta] = 4\alpha\beta$

$\Rightarrow 5(\alpha + \beta)^2 = 14\alpha\beta$

$\Rightarrow 5\left(\frac{\lambda - 1}{\lambda}\right)^2 = 14 \times \frac{5}{\lambda}$ [Using (i) and (ii)]

$\Rightarrow \lambda^2 - 16\lambda + 1 = 0.$

This equation has roots λ_1 and λ_2

$\therefore \lambda_1 + \lambda_2 = 16$ and $\lambda_1 \lambda_2 = 1$

Now,

$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$

$= \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2}$

$= \frac{(\lambda_1 + \lambda_2)^2 - 2\lambda_1 \lambda_2}{\lambda_1 \lambda_2}$

$= \frac{(16)^2 - 2 \times 1}{1}$

$= 256 - 2 = 254$

EXAMPLE 16 *Let p and q be the two roots of the equation*

$$mx^2 + x(2 - m) + 3 = 0$$

Let m_1, m_2 be the two values of m satisfying

$$\frac{p}{q} + \frac{q}{p} = \frac{2}{3}.$$

Determine the numerical value of $\frac{m_1}{m_2^2} + \frac{m_2}{m_1^2}$.

SOLUTION We have,

$$mx^2 + x(2 - m) + 3 = 0 \quad ...(i)$$

Since p and q are the roots of (i). Therefore,

$p + q = \frac{m - 2}{m}$ and $pq = \frac{3}{m}$ ...(ii)

Now,

$\frac{p}{q} + \frac{q}{p} = \frac{2}{3}$

$\Rightarrow 3(p^2+q^2) = 2pq$

$\Rightarrow 3\{(p+q)^2 - 2pq\} = 2pq$

$\Rightarrow 3(p+q)^2 - 8pq = 0$

$\Rightarrow 3\left(\frac{m-2}{m}\right)^2 - 8\times\frac{3}{m} = 0$ [Using (ii)]

$\Rightarrow 3(m-2)^2 - 24m = 0$

$\Rightarrow m^2 - 12m + 4 = 0$

Since m_1 and m_2 are the roots of this equation. Therefore,

$m_1 + m_2 = 12$ and $m_1 m_2 = 4$.

Now,

$$\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}$$

$$= \frac{m_1^3 + m_2^3}{(m_1 m_2)^2}$$

$$= \frac{(m_1+m_2)^3 - 3m_1 m_2 (m_1+m_2)}{(m_1 m_2)^2}$$

$$= \frac{12^3 - 3\times 4\times 12}{4^2}$$

$= 108 - 9 = 99$

EXAMPLE 17 *If a, b, c are in HP and α, β are the roots of $ax^2+bx+c=0$, then prove that the HM of α, β is $-(1+\alpha\beta)$.*

SOLUTION Since α, β are the roots of the equation $ax^2+bx+c=0$. Therefore,

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Now,

a, b, c are in HP.

$\Rightarrow b = \frac{2ac}{a+c}$

$\Rightarrow \frac{b}{a} = \frac{2c}{a+c}$ [Dividing both sides by a]

$\Rightarrow \frac{b}{a} = \frac{2(c/a)}{1+\frac{c}{a}}$ [Dividing N^r and D^r of RHS by a]

$\Rightarrow -(\alpha+\beta) = \frac{2\alpha\beta}{1+\alpha\beta}$

$\Rightarrow -(1+\alpha\beta) = \frac{2\alpha\beta}{\alpha+\beta}$

$\Rightarrow$ HM of α, β is $-(1+\alpha\beta)$.

EXAMPLE 18 *If α, β are the roots of the equation $x^2+px+q=0$ and also of $x^{2n}+p^n x^n+q^n=0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $x^n+1+(x+1)^n=0$, then prove that n is an even integer.*

SOLUTION Since α, β are the roots of $x^2+px+q=0$. Therefore,

$$\alpha+\beta = -p \quad ...(i)$$

and, $$\alpha\beta = q \quad ...(ii)$$

It is given that α, β are also the roots of $x^{2n}+p^n x^n+q^n=0$. Therefore,

$$\alpha^{2n}+p^n\alpha^n+q^n = 0 \quad ...(iii)$$

and, $$\beta^{2n}+p^n\beta^n+q^n = 0 \quad ...(iv)$$

Substracting (iv) from (iii), we get

$\Rightarrow \alpha^{2n} - \beta^{2n} + p^n(\alpha^n - \beta^n) = 0$

$\Rightarrow (\alpha^n - \beta^n)\{\alpha^n + \beta^n + p^n\} = 0$

$\Rightarrow \alpha^n + \beta^n + p^n = 0$

$\Rightarrow \alpha^n + \beta^n = -p^n$ ...(v)

Since $\frac{\alpha}{\beta}$ is a root of the equation $x^n+1+(x+1)^n=0$. Therefore,

$$\left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta}+1\right)^n = 0$$

$\Rightarrow \alpha^n + \beta^n + (\alpha+\beta)^n = 0$

$\Rightarrow -p^n + (-p)^n = 0$ [Using (i) and (v)]

$\Rightarrow (-p)^n = p^n$

$\Rightarrow n$ is an even integer.

EXAMPLE 19 *If $\sin\alpha, \sin\beta$ are the roots of the equation $ax^2+bx+c=0$ and also of the equation $a^n x^{2n}+b^n x^n+c^n=0$, show that when n is an even integer and $\alpha, \beta \neq k\pi, k \in Z$, $\frac{\sin\alpha}{\sin\beta}$ is a root of the equation $(x+1)^n + x^n + 1 = 0$. What is the other root ?*

SOLUTION It is given that $\sin\alpha, \sin\beta$ are the roots of the equation $ax^2+bx+c=0$. Therefore,

$$\left.\begin{aligned} \sin\alpha+\sin\beta &= -\frac{b}{a} \\ \text{and,} \quad \sin\alpha\sin\beta &= \frac{c}{a} \end{aligned}\right\} \quad ...(i)$$

Since $\sin\alpha, \sin\beta$ are also the roots of $a^n x^{2n}+b^n x^n+c^n=0$. Therefore,

$$a^n(\sin\alpha)^{2n} + b^n(\sin\alpha)^n + c^n = 0 \quad ...(ii)$$

and,

$$a^n(\sin\beta)^{2n} + b^n(\sin\beta)^n + c^n = 0 \quad ...(iii)$$

Subtracting (iii) from (ii), we get

$$a^n\{(\sin\alpha)^{2n}-(\sin\beta)^{2n}\}+b^n\{(\sin\alpha)^n-(\sin\beta)^n\}=0$$

$$\Rightarrow a^n\{(\sin\alpha)^n+(\sin\beta)^n\}+b^n=0$$

$$\Rightarrow (\sin\alpha)^n+(\sin\beta)^n=-\frac{b^n}{a^n}$$

$$\Rightarrow (\sin\alpha)^n+(\sin\beta)^n=-\left(\frac{b}{a}\right)^n \quad ...(iv)$$

We have to prove that $\frac{\sin\alpha}{\sin\beta}$ is a root of the equation

$$(x+1)^n+x^n+1=0 \quad ...(v)$$

We have,

$$\left\{\frac{\sin\alpha}{\sin\beta}+1\right\}^n+\left\{\frac{\sin\alpha}{\sin\beta}\right\}^n+1$$

$$=\frac{(\sin\alpha+\sin\beta)^n+(\sin\alpha)^n+(\sin\beta)^n}{(\sin\beta)^n}$$

$$=\frac{\left(-\frac{b}{a}\right)^n+\left\{-\left(\frac{b}{a}\right)^n\right\}}{(\sin\beta)^n} \quad \text{[Using (i) and (iv)]}$$

$$=\frac{\left(\frac{b}{a}\right)^n-\left(\frac{b}{a}\right)^n}{(\sin\beta)^n} \quad [\because n \text{ is an even integer}]$$

$$=0$$

$\therefore\ x=\frac{\sin\alpha}{\sin\beta}$ is a root of the equation (v).

From the symmetry, we can also say that the other root is $\frac{\sin\beta}{\sin\alpha}$.

EXAMPLE 20 *If α, β are the roots of the equation $x^2-bx+c=0$, then find the equation whose roots are $(\alpha^2+\beta^2)(\alpha^3+\beta^3)$ and $\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4$.*

SOLUTION Since α, β are the roots of the equation $x^2-bx+c=0$. Therefore,

$$\alpha+\beta=b \quad ...(i)$$

and,

$$\alpha\beta=c \quad ...(ii)$$

Let $A=(\alpha^2+\beta^2)(\alpha^3+\beta^3)$ and $B=\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4$. Then,

$$A=(\alpha^2+\beta^2)(\alpha^3+\beta^3)$$

$$=\{(\alpha+\beta)^2-2\alpha\beta\}\{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\}$$

$$=(b^2-2c)(b^3-3bc) \quad \text{[Using (i) and (ii)]}$$

and,

$$B=\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4$$

$$=\alpha^3\beta^3(\alpha^2+\beta^2-2\alpha\beta)$$

$$=(\alpha\beta)^3\{(\alpha+\beta)^2-4\alpha\beta\}$$

$$=c^3(b^2-4c) \quad \text{[Using (i) and (i}$$

$$\therefore\ A+B=(b^2-2c)(b^3-3bc)+c^3(b^2-4c) \quad ...(i$$

and,

$$AB=(b^2-2c)(b^3-3bc)c^3(b^2-4c) \quad ...(i$$

Hence, required equation is

$x^2-x(A+B)+AB=0$, where $A+B$ and AB are given by (ii and (iv) respectively.

EXAMPLE 21 *If α, β be the roots of the equation $x^2-ax+b=0$ an $V_n=\alpha^n+\beta^n$, then show that $V_{n+1}=aV_n-bV_{n-1}$. Hence, obtai the value of $\alpha^5+\beta^5$.*

SOLUTION Since α, β are the roots of the equatio $x^2-ax+b=0$. Therefore,

$$\alpha^2-a\alpha+b=0 \text{ and } \beta^2-a\beta+b=0$$

$$\Rightarrow \alpha^{n-1}(\alpha^2-a\alpha+b)=\alpha^{n-1}\times 0$$

and, $\beta^{n-1}(\beta^2-a\beta+b)=\beta^{n-1}\times 0$

$$\Rightarrow \alpha^{n+1}-a\alpha^n+b\alpha^{n-1}=0 \text{ and, } \beta^{n+1}-a\beta^n+b\beta^{n-1}=0$$

$$\Rightarrow (\alpha^{n+1}+\beta^{n+1})-a(\alpha^n+\beta^n)+b(\alpha^{n-1}+\beta^{n-1})=0$$

$$\Rightarrow V_{n+1}-aV_n+bV_{n-1}=0 \quad [\because V_n=\alpha^n+\beta^n$$

$$\Rightarrow V_{n+1}=aV_n-bV_{n-1} \quad ...(i$$

Since α, β are the roots of the given equation. Therefore,

$$\alpha+\beta=a \text{ and } \alpha\beta=b.$$

Now,

$$V_{n+1}=aV_n-bV_{n-1}$$

$$\Rightarrow V_2=aV_1-bV_0 \quad \text{[Putting } n=1\text{]}$$

$$\Rightarrow V_2=a(\alpha+\beta)-b(\alpha^0+\beta^0)$$

$$\Rightarrow V_2=a^2-2b \quad ...(ii)$$

Again, putting $n=2$ in (i), we obtain

$$V_3=aV_2-bV_1$$

$$\Rightarrow V_3=a(a^2-2b)-b(a) \quad \text{[Using (ii) and } V_1=\alpha+\beta=a\text{]}$$

$$\Rightarrow V_3=a^3-3ab \quad ...(iii)$$

Now, putting $n=3$ in (i), we obtain

$$V_4=aV_3-bV_2$$

$$\Rightarrow V_4=a(a^3-3ab)-b(a^2-2b) \quad \text{[Using (ii) and (iii)]}$$

$$\Rightarrow V_4=a^4-4a^2b+2b^2 \quad ...(iv)$$

Finally, putting $n=4$ in (i), we get

$$V_5=aV_4-bV_3$$

$$\Rightarrow V_5=a(a^4-4a^2b+2b^2)-b(a^3-3ab)$$

$$\Rightarrow V_5=a^5-5a^3b+2ab^2+3ab^2$$

$\Rightarrow V_5 = a^5 - 5a^3b + 5ab^2$

Hence, $\alpha^5 + \beta^5 = V_5 = a^5 - 5a^3 b + 5ab^2$.

EXAMPLE 22 *Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that* $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ [IIT 1995]

SOLUTION We have,

$\alpha < -1$ and $\beta > 1$

$\Rightarrow \alpha + \lambda = -1$ and $\beta = 1 + \mu$, where $\lambda, \mu > 0$ ...(i)

Since α, β are the roots of the equation $ax^2 + bx + c = 0$. Therefore,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a} \quad ...(ii)$$

Now,

$$1 + \frac{c}{a} + \left|\frac{b}{a}\right|$$

$= 1 + \alpha\beta + |\alpha + \beta|$ [Using (ii)]

$= 1 + (-1 - \lambda)(1 + \mu) + |-1 - \lambda + 1 + \mu|$ [Using (i)]

$= 1 - (1 + \lambda + \mu + \lambda\mu) + |\mu - \lambda|$

$= -\lambda - \mu - \lambda\mu + |\mu - \lambda|$

Now, two cases arise :

CASE I *When $\mu > \lambda$:*

In this case, we have

$|\mu - \lambda| = \mu - \lambda$

$\therefore 1 + \frac{c}{a} + \left|\frac{b}{a}\right| = -\lambda - \mu - \lambda\mu + \mu - \lambda$

$\Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| = -\lambda\mu - 2\lambda$

$\Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ [$\because \lambda, \mu > 0$]

CASE II *When $\mu < \lambda$:*

In this case, we have

$|\mu - \lambda| = \lambda - \mu$

$\therefore 1 + \frac{c}{a} + \left|\frac{b}{a}\right| = -\lambda - \mu - \lambda\mu + \lambda - \mu$

$\Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| = -2\mu - \lambda\mu$

$\Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ [$\because \lambda, \mu > 0$]

Hence, in both the case, we have

$$1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0.$$

ALITER We have,

$\alpha < -1$ and $\beta > 1$

$\Rightarrow \alpha\beta < -1 \Rightarrow \frac{c}{a} < -1 \Rightarrow 1 + \frac{c}{a} < 0.$

Let $f(x) = ax^2 + bx + c$.

Clearly, 1 and -1 lie between the roots of $ax^2 + bx + c = 0$.

$\therefore f(1) f(-1) > 0$

$\Rightarrow (a + b + c)(a - b + c) > 0$

$\Rightarrow (a + c)^2 - b^2 > 0$

$\Rightarrow \left(1 + \frac{c}{a}\right)^2 - \left(\frac{b}{a}\right)^2 > 0$

$\Rightarrow \left(\frac{b}{a}\right)^2 - \left(1 + \frac{c}{a}\right)^2 < 0$

$\Rightarrow \left|\frac{b}{a}\right| < \left|1 + \frac{c}{a}\right|$

$\Rightarrow \left|\frac{b}{a}\right| < -\left(1 + \frac{c}{a}\right)$ $\left[\because 1 + \frac{c}{a} < 0\right]$

$\Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$

EXAMPLE 23 *Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, then show that* $4 + \frac{c}{a} + 2\left|\frac{b}{a}\right| < 0.$

SOLUTION Since α, β are the roots of the equation $ax^2 + bx + c = 0$. Therefore,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Also,

$\alpha < -2$ and $\beta > 2$

$\Rightarrow \alpha = -2 - \lambda$ and $\beta - \mu = 2$, where $\lambda, \mu > 0$.

Now,

$$4 + \frac{c}{a} + 2\left|\frac{b}{a}\right|$$

$= 4 + \alpha\beta + 2|\alpha + \beta|$

$= 4 + (-\lambda - 2)(\mu + 2) + 2|\mu - \lambda|$

Now, two cases arise :

CASE I *When $\mu \geq \lambda$.*

In this case, we have

$|\mu - \lambda| = \mu - \lambda$

$\therefore 4 + \frac{c}{a} + 2\left|\frac{b}{a}\right|$

$= 4 + (-\lambda - 2)(\mu + 2) + 2(\mu - \lambda)$

$= 4-(4+2\mu+2\lambda+\mu\lambda)+2\mu-2\lambda$

$= -2\mu-2\lambda-\mu\lambda+2\mu-2\lambda = -\mu\lambda-4\lambda<0$

CASE II *When* $\mu<\lambda$:

In this case, we have

$|\mu-\lambda| = -(\mu-\lambda)$

$\therefore\ 4+\frac{c}{a}+2\left|\frac{b}{a}\right|$

$= 4+(-\lambda-2)(\mu+2)-2(\mu-\lambda)$

$= 4-(\mu\lambda+2\lambda+2\mu+4)-2\mu+2\lambda$

$= -\mu\lambda-4\mu<0.$

Thus, in both the cases, we have

$$4+\frac{c}{a}+2\left|\frac{b}{a}\right|<0.$$

EXAMPLE 24 *Let a, b, c be real numbers and $ax^2+bx+c=0$ has real roots α and β such that $\alpha>\beta$. Find the condition on the roots, in relation to p, that $p^2+\frac{c}{a}+p\left|\frac{b}{a}\right|<0$*

SOLUTION Since α, β are the roots of $ax^2+bx+c=0$. Therefore,

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Now,

$$p^2+\frac{c}{a}+p\left|\frac{b}{a}\right|<0$$

$\Rightarrow p^2+\alpha\beta+p|\alpha+\beta|<0$

Now, two cases arise :

CASE I *When* $\alpha+\beta>0$:

In this case, we have

$|\alpha+\beta| = \alpha+\beta$

$\therefore\ p^2+\alpha\beta+p|\alpha+\beta|<0$

$\Rightarrow p^2+\alpha\beta+p(\alpha+\beta)<0$

$\Rightarrow (p+\alpha)(p+\beta)<0$

$\Rightarrow -\beta<p<-\alpha$ $\quad[\because \alpha<\beta \therefore -\alpha>-\beta]$

$\Rightarrow p\in(-\beta,-\alpha)$

CASE II *When* $\alpha+\beta<0$:

In this case, we have

$|\alpha+\beta| = -(\alpha+\beta)$

$\therefore\ p^2+\alpha\beta+p|\alpha+\beta|<0$

$\Rightarrow p^2+\alpha\beta-p(\alpha+\beta)<0$

$\Rightarrow (p-\alpha)(p-\beta)<0$

$\Rightarrow \alpha<p<\beta$

$\Rightarrow p\in(\alpha,\beta).$

EXAMPLE 25 *Find a quadratic equation the sum of whose roots α and β is equal to 2 and $\frac{1-\alpha}{1+\beta}+\frac{1-\beta}{1+\alpha} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$.*

SOLUTION We have, $\alpha+\beta=2$. Let $\alpha\beta=\lambda$.

Now,

$$\frac{1-\alpha}{1+\beta}+\frac{1-\beta}{1+\alpha} = \frac{2(4a^2+15)}{4a^2-1}$$

$$\Rightarrow \frac{1-\alpha^2+1-\beta^2}{(1+\beta)(1+\alpha)} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$$\Rightarrow \frac{2-(\alpha^2+\beta^2)}{1+(\alpha+\beta)+\alpha\beta} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$$\Rightarrow \frac{2-\{(\alpha+\beta)^2-2\alpha\beta\}}{1+(\alpha+\beta)+\alpha\beta} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$$\Rightarrow \frac{2-(4-2\alpha\beta)}{1+2+\alpha\beta} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$$\Rightarrow \frac{2\alpha\beta-2}{\alpha\beta+3} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$$\Rightarrow \frac{2(\lambda-1)}{\lambda+3} = 2\left(\frac{4a^2+15}{4a^2-1}\right)$$

$\Rightarrow \lambda(4a^2+15-4a^2+1) = -4a^2+1-3(4a^2+15)$

$\Rightarrow 16\lambda = -16a^2-44$

$$\Rightarrow \lambda = -\left(\frac{4a^2+11}{4}\right)$$

$$\Rightarrow \alpha\beta = -\left(\frac{4a^2+11}{4}\right)$$

Hence, the required equation is

$$x^2-x(\alpha+\beta)+\alpha\beta=0$$

or, $$x^2-2x-\left(\frac{4a^2+11}{4}\right)=0$$

EXAMPLE 26 *Let a, b, c, d be real numbers in G.P., if u, v, w satisfy the system of equations*

$$u+2v+3w = 6$$

$$4u+5v+6w = 12$$

$$6u+9v = 4$$

then show that the roots of the equations

$$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+\{(b-c)^2+(c-a)^2+(d-b)^2\}x+u+v+w=0$$

and,

$$20x^2+10(a-d)^2x-9 = 0$$

are reciprocals of each other.

[IIT 1999]

SOLUTION Solving the given system of equations, we get

$$u = -\frac{1}{3}, v = \frac{2}{3} \text{ and } w = \frac{5}{3}.$$

$$\therefore u+v+w = 2 \text{ and } \frac{1}{u}+\frac{1}{v}+\frac{1}{w} = -\frac{9}{10}.$$

Now,

$$(b-c)^2+(c-a)^2+(d-b)^2$$

$$= 2b^2+2c^2+a^2+d^2-2bc-2ca-2bd$$

$$= 2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$

$$= a^2+d^2-2ad \qquad \left[\because a, b, c, d \text{ are in G.P.} \Rightarrow b^2 = ac, c^2 = bd \text{ and } ad = bc\right]$$

$$= (a-d)^2$$

$$\therefore \left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+[(b-c)^2+(c-a)^2+(d-b)^2]x+u+v+w = 0$$

$$\Rightarrow -\frac{9}{10}x^2+(a-d)^2x+2 = 0 \qquad \left[\because a, b, c \text{ are in G.P.} \Rightarrow (b-c)^2+(c-a)^2+(d-b)^2 = (a-d)^2\right]$$

$$\Rightarrow 9x^2-10(a-d)^2x-20 = 0 \qquad \text{...(i)}$$

The equation whose roots are the reciprocals of the roots of the above equation is obtained by replacing x by $\frac{1}{x}$ in (i).

Replacing x by $\frac{1}{x}$ in (i), we get

$$\frac{9}{x^2}-10(a-d)^2\frac{1}{x}-20 = 0$$

$$\Rightarrow 9-10(a-d)^2x-20x^2 = 0$$

$\Rightarrow 20x^2+10(a-d)^2x-9 = 0$, which is same as the second equation.

Hence, the roots of the two equation are reciprocal of each other.

EXAMPLE 27 *Let $f(x) = Ax^2+Bx+C$ where A, B, C are real numbers. Prove that if $f(x)$ is an integer then the numbers $2A$, $A+B$ and C are all integers. Conversely, prove that if the numbers $2A$, $A+B$ and C are all integers, then $f(x)$ is an integer whenever x is an integer.*

[IIT 1998]

SOLUTION It is given that $f(x) = Ax^2+Bx+C$ is an integer for all integer values of x. Therefore,

$\therefore f(0), f(1)$ and $f(-1)$ are intgers

$\Rightarrow f(0) = C, f(1) = A+B+C$

and, $f(-1) = A-B+C$ are integers.

$\Rightarrow A+B = f(1)-C$ and $A-B = f(-1)-C$ are integers.

$\Rightarrow 2A = f(1)-C+f(-1)-C$ is an integer

Hence, $A+B = f(1)-C$, $2A = f(1)+f(-1)-2C$ and C are integers.

Conversely, let the numbers $2A$, $A+B$ and C be integers. Then, we have to prove that $f(x)$ is an integer for all integer x.

We have,

$$2B = 2(A+B)-2A$$

$\Rightarrow 2B$ is an integer. $[\because A+B \text{ and } 2A \text{ are integers}]$

Now, two cases arise :

CASE I *When A and B are integers :*

In this case, $f(x) = Ax^2+Bx+C$ is an integer for all integer x.

CASE II *When A and B are not integers :*

If $2A$ and $2B$ are integers but A and B are not integers, then $A = \frac{2m+1}{2}$, $B = \frac{2n+1}{2}$ for some $m, n \in Z$.

$$\therefore f(x) = \left(\frac{2m+1}{2}\right)x^2+\left(\frac{2n+1}{2}\right)x+C \qquad \text{...(i)}$$

Let x be an even integer. Then, $x = 2\lambda, \lambda \in Z$.

$$\therefore f(x) = \left(\frac{2m+1}{2}\right)4\lambda^2+\left(\frac{2n+1}{2}\right)2\lambda+C$$

$= 2\lambda^2(2m+1)+(2n+1)\lambda+C$, which is an integer.

Let x be an odd integer. Then, $x = 2\lambda+1$ for some $\lambda \in Z$.

$$\therefore f(x) = \left(\frac{2m+1}{2}\right)(2\lambda+1)^2+\left(\frac{2n+1}{2}\right)(2\lambda+1)+C$$

$$= (2\lambda+1)(2m\lambda+m+n+\lambda+1)+C,$$

which is also an integer.

Hence, $f(x)$ is an integer for all integer values of x.

EXAMPLE 28 *A quadratic equation with roots α and β where $\alpha\beta = 4$, also has the condition*

$$\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1} = \frac{a^2-7}{a^2-4}$$

For what values of 'a' does the equation have equal roots.

SOLUTION We have,

$$\alpha\beta = 4 \text{ and } \frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1} = \frac{a^2-7}{a^2-4}$$

Now,

$$\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1} = \frac{a^2-7}{a^2-4}$$

$$\Rightarrow \frac{2\alpha\beta-(\alpha+\beta)}{\alpha\beta-(\alpha+\beta)+1} = \frac{a^2-7}{a^2-4} \qquad [\because \alpha\beta = 4]$$

$$\Rightarrow \frac{8-(\alpha+\beta)}{4-(\alpha+\beta)+1} = \frac{a^2-7}{a^2-4}$$

$$\Rightarrow \frac{8-(\alpha+\beta)}{5-(\alpha+\beta)} = \frac{a^2-7}{a^2-4}$$

$$\Rightarrow 8(a^2-4)-(\alpha+\beta)(a^2-4) = 5(a^2-7)-(a^2-7)(\alpha+\beta)$$

$$\Rightarrow 3(\alpha+\beta) = 3(a^2+1)$$

$$\Rightarrow \alpha+\beta = a^2+1$$

The equation having α, β as its roots is

$$x^2-x(\alpha+\beta)+\alpha\beta = 0$$

or, $x^2-(a^2+1)x+4 = 0$

This equation will have equal roots, if

$$(a^2+1)^2-16 = 0$$

$$\Rightarrow a^2+1 = \pm 4$$

$$\Rightarrow a^2 = 3$$

$$\Rightarrow a = \pm\sqrt{3}.$$

EXAMPLE 29 *If a root of $ax^2+bx+c=0$ is numerically equal but opposite in sign of a root of $a_1x^2+b_1x+c_1=0$, then prove that the quadratic equation whose roots are the other roots of the two equations is*

$$\frac{x^2}{\frac{b}{a}+\frac{b_1}{a_1}}+x+\frac{1}{\frac{b}{c}+\frac{b_1}{c_1}} = 0$$

SOLUTION Let α, β be the roots of the equation $ax^2+bx+c=0$. Then as per the question $-\alpha, \gamma$ are the roots of the equation $a_1x^2+b_1x+c_1=0$.

$$\therefore \quad \alpha+\beta = -\frac{b}{a} \qquad ...(i)$$

$$\alpha\beta = \frac{c}{a} \qquad ...(ii)$$

and,

$$-\alpha+\gamma = -\frac{b_1}{a_1} \qquad ...(iii)$$

$$-\alpha\gamma = \frac{c_1}{a_1} \qquad ...(iv)$$

We have to find equation having β and γ as its roots. For it we require the values of $\beta+\gamma$ and $\beta\gamma$.

Adding (i) and (iii), we get

$$\beta+\gamma = -\left(\frac{b}{a}+\frac{b_1}{a_1}\right) \qquad ...(v)$$

Now,

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$\Rightarrow \frac{1}{\alpha}+\frac{1}{\beta} = -\frac{b}{a} \qquad \text{[Using (i) and (ii)]}$$

and,

$$\frac{-1}{\alpha}+\frac{1}{\gamma} = \frac{-\gamma+\alpha}{\alpha\gamma}$$

$$\Rightarrow \frac{-1}{\alpha}+\frac{1}{\gamma} = \frac{-b_1}{c_1} \qquad \text{[Using (iii) and (iv)]}$$

Adding these two, we obtain

$$\frac{1}{\beta}+\frac{1}{\gamma} = -\frac{b}{c}-\frac{b_1}{c_1}$$

$$\Rightarrow \frac{\beta+\gamma}{\beta\gamma} = -\left(\frac{b}{c}+\frac{b_1}{c_1}\right)$$

$$\Rightarrow \beta\gamma = -\frac{(\beta+\gamma)}{\frac{b}{c}+\frac{b_1}{c_1}}$$

$$\Rightarrow \beta\gamma = \frac{\frac{b}{a}+\frac{b_1}{a_1}}{\frac{b}{c}+\frac{b_1}{c_1}} \qquad ...(vi) \qquad \text{[Using (v)]}$$

The equation having β and γ as its roots is

$$x^2-x(\beta+\gamma)+\beta\gamma = 0$$

$$\text{or, } x^2+\left(\frac{b}{a}+\frac{b_1}{a_1}\right)x+\left(\frac{\frac{b}{a}+\frac{b_1}{a_1}}{\frac{b}{c}+\frac{b_1}{c_1}}\right) = 0 \qquad \text{[Using (v) and (vi)]}$$

$$\text{or, } \frac{x^2}{\frac{b}{a}+\frac{b_1}{a_1}}+x+\frac{1}{\frac{b}{c}+\frac{b_1}{c_1}} = 0$$

EXAMPLE 30 *If one root of the equation $x^2+ax+b=0$ is also a root of the equation $x^2+mx+n=0$, show that its other root is a root of $x^2+(2a-m)x+a^2-am+n=0$.*

SOLUTION Let α be a root of the equation $x^2+ax+b=0$ which is also a root of the equation $x^2+mx+n=0$. Let β be the other root of the equation $x^2+ax+b=0$. Then,

$$\alpha+\beta = -a \Rightarrow \alpha = -a-\beta.$$

Since α is a root of the equation $x^2+mx+n=0$. Therefore,

$$(-a-\beta)^2+m(-a-\beta)+n = 0$$

$$\Rightarrow \beta^2+2a\beta+a^2-am-m\beta+n = 0$$

$$\Rightarrow \beta^2+(2a-m)\beta+a^2-ma+n = 0$$

$\Rightarrow \beta$ is a root of the equation $x^2+(2a-m)x+a^2-ma+n=0$

EXAMPLE 31 *Let α, β be two real numbers satisfying the following relations :*

$$\alpha^2 + \beta^2 = 5, 3(\alpha^5 + \beta^5) = 11(\alpha^3 + \beta^3).$$

Find the quadratic equation having α and β as its roots.

SOLUTION We have,

$$3(\alpha^5 + \beta^5) = 11(\alpha^3 + \beta^3)$$

$$\Rightarrow \frac{\alpha^5 + \beta^5}{\alpha^3 + \beta^3} = \frac{11}{3}$$

$$\Rightarrow \frac{(\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)}{\alpha^3 + \beta^3} = \frac{11}{3}$$

$$\Rightarrow (\alpha^2 + \beta^2) - \frac{\alpha^2\beta^2(\alpha + \beta)}{(\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)} = \frac{11}{3}$$

$$\Rightarrow 5 - \frac{\alpha^2\beta^2}{5 - \alpha\beta} = \frac{11}{3}$$

$$\Rightarrow \frac{\alpha^2\beta^2}{5 - \alpha\beta} = 5 - \frac{11}{3}$$

$$\Rightarrow \frac{\alpha^2\beta^2}{5 - \alpha\beta} = \frac{4}{3}$$

$$\Rightarrow 3(\alpha\beta)^2 + 4(\alpha\beta) - 20 = 0$$

$$\Rightarrow (\alpha\beta - 2)(3\alpha\beta + 10) = 0$$

$$\Rightarrow \alpha\beta = 2, -\frac{10}{3}$$

Now,

$$\alpha^2 + \beta^2 = 5$$

$$\Rightarrow (\alpha + \beta)^2 - 2\alpha\beta = 5$$

$$\Rightarrow (\alpha + \beta)^2 = 5 + 2\alpha\beta.$$

When $\alpha\beta = 2$, we have

$$(\alpha + \beta)^2 = 5 + 4$$

$$\Rightarrow \alpha + \beta = \pm 3$$

When $\alpha\beta = -\frac{10}{3}$, we have

$(\alpha + \beta)^2 = 5 - \frac{20}{3} = -\frac{5}{3}$, which is not possible as α, β are real.

Hence, the required equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or, $$x^2 \pm 3x + 2 = 0.$$

EXAMPLE 32 *If the sum of two roots of the equation $x^3 - px^2 + qx - r = 0$ is zero, then prove that $pq = r$.*

SOLUTION Let the roots of the given equation be α, β, γ such that $\alpha + \beta = 0$. Then,

$$\alpha + \beta + \gamma = -\frac{(-p)}{1}$$

$$\Rightarrow \alpha + \beta + \gamma = p$$

$$\Rightarrow \gamma = p \qquad [\because \alpha + \beta = 0]$$

But, γ is a root of the given equation. Therefore,

$$\gamma^3 - p\gamma^2 + q\gamma - r = 0,$$

$$\Rightarrow p^3 - p^3 + qp - r = 0$$

$$\Rightarrow pq = r \qquad [\because r = p]$$

EXAMPLE 33 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P.*

SOLUTION Let the roots of the given equation be $a - d, a, a + d$. Then,

$$(a - d) + a + (a + d) = -\frac{(-p)}{1}$$

$$\Rightarrow a = p/3$$

Since a is a root of the given equation. Therefore,

$$a^3 - pa^2 + qa - r = 0$$

$$\Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{qp}{3} - r = 0 \qquad \left[\because a = \frac{p}{3}\right]$$

$$\Rightarrow 2p^3 - 9pq + 27r = 0$$

This is the required condition.

EXAMPLE 34 *If $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$, then find the real root of $ax^3 + bx^2 + cx + d = 0$*

SOLUTION $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$. So roots of $x^2 + x + 1 = 0$ are also the roots of the equation $ax^3 + bx^2 + cx + d = 0$. But, $x^2 + x + 1 = 0$ has its roots as ω and ω^2. So, two imaginary roots of $ax^3 + bx^2 + cx + d = 0$ are ω and ω^2. Let the third real root be α. Then,

$$\omega \cdot \omega^2 \alpha = (-1)^3 \frac{d}{a}$$

$$\Rightarrow \alpha = -\frac{d}{a}$$

Hence, the real root of $ax^3 + bx^2 + cx + d = 0$ is $-\frac{d}{a}$.

EXAMPLE 35 *If A, G and H are respectively the A.M., G.M. and H.M. of three positive numbers a, b and c, then prove that the equation whose roots are a, b, c is*

$$x^3 - 3Ax^2 + 3\frac{G^3}{H}x - G^3 = 0$$

SOLUTION We have,

$$A = \frac{a + b + c}{3}$$

$$\Rightarrow a + b + c = 3A \qquad \ldots(i)$$

$$G = (abc)^{1/3}$$

$$\Rightarrow abc = G^3 \qquad \ldots(ii)$$

and,

$$\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

$$\Rightarrow \frac{3}{H} = \frac{bc+ca+ab}{abc}$$

$$\Rightarrow \frac{3}{H} = \frac{ab+bc+ca}{G^3}$$

$$\Rightarrow ab+bc+ca = \frac{3G^3}{H} \quad ...(iii)$$

The equation whose roots are a, b and c is

$$x^3 - S_1 x^2 + S_2 x - S_3 = 0$$

or, $x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$

or, $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$ [Using (i), (ii) and (iii)]

EXAMPLE 36 *The real numbers satisfying the equation* $x^3 - x^2 + \beta x + \gamma = 0$ *are in A.P. Find the intervals in which* β *and* γ *lie.* [IIT 1996]

SOLUTION It is given that x_1, x_2, x_3 are the roots of the equation $x^3 - x^2 + \beta x + \gamma = 0$. Therefore,

$$x_1 + x_2 + x_3 = 1 \quad ...(i)$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \beta \quad ...(ii)$$

and, $$x_1x_2x_3 = -\gamma \quad ...(iii)$$

Also, x_1, x_2, x_3 are in AP. Therefore,

$$2x_2 = x_1 + x_3 \quad ...(iv)$$

$$\Rightarrow 3x_2 = x_1 + x_2 + x_3$$

$$\Rightarrow 3x_2 = 1 \quad \text{[Using (i)]}$$

$$\Rightarrow x_2 = \frac{1}{3}.$$

Putting $x_2 = \frac{1}{3}$ in (iv) and (iii), we get

$$x_1 + x_3 = \frac{2}{3} \text{ and } x_1x_3 = -3\gamma$$

The quadratic equation having x_1 and x_3 as its roots is

$$x^2 - x(x_1 + x_3) + x_1x_3 = 0$$

or, $x^2 - \frac{2}{3}x - 3\gamma = 0$

or, $3x^2 - 2x - 9\gamma = 0$

Since x_1 and x_3 are real numbers i.e. the above equation has real roots. Therefore,

$$4 + 108\gamma \geq 0 \quad \text{[Putting Disc} \geq 0\text{]}$$

$$\Rightarrow \gamma \geq -\frac{1}{27}.$$

Since $x_2 = \frac{1}{3}$ is a root of the given equation. Therefore,

$$\frac{1}{27} - \frac{1}{9} + \frac{\beta}{3} + \gamma = 0 \quad \text{[Putting } x_2 = \frac{1}{3} \text{ in the given equation]}$$

$$\Rightarrow \frac{\beta}{3} + \gamma = \frac{2}{27}$$

$$\Rightarrow \gamma = \frac{2}{27} - \frac{\beta}{3}$$

$$\Rightarrow \frac{2}{27} - \frac{\beta}{3} \geq -\frac{1}{27} \quad \left[\because \gamma \geq -\frac{1}{27}\right]$$

$$\Rightarrow \frac{2}{27} + \frac{1}{27} \geq \frac{\beta}{3}$$

$$\Rightarrow \frac{1}{9} \geq \frac{\beta}{3}$$

$$\Rightarrow \beta \leq \frac{1}{3}.$$

Hence, $\beta \in (-\infty, 1/3]$ and $\gamma \in [-1/27, \infty)$.

EXAMPLE 37 *If* $f(x) = x^3 + bx^2 + cx + d$ *and* $f(0), f(-1)$ *are odd integers, prove that* $f(x) = 0$ *cannot have all integral roots.*

SOLUTION We have,

$$f(x) = x^3 + bx^2 + cx + d$$

$$\Rightarrow f(0) = d \text{ and } f(-1) = -1 + b - c + d$$

Since $f(0)$ and $f(-1)$ are odd integers. Therefore,

d and $-1 + b - c + d$ are odd integers

Now,

$-1 + b - c + d$ is an odd integer, say m.

$$\Rightarrow -1 + b - c + d = m$$

$$\Rightarrow b - c = 1 + m - d$$

$\Rightarrow b - c$ is an odd integer $\left[\begin{array}{l}\because m \text{ and } d \text{ are odd integers} \\ \therefore m - d \text{ is an even integer} \\ \Rightarrow m - d + 1 \text{ is an odd integer}\end{array}\right]$

Thus, d and $(b - c)$ are odd integers.

Let α, β, γ be three roots of

$$ax^3 + bx^2 + cx + d = 0$$

Then,

$$\alpha + \beta + \gamma = -b \quad ...(i)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = c \quad ...(ii)$$

$$\alpha\beta\gamma = -d \quad ...(iii)$$

If possible, let α, β, γ be all integers.

From (iii), we have

$\alpha\beta\gamma = -d$ and d is an odd integer

$\Rightarrow \alpha\beta\gamma$ is an odd integer

$\Rightarrow \alpha, \beta, \gamma$ are all odd integers.

From (i) and (ii), we have

$\alpha + \beta + \gamma = -b$ and $\alpha\beta + \beta\gamma + \gamma\alpha = c$

$\Rightarrow$ b and c are odd integers

$\Rightarrow$ $b-c$ is an even integer, which is a contradiction.

Therefore, α, β, γ are not all odd integers.

EXAMPLE 38 *Let a, b, c be real numbers and $a \neq 0$. If α is a root of $a^2x^2+bx+c=0$ and β is a root of $a^2x^2-bx-c=0$, then prove that the equation $a^2x^2+2bx+2c=0$ has a root lying between α and β.*

SOLUTION Since α and β are the roots of the equations $a^2x^2+bx+c=0$ and $a^2x^2-bx-c=0$ respectively. Therefore,

$$a^2\alpha^2+b\alpha+c = 0 \quad ...(i)$$

and,
$$a^2\beta^2-b\beta-c = 0 \quad ...(ii)$$

Let $f(x) = a^2x^2+2bx+2c$.

In order to prove that $f(x)=0$ has a root lying between α and β, it is sufficient to show that $f(\alpha)$ and $f(\beta)$ are of opposite signs.

Now,

$$\begin{aligned} f(\alpha) &= a^2\alpha^2+2b\alpha+2c \\ &= a^2\alpha^2+2(b\alpha+c) \\ &= a^2\alpha^2-2\alpha^2\alpha^2 \quad [\text{From (i), we have } b\alpha+c=-a^2\alpha^2] \\ &= -a^2\alpha^2 \end{aligned}$$

and,

$$\begin{aligned} f(\beta) &= a^2\beta^2+2b\beta+2c \\ &= a^2\beta^2+2(b\beta+c) \\ &= a^2\beta^2+2a^2\beta^2 \quad [\text{From (ii), we have } b\beta+c=a^2\beta^2] \\ &= 3a^2\beta^2 \end{aligned}$$

Clearly, $f(\alpha)<0$ and $f(\beta)>0$.

Hence, $f(x)=0$ i.e. $a^2x^2+2bx+2c=0$ has a root lying between α and β.

EXAMPLE 39 *If a, b, c, A, B, C are real number such that $a<b<c$, $f(x)=(x-a)(x-b)(x-c)-A^2(x-a)-B^2(x-b)-C^2(x-c)$, then show that $f(x)=0$ has all real roots.*

SOLUTION We have,

$$f(x)=(x-a)(x-b)(x-c)-A^2(x-a)-B^2(x-b)-C^2(x-c)$$

$$\Rightarrow f(a) = -B^2(a-b)-C^2(a-c)$$

$$\Rightarrow f(a) = -[B^2(a-b)+C^2(a-c)]>0 \quad [\because a<b<c]$$

and,

$$f(c) = -A^2(c-a)-B^2(c-b)$$

$$\Rightarrow f(c) = -[A^2(c-a)+B^2(c-b)]<0 \quad [\because a<b<c]$$

Also, $f(x)\to-\infty$ as $x\to-\infty$ and, $f(x)\to\infty$ as $x\to\infty$.

Thus, $f(x)$ changes its signs from positive to negative or negative to positive in the intervals : $(-\infty, a)$, (a, c) and (c, ∞). Hence, $f(x)=0$ has three real roots.

EXAMPLE 40 *If α is a real root of the quadratic equation $ax^2+bx+c=0$ and β is a real root of $-ax^2+bx+c=0$, then show that there is a root γ of the equation $\frac{a}{2}x^2+bx+c=0$ which lies between α and β.*

SOLUTION Let $f(x)=\frac{a}{2}x^2+bx+c$.

Since α and β are real roots of equations $ax^2+bx+c=0$ and $-ax^2+bx+c=0$ respectively. Therefore,

$$a\alpha^2+b\alpha+c=0 \text{ and, } -a\beta^2+b\beta+c=0$$

$$\Rightarrow b\alpha+c=-a\alpha^2 \quad \text{and, } b\beta+c=a\beta^2 \quad ...(i)$$

Now,

$$\begin{aligned} f(\alpha) &= \frac{a}{2}\alpha^2+b\alpha+c \\ &= \frac{1}{2}(a\alpha^2+2b\alpha+2c) \\ &= \frac{1}{2}(a\alpha^2-2a\alpha^2) \quad [\text{Using (i)}] \\ &= \frac{-a\alpha^2}{2} \end{aligned}$$

and,

$$\begin{aligned} f(\beta) &= \frac{a}{2}\beta^2+b\beta+c \\ &= \frac{1}{2}(a\beta^2+2b\beta+2c) \\ &= \frac{1}{2}(a\beta^2+2a\beta^2) \quad [\text{Using (i)}] \\ &= \frac{3}{2}a\beta^2. \end{aligned}$$

$$\therefore f(\alpha)f(\beta) = -\frac{3}{4}a^2\alpha^2\beta^2<0$$

$\Rightarrow$ $f(\alpha)$ and $f(\beta)$ are of opposite signs

$\Rightarrow$ There is γ lying between α and β such that $f(\gamma)=0$.

Hence, γ is a root of $\frac{a}{2}x^2+bx+c=0$, where γ lies between α and β.

EXAMPLE 41 *Solve the equation $x^4-2x^3+4x^2+6x-21=0$, if it is given that two of its roots are equal in magnitude but opposite in sign.*

SOLUTION Let $\alpha, \beta, \gamma, \delta$ be the roots of the given biquadratic. Then,

$$\alpha+\beta+\gamma+\delta = 2 \quad ...(i)$$

$$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta = 4 \quad ...(ii)$$

$$(\alpha+\beta)\gamma\delta+(\gamma+\delta)\alpha\beta = -6 \quad ...(iii)$$

$$\alpha\beta\gamma\delta = -21 \quad ...(iv)$$

It is given that two roots of the biquadratic are equal in magnitude but opposite in sign. Therefore, let $\alpha+\beta=0$.

From (i), we have

$$\gamma+\delta = 2 \qquad [\because \alpha+\beta = 0]$$

From (iii), we have

$$\alpha\beta+\gamma\delta = 4 \qquad ...(v)$$

From (iv), we have

$$0\,.\,\gamma\delta+2\times\alpha\beta = -6 \quad [\because \alpha+\beta=0 \text{ and } \gamma+\delta = 2]$$

$$\Rightarrow \quad \alpha\beta = -3$$

Putting $\alpha\beta = -3$ in (v), we get

$$\gamma\delta = 7.$$

The equation having α, β as its roots is

$$x^2-x(\alpha+\beta)+\alpha\beta = 0$$

$$\Rightarrow x^2-3 = 0$$

$$\Rightarrow x = \pm\sqrt{3}$$

The equation having γ, δ as its roots is

$$x^2-(\gamma+\delta)x+\gamma\delta = 0$$

$$\Rightarrow \quad x^2-2x+7 = 0$$

$$\Rightarrow x = \frac{2\pm\sqrt{4-28}}{2} = 1\pm i\sqrt{6}$$

Thus, the roots of the biquadratic are $\pm\sqrt{3}, 1\pm i\sqrt{6}$.

EXAMPLE 42 *Let a be a real number and α and β the roots of $x^2-2a^2x+1=0$. Show that $\frac{\alpha}{\beta^3}+\frac{\beta}{\alpha^3}$ attains a minimum value. Find the minimum value and the roots for which it has attained.*

SOLUTION Since α, β are the roots of the equation $x^2-2ax+1=0$. Therefore,

$$\alpha+\beta = 2a^2 \text{ and } \alpha\beta = 1.$$

Now,

$$\frac{\alpha}{\beta^3}+\frac{\beta}{\alpha^3}$$

$$= \frac{\alpha^4+\beta^4}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)^4-4\alpha\beta(\alpha+\beta)^2+2\alpha^2\beta^2}{(\alpha\beta)^3}$$

$$= 16a^4-8a^2+2$$

$$= 16\left(a^2-\frac{1}{2}\right)^2-2$$

$$\Rightarrow \frac{\alpha}{\beta^3}+\frac{\beta}{\alpha^3} \geq -2$$

Thus, $\frac{\alpha}{\beta^3}+\frac{\beta}{\alpha^3}$ attain the minimum value -2 when $a^2-\frac{1}{2}=0$ i.e. $a^2 = \frac{1}{\sqrt{2}}$.

Putting $a^2 = \frac{1}{2}$ in the given equation, we obtain

$$x^2-x+1 = 0$$

Clearly, roots of this equation are $-w$ and $-w^2$

EXAMPLE 43 *Let α and β be the roots of $x^2-2px+pq=0$ $(p>0, q>0)$. If $S_n = 1+2\frac{q}{p}+3\left(\frac{q}{p}\right)^2+\ldots+n\left(\frac{q}{p}\right)^{n-1}$, show that $S_n \not> \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$.*

SOLUTION Since α and β are the roots of the equation $x^2-2px+pq=0$

$\therefore\ \alpha+\beta = 2p$ and $\alpha\beta = pq$.

Now,

$$\frac{q}{p} = \frac{qp}{p^2} = \frac{\alpha\beta}{\left(\frac{\alpha+\beta}{2}\right)^2} = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

We know that :

$$AM > GM$$

$$\Rightarrow \quad \frac{\alpha+\beta}{2} > \sqrt{\alpha\beta}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2}{4} > \alpha\beta$$

$$\Rightarrow \quad 1 > \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$\Rightarrow \quad 0 < \frac{q}{p} < 1.$$

Now,

$$S_n = 1+2\frac{q}{p}+3\left(\frac{q}{p}\right)^2+\ldots+n\left(\frac{q}{p}\right)^{n-1} \qquad ...(i)$$

$$\Rightarrow \frac{q}{p}S_n = \frac{q}{p}+2\left(\frac{q}{p}\right)^2+\ldots+(n-1)\left(\frac{q}{p}\right)^n+n\left(\frac{q}{p}\right)^n \qquad ...(ii)$$

Subtracting (ii) from (i), we get

$$S_n\left(1-\frac{q}{p}\right) = 1+\frac{q}{p}+\left(\frac{q}{p}\right)^2+\ldots+\left(\frac{q}{p}\right)^n-n\left(\frac{q}{p}\right)^n$$

$$\Rightarrow S_n\left(1-\frac{q}{p}\right) = \frac{1-\left(\frac{q}{p}\right)^n}{1-\left(\frac{q}{p}\right)}-n\left(\frac{q}{p}\right)^n$$

$$\Rightarrow S_n = \frac{1}{\left(1-\frac{q}{p}\right)^2}-\frac{\left(\frac{q}{p}\right)^n}{\left(1-\frac{q}{p}\right)^2}-\frac{n\left(\frac{q}{p}\right)^n}{\left(1-\frac{q}{p}\right)}$$

Since $0<\frac{q}{p}<1$. Therefore, $\left(\frac{q}{p}\right)^n \to 0$ as $n\to\infty$.

Hence, $\lim_{n\to\infty} S_n = \dfrac{1}{\left(1-\dfrac{q}{p}\right)^2}$

$$= \frac{1}{\left(1-\dfrac{4\alpha\beta}{(\alpha+\beta)^2}\right)^2} = \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$$

Since $S_1 < S_2 < S_3 < \ldots S_n < \ldots$ Therefore,

$$S_n \le \lim_{n\to\infty} S_n \Rightarrow S_n \le \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4 \Rightarrow S_n \not> \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$$

EXAMPLE 44 *If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b > 12\,(2^{1/3})$*

SOLUTION Let α, β, γ be the roots of the given equation. Then,

$$\alpha+\beta+\gamma = -\frac{a}{2} \qquad \ldots(i)$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{b}{2} \qquad \ldots(ii)$$

and, $$\alpha\beta\gamma = -2 \qquad \ldots(iii)$$

Since all the coefficients of the given equation are positive, so its all roots are negative.

Let $\alpha = -\alpha_1$, $\beta = -\beta_1$ and $\gamma = -\gamma_1$. Therefore, from (i), (ii) and (iii), we obtain

$$\alpha_1+\beta_1+\gamma_1 = \frac{a}{2} \qquad \ldots(iv)$$

$$\alpha_1\beta_1+\beta_1\gamma_1+\gamma_1\alpha_1 = \frac{b}{2} \qquad \ldots(v)$$

and, $$\alpha_1\beta_1\gamma_1 = 2 \qquad \ldots(vi)$$

Now,

A.M ≥ G.M.

$$\Rightarrow \frac{\alpha_1+\beta_1+\gamma_1}{3} \ge (\alpha_1\beta_1\gamma_1)^{1/3}$$

$$\Rightarrow \frac{a}{6} \ge 2^{1/3}$$

$$\Rightarrow a \ge 6\,(2^{1/3}) \qquad \ldots(vii)$$

Again,

A.M. ≥ G.M.

$$\Rightarrow \frac{\alpha_1\beta_1+\beta_1\gamma_1+\gamma_1\alpha_1}{3} \ge [(\alpha_1\beta_1)(\beta_1\gamma_1)(\gamma_1\alpha_1)]^{1/3}$$

$$\Rightarrow \frac{\alpha_1\beta_1+\beta_1\gamma_1+\gamma_1\alpha_1}{3} \ge (\alpha_1\beta_1\gamma_1)^{2/3}$$

$$\Rightarrow \frac{b}{6} \ge 2^{2/3}$$

$$\Rightarrow b \ge 6\,(2^{2/3}) \qquad \ldots(viii)$$

From (vii) and (viii), we obtain

$$a+b = 6\,(2^{1/3}+2^{2/3})$$

$$\Rightarrow a+b > 6\,(2^{1/3}+2^{1/3})$$

$$\Rightarrow a+b > 12\,(2^{1/3})$$

EXAMPLE 45 *If α and β are roots of the equation $ax^2+bx+c=0$ and α^4 and β^4 are the roots of the equation $lx^2+mx+n=0$, then prove that the roots of the equation $a^2lx^2 - 4aclx + 2c^2l + a^2m = 0$ are always real and opposite in sign (α, β are real and distinct).*

SOLUTION It is given that α and β are the roots of $ax^2+bx+c=0$ and α^4, β^4 are the roots of $lx^2+mx+n=0$. Therefore,

$$\alpha+\beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$$

and, $\alpha^4+\beta^4 = \dfrac{-m}{l}, \alpha^4\beta^4 = \dfrac{n}{l}$.

Now,

$$\alpha^4+\beta^4 = \frac{-m}{l}$$

$$\Rightarrow (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = \frac{-m}{l}$$

$$\Rightarrow [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2 = -\frac{m}{l}$$

$$\Rightarrow \left(\frac{b^2}{a^2}-\frac{2c}{a}\right)^2 - 2\frac{c^2}{a^2} = -\frac{m}{l}$$

$$\Rightarrow \left(\frac{b^2}{a^2}\right)^2 - 4\left(\frac{b^2}{a^2}\right)\left(\frac{c}{a}\right) + 2\left(\frac{c^2}{a^2}\right) + \frac{m}{l} = 0$$

$$\Rightarrow a^2\left(\frac{b^2}{a^2}\right)^2 - 4ac\left(\frac{b^2}{a^2}\right) + 2c^2 + \frac{a^2m}{l} = 0$$

$$\Rightarrow a^2l\left(\frac{b^2}{a^2}\right)^2 - 4acl\left(\frac{b^2}{a^2}\right) + 2c^2l + a^2m = 0$$

$\Rightarrow \dfrac{b^2}{a^2}$ is a root of the equation $a^2lx^2 - 4acl\,x + 2c^2l + a^2m = 0$.

Also, $\dfrac{b^2}{a^2} > 0$.

Thus, the equation $a^2lx^2 - 4acl\,x + 2c^2l + a^2m = 0$ has a positive real root.

Let γ be the other root of this equation. Then,

$$\gamma + \frac{b^2}{a^2} = \frac{4acl}{a^2l}$$

$$\Rightarrow \gamma = \frac{4c}{a} - \frac{b^2}{a^2} = \frac{4ac-b^2}{a^2}$$

$\Rightarrow \gamma < 0$ [$\because ax^2 + bx + c = 0$ has real roots]

Hence, the roots of the equation $a^2l\,x^2 - 4aclx + 2c^2l + a^2m = 0$ are always real and opposite in sign.

EXAMPLE 46 *Find out minimum non-negative real values of a, b and c for which $x^4 + ax^3 + bx^2 + cx + 1 = 0$ have only real roots.*

SOLUTION Since a, b, c are non-negative real numbers. So, all roots of the given equation will be negative. Let its roots be $\alpha_1, \beta_1, \gamma_1$ and δ_1. Then,

$\alpha_1 = -\alpha, \beta_1 = -\beta, \gamma_1 = -\gamma, \delta_1 = -\delta$, where $\alpha, \beta, \gamma, \delta > 0$.

Now,

$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = -a$

$\Rightarrow \alpha + \beta + \gamma + \delta = a$ ...(i)

$(\alpha_1 + \beta_1)(\gamma_1 + \delta_1) + \alpha_1\beta_1 + \gamma_1\delta_1 = b$

$\Rightarrow (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = b$ ...(ii)

$\alpha_1\beta_1(\gamma_1 + \delta_1) + \gamma_1\delta_1(\alpha_1 + \beta_1) = -c$

$\Rightarrow \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = c$ ...(iii)

$\alpha_1\beta_1\gamma_1\delta_1 = 1$

$\Rightarrow \alpha\beta\gamma\delta = 1$ ...(iv)

Now,

AM $\geq$ GM

$\Rightarrow \dfrac{\alpha + \beta + \gamma + \delta}{4} \geq (\alpha\beta\gamma\delta)^{1/4}$

$\Rightarrow \dfrac{a}{4} \geq 1$ [Using (i) and (iv)]

$\Rightarrow a \geq 4$

$\dfrac{\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta)}{6} \geq (\alpha\beta\gamma\delta)^{3/6}$

$\Rightarrow \dfrac{b}{6} \geq 1$ [Using (ii) and (iv)]

$\Rightarrow b \geq 6$

and,

$$\frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}}{4} \geq \left(\frac{1}{\alpha\beta\gamma\delta}\right)^{1/4}$$

$$\Rightarrow \frac{\alpha\beta(\gamma + \delta) + (\alpha + \beta)\gamma\delta}{4\alpha\beta\gamma\delta} \geq \left(\frac{1}{\alpha\beta\gamma\delta}\right)^{1/4}$$

$\Rightarrow \dfrac{c}{4} \geq 1 \Rightarrow c \geq 4.$

Hence, the minimum values of a, b, c are 4, 6, 4 respectively.

EXAMPLE 47 *If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then prove that the equation $x^2 - 4qx + 2q^2 - r = 0$ has real roots.*

SOLUTION Since α, β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$. Therefore,

$\alpha + \beta = -p, \alpha\beta = q$

$\alpha^4 + \beta^4 = r$ and $\alpha^4\beta^4 = s$.

Now,

$\alpha^4 + \beta^4 = r$

$\Rightarrow (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = r$

$\Rightarrow \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2 = r$

$\Rightarrow (p^2 - 2q)^2 - 2q^2 = r$ ...(i)

$\Rightarrow (p^2)^2 - 4p^2q + 2q^2 - r = 0$

$\Rightarrow p^2$ is a root of the equation $x^2 - 4qx + 2q^2 - r = 0$.

Let D be the discriminant of the above equation. Then,

$D = 16q^2 - 4(2q^2 - r)$

$= 8q^2 + 4r$

$= 4(2q^2 + r)$

$= 4(p^2 - 2q)^2$ [Using (i)]

$\Rightarrow D > 0.$

Hence, the roots of the equation $x^2 - 4qx + 2q^2 - r = 0$ are real.

EXAMPLE 48 *Show that the roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ cannot be all real if $(n-1)a_1^2 - 2na_2 < 0$.*

SOLUTION Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the given equation. Then,

$$\alpha_1 + \alpha_1 + \ldots + \alpha_n = -a_1$$

and, $$\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j = a_2$$

and so on.

Now,

$(n-1)a_1^2 - 2na_2$

$= (n-1)(\alpha_1 + \alpha_2 + \ldots + \alpha_n)^2 - 2n\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j$

$= (n-1)\left\{\sum_{i=1}^{n} \alpha_i^2 + 2\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j\right\} - 2n\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j$

$= (n-1)\sum_{i=1}^{n} \alpha_i^2 - 2\left\{\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j\right\}$

$= \sum_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$

$\therefore (n-1)a_1^2 - 2na_2 < 0$

$\Rightarrow \sum_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2 < 0$

This cannot be possible if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real.

Hence, the roots of the given equation cannot be all real if $(n-1)a_1^2 - 2na_2 < 0$.

EXAMPLE 49 *If a, b, c are the roots of the equation $x^3 + qx + r = 0$ form an equation whose roots are $a+b+c, a-b+c$ and $b+c-a$.*

SOLUTION Since a, b, c are the roots of the equation $x^3 + qx + r = 0$. Therefore,

$$a + b + c = 0 \quad \ldots(i)$$

$$ab + bc + ca = q \quad \ldots(ii)$$

and, $$abc = -r \quad \ldots(iii)$$

Let $\alpha_1 = a + b - c$, $\beta_1 = a - b + c$ and $\gamma_1 = b + c - a$. Then the cubic equation having $\alpha_1, \beta_1, \gamma_1$ as its roots is

$$x^3 - x^2(\alpha_1 + \beta_1 + \gamma_1) + (\alpha_1\beta_1 + \beta_1\gamma_1 + \gamma_1\alpha_1)x - \alpha_1\beta_1\gamma_1 = 0 \quad \ldots(iv)$$

Now,

$$\alpha_1 + \beta_1 + \gamma_1 = (a+b-c) + (a-b+c) + (b+c-a)$$

$$= a + b + c = 0 \quad \text{[Using (i)]}$$

$$\alpha_1\beta_1 + \beta_1\gamma_1 + \gamma_1\alpha_1$$

$$= (a+b-c)(b+c-a) + (b+c-a)(c+a-b) + (c+a-b)(a+b-c)$$

$$= (-2c)(-2a) + (-2a)(-2b) + (-2b)(-2c) \quad \text{[Using (i)]}$$

$$= 4(ab + bc + ca) = 4q$$

and,

$$\alpha_1\beta_1\gamma_1$$

$$= (a+b-c)(b+c-a)(c+a-b)$$

$$= (-2c)(-2a)(-2b) \quad \text{[Using (i)]}$$

$$= -8abc.$$

$$= 8r \quad \text{[Using (iii)]}$$

Substituting these values in (iv), we obtain the required equation as

$$x^3 + 4qx - 8r = 0.$$

EXAMPLE 50 *If the sum of two roots of the equation $4x^4 - 8x^3 - 13x^2 + 2x + 3 = 0$ is zero, find all of its roots.*

SOLUTION Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation such that $\alpha + \beta = 0$. Then,

$$\alpha + \beta + \gamma + \delta = 2 \quad \ldots(i)$$

$$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = \frac{-13}{4} \quad \ldots(ii)$$

$$\alpha\beta(\gamma+\delta) + (\alpha+\beta)\gamma\delta = \frac{-2}{4} = -\frac{1}{2} \quad \ldots(iii)$$

and, $$\alpha\beta\gamma\delta = \frac{3}{4} \quad \ldots(iv)$$

Putting $(\alpha + \beta) = 0$ in (i), we get

$$\gamma + \delta = 2.$$

Putting $\alpha + \beta = 0$ and $\gamma + \delta = 2$ in (iii), we get

$$\alpha\beta = -\frac{1}{4}$$

Putting $\alpha\beta = -\frac{1}{4}$ in (iv), we get

$$\gamma\delta = -3.$$

The equation having α, β as its roots is

$$x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

or, $$x^2 - \frac{1}{4} = 0$$

$\Rightarrow$ $$x = \pm\frac{1}{2}$$

The equation having γ, δ as its roots is

$$x^2 - x(\gamma + \delta) + \gamma\delta = 0$$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3, -1.$

Hence, the roots of the given equation are : $-1, -\frac{1}{2}, \frac{1}{2}, 3$.

EXAMPLE 51 *Let r, s, t, u be the roots of the equation $x^4 + Ax^3 + Bx^2 + Cx + D = 0$. Prove that, if $rs = tu$, then $A^2D = C^2$.*

SOLUTION We have,

$$r + s + t + u = -A \quad \ldots(i)$$

$$(r+s)(t+u) + rs + tu = B \quad \ldots(ii)$$

$$rs(t+u) + tu(r+s) = -C \quad \ldots(iii)$$

and, $$rstu = D.$$

We have, $rs = tu$. Therefore, from (iii) and (iv), we get

$rs(r+s+t+u) = -C$ and $(rs)^2 = D$

$\Rightarrow rs(-A) = -C$ and $(rs)^2 = D$

$\Rightarrow \left(\frac{C}{A}\right)^2 = D$

$\Rightarrow C^2 = A^2D.$

EXAMPLE 52 *Let α and β be two real roots of the equation $x^3 + px^2 + qx + r = 0$ satisfying the relation $\alpha\beta + 1 = 0$. Prove that $r^2 + pr + q + 1 = 0$.*

SOLUTION Let δ be the third root of the given equation. Then,

$\alpha\beta\delta = -r$

$\Rightarrow -\delta = -r \quad [\because \alpha\beta + 1 = 0 \text{ (given)} \Rightarrow \alpha\beta = -1]$

$\Rightarrow \delta = r$

Since δ is a root of the given equation. Therefore,

$\delta^3 + p\delta^2 + q\delta + r = 0$

$\Rightarrow r^3 + pr^2 + qr + r = 0$

$\Rightarrow r^2+pr+q+1=0.$

EXAMPLE 53 *Show that* $x^3+2x^2+x+5=0$ *has only one real root* α *such that* $[\alpha]=-3$.

SOLUTION Let $f(x)=x^3+2x^2+x+5$. Then,

$f(-2)=3$ and $f(-3)=-7$

$\therefore f(-2)\,f(3)<0$

$\Rightarrow f(x)$ has an odd number of real roots between -2 and -3.

But, if the given equation has an odd number of real roots between -2 and -3, then their product will be less than -8. However, the product of the roots is -5. So, there is exactly one real root between -3 and -2. Let it be α. Then,

$$-3<\alpha<-2\Rightarrow[\alpha]=-3.$$

EXAMPLE 54 *Find the value of positive integer* n *for which the equation*

$$\sum_{r=1}^{n}(x+r-1)(x+r)=10n$$

has solutions α and $\alpha+1$ *for some* α.

SOLUTION We have,

$$\sum_{r=1}^{n}(x+r-1)(x+r)=10n$$

$$\Rightarrow \sum_{r=1}^{n}[x^2+(2r-1)x+r^2-r]=10n$$

$$\Rightarrow nx^2+\left|\sum_{r=1}^{n}(2r-1)\right|x+\sum_{r=1}^{n}(r^2-r)=10n$$

$$\Rightarrow nx^2+n^2x+\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=10n$$

$$\Rightarrow nx^2+n^2x+\frac{1}{3}n(n^2-1)=10n$$

$$\Rightarrow nx^2+n^2x+n\frac{(n^2-31)}{3}=0$$

$$\Rightarrow x^2+nx+\frac{n^2-31}{3}=0$$

This equation has two roots α and $\alpha+1$.

$$\therefore \alpha+(\alpha+1)=-n \text{ and } \alpha(\alpha+1)=\frac{n^2-31}{3}$$

$$\Rightarrow \alpha=-\left(\frac{n+1}{2}\right) \text{ and } \alpha^2+\alpha=\frac{n^2-31}{3}$$

$$\Rightarrow \frac{(n+1)^2}{4}-\left(\frac{n+1}{2}\right)=\frac{n^2-31}{3}$$

$$\Rightarrow 3(n+1)^2-6(n+1)=4(n^2-31)$$

$$\Rightarrow n^2=121$$

$$\Rightarrow n=11.$$

EXAMPLE 55 *For what values of* $a\in R$ *does the equation* $ax^2+x+a-1=0$ *possess two distinct roots* x_1 *and* x_2 *satisfying the condition* $\left|\frac{1}{x_1}-\frac{1}{x_2}\right|>1$ *?*

SOLUTION Since x_1 and x_2 are the roots of the equation $ax^2+x+a-1=0$. Therefore,

$$x_1+x_2=-\frac{1}{a} \text{ and, } x_1x_2=\frac{a-1}{a}.$$

For the roots to be distinct, we must have

Discriminant >0

$$\Rightarrow 1-4a(a-1)>0$$

$$\Rightarrow 4a^2-4a-1<0$$

$$\Rightarrow (2a-1)^2-2<0$$

$$\Rightarrow (2a-1-\sqrt{2})(2a-1+\sqrt{2})<0$$

$$\Rightarrow \frac{1-\sqrt{2}}{2}<a<\frac{1+\sqrt{2}}{2}$$

Thus, the roots are real and distinct, if

$$a\in\left(\frac{1-\sqrt{2}}{2},\frac{1+\sqrt{2}}{2}\right) \qquad ...(i)$$

Without any loss of generality, let us assume that $x_2>x_1$.

Now,

$$\left|\frac{1}{x_1}-\frac{1}{x_2}\right|>1$$

$$\Rightarrow \frac{|x_2-x_1|}{|x_1x_2|}>1$$

$$\Rightarrow \frac{x_2-x_1}{|x_1x_2|}>1 \qquad ...(ii) \qquad [\because x_2>x_1\ \therefore x_2-x_1>0]$$

Now two cases arise :

CASE I *When* $x_1x_2>0$:

In this case, we have

$$x_1x_2>0\Rightarrow\frac{a-1}{a}>0\Rightarrow a\in(-\infty,0)\cup(1,\infty) \qquad ...(iii)$$

From (ii), we have

$$\frac{x_2-x_1}{x_1x_2}>1$$

$$\Rightarrow x_2-x_1>x_1x_2$$

$$\Rightarrow (x_2-x_1)^2>(x_1x_2)^2$$

$$\Rightarrow (x_2+x_1)^2-4x_1x_2-(x_1x_2)^2>0$$

$$\Rightarrow \frac{1}{a^2}-4\left(\frac{a-1}{a}\right)-\left(\frac{a-1}{a}\right)^2>0$$

$\Rightarrow \quad 1-4a(a-1)-(a-1)^2 > 0$

$\Rightarrow 1-4a^2+4a-a^2+2a-1 > 0$

$\Rightarrow -5a^2+6a > 0$

$\Rightarrow (5a-6)\,a < 0$

$\Rightarrow \quad 0 < a < \frac{6}{5}$

$\Rightarrow a \in (0, 6/5)$ ...(iv)

From a (i) and (iii), we obtain

$$a \in (1, 6/5) \qquad \text{...(v)}$$

CASE II *When* $x_1 x_2 < 0$:

In this case, we have

$$x_1 x_2 < 0 \Rightarrow \frac{a-1}{a} < 0 \Rightarrow a \in (0, 1) \qquad \text{...(vi)}$$

From (ii), we have

$$\frac{x_2 - x_1}{-x_1 x_2} > 1$$

$\Rightarrow (x_2 - x_1) > (-x_1 x_2)$ $[\because -x_1 x_2 > 0]$

$\Rightarrow (x_2 - x_1)^2 > (-x_1 x_2)^2$

$\Rightarrow (x_2 + x_1)^2 - 4x_1 x_2 - x_1^2 x_2^2 > 0$

$\Rightarrow a \in (0, 6/5)$ [Proceed as in case I] ...(vii)

From (i), (v) and (vi), we get

$$a \in (0, 1) \qquad \text{...(viii)}$$

From (v) and (viii), we get

$$a \in (0, 1) \cup (1, 6/5).$$

EXAMPLE 56 *Find the solutions of the equation* $x^2 - 3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 3$, *where* $[\cdot]$ *denotes the greatest integer function.*

SOLUTION The given equation can be written as

$$x^2 - 3 = 3\left[\sin\left(x - \frac{\pi}{6}\right)\right] \qquad \text{...(i)}$$

Clearly, right hand side can take only three values $-3, 0, 3$.

CASE I *When* $3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 3$:

In this case, equation (i) reduces to

$$x^2 - 3 = 3 \Rightarrow x = \pm\sqrt{6}.$$

But, for $x = \pm\sqrt{6}$, we have

$$3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 3\left[\sin\left(\pm\sqrt{6} - \frac{\pi}{6}\right)\right] \neq 3.$$

So, $x = \pm\sqrt{6}$ is not a solution of the given equation.

CASE II *When* $3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 0$:

In this case, the given equation reduces to

$$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$$

When $x = \sqrt{3}$, we have,

$$x - \frac{\pi}{6} = \sqrt{3} - \frac{\pi}{6} < \frac{\pi}{2} \Rightarrow \left[\sin\left(x - \frac{\pi}{6}\right)\right] = 0$$

$\therefore$ $x = \sqrt{3}$ is a solution.

When $x = -\sqrt{3}$, we have

$$x - \frac{\pi}{6} = -\sqrt{3} - \frac{\pi}{6}$$

$$\Rightarrow -\pi < x - \frac{\pi}{6} < 0$$

$$\Rightarrow \left[\sin\left(x - \frac{\pi}{6}\right)\right] = -3$$

$\therefore$ $x = -\sqrt{3}$ is not a solution.

CASE III *When* $3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = -3$.

In this case, the given equation reduces to

$$x^2 - 3 = -3 \Rightarrow x = 0$$

When $x = 0$, we have

$$\left[\sin\left(x - \frac{\pi}{6}\right)\right] = \left[\sin\left(-\frac{\pi}{6}\right)\right] = -1.$$

$$\Rightarrow 3\left[\sin\left(x - \frac{\pi}{6}\right)\right] = -3.$$

So, $x = 0$ is a solution of the given equation.

Hence, the given equation has only two solutions $x = 0, \sqrt{3}$.

EXAMPLE 57 *If* α and β *are roots of* $ax^2 + (a-1)x + 1 = 0$ *such that* $\alpha\cos\theta - \beta\sin\theta = \sin\theta - \cos\theta$, *then prove that* $a \in (-\infty, -3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$.

SOLUTION Since α and β are roots of $ax^2 + (a-1)x + 1 = 0$. Therefore,

$$\alpha = (1-a) + \sqrt{(a-1)^2 - 4a}$$

and, $\beta = (1-a) - \sqrt{(a-1)^2 - 4a}$

Now, $\alpha\cos\theta - \beta\sin\theta = \sin\theta - \cos\theta$

$\Rightarrow (\alpha + 1)\cos\theta = (\beta + 1)\sin\theta$

$$\Rightarrow \tan\theta = \frac{\alpha + 1}{\beta + 1}$$

$$\Rightarrow \tan\theta = \frac{2 - a + \sqrt{(a-1)^2 - 4a}}{2 - a - \sqrt{(a-1)^2 - 4a}}$$

$$\Rightarrow \tan\theta = \frac{2 - a + \sqrt{a^2 - 6a + 1}}{2 - a - \sqrt{a^2 - 6a + 1}}$$

Since $\tan\theta \in R$. Therefore,

$$\frac{2-a+\sqrt{a^2-6a+1}}{2-a-\sqrt{a^2-6a+1}} \in R$$

$\Rightarrow a^2-6a+1 \geq 0$

$\Rightarrow \{a-(3+2\sqrt{2})\}\{a-(3-2\sqrt{2})\} \geq 0$

$\Rightarrow a \leq 3-2\sqrt{2}$ or $a \geq 3+2\sqrt{2}$

$\Rightarrow a \in (-\infty, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$

EXERCISE 20.2

1. If the sum of the roots of the equation $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{c}$ is zero, prove that the product of the roots is $-\frac{1}{2}(a^2+b^2)$.

2. Find the value of 'a' for which the roots α, β of the equation $x^2-6x+a=0$ satisfy the relation $3\alpha+2\beta=20$.

3. If the roots of the equation $x^2-lx+m=0$ differ by unity, then prove that $l^2=4m+1$.

4. If one root of the equation $(l-m)x^2+lx+1=0$ is double the other and l is real, find the greatest value of m.

5. Prove that $\frac{b^2-4ac}{a^2}$ is invariant for the family of equations $ax^2+bx+c=0$ for which the difference of the roots is a constant.

6. If the ratio of the roots of $ax^2+bx+c=0$ is invariant, then prove that $b^2 : ac$ is also invariant.

7. Show that the A.M. of the roots of $x^2-2ax+b^2=0$ is equal to the GM of the roots of the equation $x^2-2bx+a^2=0$ and vice-versa.

8. If a, b, c are in GP, find the ratio of the roots of the equation $ax^2+bx+c=0$.

9. Let α_1, α_2 and β_1, β_2 be the roots of $ax^2+bx+c=0$ and $px^2+qx+r=0$ respectively. If the system of equations $\alpha_1 y+\alpha_2 z=0$ and $\beta_1 y+\beta_2 z=0$ has a non-trivial solution, then prove that $\frac{b^2}{q^2}=\frac{ac}{pr}$.

10. Find the value of k for which the roots α, β of the equation $5x^2+(2k+1)x+k-2=0$ satisfy the relation $2\alpha+5\beta=1$.

11. If the difference of the roots of $x^2+2px+q=0$ and $x^2+2qx+p=0$ be equal, prove that $p+q+1=0$, where $p \neq q$.

12. If the ratio of the roots of the equation $ax^2+bx+c=0$ is r, then prove that

$$\frac{(r+1)^2}{r}=\frac{b^2}{ac}$$

13. If one root of $x^2+px+q=0$ be the square of the other, then prove that $p^3+q^2+q=3pq$.

14. If the ratio of the roots of the equation $a_1x^2+b_1x+c_1=0$ be equal to the ratio of the roots of $a_2x^2+b_2x+c_2=0$, then prove that

$$\left(\frac{b_1}{b_2}\right)^2=\frac{a_1c_1}{a_2c_2}$$

15. If α, β be the roots of the equation $\lambda(x^2-x)+2x+3=0$ and λ_1, λ_2 be the two values of λ for which α and β are connected by the relation

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{3}, \text{ then find } \sqrt{\frac{\lambda_1}{\lambda_2}}+\sqrt{\frac{\lambda_2}{\lambda_1}}.$$

16. If α, β are the roots of the equation $x+1=\lambda x(1-\lambda x)$ and λ_1, λ_2 be two values of λ determined from the equation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\pi-2$, show that

$$\frac{\lambda_1^2}{\lambda_2^2}+\frac{\lambda_2^2}{\lambda_1^2}+2=4\left(\frac{\pi+1}{\pi-1}\right)^2.$$

17. If α, β are the roots of the equation $\lambda(x^2-x)+x+5=0$ and if λ_1, λ_2 are the two values of λ obtained from the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, show that

$$\frac{\lambda_1}{\lambda_2^2}+\frac{\lambda_2}{\lambda_1^2}=4048$$

18. If $\sin\alpha$ and $\cos\alpha$ are roots of the equation $px^2+qx+r=0$, then prove that $p^2-q^2+2pr=0$

19. Find the values of the parameter $a \neq 0$ for which one of the roots of equation $x^2-x+3a=0$ is double of one of the roots of the equation $x^2-x+a=0$.

20. If α, β be the roots of $x^2-9x+A=0$ and γ, δ are the roots of $x^2-36x+B=0$ and if $\alpha, \beta, \gamma, \delta$ form an increasing G.P., find the equation whose roots are A and B.

21. If α, β are the roots of the equation $6x^2-6x+1=0$, then prove that

$$\frac{1}{2}(p+q\alpha+r\alpha^2+s\alpha^3)+\frac{1}{2}(p+q\beta+r\beta^2+s\beta^3)=\frac{p}{1}+\frac{q}{2}+\frac{r}{3}+\frac{s}{4}$$

22. If α, β be the roots of the equation $\lambda^2(x^2-x)+2\lambda x+3=0$ and λ_1, λ_2 be two values of λ for which α and β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{3}$, then find the equation whose roots are $\frac{\lambda_1^2}{\lambda_2}$ and $\frac{\lambda_2^2}{\lambda_1}$.

23. If α, β are the roots of the equation $x^2 - px + q = 0$, then find the equation whose roots are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^3\beta^2 + \alpha^2\beta^3$.

24. If the roots of the equation $x^2 - ax + b = 0$ are real and differ by a quantity less than unity, prove that $\frac{a^2-1}{4} < b < \frac{a^2}{4}$.

25. If α, β, γ be the roots of $2x^3 + x^2 - 7 = 0$ then find the value of $\Sigma\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$.

26. If α, β are the roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, prove that $a S_{n+1} + b S_n + c S_{n-1} = 0$ and hence find S_4.

27. The roots of the equation $x^2 + px + q = 0$ are α and β. If $S_n = \alpha^n + \beta^n$, prove that

(i) $S_{2n} = S_n^2 - 2q^n$ (ii) $S_{2n+1} = S_n \cdot S_{n+1} + pq^n$ and hence find S_7.

28. If $1, \alpha, \beta$ are the roots of $x^3 + ax^2 - 1 - a = 0$, find the equation whose roots are $1, \frac{\alpha}{2-\alpha}, \frac{\beta}{2-\beta}$.

29. Find the condition that one root of the equation
$(a+b)(ax+b)(a-bx) = (a^2x - b^2)(a+bx), ab \neq 0$
is the twice the other.

30. If α, β are the roots of the equation $x^2 - 2x + 3 = 0$. Determine the equation whose roots are $\alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $\beta^2 - \beta^2 + \beta + 5$.

31. If α, β be the roots of $ax^2 + bx + c = 0$ and γ, δ those of $lx^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

32. If α, β be the roots of $x^2 - px + q = 0$ and α', β' be those of $x^2 - p'x + q' = 0$, find the value of

$$(\alpha - \alpha')^2 + (\beta - \alpha')^2 + (\alpha - \beta')^2 + (\beta - \beta')^2$$

33. If α, β be the roots of $x^2 + px - q = 0$ and γ, δ be those of $x^2 + px + r = 0$, prove that

$$(\alpha - \gamma)(\alpha - \delta) = (\beta - \gamma)(\beta - \delta) = q + r$$

34. If α and β be the roots of $x^2 + px + 1 = 0$ and γ, δ be those of $x^2 + qx + 1 = 0$, prove that

$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = q^2 - p^2.$$

35. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, prove that $(a+b+c)^2 = b^2 - 4ac$.

36. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then show that bc^2, ca^2, ab^2 are in AP.

37. If a, b, c are integers and $ax^2 + bx + c = 0$ has an irrational root. Prove that $|f(\lambda)| \geq \frac{1}{q^2}$, where $\lambda = \frac{p}{q} \in Q$ and $f(x) = ax^2 + bx + c$.

38. If α, β and γ are the roots of $x^3 + qx + r = 0$, then find $\Sigma(\alpha - \beta)^2$ in terms of q and r.

39. If α, β be the roots of the equation $(x-a)(x-b) + c = 0$, $c \neq 0$, then find the roots of the equation $(x - c - \alpha)(x - c - \beta) = c$ in terms of a, b and c.

40. If $f(x) = a_1x^2 + b_1x + c_1$ and α, β are the roots of $ax^2 + bx + c = 0$.

Prove that

$$f(\alpha)f(\beta) = \frac{(c_1a - ca_1)^2 - (ab_1 - a_1b)(bc_1 - b_1c)}{a^2}$$

41. Find all integral values of a for which $(a+2)x^2 + 2(a+1)x + a = 0$ will have both roots integers.

42. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in HP, find c.

43. If the roots of $x^3 - ax^2 + x - b = 0$ are in HP, show that the mean root is $3b$.

44. If two roots of the equation $x^3 + ax^2 + bx + c = 0$ be equal in magnitude and of opposite signs, show that $ab = c$.

45. Solve the equation $x^3 - 11x^2 + 36x - 36 = 0$, if its roots are in HP.

46. If the roots of the equation $x^3 + ax^2 + bx + c = 0$ are in G.P. prove that $a^3c = b^3$.

ANSWERS

2. -16 4. $\frac{9}{8}$ 8. ω, ω^2

10. -4 or $\frac{1}{8}$ 15. 7 19. -2

20. $x^2 - 306x + 5184 = 0$ 22. $3x^2 + 68x - 18 = 0$

23. $x^2 - x(p^4 - 5p^2q + 5q^2)p + p^2q^2(p^4 - 5p^2q + 4q^2) = 0$

25. -3 26. $\frac{1}{a^4}(b^4 - 4ab^2c + 2c^2a^2)$

27. $-p^7 + 7p^5q - 14p^3q^2 + 7pq^3$

28. $8x^3 + 4ax^2(1+x) - (1+a)(1+x)^3 = 0$

29. $5a + 4b = 0$ or $4a + 5b = 0$

30. $x^2 - 3x + 2 = 0$

31. $x^2 - \frac{bm}{al}x + \frac{b^2nl + m^2ac - 4acnl}{a^2l^2} = 0$

32. $2[p^2 + p'^2 - 2q - 2q' - pp']$

39. $c+a, c+b$ 41. $0, -1, -3, -4$

42. 9 45. 6, 3, 2

HINTS TO SELECTED PROBLEMS

18. We have, $\sin\alpha + \cos\alpha = -\frac{q}{p}$ and $\sin\cos\alpha = \frac{r}{p}$.

Now, $\sin^2\alpha + \cos^2\alpha = (\sin\alpha + \cos\alpha)^2 - 2\sin\alpha\cos\alpha$

$$\Rightarrow 1 = \frac{q^2}{p^2} - \frac{2r}{p}$$

42. If the roots of the given equation are in HP, then roots of the equation

$$-27x^3 - 54x^2 - cx + 10 = 0 \text{ are in AP.}$$

Let the roots be, $a - d$, a and $a + d$. Then,

$$3a = -\left(\frac{-54}{-27}\right) \Rightarrow a = -\frac{2}{3}$$

Since 'a' is a root of the equation

$$-27x^3 - 54x^2 - cx + 10 = 0$$

$$-27a^3 - 54a^2 - ac + 10 = 0.$$

—❑❑❑—

20.9 COMMON ROOTS

Let $a_1 x^2 + b_1 x + c_1 = 0$ and $a_2 x^2 + b_2 x + c_2 = 0$ be two quadratic equations such that $a_1, a_2 \neq 0$ and $a_1 b_2 \neq a_2 b_1$. Let α be the common root of these two equations. Then,

$$a_1 \alpha^2 + b_1 \alpha + c_1 = 0$$

and,
$$a_2 \alpha^2 + b_2 \alpha + c_2 = 0$$

Solving these two equations by cross-multiplication, we get

$$\frac{\alpha^2}{b_1 c_2 - b_2 c_1} = \frac{\alpha}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \text{ and } \alpha = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

Eliminating α, we get

$$\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} = \left(\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)^2$$

$$\Rightarrow (b_1 c_2 - b_2 c_1)(a_1 b_2 - a_2 b_1) = (c_1 a_2 - c_2 a_1)^2$$

This is the required condition for the two equations to have a common root.

The common root is given by

$$\alpha = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \text{ or, } \alpha = \frac{b_1 c_2 - b_2 c_1}{c_1 a_2 - c_2 a_1}$$

NOTE *Sometimes, to find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of x so obtained is the required common root.*

If the two equations have both roots common, then we have

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the value of k, so that the equations* $2x^2 + kx - 5 = 0$ *and* $x^2 - 3x - 4 = 0$ *have one root in common.*

SOLUTION Let α be the common root of the two equations. Then,

$$2\alpha^2 + k\alpha - 5 = 0$$

$$\alpha^2 - 3\alpha - 4 = 0$$

Solving these two equations, we get

$$\frac{\alpha^2}{-4k - 15} = \frac{\alpha}{-5 + 8} = \frac{1}{-6 - k}$$

$$\Rightarrow \alpha^2 = \frac{4k + 15}{k + 6} \text{ and } \alpha = \frac{-3}{k + 6}$$

$$\Rightarrow \left(\frac{-3}{k+6}\right)^2 = \frac{4k + 15}{k + 6} \qquad [\because \alpha^2 = (\alpha)^2]$$

$$\Rightarrow (4k + 15)(k + 6) = 9$$

$$\Rightarrow 4k^2 + 39k + 81 = 0$$

$$\Rightarrow k = -3 \text{ or, } k = -\frac{27}{4}.$$

EXAMPLE 2 *If* $ax^2 + bx + c = 0$ *and* $bx^2 + cx + a = 0$, $a \neq 0, b \neq 0$ *have a common root, prove that* $a^3 + b^3 + c^3 = 3abc$.

SOLUTION Let α be the common root of the two equations. Then,

$$a\alpha^2 + b\alpha + c = 0$$

and,
$$b\alpha^2 + c\alpha + a = 0$$

Solving these two equations, we get

$$\frac{\alpha^2}{ab - c^2} = \frac{\alpha}{bc - a^2} = \frac{1}{ac - b^2}$$

$$\Rightarrow \quad \alpha^2 = \frac{ab - c^2}{ac - b^2} \text{ and } \alpha = \frac{bc - a^2}{ac - b^2}$$

$$\Rightarrow \frac{ab - c^2}{ac - b^2} = \left(\frac{bc - a^2}{ac - b^2}\right)^2 \qquad [\because \alpha^2 = (\alpha)^2]$$

$$\Rightarrow (ab - c^2)(ac - b^2) = (bc - a^2)^2$$

$$\Rightarrow a(a^3 + b^3 + c^3 - 3abc) = 0$$

$\Rightarrow a^3+b^3+c^3-3abc = 0$ $[\because a \neq 0]$

$\Rightarrow a^3+b^3+c^3 = 3abc.$

EXAMPLE 3 *If* $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$ *are in AP and the equations* $a_1x^2+2b_1x+c_1=0$ *and* $a_2x^2+2b_2x+c_2=0$ *have a common root, prove that* a_2, b_2, c_2 *are in G.P.*

SOLUTION It is given that the equations $a_1x^2+2b_1x+a=0$ and $a_2x^2+2b_2x+c_2=0$ have a common root. So, let α be the common root. Then,

$$a_1\alpha^2+2b_1\alpha+c_1 = 0$$

and, $a_2\alpha^2+2b_2\alpha+c_2 = 0$

$$\Rightarrow \frac{\alpha^2}{2(b_1c_2-b_2c_1)} = \frac{\alpha}{c_1a_2-c_2a_1} = \frac{1}{2(a_1b_2-a_2b_1)}$$

$$\Rightarrow \alpha^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } \alpha = \frac{c_1a_2-c_2a_1}{2(a_1b_2-a_2b_1)}$$

Now,

$$\alpha^2 = (\alpha)^2$$

$$\Rightarrow \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} = \left|\frac{c_1a_2-c_2a_1}{2(a_1b_2-a_2b_1)}\right|^2$$

$$\Rightarrow 4(b_1c_2-b_2c_1)(a_1b_2-a_2b_1) = (c_1a_2-c_2a_1)^2$$

$$\Rightarrow 4b_2c_2\left(\frac{b_1}{b_2}-\frac{c_1}{c_2}\right).a_2b_2\left(\frac{a_1}{a_2}-\frac{b_1}{b_2}\right) = (c_2a_2)^2\left(\frac{c_1}{c_2}-\frac{a_1}{a_2}\right)^2$$

$$\Rightarrow 4a_2b_2^2c_2\left(\frac{b_1}{b_2}-\frac{c_1}{c_2}\right)\left(\frac{a_1}{a_2}-\frac{b_1}{b_2}\right) = a_2^2c_2^2\left(\frac{c_1}{c_2}-\frac{a_1}{a_2}\right)^2 \quad ...(i)$$

It is given that $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$ are in AP. Let D be the common difference of this AP. Then,

$$\therefore \frac{b_1}{b_2}-\frac{a_1}{a_2} = \frac{c_1}{c_2}-\frac{b_1}{b_2} = D \text{ and } \frac{c_1}{c_2}-\frac{a_1}{a_2} = 2D.$$

Putting these values in (i), we get

$$4a_2b_2^2c_2\times(-D)\times-(D) = a_2^2c_2^2(2D)^2$$

$$\Rightarrow b_2^2 = a_2c_2$$

$\Rightarrow a_2, b_2, c_2$ are in GP.

EXAMPLE 4 *If* a, b, c *are in G.P., then prove that the equations* $ax^2+2bx+c = 0$ *and* $dx^2+2ex+f = 0$ *have a common root if* $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ *are in A.P.*

SOLUTION Since a, b, c are in G.P. Therefore,

$$b^2 = ac.$$

Now, $ax^2+2bx+c = 0$

$$\Rightarrow ax^2+2\sqrt{ac}\,x+c = 0$$

$$\Rightarrow (\sqrt{a}x+\sqrt{c})^2 = 0$$

$$\Rightarrow x = \frac{-\sqrt{c}}{\sqrt{a}}$$

Thus, both the roots of $ax^2+2bx+c = 0$ are equal and equal to $-\sqrt{\frac{c}{a}}$.

Since $dx^2+2ex+f = 0$ and $ax^2+2bx+c = 0$ have a common root. So, the common root must be $-\sqrt{\frac{c}{a}}$.

Now, $-\sqrt{\frac{c}{a}}$ is a root of $dx^2+2ex+f = 0$. Therefore,

$$d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f = 0$$

$$\Rightarrow \frac{dc}{a}+f = 2e\sqrt{\frac{c}{a}}$$

$$\Rightarrow \frac{d}{a}+\frac{f}{c} = 2e\sqrt{\frac{1}{ac}} \quad \text{[Dividing both sides by c]}$$

$$\Rightarrow \frac{d}{a}+\frac{f}{c} = \frac{2e}{b} \quad [\because b^2 = ac]$$

$$\Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

EXAMPLE 5 *If* $ax^2+bx+c=0$ and $bx^2+cx+a=0$ *have a common root in common, find the relation between a, b and c.*

SOLUTION Let α be the common root of the two equations. Then,

$$a\alpha^2+b\alpha+c = 0$$

and, $b\alpha^2+c\alpha+a = 0$

Solving these equations by cross-multiplication, we get

$$\frac{\alpha^2}{ab-c^2} = \frac{\alpha}{cb-a^2} = \frac{1}{ac-b^2}$$

$$\Rightarrow \alpha = \frac{ab-c^2}{cb-a^2} \text{ or, } \alpha = \frac{cb-a^2}{ac-b^2}$$

$$\Rightarrow \frac{ab-c^2}{cb-a^2} = \frac{cb-a^2}{ac-b^2}$$

$$\Rightarrow (cb-a^2)^2 = (ab-c^2)(ac-b^2)$$

$$\Rightarrow a(a^3+b^3+c^3-3abc) = 0$$

$$\Rightarrow a = 0 \text{ or, } a^3+b^3+c^3 = 3abc.$$

This is the required relation between a, b and c.

EXAMPLE 6 *If each pair of three equations* $x^2+px+qr=0$, $x^2+qx+pr=0$ *and* $x^2+rx+pq=0$ *have a common root, find the sum and product of the common roots.*

SOLUTION We have :

$$x^2 + px + qr = 0 \quad ...(i)$$

$$x^2 + qx + pr = 0 \quad ...(ii)$$

$$x^2 + rx + pq = 0 \quad ...(iii)$$

Since each pair of three equations have a common root. So, let the roots of (i), (ii) and (iii) be α, β; β, γ and γ, α respectively.

$$\therefore \quad \alpha + \beta = -p, \alpha\beta = qr; \quad ...(iv)$$

$$\beta + \gamma = -q, \beta\gamma = pr; \quad ...(v)$$

and, $$\gamma + \alpha = -r, \gamma\alpha = pq; \quad ...(vi)$$

Adding the first sets of (iv), (v) and (vi) we get

$$(\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = -(p + q + r)$$

$$\Rightarrow \quad 2(\alpha + \beta + \gamma) = -(p + q + r)$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\left(\frac{p + q + r}{2}\right)$$

Multiplying the second sets of (iv), (v) and (vi), we get

$$(\alpha\beta)(\beta\gamma)(\gamma\alpha) = (qr)(pr)(pq)$$

$$\Rightarrow \quad \alpha\beta\gamma = pqr.$$

EXAMPLE 7 *If each pair of the three equations* $x^2 + p_1 x + q_1 = 0, x^2 + p_2 x + q_2 = 0$ *and* $x^2 + p_3 x + q_3 = 0$ *have a common root, then prove that*

$$p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1 p_2 + p_2 p_3 + p_3 p_1).$$

SOLUTION The three equations are :

$$x^2 + p_1 x + q_1 = 0 \quad ...(i)$$

$$x^2 + p_2 x + q_2 = 0 \quad ...(ii)$$

and, $$x^2 + p_3 x + q_3 = 0 \quad ...(iii)$$

Since each pair has a common root, the roots of (i), (ii) and (iii) can be taken as α, β ; β, γ, and γ, α respectively.

$$\therefore \quad \alpha + \beta = -p_1, \alpha\beta = q_1 \quad ...(iv)$$

$$\beta + \gamma = -p_2, \beta\gamma = q_2 \quad ...(v)$$

and, $$\gamma + \alpha = -p_3, \gamma\alpha = q_3 \quad ...(vi)$$

Now, $(\alpha + \beta) - (\beta + \gamma) = (-p_1 + p_2)$

$\Rightarrow \alpha - \gamma = (p_2 - p_1)$

$\Rightarrow (\alpha - \gamma)^2 = (p_2 - p_1)^2$

$\Rightarrow (\alpha + \gamma)^2 - 4\alpha\gamma = (p_2 - p_1)^2$

$\Rightarrow (-p_3)^2 - 4q_3 = (p_2 - p_1)^2$

$\Rightarrow (p_2 - p_1)^2 = p_3^2 - 4q_3 \quad ...(vii)$

Similarly, $(p_3 - p_1)^2 = p_2^2 - 4q_2 \quad ...(viii)$

and, $$(p_2 - p_3)^2 = p_1^2 - 4q_1 \quad ...(ix)$$

Adding (vii), (viii) and (ix), we get

$$(p_2 - p_1)^2 + (p_3 - p_1)^2 + (p_2 - p_3)^2$$

$$= (p_3^2 - 4q_3) + (p_2^2 - 4q_2) + (p_1^2 - 4q_1)$$

$$\Rightarrow p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1 p_2 + p_2 p_3 + p_3 p_1).$$

EXAMPLE 8 *If the equations* $ax^3 + 3bx^2 + 3cx + d = 0$ *and* $ax^2 + 2bx + c = 0$ *have common root, prove that*

$$(bc - ad)^2 = 4(ac - b^2)(bd - c^2).$$

SOLUTION Let α be a common root of the two equations. Then,

$$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \quad ...(i)$$

and, $$a\alpha^2 + 2b\alpha + c = 0 \quad ...(ii)$$

Multiplying (ii) by α and subtracting it from (i), we get

$$b\alpha^2 + 2c\alpha + d = 0 \quad ...(iii)$$

From (ii) and (iii), we obtain

$$\frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)} \text{ [By cross multiplication]}$$

$$\Rightarrow \frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} \text{ and } \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)}$$

$$\Rightarrow \alpha = \frac{2(bd - c^2)}{bc - ad} \text{ and } \alpha = \frac{bc - ad}{2(ac - b^2)}$$

$$\Rightarrow \frac{2(bd - c^2)}{bc - ad} = \frac{bc - ad}{2(ac - b^2)}$$

$$\Rightarrow 4(bd - c^2)(ac - b^2) = (bc - ad)^2.$$

EXAMPLE 9 *If* $a_1 x^3 + b_1 x^2 + c_1 x + d_1 = 0$ *and* $a_2 x^3 + b_2 x^2 + c_2 x + d_2 = 0$ *have a pair of repeated common roots, then prove that*

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2 b_1 - a_1 b_2 & c_1 a_2 - c_2 a_1 & d_1 a_2 - d_2 a_1 \end{vmatrix} = 0.$$

SOLUTION Let α be a repeated root common to both the given equations. Then, α is a common root between their derived equations i.e. α is a common root between

$$3a_1 x^2 + 2b_1 x + c_1 = 0$$

and, $$3a_2 x^2 + 2b_2 x + c_2 = 0 \quad \left[\text{Diff. both sides of the given equations}\right]$$

$$\Rightarrow \quad 3a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \quad ...(i)$$

and, $$3a_2\alpha^2 + 2b_2\alpha + c_2 = 0 \quad ...(ii)$$

Since α is a common root between the given equations. Therefore,

$a_1\alpha^3+b_1\alpha^2+c_1\alpha+d_1=0$ and $a_2\alpha^3+b_2\alpha^2+c_2\alpha+d_2=0$

Multiplying first by a_2, second by a_1 and subtracting, we get

$$(a_2b_1-a_1b_2)\alpha^2+(a_2c_1-a_1c_2)\alpha+(a_2d_1-a_1d_2)=0 \quad ...(iii)$$

Eliminating α^2, α and α^0 from (i), (ii) and (iii), we get

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ (a_2b_1-a_1b_2) & (a_2c_1-a_1c_2) & (a_2d_1-a_1d_2) \end{vmatrix} = 0.$$

EXAMPLE 10 *If the equations* $x^2+abx+c=0$ *and* $x^2+acx+b=0$ *have a common root, then prove that their other roots are the roots of the equation*

$$x^2-a(b+c)x+a^2bc=0$$

SOLUTION Let α be a common root of the given equations and let β and γ be their other roots. Then,

$$\alpha+\beta=-ab,\ \alpha\beta=c \quad ...(i)$$

and,
$$\alpha+\gamma=-ac,\ \alpha\gamma=b \quad ...(ii)$$

Since α is a common root of the given equations. Therefore,

$\alpha^2+ab\alpha+c=0$ and $\alpha^2+ac\alpha+b=0$

$\Rightarrow (ab-ac)\alpha+c-b=0$

$\Rightarrow a(b-c)\alpha-(b-c)=0$

$\Rightarrow \alpha=\frac{1}{a}$.

Putting the value of α in (i) and (ii), we get

$$\beta=ac \text{ and } \gamma=ab$$

$\Rightarrow \beta+\gamma=ac+ab=a(b+c)$ and $\beta\gamma=a^2bc$

So, the equation having β and γ as its roots is

$$x^2-x(\beta+\gamma)+\beta\gamma=0$$

or,
$$x^2-a(b+c)x+a^2bc=0.$$

EXAMPLE 11 *If the three equations* $x^2+ax+12=0$, $x^2+bx+15=0$ *and* $x^2+(a+b)x+36=0$ *have a common positive root, then find* a *and* b *and the roots.*

SOLUTION The given equations are:

$$x^2+ax+12=0 \quad ...(i)$$

$$x^2+bx+15=0 \quad ...(ii)$$

$$x^2+(a+b)x+36=0 \quad ....(iii)$$

Let $\alpha, \beta, \gamma; \alpha, \delta$ be the roots of (i), (ii) and (iii) respectively. Then

$$\left.\begin{aligned} \alpha+\beta=-a,\ \alpha\beta &= 12 \\ \alpha+\gamma=-b,\ \alpha\gamma &= 15 \\ \alpha+\delta=-(a+b),\ \alpha\delta &= 36 \end{aligned}\right| \quad ...(iv)$$

Now, $\alpha+\beta=-a, \alpha+\gamma=-b$ and $\alpha+\delta=-(a+b)$

$\Rightarrow \alpha+\beta+\alpha+\gamma=-(a+b)$ and $\alpha+\delta=-(a+b)$

$\Rightarrow 2\alpha+\beta+\gamma=\alpha+\delta$

$\Rightarrow \alpha+\beta+\gamma=\delta \quad ...(v)$

and, $\alpha\beta=12, \alpha\gamma=15, \alpha\delta=36$

$\Rightarrow \alpha\beta+\alpha\gamma+\alpha\delta=12+15+36$

$\Rightarrow \alpha(\beta+\gamma+\delta)=63$

$\Rightarrow \alpha(\delta-\alpha+\delta)=63$ [Using (v)]

$\Rightarrow 2\alpha\delta-\alpha^2=63$

$\Rightarrow 2\times36-\alpha^2=63$ [$\because \alpha\delta=36$]

$\Rightarrow \alpha^2=9$

$\Rightarrow \alpha=3$ [$\because \alpha>0$]

Putting the value of α in (iv), we get

$$\beta=4,\ \gamma=5 \text{ and } \delta=12$$

Substituting the values of $\alpha, \beta, \gamma, \delta$ in (iv), we get $a=-7$ and $b=-8$.

EXAMPLE 12 *Find all values of the parameter* a *for which the quadratic equations* $(1-2a)x^2-6ax-1=0$ *and* $ax^2-x+1=0$ *have at least one root in common.*

SOLUTION For the given equations to be quadratic equations, we must have $1-2a\neq0$ and $a\neq0$.

i.e. $a\neq0, \frac{1}{2}$.

If α is a common root of the given equations, we must have

$$(1-2a)\alpha^2-6a\alpha-1=0 \quad ...(i)$$

and
$$a\alpha^2-\alpha+1=0 \quad ...(ii)$$

Adding these two, we get

$$(1-a)\alpha^2-(1+6a)\alpha=0$$

$\Rightarrow \alpha=0$ or, $\alpha=\frac{1+6a}{1-a}$

But $\alpha\neq0$ as it is not a root of any of the given equations.

Therefore, $\alpha=\frac{1+6a}{1-a}$.

Putting $\alpha=\frac{1+6a}{1-a}$ in (ii), we get

$$a\left(\frac{1+6a}{1-a}\right)^2-\left(\frac{1+6a}{1-a}\right)+1=0$$

$\Rightarrow a(1+6a)^2-(1+6a)(1-a)+(1-a)^2=0$

$\Rightarrow a(36a^2+19a-6)=0$

$\Rightarrow 36a^2+19a-6=0$ [$\because a\neq0$]

$\Rightarrow (9a-2)(4a+3)=0$

$\Rightarrow a=\frac{2}{9}, \frac{-3}{4}$

When $a = \frac{-3}{4}$, the equation $ax^2 - x + 1 = 0$ becomes

$$\frac{3x^2}{4} - x + 1 = 0 \text{ or, } 3x^2 - 4x + 4 = 0$$

Clearly, it has no real roots. Therefore, $a = 2/9$.

EXAMPLE 13 *If each pair of equations* $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ *and* $a_3x^2 + b_3x + c_3 = 0$ *have a common root, then prove that*

(i) $\left(\frac{a_1b_2 - a_2b_1}{a_1c_2 - a_2c_1}\right)^2 = \frac{a_1a_2c_3}{c_1c_2a_3}$.

(ii) $\frac{c_1a_2 + c_2a_1}{c_1a_2 - c_2a_1} + \frac{a_1a_2b_3}{a_3(a_1b_2 - a_2b_1)} = 0$.

SOLUTION The given equations are:

$$a_1x^2 + b_1x + c_1 = 0 \quad ...(i)$$

$$a_2x^2 + b_2x + c_2 = 0 \quad ...(ii)$$

$$a_3x^2 + b_3x + c_3 = 0 \quad ...(iii)$$

Since each pair has a common root, the roots of (i), (ii) and (iii) can be taken as α, β; β, γ and γ, α respectively.

$$\therefore \quad \alpha + \beta = \frac{-b_1}{a_1}, \alpha\beta = \frac{c_1}{a_1}$$

$$\beta + \gamma = \frac{-b_2}{a_2}, \beta\gamma = \frac{c_2}{a_2}$$

$$\text{and,} \quad \gamma + \alpha = \frac{-b_3}{a_3}, \gamma\alpha = \frac{c_3}{a_3}$$

(i) Now,

$$(\alpha + \beta) - (\beta + \gamma) = \left(\frac{-b_1}{a_1} + \frac{b_2}{a_2}\right)$$

$$\Rightarrow \quad \alpha - \gamma = \left(\frac{a_1b_2 - a_2b_1}{a_1a_2}\right) \quad ...(iv)$$

and,

$$\alpha\beta = \frac{c_1}{a_1}, \ \beta\gamma = \frac{c_2}{a_2}$$

$$\Rightarrow \alpha\beta - \beta\gamma = \frac{c_1}{a_1} - \frac{c_2}{a_2}$$

$$\Rightarrow (\alpha - \gamma)\beta = \frac{c_1a_2 - c_2a_1}{a_1a_2} \quad ...(v)$$

From (iv) and (v), we get

$$\left(\frac{a_1b_2 - a_2b_1}{a_1a_2}\right)\beta = \frac{c_1a_2 - c_2a_1}{a_1a_2}$$

$$\Rightarrow \quad \beta = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad ...(vi)$$

Again, $\alpha\beta = \frac{c_1}{a_1}$, $\beta\gamma = \frac{c_2}{a_2}$ and $\gamma\alpha = \frac{c_3}{a_3}$

$$\Rightarrow (\alpha\beta)(\beta\gamma)(\gamma\alpha) = \frac{c_1c_2c_3}{a_1a_2a_3}$$

$$\Rightarrow (\alpha\beta\gamma)^2 = \frac{c_1c_2c_3}{a_1a_2a_3}$$

$$\Rightarrow \beta^2(\alpha\gamma)^2 = \frac{c_1c_2c_3}{a_1a_2a_3}$$

$$\Rightarrow \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2 \frac{c_3^2}{a_3^2} = \frac{c_1c_2c_3}{a_1a_2a_3} \quad \left[\text{Using (vi) and } \alpha\gamma = \frac{c_3}{a_3}\right]$$

$$\Rightarrow \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2 = \frac{c_1c_2a_3}{a_1a_2c_3}$$

(ii) We have,

$$\alpha\beta = \frac{c_1}{a_1} \text{ and } \beta\gamma = \frac{c_2}{a_2}$$

$$\therefore \ \alpha\beta + \beta\gamma = \frac{c_1}{a_1} + \frac{c_2}{a_2}$$

$$\Rightarrow \beta(\alpha + \gamma) = \frac{c_1a_2 + c_2a_1}{a_1a_2}$$

$$\Rightarrow \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \times \frac{-b_3}{a_3} = \frac{c_1a_2 + c_2a_1}{a_1a_2} \quad [\text{Using (v) and } \gamma + \alpha = \frac{-b_3}{a_3}]$$

$$\Rightarrow \frac{c_1a_2 + c_2a_1}{c_1a_2 - c_2a_1} + \frac{a_1a_2b_3}{a_3(a_1b_2 - a_2b_1)} = 0$$

EXAMPLE 14 *If* $ax^2 + 2bx + c = 0$ *has a common root with* $2x^3 + x^2 - 5x + 2 = 0$ *and* $b^2 = ac$, *find the possible value(s) of* $a : b : c$.

SOLUTION We have, $b^2 = ac$ i.e. $b^2 - ac = 0$. Therefore, discriminant of $ax^2 + 2bx + c = 0$ is zero. So, it has equal roots. Let α, α be the roots of this equation. Then,

$$\alpha + \alpha = \frac{-2b}{a} \Rightarrow \alpha = \frac{-b}{a}.$$

Now $2x^3 + x^2 - 5x + 2 = 0$

$$\Rightarrow (x - 1)(x + 2)(2x - 1) = 0$$

$$\Rightarrow x = 1, -2, 1/2.$$

CASE I *When* $x = 1$ *is a common root;*

In this case, we have

$$\frac{-b}{a} = 1 \Rightarrow b = -a.$$

But, $b^2 = ac$. Therefore,

$$c = \frac{b^2}{a} = \frac{a^2}{a} = a.$$

$$\therefore a : b : c = a : -a : a = 1 : -1 : 1$$

CASE II *When $x = -2$ is a common root:*

In this case, we have

$$\frac{-b}{a} = -2 \Rightarrow b = 2a$$

But, $b^2 = ac$. Therefore,

$$(2a)^2 = ac \Rightarrow 4a^2 = ac \Rightarrow 4a = c$$

$$\therefore a : b : c = a : 2a : 4a = 1 : 2 : 4$$

CASE III *When $x = \frac{1}{2}$ is a common root:*

In this case, we have

$$\frac{-b}{a} = \frac{1}{2} \Rightarrow b = \frac{-a}{2}.$$

But, $b^2 = ac$. Therefore,

$$\left(\frac{-a}{2}\right)^2 = ac \Rightarrow \frac{a}{4} = c$$

$$\therefore a : b : c = a : \frac{-a}{2} : \frac{a}{4} = 1 : -\frac{1}{2} : \frac{1}{4} = 4 : -2 : 1$$

EXERCISE 20.3

1. If the equations $x^2 + a_1 x + b_1 = 0$ and $x^2 + a_2 x + b_2 = 0$ have a common root, show that it is equal to

$$\frac{a_1 b_2 - a_2 b_1}{b_1 - b_2} \text{ or } \frac{b_1 - b_2}{a_2 - a_1}.$$

2. If the equations $x^2 - 3x - k = 0$ and $4x^2 - 11x + 2k = 0$ have a common root, then find the value of k. Also, find the common root.

3. If $ax^2 + 2bx + c = 0$ and $px^2 + 2qx + r = 0$ have a common root and $\frac{a}{p}, \frac{b}{q}, \frac{c}{r}$ are in *A.P.*, then prove that p, q, r are in *G.P.*

4. Find the value of a so that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

5. If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have one root in common, prove that $a + b + c = 0$ or $a - b + c = 0$.

6. Find a common root of the equations

$$(b - c)x^2 + (c - a)x + (a - b) = 0$$

$$and (c - a)x^2 + (a - b)x + (b - c) = 0$$

7. If the equations $ax^2 + 2bx + c = 0$, $a'x^2 - 2b'x + c' = 0$ have a common root, prove that the equation

$$(b^2 - ac)x^2 + (2bb' - ac' - a'c)x + (b'^2 - a'c') = 0$$

has equal roots.

8. If the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have a common root, prove that $a + b + 1 = 0$.

9. If the equations $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root, then prove that their other roots are the roots of the equation $x^2 + ax + bc = 0$.

10. If the equations $a_1x^2 + 2b_1x + c_1 = 0$ and $a_2x^2 + 2b_2x + c_2 = 0$ have a common root, then prove that the euqation

$$(b_1^2 - a_1c_1)x^2 + (2b_1b_2 - a_1c_2 - a_2c_1)x + (b_2^2 - a_2c_2) = 0$$

has equal roots.

11. Find the value of a so that the equations $(2a - 5)x^2 - 4x - 15 = 0$ and $(3a - 8)x^2 - 5x - 21 = 0$ have a common root.

12. If the equations $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root, then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

13. If each pair of equations : $x^2 + a_1x + b_1 = 0$, $x^2 + a_2x + b_2 = 0$ and $x^2 + a_3x + b_3 = 0$ has exactly one root in common, then show that

$$(a_1 + a_2 + a_3)^2 = 4(a_1a_2 + a_2a_3 + a_3a_1 - b_1 - b_2 - b_3).$$

14. Prove that $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ cannot have a common root unless $a : b : c = 1 : 2 : 3$.

15. Find a, b so that the equations $x^2 + ax - 1 = 0$, $2x^2 + x + b = 0$ and $4x^2 - 8x + 3 = 0$ may have a common root for each pair of equations.

16. If the equations $a_1x^2 + 2b_1x + c_1 = 0$ and $a_2x^2 + 2b_2x + c_2 = 0$ have one and only one root in common, then prove that $b_1^2 - a_1c_1$ and $b_2^2 - a_2c_2$ will both be perfect squares.

17. Find the condition for the three equations

$$a_1x^2 + b_1x + c_1 = 0,$$

and $a_2x^2 + b_2x + c_2 = 0$

$a_3x^2 + b_3x + c_3 = 0$ to have a common root.

18. If each pair of equations $x^2 = b_1x + c_1$, $x^2 = b_2x + c_2$ and $x^2 + b_3x = c_3$ have a common root, prove that

$$(b_1 + b_2 + b_3)^2 = 4(c_1 + c_2 + c_3 + b_1 b_2).$$

19. If the equations $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and roots of the second equation are equal, then prove that $ap = 2(b + q)$.

20. Find the value of m for which $x^2 - x - m = 0$ and $x^2 + 2mx - 12 = 0$ have a common root.

ANSWERS

2. $k = 0, \frac{-17}{36}$, common root $= 0, \frac{17}{6}$

11. 4

15. $a = 3/2, b = -6$

17. $(c_1 a_2 - c_2 a_1)^2 = (ab_2 - a_2 b_1)(b_1 c_2 - b_2 c_1)$

and, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

—❑❑❑—

20.10 QUADRATIC EXPRESSION AND ITS GRAPH

Let a, b, c be real numbers and $a \neq 0$. Then, $f(x) = ax^2 + bx + c$ is known as a quadratic expression or a quadratic polynomial in x. In this section, we shall discuss the graph of the quadratic polynomial i.e. the curve whose equation is $y = ax^2 + bx + c$, $a \neq 0$.

First we shall show that the graph of a quadratic expression is a parabola.

Let $y = ax^2 + bx + c$, where $a \neq 0$. Then,

$$y = ax^2 + bx + c$$

$$\Rightarrow \quad y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$\Rightarrow \quad y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$\Rightarrow \quad y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$\Rightarrow \left(y + \frac{b^2 - 4ac}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 \qquad ...(i)$$

where $D = b^2 - 4ac$ is the discriminant of the quadratic.

Shifting the origin at $(-b/2a, -D/4a)$, we have

$$x = X + \left(\frac{-b}{2a}\right) \text{and } y = Y + \left(\frac{-D}{4a}\right)$$

Substituting these values in (i), we get

$$Y = aX^2$$

Clearly, it is the equation of a parabola having its vertex at $(-b/2a, -D/4a)$.

Thus, the shape of the curve $y = f(x)$, where $f(x)$ is a quadratic expression, is a parabola having vertex at $\left(-\frac{b}{2a}, \frac{-D}{4a}\right)$.

The parabola opens upwards or downwards according as $a > 0$ or $a < 0$ as shown in Fig. 20.6 and 20.7.

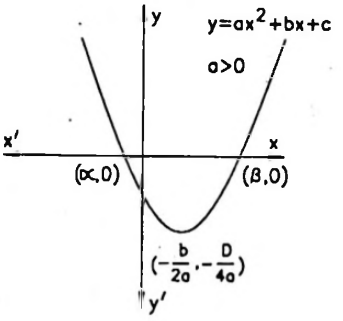


Fig. 20.6

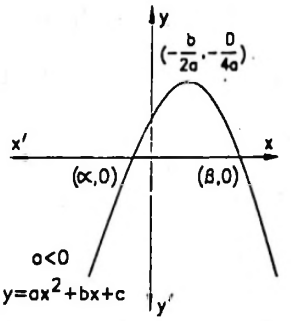


Fig. 20.7

We observe the following :

When $a > 0$:

The curve $y = ax^2 + bx + c$ is a parabola opening upwards such that

$$y_{min} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a} \text{ and } y_{max} \to \infty$$

When $a < 0$:

The curve $y = ax^2 + bx + c$ is a parabola opening downwards such that

$$y_{max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a} \text{ and } y_{min} \to -\infty$$

Now, let α be a root of the quadratic equation $ax^2 + bx + c = 0$. Then,

$$a\alpha^2 + b\alpha + c = 0$$

$\Rightarrow$ Point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$

It follows from this that every real root of $ax^2+bx+c=0$ is represented by the point of intersecton of the parabola with x-axis and vice-versa.

Thus, we may draw the following conclusions :

(I) *The parabola will intersect the x-axis in two distinct points iff* $D>0$.

In this case, the parabola cuts x-axis at $(\alpha, 0)$ and $(\beta, 0)$, where

$$\alpha=\frac{-b-\sqrt{D}}{2a} \text{ and } \beta=\frac{-b+\sqrt{D}}{2a}.$$

For $a>0$, *we have :*

$$y_{\min}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a}, \text{ and } y_{\max}\to\infty$$

Also, $$y=f(x)\begin{cases}<0 & \text{for } \alpha<x<\beta\\ 0 & \text{for } x=\alpha,\beta\\ >0 & \text{for } x<\alpha \text{ or } x>\beta\end{cases}$$

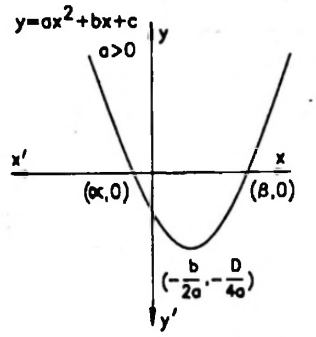


Fig. 20.8

For $a<0$, *we have :*

$$y_{\max}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a},\ y_{\min}\to-\infty$$

and, $$y=f(x)\begin{cases}<0 & \text{for } x<\alpha \text{ or } x>\beta\\ 0 & \text{for } x=\alpha,\beta\\ >0 & \text{for } \alpha<x<\beta\end{cases}$$

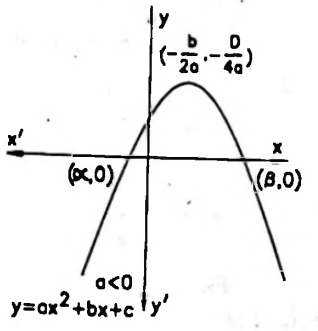


Fig. 20.9

(II) *The parabola will just touch the x-axis at one point iff* $D=0$

In this case, the parabola touches x-axis at $(\alpha, 0)$, when $\alpha=\beta=\frac{-b}{2a}$.

For $a>0$, *we have :*

$$y_{\min}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a},\ y_{\max}\to\infty$$

and, $$y=f(x)\begin{cases}>0 & \text{for } x\neq\alpha\\ 0 & \text{for } x=\alpha\end{cases}$$

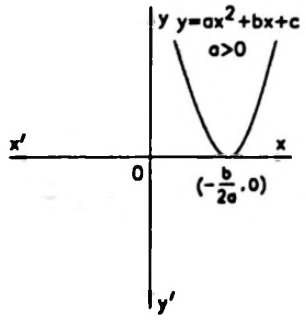


Fig. 20.10

For $a<0$, *we have :*

$$y_{\max}=0 \text{ at } x=\frac{-b}{2a},\ y_{\min}\to-\infty$$

and, $$y=f(x)\begin{cases}<0 & \text{for } x\neq\alpha\\ 0 & \text{for } x=\alpha\end{cases}$$

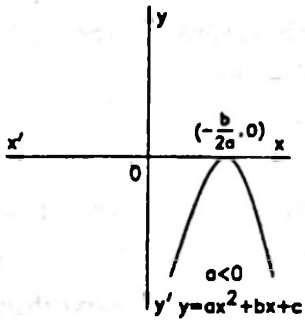


Fig. 20.11

(III) *The parabola will not interssect x-axis iff* $D<0$

In this case, the parabola remains either completely above x-axis or completely below x-axis according as $a>0$ or $a<0$.

For $a>0$, *we have :*

$$y_{\min}=\frac{-D}{4a} \text{ at } x=\frac{-b}{2a},\ y_{\max}\to\infty$$

and, $$y=f(x)>0 \quad \text{for all } x$$

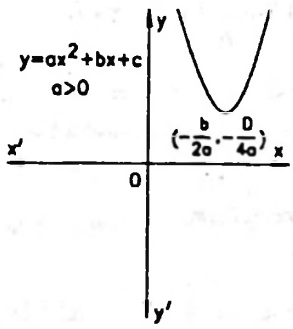


Fig. 20.12

For $a<0$, we have :

$$y_{max} = -\frac{D}{4a} \text{ at } x = \frac{-b}{2a},\ y_{min} \to -\infty$$

and, $y = f(x) < 0$ for all x

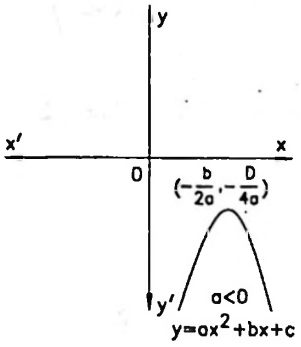


Fig. 20.13

20.11 SIGN OF A QUADRATIC EXPRESSION

Let $f(x) = ax^2 + bx + c$ be a quadratic expression, where $a, b, c \in R$ and $a \neq 0$. In this section, we shall determine the sign of $f(x) = ax^2 + bx + c$ for real values of x. As the discriminant of $f(x) = ax^2 + bx + c$ can be positive, zero or negative. So, we shall discuss the following three cases.

CASE I *When* $D = b^2 - 4ac < 0$

If $D < 0$, then it is evident from Figs 20.12 and 20.13 that

$f(x) > 0$ iff $a > 0$ and $f(x) < 0$ iff $a < 0$.

CASE II *When* $D = b^2 - 4ac = 0$

From Figs. 20.10 and 20.11, we observe that:

When $D = 0$, we have

$f(x) \geq 0$ iff $a > 0$ and $f(x) \leq 0$ iff $a < 0$.

CASE III *When* $D = b^2 - 4ac > 0$

From Fig. 20.8 and 20.9, we observe the following:

If $D = b^2 - 4ac > 0$ and $a > 0$, then

$$f(x) \begin{cases} > 0 \text{ for } x < \alpha \text{ or } x > \beta \\ < 0 \text{ for } \alpha < x < \beta \\ = 0 \text{ for } x = \alpha, \beta \end{cases}$$

If $D = b^2 - 4ac > 0$ and $a < 0$, then

$$f(x) \begin{cases} < 0 \text{ for } x < \alpha \text{ or } x > \beta \\ > 0 \text{ for } \alpha < x < \beta \\ 0 \text{ for } x = \alpha, \beta \end{cases}$$

If follows from the above discussion that, for real values of x, the sign of the quadratic expression $f(x) = ax^2 + bx + c$ *is same as that of 'a', except when the roots of the equation* $ax^2 + bx + c = 0$ *are real and distinct and x lies between them.*

It also follows that

$ax^2 + bx + c > 0$ *for all* $x \in R$ *iff* $a > 0$ and $D < 0$,

and,

$ax^2 + bx + c < 0$ *for all* $x \in R$ *iff* $a < 0$ and $D < 0$.

20.12 SOLUTION OF INEQUATIONS

In this section, we shall discuss solutions of quadratic inequations and rational algebraic inequalities.

SOLUTION OF QUADRATIC INEQUATIONS

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$.

Then, $f(x) \geq 0, f(x) < 0, f(x) \leq 0, f(x) > 0$ are quadratic inequations. The set of all real values of x, which satisfy an equation is called its solution set. To find the solution set of a quadratic inequation we may use the following algorithm.

ALGORITHM

STEP I *Obtain the discriminant* $D = b^2 - 4ac$ *of the quadratic inequation.*

STEP II *Determine the sign of D.*

STEP III *If* $D < 0$, *then determine the sign of 'a'.*

If $a < 0$, *then* $f(x) < 0$ *for all* $x \in R$

If $a > 0$, *then* $f(x) > 0$ *for all* $x \in R$

If $D = 0$, *then determine the sign of 'a'.*

If $a < 0$, *then* $f(x) < 0$ *for all* $x \in R$.

If $a > 0$, *then* $f(x) \geq 0$ *for all* $x \in R$

If $D > 0$, *then go to step IV.*

STEP IV *Make the coefficient of* x^2 *positive, if it is not so, by multiplying both sides of the inequation by* -1.

STEP V *Factorise the expression and represent the left hand side of the inequality in the form* $(x - \alpha)(x - \beta)$, *where* $\alpha < \beta$.

STEP VI *If* $(x - \alpha)(x - \beta) > 0$, *then x lies outside the interval* $[\alpha, \beta]$ *i.e.* $x \in (-\infty, \alpha) \cup (\beta, \infty)$

If $(x - \alpha)(x - \beta) \geq 0$, *then x lies on and outside the interval* $[\alpha, \beta]$ *i.e.* $x \in (-\infty, \alpha] \cup [\beta, \infty)$.

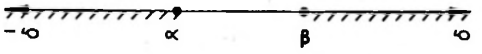


Fig. 20.14

If $(x - \alpha)(x - \beta) < 0$, *then x lies inside the interval* (α, β)

If $(x - \alpha)(x - \beta) \leq 0$, *then x lies inside the interval* $[\alpha, \beta]$ *i.e.* $x \in [\alpha, \beta]$.

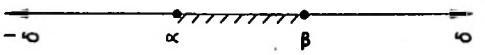


Fig. 20.15

ILLUSTRATION 1 *Solve the following in equations :*

(i) $2x^2 - x - 1 < 0$ (ii) $-x^2 + 3x + 4 < 0$

SOLUTION (i) We have,

$2x^2 - x - 1 < 0$

$\Rightarrow (2x + 1)(x - 1) < 0$

$\Rightarrow -1/2 < x < 1 \Rightarrow x \in (-1/2, 1)$

(ii) We have,

$-x^2 + 3x + 4 < 0$

$\Rightarrow x^2 - 3x - 4 > 0$

$\Rightarrow (x-4)(x+1) > 0$

$\Rightarrow x < -1$ or $x > 4$

$\Rightarrow x \in (-\infty, -1) \cup (4, \infty)$

ILLUSTRATION 2 *Determine the set of values of 'a' for which* $x^2 + ax + 4 > 0$ *for all* $x \in R$.

SOLUTION Here, coefficient of x^2 is positive. So, the given inequality holds for all $x \in R$, iff

$\text{Disc} < 0$

$\Rightarrow a^2 - 16 < 0$

$\Rightarrow (a-4)(a+4) < 0$

$\Rightarrow -4 < a < 4 \Rightarrow a \in (-4, 4)$

SOLUTION OF RATIONAL ALGEBRAIC INEQUATIONS

If $P(x)$ and $Q(x)$ are polynomials in x, then the inequations $\frac{P(x)}{Q(x)} > 0$, $\frac{P(x)}{Q(x)} < 0$, $\frac{P(x)}{Q(x)} \geq 0$, and $\frac{P(x)}{Q(x)} \leq 0$ are known as rational algebraic inequations. To solve these inequations we use the sign method as explained in the following algorithm.

ALGORITHM

STEP I *Obtain* $P(x)$ *and* $Q(x)$.

STEP II *Factorise* $P(x)$ *and* $Q(x)$ *into linear factors.*

STEP III *Make the coefficient of* x *positive in all factors.*

STEP IV *Obtain critical points by equating all factors to zero.*

STEP V *Plot the critical points on the number line. If there are* n *critical points, they divide the number line into* $(n+1)$ *regions.*

STEP VI *In the right most region the expression* $\frac{P(x)}{Q(x)}$ *bears positive sign and in other regions the expression bears altenate positive and negative signs.*

ILLUSTRATION 1 *Solve :* $\frac{x^2 + 4x + 3}{x^3 - 6x^2 + 11x - 6} > 0.$

SOLUTION We have,

$$\frac{x^2 + 4x + 3}{x^3 - 6x^2 + 11x - 6} = \frac{(x+1)(x+3)}{(x-1)(x-2)(x-3)}$$

Critical points are : $x = -3, -1, 1, 2, 3$.

$\therefore \frac{x^2 + 4x + 3}{x^3 - 6x^2 + 11x - 6} > 0$

$\Rightarrow \frac{(x+1)(x+3)}{(x-1)(x-2)(x-3)} > 0$

$\Rightarrow x \in (-3, -1) \cup (1, 2) \cup (3, \infty)$

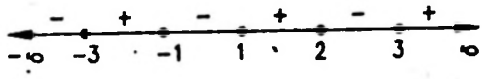


Fig. 20.16

ILLUSTRATION 2 *Solve :* $\frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3$

SOLUTION $\frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3$

$\Rightarrow \frac{8x^2 + 16x - 51}{(2x-3)(x+4)} - 3 > 0$

$\Rightarrow \frac{8x^2 + 16x - 51 - 3(2x-3)(x+4)}{(2x-3)(x+4)} > 0$

$\Rightarrow \frac{2x^2 + x - 15}{(2x-3)(x+4)} > 0$

$\Rightarrow \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$

Critical points are : $x = -4, -3, 3/2, 5/2$

$\therefore \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$

$\Rightarrow x \in (-\infty, -4) \cup (-3, 3/2) \cup (5/2, \infty)$.

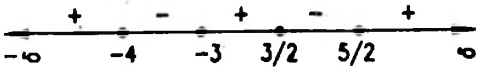


Fig. 20.17

20.13 MAXIMUM AND MINIMUM VALUES OF RATIONAL EXPRESSIONS

In this section, we shall find the values attained by a rational expression of the form $\frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ for real values of x.

The following algorithm will explain the procedure.

ALGORITHM

STEP I *Equate the given rational expression to* y.

STEP II *Obtain a quadratic equation in* x *by simplifying the expression in step I.*

STEP III *Obtain the discriminant of the quadratic in step II.*

STEP IV *Put* $\text{Disc} \geq 0$ *and solve the inequation for* y. *The values of* y *so obtained determine the set of values attained by the given rational expression.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If x is real, prove that the values of $\frac{x^2-3x+4}{x^2+3x+4}$ lies between $1/7$ and 7.*

SOLUTION Let $y=\frac{x^2-3x+4}{x^2+3x+4}$. Then,

$$x^2(y-1)+3x(y+1)+4(y-1)=0$$

This equation gives the values of x for given values of y. But y is the value when x is real. So, the roots of this equation are real.

$\therefore$ Disc ≥ 0

$\Rightarrow 9(y+1)^2-16(y-1)^2\geq 0$

$\Rightarrow -7y^2+50y-7\geq 0$

$\Rightarrow 7y^2-50y+7\leq 0$

$\Rightarrow (7y-1)(y-7)\leq 0$

$\Rightarrow 1/7\leq y\leq 7.$

Hence, the given expression lies between $1/7$ and 7.

EXAMPLE 2 *If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ cannot have value between 5 and 9.*

SOLUTION Let $y=\frac{x^2+34x-71}{x^2+2x-7}$. Then,

$$x^2(y-1)+2x(y-17)-(7y-71)=0$$

Since x is real. Therefore, the above equation has real roots. So,

Disc ≥ 0

$\Rightarrow 4(y-17)^2+4(y-1)(7y-71)\geq 0$

$\Rightarrow y^2-14y+45\geq 0$

$\Rightarrow (y-5)(y-9)\geq 0$

$\Rightarrow y\leq 5$ or, $y\geq 9$

Hence, the values of the given expression cannot lie between 5 and 9.

EXAMPLE 3 *Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.*

SOLUTION Let $\frac{ax^2+3x-4}{3x-4x^2+a}$. Then,

$$x^2(a+4y)+3(1-y)x-(4+ay)=0$$

Since x is real. So, roots of the above equation are real.

$\therefore$ Disc ≥ 0 for all $y\in R$

$\Rightarrow 9(1-y)^2+4(a+4y)(4+ay)\geq 0$ for all $y\in R$

$\Rightarrow (9+16a)y^2+(4a^2+46)y+(9+16a)\geq 0$ for all $y\in R$

$\Rightarrow 9+16a>0$ and Disc ≤ 0

Now,

Disc ≤ 0

$\Rightarrow (4a^2+46)^2-4(9+16a)(9+16a)\leq 0$

$\Rightarrow 4(2a^2+23)^2-4(9+16a)^2\leq 0$

$\Rightarrow (2a^2+23)^2-(9+16a)^2\leq 0$

$\Rightarrow (2a^2+16a+32)(2a^2-16a+14)\leq 0$

$\Rightarrow 4(a+4)^2(a^2-8a+7)\leq 0$

$\Rightarrow a^2-8a+7\leq 0$

$\Rightarrow (a-1)(a-7)\leq 0$

$\Rightarrow 1\leq a\leq 7.$

Thus, $9+16a>0$ and $1\leq a\leq 7$.

$\Rightarrow 1\leq a\leq 7.$

Thus, the expression will assume all real values for all real values of x, if $a\in[1,7]$

EXAMPLE 4 *If β is such that $\sin 2\beta\neq 0$, then show that the expression*

$$\frac{x^2+2x\cos 2\alpha+1}{x^2+2x\cos 2\beta+1},\ x\in R$$

always lies between $\frac{\cos^2\alpha}{\cos^2\beta}$ and $\frac{\sin^2\alpha}{\sin^2\beta}$.

SOLUTION Let $y=\frac{x^2+2x\cos 2\alpha+1}{x^2+2x\cos 2\beta+1}$. Then

$$(x^2+2x\cos 2\beta+1)y=x^2+2x\cos 2\alpha+1$$

$\Rightarrow x^2(y-1)+2x(y\cos 2\beta-\cos 2\alpha)+y-1=0$

As x is real, therefore, the discriminant must be greater than or equal to zero i.e.

$$4(y\cos 2\beta-\cos 2\alpha)^2-4(y-1)^2\geq 0$$

$\Rightarrow (y\cos 2\beta-\cos 2\alpha)-(y-1)^2\geq 0$

$\Rightarrow (y\cos 2\beta-\cos 2\alpha+y-1)(y\cos 2\beta-\cos 2\alpha-y+1)\geq 0$

$\Rightarrow \{y(1+\cos 2\beta)-(1+\cos 2\alpha)\}\{y(-1+\cos 2\beta)+(1-\cos 2\alpha)\}\geq 0$

$\Rightarrow (2y\cos^2\beta-2\cos^2\alpha)(-2y\sin^2\beta+2\sin^2\alpha)\geq 0$

$$\Rightarrow -4\cos^2\beta\sin^2\beta\left(y-\frac{\cos^2\alpha}{\cos^2\beta}\right)\left(y-\frac{\sin^2\alpha}{\sin^2\beta}\right)\geq 0$$

$$\Rightarrow \left(y-\frac{\cos^2\alpha}{\cos^2\beta}\right)\left(\frac{y-\sin^2\alpha}{\sin^2\beta}\right)\le 0$$

$\Rightarrow$ y lies between $\frac{\cos^2\alpha}{\cos^2\beta}$ and $\frac{\sin^2\alpha}{\sin^2\beta}$.

EXAMPLE 5 *Show that the expression* $\frac{\tan(x+\alpha)}{\tan(x-\alpha)}$, *where* $0<\alpha<\pi/4$ *cannot lie between the values* $\tan^2\left(\frac{\pi}{4}-\alpha\right)$ *and* $\tan^2\left(\frac{\pi}{4}+\alpha\right)$.

SOLUTION Let $y=\frac{\tan(x+\alpha)}{\tan(x-\alpha)}$, $\alpha\in(0,\pi/4)$.

$$\Rightarrow y=\frac{\tan x+\tan\alpha}{1-\tan x\tan\alpha}\times\frac{1+\tan x\tan\alpha}{\tan x-\tan\alpha}$$

$$\Rightarrow y(1-\tan x\tan\alpha)(\tan x-\tan\alpha)=(\tan x+\tan\alpha)(1+\tan x\tan\alpha)$$

$$\Rightarrow (y+1)\tan^2 x\tan\alpha+\tan x\,.\,\{-y(1+\tan^2\alpha)+(1+\tan^2\alpha)\}+(y+1)\tan\alpha=0$$

$$\Rightarrow (y+1)\tan\alpha\,.\,\tan^2 x+(-y+1)(1+\tan^2\alpha)\tan x+(y+1)\tan\alpha=0$$

As $\tan x$ is real for $x\in R$. Therefore,

$$(1-y)^2(1+\tan^2\alpha)^2-4(y+1)^2\tan^2\alpha\ge 0$$

[Using : Disc . ≥ 0]

$$\Rightarrow \{(1-y)(1+\tan^2\alpha)+2(1+y)\tan\alpha\}\times\{(1-y)(1+\tan^2\alpha)-2(y+1)\tan\alpha\}\ge 0$$

$$\Rightarrow \{(1+\tan\alpha)^2-y(1-\tan\alpha)^2\}\times\{(1-\tan\alpha)^2-y(1+\tan\alpha)^2\}\ge 0$$

$$\Rightarrow (1-\tan\alpha)^2(1+\tan\alpha)^2\left\{y-\left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)^2\right\}\times\left\{y-\left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2\right\}\ge 0$$

$$\Rightarrow \left\{y-\left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)^2\right\}\left\{y-\left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2\right\}\ge 0$$

$$\Rightarrow \left\{y-\tan^2\left(\frac{\pi}{4}+\alpha\right)\right\}\left\{y-\tan^2\left(\frac{\pi}{4}-\alpha\right)\right\}\ge 0$$

$\Rightarrow$ $y\le\tan^2(\pi/4-\alpha)$ or, $y\ge\tan^2(\pi/4+\alpha)$

$\Rightarrow$ y cannot lie between the values $\tan^2(\pi/4-\alpha)$ and $\tan^2(\pi/4+\alpha)$.

EXAMPLE 6 *Show that for all real values, the expression* $\frac{2a(x-1)\sin^2\alpha}{x^2-\sin^2\alpha}$ *cannot lie between* $2a\sin^2(\alpha/2)$ *and* $2a\cos^2(\alpha/2)$.

SOLUTION Let $y=\frac{2a(x-1)\sin^2\alpha}{x^2-\sin^2\alpha}$. Then,

$$y(x^2-\sin^2\alpha)=2a(x-1)\sin^2\alpha$$

$$\Rightarrow x^2y-2ax\sin^2\alpha+(2a-y)\sin^2\alpha=0$$

As x is real. Therefore,

Discriminant ≥ 0

$$\Rightarrow 4a^2\sin^4\alpha-4y(2a-y)\sin^2\alpha\ge 0$$

$$\Rightarrow y^2-2ay+a^2\sin^2\alpha\ge 0$$

$$\Rightarrow (y-a)^2-a^2+a^2\sin^2\alpha\ge 0$$

$$\Rightarrow (y-a)^2-a^2\cos^2\alpha\ge 0$$

$$\Rightarrow (y-a-a\cos\alpha)(y-a+a\cos\alpha)\ge 0$$

$$\Rightarrow \{y-a(1+\cos\alpha)\}\{y-a(1-\cos\alpha)\}\ge 0$$

$$\Rightarrow (y-2a\cos^2\alpha/2)(y-2a\sin^2\alpha/2)\ge 0$$

$\Rightarrow$ y does not lie between $2a\sin^2\alpha/2$ and $2a\cos^2\alpha/2$.

EXAMPLE 7 *Prove that if* x *is real, the minimum value of* $\frac{(a+x)(b+x)}{(c+x)}$ *for* $x>-c, a>c, b>c$ *is* $(\sqrt{a-c}+\sqrt{b-c})^2$.

SOLUTION Let $y=\frac{(a+x)(b+x)}{(c+x)}$. Then,

$$x^2+x(a+b)+ab=cy+xy$$

$$\Rightarrow x^2+x(a+b-y)+ab-cy=0$$

As x is real, therefore,

Discriminant ≥ 0

$$\Rightarrow (a+b-y)^2-4(ab-cy)\ge 0$$

$$\Rightarrow y^2-2y(a+b-2c)+(a-b)^2\ge 0 \qquad ...(i)$$

To solve this inequation, we require the roots of the equation $y^2-2y(a+b-2c)+(a-b)^2=0$.

The roots of this equation are given by

$$y=\frac{2(a+b-2c)\pm\sqrt{4(a+b-2c)^2-4(a-b)^2}}{2}$$

$$=(a+b-2c)\pm\sqrt{(a+b-2c)^2-(a-b)^2}$$

$$=(a+b-2c)\pm\sqrt{(a+b-2c-a+b)(a+b-2c+a-b)}$$

$$=(a+b-2c)\pm\sqrt{4(b-c)(a-c)}$$

$$=(a+b-2c)\pm 2\sqrt{(b-c)(a-c)}$$

$$=(a-c)+(b-c)\pm 2\sqrt{(a-c)(b-c)}$$

$$=(\sqrt{a-c}\pm\sqrt{b-c})^2$$

Clearly, y is real if $a>c$ and $b>c$.

From (i), we obtain

$$\left[y-\left\{\sqrt{a-c}+\sqrt{b-c}\right\}^2\right]\left[y-\left\{\sqrt{a-c}-\sqrt{b-c}\right\}^2\right]\geq 0$$

$\Rightarrow y \leq \left\{\sqrt{a-c}-\sqrt{b-c}\right\}^2$ or, $y \geq \left\{\sqrt{a-c}+\sqrt{b-c}\right\}^2$

Now,

$$y \leq \left\{\sqrt{a-c}-\sqrt{b-c}\right\}^2$$

$$\Rightarrow \frac{(a+x)(b+x)}{c+x} \leq \left\{\sqrt{a-c}-\sqrt{b-c}\right\}^2$$

$$\Rightarrow x^2+x(a+b)+ab \leq (c+x)\left\{\sqrt{a-c}-\sqrt{b-c}\right\}^2 \quad [\because c+x>0]$$

$$\Rightarrow x^2+2\left[c+\sqrt{a-c}\sqrt{b-c}\right]x + \left[c^2+(a-c)(b-c)+2c\sqrt{(a-c)(b-c)}\right] \leq 0$$

$$\Rightarrow \left\{x+c+\sqrt{(a-c)(b-c)}\right\}^2 \leq 0$$

$$\Rightarrow x+c+\sqrt{(a-c)(b-c)} = 0$$

$$\Rightarrow x = -c-\sqrt{(a-c)(b-c)}$$

$\Rightarrow x < -c$, which contradicts the hypothesis that $x > -c$.

Thus, $y \geq \left\{\sqrt{(a-c)}+\sqrt{(b-c)}\right\}^2$.

Hence, the minimum value of y is $\left\{\sqrt{a-c}+\sqrt{b-c}\right\}^2$.

EXAMPLE 8 *If the expression* $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ *assumes all real values for real values of* x, *then prove that* (a^2-b^2) *and* (c^2-d^2) *have the same sign.*

SOLUTION Let $y = \frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$. Then,

$$y = \frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$$

$$\Rightarrow (ad-bcy)x^2+(ac+bd)(y-1)x+(bc-ady) = 0$$

As x is real, therefore,

$$(ac+bd)^2(y-1)^2-4(ad-bcy)(bc-ady) \geq 0 \quad \text{for all } y \in R$$

$$\Rightarrow (ac-bd)^2y^2+2\left\{2(a^2d^2+b^2c^2)-(ac+bd)^2\right\}y + (ac-bd)^2 \geq 0 \text{ for all } y \in R$$

$$\Rightarrow 4\left\{2(a^2d^2+b^2c^2)-(ac+bd)^2\right\}^2-4(ac-bd)^4 \leq 0$$

$$\Rightarrow \left\{2(a^2d^2+b^2c^2)-(ac+bd)^2-(ac-bd)^2\right\} \left\{2(a^2d^2+b^2c^2)-(ac+bd)^2\right\}+(ac-bd)^2 \leq 0$$

$$\Rightarrow \left\{2a^2d^2+2b^2c^2-2a^2c^2-2b^2d^2\right\}\left\{2a^2d^2+2b^2c^2-4abcd\right\} \leq 0$$

$$\Rightarrow 4(a^2-b^2)(d^2-c^2)(ad-bc)^2 \leq 0$$

$$\Rightarrow (a^2-b^2)(d^2-c^2) \leq 0$$

$$\Rightarrow (a^2-b^2)(c^2-d^2) \geq 0$$

$\Rightarrow (a^2-b^2)$ and (c^2-d^2) are of the same sign.

EXAMPLE 9 *Let* $a_1, a_2, a_3, \ldots, a_n$ *and* x *be distinct real numbers such that*

$$\left(\sum_{r=1}^{n-1} a_r^2\right)x^2+2\left(\sum_{r=1}^{n-1} a_r a_{r+1}\right)x+\sum_{r=2}^{n} a_r^2 \leq 0.$$

Show that $a_1, a_2, a_3, \ldots, a_n$ are in *G.P.*

SOLUTION We have,

$$\left(\sum_{r=1}^{n-1} a_r^2\right)x^2+2\left(\sum_{r=1}^{n-1} a_r a_{r+1}\right)x+\sum_{r=2}^{n} a_r^2 \leq 0.$$

$$\Rightarrow (a_1^2+a_2^2+\ldots+a_n^2)x^2+2(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)x + (a_2^2+\ldots+a_n^2) \leq 0$$

$$\Rightarrow (a_1x+a_2)^2+(a_2x+a_3)^2+\ldots+(a_{n-1}x+a_n)^2 \leq 0$$

$$\Rightarrow a_1x+a_2 = 0,\ a_2x+a_3 = 0, \ldots, a_{n-1}x+a_n = 0$$

$$\Rightarrow \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}} = -x$$

$\Rightarrow a_1, a_2, a_3, \ldots, a_n$ are in *G.P.*

EXAMPLE 10 *Let* $f(x)$ *be a quadratic expression such that it is positive for all real* x. *If* $g(x) = f(x)+f'(x)+f''(x)$, *then prove that* $g(x) > 0$ *for all real* x.

SOLUTION Let $f(x) = ax^2+bx+c$.

It is given that

$f(x) > 0$ for all $x \in R$

$$\Rightarrow a > 0 \text{ and } b^2-4ac < 0 \quad \ldots(i)$$

Now,

$$g(x) = f(x)+f'(x)+f''(x)$$

$$\Rightarrow g(x) = ax^2+bx+c+2ax+b+2a$$

$$= ax^2+x(2a+b)+(b+c+2a)$$

Let D be the discriminant of $g(x)$. Then,

$$D = (2a+b)^2-4a(b+c+2a)$$

$$= -4a^2-4ac+b^2$$

$$= (b^2-4ac)-4 < 0 \quad \text{[Using (i)]}$$

Also, $a > 0$. Therefore,

$$g(x) > 0 \text{ for all } x \in R.$$

EXAMPLE 11 *Find all, real values of 'a' for which* $-3 < \frac{x^2+ax-2}{x^2-x+1} < 2$ *holds for all* $x \in R$.

SOLUTION Since x^2-x+1 has imaginary roots and coefficient of x^2 is positive. Therefore, $x^2-x+1>0$.

Now,

$$-3 < \frac{x^2+ax-2}{x^2-x+1} < 2 \text{ for all } x \in R \qquad \left[\because x^2-x+1>0 \text{ for all } x \in R\right]$$

$$\Rightarrow -3(x^2-x+1) < x^2+ax-2$$

and $x^2+ax-2 < 2(x^2-x+1)$ for all $x \in R$

Now,

$-3(x^2-x+1) < x^2+ax-2$ for all $x \in R$

$\Rightarrow 4x^2+x(a-3)+1 > 0$ for all $x \in R$

$\Rightarrow (a-3)^2-16 < 0$

$\Rightarrow (a-3-4)(a-3+4) < 0$

$\Rightarrow (a-7)(a+1) < 0$

$\Rightarrow -1 < a < 7$ ... (i)

and,

$x^2+ax-2 < 2(x^2-x+1)$ for all $x \in R$

$\Rightarrow x^2-(a+2)x+4 > 0$ for all $x \in R$

$\Rightarrow (a+2)^2-16 < 0$

$\Rightarrow (a+2-4)(a+2+4) < 0$

$\Rightarrow (a-2)(a+6) < 0$

$\Rightarrow -6 < a < 2$ ...(ii)

Thus, $-3 < \dfrac{x^2+ax-2}{x^2-x+1} < 2$ for all $x \in R$

If $-1 < a < 7$ and $-6 < a < 2$

$\Rightarrow -1 < a < 2$

$\Rightarrow a \in (-1, 2)$.

EXAMPLE 12 *If x is real, find the values of k for which*

$$\left|\frac{x^2+kx+1}{x^2+x+1}\right| < 2 \text{ is valid.}$$

SOLUTION We have,

$\left|\dfrac{x^2+kx+1}{x^2+x+1}\right| < 2$ for all $x \in R$

$\Rightarrow -2 < \dfrac{x^2+kx+1}{x^2+x+1} < 2$ for all $x \in R$

$\Rightarrow -2(x^2+x+1) < x^2+kx+1 < 2(x^2+x+1)$ for all $x \in R$

$[\because\ x^2+x+1 > 0$ for all $x \in R]$

$\Rightarrow -2(x^2+x+1) < x^2+kx+1$

and,

$x^2+kx+1 < 2(x^2+x+1)$ for all $x \in R$

$\Rightarrow 3x^2+(k+2)x+3 > 0$

and, $x^2+(2-k)x+1 > 0$ for all $x \in R$

$\Rightarrow (k+2)^2-36 < 0$ and, $(2-k)^2-4 < 0$

$\Rightarrow (k+2-6)(k+2+6) < 0$ and, $(2-k-2)(2-k+2) < 0$

$\Rightarrow (k-4)(k+8) < 0$ and, $k(k-4) < 0$

$\Rightarrow -8 < k < 4$ and, $0 < k < 4$

$\Rightarrow 0 < k < 4$.

EXAMPLE 13 *A real number 'a' is called a good number if the inequality* $\dfrac{2x^2+2x+3}{x^2+x+1} \le a$ *is satisfied for all real x. Find the set of all good real numbers.*

SOLUTION We have,

$\dfrac{2x^2+2x+3}{x^2+x+1} \le a$ for all $x \in R$

$\Rightarrow 2x^2+2x+3 \le a(x^2+x+1)$ for all $x \in R$

$[\because\ x^2+x+1 > 0$ for all $x \in R]$

$\Rightarrow x^2(2-a)+x(2-a)+(3-a) \le 0$ for all $x \in R$

$\Rightarrow 2-a < 0$ and $(2-a)^2-4(2-a)(3-a) \le 0$

$\Rightarrow a > 2$ and $(2-a)(3a-10) \le 0$

$\Rightarrow a > 2$ and $(a-2)(3a-10) \ge 0$

$\Rightarrow a > 2$ and $a \le 2$ or $a \ge \dfrac{10}{3}$

$\Rightarrow a \ge \dfrac{10}{3}$

$\Rightarrow a \in [10/3, \infty)$.

EXAMPLE 14 *If a, b, c ∈ R, where a ≠ 0, then solve the system of equations*

$$ax_1^2+bx_1+c = x_2$$
$$ax_2^2+bx_2+c = x_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$ax_{n-1}^2+bx_{n-1}+c = x_n$$
$$ax_n^2+bx_n+c = x_1$$

in n unknowns $x_1, x_2, \ldots, x_n$ *when:*

(i) $(b-1)^2 < 4ac$ (ii) $(b-1)^2 = 4ac$

SOLUTION Let us define n variables $x_1, x_2, \ldots, x_n$ as follows:

$$X_1 = x_2-x_1,\ X_2 = x_3-x_2,\ X_3 = x_4-x_3, \ldots, X_{n-1} = x_n-x_{n-1}$$

and $X_n = x_1-x_n$.

Clearly, $X_1+X_2+X_3+\ldots+X_n = 0$

Now, the given system of equations can be written as

$$ax_1^2+(b-1)x_1+c = x_2-x_1$$
$$ax_2^2+(b-1)x_2+c = x_3-x_2$$
$$\vdots \quad \vdots \quad \vdots$$
$$ax_{n-1}^2+(b-1)x_{n-1}+c = x_n-x_{n-1}$$
$$ax_n^2+(b-1)x_n+c = x_1-x_n$$

or,

$ax_1^2 + (b-1)x_1 + c = X_1$

$ax_2^2 + (b-1)x_2 + c = X_2$

$\vdots \quad \vdots \quad \vdots$

$ax_n^2 + (b-1)x_n + c = X_n$

(i) *When* $(b-1)^2 < 4ac$:

If $(b-1)^2 < 4ac$, then each of the equations

$ax_i^2 + (b-1)x_i + c = 0,\ i = 1, 2, ..., n$

has imaginary roots

Therefore, if $a > 0$, then

$ax_i^2 + (b-1)x_i + c > 0$ for $i = 1, 2, ..., n$

$\Rightarrow X_i > 0$ for $i = 1, 2, ..., n$

$\Rightarrow X_1 + X_2 + ... + X_n > 0$

If $a > 0$, then

$ax_i^2 + (b-1)x_i + c < 0$ for $i = 1, 2, ..., n$

$\Rightarrow X_i < 0$ for $i = 1, 2, ... n$

$\Rightarrow X_1 + X_2 + ... + X_n < 0$

But, $X_1 + X_2 + X_3 + ... + X_n = 0$

Hence, the system has no solution in this case.

(ii) When $(b-1)^2 = 4ac$:

If $(b-1)^2 = 4ac$, then roots of each of the equations $ax_i^2 + (b-1)x_i + c = 0$ are real and equal.

$\therefore\ ax_i^2 + (b-1)x_i + c \geq 0$ for all $i = 1, 2, ... n$

$\Rightarrow X_i \geq 0$ for all $i = 1, 2, ..., n$

$\Rightarrow X_i = X_2 = ... = X_n = 0 \qquad [\because x_1 + x_2 + ... + x_n = 0]$

$\Rightarrow X_1 = X_2 = ... = X_n$

Putting $X_i = 0$ in $ax_i^2 + (b-1)x_i + c = X_i$, we get

$ax_i^2 + (b-1)x_i + c = 0$

$\Rightarrow x_i = \dfrac{(1-b) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$

$\Rightarrow x_i = \dfrac{1-b}{2a} \qquad [\because (b-1)^2 = 4ac]$

$\therefore\ x_1 = x_2 = ... = x_n = \dfrac{1-b}{2a}$

EXERCISE 20.4

1. Find the greatest and the least values of $\dfrac{x^2+2x+1}{x^2+2x+7}$, where x is real.

2. Find the condition(s) that the expression $\dfrac{(x-a)(x-b)}{x(x-c)}$, x being real takes all real values.

3. For real values of x, prove that the values of the expression $\dfrac{11x^2+12x+6}{x^2+4x+2}$ cannot lie between -5 and 3.

4. If x is real, show that $\dfrac{x^2-x+1}{x^2+x+1}$ takes values between 1/3 and 3.

5. If x is real, show that the expression $\dfrac{x^2-bc}{2x-b-c}$ has no real values between b and c.

6. Show that $-1 < \dfrac{2x^2-14x+11}{2x^2-2x+5} < 3$ for all $x \in R$.

7. Find the real values m for which $\left|\dfrac{x^2+mx+1}{x^2+x+1}\right| < 3$ holds for all real values of x.

ANSWERS

1. Least value = 0, Greatest value = 1.

2. If $ab > 0$ then c lies between a and b.
If $ab < 0, c > \max(a, b)$ or $c < \min.(a, b)$.

7. $-1 < m < 5$

—❑❑❑—

20.14 LOCATION OF THE ROOTS OF A QUADRATIC EQUATION

In this section, we shall disucss various conditions satisfied by the coefficients of a quadratic when the location of its roots is marked.

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, be a quadratic expression and let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$ i.e. $f(x) = 0$. Let D be the discriminant of $f(x) = 0$. Then,

$$\alpha = \frac{-b-\sqrt{D}}{2a} \text{ and } \beta = \frac{-b+\sqrt{D}}{2a}$$

Clearly, if $D > 0$, then $\alpha < \beta$ and $\alpha = \beta$ for $D = 0$

Now, we discuss the following cases:

CASE I *When a real number k lies between the roots* α and β *i.e.* $\alpha < k < \beta$:

In this case, as is evident from the graph of the curve $y = f(x)$, a i.e. the coefficient of x^2 and $f(k)$ are of opposite signs. Thus, the conditions are :

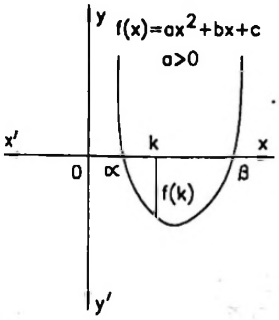


Fig. 20.18

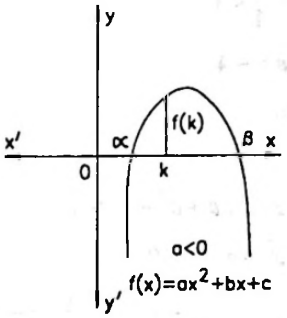


Fig. 20.19

(i) $D > 0$

(ii) $af(k) > 0$

CASE II *When both the roots* α and β *lie in the interval* (k_1, k_2):

If both the roots α and β of $ax^2 + bx + c = 0$ lie between numbers k_1 and k_2, then we have the following conditions:

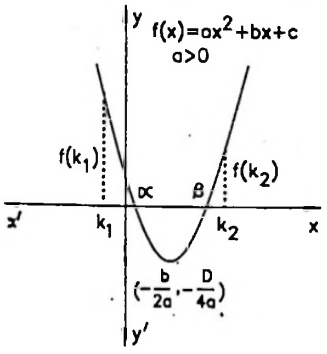


Fig. 20.20

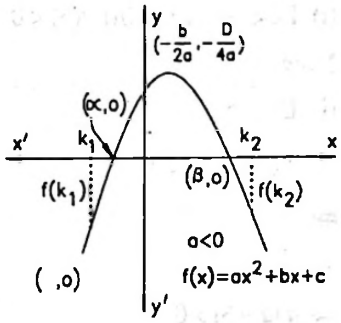


Fig. 20.21

(i) $D \geq 0$ i.e. roots are real

(ii) $af(k_1) > 0$, $af(k_2) > 0$

(iii) $k_1 < -\frac{b}{2a} < k_2$

i.e. vertex of the parabola $y = ax^2 + bx + c$ lies between k_1 and k_2.

CASE III *When a real number k is less than the roots* α and β.

If a real number k is less than α and β, then it is evident from the graph of $y = f(x)$ that a and $f(k)$ are of the same sign and point representing k lies on the left hand side of the vertex of the parabola $y = f(x)$. Thus, we have the following conditions.

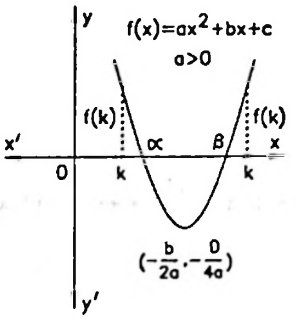


Fig. 20.22

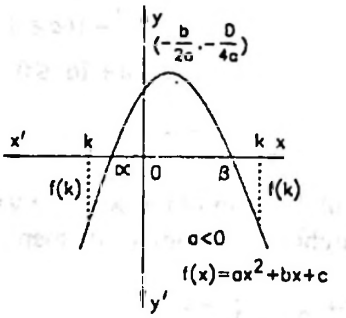


Fig. 20.23

(i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k < \frac{-b}{2a}$

CASE IV *When a real number k is greater than the roots* α and β.

If a real number k is greater than α and β, then the point representing k on x-axis is on the right side of the vertex of the parabola $y = f(x)$. Thus, we have the following conditions:

(i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k > \frac{-b}{2a}$

CASE V *When exactly one root of the quadratic lies in the interval* (k_1, k_2), $k_1 < k_2$

If exactly one root of the quadratic equations $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2), then $f(k_1)$ and $f(k_2)$ are opposite signs as shown in Fig. 20.24 and 20.25. Thus, we have the following conditions:

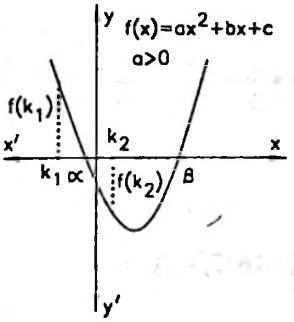


Fig. 20.24

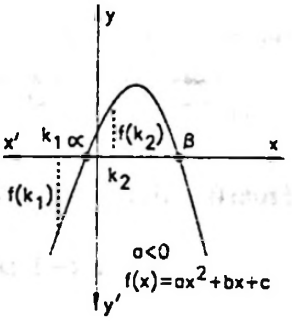


Fig. 20.25

(i) $D > 0$ (ii) $f(k_1)f(k_2) < 0$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *For what values of 'a' are the roots of the equation* $(a+1)x^2 - 3ax + 4a = 0$ $(a \neq -1)$ *greater than unity ?*

SOLUTION Let $f(x) = (a+1)x^2 - 3ax + 4a$ and let α, β be the roots of the equation $f(x) = 0$. The equation will have roots greater than 1 iff

(i) Disc ≥ 0

(ii) Vertex lies on the right side of the point (1, 0)

(iii) $(a+1)f(1) > 0$

Now,

(i) Disc $\geq 0 \Rightarrow 9a^2 - 16a(a+1) \geq 0$

$\Rightarrow -7a^2 - 16a \geq 0$

$\Rightarrow a(7a+16) \leq 0$

$\Rightarrow -\frac{16}{7} \leq a \leq 0$ ...(i)

(ii) If vertex of the parabola $y = (a+1)x^2 - 3ax + 4a$ is on the right side of point (1, 0), then

$\Rightarrow \frac{3a}{2(a+1)} > 1$

$\Rightarrow \frac{3a}{a+1} - 2 > 0$

$\Rightarrow \frac{3a - 2a - 2}{a+1} > 0$

$\Rightarrow \frac{a-2}{a+1} > 0$

$\Rightarrow a < -1$ or, $a > 2$ ...(ii)

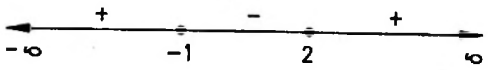


Fig. 20.26

(iii) $(a+1)f(1) > 0$

$\Rightarrow (a+1)(a+1-3a+4a) > 0$

$\Rightarrow (a+1)(2a+1) > 0$

$\Rightarrow a < -1$ or $a > -1/2$ ...(iii)

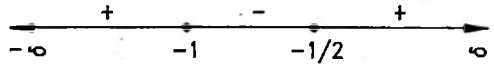


Fig. 20.27

From (i), (ii) and (iii), we get :

$$\frac{-16}{7} \leq a < -1 \text{ i.e. } a \in [-16/7, -1)$$

EXAMPLE 2 *Find the values of a, for which both the roots of the equation $2x^2 + ax + a^2 - 5 = 0$ are less than unity.*

SOLUTION Let $f(x) = 2x^2 + ax + a^2 - 5$ and let α, β be the roots of $f(x) = 0$. Then, both the roots of $f(x) = 0$ will be less than one, iff

(i) Disc ≥ 0

(ii) Vertex of parabola $y = f(x)$ is on the left side of the point (1, 0).

(iii) $2f(1) > 0$

Now,

(i) Disc ≥ 0

$\Rightarrow a^2 - 8(a^2 - 5) \geq 0$

$\Rightarrow 7a^2 - 40 \leq 0$

$\Rightarrow (\sqrt{7}a - \sqrt{40})(\sqrt{7}a + \sqrt{40}) \leq 0$

$\Rightarrow a \in \left[-\sqrt{\frac{40}{7}}, \sqrt{\frac{40}{7}}\right]$ ...(i)

(ii) Vertex of $y = f(x)$ is on the left side of the point (1, 0)

$\Rightarrow -\frac{a}{4} < 1$

$\Rightarrow a > -4$ ...(ii)

and,

(iii) $2f(1) > 0$

$\Rightarrow 2(2 + a + a^2 - 5) > 0$

$\Rightarrow a^2 + a - 3 > 0$

$\Rightarrow a \in \left(-\infty, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{-1+\sqrt{13}}{2}, \infty\right)$ ...(iii)

From (i), (ii) and (iii), we get

$$a \in \left[-\sqrt{\frac{40}{7}}, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{\sqrt{13}-1}{2}, \sqrt{\frac{40}{7}}\right]$$

EXAMPLE 3 *Find all the values of a, so that 6 lies between the roots of the equation $x^2 + 2(a-3)x + 9 = 0$.*

SOLUTION Let $f(x) = x^2 + 2(a-3)x + 9$. If 6 lies between the roots of $f(x) = 0$, then we must have the following conditions :

(i) Disc > 0, and (ii) $f(6) < 0$ [$\because$ Coeff. of x^2 is positive)]

Now,

(i) Disc > 0

$\Rightarrow 4(a-3)^2 - 36 > 0$

$\Rightarrow (a-3)^2 - 9 > 0$

$\Rightarrow a^2 - 6a > 0$

$\Rightarrow a(a-6) > 0$ ...(i)

$\Rightarrow a < 0$ or, $a > 6$

and,

(ii) $f(6) < 0$

$\Rightarrow 36 + 12(a-3) + 9 < 0$

$\Rightarrow 12a + 9 < 0$

$\Rightarrow a < -\frac{3}{4}$ ...(ii)

From (i) and (ii), we get :

$$a < -3/4 \text{ i.e. } a \in (-\infty, -3/4).$$

EXAMPLE 4 *Find all the values of 'a' for which the roots of the equation $(a-3)x^2 - 2ax + 5a = 0$ are positive.*

SOLUTION Let $f(x) = (a-3)x^2 - 2ax + 5a$. For the roots of $f(x) = 0$ to be positive, we must have

(i) Disc ≥ 0

(ii) Vertex of $y = f(x)$ is on the right side of $(0, 0)$

and, (iii) $(a-3) f(0) > 0$

Now,

(i) Disc ≥ 0

$\Rightarrow 4a^2 - 20a(a-3) > 0$

$\Rightarrow -16a^2 + 60a \geq 0$

$\Rightarrow 4a(4a - 15) \leq 0$

$\Rightarrow 0 \leq a \leq 15/4$ ...(i)

(ii) Vertex of $y = f(x)$ is on the right sides of the point $(0, 0)$

$\Rightarrow \frac{a}{a-3} > 0$

$\Rightarrow a < 0$ or, $a > 3$ ...(ii)

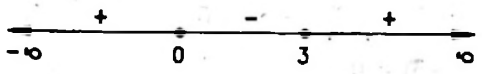


Fig. 20.28

and,

(iii) $(a-3) f(0) > 0$

$\Rightarrow (a-3) 5a > 0$

$\Rightarrow a(a-3) > 0$

$\Rightarrow a < 0$ or $a > 3$ ...(iii)

From (i), (ii) and (iii), we get :

$$3 < a \leq 15/4 \text{ i.e. } a \in (3, 15/4]$$

EXAMPLE 5 *Find all values of m such that the roots of the equation* $2x^2 - x - 1 = 0$ *lie inside the roots of the equation*

$$x^2 + (2m - m^2)x - 2m^3 = 0.$$

SOLUTION The roots of the equation $2x^2 - x - 1 = 0$ are given by:

$2x^2 - x - 1 = 0$

$\Rightarrow (2x+1)(x-1) = 0$

$\Rightarrow x = -1/2, 1.$

Let $f(x) = x^2 + (2m - m^2)x - 2m^3$.

Clearly, $y = f(x)$ represents a parabola opening upwards. Therefore, $-1/2$ and 1 will lie between the roots of $f(x) = 0$, if the following conditions hold simultaneously:

(i) $D > 0$, where D is the discriminant of $f(x) = 0$.

(ii) $f\left(\frac{-1}{2}\right) < 0$ and $f(1) < 0$.

Now

$D > 0 \Rightarrow (2m - m^2) + 8m^3 > 0$

$\Rightarrow 4m^2 - 4m^3 + m^4 + 8m^3 > 0$

$\Rightarrow m^4 + 4m^3 + 4m^2 > 0,$

$\Rightarrow (m^2 + 2m)^2$, which is true for all values of m.

$f\left(\frac{-1}{2}\right) < 0$

$\Rightarrow \frac{1}{4} - \left(\frac{2m - m^2}{2}\right) - 2m^3 < 0$

$\Rightarrow -8m^3 + 2m^2 - 4m + 1 < 0$

$\Rightarrow 8m^3 - 2m^2 + 4m - 1 > 0$

$\Rightarrow (4m-1)(2m^2+1) > 0$

$\Rightarrow 4m - 1 > 0$

$\Rightarrow m > \frac{1}{4}$

and,

$f(1) < 0$

$\Rightarrow 1 + 2m - m^2 - 2m^3 < 0$

$\Rightarrow 2m^3 + m^2 - 2m - 1 < 0$

$\Rightarrow (m^2 - 1)(2m + 1) < 0$

$\Rightarrow (m-1)(m+1)(2m+1) < 0$

$\Rightarrow m \in (-\infty, -1) \cup (-1/2, 1)$

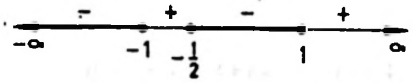


Fig. 20.29

Thus, $D > 0, f\left(\frac{-1}{2}\right) < 0$ and $f(1) < 0$

$\Rightarrow m \in (1/4, 1)$.

EXAMPLE 6 *For what real values of 'a' do the roots of the equation* $x^2 - 2x - a^2 + 1 = 0$ *lie between the roots of the equation*

$$x^2 - 2(a+1)x + a(a-1) = 0.$$

SOLUTION We have,

$x^2 - 2x - a^2 + 1 = 0$

$\Rightarrow (x^2 - 2x + 1) - a^2 = 0$

$\Rightarrow (x-1)^2 - a^2 = 0$

$\Rightarrow (x - 1 + a)(x - 1 - a) = 0$

$\Rightarrow x = 1 + a, 1 - a$

Thus, the roots of the equation $x^2 - 2x - a^2 + 1 = 0$ are $1 + a$ and $1 - a$.

Let $f(x) = x^2 - 2(a+1)x + a(a-1)$.

If the roots of the first equation i.e. $1 \pm a$ lie between the roots of the second equation i.e. $f(x) = 0$, then the following conditions must hold good simultaneously:

(i) $D > 0$, where D is the discriminant of $f(x) = 0$

(ii) $f(1 \pm a) < 0$ [$\because$ Coefficient of x^2 in $f(x)$ is positive]

Now,

$D > 0$

$\Rightarrow 4(a+1)^2 - 4a(a-1) > 0$

$\Rightarrow 3a + 1 > 0$

$\Rightarrow a > \frac{-1}{3}$.

$f(1-a) < 0$

$\Rightarrow (1-a)^2 - 2(a+1)(1-a) + a(a-1) < 0$

$\Rightarrow (1-a)^2 - 2(1-a^2) + a(a-1) < 0$

$\Rightarrow -1 - 2a + 3a^2 + a^2 - a < 0$

$\Rightarrow 4a^2 - 3a - 1 < 0$

$\Rightarrow (4a+1)(a-1) < 0$

$\Rightarrow -\frac{1}{4} < a < 1$

and,

$f(1+a) < 0$

$\Rightarrow (1+a)^2 - 2(a+1)^2 + a(a-1) < 0$

$\Rightarrow -(1+a)^2 + a(a-1) < 0$

$\Rightarrow -3a - 1 < 0$

$\Rightarrow a > -\frac{1}{3}$.

Thus, $D > 0, f(1-a) < 0$ and $f(1+a) < 0$

$\Rightarrow \quad a \in (-1/4, 1)$.

EXAMPLE 7 *Find the values of 'a' for which the roots of the equation* $x^2 - 2ax + a^2 - 1 = 0$ *lie between the numbers 2 and 4. Also, find the values of 'a' for which exactly one root lies between the numbers 2 and 4.*

SOLUTION Let $f(x) = x^2 - 2ax + a^2 - 1$.

If both the roots of the equation $f(x) = 0$ lie between 2 and 4, then the following conditions must hold simultaneously:

(i) $D \geq 0$, where D is the discriminant of $f(x) = 0$.

(ii) $f(2) > 0$ [$\because y = f(x)$ is a parabola opening upwards]

(iii) $f(4) > 0$ [$\because y = f(x)$ is a parabola opening upwards]

(iv) Vertex of the parabola $y = f(x)$ lies between (2,0) and (4,0) i.e. $2 < a < 4$.

Now, $D = 4a^2 - 4(a^2 - 1) = 4 \geq 0$, which is always true.

$f(2) > 0$

$\Rightarrow 4 - 4a + a^2 - 1 > 0$

$\Rightarrow a^2 - 4a + 3 > 0$

$\Rightarrow (a-1)(a-3) > 0$

$\Rightarrow a < 1$ or $a > 3$

$\Rightarrow a \in (-\infty, 1) \cup (3, \infty)$.

and, $f(4) > 0$

$\Rightarrow 16 - 8a + a^2 - 1 > 0$

$\Rightarrow a^2 - 8a + 15 > 0$

$\Rightarrow (a-5)(a-3) > 0$

$\Rightarrow a < 3$ or $a > 5$

$\Rightarrow a \in (-\infty, 3) \cup (5, \infty)$

We find that there is no value of a satisfying all these conditions simultaneously.

Hence, there is no value of 'a' for which the roots of the equation can lie between 2 and 4.

If exactly one root of $f(x) = 0$ lies between 2 and 4, then, $f(2)$ and $f(4)$ are opposite signs.

$\therefore f(2) f(4) < 0$

$\Rightarrow (4 - 4a + a^2 - 1)(16 - 8a + a^2 - 1) < 0$

$\Rightarrow (a^2 - 4a + 3)(a^2 - 8a + 15) < 0$

$\Rightarrow (a-1)(a-3)(a-3)(a-5) < 0$

$\Rightarrow (a-1)(a-5) < 0$

$\Rightarrow 1 < a < 5$

$\Rightarrow a \in (1, 5)$.

EXAMPLE 8 *Find the values of p for which both the roots of the equation* $4x^2 - 20px + 25p^2 + 15p - 66 = 0$ *are greater than 2.*

SOLUTION Let $f(x) = 4x^2 - 20px + 25p^2 + 15p - 66$.

In the equation $f(x) = 0$, we observe that the coefficient of x^2 is positive. So, the curve $y = f(x)$ represents a parabola opening upward. If both the roots of the equation $f(x) = 0$ are greater than 2, then following conditions must hold simultaneously:

(i) $D > 0$, where D denotes the discriminant of $f(x) = 0$

(ii) $f(2) > 0$

(iii) 2 is on the left hand side of the vertex of $y = f(x)$ i.e. 2 is less than the x-coordinate of the vertex of the parabola $y = f(x)$.

Now,

$D > 0$

$\Rightarrow 400p^2 - 16(25p^2 + 15p - 66) > 0$

$\Rightarrow -16(15p - 66) > 0$

$\Rightarrow 5p - 22 < 0$

$\Rightarrow p < \frac{22}{5}$ ...(i)

$f(2) > 0$

$\Rightarrow 16 - 40p + 25p^2 + 15p - 66 > 0$

$\Rightarrow 25p^2 - 25p - 50 > 0$

$\Rightarrow p^2 - p - 2 > 0$

$\Rightarrow (p-2)(p+1) > 0$

$\Rightarrow p < -1$ or $p > 2$

$\Rightarrow p \in (-\infty, -1) \cup (2, \infty)$ ...(ii)

Clearly, x-coordinate of the vertex is $\frac{20p}{8} = \frac{5p}{2}$

$\therefore 2 < \frac{5p}{2} \Rightarrow p > \frac{4}{5}$ ...(iii)

From (i), (ii) and (iii), we obtain

$$p \in (2, 22/5).$$

EXAMPLE 9 *Suppose* α and β *are the roots of the equation* $x^2 + 2(m-3)x + 9 = 0$.

(i) *Find all the values of* m *such that both 6 and 1 lie between* α and β.

(ii) *Find all the values of* m *such that at most one of 6 and 1 lie between* α and β.

SOLUTION Let $f(x) = x^2 + 2(m-3)x + 9$

Clearly, $y = f(x)$ represents a parabola opening upward.

(i) If 6 and 1 both lie between the roots of $f(x) = 0$ then

$f(6) < 0$ and $f(1) < 0$

$\Rightarrow 36 + 12(m-3) + 9 < 0$ and $1 + 2(m-3) + 9 < 0$

$\Rightarrow 12m + 9 < 0$ and $2m + 4 < 0$

$\Rightarrow 4m + 3 < 0$ and $m + 2 < 0$

$\Rightarrow m < \frac{-3}{4}$, and $m < -2$

$\Rightarrow m < -2$.

(ii) If at most one of 6 and 1 lie between α and β, then 6 and 1 both must not lie between α and β.

$\therefore m \nless -2 \Rightarrow m \geq -2$.

EXAMPLE 10 *Find all values of 'a' for which exactly one root of the equation* $e^a x^2 - e^{2a} x + e^a - 1 = 0$, *lies between 1 and 2.*

SOLUTION Let $f(x) = e^a x^2 - e^{2a} x + e^a - 1$

If exactly one root of $f(x) = 0$ lies between 1 and 2, then

$f(1) f(2) < 0$

$\Rightarrow (e^a - e^{2a} + e^a - 1)(4e^a - 2e^{2a} + e^a - 1) < 0$

$\Rightarrow (2e^a - e^{2a} - 1)(5e^a - 2e^{2a} - 1) < 0$

$\Rightarrow (e^{2a} - 2e^a + 1)(2e^{2a} - 5e^a + 1) < 0$

$\Rightarrow (e^a - 1)^2 (2e^{2a} - 5e^a + 1) < 0$

$\Rightarrow 2e^{2a} - 5e^a + 1 < 0$ $[\because (e^a - 1)^2 > 0]$

$\Rightarrow \frac{5 - \sqrt{17}}{4} < e^a < \frac{5 + \sqrt{17}}{4}$ $\left[\because 2e^{2a} - 5e^a + 1 = 0 \Rightarrow e^a = \frac{5 \pm \sqrt{17}}{4}\right]$

$\Rightarrow \log_e\left(\frac{5 - \sqrt{17}}{4}\right) < a < \log_e\left(\frac{5 + \sqrt{17}}{4}\right)$ $[\because \log_e x$ is an increasing function$]$

$\Rightarrow a \in \left(\log_e\left(\frac{5 - \sqrt{17}}{4}\right), \log_e\left(\frac{5 + \sqrt{17}}{4}\right)\right)$

EXAMPLE 11 *For what real values of 'a' is one of the roots of the equation* $(2a+1)x^2 - ax + a - 2 = 0$ *greater and the other is smaller than unity?*

SOLUTION Let $f(x) = (2a+1)x^2 - ax + a - 2$.

If one root of $f(x) = 0$ is greater than unity and the other is less than unity, then the following conditions must hold good simultaneously:

(i) $D > 0$, where D is the discriminant of $f(x) = 0$

(ii) 1 lies between the roots, therefore, $(2a+1) f(1) < 0$.

Now,

$D > 0$

$\Rightarrow a^2 - 4(2a+1)(a-2) > 0$

$\Rightarrow a^2 - 4(2a^2 - 3a - 2) > 0$

$\Rightarrow 7a^2 - 12a - 8 < 0$

$\Rightarrow \frac{6 - 2\sqrt{23}}{7} < a < \frac{6 + 2\sqrt{23}}{7}$ $\left[\because 7a^2 - 12a - 8 = 0 \Rightarrow a = \frac{6 \pm 2\sqrt{23}}{7}\right]$

and,

$(2a+1) f(1) < 0$

$\Rightarrow (2a+1)(2a+1-a+a-2) < 0$

$\Rightarrow (2a+1)(2a-1) < 0$

$\Rightarrow \frac{-1}{2} < a < \frac{1}{2}$.

Since, $\frac{6 - 2\sqrt{23}}{7} < \frac{-1}{2}$ and $\frac{6 + 2\sqrt{23}}{7} > \frac{1}{2}$.

Hence, the required values of a lie in the interal $(-1/2, 1/2)$.

EXAMPLE 12 *Find all possible values of the real parameter 'a' so that* $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has

(i) *at least two distinct positive real roots.*

(ii) *at least two distinct negative real roots.*

SOLUTION We have,

$x^4 + 2ax^3 + x^2 + 2ax + 1 = 0, x \neq 0$ ...(i)

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) + 2a\left(x + \frac{1}{x}\right) + 1 = 0$ [Dividing throughout by x^2]

$\Rightarrow \left(x + \frac{1}{x}\right)^2 + 2a\left(x + \frac{1}{x}\right) - 1 = 0$

$\Rightarrow t^2 + 2at - 1 = 0,$ ...(ii)

where $t = x + \frac{1}{x}$.

CASE I *When* $x > 0$

We know that the sum of a positive real number and its reciprocal is always greater than or equal to two. Therefore, $t \geq 2$.

The roots of (ii) are real and distinct as its discriminant $D = 4a^2 + 4 > 0$ for all $a \in R$. Consequently $t > 2$.

Thus, the given equation will have at least two distinct positive roots, if equation (ii) has at least one root greater than 2.

If exactly one root of (ii) is greater than 2, then 2 must lie between the roots.

$\therefore f(2) < 0$, where $f(t) = t^2 + 2at - 1$

$\Rightarrow 4 + 4a - 1 < 0$

$\Rightarrow a < \frac{-3}{4}$

$\Rightarrow a \in (-\infty, -3/4)$

If both the roots of equation (ii) are greater than 2, then the point representing (2, 0) on t-axis is on the left hand side of the vertex of the parabola $f(t) = t^2 + 2at - 1$ and $f(2) > 0$. That is,

$2 < -a$ and $4 + 4a - 1 > 0$

$\Rightarrow a < -2$ and $a > -\frac{3}{4}$, which is not possible.

Hence, $a \in (-\infty, -3/4)$.

CASE II *When* $x < 0$

Since the sum of a negative real number and its reciprocal is less than or equal to -2. Therefore, $t \leq -2$. As the discriminant of (ii) is positive so equation (ii) has distinct real roots. Therefore, $t \neq -2$. Thus, $t > -2$.

If the given equation has at least two distinct negative real roots, then the equation (ii) has at least one root less than -2. The conditions for the same are

(i) $f(-2) < 0$ or, (ii) $f(-2) > 0$ and the vertex of the parabola $f(t) = t^2 + 2at - 1$ is on the left hand side of $t = -2$.

Now

$$f(-2) < 0 \Rightarrow 4 - 4a - 1 < 0 \Rightarrow a > \frac{3}{4}$$

and, $f(-2) > 0$ and $-a < -2$

$\Rightarrow a < \frac{3}{4}$ and $a > 2$, which is not possible.

Hence, $a > \frac{3}{4} \Rightarrow a \in \left(\frac{3}{4}, \infty\right)$.

EXAMPLE 13 *Find the values of k for which the inequality* $(x - 3k)(x - k - 3) < 0$ *satisfies for all x such that* $1 \leq x \leq 3$.

SOLUTION Let $f(x) = (x - 3k)(x - k - 3)$

Clearly, $\alpha = 3k$ and $\beta = k + 3$ are the roots of $f(x) = 0$.

Also, coefficient of x^2 is positive. Therefore, $y = f(x)$ represents a parabola opening upward.

Thus, $f(x) < 0$ for all $x \in (1, 3)$

$\Rightarrow$ 1, 3 lie between the roots of $f(x)$.

$\Rightarrow f(1) < 0$ and $f(3) < 0$

$\Rightarrow (1 - 3k)(1 - k - 3) < 0$ and $(3 - 3k)(3 - k - 3) < 0$

$\Rightarrow (3k - 1)(k + 2) < 0$ and $3(k - 1)k < 0$

$\Rightarrow -2 < k < \frac{1}{3}$ and $0 < k < 1$

$\Rightarrow k \in (0, 1/3)$

EXAMPLE 14 *Find all the values of parameter 'a' for which the inequality* $4^x - a.2^x - a + 3 \leq 0$ *is satisfied by at least one real value of x.*

SOLUTION Let $t = 2^x$. Then, $t > 0$ for all $x \in R$ and

$4^x - a.2^x - a + 3 = t^2 - at - a + 3$

$\therefore 4^x - a.2^x - a + 3 \leq 0$ for at least one real x.

$\Rightarrow t^2 - at - a + 3 \leq 0$ for at least one $t > 0$

$\Rightarrow f(t) \leq 0$ for at least one $t > 0$, where $f(t) = t^2 - at - a + 3$.

Let D be the discriminant of $f(t) = 0$. Then,

$D < 0 \Rightarrow f(t) > 0$ for all t. [$\because$ Coefficient of $t^2 > 0$]

So, $f(t) \leq 0$ for at least one $t > 0$ can be true only when $D \geq 0$. It is evident from the following graphs that $f(t) \leq 0$ will be true for at least one $t > 0$ in the following two cases:

CASE I *When* $D \geq 0$ *and vertex of the parabola* $y = f(t)$ *is on the right side of the origin.*

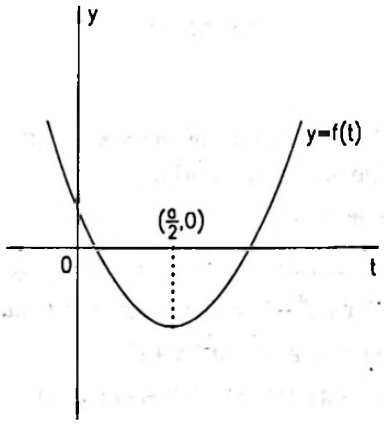


Fig. 20.30

In this case

$f(t) \geq 0$ for at least one $t > 0$

Now, $D \geq 0$ and $a/2 > 0$

$\Rightarrow a^2 + 4(a - 3) \geq 0$ and $a > 0$

$\Rightarrow (a + 6)(a - 2) \geq 0$ and $a > 0$

$\Rightarrow a \in (-\infty, -6] \cup [2, \infty)$ and $a > 0$

$\Rightarrow a \in [2, \infty)$.

CASE II *When* $D \geq 0$ *and vertex of the parabola* $y = f(t)$ *is on the left side of the origin.*

In this case, $f(t) \geq 0$ will be true for at least one $t > 0$, if $f(0) < 0$.

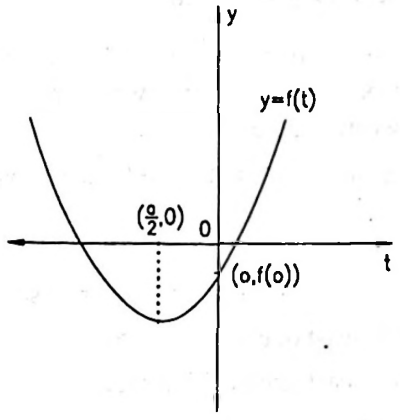


Fig. 20.31

Now, $D \geq 0 \Rightarrow a \in (-\infty, -6] \cup [2, \infty)$ [See Case I]

$a/2 < 0 \Rightarrow a < 0$

and, $f(0) < 0 \Rightarrow -a + 3 < 0$

$\Rightarrow a > 3$

$\therefore D \geq 0, a/2 < 0$ and $f(0) < 0$

$\Rightarrow a \in (-\infty, -6] \cup [2, \infty)$, $a < 0$ and $a > 3$

Clearly, these three are not true simultaneously. So, there is no value of a in this case.

Hence, $a \in [2, \infty)$.

EXAMPLE 15 *Consider the inequality* $9^x - a \cdot 3^x - a + 3 \leq 0$, *where 'a' is a real parameter. Find the values of 'a' so that the given inequality has:*

(i) *at least one negative solution.*

(ii) *at least one positive solution.*

(iii) *at least one solution in (1, 2).*

(iv) *at least one real solution.*

SOLUTION Let $t = 3^x$. Then,

$$9^x - a \cdot 3^x - a + 3 = t^2 - at - (a - 3)$$

Also, let $f(t) = t^2 - at - (a - 3)$.

$\therefore 9^x - a \cdot 3^x - a + 3 \leq 0$

$\Rightarrow$ $f(t) \leq 0$, where $t = 3^x > 0$ for all real x.

Clearly, the graph of $f(t)$ is a parabola opening upwards.

Let D be the discriminant of $f(t)$. Then,

$$D = a^2 + 4(a - 3) = a^2 + 4a - 12 = (a + 6)(a - 2)$$

Clearly, $f(t) = 0$ will have real roots, if

$D \geq 0 \Rightarrow a \leq -6$ or $a \geq 2$.

(i) $9^x - a \cdot 3^x - a + 3 \leq 0$ has at least one negative solution

$\Rightarrow f(t) \leq 0$ has at least one solution such that $t < 1$

$[\because x < 0 \Rightarrow t = 3^x < 1]$

$\Rightarrow f(t) \leq 0$ for at least one $t \in (0, 1)$ $[\because t > 0]$

$\Rightarrow f(t) = 0$ has either a root between 0 and 1 or it has both roots between 0 and 1.

CASE I *When $f(t) = 0$ has a root between 0 and 1:*

In this case,

$D \geq 0$ and $f(0) f(1) < 0$

$\Rightarrow a \in (-\infty, -6] \cup [2, \infty)$ and $-(a - 3)(4 - 2a) < 0$

$\Rightarrow a \in (-\infty, -6] \cup [2, \infty)$ and $(a - 3)(a - 2) < 0$

$\Rightarrow a \in (-\infty, -6] \cup [2, \infty)$ and $2 < a < 3$

$\Rightarrow a \in (2, 3)$

CASE II *When $f(t) = 0$ has both roots between 0 and 1.*

In this case, the following conditions must hold good simultaneously:

(i) $D \geq 0$

(ii) $f(0) > 0$

(iii) $f(1) > 0$

(iv) Vertex of the parabola is between 0 and 1 i.e. $0 < \frac{a}{2} < 1$.

Now,

$D \geq 0 \Rightarrow a \in (-\infty, -6] \cup [2, \infty)$

$f(0) > 0 \Rightarrow -(a - 3) > 0 \Rightarrow a < 3 \Rightarrow a \in (-\infty, 3)$

$f(1) > 0 \Rightarrow 4 - 2a > 0 \Rightarrow a < 2 \Rightarrow a \in (-\infty, 2)$

and,

$0 < a/2 < 1 \Rightarrow a \in (0, 2)$.

Clearly, all these conditions cannot hold good simultaneously. So, in this there is no value of a for which $f(t) = 0$ has both roots between 0 and 1.

Hence, $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one negative values of x, if $a \in (2, 3)$

(ii) $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one positive value of x

$\Rightarrow f(t) \leq 0$ for at least one $t > 1$ $[\because x > 0 \Rightarrow t = 3^x > 1]$

$\Rightarrow f(t) = 0$ has at least one root greater than 1.

$\Rightarrow f(t) = 0$ has either one root greater than 1 or both roots greater than 1.

CASE I *When $f(t) = 0$ has one root greater than 1.*

In this case, the following conditions must hold simultaneously.

(i) $D < 0$ (ii) $f(1) < 0$

Now,

(i) $D > 0 \Rightarrow a \in (-\infty, 6] \cup (2, \infty)$

(ii) $1 - a - a + 3 < 0 \Rightarrow 4 - 2a < 0 \Rightarrow a - 2 > 0 \Rightarrow a > 2$.

Combining these two, we obtain

$a \in (2, \infty)$

CASE II *When $f(t) = 0$ has both roots greater than 1 :*

In this case, the following conditions must hold simultaneously :

(i) $D \geq 0$ (ii) $f(1) > 0$

(iii) Vertex of the parabola is on the right side of 1 i.e. $\frac{a}{2} > 1$

Now,

(i) $D \geq 0 \Rightarrow a \in (-\infty, 6] \cup [2, \infty)$

(ii) $f(1) = 0 \Rightarrow a < 2$

(iii) $\frac{a}{2} > 1 \Rightarrow a > 2$

Clearly, there is no value of a sasitfying these three conditions simultaneously.

Hence, $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one positive value of x, if $a \in (2, \infty)$

(iii) $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one $x \in (1, 2)$

$\Rightarrow t^2 - at - a + 3 \leq 0$ for at least one $t \in (3, 9)$ $[\because t = 3^x]$

$\Rightarrow$ At least one root of $f(t) = 0$ lies between 3 and 9.

$\Rightarrow f(t) = 0$ has exactly one root between 3 and 9 or it has both roots between 3 and 9.

CASE I *When $f(t) = 0$ has one root between 3 and 9.*

In this case, the following conditions must hold good simultaneously.

(i) $D > 0$ (ii) $f(3) f(9) < 0$

Now,

(i) $D > 0 \Rightarrow a \in (-\infty, -6) \cup (2, \infty)$

and, (ii) $f(3) f(9) < 0$

$\Rightarrow (12 - 4a)(84 - 10a) < 0$

$\Rightarrow (4a - 12)(10a - 84) < 0$

$\Rightarrow 3 < a < \frac{84}{10}$

Clearly, (i) and (ii) are true for all $a \in (3, 84/10)$

CASE II *When $f(t) = 0$ has both roots between 3 and 9 :*

In this case, the following conditions must hold simultaneously :

(i) $D \geq 0$ (ii) $f(3) > 0$ (iii) $f(9) > 0$

(iv) Vertex of the parabola lies between 3 and 9 i.e. $3 < a/2 < 9$

Now,

(i) $D \geq 0 \Rightarrow a \leq -6$ or $a \geq 2$

(ii) $f(3) = 0 \Rightarrow 9 - 3a - a + 3 > 0$

$\Rightarrow -4a + 12 > 0$

$\Rightarrow a < 3$

(iii) $f(9) > 0 \Rightarrow 81 - 9a - a + 3 > 0$

$\Rightarrow 84 - 10a > 0$

$\Rightarrow a < \frac{84}{10}$

(iv) $3 < \frac{a}{2} < 9 \Rightarrow 6 < a < 18$

Clearly, all these inequaties do not hold good simultaneously. So, there is no value of 'a' for which both the roots of $f(t) = 0$ lie between 3 and 9.

Hence, $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one $x \in (1, 2)$

if $a \in (3, 84/10)$

(iv) $9^x - a \cdot 3^x - a + 3 \leq 0$ has at least one real solution.

$\Rightarrow f(t) \leq 0$ has at least one solution $t > 0$

$\Rightarrow f(t) = 0$ has at least one positive root.

$\Rightarrow f(t) = 0$ has one positive root or it has both positive roots.

CASE I *When $f(t) = 0$ has one positive root.*

In this case, the following conditions must hold simultaneously.

(i) $D > 0$ (ii) $f(0) < 0$

Now,

(i) $D > 0 \Rightarrow a \in (-\infty, 6) \cup (2, \infty)$

and, (ii) $f(0) < 0 \Rightarrow -(a - 3) < 0 \Rightarrow a > 3$.

These two conditions will hold simultaneously, if $a \in (3, \infty)$

CASE II *When $f(t) = 0$ has both positive roots :*

In this case, the following conditions must hold simultaneously :

(i) $D \geq 0$ (ii) $f(0) > 0$

and,

(iii) Vertex of the parabola is on the right side of the origin

i.e. $\frac{a}{2} > 0$

Now,

(i) $D \geq 0 \Rightarrow a \in (-\infty, -6] \cup [2, \infty)$

(ii) $f(0) > 0 \Rightarrow -(a - 3) > 0 \Rightarrow a < 3$

and, (iii) $\frac{a}{2} > 0 \Rightarrow a > 0$

Clearly, these three conditions do not hold simultaneously any value of a. So, there is no value of 'a' in this case.

Hence $9^x - a \cdot 3^x - a + 3 \leq 0$ for at least one real value of x, if $a \in (3, \infty)$

EXAMPLE 16 *Find all the values of the parameter a for which the inequality $a \cdot 9^x + 4(a - 1) \cdot 3^x + a > 1$ is satisfied for all real values of x.*

SOLUTION Let $t = 3^x$. Then,

$$a \cdot 9^x + 4(a - 1) 3^x + a - 1 = at^2 + 4(a - 1) t + a - 1$$

Let

$$f(t) = at^2 + 4(a - 1) t + (a - 1)$$

$\therefore a \cdot 9^x + 4(a - 1) 3^x + a > 1$

$\Rightarrow a \cdot 9^x + 4(a - 1) 3^x + (a - 1) > 0$

$\Rightarrow f(t) > 0$, where $t = 3^x$.

Thus,

$a.9^x + 4(a-1)3^x + a > 1$ for all $x \in R$

$at^2 + 4(a-1)t + a - 1 > 0$

$\Rightarrow$ for all $t > 0$ $[\because x \in R \Rightarrow t = 3^x > 0]$

Now two cases arise:

CASE I *When $f(t) = 0$ i.e. $at^2 + 4(a-1)t - 1 = 0$ has imaginary roots.*

In this case, we have

$f(t) > 0$ for all $t > 0$, if the following conditions hold simultaneously:

(i) $a > 0$ (ii) Discriminant D of $f(t) = 0$ is less than zero.

$\Rightarrow a > 0$ and $16(a-1)^2 - 4a(a-1) < 0$

$\Rightarrow a > 0$ and $4(a-1)(3a-4) < 0$

$\Rightarrow a > 0$ and $(a-1)(3a-4) < 0$

$\Rightarrow a > 0$ and $1 < a < \frac{4}{3}$

$\Rightarrow a \in (1, 4/3)$

CASE II *When $f(t) = 0$ has real roots:*

In this case, $f(t) > 0$ for all $t > 0$

If both the roots of $f(t) = 0$ are less than zero and the parabola opens upwards.

This is possible only when each of the following conditions are satisfied simultaneously.

(i) $D \geq 0$

(ii) $a > 0$

(iii) $f(0) \geq 0$

(iv) Vertex of the parabola is on the negative side of t-axis or it is at the origin.

Now,

(i) $D \geq 0$

$\Rightarrow 4(a-1)(3a-4) \geq 0$

$\Rightarrow a \leq 1$ or, $a \geq \frac{4}{3}$

(ii) $a > 0$

(iii) $f(0) \geq 0$

$\Rightarrow a - 1 \geq 0$

$\Rightarrow a \geq 1$.

(iv) $-\frac{4(a-1)}{2a} \leq 0$

$\Rightarrow \frac{a-1}{a} \geq 0$

$\Rightarrow a \leq 0$ or, $a \geq 1$

Clearly, all these inequalities will hold good simultaneously for all $a \geq \frac{4}{3}$ i.e. for all $a \in [4/3, \infty)$.

Thus, $f(t) > 0$ for all $t > 0$ if $a \in (1, \infty)$. Clearly, for $a = 1, f(t) = t^2 > 0$ for all $t > 0$

Hence, $a \geq 1$

EXAMPLE 17 *Find all the values of parameter c for which the inequality*

$$1 + \log_2\left(2x^2 + 2x + \frac{7}{2}\right) \geq \log_2(cx^2 + c)$$

possesses at least one solution.

SOLUTION The given inequality is meaningful, if

$$c > 0 \text{ and } 2x^2 + 2x + \frac{7}{2} > 0$$

Clearly, $2x^2 + 2x + \frac{7}{2} > 0$ for all $x \in R$ $[\because$ Disc $< 0]$

Thus, the given inequality is meaningful for $c > 0$.

Now,

$$1 + \log_2\left(2x^2 + 2x + \frac{7}{2}\right) \geq \log_2(cx^2 + c) \quad \ldots(i)$$

$$\Rightarrow \log_2 2 + \log_2\left(2x^2 + 2x + \frac{7}{2}\right) \geq \log_2(cx^2 + c)$$

$$\Rightarrow \log_2\left\{2\left(2x^2 + 2x + \frac{7}{2}\right)\right\} \geq \log_2(cx^2 + c)$$

$\Rightarrow 4x^2 + 4x + 7 \geq cx^2 + c$ $[\because \log_2 x$ is an increasing function$]$

$\Rightarrow (c-4)x^2 - 4x + (c-7) \leq 0$ ...(ii)

Let D be the discriminant of $(c-4)x^2 - 4x + (c-7) = 0$. Then,

$$D = 16 - 4(c-4)(c-7)$$
$$= -4(c-3)(c-8)$$

Now we have the following cases:

CASE I *When $D < 0$*

$D < 0$

$\Rightarrow -4(c-3)(c-8) < 0$

$\Rightarrow (c-3)(c-8) > 0$

$\Rightarrow c \in (-\infty, 3) \cup (8, \infty)$

$\Rightarrow c \in (0, 3) \cup (8, \infty)$ $[\because c > 0]$

Coefficient of x^2 in (ii) $= (c-4)$

Thus, if $D < 0$ i.e. $c \in (0, 3) \cup (8, \infty)$ and $c - 4 < 0$, then the given inequality holds for all $x \in R$.

Clearly, $c \in (0, 3) \cup (8, \infty)$ and $c - 4 < 0$

$\Rightarrow$ $c \in (0, 3)$.

Thus, if $c \in (0, 3)$, then the given inequality holds for all $x \in R$.

CASE II *When $D \geq 0$ and coefficient of x^2 in (ii) < 0.*

$D \geq 0$

$\Rightarrow -4(c-3)(c-8) \geq 0$

$\Rightarrow (c-3)(c-8) \le 0$

$\Rightarrow 3 \le c \le 8.$

Coefficient of $x^2 = c-4$

If $D \ge 0$ and $c-4 < 0$, then the inequation (ii) holds for all

$$x \in -\left(\frac{2+\sqrt{D}}{4-c}\right) \text{ and for all } x \ge \left(\frac{\sqrt{D}-2}{4-c}\right)$$

But,

$D \ge 0$ and $c-4 < 0$

$\Rightarrow 3 \le c \le 8$ and $c < 4$

$\Rightarrow c \in [3, 4)$

Thus, if $c \in [3, 4)$, then the given inequality holds for some x.

CASE III *When $D \ge 0$ and coefficient of x^2 in (ii) > 0:*

In this case, inequation (ii) holds for all x satisfying.

$$x \in \left(\frac{2-\sqrt{D}}{c-4}, \frac{2+\sqrt{D}}{c-4}\right).$$

Now,

$D \ge 0 \Rightarrow 3 \le c \le 8$

and,

Coeff. of $x^2 > 0 \Rightarrow c-4 > 0 \Rightarrow c > 4$

$\therefore$ $D \ge 0$ and coefficient of $x^2 > 0$

$\Rightarrow 3 \le c \le 8$ and $c > 4$

$\Rightarrow c \in (4, 8]$.

Thus, if $c \in (4, 8]$, then the given equation holds for some x.

It follows from all the above cases that the given inequation holds for at least one x, if

$$c \in (0, 3) \cup [3, 4) \cup (4, 8]$$

Also, for $c = 4$, (ii) is satisfied for all $x \ge \frac{-3}{4}$.

Hence, if $0 < c \le 8$, there is at least one solution of the given inequation.

EXERCISE 20.5

1. Find all the values of m for which both roots of the equation $2x^2 + mx + m^2 - 5 = 0$ are (i) less than 1 (ii) exceed -1

2. Find all values of a for which 1 and 5 lie between the roots of the equation $x^2 + 2(a-5)x + 3 = 0$

3. For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$ lies between 1 and 2.

4. Find all values of 'a' for which the equation $2x^2 - 2(2a+1)x + a(a+1) = 0$ has two roots one of which is greater than and the other is smaller than 'a'

5. Find the values of parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a-1)x + a + 5 = 0$ are:

(i) positive

(ii) negative

(iii) such that one root is greater than 3, and the other is smaller than 3.

(iv) greater than 3

(v) smaller than 3

(vi) such that exactly one root lies in the interval (1, 3)

(vii) such that both the roots lies in the interval (1, 3)

(viii) such that one root is greater than 3 and the other root is smaller than 1.

6. For what real values of m both the roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?

7. Find the values of 'a' for which the inequation $x^2 + ax + a^2 + 6a < 0$ is satisfied for all $x \in (1, 2)$.

8. If $x^2 - (a+1)x + a^2 + a - 8 = 0$ has one root greater than 1 and the other root less than 1, find a.

9. For what values of 'a' does the equation $ax^2 - (a+1)x + 3 = 0$ have roots in the interval (1, 2)?

10. Find the real values of the parameter 'a' for which the equation $(2a+1)x^2 - a(x-1) + 2 = 0$ has one root greater than unity and the other less than unity.

11. Find the set of values of k for which the curve $y = f(x)$ cuts, x-axis at atleast one point on the positive side, where

$f(x) = x^2 + 2(k-1)x + k + 5, \quad k \in R.$

12. Find all values of parameter 'a' for which all solutions of the inequation $ax^2 - 2(a^2-3)x - 12a \ge 0$ are the solutions of the inequation $x^2 - 49 \ge 0$

13. Determine all values of 'a' for which the equation $2\sec^4 x - 2(2a-1)\tan^2 x - (2a-1) = 0$ possesses solutions, and find these solutions.

14. Find all values of 'a' for which at least one root of $5x^2 + (a+1)x + a = 0$ lie in the interval $1 < x < 3$.

ANSWERS

1. (i) $m \in \left[-\sqrt{\frac{40}{7}}, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{-1+\sqrt{13}}{2}, \sqrt{\frac{40}{7}}\right]$

(ii) $m \in \left[-\sqrt{\frac{40}{7}}, \frac{1-\sqrt{13}}{2}\right) \cup \left(\frac{1+\sqrt{13}}{2}, \sqrt{\frac{40}{7}}\right]$

2. $a \in (-\infty, 11/5)$ **3.** $(-1, 0)$

4. $a \in (-\infty, -1) \cup (0, \infty)$

5. (i) $a \in (-5, -1]$ (ii) $a \in [4, \infty)$

(iii) $a \in (-\infty, -8/7)$ (iv) No value of a

(v) $a \in (-8/7, -1] \cup [4, \infty)$ (vi) $a \in (-4/3, -8/7]$

(vii) $a \in (-8/7, -1]$ (viii) $a \in (-\infty, -4/3)$.

6. $m \in (11/9, \infty)$ **7.** $a \in \left(\frac{-7-\sqrt{45}}{2}, -4+2\sqrt{3}\right)$

8. $a \in (-2\sqrt{2}, 2\sqrt{2})$ 9. No real value 12. $a \in (0, 7/6]$ 13. $a = \frac{3}{2}, x = 0.$

10. $a \in \left(\frac{-1}{2}, \frac{1}{2}\right)$ 11. $k \in (-\infty, -1]$

—❑❑❑—

20.15 CONDITION FOR RESOLUTION INTO LINEAR FACTORS

THEOREM *The quadratic function*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

is resolvable into linear rational factors iff

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ i.e. } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

PROOF The equation corresponding to the given quadratic function $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow \quad ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0$$

Solving for x, we get

$$x = \frac{-2(hy + g) \pm \sqrt{4(hy + g)^2 - 4a(by^2 + 2fy + c)}}{2a}, \text{ where } a \neq 0.$$

The given expression will have linear rational factors only if the values of x are rational linear expression in y. This is possible only when the roots of the above equation are real and equal i.e. Disc = 0.

or, $$4(hy + g)^2 - 4a(by^2 + 2fy + c) = 0$$

$$\Rightarrow \quad y^2(h^2 - ab) + 2(hg - af)y + g^2 - ac = 0$$

The roots of this equation are equal, if

Disc = 0

$$\Rightarrow 4(hg - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

which is the required condition.

20.16 ALGEBRAIC INTERPRETATION OF ROLLE'S THEOREM

Let $f(x)$ be a polynomial having α and β as its roots such that $\alpha < \beta$. Then, $f(\alpha) = f(\beta) = 0$. Also, a polynomial function is everywhere continuous and differentiable. Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem. Consequently, there exists $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$ i.e. $f'(x) = 0$ at $x = \gamma$. In other words $x = \gamma$ is a root of $f'(x) = 0$. Thus, algebraically Rolle's theorem can be interpreted as follows :

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *For what values of m can the expression*

$$2x^2 + mxy + 3y^2 - 5y - 2$$

be expressed as the product of two linear factors ?

SOLUTION Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have :

$$a = 2, h = m/2, b = 3, c = -2, f = -5/2, g = 0$$

The given expression is resolvable into linear factors, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow -12 - \frac{25}{2} + 2\left(\frac{m^2}{4}\right) = 0$$

$$\Rightarrow m^2 = 49 \Rightarrow m = \pm 7.$$

EXAMPLE 2 *If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ can be resolved into rational factors, prove that $a^3 + b^3 + c^3 = 3abc$.*

SOLUTION The given expression is

$$ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$$

$$= z^2\left[a\left(\frac{x}{z}\right)^2 + b\left(\frac{y}{z}\right)^2 + c + 2a\left(\frac{y}{z}\right) + 2b\left(\frac{x}{z}\right) + 2c\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)\right]$$

$$= z^2[aX^2 + bY^2 + 2cXY + 2bX + 2aY + c] \quad \ldots(i)$$

where $X = \frac{x}{z}$ and $Y = \frac{y}{z}$.

The given expression can be resolved into rational factors if the expression within brackets in (i) is expressible into rational facors, the condition for which is

$$abc + 2abc - a \cdot a^2 - b \cdot b^2 - c \cdot c^2 = 0$$

$$\Rightarrow a^3 + b^3 + c^3 = 3abc$$

EXAMPLE 3 *If $a, b, c \in R$ such that $2a + 3b + 6c = 0$, show that the quadratic equatoin $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.*

SOLUTION Consider the polynomial

$$f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx.$$

We have,

$$f(0) = 0$$

$$\text{and, } f(1) = \frac{a}{3} + \frac{b}{2} + c$$

$= \frac{2a+3b+6c}{6} = 0$ $[\because 2a+3b+6c=0]$

So, 0 and 1 are two roots of $f(x) = 0$. Therefore, $f'(x) = 0$ i.e. $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

EXAMPLE 4 *If $a, b, c \in R$ and $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $[-1, 1]$.*

SOLUTION Consider $f(x) = ax^3 + bx^2 + cx$.

We have,

$$f(0) = 0 \text{ and } f(1) = a + b + c = 0 \quad [\because a + b + c = 0]$$

So, 0 and 1 are roots of $f(x) = 0$. Therefore, $f'(x) = 0$ i.e. $3ax^2 + 2bx + c = 0$ has at least one real root between 0 and 1. Also, $(0, 1) \in [-1, 1]$.

Hence, $3ax^2 + 2bx + c = 0$ has at least one real root in $[-1, 1]$.

EXAMPLE 5 *Suppose n is an odd positive integer and $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + a_n = 0$, where $a_0, a_1, \ldots, a_n$ are real numbers. Show that the equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots, + a_n = 0$ has at least one root lying in (0, 1).*

SOLUTION Let us define

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \frac{a_2}{n-1} x^{n-1} + \ldots + \frac{a_{n-1}}{2} x^2 + a_n x$$

Clearly, $f(x)$ is continuous and differentiable on R. So, it is continuous on [0, 1] and differentiable on (0, 1). Also

$$f(0) = 0 \text{ and } f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0 \quad \text{[Given]}$$

Therefore, all the three conditions in the hypothesis of Rolle's theorem are satisfied. Thus, there exists at least one $\alpha \in (0, 1)$ such that $f'(x) = 0$

But, $f'(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$

$\therefore \quad f'(\alpha) = 0$

$\Rightarrow \alpha$ is a root of $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$

Hence, $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$ has at least one root in (0, 1).

20.17 SOME APPLICATIONS OF GRAPHS TO FIND THE ROOTS OF EQUATIONS

In chapter 2, we have learnt how to draw the graphs of functions. In this section, we shall discuss some examples to find the roots of equations with the help of graphs.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the equation $x^3 - [x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x.*

SOLUTION We have,

$$x^3 - [x] = 3 \quad \ldots(i)$$

$$\Rightarrow \quad x^3 - 3 = [x]$$

Let $f(x) = x^3 - 3$ and $g(x) = [x]$

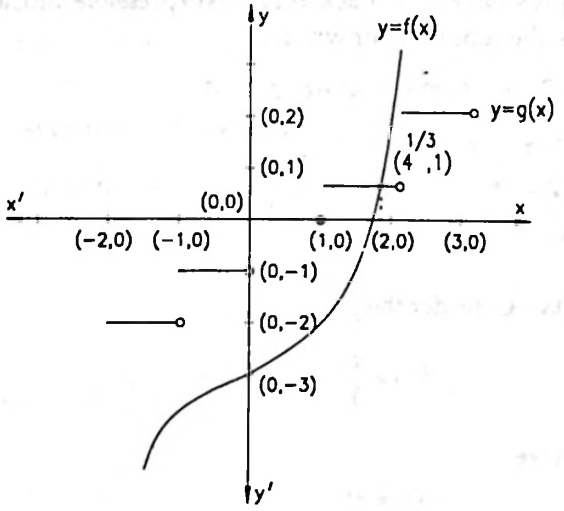


Fig. 20.32

Solving the given equations means finding the x-coordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$.

The graphs of these two curves are shown in Fig. 20.28. Clearly, the x-coordinate of the point of intersection of the two curves lies between (1, 0) and (2, 0).

If $1 < x < 2$, we have $g(x) = 1$ and $f(x) = x^3 - 3$.

At the point of intersection of the two curves, we have

$$x^3 - 3 = 1 \Rightarrow x = 4^{1/3}.$$

Hence, $x = 4^{1/3}$ is the solution of the equation $x^3 - [x] = 3$.

EXAMPLE 2 *Solve the equation $x^3 - 3x - k = 0$ for different values of k.*

SOLUTION We have,

$$x^3 - 3x - k = 0$$

$$\Rightarrow x^3 - 3x = k$$

Let $f(x) = x^3 - 3x$ and $g(x) = k$.

The graphs of $y = f(x)$ and $y = k$ are shonw in Fig. 20.33.

Cleraly, solutions of $x^3 - 3x - k = 0$ are the x-coordinates of the points of intersection of the curves $y = x^3 - 3x$ and $y = k$.

We observe that $x=\pm 1$ are extreme points of $f(x)=x^3-3x$ and -2 and 2 are its local minimum and maximum values.

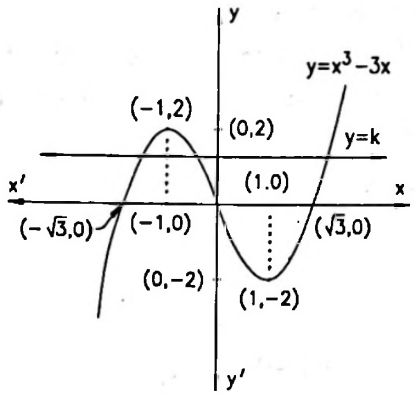


Fig. 20.33

Following cases arise :

CASE I *When,* $-2 \le k \le 2$:

In this case, $y = f(x)$ and $y = g(x)$ intersect at three points. So, all the roots of $x^3 - 3 - k = 0$ are real.

CASE II *When* $k < -2$:

In this case, $y = f(x)$ and $y = g(x)$ intersects at only one point. So, $x^3 - 3x - k = 0$ has only one real root.

CASE III *When* $k > 2$:

In this case, $y = f(x)$ and $y = g(x)$ intersect at only one point whose x-coordinate is greater than $\sqrt{3}$. So, the given equation has one real root greater than $\sqrt{3}$ and other two roots are imaginary.

EXAMPLE 3 *Find all values of the parameter k for which all the roots of the equation* $x^4 + 4x^3 - 8x^3 + k = 0$ *are real.*

SOLUTION We have,

$$x^4 + 4x^3 - 8x^3 + k = 0$$

$$\Rightarrow \quad x^4 + 4x^3 - 8x^2 = -k$$

Let $f(x) = x^4 + 4x^3 - 8x^2$ and $g(x) = -k$

The graph of $f(x)$ and $g(x)$ is shown in Fig. 20.34

Following cases arise:

CASE I *When* $-3 \le -k \le 0$, i.e. $0 \le k \le 3$

In this case, $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four points, so, $x^4 + 4x^3 - 8x^2 + k = 0$ has all real roots.

CASE II *When* $-4 \le -k \le -3$ i.e. $3 < k \le 4$:

In this case, $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at two points, so the given equation has two real roots.

CASE III *When* $-k > 0$ i.e. $k < 0$:

In this case, there are two points of intersection. So, the equation has two real roots.

CASE IV *When* $-k < -4$ i.e. $k > 4$

In this case, two curves donot intersect, so there is no real root.

EXAMPLE 4 *For what real value of k does the equation* $x^3 - 2x^2 - 4x + k = 0$ *have at least one root strictly between 2 and 0?*

SOLUTION We have,

$$x^3 - 2x^2 - 4x + k = 0$$

$$\Rightarrow x^3 - 2x^2 - 4x = -k$$

Let $f(x) = x^3 - 2x^2 - 4x$ and $g(x) = -k$

Then,

$$f(x) = x(x^2 - 2x - 4)$$
$$= x(x - 1 - \sqrt{5})(x - 1 + \sqrt{5})$$

and,

$$f'(x) = 3x^2 - 4x - 4$$
$$= (x - 2)(3x + 2)$$

$$\therefore \quad f'(x) = 0 \Rightarrow x = 2, \ -2/3$$

$$f''(x) = 6x - 4$$

$$\Rightarrow f''(2) = 8 > 0 \text{ and } f''(-3/3) = -8 < 0$$

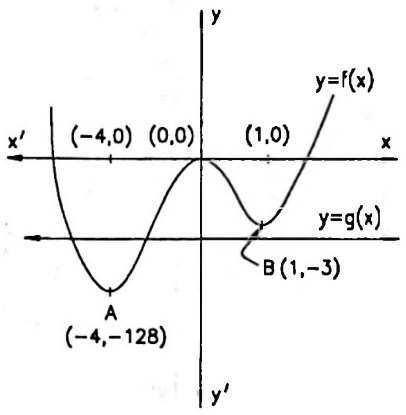


Fig. 20.34

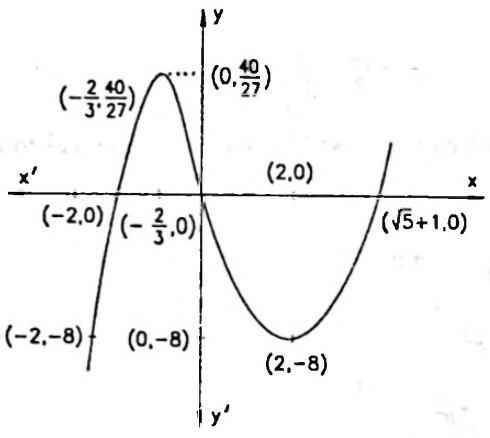


Fig. 20.35

So, $x = 2$ and $x = -\frac{2}{3}$ are points of local minimum and maximum respectively.

Thus, the graph of $y = f(x)$ is shown in Fig. 20.35. The graph of $y = g(x)$ is a straight line parallel to x-axis. The points of intersecion of $y = f(x)$ and $y = g(x)$ determine the roots of the given equation.

Clearly, $y = f(x)$ and $y = g(x)$ i.e. $y = x^3 - 2x^2 - 4x$ and $y = -k$ will intersect at least one point between -2 and 0, if

$$-8 < -k \leq \frac{40}{27} \Rightarrow k \in \left[-\frac{40}{27}, 8\right]$$

Hence, the given equation will have at least one real root between -2 and 0, if $k \in \left[-\frac{40}{27}, 8\right]$

EXAMPLE 5 *If $x^3 + ax + b = 0$ has only one real root, then prove that $4a^3 + 27b^2 \leq 0$.*

SOLUTION Let $y = x^3 + ax + b$.

We observe that $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$. If $x^3 + ax + b = 0$ has only one real root, then $y = x^3 + ax + b$ has only one point of intersection with x-axis. This is possible only when local maximum and local minimum values of $f(x) = x^3 + ax + b$ are positive.

Now,

$$y = x^3 + ax + b$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + a$$

For local maximum and minimum, we must have

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + a = 0 \Rightarrow x^2 = \left(\frac{-a}{3}\right) \Rightarrow x + \pm\left(\frac{-a}{3}\right)^{1/2}$$

Thus, $x = \pm\left(\frac{-a}{3}\right)^{1/2}$ are the points of local maximum.

Local maximum and minimum values of $f(x)$ are given by

$$f\left\{\pm\left(\frac{-a}{3}\right)^{1/2}\right\} = \pm\left(\frac{-a}{3}\right)^{1/2}\left(\frac{-a}{3} + a\right) + b$$

$$= \pm\frac{2a}{3}\left(\frac{-a}{3}\right)^{1/2} + b$$

The given equation will have exactly one real root if local maximum and minimum values of $f(x)$ are positive.

$$\therefore \pm\frac{2a}{3}\left(\frac{-a}{3}\right)^{1/2} + b \geq 0$$

$$\Rightarrow \frac{-4a^3}{27} \geq b^2$$

$$\Rightarrow -4a^3 \geq 27b^2$$

$$\Rightarrow 4a^3 + 27b^2 \leq 0$$

EXAMPLE 6 *Find the values of 'a' for which the inequality $3 - |x - a| > x^2$ is satisfied by at least one negative value of x.*

SOLUTION We have,

$$3 - |x - a| > x^2$$

$$\Rightarrow \quad 3 - x^2 > |x - a|.$$

Thus, we have to find the values of 'a' for which the graph of $y = 3 - x^2$ is above the graph of $y = |x - a|$ for at least one negative value of x.

The graphs of $y = 3 - x^2$ and $y = |x - a|$ are shown in Fig. 20.36.

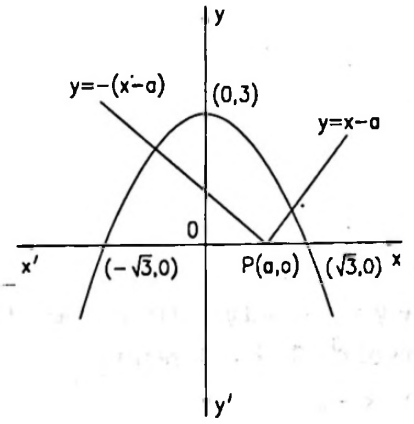


Fig. 20.36

Clearly, the graph of $y = 3 - x^2$ will be above the graph of $y = |x - a|$, if at leat one of the lines given by $y = |x - a|$ intersect $y = 3 - x^2$ in the second quadrant. Consequently, the point $P(a, o)$ will vary between two extreme positions A and B as shown in Fig. 20.37 and 20.38.

To find the coordinates of A, we proceed as follows:

Line $y = x - a$ touches $y = 3 - x^2$. Therefore,

$$\left(\frac{dy}{dx}\right)_{\text{at the point of contact}} = 1$$

$$\Rightarrow \quad -2x = 1$$

$$\Rightarrow \quad x = -\frac{1}{2}$$

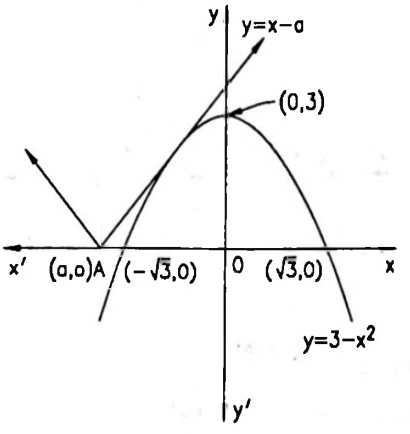


Fig. 20.37

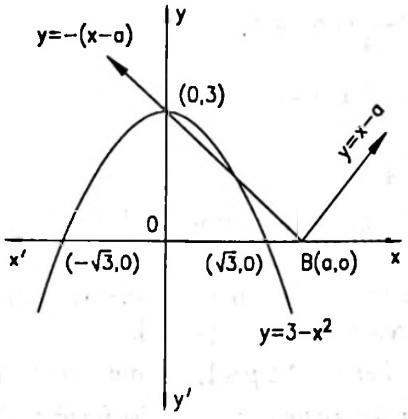


Fig. 20.38

Putting $x = \frac{-1}{2}$ in $y = 3 - x^2$, we get

$$y = 3 - \frac{1}{4} = \frac{11}{4}$$

Since the point of contact lies on $y = x - a$. Therefore,

$$\frac{11}{4} = -\frac{1}{2} - a \Rightarrow a = \frac{-13}{4}$$

So, the coordinates of A are $\left(\frac{-13}{4}, 0\right)$.

To find the coordinates of B we proceed as follows:

Clearly, $y = -(x - a)$, passes through $(0, 3)$. Therefore,

$$3 = a.$$

The vertex $P(a, 0)$ will lie between A and B, if

$$\frac{-13}{4} < a < 3.$$

EXAMPLE 7 *Show that the equation* $x \log_e x = a$ *has*

(i) *no root for* $a < -1/e$

(ii) *one real root for* $a = -1/e$

(iii) *two roots for* $-1/e < a < 0$

(iv) *one root for* $a \geq 0$

SOLUTION Let $f(x) = x \log_e x$ and $g(x) = a$.

The graphs $y = f(x)$ and $y = g(x)$ are shown in Fig. 20.39 We have,

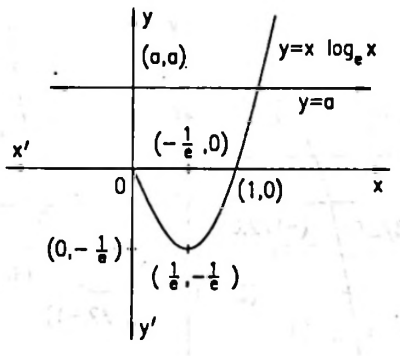


Fig. 20.39

$$f'(x) = 1 + \log_e x$$

$$\therefore f'(x) = 0$$

$$\Rightarrow \quad x = 1/e.$$

So, $1/e$ is the point of local minimum. The local minimum value of $f(x)$ is $(-1/e)$.

(i) Clearly, two curves $y = a$ and $y = x \log_e x$ do not intersect, if $a < -1/e$.

Hence, the given equation has no solution for $a < -1/e$.

(ii) We observe that the curves $y = x \log_e x$ and $y = a$ touch each other at $(1/e, -1/e)$. So, the given equation has only one real root, if $a = -1/e$.

(iii) Clearly, the curves $y = x \log_e x$ and $y = a$ will intersect at two points if $-1/e < a < 0$. So, the given equation will have two real roots for $-1/e < a < 0$.

(iv) The line $y = a$ will cut the curve $y = x \log_e x$ at only one point, if $a \geq 0$. So, the given equation has only one real root, if $a \geq 0$.

EXAMPLE 8 *Find the values of* α *for which the equation*

$$|x|^2(|x|^2 - 2\alpha + 1) = 1 - \alpha^2$$

(i) *has no real roots*

(ii) *exactly two real roots.*

SOLUTION We have, $|x|^2 = x^2$. So, let us consider the following curves.

$$y = x^2(x^2 - 2\alpha + 1) \text{ and } y = 1 - \alpha^2$$

Clearly, roots of the given equation are the points of intersection of these two curves.

We have,

$$y = x^2(x^2 - 2\alpha + 1)$$

$$\Rightarrow y = x^2\{x^2 - 2(\alpha - 1/2)\}$$

So, let us consider the following cases:

CASE I *When* $\alpha \leq \frac{1}{2}$:

In this case, the graph of $y = x^2(x^2 - 2\alpha + 1)$ is as shown in Fig. 20.40. The equation $y = 1 - \alpha^2$ represents a line parallel to

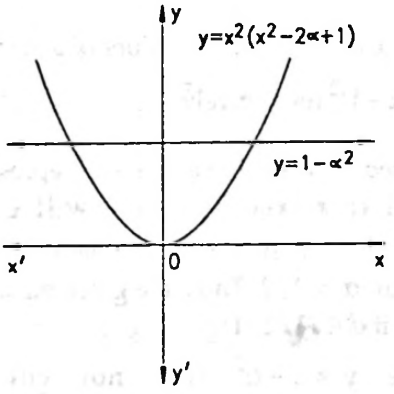


Fig. 20.40

x-axis which will be below x-axis if $1-\alpha^2 < 0$ and above x-axis, if $1-\alpha^2 \geq 0$.

Thus, $y = 1-\alpha^2$ will not intersect with the curve $y = x^2(x^2-2\alpha+1)$, if $1-\alpha^2 < 0$ i.e., $\alpha \in (-\infty,-1)\cup(1,\infty)$. But, $\alpha \leq \frac{1}{2}$. So, the given equation will have no real root, if $\alpha \in (-\infty,-1)$.

The line $y = 1-\alpha^2$ will intersect the curve at two points (coincident or distinct) if $1-\alpha^2 \geq 0$ i.e. $\alpha \in [-1,1]$. But, $\alpha \leq \frac{1}{2}$. Therefore, for $\alpha \in [-1, 1/2]$, the given equation will have two real roots.

CASE II *When* $\alpha > 1/2$

In this case, the graph of $y = x^2(x^2-2\alpha+1)$ is as shown in Fig. 20.41.

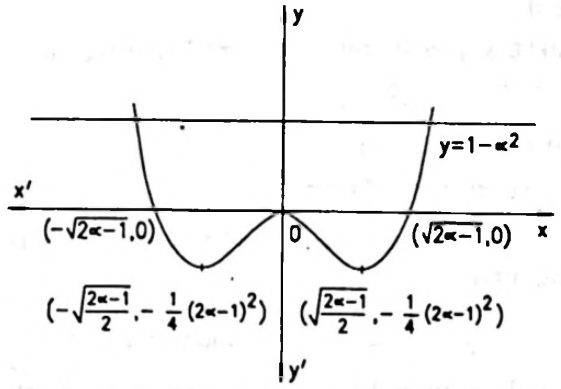


Fig. 20.41

Now,

$$y = x^2(x^2-2\alpha+1)$$

$$\Rightarrow \frac{dy}{dx} = 4x^3-2x(2\alpha-1)$$

$$\therefore \frac{dy}{dx} = 0$$

$$\Rightarrow x = 0, x = \pm\sqrt{\frac{2\alpha-1}{2}}$$

Thus, $x = 0, \pm\sqrt{\frac{2\alpha-1}{2}}$ are critical points of $y = x^2(x^2-2\alpha+1)$. The critical values of y at these points are 0 and $-\frac{1}{4}(2\alpha-1)^2$ respectively.

As discussed in case I that $y = 1-\alpha^2$ represents a straight line parallel to x-axis. This line will cut the curve $y = x^2(x^2-2\alpha+1)$ at two distinct points, if $1-\alpha^2 > 0$ i.e. $\alpha \in (-1, 1)$. But, $\alpha > 1/2$. Thus, the given equation will have two real roots if $\alpha \in (1/2, 1)$.

The line $y = 1-\alpha^2$ will not cut the curve $y = x^2(x^2-2\alpha+1)$, if

$$1-\alpha^2 < -\frac{1}{4}(2\alpha-1)^2.$$

$$\Rightarrow 4-4\alpha^2 < -4\alpha^2+4\alpha-1$$

$$\Rightarrow \alpha > \frac{5}{4}$$

Thus, the given equation will have no real roots, if $\alpha > 5/4$. It follows from the above two cases that the given equation has no real roots, if $\alpha \in (-\infty,-1)\cup(5/4,\infty)$ and it has exactly two real roots, if $\alpha \in [-1, 1]$.

EXAMPLE 9 *Let* $-1 \leq p \leq 1$, *show that the equation* $4x^3-3x-p=0$ *has a unique root in the interval* $[1/2, 1]$ *and identify it.*

[IIT 2001]

SOLUTION We have,

$$4x^3-3x-p = 0 \quad \ldots(i)$$

$$\Rightarrow 4x^3-3x = p$$

Let $f(x) = 4x^3-3x$ and $g(x) = p$.

Clearly, roots of the equation (i) are the x-coordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$.

The graphs of $y = f(x)$ and $y = g(x)$ are shown in Fig. 20.42.

We have,

$$f'(x) = 12x^2-3$$

Clearly, $f'x = 0$ at $x = \pm\frac{1}{2}$.

We observe that $x = \frac{1}{2}$ is the point of local minimum and $x = -\frac{1}{2}$ is the point of local maximum of $f(x)$. Also,

$$f\left(\frac{1}{2}\right) = -1 \text{ and } f\left(-\frac{1}{2}\right) = \frac{1}{2}.$$

We observe that the line $y = p$, where $-1 \leq p \leq 1$, intersects the curve $y = f(x)$ exactly at point $\alpha \in [1/2, 1]$.

Thus, $f(x) = p$ for exactly one value of $\alpha \in [1/2, 1]$

$$\Rightarrow 4x^3-3x = p \text{ for exactly one value of } \alpha \in [1/2, 1]$$

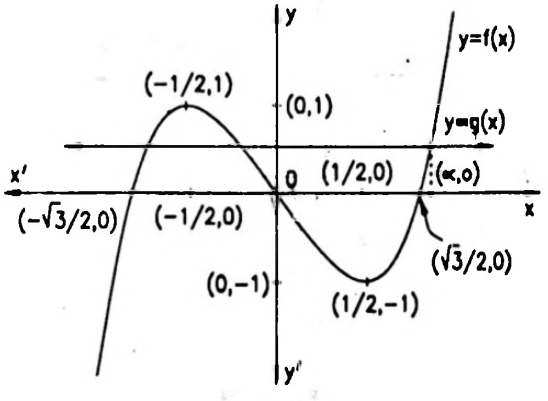


Fig. 20.42

Hence, $4x^3 - 3x - p = 0$ has exactly one root in the interval [1/2, 1].

In order to find the value of root α. Let $\alpha = \cos\theta$. Then,

$f(x) = 0$

$\Rightarrow 4\cos^3\theta - 3\cos\theta - p = 0$

$\Rightarrow \cos 3\theta = p$

$\Rightarrow \theta = \frac{1}{3}\cos^{-1}(p)$

$\Rightarrow \alpha = \cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$.

ALITER *Let* $f(x) = 4x^3 - 3x - p$. *Then,*

$f'(x) = 12x^2 - 3$

$\Rightarrow f'x = 12\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)$

Clearly, $f'(x) > 0$ for $x \in [1/2, 1]$.

$\Rightarrow f(x)$ is an increasing function on [1/2, 1]

$\Rightarrow f(x)$ can have at most one root in [1/2, 1].

Also,

$f\left(\frac{1}{2}\right) = -1 - p$ and $f(1) = 1 - p$

$\therefore f\left(\frac{1}{2}\right)f(1) = -(1-p^2)\ p^2 - 1 \le 0$ $[\because -1 \le p \le 1]$

Since $f(x)$, being a polynomial, continuous on [1/2, 1] and $f\left(\frac{1}{2}\right)f(1) \le 0$. Therefore, by intermediate value theorem, $f(x)$ has at least one root in [1/2, 1].

Hence, $f(x)$ has exactly one root in [1/2, 1].

20.17 SOLUTIONS OF MISCELLANEOUS EQUATIONS

In this section, we shall study various methods of solving equations which are either reducible to quadratic equations or to some simple equations which are easy to slove. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

TYPE I **EQUATIONS OF THE FORM:**

$$(x-a)(x-b)(x-c)(x-d) = A,$$

where $a < b < c < d$ and $b - a = d - c$

This type of equations can be solved by using the following algorithm:

ALGORITHM

STEP I *Obtain the equation and check whether* $a < b < c < d$ *or not. If* $a < b < c < d$ *then check whether* $b - a = d - c$. *If* $b - a = d - c$ *go to step II.*

STEP II *Put* $y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$

or, $x = y + \frac{a+b+c+d}{4}$

in the given equation.

Following examples will illustrate the above procedure:

EXAMPLE 1 *Solve the equation:*

$$(12x-1)(6x-1)(4x-1)(3x-1) = 5$$

SOLUTION The given equation can be written as:

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right) = \frac{5}{12.6.4.3} \quad ...(i)$$

Since $\frac{1}{12} < \frac{1}{6} < \frac{1}{4} < \frac{1}{3}$ and $\frac{1}{6} - \frac{1}{12} = \frac{1}{3} - \frac{1}{4}$. So, we introduce a new variable y given by

$$y = \frac{1}{4}\left[\left(x-\frac{1}{12}\right)+\left(x-\frac{1}{6}\right)+\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{3}\right)\right]$$

$$= x - \frac{5}{24}$$

$$\Rightarrow \quad x = y + \frac{5}{24}$$

Putting $x = y + \frac{5}{24}$ in (i), we get

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right) = \frac{5}{3.4.6.12}$$

$$\Rightarrow \left\{y^2 - \left(\frac{1}{24}\right)^2\right\}\left\{y^2 - \left(\frac{3}{24}\right)^2\right\} = \frac{5}{3.4.6.12}$$

$$\Rightarrow \left(y^2 - \frac{1}{24^2}\right)\left(y^2 - \frac{9}{24^2}\right) = \frac{5}{3.4.6.12}$$

$$\Rightarrow \left(y^2 - \frac{5}{24^2} + \frac{4}{24^2}\right)\left(y^2 - \frac{5}{24^2} - \frac{4}{24^2}\right) = \frac{5}{3.4.6.12}$$

$$\Rightarrow \left(t + \frac{4}{24^2}\right)\left(t - \frac{4}{24^2}\right) = \frac{5}{3.4.6.12}, \text{ where } t = y^2 - \frac{5}{24^2}$$

$$\Rightarrow t^2 - \frac{16}{24^4} = \frac{5}{3.4.6.12}$$

$$\Rightarrow t^2 = \frac{16}{24^4} + \frac{5}{3.4.6.12} = \frac{121}{144^2}$$

$$\Rightarrow t = \frac{11}{144}$$

$$\Rightarrow y^2 - \frac{5}{24^2} = \frac{11}{144}$$

$$\Rightarrow y^2 = \frac{11}{144} + \frac{5}{24^2} = \frac{49}{24^2}$$

$$\Rightarrow y = \frac{\pm 7}{24}$$

$$\Rightarrow x - \frac{5}{24} = \pm\frac{7}{24}$$

$$\Rightarrow x = \frac{5 \pm 7}{24} = \frac{1}{2}, -\frac{1}{12}.$$

TYPE II EQUATIONS OF THE FORM:

$(x-a)(x-b)(x-c)(x-d) = Ax^2$, where $ab = cd$.

In order to solve this type of equations, we use the following algorithm:

ALGORITHM

STEP I *Obtain the equation and express it in the following form:*

$$\{(x-a)(x-b)\}\{(x-c)(x-d)\} = Ax^2$$

or, $\{x^2 - x(a+b) + ab\}\{x^2 - x(c+d) + cd\} = Ax^2$

STEP II *Divide both sides by x^2 to get*

$$\left\{x + \frac{ab}{x} - (a+b)\right\}\left\{x + \frac{cd}{x} - (c+d)\right\} = A.$$

STEP III *Put $x + \frac{ab}{x} = x + \frac{cd}{x} = y$ to obtain*

$$\{y - (a+b)\}\{y - (c+d)\} = A$$

STEP IV *Solve the equation in step III for y. Let the values of y be y_1 and y_2.*

STEP V *Solve the equations $x + \frac{ab}{x} = y_1$ and $x + \frac{ab}{x} = y_2$ to get the values of x.*

Following examples will illustrate the above procedure :

EXAMPLE 2 *Solve the equation:*

$$(x+2)(x+3)(x+8)(x+12) = 4x^2$$

SOLUTION *We have,*

$$(x+2)(x+3)(x+8)(x+12) = 4x^2$$

$$\Rightarrow (x-(-2))(x-(-3))(x-(-8))(x-(-12)) = 4x^2$$

Since $(-2)(-12) = (-3)(-8)$, so we write the above equation in the following form:

$$\{(x+2)(x+12)\}\{(x+3)(x+8)\} = 4x^2$$

$$\Rightarrow (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$$

Clearly, $x = 0$ does not satisfy this equation. So, we divide both sides by x^2 to obtain the following equation:

$$\left(x + \frac{24}{x} + 14\right)\left(x + \frac{24}{x} + 11\right) = 4$$

Let $x + \frac{24}{x} = y$. Then,

$$(y+14)(y+11) = 4$$

$$\Rightarrow y^2 + 25y + 150 = 0$$

$$\Rightarrow (y+15)(y+10) = 0$$

$$\Rightarrow y = -10, -15$$

CASE I *When* $y = -10$

$$y = -10 \Rightarrow x + \frac{24}{x} = -10$$

$$\Rightarrow x^2 + 24 = -10x$$

$$\Rightarrow x^2 + 10x + 24 = 0$$

$$\Rightarrow (x+6)(x+4) = 0$$

$$\Rightarrow x = -4, -6$$

CASE II *When* $y = -15$

$$y = -15 \Rightarrow x + \frac{24}{x} = -15$$

$$\Rightarrow x^2 + 15x + 24 = 0$$

$$\Rightarrow x = \frac{-15 \pm \sqrt{129}}{2}$$

Hence, the solutions of the given equation are:

$$x = -4, -6, \frac{-15 \pm \sqrt{129}}{2}.$$

EXAMPLE 3 *Solve the equation:*

$$(2x^2 - 3x + 1)(2x^2 + 5x + 1) = 9x^2$$

SOLUTION We have,

$$(2x^2 - 3x + 1)(2x^2 + 5x + 1) = 9x^2$$

Clearly, $x = 0$ does not satisfy the above equation. So, we divide both sides by x^2 to get

$$\left(2x + \frac{1}{x} - 3\right)\left(2x + \frac{1}{x} + 5\right) = 9$$

$$\Rightarrow (y-3)(y+5) = 9, \text{where } y = 2x + \frac{1}{x}$$

$$\Rightarrow y^2 + 2y - 24 = 0$$

$$\Rightarrow (y+6)(y-4) = 0$$

$$y = -6, 4$$

CASE I *When* $y = -6$

In this case, we have

$$2x + \frac{1}{x} = -6$$

$$\Rightarrow 2x^2 + 6x + 1 = 0$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36-8}}{4}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{28}}{4}$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{7}}{2}$$

CASE II *When* $y = 4$

In this case, we have

$$2x + \frac{1}{x} = 4$$

$$\Rightarrow 2x^2 - 4x + 1 = 0$$

$$\Rightarrow x = \frac{-4 \pm \sqrt{16-8}}{4} = \frac{-2 \pm \sqrt{2}}{2}$$

Hence, the solutions of the given equation are:

$$\frac{-3 \pm \sqrt{7}}{2}, \frac{2 \pm \sqrt{2}}{2}.$$

TYPE III **EQUATIONS OF THE FORM**

$$(x-a)^4 + (x-b)^4 = A$$

In order to solve the above type of equations, we change the variable by making the substitution:

$$y = \frac{(x-a)+(x-b)}{2}.$$

EXAMPLE 4 *Solve the equation:*

$$(x-1)^4 + (x-5)^4 = 82$$

SOLUTION Let $y = \frac{(x-1)+(x-5)}{2} = x-3$

$$\Rightarrow x = y+3$$

Putting $x = y+3$ in the given equation, we get

$$(y+3-1)^4 + (y+3-5)^4 = 82$$

$$\Rightarrow (y+2)^4 + (y-2)^4 = 82$$

$$\Rightarrow 2(y^4 + 24y^2 + 16) = 82$$

$$\Rightarrow y^4 + 24y^2 - 25 = 0$$

$$\Rightarrow (y^2+25)(y^2-1) = 0$$

$$\Rightarrow y^2 + 25 = 0 \text{ or } y^2 - 1 = 0$$

$$\Rightarrow y = \pm 5i \quad \text{or } y = \pm 1$$

$$\Rightarrow x = 3 \pm 5i \quad \text{or } x = 3 \pm 1 \qquad [\because x = y+3]$$

$$\Rightarrow x = 3 \pm 5i, 4, 2.$$

EXAMPLE 5 *Solve:*

$$(x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 + (x-5)^3 = 0$$

SOLUTION We have,

[A.M. of $(x-1)$ and $(x-5)$] = [A.M. of $(x-2)$ and $(x-4)$]

So, we substitute

$$y = \frac{(x-1)+(x-5)}{2} = x-3$$

$$\Rightarrow x = y+3$$

Substituting $x = y+3$ in the given equation, we obtain

$$(y+2)^3 + (y+1)^3 + y^3 + (y-1)^3 + (y-2)^3 = 0$$

$$\Rightarrow \{(y+2)^3 + (y-2)^3\} + \{(y+1)^3 + (y-1)^3\} + y^3 = 0$$

$$\Rightarrow 5y^3 + 30y = 0$$

$$\Rightarrow 5y(y^2+6) = 0$$

$$\Rightarrow y = 0, y = \pm\sqrt{6}i$$

$$\Rightarrow x-3 = 0 \text{ or, } x-3 = \pm\sqrt{6}i$$

$$\Rightarrow x = 3, 3 \pm \sqrt{6}i$$

20.18 SOLUTIONS OF EXPONENTIAL EQUATIONS

If a, b are positive real numbers such that $a \neq 1$, then an equation of the form $a^x = b$ is known as an exponential equation.

Following examples will illustrate the solutions of the exponential equations.

ILLUSTRATIVE EXAMPLES

TYPE I **EXPONENTIAL EQUATION OF THE FORM:**

$$a^{f(x)} = a^b, a > 0, a \neq 1$$

is equivalent to the equation $f(x) = b$ *which can be solved by the methods studied earlier as discussed in the following examples.*

EXAMPLE 1 *Solve the equation:*

$$5^{x^2+3x} = \frac{1}{25}$$

SOLUTION We have,

$$5^{x^2+3x} = \frac{1}{25}$$

$$\Rightarrow 5^{x^2+3x} = 5^{-2}$$

$$\Rightarrow x^2 + 3x = -2$$

$$\Rightarrow x^2 + 3x + 2 = 0$$

$$\Rightarrow (x+2)(x+1) = 0$$

$$\Rightarrow x = -1, -2.$$

EXAMPLE 2 *Solve the equation* : $2^{x^2} : 2^{2x} = 8 : 1$

SOLUTION We have,

$$\frac{2^{x^2}}{2^{2x}} = \frac{8}{1}.$$

$\Rightarrow 2^{x^2-2x} = 2^3$

$\Rightarrow x^2 - 2x = 3$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = -1, 3$

EXAMPLE 3 *Solve the equation:*

$$2^{|x+1|} - 2^x = |2^x - 1| + 1$$

SOLUTION We have,

$$|2^x - 1| = \begin{cases} 2^x - 1, & \text{for all } x \geq 0 \\ -(2^x - 1), & \text{for all } x < 0 \end{cases}$$

Also, $|x+1| = \begin{cases} x+1, & \text{for all } x \geq -1 \\ -(x+1), & \text{for all } x < -1 \end{cases}$

Thus, the critical points are $x = 0$ and -1. Consequently, we have the following cases:

CASE I *When* $x < -1$

In this case, we have

$|x+1| = -(x+1)$ and $|2^x - 1| = -(2^x - 1)$.

$\therefore 2^{|x+1|} - 2^x = |2^x - 1| + 1$

$\Rightarrow 2^{-(x+1)} - 2^x = -(2^x - 1) + 1$

$\Rightarrow \frac{2^{-x}}{2} - 2^x = -2^x + 2$

$\Rightarrow \frac{2^{-x}}{2} = 2$

$\Rightarrow 2^{-x} = 2^2$

$\Rightarrow -x = 2$

$\Rightarrow x = -2$

CASE II *When* $-1 \leq x < 0$

In this case, we have

$|x+1| = x+1$ and $|2^x - 1| = -(2^x - 1)$

$\therefore 2^{|x+1|} - 2^x = |2^x - 1| + 1$

$\Rightarrow 2^{x+1} - 2^x = -(2^x - 1) + 1$

$\Rightarrow 2 \cdot 2^x - 2^x = -2^x + 2$

$\Rightarrow 2^x = 1$

$\Rightarrow x = 0$

But, x does not satisfy $-1 \leq x < 0$.

So, there is no solution in this case.

CASE III *When* $x \geq 0$

In this case, we have

$|x+1| = x+1$ and $|2^x - 1| = 2^x - 1$.

$\therefore 2^{|x+1|} - 2^x = |2^x - 1| + 1$

$\Rightarrow 2^{x+1} - 2^x = 2^x - 1 + 1$

$\Rightarrow 2^{x+1} = 2 \cdot 2^x$

$\Rightarrow 2^{x+1} = 2^{x+1}$, which is an identity

Thus, in this case all non-negative real values of x are the solutions.

Hence, the solution set of the given equation is

$$\{x \in R \mid x \geq 0\} \cup \{-2\}.$$

EXAMPLE 4 *Solve the equation*

$$2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$$

SOLUTION We have,

$$|x+2| = \begin{cases} x+2, & \text{if } x \geq -2 \\ -(x+2), & \text{if } x < -2 \end{cases}$$

and, $|2^{x+1} - 1| = \begin{cases} (2^{x+1} - 1), & \text{if } x \geq -1 \\ -(2^{x+1} - 1), & \text{if } x < -1 \end{cases}$

Clearly, $x = -2, -1$ are two critical points which divide the number line in three parts. Consequently, we have the following cases:

CASE I *When* $x < -2$

In this case, we have

$|x+2| = -(x+2)$ and $2^{x+1} - 1 = -(2^{x+1} - 1)$.

$\therefore 2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$

$\Rightarrow 2^{-(x+2)} + 2^{x+1} - 1 = 2^{x+1} + 1$

$\Rightarrow 2^{-(x+2)} = 2^1$

$\Rightarrow -(x+2) = 1$

$\Rightarrow x = -3$.

Clearly, it satisfies the condition $x < -2$.

CASE II *When* $-2 \leq x < -1$

In this case, we have

$|x+2| = x+2$ and $|2^{x+1} - 1| = -(2^{x+1} - 1)$

$\therefore 2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$

$\Rightarrow 2^{x+2} + 2^{x+1} - 1 = 2^{x+1} + 1$

$\Rightarrow 2^{x+2} = 2^1$

$\Rightarrow x + 2 = 1$

$\Rightarrow x = -1$

But this value of x does not satisfy the condition $-2 \leq x < -1$.

So, there is no solution in this case.

CASE III *When* $x \geq -1$

In this case, we have

$|x+2| = x+2$ and $|2^{x+1} - 1| = 2^{x+1} - 1$

$\therefore 2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$.

$\Rightarrow 2^{x+2} - (2^{x+1} - 1) = 2^{x+1} + 1$

$\Rightarrow 2^{x+2} = 2 \cdot 2^{x+1}$

$\Rightarrow 2^{x+2} = 2^{x+2}$, which is an identity.

So, in this case every $x \geq -1$ is a solution of the given equation.

Hence, the solution set of the given equation is

$$\{-2\} \cup [-1, \infty)$$

EXAMPLE 5 *Find the set of all solutions of the equation:*

$$2^{|x|} - |2^{x-1} - 1| = 2^{x-1} + 1$$

SOLUTION We have,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

and,

$$|2^{x-1} - 1| = \begin{cases} (2^{x-1} - 1), & \text{if } x \geq 1 \\ -(2^{x-1} - 1), & \text{if } x < 1 \end{cases}$$

Clearly, $x = 0$ and $x = 1$ are two critical points which divide the real line in three parts. Consequently we have the following cases:

CASE I *When* $x < 0$

In this case, we have

$|x| = -x$ and $|2^{x-1} - 1| = -(2^{x-1} - 1)$.

$\therefore 2^{|x|} - |2^{x-1} - 1| = 2^{x-1} + 1$

$\Rightarrow 2^{-x} + 2^{x-1} - 1 = 2^{x-1} + 1$

$\Rightarrow 2^{-x} = 2^1$

$\Rightarrow -x = 1$

$\Rightarrow x = -1$.

CASE II *When* $0 \leq x < 1$

In this case, we have

$|x| = x$ and $|2^{x-1} - 1| = -(2^{x-1} - 1)$

$\therefore 2^{|x|} - |2^{x-1} - 1| = 2^{x-1} + 1$

$\Rightarrow 2^x + (2^{x-1} - 1) = 2^{x-1} + 1$

$\Rightarrow 2^x = 2^1$

$\Rightarrow x = 1$

But $x = 1$ does not satisfy the condition $0 \leq x < 1$. So, there is no solution in this case.

CASE III *When* $x \geq -1$

In this case, we have

$|x| = x$ and $|2^{x-1} - 1| = 2^{x-1} - 1$

$\therefore 2^{|x|} - |2^{x-1} - 1| = 2^{x-1} + 1$.

$\Rightarrow 2^x - (2^{x-1} - 1) = 2^{x-1} + 1$

$\Rightarrow 2^x = 2^{x-1} + 2^{x-1}$

$\Rightarrow 2^x = 2^x$, which is an identity.

So, in this case every value of x satisfying $x \geq 1$ is a solution of the given equation.

Hence, the solution set of the given equation is

$$\{-1\} \cup [1, \infty).$$

TYPE II **EXPONENTIAL EQUATIONS OF THE FORM**

$$\alpha \cdot a^{f(x)} + \beta \cdot b^{f(x)} + \gamma \cdot c^{f(x)} = 0$$

where α, β, γ are non-zero real numbers and a, b, c satisfy the condition $b^2 = ac$ can be reduced to the equation

$$\alpha t^2 + \beta t + \gamma = 0, \text{ where } t = \left(\frac{a}{b}\right)^{f(x)} \qquad [\text{Dividing by } b^x]$$

This equation can be solved by the usual methods. Let its roots be t_1 and t_2. Then, we have

$$\left(\frac{a}{b}\right)^{f(x)} = t_1 \text{ and, } \left(\frac{a}{b}\right)^{f(x)} = t_2$$

These equations can be solved by the methods studied earlier.

EXAMPLE 6 *Solve the equation:*

$$64 \cdot 9^x - 84 \cdot 12^x + 27 \cdot 16^x = 0$$

SOLUTION We have, $9 \times 16 = (12)^2$. So, we divide both sides by 12^x and obtain

$$64\left(\frac{3}{4}\right)^x - 84 + 27 \cdot \left(\frac{4}{3}\right)^x = 0$$

$$\Rightarrow 64t - 84 + \frac{27}{t} = 0, \text{ where } t = \left(\frac{3}{4}\right)^x$$

$\Rightarrow 64t^2 - 84t + 27 = 0$

$\Rightarrow (4t - 3)(16t - 9) = 0$

$\Rightarrow t = \frac{3}{4}, \frac{9}{16}$.

CASE I *When* $t = \frac{3}{4}$

In this case, we have

$$t = \frac{3}{4} \Rightarrow \left(\frac{3}{4}\right)^x = \frac{3}{4} \Rightarrow x = 1$$

CASE II *When* $t = \frac{9}{16}$

In this case, we have

$$t = \frac{9}{16} \Rightarrow \left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2 \Rightarrow x = 2.$$

Hence, the roots of the given equation are : $x = 1, 2$.

EXAMPLE 7 *Solve the equation:*

$$2^{2x+2} - 6^x - 2 \cdot 3^{2x+2} = 0.$$

SOLUTION We have,

$$2^{2x+2} - 6^x - 2 \cdot 3^{2x+2} = 0$$

$\Rightarrow 2^2 . 2^{2x} - 2^x . 3^x - 2 . 3^2 . 3^{2x} = 0$

$\Rightarrow 4 . 2^{2x} - 2^x . 3^x - 18 . 3^{2x} = 0$

$\Rightarrow 4\left(\frac{2}{3}\right)^x - 1 - 18\left(\frac{3}{2}\right)^x = 0$

$\Rightarrow 4t - 1 - \frac{18}{t} = 0$, where $t = \left(\frac{2}{3}\right)^x$

$\Rightarrow 4t^2 - t - 18 = 0$

$\Rightarrow 4t^2 - 9t + 8t - 18 = 0$

$\Rightarrow (4t - 9)(t + 2) = 0$

$\Rightarrow t = \frac{9}{4}, -2$

$\Rightarrow t = \frac{9}{4}$ $[\because t = (2/3)^x > 0 \text{ for all } x]$

$\Rightarrow \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2}$

$\Rightarrow x = -2.$

Hence, $x = -2$ is the solution of the given equation.

EXAMPLE 8 *Solve the equation.*

$$4^{-|x-2|} - 4 . 2^{-|x-2|} + 2 = 0$$

SOLUTION We have,

$$|x-2| = \begin{cases} x-2, & \text{for } x \geq 2 \\ -(x-2), & \text{for } x < 2 \end{cases}.$$

So, we have the following cases:

CASE I *When* $x \geq 2$:

In this case, we have

$|x-2| = x-2.$

$\therefore 4^{-|x-2|} - 4 . 2^{-|x-2|} + 2 = 0$

$\Rightarrow 4^{-(x-2)} - 4 . 2^{-(x-2)} + 2 = 0$

$\Rightarrow t^2 - 4t + 2 = 0$, where $t = 2^{-(x-2)}$.

$\Rightarrow t = 2 \pm \sqrt{2}$

$\Rightarrow t = 2 - \sqrt{2}$ $[\because t = 2^{-(x-2)} < 1 \text{ for all } x \geq 2]$

$\Rightarrow 2^{-(x-2)} = 2 - \sqrt{2}$

$\Rightarrow -(x-2) = \log_2 (2 - \sqrt{2})$

$\Rightarrow x = 2 - \log_2 (2 - \sqrt{2})$

CASE II *When* $x < 2$

In this case, we have

$|x-2| = -(x-2).$

$\therefore 4^{-|x-2|} - 4 . 2^{-|x-2|} + 2 = 0$

$\Rightarrow 4^{x-2} - 4 . 2^{x-2} + 2 = 0$

$\Rightarrow t^2 - 4t + 2 = 0$, where $t = 2^{x-2}$

$\Rightarrow t = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

$\Rightarrow t = 2 - \sqrt{2}$ $[\because t = 2^{x-2} < 1 \text{ for all } x < 2]$

$\Rightarrow 2^{x-2} = 2 - \sqrt{2}$

$\Rightarrow x - 2 = \log_2 (2 - \sqrt{2})$

$\Rightarrow x = 2 + \log_2 (2 - \sqrt{2})$

Hence, lthe solutions of the given equation are:

$x = 2 \pm \log_2 (2 - \sqrt{2}).$

EXAMPLE 9 *Solve:*

$144^{|x|} - 2 . 12^{|x|} + a = 0$, where $a \in R$.

SOLUTION We have,

$144^{|x|} - 2 . 12^{|x|} + a = 0$

$\Rightarrow (12)^{2|x|} - 2 . 12^{|x|} + a = 0$

$\Rightarrow t^2 - 2t + a = 0$, where $t = 12^{|x|}$

$\Rightarrow t = \frac{2 \pm \sqrt{4 - 4a}}{2}$

$\Rightarrow t = 1 \pm \sqrt{1 - a}$

Since t is real. Therefore, we must have $1 - a \geq 0$ i.e. $a \leq 1$.

Also, $t = 12^{|x|} > 1$ for all $x \in R$

$\therefore t = 1 + \sqrt{1 - a},$ for all $x \in R$

$\Rightarrow 12^{|x|} = 1 + \sqrt{1 - a},$ for all $a \leq 1$

$\Rightarrow |x| = \log_{12} (1 + \sqrt{1 - a})$, for all $a \leq 1$

$\Rightarrow x = \pm \log_{12} (1 + \sqrt{1 - a})$, for all $a \leq 1$.

EXAMPLE 10 *Solve* $2^{\sin^2 x} + 5 . 2^{\cos^2 x} = 7.$

SOLUTION We have,

$2^{\sin^2 x} + 5 . 2^{\cos^2 x} = 7$

$\Rightarrow 2^{\sin^2 x} + 5 . 2^{1 - \sin^2 x} = 7$

$\Rightarrow t + 5\left(\frac{2}{t}\right) = 7$, where $t = 2^{\sin^2 x}$

$\Rightarrow t^2 - 7t + 10 = 0$

$\Rightarrow (t - 2)(t - 5) = 0$

$\Rightarrow t = 2, 5$

$\Rightarrow t = 2$ $[\because 0 \leq \sin^2 x \leq 1, \therefore t = 2^{\sin^2 x} \leq 2]$

$\Rightarrow 2^{\sin^2 x} = 2$

$\Rightarrow \sin^2 x = 1$

$\Rightarrow \sin^2 x = \sin^2 \pi/2$

$\Rightarrow x = n\pi \pm \pi/2, n \in Z$

EXAMPLE 11. *Solve :* $4^{x+1.5} + 9^{(x+0.5)} = 10 . 6^x.$

SOLUTION We have,

$$4^{x+1.5}+9^{(x+0.5)}=10.6^x$$

$$\Rightarrow 2^{2(x+1.5)}+3^{2(x+0.5)}=10.3^x.2^x$$

$$\Rightarrow 2^{2x+3}+3^{2x+1}=10.3^x.2^x$$

$$\Rightarrow 8.2^{2x}+3.3^{2x}-10.3^x.2^x=0$$

$$\Rightarrow 8\left(\frac{2}{3}\right)^{2x}-10\left(\frac{2}{3}\right)^{x}+3=0 \qquad \text{[Dividing both sides by } 3^{2x}\text{]}$$

$$\Rightarrow 8t^2-10t+3=0, \text{ where } t=(2/3)^x$$

$$\Rightarrow (4t-3)(2t-1)=0$$

$$\Rightarrow t=\frac{3}{4},\frac{1}{2}$$

$$\Rightarrow \left(\frac{2}{3}\right)^x=\frac{3}{4},\left(\frac{2}{3}\right)^x=\frac{1}{2}.$$

$$\Rightarrow x=\log_{2/3}\left(\frac{3}{4}\right), x=\log_{2/3}\left(\frac{1}{2}\right).$$

EXAMPLE 12 *Solve the equation* $64^{|x|}-2.8^{|x|}+a=0$ *for every real value of the parameter a.*

SOLUTION We have,

$$64^{|x|}-2.8^{|x|}+a=0$$

$$\Rightarrow (8^{|x|})^2-2.8^{|x|}+1=1-a$$

$$\Rightarrow (8^{|x|}-1)^2=1-a$$

Clearly, LHS ≥ 0 for all $x \in R$. So, the given equality will hold good, if

$$1-a\geq 0 \Rightarrow a\leq 1.$$

Now,

$$\left(8^{|x|}-1\right)^2=1-a$$

$$\Rightarrow 8^{|x|}-1=\pm\sqrt{1-a}$$

$$\Rightarrow 8^{|x|}=1\pm\sqrt{1-a}$$

Clearly, LHS of this equation is always greater than or equal to unity. Therefore, for the equation to exist, its RHS must also be greater than or equal to unity.

Clearly, $1-\sqrt{1-a}$ is less than 1 for all $a<1$.

$$\therefore 8^{|x|}=1+\sqrt{1-a} \text{ for all } a\leq 1.$$

$$\Rightarrow |x|=\log_8(1+\sqrt{1-a})$$

$$\Rightarrow x=\pm\log_8(1+\sqrt{1-a})$$

Hence, $x=\pm\log_8(1+\sqrt{1-a})$ for all $a\leq 1$ and there is no solution for $a>1$.

EXAMPLE 13 *Solve the equation:*

$\sqrt{a(2^x-2)+1}=1-2^x$ *for every value of the parameter a.*

SOLUTION Let $y=2^x$. Then,

$$\sqrt{a(2^x-2)+1}=1-2^x$$

$$\Rightarrow \sqrt{a(y-2)+1}=1-y$$

$$\Rightarrow a(y-2)+1=(1-y)^2$$

$$\Rightarrow y^2-2y=a(y-2)$$

$$\Rightarrow (y-2)(y-a)=0$$

$$\Rightarrow y=2 \text{ or } y=a.$$

Now, $y=2 \Rightarrow 2^x=2 \Rightarrow x=1$.

But, $x=1$ is not a solution of the given equation, because for $x=1$, LHS $=1$ and RHS $=-1$.

$$y=a$$

$$\Rightarrow 2^x=a \Rightarrow a>0 \text{ and } x=\log_2 a \qquad [\because 2^x>0 \text{ for all } x\in R]$$

Putting $2^x=a$ in the given equation, we get

$$\sqrt{a(a-2)+1}=1-a$$

$$\Rightarrow \sqrt{(a-1)^2}=1-a$$

$$\Rightarrow |a-1|=1-a$$

$$\Rightarrow a-1\leq 0$$

$$\Rightarrow a\leq 1.$$

Also, $a>0$. Therefore, $a\in(0,1]$.

Hence, $x=\log_2 a$ is the solution of the given equation for all $a\in(0,1]$. For $a\leq 0$ and for $a>1$, the equation has no solution.

TYPE V **EQUATIONS OF THE FORM**

$a^{f(x)}+b^{f(x)}=a^n+b^n$, where $a, b \in R$.

This type of equation has only one solution given by $f(x)=n$.

EXAMPLE 14 *Solve the equation*

$$3^{2x-4}+4^{2x-4}=25.$$

SOLUTION We have,

$$3^{2x-4}+4^{2x-4}=25$$

$$3^{2x-4}+4^{2x-4}=3^2+4^2$$

$$\Rightarrow 2x-4=2$$

$$\Rightarrow x=3.$$

EXAMPLE 15 *Solve:*

$$2^{x-1}+3^{x-1}=35$$

SOLUTION We have,

$$2^{x-1}+3^{x-1}=35$$

$$\Rightarrow 2^{x-1}+3^{x-1}=2^3+3^3$$

$$\Rightarrow x-1=3$$

$$\Rightarrow x=4.$$

EXAMPLE 16 *Solve:* $(\sqrt{2}+1)^x+(\sqrt{2}-1)^x=6$

SOLUTION We have,

$$(\sqrt{2}+1)^x+(\sqrt{2}-1)^x=6$$

$\Rightarrow \left[(\sqrt{2}+1)^2\right]^{x/2} + \left[(\sqrt{2}-1)^2\right]^{x/2} = 6$

$\Rightarrow (\sqrt{3+2\sqrt{2}})^x + (\sqrt{3-2\sqrt{2}})^x = [(\sqrt{3+2\sqrt{2}})^2 + (\sqrt{3-2\sqrt{2}})^2]$

$\Rightarrow x = 2.$

EXERCISE 20.6

1. Solve : $5^{x+1} + 5^{1-x} = 26.$
2. Solve : $16^{\cos^2 x} + 16^{\sin^2 x} = 10.$
3. Solve : $5^{2x} - 24 \cdot 5^x - 25 = 0.$
4. Solve : $3^{x/5} + 3^{\frac{x-10}{10}} = 84.$
5. Solve : (i) $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10.$
   (ii) $(7-4\sqrt{3})^{x^2-4x+3} + (7+4\sqrt{3})^{x^2-4x+3} = 14.$
   (iii) $(\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = 10.$
   (iv) $(15+4\sqrt{14})^{x^2-2|x|} + (15-4\sqrt{14})^{x^2-|x|} = 30.$
6. Solve : $(\sqrt{2}+1)^x + (\sqrt{2}-1)^x = 6^{x/2}.$
7. Solve : $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{2}{2-\sqrt{3}}$
8. Solve : $7 \cdot 3^{x+1} - 5^{x+3} = 3^{x+1} - 5^{x+2}.$

ANSWERS

1. $1, -1$
2. $2n\pi \pm \pi/3, 2n\pi \pm 2\pi/3, 2n\pi \pm \pi/6, 2n\pi \pm 5\pi/6$
3. 2.
4. 20.
5. (i) $\pm 2, \pm\sqrt{2}$ (ii) $2, 2 \pm \sqrt{2}$ (iii) ± 2 (iv) $\pm 1, \pm(1+\sqrt{2})$
6. 2
7. 0, 2
8. $\frac{\log_{10}(50/9)}{\log_{10}(3/5)}$

—❑❑❑—

20.19 SOLUTIONS OF LOGARITHMIC EQUATIONS

In this section, we shall discuss some equations containing logarithms of unkonwn quantities. In order to solve such type of equations, we first find the values of unknown for which the two sides are defined or meaningful. The following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *For $a > 0$, solve the equation:*

$$2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0.$$

SOLUTION Clearly, various terms on the LHS of the given equation are meaningful for $x > 0$. Also, we assume that $a \neq 1$, because for $a = 1$ the LHS of the equation reduces to zero. Thus, for $x > 0$, $a > 0$, and $a \neq 1$, we have

$$2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$$

$$\Rightarrow \frac{2}{\log_a x} + \frac{1}{\log_a ax} + \frac{3}{\log_a a^2 x} = 0$$

$$\Rightarrow \frac{2}{\log_a x} + \frac{1}{\log_a a + \log_a x} + \frac{3}{\log_a a^2 + \log_a x} = 0$$

$$\Rightarrow \frac{2}{y} + \frac{1}{1+y} + \frac{3}{2+y} = 0$$

$$\Rightarrow 2(1+y)(2+y) + y(2+y) + 3y(1+y) = 0$$

$$\Rightarrow 6y^2 + 11y + 4 = 0$$

$$\Rightarrow (3y+4)(2y+1) = 0$$

$$\Rightarrow y = \frac{-1}{2}, -\frac{4}{3}$$

$$\Rightarrow \log_a x = \frac{-1}{2}, \frac{-4}{3} \Rightarrow x = a^{-1/2}, a^{-4/3}.$$

EXAMPLE 2 *Find all real numbers x which satisfy the equation:*

$$2\log_2(\log_2 x) + \log_{1/2}\{\log_2(2\sqrt{2}\, x)\} = 1.$$

SOLUTION Clearly, LHS of the given equation is defined for all $x > 0$.

We have,

$$2\log_2(\log_2 x) + \log_{1/2}\{\log_2(2\sqrt{2}\, x)\} = 1.$$

$$\Rightarrow 2\log_2(\log_2 x) - \log_2\{\log_2(2\sqrt{2}) + \log_2 x\} = 1$$

$$[\because \log_{1/2}\alpha = -\log_2\alpha]$$

$$\Rightarrow 2\log_2(\log_2 x) - \log_2(\log_2 2^{3/2} + \log_2 x) = 1$$

$$\Rightarrow 2\log_2(\log_2 x) - \log_2\left(\frac{3}{2} + \log_2 x\right) = 1$$

$$\Rightarrow 2\log_2 y - \log_2\left(\frac{3}{2} + y\right) = 1, \text{where } y = \log_2 x$$

$\Rightarrow \log_2\left(\frac{y^2}{\frac{3}{2}+y}\right) = 1$

$\Rightarrow \frac{y^2}{\frac{3}{2}+y} = 2^1$

$\Rightarrow \frac{2y^2}{3+2y} = 2$

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y-3)(y+1) = 0$

$\Rightarrow y = 3, -1$

$\Rightarrow \log_2 x = 3, \log_2 x = -1.$

$\Rightarrow x = 2^3, 2^{-1}.$

EXAMPLE 3 *Solve:* $\log_{10}(2^x+1) + x = \log_{10} 6 + x\log_{10} 5$

SOLUTION We have,

$\log_{10}(2^x+1) + x = \log_{10} 6 + x\log_{10} 5$

$\Rightarrow \log_{10}(2^x+1) + x - x\log_{10}\left(\frac{10}{2}\right) = \log_{10} 6$

$\Rightarrow \log_{10}(2^x+1) + x - x(\log_{10} 10 - \log_{10} 2) = \log_{10} 6$

$\Rightarrow \log_{10}(2^x+1) + x\log_{10} 2 = \log_{10} 6$

$\Rightarrow \log_{10}(2^x+1) + \log_{10} 2^x = \log_{10} 6$

$\Rightarrow \log_{10}\{(2^x+1)2^x\} = \log_{10} 6$

$\Rightarrow 2^x(2^x+1) = 6$

$\Rightarrow (2^x)^2 + 2^x - 6 = 0$

$\Rightarrow (2^x+3)(2^x-2) = 0$

$\Rightarrow 2^x - 2 = 0 \qquad [\because 2^x + 3 \neq 0 \text{ as } 2^x > 0 \text{ for all } x]$

$\Rightarrow 2^x = 2$

$\Rightarrow x = 1$

EXAMPLE 4 *Solve:*

$\log_{2x+3}(6x^2+23x+21) = 4 - \log_{3x+7}(4x^2+12x+9)$

SOLUTION The LHS and RHS of the given equation are defined or meaningful if

(i) $2x+3 > 0, 2x+3 \neq 1$

(ii) $6x^2+23x+21 > 0$

(iii) $3x+7 > 0, 3x+7 \neq 1$

and,

(iv) $4x^2+12x+9 > 0$

Now,

(i) $2x+3 > 0, 2x+3 \neq 1$

$\Rightarrow \quad x > \frac{-3}{2}, x \neq -1$

(ii) $6x^2+23x+21 > 0$

$\Rightarrow (2x+3)(3x+7) > 0$

$\Rightarrow x < \frac{-7}{3}$ or $x > \frac{-3}{2}$

(iii) $3x+7 > 0, 3x+7 \neq 1$

$\Rightarrow x > \frac{-7}{3}, x \neq -2$

and,

(iv) $4x^2+12x+9 > 0$

$\Rightarrow (2x+3)^2 > 0$ which is true for all $x \neq \frac{-3}{2}$.

It follows from the above results that the two sides of the given equation are meaningful for all $x > -3/2$, $x \neq -1$.

Now, for all $x > -3/2$, $x \neq -1$, we have,

$\log_{2x+3}(6x^2+23x+21) = 4 - \log_{3x+7}(4x^2+12x+9)$

$\Rightarrow \log_{2x+3}(3x+7)(2x+3) = 4 - \log_{3x+7}(2x+3)^2$

$\Rightarrow \log_{(2x+3)}(3x+7) + \log_{(2x+3)}(2x+3) = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow \log_{(2x+3)}(3x+7) + 1 = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow y + 1 = 4 - \frac{2}{y}$, where $y = \log_{(2x+3)}(3x+7)$

$\Rightarrow y^2 + y = 4y - 2$

$\Rightarrow y^2 - 3y + 2 = 0$

$\Rightarrow (y-2)(y-1) = 0$

$\Rightarrow y = 1, 2$

$\Rightarrow \log_{(2x+3)}(3x+7) = 1, 2$

$\Rightarrow 3x+7 = (2x+3)^1$ or $(3x+7) = (2x+3)^2$

$\Rightarrow x+4 = 0$ or $4x^2+9x+2 = 0$

$\Rightarrow x+4 = 0$ or $(4x+1)(x+2) = 0$

$\Rightarrow x = -4, -\frac{1}{4}, -2.$

$\Rightarrow x = -\frac{1}{4} \qquad \left[\begin{array}{l}\because x \text{ must be greater than } \frac{-3}{2} \\ \text{and should not be equal to } -1\end{array}\right]$

Hence, $x = \frac{-1}{4}$ is the only solution of the given equation.

EXAMPLE 5 *Solve:*

$$\log_{(x^2-1)}(x^3+6) = \log_{(x^2-1)}(2x^2+5x)$$

SOLUTION The given equation is meaninfgul if each of the following conditions hold:

(i) $x^2 - 1 > 0$ and $x^2 - 1 \neq 1$

(ii) $x^3 + 6 > 0$

(iii) $2x^2 + 5x > 0$

Now,

(i) $x^2-1>0$ and $x^2-1\neq 1$

$\Rightarrow x\in(\infty,-1)\cup(1,\infty)$ and $x\neq\pm\sqrt{2}$

(ii) $x^3+6>0\Rightarrow x>(-6)^{1/3}$.

and,

(iii) $2x^2+5x>0\Rightarrow x\in(-\infty,-5/2)\cup(0,\infty)$

It follows from the above conditions that the given equation is meaningful, if

$x\in(1,\infty)-\{\sqrt{2}\}$.

Now, for $x\in(1,\infty)-\{\sqrt{2}\}$, we have

$\log_{(x^2-1)}(x^3+6)=\log_{(x^2-1)}(2x^2+5x)$

$\Rightarrow x^3+6=2x^2+5x$

$\Rightarrow x^3-2x^2-5x+6=0$

$\Rightarrow (x+2)(x^2-4x+3)=0$

$\Rightarrow (x+2)(x-1)(x-3)=0$

$\Rightarrow x=-2,1,3$

Clearly, $x=3$ lies in the set $(1,\infty)-\{\sqrt{2}\}$.

Hence, $x=3$ is the only root of the given equation.

EXAMPLE 6 *Solve:* $\log_3(3^x-8)=(2-x)$

SOLUTION The given equation is meaningful, if

$$3^x-8>0\Rightarrow 3^x>8\Rightarrow x>\log_3 8.$$

For $x>\log_3 8$, We have

$\log_3(3^x-8)=(2-x)$

$\Rightarrow 3^x-8=3^{2-x}$

$\Rightarrow 3^x-8=3^2.3^{-x}$

$\Rightarrow 3^x-8=\dfrac{9}{3^x}$

$\Rightarrow y-8=\dfrac{9}{y}$, where $y=3^x$

$\Rightarrow y^2-8y-9=0$

$\Rightarrow (y-9)(y+1)=0$

$\Rightarrow y=9-1$

$\Rightarrow 3^x=9$ $[\because 3^x<0]$

$\Rightarrow x=2.$

Clearly, $2>\log_3 8$.

Hence, $x=2$ is the solution of the given equation.

EXAMPLE 7 *Solve:*

$$5^{1+\log_4 x}+5^{\log_{0.25}x-1}=\frac{26}{5}$$

SOLUTION The given equation is meaningful for all $x>0$. For $x>0$, we have,

$$5^{1+\log_4 x}+5^{\log_{0.25}x-1}=\frac{26}{5}$$

$\Rightarrow 5^1.5^{\log_4 x}+5^{\log_{(1/4)}x}.5^{-1}=\dfrac{26}{5}$

$\Rightarrow 5.5^{\log_4 x}+5^{-\log_4 x}.\dfrac{1}{5}=\dfrac{26}{5}$

$[\because \log_{1/4}x=\log_{4^{-1}}x=-\log_4 x]$

$\Rightarrow 5t+\dfrac{1}{5t}=\dfrac{26}{5}$, where $t=5^{\log_4 x}$.

$\Rightarrow 25t^2-26t+1=0$

$\Rightarrow (25t-1)(t-1)=0$

$\Rightarrow t=1,\dfrac{1}{25}$

$\Rightarrow t=1,5^{-2}$

$\Rightarrow {}_5\log_4 x=5^0,5^{-2}$

$\Rightarrow \log_4 x=0,-2$

$\Rightarrow x=4^0,4^{-2}$

$\Rightarrow x=1,\dfrac{1}{16}$

EXAMPLE 8 *Solve:* $|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7$

SOLUTION On the LHS of the given equation there is a term containing $\log_x 9$ which is meaningful, if $x>0$ and $x\neq 1$. Also, for $x>0$ and $x\neq 1$, LHS >0. So, RHS must also be positive, which is true for all $x>1$. Thus, the given equation exists, if $x>1$.

For $x>1$, we have,

$|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7$

$\Rightarrow (x-1)^{2\log_3 x-4\log_x 3}=(x-1)^7$ $\left[\because |x-1|=x-1 \text{ for } x>1\right]$

$\Rightarrow 2\log_3 x-4\log_x 3=7$

$\Rightarrow 2y^2-7y-4=0$, where $y=\log_3 x$

$\Rightarrow (2y+1)(y-4)=0$

$\Rightarrow y=-\dfrac{1}{2},4$

$\Rightarrow \log_3 x=-\dfrac{1}{2},4$

$\Rightarrow x=3^{-1/2},3^4$

$\Rightarrow x=\dfrac{1}{\sqrt{3}},3^4$

$\Rightarrow x=3^4$ $\left[\because x=\dfrac{1}{\sqrt{3}}\neq 1\right]$

EXAMPLE 9 *Solve:*

$$\log_{3/4}\{\log_8(x^2+7)\}+\log_{1/2}\{\log_{1/4}(x^2+7)^{-1}\}=-2.$$

SOLUTION We have,

$$\log_{3/4}\{\log_8(x^2+7)\}+\log_{1/2}\{\log_{1/4}(x^2+7)^{-1}\}=-2.$$

$$\Rightarrow \log_{3/4}\{\log_{2^3}(x^2+7)\}+\log_{1/2}\{\log_{2^{-2}}(x^2+7)^{-1}\}=-2$$

$$\Rightarrow \log_{3/4}\left\{\frac{1}{3}\log_2(x^2+7)\right\}+\log_{1/2}\left\{\frac{-1}{-2}\log_2(x^2+7)\right\}=-2$$

$$[\because \log_{a^b} m^n=\frac{n}{b}\log_a m]$$

$$\Rightarrow \log_{3/4}\left(\frac{t}{3}\right)+\log_{1/2}\left(\frac{t}{2}\right)=-2, \text{ where } t=\log_2(x^2+7)$$

$$\Rightarrow \frac{\log_{10}\left(\frac{t}{3}\right)}{\log_{10}\left(\frac{3}{4}\right)}+\frac{\log_{10}\left(\frac{t}{2}\right)}{\log_{10}\left(\frac{1}{2}\right)}=-2$$

$$\Rightarrow \frac{\log_{10}t-\log_{10}3}{\log_{10}3-\log_{10}4}+\frac{\log_{10}t-\log_{10}2}{-\log_{10}2}=-2$$

$$\Rightarrow (\log_{10}t)\left\{\frac{1}{\log_{10}3-2\log_{10}2}-\frac{1}{\log_{10}2}\right\}$$

$$=\frac{\log_{10}3}{\log_{10}3-2\log_{10}2}-3.$$

$$\Rightarrow (\log_{10}t)\{3\log_{10}2-\log_{10}3\}=(6\log_{10}2-2\log_{10}3)\log_{10}2$$

$$\Rightarrow \log_{10}t=2\log_{10}2=\log_{10}4$$

$$\Rightarrow t=4$$

$$\Rightarrow \log_2(x^2+7)=4$$

$$\Rightarrow x^2+7=2^4$$

$$\Rightarrow x^2=9$$

$$\Rightarrow x=\pm 3.$$

EXAMPLE 10 *Solve:*

$$x^{\left\{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right\}}=\sqrt{2}.$$

SOLUTION. Clearly, the given equation is meaningful for all $x>0$ and $x\neq 1$. For such values of x, we have

$$x^{\left\{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right\}}=\sqrt{2}.$$

$$\Rightarrow \frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}=\log_x\sqrt{2}$$

$$\Rightarrow \frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}=\frac{1}{2}\log_x 2$$

$$\Rightarrow \frac{3}{4}y^2+y-\frac{5}{4}=\frac{1}{2y}, \text{ where } y=\log_2 x$$

$$\Rightarrow 3y^3+4y^2-5y=2$$

$$\Rightarrow 3y^3+4y^2-5y-2=0$$

$$\Rightarrow (y-1)(y+2)(3y+1)=0$$

$$\Rightarrow y=-\frac{1}{3},-2,1$$

$$\Rightarrow \log_2 x=-\frac{1}{3},-2,1$$

$$\Rightarrow x=2^{-1/3},2^{-2},2$$

$$\Rightarrow x=2,\frac{1}{4},\frac{1}{2^{1/3}}.$$

EXAMPLE 11 *Solve:*

$$\frac{3}{2}\cdot\{\log_{1/4}(x+2)^2\}-3=\log_{1/4}(4-x)^3+\log_{1/4}(x+6)^3.$$

SOLUTION Clearly, RHS of the given equation is meaningful when $4-x>0$ and $x+6>0$ i.e. when $-6<x<4$.

Now,

$$\frac{3}{2}\cdot\log_{1/4}(x+2)^2-3=\log_{1/4}(4-x)^3+\log_{1/4}(x+6)^3.$$

$$\Rightarrow \frac{3}{2}\times 2\log_{1/4}|x+2|-3=3\log_{1/4}(4-x)+3\log_{1/4}(x+6)$$

$$\Rightarrow 3\log_{2^{-2}}|x+2|-3=3\log_{2^{-2}}(4-x)+3\log_{2^{-2}}(x+6)$$

$$\Rightarrow -\frac{3}{2}\log_2|x+2|-3=-\frac{3}{2}\log_2(4-x)-\frac{3}{2}\log_2(x+6)$$

$$\Rightarrow \log_2|x+2|+2=\log_2(4-x)+\log_2(x+6)$$

$$\Rightarrow \log_2|x+2|+\log_2 4=\log_2(4-x)+\log_2(x+6)$$

$$\Rightarrow \log\{4|x+2|\}=\log_2\{(4-x)(x+6)\}$$

$$\Rightarrow 4|x+2|=(4-x)(x+6) \quad \ldots(i)$$

Now, two cases arise:

CASE I *When* $-6<x<-2$:

In this case, we have

$$|x+2|=-(x+2).$$

Therefore, equation reduces to

$$-4(x+2)=(4-x)(x+6)$$

$$\Rightarrow x^2-2x-32=0$$

$$\Rightarrow x=1\pm\sqrt{33}$$

$$\Rightarrow x=1-\sqrt{33} \quad [\because 1+\sqrt{33} \text{ does not lie between } -6 \text{ and } -2]$$

CASE II *When* $-2\leq x<4$

In this case, we have

$$|x+2|=x+2.$$

$$\therefore 4|x+2|=(4-x)(x+6)$$

$$\Rightarrow 4(x+2)=(4-x)(x+6)$$

$$\Rightarrow x^2+6x-16=0$$

$$\Rightarrow (x+8)(x-2)=0$$

$$\Rightarrow x=2,-8.$$

$$\Rightarrow x=2 \quad [\because -8\notin[-2,4)]$$

Hence, the roots of the given equations are $x = 1 - \sqrt{33}$ and $x = 2$.

EXAMPLE 12 *Solve:*

$$\frac{6}{5}a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} - 3^{\log_{10}\left(\frac{x}{10}\right)} = 9^{\log_{100} x + \log_4 2}$$

SOLUTION The given equation exists for all $x > 0$.

Also,

$$\log_a x \cdot \log_{10} a = \log_{10} x, \quad \log_{10}\left(\frac{x}{10}\right) = \log_{10} x - 1$$

and,

$$\log_{100} x + \log_4 2 = \log_{10^2} x + \log_{2^2} 2$$

$$= \frac{1}{2}\log_{10} x + \frac{1}{2}.$$

So, the given equation reduces to

$$\frac{6}{5} \cdot a^{\log_{10} x \cdot \log_a 5} - 3^{\log_{10} x - 1} = 9^{1/2(\log_{10} x + 1)}$$

$$\Rightarrow \frac{6}{5}a^{\log_a (5^{(\log_{10} x)})} - 3^{\log_{10} x} \cdot 3^{-1} = 3^{2\left(\frac{1}{2}(\log_{10} x + 1)\right)}$$

$$\Rightarrow \frac{6}{5} \cdot 5^{(\log_{10} x)} - \frac{3^{\log_{10} x}}{3} = 3^{(\log_{10} x + 1)}$$

$$\Rightarrow \frac{6}{5} \cdot 5^{\log_{10} x} - \frac{3^{\log_{10} x}}{3} = 3^{\log_{10} x} \cdot 3$$

$$\Rightarrow \frac{2}{5} \cdot 5^{\log_{10} x} - \frac{3^{\log_{10} x}}{9} = 3^{\log_{10} x}$$

$$\Rightarrow \frac{2}{5} \cdot 5^{\log_{10} x} = \frac{10}{9} \cdot (3^{\log_{10} x})$$

$$\Rightarrow \frac{1}{5^2} \cdot (5^{\log_{10} x}) = \frac{1}{3^2} \cdot (3^{\log_{10} x})$$

$$\Rightarrow 5^{\log_{10} x - 2} = 3^{\log_{10} x - 2}$$

$$\Rightarrow \log_{10} x - 2 = 0 \Rightarrow \log_{10} x = 2 \Rightarrow x = 10^2 = 100.$$

EXAMPLE 13 *Find all values of the parameter 'a' for which the inequality*

$$1 + \log_5 (x^2 + 1) \geq \log_5 (ax^2 + 4x + a)$$

is valid for all real x.

SOLUTION Clearly, RHS of the given inequality is meaningful, if

$$ax^2 + 4x + a > 0 \text{ for all } x \in R$$

$$\Rightarrow 16 - 4a^2 < 0 \text{ and } a > 0$$

$$\Rightarrow 4 - a^2 < 0 \quad \text{and } a > 0$$

$$\Rightarrow a^2 - 4 > 0 \quad \text{and } a > 0.$$

$$\Rightarrow a > 2 \qquad \ldots(i)$$

Now,

$$1 + \log_5 (x^2 + 1) \geq \log_5 (ax^2 + 4x + a) \qquad \text{for all } x \in R$$

$$\Rightarrow \log_5 5 + \log_5 (x^2 + 1) \geq \log_5 (ax^2 + 4x + a) \text{ for all } x \in R$$

$$\Rightarrow \log_5 [5(x^2 + 1)] \geq \log_5 (ax^2 + 4x + a) \qquad \text{for all } x \in R$$

$$\Rightarrow 5(x^2 + 1) \geq (ax^2 + 4x + a) \qquad \text{for all } x \in R$$

$$\Rightarrow (5 - a)x^2 - 4x + (5 - a) \geq 0 \qquad \text{for all } x \in R$$

$$\Rightarrow 5 - a > 0 \text{ and } 16 - 4(5 - a)^2 \leq 0$$

$$\Rightarrow a < 5 \quad \text{and } 2^2 - (5 - a)^2 \leq 0$$

$$\Rightarrow a < 5 \quad \text{and } (7 - a)(-3 + a) \leq 0$$

$$\Rightarrow a < 5 \quad \text{and } (a - 3)(a - 7) \geq 0$$

$$\Rightarrow a < 5 \quad \text{and } a \in (-\infty, 3] \cup [7, \infty)$$

$$\Rightarrow a \in (-\infty 3] \qquad \ldots(ii)$$

From (i) and (ii), we obtain that $a \in (2, 3]$.

EXAMPLE 14 *If* $x = \frac{4}{9}$ *satisfies the inequation*

$$\log_a (x^2 - x + 2) > \log_a (-x^2 + 2x + 3)$$

then find its all real solutions.

SOLUTION The given inequation is defined if

(i) $a > 0, a \neq 1$, (ii) $x^2 - x + 2 > 0$ and (iii) $-x^2 + 2x + 3 > 0$

Clearly, $x^2 - x + 2 > 0$ for all x [$\because$ Disc < 0]

Now,

$$-x^2 + 2x + 3 > 0$$

$$\Rightarrow x^2 - 2x - 3 < 0$$

$$\Rightarrow (x - 3)(x + 1) < 0$$

$$\Rightarrow -1 < x < 3. \qquad \ldots(i)$$

Thus, the given inequation is meaningful if $a > 0, a \neq 1$ and $x \in (-1, 3)$.

It is given that $x = \frac{4}{9}$ is a solution of the given inequation.

Therefore,

$$\log_a\left(\frac{16}{81} - \frac{4}{9} + 2\right) > \log_a\left(\frac{-16}{81} + \frac{8}{9} + 3\right)$$

$$\Rightarrow \log_a\left(\frac{142}{81}\right) > \log_a\left(\frac{299}{81}\right)$$

$$\Rightarrow 0 < a < 1 \qquad \left[\because \frac{142}{81} < \frac{299}{81}\right]$$

Now,

$$\log_a (x^2 - x + 2) > \log_a (-x^2 + 2x + 3)$$

$$\Rightarrow x^2 - x + 2 < -x^2 + 2x + 3 \qquad [\because 0 < a < 1]$$

$\Rightarrow 2x^2 - 3x - 1 < 0$

$\Rightarrow \quad \frac{3-\sqrt{17}}{4} < x < \frac{3+\sqrt{17}}{4}$ ...(ii)

From (i) and (ii), we obtain

$$x \in \left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right)$$

Hence, the solution set of the given inequation is

$$\left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right).$$

EXAMPLE 15 *Solve:*

$$\log_{3x+5}(9x^2+8x+8) > 2$$

SOLUTION The given inequation is meaningful, if

(i) $3x+5 > 0, 3x+5 \neq 1$

and,

(ii) $9x^2+8x+8 > 0$

Now,

$3x+5 > 0$ and $3x+5 \neq 1 \Rightarrow x > -5/3$ and $x \neq -4/3$

Clearly, $9x^2+8x+8 > 0$ for all x [$\because$ Disc < 0]

Thus, the given inequation is defined for all $x \in (-5/3, \infty) - \left\{\frac{-4}{3}\right\}$.

Now, two cases arise :

CASE I *When* $0 < 3x+5 < 1$ *i.e.* $x \in \left(\frac{-5}{3}, \frac{-4}{3}\right)$

In this case, logarithm to the base $3x+5$ will be a decreasing function. Therefore,

$$\log_{3x+5}(9x^2+8x+8) > 2$$

$\Rightarrow (9x^2+8x+8) < (3x+5)^2$

$\Rightarrow 22x+17 > 0$

$\Rightarrow x > \frac{-17}{22}$

$\Rightarrow x \in \left(\frac{-17}{22}, \infty\right)$

But, $\left(\frac{-5}{3}, \frac{-4}{3}\right) \cap \left(\frac{-17}{22}, \infty\right) = \phi$

So, there is no solution in this case.

CASE II *When* $3x+5 > 1$ *i.e.* $x > \frac{-4}{3}$

In this case, the base of the logarithm is greater than 1. So, it is an increasing function. Therefore,

$$\log_{3x+5}(9x^2+8x+8) > 2$$

$\Rightarrow (9x^2+8x+8) > (3x+5)^2$

$\Rightarrow 22x+17 < 0$

$\Rightarrow x < \frac{-17}{22}$

$\Rightarrow x \in \left(-\infty, \frac{-17}{22}\right)$

But x must also be greater than $\frac{-4}{3}$. Therefore,

$$x \in \left(\frac{-4}{3}, \frac{-17}{22}\right)$$

Hence, the required solution is $\left(\frac{-4}{3}, \frac{-17}{22}\right)$

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20.20 SOLUTIONS OF SIMULTANEOUS EQUATIONS

In this section, we shall discuss solutions of simultaneous equations through following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the equations:*

$$\log_3 x + \log_3 y = 2 + \log_3 2$$

$$\log_{27}(x+y) = \frac{2}{3}$$

SOLUTION We have,

$\log_3 x + \log_3 y = 2 + \log_3 2$

$\Rightarrow \log_3(xy) = \log_3 3^2 + \log_3 2$

$\Rightarrow \log_3(xy) = \log_3(18)$

$\Rightarrow xy = 18$ ...(i)

Also, we have

$$\log_{27}(x+y) = \frac{2}{3}$$

$\Rightarrow x+y = (27)^{2/3}$

$\Rightarrow x+y = 9$ ...(ii)

The equation having x and y as its root is

$t^2 - (x+y)t + xy = 0$

$\Rightarrow t^2 - 9t + 18 = 0$

$\Rightarrow (t-3)(t-6) = 0$

$\Rightarrow t = 3, 6.$

Hence, $x = 3$ and $y = 6$ or, $x = 6$ and $y = 3$.

EXAMPLE 2 *Solve the simultaneous equations:*

$$\log_2 x + \log_4 y + \log_4 z = 2$$

$$\log_3 y + \log_9 z + \log_9 x = 2$$

$$\log_4 z + \log_{16} x + \log_{16} y = 2.$$

SOLUTION Clearly, given equations are defined for $x > 0, y > 0, z > 0$. We have,

$$\log_2 x + \log_4 y + \log_4 z = 2$$

$$\Rightarrow \log_2 x + \frac{1}{2}\log_2 y + \frac{1}{2}\log_2 z = 2$$

$$\Rightarrow 2\log_2 x + \log_2 y + \log_2 z = 4$$

$$\Rightarrow \log_2 (x^2yz) = 4$$

$$\Rightarrow x^2yz = 2^4 = 16 \quad \ldots(i)$$

$$\log_3 y + \log_9 z + \log_9 x = 2$$

$$\Rightarrow \log_3 y + \frac{1}{2}\log_3 z + \frac{1}{2}\log_3 x = 2$$

$$\Rightarrow 2\log_2 y + \log_3 z + \log_3 x = 4$$

$$\Rightarrow \log_3 (xy^2z) = 4$$

$$\Rightarrow xy^2z = 3^4 = 81 \quad \ldots(ii)$$

and,

$$\log_4 z + \log_{16} x + \log_{16} y = 2$$

$$\Rightarrow \frac{1}{2}\log_2 z + \frac{1}{4}\log_2 x + \frac{1}{4}\log_2 y = 2$$

$$\Rightarrow 2\log_2 z + \log_2 x + \log_2 y = 8$$

$$\Rightarrow \log_2 (xyz^2) = 8$$

$$\Rightarrow xyz^2 = 2^8 \quad \ldots(iii)$$

Multiplying (i), (ii) and (iii), we get

$$x^4y^4z^4 = 16 \times 81 \times 2^8$$

$$\Rightarrow (xyz)^4 = (2 \times 3 \times 2^2)^4$$

$$\Rightarrow xyz = 2 \times 3 \times 2^2 \quad [\because x > 0, y > 0, z > 0]$$

$$\Rightarrow xyz = 24 \quad \ldots(iv)$$

Dividing (i) by (iv), we get $x = \frac{2}{3}$

Similarly, dividing (ii) and (iii) by (iv), we get

$$y = \frac{81}{24} = \frac{27}{8} \text{ and } z = \frac{32}{3}.$$

EXAMPLE 3 *Solve the equations:*

$$5(\log_y x + \log_x y) = 26, xy = 64$$

SOLUTION Clearly, $\log_y x$ and $\log_x y$ are defined for x, $y > 0$ and $x, y \neq 1$.

Now,

$$5(\log_y x + \log_x y) = 26$$

$$\Rightarrow 5\left(t + \frac{1}{t}\right) = 26, \text{ where } t = \log_y x$$

$$\Rightarrow 5t^2 - 26t + 5 = 0$$

$$\Rightarrow (5t - 1)(t - 5) = 0$$

$$\Rightarrow t = 5, 5^{-1}.$$

$$\Rightarrow \log_y x = 5 \text{ or, } \log_y x = 5^{-1}$$

$$\Rightarrow x = y^5 \text{ or } x = y^{1/5}.$$

CASE I *When* $x = y^5$

The second equation is

$$xy = 64.$$

$$\Rightarrow y^6 = 64$$

$$\Rightarrow y^6 = 2^6$$

$$\Rightarrow y = 2$$

$$\therefore x = y^5 \Rightarrow x = 32.$$

CASE II *When* $x = y^{1/5}$

The second equation is

$$xy = 64$$

$$\Rightarrow y^{1/5}y = 64$$

$$\Rightarrow y^{6/5} = 64$$

$$\Rightarrow y = (64)^{5/6} = (2^6)^{5/6} = 2^5 = 32$$

$$\therefore x = y^{1/5} \Rightarrow x = (32)1/5 = 2.$$

Hence, the solutions of the given system of equations are:

$x = 32, y = 2$ and $x = 2, y = 32$.

EXAMPLE 4 *Solve the equations:*

$$3^x \cdot 5^y = 75 \text{ and } 3^y \cdot 5^x = 45$$

SOLUTION We have

$$3^x \cdot 5^y = 75 \quad \ldots(i)$$

and, $$3^y \cdot 5^x = 45 \quad \ldots(ii)$$

Multiplying (i) and (ii), we get

$$(3^x \cdot 5^y)(3^y \cdot 5^x) = 75 \times 45$$

$$\Rightarrow (3 \times 5)^x \cdot (3 \times 5)^y = 15^3$$

$$15^x \cdot 15^y = 15^3$$

$$\Rightarrow 15^{x+y} = 15^3$$

$$\Rightarrow x + y = 3 \quad \ldots(iii)$$

Dividing (i) by (ii), we get

$$\frac{3^x \cdot 5^y}{3^y \cdot 5^x} = \frac{75}{45}$$

$$\Rightarrow \left(\frac{3}{5}\right)^{x-y} = \left(\frac{3}{5}\right)^{-1}$$

$$\Rightarrow x-y = -1 \qquad \text{...(iv)}$$

Solving (iii) and (iv), we get $x = 1, y = 2$

EXAMPLE 5 *Solve the equations:*

$$\log_y x + \log_x y = 2 \text{ and } x^2 + y = 12.$$

SOLUTION Clearly, $\log_y x$ and $\log_x y$ are defined for $x, y > 0$ and $x, y \neq 1$

We have,

$$\log_y x + \log_x y = 2$$

$$\Rightarrow t + \frac{1}{t} = 2, \text{ where } t = \log_y x$$

$$\Rightarrow (t-1)^2 = 0$$

$$\Rightarrow t = 1$$

$$\Rightarrow \log_y x = 1$$

$$\Rightarrow x = y.$$

We also have,

$$x^2 + y = 12$$

$$\Rightarrow x^2 + x = 12 \qquad [\because y = x]$$

$$\Rightarrow x^2 + x - 12 = 0$$

$$\Rightarrow (x+4)(x-4) = 0$$

$$\Rightarrow x = 3 \qquad [\because x > 0 \ \therefore \ x+4 \neq 0]$$

$$\Rightarrow y = 3 \qquad [\because y = x]$$

Hence, $x = 3, y = 3$ is the solution of the given equation.

EXAMPLE 6 *Solve for real values of* x and y:

$$\log_2 xy = 5, \ \log_{1/2}\left(\frac{x}{y}\right) = 1.$$

SOLUTION The given equations are defined of $xy > 0$ and $\frac{x}{y} > 0$. This is possible when x and y are of the same signs.

Therefore, we have the following cases:

CASE I *When* $x > 0, \ y > 0$

In this case, we have

$$\log_2 xy = 5 \text{ and } \log_{1/2}\left(\frac{x}{y}\right) = 1$$

$$\Rightarrow xy = 2^5 \text{ and } \frac{x}{y} = \frac{1}{2}$$

$$\Rightarrow xy \times \frac{x}{y} = 2^5 \times \frac{1}{2}$$

$$\Rightarrow x^2 = 2^4$$

$$\Rightarrow x = 4$$

Putting $x = 4$ in $\frac{x}{y} = \frac{1}{2}$, we get $y = 8$.

Thus, $x = 4, y = 8$ is the solution in this case.

CASE II *When* $x < 0, \ y < 0$.

In this case, the given equations reduce to

$$xy = 2^5 \text{ and } \frac{x}{y} = \frac{1}{2} \qquad \text{[See case I]}$$

$$\Rightarrow x^2 = 2^4 \qquad \text{[See case I]}$$

$$\Rightarrow x = -4 \qquad [\because x < 0]$$

Putting $x = -4$ in $\frac{x}{y} = \frac{1}{2}$, we get $y = -8$.

Thus $x = -4, y = -8$ is the solution in this case.

Hence, $x = 2, y = 8$ and $x = -4, y = -8$ are two solutions of the given equation.

EXAMPLE 7 *Solve the system of equations:*

$$|x^2 - 2x| + y = 1, x^2 + |y| = 1.$$

SOLUTION We have the following four cases:

CASE I *When* $x^2 - 2x \geq 0$ and $y \geq 0$

In this case, we have

$$|x^2 - 2x| = x^2 - 2x \text{ and } |y| = y.$$

Also, $x^2 - 2x \geq 0 \Rightarrow x \leq 0$ or $x \geq 2 \Rightarrow x \in (-\infty, 0] \cup [2, \infty)$

Now, $|x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$

$$\Rightarrow x^2 - 2x + y = 1 \qquad \text{and } x^2 + y = 1$$

$$\Rightarrow x^2 - 2x + 1 - x^2 = 1$$

$$\Rightarrow x = 0$$

Putting $x = 0$ in $x^2 + y = 1$, we obtain $y = 1$.

Clearly, the pair $x = 0, y = 1$ satisfies the two conditions i.e. $x^2 - 2x > 0$ and $y \geq 0$.

Hence, $x = 0, y = 1$ is the solution in this case.

CASE II *When* $x^2 - 2x \geq 0$ and $y < 0$:

In this case, we have

$$|x^2 - 2x| = x^2 - 2x \text{ and } |y| = -y.$$

Also, $x^2 - 2x \geq 0 \Rightarrow x \in (-\infty, 0] \cup [2, \infty)$

$\therefore \ |x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$

$$\Rightarrow x^2 - 2x + y = 1 \qquad \text{and } x^2 - y = 1$$

$\Rightarrow 2x^2 - 2x - 2 = 0$ [Eliminating y]

$\Rightarrow x^2 - x - 1 = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{5}}{2}$

$\Rightarrow x = \dfrac{1 - \sqrt{5}}{2}$ $\left[\because x = \dfrac{1 + \sqrt{5}}{2} \notin (-\infty, 0] \cup [2, \infty)\right]$

Putting this value of x in $x^2 - y = 1$, we get

$$y = \frac{1 - \sqrt{5}}{2}$$

Hence, the pair $x = \dfrac{1 - \sqrt{5}}{2}, y = \dfrac{1 - \sqrt{5}}{2}$ is the solution in this case.

CASE III *When* $x^2 - 2x < 0$ *and* $y \geq 0$

In this case, we have

$|x^2 - 2x| = -(x^2 - 2x)$ and $|y| = y$.

Now, $x^2 - 2x < 0 \Rightarrow x \in (0, 2)$.

$\therefore |x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$

$\Rightarrow -(x^2 - 2x) + y = 1$ and $x^2 + y = 1$

$\Rightarrow -2x^2 + 2x + 1 = 1$ [Eliminating y]

$\Rightarrow x = 0, 1$

$\Rightarrow x = 1$ $[\because 0 \notin (0, 2)]$

Putting $x = 1$ in $x^2 + y = 1$, we get $y = 0$.

Hence, $x = 1, y = 0$ is the only solution in this case.

CASE IV *When* $x^2 - 2x < 0$ and $y < 0$:

In this case, we have

$|x^2 - 2x| = -(x^2 - 2x)$ and $|y| = -y$.

Also, $x^2 - 2x < 0 \Rightarrow x \in (0, 2)$.

$\therefore |x^2 - 2x| + y = 1$ and $x^2 + |y| = 1$

$\Rightarrow -(x^2 - 2x) + y = 1$ and $x^2 - y = 1$

$\Rightarrow -(x^2 - 2x) + x^2 - 1 = 1$ [Eliminating y]

$\Rightarrow 2x = 2$

$\Rightarrow x = 1$.

Clearly, $x = 1 \in (0, 2)$.

Putting $x = 1$ in $x^2 - y = 1$, we get $y = 0$.

But the values of y must be negative.

So, there is no solution in this case.

If follows from all these cases that the given system of equations has the following solutions:

$$x = 0, y = 1; x = 1, y = 0; x = y = \frac{1 - \sqrt{5}}{2}$$

EXERCISE 20.7

Solve each of the following equations for real roots:

1. $\log_x (3x^2 + 10x) = 3$
2. $\log_9 x + \log_{x^2} 3 = 1$
3. $\log_5 x + \log_x 5 = \dfrac{5}{2}$
4. $\log_{16} x + \log_4 x + \log_2 x = 7$
5. $\log_{10} \left\{98 + \sqrt{x^3 - x^2 - 12x + 36}\right\} = 2$
6. $\log_{(5-x)} (x^2 - 2x + 65) = 2$
7. $\log_{(1-x)} (3 - x) = \log_{(3-x)} (1 - x)$
8. $1 + 2 \log_{(x+2)} 5 = \log_5 (x + 2)$
9. $6\left(\log_x 2 - \log_4 x\right) - 7 = 0$
10. $\log_4 8 + \log_4 (x + 3) - \log_4 (x - 1) = 2$.
11. $x^x = x$
12. $x^{\sqrt{x}} = (\sqrt{x})^x$
13. $\log_2 (25^{x+3} - 1) = 2 + \log_2 (5^{x+3} + 1)$
14. $\log_4 (2 \cdot 4^{x-2} - 1) + 4 = 2x$
15. $\log_{10} (2^x + x - 1)^{-1} = x (\log_{10} 5 - 1)$
16. $3 \cdot \log_x 4 + 2 \log_{4x} 4 + 3 \log_{16x} 4 = 0$
17. $\log_2 (4 \cdot 3^x - 6) - \log_2 (9^x - 6) = 1$
18. $\log_{(1-2x)} (6x^2 - 5x + 1) - \log_{(1-3x)} (4x^2 - 4x + 1) = 2$
19. $\log_{10} (x^2 - x - 6) - x = \log_{10} (x + 2) - 4$
20. $\log_{|x+6|^2} \{2 \cdot \log_2 (x^2 - x - 2)\} = 1$.
21. $x^{\frac{2}{3}\left\{(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right\}} = \sqrt{2}$
22. $25^{\log_{10} x} = 5 + 4 \cdot x^{\log_{10} 5}$
23. $\log_{x^2 - 6x + 8} \left\{\log_{2x^2 - 2x - 8} (x^2 + 5x)\right\} = 0$
24. $2 \log_3 x + \log_3 (x^2 - 3) = \log_3 0.5 + 5^{\log_5 (\log_3 8)}$

ANSWERS

1. 5 2. 3 3. $25, \sqrt{5}$
4. 16 5. -4 6. -5
7. $2-\sqrt{2}$ 8. $23, -9/5$ 9. $8, 2^{-2/3}$
10. 5 11. ± 1 12. $1, 4$
13. -2 14. 2 15. 1
16. $\frac{1}{2}, \frac{1}{8}$ 17. 1 18. $\frac{1}{4}$
19. 4 20. $\frac{-38}{13}$ 21. $2, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$
22. 10 23. 8 24. 2.

HINTS TO SELECTED PROBLEMS

1. Given equation is meaningful for $x > 0, x \neq 1$ and $3x^2 + 10x > 0$. It reduces to $3x^2 + 10x = x^3$.

2. It is meaningful for $x > 0, x \neq 1$. and it reduces to $\log_{3^2} x + \log_{x^2} 3 = 1$

$\Rightarrow \frac{1}{2}\log_3 x + \frac{1}{2}\log_x 3 = 1.$

$\Rightarrow y + \frac{1}{y} = 2$

$\Rightarrow y = 1$

$\Rightarrow \log_3 x = 1 \Rightarrow x = 3$

3. It is meaningful for $x > 0, x \neq 1$ and it reduces to $2y^2 - 5y + 2 = 0$, where $y = \log_5 x$

$\Rightarrow (2y-1)(y-2) = 0$

$\Rightarrow y = 2, 1/2 \Rightarrow \log_5 x = 2, \frac{1}{2}.$

4. Given equation reduces to :

$$\frac{1}{4}\log_2 x + \frac{1}{2}\log_2 x + \log_2 x = 7$$

5. It is meaningful, if $x^3 - x^2 - 12x + 36 \geq 0$ and in that case it reduces to:

$$98 + \sqrt{x^3 - x^2 - 12x + 36} = 10^2.$$

11. Clearly, $x = -1$ satisfies it. Also for $x > 0$,

$x^x = x$

$\Rightarrow x \log_{10} x = \log_{10} x.$

$\Rightarrow (x-1)\log_{10} x = 0$

$\Rightarrow x - 1 = 0$ or, $\log_{10} x = 0$

$\Rightarrow x = 1$

12. For the equation to be meaningful, we must have $x > 0$. Also, for $x > 0$

$x^{\sqrt{x}} = (\sqrt{x})^x$

$\Rightarrow \sqrt{x}\log x = x \log \sqrt{x}$

$\Rightarrow \sqrt{x}\log x = \frac{x}{2}\log x$

$\Rightarrow \sqrt{x}(\log x)\left(1 - \frac{\sqrt{x}}{2}\right) = 0$

$\Rightarrow \log x = 0$ or, $1 - \frac{\sqrt{x}}{2} = 0$

$\Rightarrow x = 1$ or, $x = 4$.

15. $x(\log_{10} 5 - 1) = x(\log_{10} 5 - \log_{10} 10)$

$= x\log_{10}(5/10) = x\log_{10} 1/2 = \log_{10} 2^{-x}$

16. Given equation reduces to

$$\frac{3}{\log_4 x} + \frac{2}{\log_4 4 + \log_4 x} + \frac{3}{\log_4 4^2 + \log_4 x} = 0$$

or, $\frac{3}{y} + \frac{2}{1+y} + \frac{3}{2+y} = 0$, where $y = \log_4 x$.

22. Use $x^{\log_{10} 5} = 5^{\log_5 (x^{\log_{10} 5})} = 5^{\log_{10} 5 . \log_5 x} = 5^{\log_{10} x}$

CHAPTER

21

PROGRESSIONS

21.1 SEQUENCE

A sequence is a function whose domain is the set N of natural numbers.

It is customary to denote a sequence by a letter 'a' and the image $a(n)$ of $n \in N$ under a by a_n. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. The images of $1, 2, 3, ..., n, ...$ under a sequence 'a' are generally denoted by $a_1, a_2, a_3, ... a_n, ...$ respectively. The images $a_1, a_2, a_3, ... a_n, ...$ are known as first term, second term, third term,..., nth term,... respectively of the sequence. If a_n is the nth term of a sequence, 'a' then we write $a = <a_n>$.

REAL SEQUENCE *A sequence whose range is a subset of R is called a real sequence.*

In other words, a real sequence is a function with domain N and the range a subset of the set R of real numbers.

REPRESENTATION OF A SEQUENCE There are several ways of representing a real sequence.

One way to represent a real sequence is to list its first few terms till the rule for writing down other terms becomes clear.

For example, 1, 3, 5, ... is a sequence whose nth term is $(2n - 1)$.

Another way to represent a real sequence is to give a rule of writing the nth term of the sequence. For example, the sequence 1, 3, 5, 7, ... can be written as

$$a_n = 2n - 1.$$

Sometimes we represent a real sequence by using a recursive relation.

For example, the Fibonacci sequence is given by $a_1 = 1$, $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$. The terms of this sequence are 1, 1, 2, 3, 5, 8, ...

SERIES *If $a_1, a_2, a_3, a_4, ..., a_n, ...$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_n + ...$ is a series.*

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

PROGRESSIONS *It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.*

21.2 ARITHMETIC PROGRESSION (A.P.)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same, i.e.

$$a_{n+1} - a_n = \text{constant } (= d) \text{ for all } n \in N$$

The constant difference, generally denoted by d is called the common difference.

ILLUSTRATION 1 *1, 4, 7, 10, ... is an A.P. whose first term is 1 and the common difference is* $4 - 1 = 3$.

ILLUSTRATION 2 *11, 7, 3, –1, ... is an A.P. whose first term is 11 and the common difference* $7 - 11 = -4$.

ILLUSTRATION 3 *Show that the sequence $<a_n>$ defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.*

SOLUTION We have,

$$a_n = 4n + 5.$$

Replacing n by $(n + 1)$, we get

$$a_{n+1} = 4(n + 1) + 5$$

$$\Rightarrow \quad a_{n+1} = 4n + 9$$

$$\therefore \quad a_{n+1} - a_n = (4n + 9) - (4n + 5) = 4$$

Clearly, $a_{n+1} - a_n$ is independent of n and is equal to 4. So, the given sequence is an A.P. with common difference 4.

ILLUSTRATION 4 *Show that the sequence $<a_n>$ defined by $a_n = 2n^2 + 1$ is not an A.P.*

SOLUTION We have,

$$a_n = 2n^2 + 1$$

Replacing n by $(n+1)$ in a_n, we obtain

$$a_{n+1} = 2(n+1)^2 + 1$$

$$\Rightarrow \quad a_{n+1} = 2n^2 + 4n + 3$$

Now,

$$a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1) = 4n + 2$$

Clearly, $a_{n+1} - a_n$ is not independent of n and is therefore not constant. So, the given sequence is not an A.P.

EXAMPLE 5 *Show that the sequence $< a_n >$ is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.*

SOLUTION Let $< a_n >$ be a sequence such that its nth term is a linear expression in n *i.e.*

$$a_n = An + B \text{ where } A, B \text{ are constants.}$$

$$\Rightarrow \quad a_{n+1} = A(n+1) + B$$

$$\therefore \quad a_{n+1} - a_n = \{A(n+1) + B\} - \{An + B\} = A$$

Clearly, $a_{n+1} - a_n$ is independent of n and is therefore a constant. So, the sequence $< a_n >$ is an A.P. with common difference A.

NOTE *Students are advised to consider the statement of the above example as a standard result.*

ILLUSTRATION 6 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots\ldots$ *forms an A.P.*

SOLUTION We have,

$$a_n = \log\left(\frac{a^n}{b^{n-1}}\right)$$

$$\Rightarrow \quad a_{n+1} = \log\left(\frac{a^{n+1}}{b^n}\right)$$

Now,

$$a_{n+1} - a_n = \log\left(\frac{a^{n+1}}{b^n}\right) - \log\left(\frac{a^n}{b^{n-1}}\right)$$

$$= \log\left(\frac{a^{n+1}}{b^n} \cdot \frac{b^{n-1}}{a^n}\right)$$

$$= \log\left(\frac{a}{b}\right)$$

Clearly, $a_{n+1} - a_n$ is constant for all values of n. So, the given sequence is an A.P. with common difference $\log\left(\frac{a}{b}\right)$.

ILLUSTRATION 7 *The nth term of a sequence is $3n - 2$. Is the sequence an A.P. ? If so, find its 10th term.*

SOLUTION Here, $a_n = 3n - 2$.

Clearly a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3.

Putting $n = 10$, we get

$$a_{10} = 3 \times 10 - 2 = 28.$$

21.3 PROPERTIES OF AN ARITHMETIC PROGRESSION

PROPERTY I *If a is the first term and d the common difference of an A.P., then its nth term a_n is given by*

$$a_n = a + (n+1)\,d$$

If the A.P. consists of n terms, then mth term from the end is $(m - n + 1)$ th term from the beginning. Therefore,

mth term from the end $= a_{m-n+1}$

$$= a + (m-n)\,d$$

Also,

mth term from the end $= l - (m-1)\,d$, where $l = a_n$.

PROPERTY II *A sequence is an A.P iff its nth term is of the form An + B i.e. a linear expression in n. The common difference in such a case is A i.e. the coefficient of n.*

PROOF Let there be an AP with first term 'a' and common difference d. Then, its nth term is given by.

$$a_n = a + (n-1)\,d$$

$$\Rightarrow \quad a_n = nd + (a - d)$$

$$\Rightarrow \quad a_n = An + B, \text{ where } A = d \text{ and } B = a - d.$$

Thus, nth term of an AP is a linear expression in n. Conversely, let there be a sequence whose nth term a_n is given by $a_n = An + B$.

Then, we have to prove that the sequence is an AP.

Now,

$$a_n = An + B$$

$$\Rightarrow \quad a_{n+1} = A(n+1) + B$$

$$\Rightarrow \quad a_{n+1} = a_n + A + B$$

$$\therefore \quad a_{n+1} - a_n = (A_n + A + B) - (An + B) = A$$

$$\Rightarrow \quad a_{n+1} - a_n = \text{Constant } (= A)$$

Hence, the sequence is an AP with common difference 'd'.

PROPERTY III *If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d, and let k be a fixed constant which is added to each term of this A.P. Then the resulting sequence is

$$a_1 + k, a_2 + k, a_3 + k \ldots$$

Let $b_n = a_n + k$, $n = 1, 2, \ldots$ Then, the new sequence is $b_1, b_2, b_3, \ldots$

We have,

$b_{n+1} - b_n = (a_{n+1} + k) - (a_n + k)$

$= a_{n+1} - a_n$

$= d$ for all $n \in N$

[$\because < a_n >$ is a sequence with common difference d]

Thus, the new sequence is also an A.P. with common difference d.

PROPERTY IV *If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d and let k be a non-zero constant. Let $b_1, b_2, b_3, \ldots$ be sequence obtained by multiplying each term of the given A.P. by k. Then,

$$b_1 = a_1 k, b_2 = a_2 k, \ldots, b_n = a_n k, \ldots$$

Now,

$b_{n+1} - b_n = a_{n+1} k - a_n k$

$= (a_{n+1} - a_n) k$

$= dk$ for all $n \in N$

This shows that the new sequence is an A.P. with common difference dk.

Similarly, it can be proved that on dividing each term of a given A.P. by a non-zero constant, we obtain a sequence which is also an A.P.

PROPERTY V *In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e.*

$$a_k + a_{n-(k-1)} = a_1 + a_n \text{ for all } k = 1, 2, 3, \ldots, n-1.$$

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference d. We have to show that

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \cdots$$

i.e $a_1 + a_n = a_k + a_{n-(k-1)}$ for all $k = 1, 2, 3, \ldots, n-1$

Now, for any $k = 1, 2, \ldots, n-1$, we have

$a_k + a_{n-(k-1)}$

$= a_k + a_{n+1-k}$

$= [a_1 + (k-1)d] + [a_1 + (n+1-k-1)d]$

$= 2a_1 + (k-1+n+1-k-1)d$

$= 2a_1 + (n-1)d$

$= a_1 + [a_1 + (n-1)d]$

$= a_1 + a_n$.

PROPERTY VI *Three numbers a, b, c are in A.P. iff* $2b = a + c$.

PROOF First, let a, b, c be in A.P. Then,

$b - a$ = Common difference

and, $c - b$ = Common difference

$\Rightarrow \quad b - a = c - b$

$\Rightarrow \quad 2b = a + c$

Conversely, let a, b, c be three numbers such that $2b = a + c$. Then, we have to show that a, b, c are in A.P.

We have,

$2b = a + c$

$\Rightarrow b - a = c - b$

$\Rightarrow$ a, b, c are in A.P.

PROPERTY VII *If the terms of an A.P. are chosen at regular intervals, then they form an A.P.*

PROPERTY VIII *If* a_n, a_{n+1} and a_{n+2} *are three consecutive terms of an A.P., then*

$$2a_{n+1} = a_n + a_{n+2}.$$

21.4 SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a-d, a+d$ and the common differences is $2d$.

21.5 SOME USEFUL RESULTS

$$\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

$$\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i, show that*

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

SOLUTION Let d be the common difference of the given A.P. Then

$$a_2-a_1=a_3-a_2=\ldots=a_n-a_{n-1}=d \qquad \ldots(i)$$

Now, $\dfrac{1}{\sqrt{a_1}+\sqrt{a_2}}+\dfrac{1}{\sqrt{a_2}+\sqrt{a_3}}+\dfrac{1}{\sqrt{a_3}+\sqrt{a_4}}+\ldots+\dfrac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$$=\frac{\sqrt{a_2}-\sqrt{a_1}}{(\sqrt{a_2}+\sqrt{a_1})(\sqrt{a_2}-\sqrt{a_1})}+\frac{\sqrt{a_3}-\sqrt{a_2}}{(\sqrt{a_3}+\sqrt{a_2})(\sqrt{a_3}-\sqrt{a_2})}+\ldots$$

$$+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{(\sqrt{a_n}+\sqrt{a_{n-1}})(\sqrt{a_n}-\sqrt{a_{n-1}})}$$

$$=\frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}+\frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

$$=\frac{\sqrt{a_2}-\sqrt{a_1}}{d}+\frac{\sqrt{a_3}-\sqrt{a_2}}{d}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d} \qquad \text{[Using (i)]}$$

$$=\frac{1}{d}\left[(\sqrt{a_2}-\sqrt{a_1})+(\sqrt{a_3}-\sqrt{a_2})+\ldots+(\sqrt{a_n}-\sqrt{a_{n-1}})\right]$$

$$=\frac{1}{d}\left[\sqrt{a_n}-\sqrt{a_1}\right]$$

$$=\frac{(\sqrt{a_n}-\sqrt{a_1})(\sqrt{a_n}+\sqrt{a_1})}{d(\sqrt{a_n}+\sqrt{a_1})}$$

$$=\frac{a_n-a_1}{d(\sqrt{a_n}+\sqrt{a_1})}$$

$$=\frac{a_1+(n-1)d-a_1}{d(\sqrt{a_n}+\sqrt{a_1})}$$

$$=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

$$=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

EXAMPLE 2 *If $a_1, a_2, a_3, \ldots, a_n$ be an A.P. of non-zero terms, prove that*

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}.$$

SOLUTION Let 'd' be the common difference of the given A.P. Then,

$$a_2-a_1=a_3-a_2=\ldots=a_n-a_{n-1}=d \qquad \ldots(i)$$

Now,

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\frac{1}{a_3a_4}+\ldots+\frac{1}{a_{n-1}a_n}$$

$$=\frac{1}{d}\left\{\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\frac{d}{a_3a_4}+\ldots+\frac{d}{a_{n-1}a_n}\right\}$$

$$=\frac{1}{d}\left\{\frac{(a_2-a_1)}{a_1a_2}+\frac{(a_3-a_2)}{a_2a_3}+\frac{(a_4-a_3)}{a_3a_4}+\ldots+\frac{(a_n-a_{n-1})}{a_{n-1}a_n}\right\}$$

$$=\frac{1}{d}\left\{\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\left(\frac{1}{a_3}-\frac{1}{a_4}\right)+\ldots+\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)\right\}$$

$$=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_n}\right\}$$

$$=\frac{1}{d}\left\{\frac{a_n-a_1}{a_1a_n}\right\}$$

$$=\frac{1}{d}\left\{\frac{a_1+(n-1)d-a_1}{a_1a_n}\right\}$$

$$=\frac{n-1}{a_1a_n}$$

EXAMPLE 3 *If $a_1, a_2, a_3, \ldots, a_n$ be an A.P. of non-zero terms, prove that*

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left\{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}.$$

SOLUTION We have to prove that

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left\{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}$$

or

$$(a_1+a_n)\left\{\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}\right\}=2\left\{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}$$

We know that the sum of the terms equidistant from the beginning and the end in an A.P. is always same i.e.

$$a_1+a_n=a_2+a_{n-1}=a_3+a_{n-2}=\ldots=a_k+a_{n-(k+1)}=\ldots$$

$$\therefore\quad (a_1+a_n)\left\{\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1}\right\}$$

$$=\frac{a_1+a_n}{a_1a_n}+\frac{a_1+a_n}{a_2a_{n-1}}+\frac{a_1+a_n}{a_3a_{n-2}}+\ldots+\frac{a_1+a_n}{a_na_1}$$

$$=\frac{a_1+a_n}{a_1a_n}+\frac{a_2+a_{n-1}}{a_2a_{n-1}}+\frac{a_3+a_{n-2}}{a_3a_{n-2}}+\ldots+\frac{a_1+a_n}{a_1a_n}$$

$$=\left(\frac{1}{a_n}+\frac{1}{a_1}\right)+\left(\frac{1}{a_{n-1}}+\frac{1}{a_2}\right)+\ldots+\left(\frac{1}{a_{n-1}}+\frac{1}{a_2}\right)+\left(\frac{1}{a_n}+\frac{1}{a_1}\right)$$

$$=2\left\{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}\right\}$$

EXAMPLE 4 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then prove that*

$$\sin d\,[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n] = \cot a_1 - \cot a_n$$

SOLUTION We have, $d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$. Thus,

$$\sin d\,[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \ldots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin (a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin (a_3 - a_2)}{\sin a_2 \sin a_3} + \ldots + \frac{\sin (a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \ldots + (\cot a_{n-1} - \cot a_n)$$

$$= \cot a_1 - \cot a_n$$

EXAMPLE 5 *Show that in an A.P. $a_1, a_2, a_3, \ldots, a_{2k}$*

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2k-1}^2 - a_{2k}^2 = \frac{k}{2k-1}(a_1^2 - a_{2k}^2)$$

SOLUTION Let d be the common difference of the given A.P. Then,

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2k-1}^2 - a_{2k}^2$$

$$= (a_1^2 - a_2^2) + (a_3^2 - a_4^2) + \ldots + (a_{2k-1}^2 - a_{2k}^2)$$

$$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \ldots\ldots\ldots\ldots + (a_{2k-1} - a_{2k})(a_{2k-1} + a_{2k})$$

$$= -d(a_1 + a_2 + \ldots + a_{2k-1} + a_{2k}) \qquad \left[\because a_2 - a_1 = a_3 - a_2 = \ldots = a_{2k} - a_{2k-1} = d\right]$$

$$= -d\,[(a_1 + a_{2k}) + (a_2 + a_{2k-1}) + \ldots + (a_k + a_{k+1})]$$

$$= -d\,[(a_1 + a_{2k}) + (a_1 + a_{2k}) + \ldots + (a_1 + a_{2k})]$$

$$= -dk(a_1 + a_{2k})$$

$$= -dk\frac{(a_1^2 - a_{2k}^2)}{a_1 - a_{2k}} \qquad \text{[Using prop. (v)]}$$

$$= dk\frac{(a_1^2 - a_{2k}^2)}{a_{2k} - a_1}$$

$$= dk\frac{(a_1^2 - a_{2k}^2)}{(2k-1)d}$$

$$= \frac{k}{2k-1}(a_1^2 - a_{2k}^2)$$

EXAMPLE 6 *If $a_1, a_2, a_3, \ldots, a_n, a_{n+1}$ are in AP with common difference d, then prove that*

$$\tan^{-1}\frac{d}{1 + a_1 a_2} + \tan^{-1}\frac{d}{1 + a_2 a_3} + \ldots + \tan^{-1}\frac{d}{1 + a_n a_{n+1}} = \tan^{-1}\frac{nd}{a + a_1 a_{n+1}}$$

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n, a_{n+1}$ are in A.P. with common differecne d. Therefore,

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_{n+1} - a_n = d$$

Now,

$$\tan^{-1}\frac{d}{1 + a_1 a_2} + \tan^{-1}\frac{d}{1 + a_2 a_3} + \ldots + \tan^{-1}\frac{d}{1 + a_n a_{n+1}}$$

$$= \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right) + \ldots + \tan^{-1}\left(\frac{a_{n+1} - a_n}{1 + a_n a_{n+1}}\right)$$

$$= (\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) + \ldots + (\tan^{-1} a_{n+1} - \tan^{-1} a_n)$$

$$= \tan^{-1} a_{n+1} - \tan^{-1} a_1$$

$$= \tan^{-1}\left(\frac{a_{n+1} - a_1}{1 + a_{n+1} a_1}\right)$$

$$= \tan^{-1}\left(\frac{nd}{1 + a_1 a_{n+1}}\right) \qquad [\because a_{n+1} - a_1 = a_1 + nd - a_1 = nd]$$

EXAMPLE 7 *If $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference d, then prove that*

$$\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} a_n = \frac{\tan a_n - \tan a_1}{\sin d}$$

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference d. Therefore,

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d.$$

Now,

$$\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n$$

$$= \frac{1}{\cos a_1 \cos a_2} + \frac{1}{\cos a_2 \cos a_3} + \ldots + \frac{1}{\cos a_{n-1} \cos a_n}$$

$$= \frac{1}{\sin d}\left\{\frac{\sin d}{\cos a_1 \cos a_2} + \frac{\sin d}{\cos a_2 \cos a_3} + \ldots + \frac{\sin d}{\cos a_{n-1} \cos a_n}\right\}$$

$$= \frac{1}{\sin d}\left\{\frac{\sin (a_2 - a_1)}{\cos a_1 \cos a_2} + \frac{\sin (a_3 - a_2)}{\cos a_2 \cos a_3} + \ldots + \frac{\sin (a_n - a_{n-1})}{\cos a_n \cos a_{n-1}}\right\}$$

$$= \frac{1}{\sin d}\{(\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \ldots + (\tan a_n - \tan a_{n-1})\}$$

$$= \frac{\tan a_n - \tan a_1}{\sin d}.$$

EXAMPLE 8 *If $a_1, a_2, a_3, ..., a_n$ are in AP with first term $a_1 = 0$ and common difference $d \neq 0$, then prove that*

$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\ldots+\frac{a_n}{a_{n-1}}-a_2\left\{\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right\} = \frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}$$

SOLUTION Since $a_1, a_2, a_3, ..., a_n$ are in A.P. with $a_1 = 0$ and common difference d $(d \neq 0)$. Therefore,

$a_2 = d, a_3 = 2d, a_4 = 3d, ..., a_n = (n-1)d$

Now,

$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right)$$

$$= \frac{a_3-a_2}{a_2}+\frac{a_4-a_2}{a_3}+\frac{a_5-a_2}{a_4}+\ldots+\frac{a_{n-1}-a_2}{a_{n-2}}+\frac{a_n}{a_{n-1}}$$

$$= \frac{2d-d}{d}+\frac{3d-d}{2d}+\frac{4d-d}{3d}+\ldots+\frac{(n-2)d-d}{(n-3)d}+\frac{(n-1)d}{(n-2)d}$$

$$= \underset{(n-3)-\text{times}}{(1+1+1+\ldots+1)}+\frac{n-1}{n-2}$$

$$= (n-3)+\frac{n-1}{n-2}$$

$$= (n-3)+\frac{(n-2)+1}{n-2}$$

$$= (n-3)+1+\frac{1}{n-2}$$

$$= \frac{(n-2)d}{d}+\frac{d}{(n-2)d}$$

$$= \frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}$$

EXAMPLE 9 *If $a_1, a_2, a_3, ..., a_n, ...$ are in AP, prove that*

(i) $$\frac{1}{a_1a_2a_3}+\frac{1}{a_2a_3a_4}+\frac{1}{a_3a_4a_5}+\ldots+\frac{1}{a_na_{n+1}a_{n+2}} = \frac{1}{2(a_2-a_1)}\left[\frac{1}{a_1a_2}-\frac{1}{a_{n+1}a_{n+2}}\right]$$

(ii) $$\frac{1}{a_1a_2a_3a_4}+\frac{1}{a_2a_3a_4a_5}+\ldots+\frac{1}{a_na_{n+1}a_{n+2}a_{n+3}} = \frac{1}{3(a_2-a_1)}\left[\frac{1}{a_1a_2a_3}-\frac{1}{a_{n+1}a_{n+2}a_{n+3}}\right]$$

(iii) $$\frac{1}{a_1a_2a_3\ldots a_r}+\frac{1}{a_2\cdot a_3\ldots a_ra_{r+1}}+\ldots+\frac{1}{a_na_{n+1}\ldots a_{n+r-1}} = \frac{1}{(r-1)(a_2-a_1)}\left\{\frac{1}{a_1a_2\ldots a_{r-1}}-\frac{1}{a_{n+1}a_{n+2}\ldots a_{n+r-1}}\right\}$$

In solving these problems, we follow the following steps :

STEP I *Write nth term T_n of the given series :*

STEP II *Define $V_n = a_n T_n$.*

STEP III *Find V_{n-1}.*

STEP IV *Find $V_n - V_{n-1}$.*

STEP V *Express $V_n - V_{n-1}$ in terms of T_n and finally obtain T_n in terms of V_n and V_{n-1}.*

STEP VI *Put $n = 1, 2, 3, ...$ in the expression obtained in step V and sum all these values to get the sum of the series.*

SOLUTION Let d be the common difference of the given AP.

(i) Let T_n be the nth term of the given series.

$$T_n = \frac{1}{a_na_{n+1}a_{n+2}},\ n = 1, 2, ..., ...$$

Let $V_n = a_nT_n$. Then,

$$V_n = a_nT_n = \frac{1}{a_{n+1}a_{n+2}}$$

$$\Rightarrow V_{n-1} = \frac{1}{a_na_{n+1}}$$

$$\therefore V_n - V_{n-1} = \frac{1}{a_{n+1}a_{n+2}}-\frac{1}{a_na_{n+1}}$$

$$\Rightarrow V_n - V_{n-1} = \frac{a_n-a_{n+2}}{a_na_{n+1}a_{n+2}}$$

$$\Rightarrow V_n - V_{n-1} = (a_n-a_{n+2})T_n$$

$$\Rightarrow V_n - V_{n-1} = 2dT_n.$$

$$\Rightarrow T_n = \frac{1}{2d}(V_{n-1}-V_n),\ n = 1, 2, ...$$

$$\therefore T_1+T_2+T_3+\ldots+T_n$$

$$= \frac{1}{2d}[(V_0-V_1)+(V_1-V_2)+(V_2-V_3)+\ldots+(V_{n-1}-V_n)]$$

$$= \frac{1}{2d}[V_0-V_n]$$

$$= \frac{1}{2d}\left[\frac{1}{a_1a_2}-\frac{1}{a_{n+1}a_{n+2}}\right]$$

$$= \frac{1}{2(a_2-a_1)}\left[\frac{1}{a_1a_2}-\frac{1}{a_{n+1}a_{n+2}}\right]$$

(ii) Let T_n be the nth term of the given series. Then,

$$T_n = \frac{1}{a_na_{n+1}a_{n+2}a_{n+3}},\ n = 1, 2, ...$$

Let

$$V_n = a_nT_n.$$

$$\Rightarrow V_n = \frac{1}{a_{n+1}a_{n+2}a_{n+3}}$$

$$\Rightarrow \quad V_{n-1} = \frac{1}{a_n a_{n+1} a_{n+2}}$$

$$\therefore \quad V_n - V_{n-1} = \frac{1}{a_{n+1} a_{n+2} a_{n+3}} - \frac{1}{a_n a_{n+1} a_{n+2}}$$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{a_n - a_{n+3}}{a_n a_{n+1} a_{n+2} a_{n+3}}$$

$$\Rightarrow \quad V_n - V_{n-1} = T_n (a_n - a_{n+3})$$

$$\Rightarrow \quad V_n - V_{n-1} = -3d\, T_n$$

$$\Rightarrow \quad T_n = \frac{V_{n-1} - V_n}{3d}, n = 1, 2, ...$$

$$\therefore \quad T_1 + T_2 + ... + T_n$$

$$= \frac{1}{3d} [(V_0 - V_1) + (V_1 - V_2) + (V_2 - V_3) + ... + (V_{n-1} - V_n)]$$

$$= \frac{1}{3d} [V_0 - V_n]$$

$$= \frac{1}{3d} \left[\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1} a_{n+2} a_{n+3}}\right]$$

$$= \frac{1}{3(a_2 - a_1)} \left[\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1} a_{n+2} a_{n+3}}\right]$$

(iii) Let T_n be the nth term of the given series. Then,

$$T_n = \frac{1}{a_n a_{n+1} a_{n+2} \cdots a_{n+r-2} a_{n+r-1}}, n = 1, 2, 3...$$

Let

$$V_n = a_n T_n$$

$$\Rightarrow \quad V_n = \frac{1}{a_{n+1} a_{n+2} \cdots a_{n+r-2} a_{n+r-1}}$$

$$\Rightarrow \quad V_{n-1} = \frac{1}{a_n a_{n+1} \cdots a_{n+r-2}}$$

$$\therefore \quad V_n - V_{n-1} = \frac{1}{a_{n+1} a_{n+2} \cdots a_{n+r-1}} - \frac{1}{a_n a_{n+1} \cdots a_{n+r-2}}$$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{a_n - a_{n+r-1}}{a_n a_{n+1} \cdots a_{n+r-1}}$$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{-(r-1)d}{a_n a_{n+1} \cdots a_{n+r-2}\ a_{n+r-1}}$$

$$\Rightarrow \quad V_n - V_{n-1} = -(r-1)d\, T_n$$

$$\Rightarrow \quad T_n = -\frac{(V_n - V_{n-1})}{(r-1)d}$$

$$\Rightarrow \quad T_n = \frac{1}{(r-1)(a_2 - a_1)} (V_{n-1} - V_n) \qquad [\because a_2 - a_1 = d]$$

Putting $n = 1, 2, ...$ and adding, we get

$$T_1 + T_2 + T_3 + ... + T_n$$

$$= \frac{1}{(r-1)(a_2 - a_1)} \{(V_0 - V_1) + (V_1 - V_2) + ... + (V_{n-1} - V_n)\}$$

$$= \frac{1}{(r-1)(a_2 - a_1)} (V_0 - V_n)$$

$$= \frac{1}{(r-1)(a_2 - a_1)} \left\{\frac{1}{a_1 a_2 ... a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \cdots a_{n+r-1}}\right\}$$

EXAMPLE 10 *If $a_1, a_2, a_3, ..., a_n, ...$ are in AP, then prove that*

(i) $a_1 a_2 + a_2 a_3 + ... + a_n a_{n+1}$

$$= \frac{1}{3(a_2 - a_1)} \{a_n a_{n+1} a_{n+2} - a_0 a_1 a_2\}$$

(ii) $a_1 a_2 a_3 + a_2 a_3 a_4 + ... + a_n a_{n+1} a_{n+2}$

$$= \frac{1}{4(a_2 - a_1)} \{a_n a_{n+1} a_{n+2} a_{n+3} - a_0 a_1 a_2 a_3\}$$

(iii) $a_1 a_2 ... a_r + a_2 a_3 ... a_{r+1} + ... + a_n a_{n+1} .... a_{n+r-1}$

$$= \frac{1}{(r+1)(a_2 - a_1)} \{a_n a_{n+1} \cdots a_{n+r} - a_0 a_1 \cdots a_r\}$$

In order to prove that results, we follow the following steps:

STEP I *Write the general term, say T_n, of the given series.*

STEP II *Define V_n by including one extra term in T_n i.e. define $V_n = T_n \cdot a_{n+r}$*

STEP III *Find V_{n-1} by replacing n by $(n-1)$ in step II.*

STEP IV *Compute $V_n - V_{n-1}$ and express it in terms of T_n.*

STEP V *Obtain T_n from step IV in terms of V_n and V_{n-1}.*

STEP VI *Put $n = 1, 2, 3, ...$ in step V and add the results so obtained to get $T_1 + T_2 + ... + T_n$.*

SOLUTION Let d be the common difference of the given A.P. Let T_n be the nth term of the given series.

Then,

$$T_n = a_n\ a_{n+1}$$

Let $V_n = a_n a_{n+1} a_{n+2}$.

$$\Rightarrow \quad V_{n-1} = a_{n-1}\ a_n a_{n+1}$$

$$\therefore \quad V_n - V_{n-1} = a_n a_{n+1} a_{n+2} - a_{n-1} a_n a_{n+1}$$

$$\Rightarrow \quad V_n - V_{n-1} = (a_{n+2} - a_{n-1}) a_n a_{n+1}$$

$$\Rightarrow \quad V_n - V_{n-1} = (a_{n+2} - a_{n-1}) T_n$$

$$\Rightarrow \quad T_n = \frac{V_n - V_{n-1}}{a_{n+2} - a_{n-1}}$$

$$\Rightarrow \quad T_n = \frac{V_n - V_{n-1}}{3d} \qquad \left[\begin{array}{c}\because a_{n+2} = a + (n+1)d \\ \text{and} \\ a_{n-1} = a + (n-2)d\end{array}\right]$$

$$\Rightarrow T_n = \frac{1}{3(a_2 - a_1)}(V_n - V_{n-1})$$

Putting $n = 1, 2, 3, ..$ and adding, we get

$T_1 + T_2 + ... + T_n$

$$= \frac{1}{3(a_2 - a_1)}\{(V_1 - V_0) + (V_2 - V_1) + ... + (V_n - V_{n-1})\}$$

$$= \frac{1}{3(a_2 - a_1)}(V_n - V_0)\}$$

$$= \frac{1}{3(a_2 - a_1)}\{a_n a_{n+1} a_{n+2} - a_0 a_1 a_2\}$$

(ii) Let T_n be nth term of the given series. Then,

$$T_n = a_n a_{n+1} a_{n+2}$$

Let $V_n = a_n a_{n+1} a_{n+2} a_{n+3}$

$$\Rightarrow V_{n-1} = a_{n-1} a_n a_{n+1} a_{n+2}.$$

$$\Rightarrow V_n - V_{n-1} = (a_{n+3} - a_{n-1}) a_n a_{n+1} a_{n+2}$$

$$\Rightarrow V_n - V_{n-1} = (a_{n+3} - a_{n-1}) T_n$$

$$\Rightarrow T_n = \frac{V_n - V_{n-1}}{a_{n+3} - a_{n-1}}$$

$$\Rightarrow T_n = \frac{V_n - V_{n-1}}{4d} \qquad \left[\because \begin{matrix} a_{n+3} = a + (n+2)d, \\ a_{n-1} = a + (n-2)d \end{matrix}\right]$$

$$\Rightarrow T_n = \frac{1}{4(a_2 - a_1)}(V_n - V_{n-1}) \qquad [\because a_2 - a_1 = d]$$

Putting $n = 1, 2, 3, ...$ and adding, we get

$$T_1 + T_2 + T_3 + ... + T_n$$

$$= \frac{1}{4(a_2 - a_1)}\{(V_1 - V_0) + (V_2 - V_1) + (V_3 - V_2) + ... + (V_n - V_{n-1})\}$$

$$= \frac{1}{4(a_2 - a_1)}\{V_n - V_0\}$$

$$= \frac{1}{4(a_2 - a_1)}\{a_n a_{n+1} a_{n+2} a_{n+3} - a_0 a_1 a_2 a_3\}$$

(iii) Proceed as in (i) and (ii).

EXAMPLE 11 *If $a_1, a_2, a_3, ..., a_n$, form an AP with common difference d, then prove that :*

$$\frac{a_1 a_2 a_3}{a_1 + a_3} + \frac{a_2 a_3 a_4}{a_2 + a_4} + \frac{a_3 a_4 a_5}{a_3 + a_5} + ... + \frac{a_n a_{n+1} a_{n+2}}{a_n + a_{n+2}}$$

$$= \frac{n}{2}\left[a_1^2 + (n+1) a_1 d + \frac{(n-1)(2n+5)}{6} d^2\right]$$

SOLUTION Since $a_1, a_2, a_3, ..., a_n, ...$ are in AP. Therefore,

$a_1 + a_3 = 2a_2, a_2 + a_4 = 2a_3, a_3 + a_5 = 2a_4, ... a_n + a_{n+2} = 2a_{n+1}.$

Now,

$$\frac{a_1 a_2 a_3}{a_1 + a_3} + \frac{a_2 a_3 a_4}{a_2 + a_4} + \frac{a_3 a_4 a_5}{a_3 + a_5} + ... + \frac{a_n a_{n+1} a_{n+2}}{a_n + a_{n+2}}$$

$$= \frac{1}{2}\left[a_1 a_3 + a_2 a_4 + a_3 a_5 + ... + a_n a_{n+2}\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n} a_k a_{k+2}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\{a_1 + (k-1)d\}\{a_1 + (k+1)d\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\{(a_1 + kd) - d\}\{(a_1 + kd) + d\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\{(a_1 + kd)^2 - d^2\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\{a_1^2 + 2a_1 d\ k + (k^2 - 1)d^2\}$$

$$= \frac{1}{2}\left[\sum_{k=1}^{n} a_1^2 + 2a_1 d\sum_{k=1}^{n} k + \left\{\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 1\right\} d^2\right]$$

$$= \frac{1}{2}\left[na_1^2 + 2a_1 d\ \frac{n(n+1)}{2} + \left\{\frac{n(n+1)(2n+1)}{6} - n\right\} d^2\right]$$

$$= \frac{1}{2}\left[na_1^2 + n(n+1) a_1 d + \frac{n(n-1)(2n+5)}{6} d^2\right]$$

$$= \frac{n}{2}\left[a_1^2 + (n+1) a_1 d + \frac{(n-1)(2n+5)}{6} d^2\right]$$

EXAMPLE 12 *Show that there is no A.P. which consists only of distinct prime numbers.*

SOLUTION Let $a_1, a_2, a_3, ..., a_n, ...$ be an A.P. consisting only of prime numbers. Let d be the common difference of the A.P. Since the difference of two consecutive prime numbers is greater than or equal to 1. Therefore, $d \geq 1$.

Now,

$(a_1 + 1)^{th}$ term of this A.P. is given by

$$a_1 + (a_1 + 1 - 1) d = a_1 (1 + d)$$

$\Rightarrow (a_1 + 1)^{th}$ term is not a prime number

This is a contradiction that the A.P. consists of only prime numbers as its terms.

Hence, there cannot be an A.P. which consists only of distinct prime numbers.

EXAMPLE 13 *Show that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannnot be the terms of a single A.P.*

SOLUTION Let $\sqrt{2}, \sqrt{3}, \sqrt{5}$ be respectively the mth, nth and pth terms of an A.P. having a as the first term and d as the common difference. Then,

$$\sqrt{2} = a + (m-1)d$$
$$\sqrt{3} = a + (n-1)d$$
and, $$\sqrt{5} = a + (p-1)d$$

$\Rightarrow \sqrt{3} - \sqrt{2} = (n-m)d$ and $\sqrt{5} - \sqrt{3} = (p-n)d$

$\Rightarrow \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \dfrac{n-m}{p-n} = k$ (say) [Here, k is a rational number]

$\Rightarrow (\sqrt{3}-\sqrt{2}) = k(\sqrt{5}-\sqrt{3})$

$\Rightarrow (\sqrt{3}-\sqrt{2})^2 = k^2(\sqrt{5}-\sqrt{3})^2$

$\Rightarrow 3 + 2 - 2\sqrt{6} = k^2(5 + 3 - 2\sqrt{15})$

$\Rightarrow 2k^2\sqrt{15} - 2\sqrt{6} = 8k^2 - 5$

$\Rightarrow 2k^2\sqrt{15} - 2\sqrt{6}$ is a rational number $[\because k \in Q]$

Let

$$2k^2\sqrt{15} - 2\sqrt{6} = l$$

$\Rightarrow (2k^2\sqrt{15} - 2\sqrt{6})^2 = l^2$

$\Rightarrow 60k^4 + 24 - 8k^2\sqrt{90} = l^2$

$\Rightarrow 60k^4 + 24 - 24k^2\sqrt{10} = l^2$

$\Rightarrow \sqrt{10} = \dfrac{60k^4 + 24 - l^2}{24k^2}$

$\Rightarrow \sqrt{10} = a$ rational number $\left[\because k, l \in Q \therefore \dfrac{60k^4 + 24 - l^2}{24k^2} \in Q\right]$

This is a contradiction.

Hence, $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be the terms of a single A.P.

EXAMPLE 14 *In the arithmetic progressions 2, 5, 8, ... upto 50 terms, and 3, 5, 7, 9, ... upto 60 terms, find how many terms are identical.*

SOLUTION Let the *m*th term of the first A.P. be equal to the *n*th term of the second A.P. Then,

$$2 + (m-1) \times 3 = 3 + (n-1) \times 2$$

$\Rightarrow 3m - 1 = 2n + 1$

$\Rightarrow 3m = 2n + 2$

$\Rightarrow \dfrac{m}{2} = \dfrac{n+1}{3} = k$ (say)

$\Rightarrow m = 2k$ and $n = 3k - 1$

$\Rightarrow 2k \le 50$ and $3k - 1 \le 60$ $[\because m \le 50$ and $n \le 60]$

$\Rightarrow k \le 25$ and $k \le 20\dfrac{1}{3}$

$\Rightarrow k \le 20$ $[\because k$ is a natural number$]$

$\Rightarrow k = 1, 2, 3, \ldots 20$

Corresponding to each value of k, we get a pair of identical terms. Hence, there are 20 identical terms in the two A.P.'s.

EXAMPLE 15 *Find the number of terms common to the two A.P.'s:* 3, 7, 11, ... 407 and 2, 9, 16, ..., 709.

SOLUTION Let the number of terms in two A.P.'s be m and n respectively. Then,

407 = *m*th term of first A.P.,

and,

709 = *n*th term of second A.P.

$\Rightarrow 407 = 3 + (m-1) \times 4$ and $709 = 2 + (n-1) \times 7$

$\Rightarrow m = 102$ and $n = 102$

So, each A.P. consists of 102 terms.

Let *p*th term of first A.P. be identical to *q*th term of the second A.P. Then,

$$3 + (p-1) \times 4 = 2 + (q-1) \times 7$$

$\Rightarrow 4p - 1 = 7q - 5$

$\Rightarrow 4p + 4 = 7q$

$\Rightarrow 4(p+1) = 7q$

$\Rightarrow \dfrac{p+1}{7} = \dfrac{q}{4} = k$ (say)

$\Rightarrow p = 7k - 1$ and $q = 4k$

Since each A.P. consists of 102 terms. Therefore

$p \le 102$ and $q \le 102$

$\Rightarrow 7k - 1 \le 102$ and $4k \le 102$

$\Rightarrow k \le 14\dfrac{5}{7}$ and $k \le 25\dfrac{1}{2}$

$\Rightarrow k \le 14$

$\Rightarrow k = 1, 2, 3, \ldots, 14$

Corresponding to each value of k, we get a pair of identical terms. Hence, there are 14 identical terms in two A.P.'s.

EXAMPLE 16 *If the fourth power of the common difference of an A.P. with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.* **[IIT 2000]**

SOLUTION Let $a-3d, a-d, a+d, a+3d$ be four consecutive terms of an A.P. with integer terms. Clearly, the common difference is $2d$.

Since the terms are integers, therefore a and d are also integers.

Also,

$$(a-3d)(a-d)(a+d)(a+3d) + (2d)^4$$
$$= (a^2 - 9d^2)(a^2 - d^2) + 16d^4$$
$$= a^4 - 10a^2d^2 + 9d^4 + 16d^4$$
$$= a^4 - 10a^2d^2 + 25d^4$$
$$= (a^2 - 5d^2)^2,$$ which is square of an integer as a and d are integers.

EXAMPLE 17 *If* $\log_{10} 2, \log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ *are in A.P., then find the value of x.*

SOLUTION $\log_{10} 2, \log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ are in A.P.

$\Rightarrow 2\log_{10}(2^x - 1) = \log_{10} 2 + \log_{10}(2^x + 3)$

$\Rightarrow \log_{10}(2^x - 1)^2 = \log_{10} 2 . (2^x + 3)$

$\Rightarrow (2^x - 1)^2 = 2 . (2^x + 3)$

$\Rightarrow (y-1)^2 = 2(y+3)$, where $y = 2^x$

$\Rightarrow y^2 - 4y - 5 = 0$

$\Rightarrow (y-5)(y+1) = 0$

$\Rightarrow y = 5$ or, $y = -1$

$\Rightarrow 2^x = 5$ $[\because 2^x \neq -1]$

$\Rightarrow x = \log_2 5$

EXAMPLE 18 *For what values of the parameter a are three values of x such that* $5^{1+x} + 5^{1-x}, a/2, 25^x + 25^{-x}$ *are three consecutive terms of an A.P. ?*

SOLUTION $5^{1+x} + 5^{1-x}, a/2, 25^x + 25^{-x}$ are in A.P.

$\Rightarrow 2(a/2) = (5^{1+x} + 5^{1-x}) + (25^x + 25^{-x})$

$\Rightarrow a = (5 . 5^x + 5.5^{-x}) + (5^{2x} + 5^{-2x})$

$\Rightarrow a = 5\left(5^x + \frac{1}{5^x}\right) + \left(5^{2x} + \frac{1}{5^{2x}}\right)$

We know that the sum of a positive real number and its reciprocal is always greater than or equal to 2.

$\therefore \left(5^x + \frac{1}{5^x}\right) \geq 2$ and $\left(5^{2x} + \frac{1}{5^{2x}}\right) \geq 2$ for all x.

$\Rightarrow 5\left(5^x + \frac{1}{5^x}\right) + \left(5^{2x} + \frac{1}{5^{2x}}\right) \geq 5 \times 2 + 2$ for all x

$\Rightarrow a \geq 12$

EXAMPLE 19 *If* $1, \log_y x, \log_z y, -15\log_x z$ *are in A.P., then prove that*

(i) $z^3 = x$ (ii) $x = y^{-1}$ (iii) $y = \frac{1}{z^3}$

SOLUTION Let d be the common difference of the given A.P. Then,

$$\log_y x = 1 + d \Rightarrow x = y^{1+d} \quad ...(i)$$

$$\log_z y = 1 + 2d \Rightarrow y = z^{1+2d} \quad ...(ii)$$

and, $$-15\log_x z = 1 + 3d \Rightarrow z = x^{-\left(\frac{1+3d}{15}\right)} \quad ...(iii)$$

From (ii) and (iii), we get

$$y = \left\{x^{-\left(\frac{1+3d}{15}\right)}\right\}^{(1+2d)} = x^{\frac{-(1+2d)(1+3d)}{15}}$$

Substituting this value of y in (i), we get

$$x = x^{\frac{-(1+d)(1+2d)(1+3d)}{15}}$$

$\Rightarrow \frac{-(1+d)(1+2d)(1+3d)}{15} = 1$

$\Rightarrow (1+d)(1+2d)(1+3d) = -15$

$\Rightarrow 6d^3 + 11d^2 + 6d + 16 = 0$

$\Rightarrow (d+2)(6d^2 - d + 8) = 0$

$\Rightarrow d = -2$ $[\because 6d^2 - d + 8 = 0$ has imaginary roots$]$

Putting $d = -2$ in (i), (ii) and (iii), we get

$x = y^{-1}, y = z^{-3}$ and $z = x^{1/3}$

$\Rightarrow x = z^3$

EXAMPLE 20 *If* $\log_5 2, \log_5(2^x - 3)$ and $\log_5\left(\frac{17}{2} + 2^{x-1}\right)$ *are in A.P., then find the value of x.*

SOLUTION $\log_5 2, \log_5(2^x - 3)$ and $\log_5\left(\frac{17}{2} + 2^{x-1}\right)$ are in A.P.

$\Rightarrow 2 . \log_5(2^x - 3) = \log_5 2 + \log_5\left(\frac{17}{2} + 2^{x-1}\right)$

$\Rightarrow \log_5(2^x - 3)^2 = \log_5 2\left(\frac{17}{2} + 2^{x-1}\right)$

$\Rightarrow (2^x - 3)^2 = (17 + 2^x)$

$\Rightarrow (y-3)^2 = 17 + y$, where $y = 2^x$

$\Rightarrow y^2 - 7y - 8 = 0$

$\Rightarrow (y-8)(y+1) = 0$

$\Rightarrow y = 8$ or, $y = -1$

$\Rightarrow 2^x = 8$ $[\because 2^x \neq -1]$

$\Rightarrow x = 3.$

EXAMPLE 21 *If* $x^{18} = y^{21} = z^{28}$, *prove that* $3, 3\log_y x, 3\log_z y, 7\log_x z$ *form an A.P.*

SOLUTION We have,

$x^{18} = y^{21} = z^{28}$

$\Rightarrow 18\log_{10} x = 21\log_{10} y = 28\log_{10} z$

$\Rightarrow \frac{\log_{10} x}{\log_{10} y} = \frac{21}{18}, \frac{\log_{10} y}{\log_{10} z} = \frac{28}{21}$ and $\frac{\log_{10} z}{\log_{10} x} = \frac{18}{28}$

$\Rightarrow \log_y x = \frac{7}{6}, \log_z y = \frac{4}{3}$ and $\log_x z = \frac{9}{14}$

$\Rightarrow 3\log_y x = \frac{7}{2}, 3\log_z y = 4$ and $7\log_x z = \frac{9}{2}$

Clearly, 3, $3\log_y x$, $3\log_z y$ and $7\log_x z$ i.e. 3, $\frac{7}{2}$, 4, $\frac{9}{2}$ are in AP with common difference 1/2.

EXAMPLE 22 *If x, y, z are real numbers satisfying the equation*

$$25(9x^2+y^2)+9z^2-15(5xy+yz+3zx)=0$$

then prove that x, y, z are in A.P.

SOLUTION We have,

$25(9x^2+y^2)+9z^2-15(5xy+yz+3zx)=0$

$\Rightarrow (15x)^2+(5y)^2+(3z)^2-(15x)(5y)-(5y)(3z)-(3z)(15x)=0$

$\Rightarrow \frac{1}{2}[(15x-5y)^2+(5y-3z)^2+(3z-15x)^2]=0$

$\Rightarrow (15x-5y)^2+(5y-3z)^2+(3z-15x)^2=0$

$\Rightarrow 15x-5y=0, 5y-3z=0, 3z-15x=0$

$\Rightarrow 15x=5y=3z$

$\Rightarrow \frac{x}{1}=\frac{y}{3}=\frac{z}{5}=k$ (say)

$\Rightarrow x=k, y=3k, z=5k$

Clearly, x, y, z are in A.P.

EXAMPLE 23 *If $a_1, a_2, a_3, \ldots, a_n, \ldots$ be in A.P. and*

$$S_k = a_{(k-1)n+1}+a_{(k-1)n+2}+a_{(k-1)n+3}+\ldots+a_{kn}$$

Prove that : $S_1, S_2, S_3 \ldots$ are in A.P. whose common difference is n^2 times the common difference of the given A.P.

SOLUTION Let d be the common difference of the given A.P. Then,

$$a_r = a_1+(r-1)d,\ r=1,2,3,\ldots.$$

Now,

$S_k = a_{(k-1)n+1}+a_{(k-1)n+2}+\ldots+a_{kn}$

$= a_{(k-1)n+1}+a_{(k-1)n+2}+a_{(k-1)n+3}+\ldots+a_{(k-1)n+n}$

$= [a_1+(k-1)nd]+[a_1+\{(k-1)n+1\}d]$

$+[a_1+\{(k-1)n+2\}d]+\ldots+[a_1+\{(k-1)n+(n-1)\}d]$

$= na_1+n(k-1)nd+\{d+2d+3d+\ldots+(n-1)d\}$

$= na_1+n^2(k-1)d+d\{1+2+3+..+(n-1)\}$

$= na_1+n^2(k-1)d+\frac{n(n-1)}{2}d.$

$S_{k+1} = na_1+n^2kd+\frac{n(n-1)}{2}d$

$\therefore S_{k+1}-S_k = n^2d, k=1,2,3,\ldots.$

$\Rightarrow S_1, S_2, S_3, \ldots$ are in A.P. with common difference n^2d i.e. n^2 (common difference of the given A.P.).

EXAMPLE 24 *If $a_1, a_2, a_3, a_4, \ldots$ are in A.P. such that*

$$a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=450$$

then find the sum $a_1+a_8+a_{17}+a_{24}$

SOLUTION We have,

$a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=450$

$\Rightarrow (a_1+a_{24})+(a_5+a_{20})+(a_{10}+a_{15})=450$

$\Rightarrow 3(a_1+a_{24})=450$ [Using property V]

$\Rightarrow a_1+a_{24}=\frac{450}{3}=150$ ...(i)

Now,

$a_1+a_8+a_{17}+a_{24} = (a_1+a_{24})+(a_8+a_{17})$

$= 2(a_1+a_{24})$

$= 2\times 150$ [Using (i)]

$= 300$

EXAMPLE 24 *If a, b, c, d are in distinct real numbers in A.P. such that*

$$2(a-b)+x(b-c)^2+(c-a)^3=2(a-d)+(b-d)^2+(c-d)^3$$

then find the values of x.

SOLUTION Let α be the common difference of the A.P. Then,

$b=a+\alpha, c=a+2\alpha$ and $d=a+3\alpha$

$\therefore 2(a-b)+x(b-c)^2+(c-a)^3 = 2(a-d)+(b-d)^2+(c-d)^3$

$\Rightarrow -2\alpha+\alpha^2x+8\alpha^3 = -6\alpha+4\alpha^2-\alpha^3$

$\Rightarrow 9\alpha^3+(x-4)\alpha^2+4\alpha=0$

$\Rightarrow 9\alpha^2+(x-4)\alpha+4=0$ [$\because \alpha \neq 0$]

But, α is real. Therefore,

$(x-4)^2-144 \geq 0$

$\Rightarrow (x-4-12)(x-4+12)\geq 0$

$\Rightarrow (x-16)(x+8)\geq 0$

$\Rightarrow x\leq -8$ or, $x\geq 16$

$\Rightarrow x\in(-\infty,-8]\cup[16,\infty).$

EXAMPLE 25 *Show that there exist no two positive integers n and r ($r<n$) such that ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}$ and ${}^nC_{r+3}$ are in A.P.*

SOLUTION If possible, let ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}$ and ${}^nC_{r+3}$ are in A.P. Then,

$2({}^nC_{r+1}) = {}^nC_r+{}^nC_{r+2}$ ...(i)

and, $2({}^nC_{r+2}) = {}^nC_{r+1}+{}^nC_{r+3}$ ...(ii)

Now,

$$2 \cdot {}^nC_{r+1} = {}^nC_r + {}^nC_{r+2}$$

$$\Rightarrow \quad 2 = \frac{{}^nC_r}{{}^nC_{r+1}} + \frac{{}^nC_{r+2}}{{}^nC_{r+1}}$$

$$\Rightarrow \quad 2 = \frac{r+1}{n-r} + \frac{n-r-1}{r+2}$$

$$\Rightarrow \quad 2(n-r)(r+2) = (r+1)(r+2) + (n-r)(n-r-1)$$

$$\Rightarrow \quad 2(nr + 2n - r^2 - 2r) = r^2 + 3r + 2 + n^2 - 2nr + r^2 - n + r$$

$$\Rightarrow \quad (n-2r)^2 + 8r - 5n + 2 = 0 \qquad \text{...(iii)}$$

and, $2({}^nC_{r+2}) = {}^nC_{r+1} + {}^nC_{r+3}$

$$\Rightarrow \quad (n-2r-2)^2 + 8(r+1) - 5n + 2 = 0 \qquad \text{...(iv)}$$

[Replacing r by $r+1$ in (iii)]

Subtracting (iv) from (iii), we get

$$2(2n - 4r - 2) - 8 = 0$$

$$\Rightarrow \quad n = 2r + 3$$

Putting $n = 2r + 3$ in (iii), we get

$r = -2$, which is a contradiction that r is a positive integer.

Hence, there exist no positive integers n and r $(r < n)$ for which ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}, {}^nC_{r+3}$ are in AP.

EXAMPLE 26 *Each of the two triplets of numbers* $\log a, \log b, \log c$; $\log a - \log 2b$, $\log 2b - \log 3c, \log 3c - \log a$ *is in A.P. Prove that* a, b, c *can be the lengths of the sides of a triangle. Also, find* $a : b : c$.

SOLUTION It is given that $\log a, \log b, \log c$ are in AP

$$\therefore \quad 2 \log b = \log a + \log c$$

$$\Rightarrow \quad \log b^2 = \log ac$$

$$\Rightarrow \quad b^2 = ac \qquad \text{...(i)}$$

Also,

$\log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a$ are in AP.

$$\Rightarrow \quad 2(\log 2b - \log 3c) = (\log a - \log 2b) + (\log 3c - \log a)$$

$$\Rightarrow \quad 3 \log 2b = 3 \log 3c$$

$$\Rightarrow \quad 2b = 3c$$

$$\Rightarrow \quad b = \frac{3c}{2}$$

Putting $b = \frac{3c}{2}$ in (i), we get

$$\left(\frac{3c}{2}\right)^2 = ac \Rightarrow 9c = 4a \Rightarrow a = \frac{9c}{4}$$

Putting $a = \frac{9c}{4}$ in (i), we get

$$b = \frac{3c}{2}$$

Thus, the given triplet of numbers is

$$a = \frac{9c}{4}, b = \frac{3c}{2}, c = c \Rightarrow a : b : c = 9 : 6 : 4.$$

Now, a, b, c will form the sides of a triangle, if

(i) $a + b > c$

(ii) $b + c > a$

and, (iii) $c + a > b$.

We have, $a + b = \frac{9c}{4} + \frac{3c}{2} = \frac{15c}{4} > c$

$$b + c = \frac{3c}{2} + c = \frac{5c}{2} > a$$

and, $$c + a = c + \frac{9c}{4} = \frac{13c}{4} > b$$

Thus, a, b, c form the sides of a triangle and $a : b : c = 9 : 6 : 4$.

EXAMPLE 27 *Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers. Find the numbers.*

SOLUTION Let the four integers be $a - d, a, a + d$ and $a + 2d$.

Clearly, a and d are integers and $d > 0$.

Since these integers form an increasing A.P. So, $a + 2d$ is the largest number.

By hypothesis,

$$(a + 2d) = (a - d)^2 + a^2 + (a + d)^2$$

$$\Rightarrow \quad 2d^2 - 2d + 3a^2 - a = 0$$

$$\Rightarrow \quad d = \frac{1}{2}\left[1 \pm \sqrt{1 + 2a - 6a^2}\right] \qquad \text{...(i)}$$

Since d is a positive integer. Therefore,

$$1 + 2a - 6a^2 > 0$$

$$\Rightarrow \quad 6a^2 - 2a - 1 < 0$$

$$\Rightarrow \quad \frac{1 - \sqrt{7}}{6} < a < \frac{1 + \sqrt{7}}{6}$$

$$\Rightarrow \quad a = 0 \qquad [\because a \text{ is an integer}]$$

Putting $a = 0$ in (i), we get

$$d = \frac{1}{2}(1 \pm 1) = 1 \text{ or } 0$$

But, $d > 0$ therefore, $d = 1$.

Hence, the numbers are $-1, 0, 1, 2$.

EXAMPLE 28 *If* $a_1, a_2, a_3, \ldots, a_n$ *are n numbers such that*

$$(n-1)x^2 - 2(a_n - a_1)x + a_1^2 + 2a_2^2 + \ldots + 2a_{n-1}^2 + a_n^2$$
$$= 2(a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n)$$

Show that $a_1, a_2, a_3, ..., a_n$ are in A.P. with common difference x.

SOLUTION We have,

$$(n-1)x^2 - 2(a_n - a_1)x + a_1^2 + 2a_2^2 + ... + 2a_{n-1}^2 + a_n^2$$
$$= 2(a_1 a_2 + a_2 a_3 + ... + a_{n-1} a_n)$$

$$\Rightarrow (n-1)x^2 - 2\{(a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1})\}x + (a_2 - a_1)^2 + (a_3 - a_2)^2 + ... + (a_n - a_{n-1})^2 = 0$$

$$\Rightarrow \{x^2 - 2(a_2 - a_1)x + (a_2 - a_1)^2\} + \{x^2 - 2(a_3 - a_2)x + (a_3 - a_2)^2\} + ... + \{x^2 - 2(a_n - a_{n-1})x + (a_n - a_{n-1})^2\} = 0$$

$$\Rightarrow \{x - (a_2 - a_1)\}^2 + \{x - (a_3 - a_2)\}^2 + ... + \{x - (a_n - a_{n-1})\}^2 = 0$$

$$\Rightarrow x - (a_2 - a_1) = 0, x - (a_3 - a_2) = 0, ..., x - (a_n - a_{n-1}) = 0$$

$$\Rightarrow a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1} = x$$

$\Rightarrow$ $a_1, a_2, a_3, ..., a_n$ are in A.P. with common difference x.

EXAMPLE 29 *If θ and α are two real numbers such that $\frac{\cos^4\theta}{\cos^2\alpha}, \frac{1}{2}, \frac{\sin^4\theta}{\sin^2\alpha}$ are in A.P.*

Prove that $\frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha}, \frac{1}{2}, \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha}$ are also in A.P. for all $n \in N$.

SOLUTION It is given that $\frac{\cos^4\theta}{\cos^2\alpha}, \frac{1}{2}$ and $\frac{\sin^4\theta}{\sin^2\alpha}$ are in A.P.

Therefore,

$$2\left(\frac{1}{2}\right) = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha} \quad ...(i)$$

$$\Rightarrow (\cos^2\theta + \sin^2\theta)^2 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow (\cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta)\cos^2\alpha\sin^2\alpha = \cos^4\theta\sin^2\alpha + \sin^4\theta\cos^2\alpha$$

$$\Rightarrow \cos^4\theta\sin^2\alpha(1 - \cos^2\alpha) + \sin^4\theta\cos^2\alpha(1 - \sin^2\alpha) - 2\sin^2\theta\cos^2\theta\sin^2\alpha\cos^2\alpha = 0$$

$$\Rightarrow \cos^4\theta\sin^4\alpha + \sin^4\theta\cos^4\alpha - 2\sin^2\theta\cos^2\theta\sin^2\alpha\cos^2\alpha = 0$$

$$\Rightarrow (\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha)^2 = 0$$

$$\Rightarrow \cos^2\theta\sin^2\alpha = \sin^2\theta\cos^2\alpha$$

$$\Rightarrow \frac{\cos^2\theta}{\cos^2\alpha} = \frac{\sin^2\theta}{\sin^2\alpha} \quad ...(ii)$$

From (i), we have

$$\frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha} = 1$$

$$\Rightarrow \frac{\cos^2\theta}{\cos^2\alpha}\cdot\cos^2\theta + \frac{\sin^2\theta}{\sin^2\alpha}\cdot\sin^2\theta = 1$$

$$\Rightarrow \frac{\sin^2\theta}{\sin^2\alpha}\cos^2\theta + \frac{\sin^4\theta}{\sin^2\alpha} = 1 \quad \text{[Using (ii)]}$$

$$\Rightarrow \frac{\sin^2\theta}{\sin^2\alpha}(\cos^2\theta + \sin^2\theta) = 1$$

$$\Rightarrow \sin^2\theta = \sin^2\alpha \quad ...(iii)$$

From (ii) and (iii), we get

$$\frac{\cos^2\theta}{\cos^2\alpha} = 1 \Rightarrow \cos^2\theta = \cos^2\alpha \quad ...(iv)$$

Now,

$$\frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha} + \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha}$$
$$= \left(\frac{\cos^2\theta}{\cos^2\alpha}\right)^n\cos^2\theta + \left(\frac{\sin^2\theta}{\sin^2\alpha}\right)^n\sin^2\theta$$
$$= \cos^2\theta + \sin^2\theta \quad \text{[Using (iii) and (iv)]}$$
$$= 1$$
$$= 2\left(\frac{1}{2}\right)$$

Hence, $\frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha}, \frac{1}{2}, \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha}$ are in A.P.

EXAMPLE 30 *Suppose x, y, z are in A.P. and that $\sin(y+z) = x$, $\sin(z+x) = y$ and $\sin(x+y) = z$. If $\sin 2y \neq 0$, then prove that $\tan x, \tan y, \tan z$ are in A.P.*

SOLUTION It is given that x, y, z are in A.P. Therefore,

$$2y = x + z \quad ...(i)$$

$$\Rightarrow 2\sin(x+z) = \sin(y+z) + \sin(x+y)$$
$$[\because x = \sin(y+z), y = \sin(z+x) \text{ and } z = \sin(x+y)]$$

$$\Rightarrow 2\sin(x+z) = 2\sin\left(\frac{2y+z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)$$

$$\Rightarrow 2\sin 2y = 2\sin 2y\cos\left(\frac{z-x}{2}\right) \quad \text{[Using (i)]}$$

$$\Rightarrow \sin 2y = \sin 2y\cos\left(\frac{z-x}{2}\right)$$

$$\Rightarrow \cos\left(\frac{z-x}{2}\right) = 1 \quad [\because \sin 2y \neq 0]$$

$$\Rightarrow z = x \quad ...(ii)$$

From (i) and (ii), we get

$$x = y = z.$$

$$\Rightarrow \tan x = \tan y = \tan z$$

$\Rightarrow$ $\tan x, \tan y, \tan z$ are in A.P. with common difference zero.

EXAMPLE 31 *There are two sets of numbers each consisting of three terms in A.P. and the sum of each set is 15. The common difference of the first set is greater by unity than the common difference of the second set. The product of the numbers in the first set is to the product in the second set as 7 to 8. Find the numbers.*

SOLUTION Let $a-b, a, a+b$ be the first set of numbers and $c-d, c, c+d$ be the second set. Then,

$$a-b+a+a+b = 15 \text{ and } c-d+c+c+d = 15$$

$\Rightarrow$ $a = 5$ and $c = 5$

It is given that the common difference of the first set is greater by unit than the common difference of the second set. Therefore,

$b = d + 1$

It is also given that

$$\frac{a(a^2 - b^2)}{c(c^2 - d^2)} = \frac{7}{8}$$

$$\Rightarrow \frac{25 - (d+1)^2}{25 - d^2} = \frac{7}{8} \qquad [\because a = c = 5 \text{ and } b = d + 1]$$

$\Rightarrow$ $d^2 + 16d - 17 = 0$

$\Rightarrow$ $(d + 17)(d - 1) = 0$

$\Rightarrow$ $d = -17, 1$

$\Rightarrow$ $b = -16, 2$ $[\because b = d + 1]$

Hence, the two sets are :

21, 5, – 11 and 22, 5, – 12 [When $b = -16, d = -17$]

or

3, 5, 7 and 4, 5, 6 [When $b = 2, d = 1$]

EXAMPLE 32 *If a, b, c, d are four distinct numbers in A.P., prove that*

$$a^{-1} + d^{-1} > b^{-1} + c^{-1} > 4(a + d)^{-1}$$

SOLUTION Let $a = \alpha - 3\beta$, $b = \alpha - \beta$, $c = \alpha + \beta$ and $d = \alpha + 3\beta$ be the four distinct numbers in A.P. Then,

$$a^{-1} + d^{-1} = \frac{1}{a} + \frac{1}{d} = \frac{1}{\alpha - 3\beta} + \frac{1}{\alpha + 3\beta} = \frac{2\alpha}{\alpha^2 - 9\beta^2}$$

and,

$$b^{-1} + c^{-1} = \frac{1}{b} + \frac{1}{c} = \frac{1}{\alpha - \beta} + \frac{1}{\alpha + \beta} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Now,

$$\alpha^2 - 9\beta^2 = (\alpha^2 - \beta^2) - 8\beta^2 < \alpha^2 - \beta^2$$

$$\therefore \quad \frac{1}{\alpha^2 - 9\beta^2} > \frac{1}{\alpha^2 - \beta^2}$$

$$\Rightarrow \frac{2\alpha}{\alpha^2 - 9\beta^2} > \frac{2\alpha}{\alpha^2 - \beta^2}$$

$$\Rightarrow a^{-1} + d^{-1} > b^{-1} + c^{-1} \qquad \text{...(i)}$$

Also,

$$4(a + d)^{-1} = \frac{4}{a + d} = \frac{4}{2\alpha} = \frac{2}{\alpha}.$$

$$\therefore \quad b^{-1} + c^{-1} > 4(a + d)^{-1}$$

If $\frac{2\alpha}{\alpha^2 - \beta^2} = \frac{2}{\alpha}$

i.e. if $2\alpha^2 > 2(\alpha^2 - \beta^2)$

i.e. if $2\beta^2 > 0$, which is always true.

$$\therefore \quad b^{-1} + c^{-1} > 4(a + d)^{-1} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$a^{-1} + d^{-1} > b^{-1} + c^{-1} > 4(a + d)^{-1}$$

EXERCISE 21.1

1. The mth term of an A.P. is n and its nth term is m. Prove that its pth term is $(m + n - p)$. Also, show that its $(m + n)$ th term is zero.

2. If the mth term of an A.P. be $1/n$ and nth term be $1/m$, then show that its (mn) *th* term is 1.

3. There are two arithmetic progressions whose common differences differ by unity, but sum of the three consecutive terms in each is 15. If P_1 and P_2 denote the products of these terms such that $P_1 : P_2 = 7 : 8$, then find the two arithmetic progressions.

4. Find the value of x for which $\log_3(2^{1-x} + 3)$, $\log_9 4$ and $\log_{27}(2^x - 1)^3$ are in A.P.

5. Prove that the square roots of three unequal prime numbers cannot be three terms of an A.P.

6. If m times the mth term of an AP is n times the nth term, then show that its $(m + n)$ *th* term is zero.

7. If $\log_a x, \log_b x, \log_c x$ are in AP where $x \neq 1$ and $x > 0$, prove that $c^2 = (ac)^{\log_a b}$.

8. If a, b, c are three terms of an A.P. such that $a \neq b$, show that $\frac{b - c}{a - b}$ is a rational number.

9. Find three digit numbers that are divisible by 5 as well as 9 and whose consecutive digits are in A.P.

10. The sides of a triangle form an A.P. whose common difference is equal to 4 cm and the area of the triangle is 24 cm^2. Determine the sides of the triangle.

11. Let $a_1, a_2, ..., a_n$ be an A.P. with non-negative terms such that :

(i) $a_4^2 + a_{12}^2 = 1170$ and (ii) $a_7 + a_{15} = 60$

Find the fist term and the common difference of the A.P.

12. If a, b, c are in A.P. prove that the following are also in A.P.

(i) $b + c, c + a, a + b$

(ii) $a^2(b + c), b^2(c + a), c^2(a + b)$

(iii) $\frac{1}{\sqrt{b} + \sqrt{c}}, \frac{1}{\sqrt{c} + \sqrt{a}}, \frac{1}{\sqrt{a} + \sqrt{b}}$

(iv) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$

(v) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$

(vi) $[(b+a)^2-a^2], [(c+a)^2-b^2], [(a+b)^2-c^2]$

13. If a^2, b^2, c^2 are in A.P., then prove that the following are in A.P.

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

14. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ an in A.P. prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

15. If $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P. show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

16. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P. prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

17. If a, b, c, are in A.P. then prove that

(i) $(a-c)^2 = 4(b^2-ac)$ (ii) $a^3+4b^3\ c^3 = 3b(a^2+c^2)$

18. If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P. Show that either a, b, c are in A.P. or $ab+bc+ca=0$.

19. An circle of one centrimetre radius is on a piece of paper and with the same centre $(3n-1)$ other circles are drawn of radii 2 cm, 3 cm, 4 cm, ... The inner circle is painted blue, the ring between them and the next circle is painted red, the next ring yellow, then the other rings blue, red, yellow and so on in this order, prove that the successive areas of each colour are in A.P.

ANSWERS

3. 7, 5, 3, ... and 6, 5, 4, ... OR $-11, 5, 21$ and.... $-12, 5, 22, \ldots$

4. 1

9. 135, 630, 765

10. $4(\sqrt{6}-1), 4\sqrt{6}, 4(\sqrt{6}+1)$

11. 0, 3

HINTS TO SELECTED PROBLEMS

14. Required sum $= 1+2+3+\ldots+n = \frac{n}{2}(1+n)$

15. Find the sum of the A.P. 3, 6, 9, ... , 99 which contains 33 terms.

16. Required sum $= 1+3+5+\ldots+(2n-1) = \frac{n}{2}[1+(2n-1)]$

17. Required sum $= 101+103+\ldots+199 = \frac{50}{2}(101+199)$.

—❑❑❑—

21.5 SUM TO n TERMS OF AN A.P.

THEOREM *The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is given by*

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Also, $S_n = \frac{n}{2}[a+l]$, where l = last term $= a+(n-1)d$.

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference 'd'. Then,

$$S_n = a_1+a_2+a_3+\ldots.+a_{n-1}+a_n$$

Also, $S_n = a_n+a_{n-1}+a_{n-2}+\ldots+a_2+a_1$

Adding these two, we get

$$2S_n = (a_1+a_n)+(a_2+a_{n-1})+(a_3+a_{n-2})+\ldots.+(a_n+a_1)$$

$\Rightarrow 2S_n = n(a_1+a_n)$ [By using property V]

$\Rightarrow S_n = \frac{n}{2}(a_1+a_n)$ ...(i)

$\Rightarrow S_n = \frac{n}{2}[a+a+(n-1)d]$ $[\because a_1 = a \text{ and } a_n = a+(n+1)d]$

$\Rightarrow S_n = \frac{n}{2}[2a+(n-1)d]$

Also, from (i), we have

$$S_n = \frac{n}{2}(a_1+a_n)$$

$\Rightarrow S_n = \frac{n}{2}(a+l)$ $[\because a_n = l \text{ and } a_1 = a]$

AN IMPORTANT PROPERTY *A sequence is an A.P. if and only if the sum of its n terms is of the form An^2+Bn, where A, B are constants. In such a case, the common difference of the A.P. is 2A.*

PROOF Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$\Rightarrow S_n = an+\frac{n^2}{2}d-\frac{n}{2}d = \left(\frac{d}{2}\right)n^2+\left(a-\frac{d}{2}\right)n$

$\Rightarrow S_n = An^2 + Bn$, where $A = \frac{d}{2}$ and $B = a - \frac{d}{2}$

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$.

Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be of the form $An^2 + Bn$. Then, we have to show that the sequence is an A.P.

We have,

$$S_n = An^2 + Bn$$

$\Rightarrow S_{n-1} = A(n-1)^2 + B(n-1)$ [On replacing n by $n-1$]

Now,

$$a_n = S_n - S_{n-1}$$

$\Rightarrow a_n = \{An^2 + Bn\} - \{A(n-1)^2 + B(n-1)\}$

$= 2An + (B - A)$

$\Rightarrow a_{n+1} = 2A(n+1) + (B - A)$ [On replacing n by $(n+1)$]

$\therefore a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B - A)\} = 2A$

Since $a_{n+1} - a_n = 2A$ for all $n \in N$. So, the sequence is an A.P. with common difference $2a$.

REMARK *It follows from this property that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2. For example, if $S_n = 3n^2 + 2n$, we can say that it is the sum of n terms of an A.P. with common difference 6.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If the mth term of an A.P. is $\frac{1}{n}$ and the nth term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn + 1)$.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$a_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n} \quad \ldots(i)$$

And, $$a_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m} \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m}$$

$$\Rightarrow (m-n)d = \frac{m-n}{mn}$$

$$\Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow a = \frac{1}{mn}$$

Now,

$$S_{mn} = \frac{mn}{2}[2a + (mn-1)d]$$

$$\Rightarrow S_{mn} = \frac{mn}{2}\left[\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right]$$

$$\Rightarrow S_{mn} = \frac{1}{2}(mn + 1)$$

EXAMPLE 2 *Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.*

SOLUTION We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e.

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$$

So, *if* an A.P. consists of 24 terms, then $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$.

Now,

$$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$$

$$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$$

$$\Rightarrow 3(a_1 + a_{24}) = 225$$

$$\Rightarrow a_1 + a_{24} = \frac{225}{3} = 75 \quad \ldots(i)$$

$$\therefore S_{24} = \frac{24}{2}(a_1 + a_{24}) \quad \left[\text{Using : } S_n = \frac{n}{2}(a_1 + a_n)\right]$$

$$= 12(75) = 900 \quad \text{[Using (i)]}$$

EXAMPLE 3 *The pth term of an A.P. is a and qth term is b. Prove that the sum of its $(p + q)$ terms is $\frac{p+q}{2}\left[a + b + \frac{a-b}{p-q}\right]$.*

SOLUTION Let A and D be the first term and common difference respectively of the given A.P. Then,

$$a = pth \text{ term} \Rightarrow a = A + (p-1)D \quad \ldots(i)$$

$$b = qth \text{ term} \Rightarrow b = A + (q-1)D \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$D = \frac{a-b}{p-q}$$

Adding (i) and (ii), we get

$$a+b = 2A+(p+q-2)D$$

$$\Rightarrow a+b = 2A+(p+q-1)D-D$$

$$\Rightarrow (a+b)+D = 2A+(p+q-1)D$$

$$\Rightarrow (a+b)+\frac{a-b}{p-q} = 2A+(p+q-1)D \quad ...(iii)$$

Now,

$$S_{p+q} = \text{Sum of } (p+q) \text{ terms}$$

$$= \frac{p+q}{2}[2A+(p+q-1)D]$$

$$= \frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right] \quad \text{[Using (iii)]}$$

EXAMPLE 4 *If in an A.P. the sum of m terms is equal to n and the sum of n terms is equal to m, then prove that the sum of (m + n) terms is − (m + n).*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = n \Rightarrow \frac{m}{2}[2a+(m-1)d] = n$$

$$\Rightarrow 2am+m(m-1)d = 2n \quad ...(i)$$

And, $S_n = m \Rightarrow \frac{n}{2}[2a+(n-1)d] = m$

$$\Rightarrow 2an+n(n-1)d = 2m \quad ...(ii)$$

Subtracting (ii) from (i), we get

$$2a(m-n)+\{m(m-1)-n(n-1)\}d = 2n-2m$$

$$\Rightarrow 2a(m-n)+\{(m^2-n^2)-(m-n)\}d = -2(m-n)$$

$$\Rightarrow 2a+(m+n-1)d = -2 \quad ...(iii)$$

[On dividing both sides by $(m-n)$]

Now,

$$S_{m+n} = \frac{m+n}{2}[2a+(m+n-1)d]$$

$$= \frac{(m+n)}{2}(-2) \quad \text{[Using (iii)]}$$

$$= -(m+n)$$

EXAMPLE 5 *If the sum of m terms of an A.P. is the same as the sum of its n terms, show that the sum of its (m + n) terms is zero.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n$$

$$\Rightarrow \frac{m}{2}[2a+(m-1)d] = \frac{n}{2}[2a+(n-1)d]$$

$$\Rightarrow 2a(m-n)+\{m(m-1)-n(n-1)\}d = 0$$

$$\Rightarrow 2a(m-n)+\{(m^2-n^2)-(m-n)\}d = 0$$

$$\Rightarrow (m-n)[2a+(m+n-1)d] = 0 \quad ...(i)$$

$$\Rightarrow 2a+(m+n-1)d = 0 \quad [\because m-n \neq 0]$$

Now,

$$S_{m+n} = \frac{m+n}{2}[2a+(m+n-1)d]$$

$$= \frac{m+n}{2} \times 0 = 0 \quad \text{[Using (i)]}$$

EXAMPLE 6 *If $S_1, S_2, S_3, \ldots S_m$ are the sums of n terms of m A.P.'s whose first terms are 1, 2, 3, ..., m and common differences are 1, 3, 5 ,..., (2m − 1) respectively. Show that*

$$S_1+S_2+\ldots+S_m = \frac{mn}{2}(mn+1).$$

SOLUTION We have,

First terms	*Common differences*	*Sums of n terms*
1	1	$S_1 = \frac{n}{2}[2\times1+(n-1)\times1]$
2	3	$S_2 = \frac{n}{2}[2\times2+(n-1)\times3]$
3	5	$S_3 = \frac{n}{2}[2\times3+(n-1)5]$
⋮	⋮	
m	$2m-1$	$S_m = \frac{n}{2}[2m+(n-1)(2m-1)]$

$$\therefore S_1+S_2+\ldots+S_m$$

$$= \frac{n}{2}[2\times1+(n-1)\times1]$$

$$+\frac{n}{2}[2\times2+(n-1)\times3]+\ldots+\frac{n}{2}[2m+(n-1)(2m-1)]$$

$$= \frac{n}{2}[2\times(1+2+3+\ldots+m)+(n-1)(1+3+5+\ldots+(2m-1))]$$

$$= \frac{n}{2}[2\times\frac{m}{2}(1+m)+(n-1)\frac{m}{2}\{1+(2m-1)\}]$$

$$= \frac{n}{2}[m(m+1)+m^2(n-1)]$$

$$= \frac{mn}{2}(mn+1)$$

EXAMPLE 7 *The ratio of the sum of n terms of two A.P.'s is (7n + 1) : (4n + 27). Find the ratio of their mth terms.*

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then the sums of their n terms are given by

$$S_n = \frac{n}{2}[2a_1+(n-1)d_1] \quad ...(i)$$

and, $S_n' = \frac{n}{2}[2a_2 + (n-1)d_2]$ ...(ii)

$$\therefore \quad \frac{S_n}{S_n'} = \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]}$$

$$= \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{7n+1}{4n+27}$$

$$\therefore \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27} \qquad \text{...(iii)}$$

To find the ratio of the *mth* terms of the two given A.P.'s, we replace n by $(2m-1)$ in (iii).

$$\therefore \quad \frac{2a_1 + (2m-2)d_1}{2a_2 + (2m-2)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27}$$

$$\Rightarrow \quad \frac{a_1 + (m-1)d_1}{a_2 + (m-1)d_2} = \frac{14m-6}{8m+23}$$

Hence, the ratio of the mth terms of the two A.P.'s is $(14m-6):(8m+23)$.

EXAMPLE 8 *The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of the mth and nth terms is $(2m-1):(2n-1)$.*

SOLUTION Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}[2a + (m-1)d] \text{ and } S_n = \frac{n}{2}[2a + (n-1)d]$$

respectively.

Now,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{\frac{m}{2}[2a + (m-1)d]}{\frac{n}{2}[2a + (n-1)d]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$$

$$\Rightarrow \{2a + (m-1)d\}n = \{2a + (n-1)d\}m$$

$$\Rightarrow 2a(n-m) = d\{(n-1)m - (m-1)n\}$$

$$\Rightarrow 2a(n-m) = d(n-m)$$

$$\Rightarrow d = 2a$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)d}{a + (n-1)d} = \frac{a + (m-1)2a}{a + (n-1)2a} = \frac{2m-1}{2n-1}$$

EXAMPLE 9 *If there are $(2n+1)$ terms in A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n+1):n$.*

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Let a_k denote the *kth* terms of the given A.P. Then, $a_k = a + (k-1)d$.

Now, S_1 = Sum of odd terms

$$= a_1 + a_3 + a_5 + \ldots + a_{2n+1}$$

$$= \frac{n+1}{2}[a_1 + a_{2n+1}]$$

$$= \frac{n+1}{2}[a + a + (2n+1-1)d]$$

$$= (n+1)(a + nd)$$

and, S_2 = Sum of even terms

$$= a_2 + a_4 + a_6 + \ldots + a_{2n}$$

$$= \frac{n}{2}[a_2 + a_{2n}]$$

$$= \frac{n}{2}[(a+d) + \{a + (2n-1)d\}]$$

$$= n(a + nd)$$

$$\therefore \quad S_1 : S_2 = (n+1)(a+nd) : n(a+nd) = (n+1):n$$

EXAMPLE 10 *The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.*

SOLUTION Let there be n sides of the polygon. Then the sum of its interior angles is given by

$$S_n = (2n-4) \text{ right angles} = (n-2) \times 180^\circ \qquad \text{...(i)}$$

Here, the interior angles form an A.P. with first term $a = 120^\circ$ and common difference $d = 5^\circ$.

$$\therefore \quad S_n = \frac{n}{2}[2 \times 120^\circ + (n-1) \times 5^\circ] \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$(n-2) \times 180^\circ = \frac{n}{2}[2 \times 120^\circ + (n-1) \times 5^\circ]$$

$$\Rightarrow \quad (n-2) \times 360 = n[5n + 235]$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0$$

$$\Rightarrow \quad (n-16)(n-9) = 0$$

$$\Rightarrow \quad n = 16 \text{ or } n = 9$$

But, when $n = 16$,

The last angle $= a_n = a + (n-1)d$

$= 120^\circ + (16-1) \times 5 = 190^\circ$, which is not possible.

Hence, $n = 9$.

EXAMPLE 11 *The first, second and the last terms of an A.P. are a, b, c respectively. Prove that the sum is*

$$\frac{(a+c)(b+c-2a)}{2(b-a)}$$

SOLUTION Let d be the common difference of the given A.P. Then,

$$d = b - a.$$

Let there be n terms in the given A.P. Then,

c = nth term

$\Rightarrow c = a + (n-1)\,d$

$\Rightarrow c = a + (n-1)\,(b-a)$ $[\because d = b-a]$

$\Rightarrow n - 1 = \dfrac{c-a}{b-a}$

$\Rightarrow n = \dfrac{c-a}{b-a} + 1$

$\Rightarrow n = \dfrac{b+c-2a}{b-a}$

Now,

Sum of the A.P.

= Sum of its n terms

$= \dfrac{n}{2}(a+c)$ $\left[\text{Using}: S_n = \dfrac{n}{2}(a+l)\right]$

$= \dfrac{(a+c)\,(b+c-2a)}{2\,(b-a)}.$

EXAMPLE 12 *Let S_k be the sum of first k terms of an A.P. What must this progression be for the ratio $\dfrac{S_{kx}}{S_x}$ to be independent of x ?*

SOLUTION Let a be the first term and d common difference of the given progression. Then,

$$\frac{S_{kx}}{S_x} = \frac{\frac{kx}{2}[2a + (kx-1)\,d]}{\frac{x}{2}[2a + (x-1)\,d]}$$

$$\Rightarrow \frac{S_{kx}}{S_x} = \frac{k\,[kxd + (2a-d)]}{[xd + (2a-d)]}$$

Clearly, the RHS of the above relation will be independend of x iff $2a - d = 0$ i.e. $d = 2a$.

Hence, the progression is $a, 3a, 5a, 7a, \ldots$, where a is any non-zero real number.

EXAMPLE 13 *Let S_n be the sum of first n terms of an A.P. with non-zero common difference. Find the ratio of first term and common difference if $\dfrac{S_{n_1\,n_2}}{S_{n_1}}$ is independent of n_1.*

SOLUTION Let the first term and common difference of the A.P. be a and d respectively. Then,

$$S_{n_1\,n_2} = \frac{n_1\,n_2}{2}\,[2a + (n_1\,n_2 - 1)\,d]$$

and,

$$S_{n_1} = \frac{n_1}{2}\,[2a + (n_1 - 1)\,d]$$

$$\therefore \frac{S_{n_1\,n_2}}{S_{n_1}} = \frac{n_2\,[2a + (n_1\,n_2 - 1)\,d]}{[2a + (n_1 - 1)\,d]}$$

$$\Rightarrow \frac{S_{n_1\,n_2}}{S_{n_1}} = \frac{n_2\,[(2a-d) + n_1\,n_2\,d]}{[(2a-d) + n_1\,d]}$$

Clearly, RHS will be independent of n_1 iff $2a - d = 0$ i.e. $d = 2a$.

Hence, $\dfrac{a}{d} = \dfrac{1}{2}$.

EXAMPLE 14 *Find an arithmetic progression, if it is given that the sum of its first n terms is a and the sum of their squares is b^2.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n$ be the required A.P. and let d be its common difference. Then,

$$a_1 + a_2 + a_3 + \ldots + a_n = a \qquad \text{[Given]}$$

and,

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = b^2 \qquad \text{[Given]}$$

Now,

$$a_1 + a_2 + \ldots + a_n = a$$

$$\Rightarrow \frac{n}{2}\,[2a_1 + (n-1)\,d] = a$$

$$\Rightarrow n\,a_1 + \frac{n\,(n-1)}{2}\,d = a \qquad \ldots(i)$$

and,

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = b^2$$

$$\Rightarrow \sum_{r=1}^{n} a_r^2 = b^2$$

$$\Rightarrow \sum_{r=1}^{n} \{a_1 + (r-1)\,d\}^2 = b^2$$

$$\Rightarrow \sum_{r=1}^{n} [a_1^2 + 2a_1\,d\,(r-1) + (r-1)^2\,d^2] = b^2$$

$$\Rightarrow \sum_{r=1}^{n} a_1^2 + 2a_1\,d \sum_{r=1}^{n} (r-1) + d^2 \sum_{r=1}^{n} (r-1)^2 = b^2$$

$$\Rightarrow n\,a_1^2 + 2a_1\,d\,\frac{n\,(n-1)}{2} + d^2 \cdot \frac{n\,(n-1)\,(2n-1)}{6} = b^2$$

$$\Rightarrow n a_1^2 + a_1\,d\,n\,(n-1) + \frac{n\,(n-1)\,(2n-1)}{6}\,d^2 = b^2 \qquad \ldots(ii)$$

Squaring (i), we get

$$n^2\,a_1^2 + n^2\,(n-1)\,a_1\,d + \frac{n^2\,(n-1)^2}{4}\,d^2 = a^2$$

$$\Rightarrow \quad n a_1^2 + n(n-1)\, a_1 d + \frac{n(n-1)^2}{4} d^2 = \frac{a^2}{n} \qquad \ldots(iii)$$

Subtracting (iii) from (ii), we get

$$\frac{n(n-1)(2n-1)}{6} d^2 - \frac{n(n-1)^2}{4} d^2 = b^2 - \frac{a^2}{n}$$

$$\Rightarrow \quad n(n-1) d^2 \left\{ \frac{2n-1}{6} - \frac{n-1}{4} \right\} = b^2 - \frac{a^2}{n}$$

$$\Rightarrow \quad n(n-1) d^2 \left(\frac{2n+2}{24} \right) = \frac{nb^2 - a^2}{n}$$

$$\Rightarrow \quad \frac{n(n^2-1)}{12} d^2 = \frac{nb^2 - a^2}{n}$$

$$\Rightarrow \quad d^2 = \frac{12(nb^2 - a^2)}{n^2(n^2-1)}$$

$$\Rightarrow \quad d = \pm \frac{2}{n} \sqrt{\frac{3(nb^2 - a^2)}{n^2 - 1}} \qquad \ldots(iv)$$

Putting the values of d in (i), we obtain

$$a_1 = \frac{a}{n} \pm \frac{n-1}{n} \sqrt{\frac{3(nb^2 - a^2)}{n^2 - 1}} \qquad \ldots(v)$$

Hence, the first term and the common difference of the required AP are given by (v) and (iv) respectively.

EXAMPLE 15 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then prove that*

$$\sum_{1 \le r < s \le n} a_r a_s = \frac{n(n-1)}{2} \left[a_1^2 + (n-1) a_1 d + \frac{1}{12} (3n^2 - 7n + 2) d^2 \right]$$

SOLUTION We have,

$$(a_1 + a_2 + \ldots + a_n)^2 = (a_1^2 + a_2^2 + \ldots + a_n^2) + 2 \sum_{1 \le r < s \le n} a_r a_s$$

$$2 \sum_{1 \le r < s \le n} a_r a_s = (a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) \qquad \ldots(i)$$

Now,

$$a_1 + a_2 + \ldots + a_n = \frac{n}{2} [2a_1 + (n-1) d]$$

$$\Rightarrow \quad (a_1 + a_2 + \ldots + a_n)^2 = \frac{n^2}{4} [4a_1^2 + 4 a_1 d (n-1) + (n-1)^2 d^2]$$

$$= n^2 a_1^2 + a_1 d\, n^2 (n-1) + \frac{n^2 (n-1)^2}{4} d^2$$

and,

$$a_1^2 + a_2^2 + \ldots + a_n^2$$

$$= \sum_{r=1}^{n} a_r^2$$

$$= \sum_{r=1}^{n} [a_1 + (r-1) d]^2 \qquad [\because a_r = a_1 + (r-1) d]$$

$$= \sum_{r=1}^{n} [a_1^2 + 2a_1 d (r-1) + (r-1)^2 d^2]$$

$$= n a_1^2 + 2a_1 d \cdot \sum_{r=1}^{n} (r-1) + d^2 \sum_{r=1}^{n} (r-1)^2$$

$$= n a_1^2 + 2a_1 d \cdot \frac{n(n-1)}{2} + d^2 \cdot \frac{n(n-1)(2n-1)}{6}$$

Substituting these values in (i), we get

$$2 \sum_{1 \le r < s \le n} a_r a_s$$

$$= \left[n^2 a_1^2 + a_1 d\, n^2 (n-1) + \frac{n^2 (n-1)^2}{4} d^2 \right]$$

$$- \left[n a_1^2 + a_1 d\, n (n-1) + d^2 \frac{n(n-1)(2n-1)}{6} \right]$$

$$= (n^2 - n) a_1^2 + a_1 d\, n (n-1)^2$$

$$+ n(n-1) d^2 \left\{ \frac{n(n-1)}{4} - \frac{2n-1}{6} \right\}$$

$$= n(n-1) a_1^2 + n(n-1)^2 a_1 d + \frac{n(n-1) d^2}{12} (3n^2 - 7n + 2)$$

$$= n(n-1) \left[a_1^2 + (n-1) a_1 d + \frac{1}{12} (3n^2 - 7n + 2) d^2 \right]$$

$$\therefore \quad \sum_{1 \le r < s \le n} a_r a_s = \frac{n(n-1)}{2} \left[a_1^2 + (n-1) a_1 d + \frac{1}{12} (3n^2 - 7n + 2) d^2 \right]$$

EXAMPLE 16 *Find the sum of the cubes of first n terms of an A.P. and show that this sum is divisible by the sum of first n terms of the A.P.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n$ be the A.P. with common difference 'd'. Then,

$$a_r = a_1 + (r-1) d, \; r = 1, 2, \ldots, n.$$

Let S_1 be the sum of first n terms of the given A.P. and S_2 denote the sum of the cubes of first n terms. Then,

$$S_1 = \frac{n}{2} [2a_1 + (n-1) d]$$

and,

$$S_2 = \sum_{r=1}^{n} a_r^3$$

$$= \sum_{r=1}^{n} [a_1 + (r-1) d]^3$$

$$= \sum_{r=1}^{n} [a_1^3 + 3a_1^2 d(r-1) + 3a_1 d^2 (r-1)^2 + (r-1)^3 d^3]$$

$$= \sum_{r=1}^{n} a_1^3 + 3a_1^2 d \sum_{r=1}^{n} (r-1) + 3a_1 d^2 \sum_{r=1}^{n} (r-1)^2 + d^3 \cdot \sum_{r=1}^{n} (r-1)^3$$

$$= na_1^3 + 3a_1^2 d \frac{n(n-1)}{2} + 3a_1 d^2 \frac{n(n-1)(2n-1)}{6} + d^3 \left(\frac{n(n-1)}{2}\right)^2$$

$$= \frac{n}{2} [2a_1^3 + 3a_1^2 d(n-1) + a_1 d^2 (n-1)(2n-1) + \frac{1}{2} n(n-1)^2 d^3]$$

$$= \frac{n}{2} \{2a_1 + (n-1)d\} \left\{a_1^2 + a_1 (n-1) d + \frac{n(n-1)}{2} d^2\right\}$$

$$= S_1 \cdot \left\{a_1^2 + a_1 (n-1) d + \frac{n(n-1)}{2} d^2\right\}$$

Hence, S_1 is a factor of S_2.

EXAMPLE 17 *Let there be a series whose terms are in A.P. If the term of the series be collected into groups of n terms and the terms in each group be added together, then show that the resulting series forms an A.P. whose common difference is to the original common difference as $n^2 : 1$.*

SOLUTION Let $a_1, a_2, a_3, a_4, \ldots$ be the A.P. with common difference d. Let the series

$$a_1 + a_2 + a_3 + \ldots + a_n + a_{n+1} + \ldots + a_{2n} + a_{2n+1} + \ldots$$

be collected into p groups, each of n-terms. Further let the sum of the terms in rth group be S_r, $r = 1, 2, \ldots, p$. Then,

$$S_1 = [a_1 + a_2 + \ldots + a_n]$$

$$= \frac{n}{2} (a_1 + a_n)$$

$$= \frac{n}{2} [2a_1 + (n-1) d]$$

$$S_2 = [a_{n+1} + a_{n+2} + \ldots + a_{2n}]$$

$$= \frac{n}{2} [a_{n+1} + a_{2n}]$$

$$= \frac{n}{2} [a_1 + nd + a_1 + (2n-1) d]$$

$$= \frac{n}{2} [2a_1 + (3n-1) d]$$

$$S_3 = [a_{2n+1} + a_{2n+2} + \ldots + a_{3n}]$$

$$= \frac{n}{2} [a_{2n+1} + a_{3n}]$$

$$= \frac{n}{2} [(a_1 + 2nd) + a_1 + (3n-1) d]$$

$$= \frac{n}{2} [2a_1 + (5n-1) d] \quad \text{and so on.}$$

In general,

$$S_r = [a_{(r-1)n+1} + \ldots + a_{rn}]$$

$$= \frac{n}{2} [a_{(r-1)n+1} + a_{rn}]$$

$$= \frac{n}{2} [a_1 + (r-1) nd + a_1 + (rn-1) d]$$

$$= \frac{n}{2} [2a_1 + \{(2r-1) n - 1\} d], \quad r = 1, 2, 3, \ldots p.$$

We have to prove that $S_1, S_2, \ldots S_p$ are also in A.P. with common difference $n^2 d$.

We have,

$$S_{r+1} - S_r$$

$$= \frac{n}{2} [2a_1 + \{(2(r+1) - 1) n - 1\} d] - \frac{n}{2} [2a_1 + \{(2r-1) n - 1\} d]$$

$$= \frac{n}{2} [\{(2r+1) n - 1\} - \{(2r-1) n - 1\}] d$$

$$= n^2 d, \; r = 1, 2, 3, \ldots; p$$

Hence, $S_1, S_2, \ldots, S_n$ are in A.P. with common difference $n^2 d$.

EXAMPLE 18 *In an A.P. of 2n terms the middle pair of terms are $\alpha + \beta$ and $\alpha - \beta$. Show that the sum of the cubes of the terms of the A.P is proportional to*

$$[2n\alpha^2 + (4n^2 - 1)\beta^2]$$

SOLUTION In a series of $2n$ terms, nth and $(n+1)$th terms form the middle pair of terms. Thus, if $a_1, a_2, a_3, \ldots, a_{2n}$ are the terms of the A.P., then

$$a_n = \alpha + \beta \text{ and } a_{n+1} = \alpha - \beta$$

$$\Rightarrow \; a_{n+1} - a_n = -2\beta$$

$$\Rightarrow \; d = -2\beta, \text{ where } d \text{ is the common difference of the A.P.}$$

Thus, the A.P. is

$$\alpha + (2n-1)\beta, \ldots, \alpha + 5\beta, \alpha + 3\beta, \alpha + \beta, \alpha - \beta, \alpha - 3\beta, \alpha - 5\beta, \ldots, \alpha - (2n-1)\beta$$

Now,

Required sum

$$= [\alpha + (2n-1)\beta]^3 + [\alpha + (2n-3)\beta]^3 + \ldots + (\alpha + \beta)^3 + (\alpha - \beta)^3 + \ldots + [\alpha - (2n-3)\beta]^3 + [\alpha - (2n-1)\beta]^3$$

$$= \sum_{r=1}^{n} [\alpha + (2r-1)\beta]^3 + [\alpha - (2r-1)\beta]^3$$

$$= \sum_{r=1}^{n} [2\alpha^3 + 6\alpha\beta^2 (2r-1)^2]$$

$$= \sum_{r=1}^{n} 2\alpha^3 + 6\alpha\beta^2 \sum_{r=1}^{n} (2r-1)^2$$

$$= n(2\alpha^3) + 6\alpha\beta^2 \left(\sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} (2r)^2 \right)$$

$$= 2n\alpha^3 + 6\alpha\beta^2 \left\{ \frac{2n(2n+1)(4n+1)}{6} - 4\,\frac{n(n+1)(2n+1)}{6} \right\}$$

$$= 2n\alpha^3 + 2n\alpha\beta^2 \left\{ (2n+1)(4n+1) - (2n+2)(2n+1) \right\}$$

$$= 2n\alpha^3 + 2n(2n+1)(2n-1)\alpha\beta^2$$

$$= 2n\alpha^3 + 2n(4n^2-1)\alpha\beta^2$$

EXAMPLE 19 *If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, prove that*

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

SOLUTION Let a denote the first term and d the common difference of the A.P. Further, let a_k denote the k*th* term of the A.P. Then,

Sum of m terms = Sum of next n terms

$$\Rightarrow a_1 + a_2 + a_3 + \ldots + a_m = a_{m+1} + a_{m+2} + \ldots + a_{m+n}$$

$$\Rightarrow 2(a_1 + a_2 + \ldots + a_m) = a_1 + a_2 + \ldots + a_m + a_{m+1} + a_{m+2} + \ldots + a_{m+n}$$

$$\Rightarrow 2S_m = S_{m+n}$$

$$\Rightarrow 2\,\frac{m}{2}[2a + (m-1)d] = \frac{m+n}{2}[2a + (m+n-1)d]$$

$$\Rightarrow \frac{2m}{m+n} = \frac{2a + (m+n-1)d}{2a + (m-1)d}$$

$$\Rightarrow \frac{2m}{m+n} - 1 = \frac{2a + (m+n-1)d}{2a + (m-1)d} - 1$$

$$\Rightarrow \frac{m-n}{m+n} = \frac{nd}{2a + (m-1)d} \qquad \ldots(i)$$

Similarly,

Sum of m terms = Sum of next p terms

$$\Rightarrow \frac{m-p}{m+p} = \frac{pd}{2a + (m-1)d} \qquad \ldots(ii)$$

Dividing (i) by (ii), we get

$$\frac{m-n}{m+n} \cdot \frac{m+p}{m-p} = \frac{n}{p}$$

$$\Rightarrow \frac{(m-n)(m+p)}{n} = \frac{(m+n)(m-p)}{p}$$

$$\Rightarrow \frac{(m-n)(m+p)}{nm} = \frac{(m+n)(m-p)}{mp}$$

$$\Rightarrow (m+p)\left(\frac{m-n}{mn}\right) = (m+n)\left(\frac{m-p}{mp}\right)$$

$$\Rightarrow (m+n)\left(\frac{1}{p} - \frac{1}{m}\right) = (m+p)\left(\frac{1}{n} - \frac{1}{m}\right)$$

$$\Rightarrow (m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

EXAMPLE 20 *Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3S_n$, then prove that $\frac{S_{3n}}{S_n} = 6$.*

SOLUTION Let a be the first term and d the common difference of the given A.P. Then,

$$S_{2n} = 3S_n$$

$$\Rightarrow \frac{2n}{2}[a + (2n-1)d] = 3 \cdot \frac{n}{2}[a + (n-1)d]$$

$$\Rightarrow 2[a + (2n-1)d] = 3[a(n-1)d]$$

$$\Rightarrow 2a - (3n - 3 - 4n + 2)d = 0$$

$$\Rightarrow 2a - (n+1)d = 0$$

$$\Rightarrow 2a = (n+1)d \qquad \ldots(i)$$

Now,

$$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)d]}{\frac{n}{2}[2a + (n-1)d]}$$

$$= \frac{3[(n+1)d + (3n-1)d]}{[(n+1)d + (n-1)d]}$$

$$= \frac{12nd}{2nd} = 6.$$

EXAMPLE 21 *Let there be two arithmetic progressions each of n terms :*

$$a, a+d, a+2d, \ldots, L \text{ and } p, p+q, p+2q, \ldots, L'$$

If these A.P's satisfy the following conditions :

$$\frac{L}{p} = \frac{L'}{a} = 4 \text{ and } \frac{S_n}{S_n'} = 2, \text{ find } \frac{d}{q} \text{ and } \frac{L}{L'}$$

SOLUTION We have,

$$L = a + (n+1)d \text{ and } L' = p + (n-1)q$$

$$\therefore \frac{L}{p} = \frac{L'}{a} = 4$$

$$\Rightarrow L = 4p \text{ and } L' = 4a$$

$$\Rightarrow a + (n-1)d = 4p \text{ and } p + (n-1)q = 4a$$

$$\Rightarrow 4p - a = (n-1)d \text{ and } 4a - p = (n-1)q$$

$$\Rightarrow \frac{4p-a}{4a-p} = \frac{(n-1)\,d}{(n-1)\,q}$$

$$\Rightarrow \frac{4p-a}{4a-p} = \frac{d}{q} \qquad \ldots(i)$$

Now,

$$\frac{S_n}{S_n'} = 2$$

$$\Rightarrow \frac{\frac{n}{2}(a+L)}{\frac{n}{2}(p+L')} = 2$$

$$\Rightarrow \frac{a+L}{p+L'} = 2$$

$$\Rightarrow \frac{a+4p}{p+4a} = 2 \qquad \left[\because \frac{L}{p} = \frac{L'}{a} = 4\right]$$

$$\Rightarrow 2p = 7a$$

$$\Rightarrow p = \frac{7a}{2}$$

Putting the value of p in (i), we get

$$\frac{14a-a}{4a-\frac{7a}{2}} = \frac{d}{q}$$

$$\Rightarrow \frac{d}{q} = 26.$$

Now,

$$\frac{L}{p} = \frac{L'}{a} \Rightarrow \frac{L}{L'} = \frac{p}{a} \Rightarrow \frac{L}{L'} = \frac{7}{2} \qquad \left[\because p = \frac{7a}{2}\right]$$

Hence, $\frac{d}{q} = 26$ and $\frac{L}{L'} = \frac{7}{2}$.

EXAMPLE 22 *Prove that in the series*

$ab + (a+x)(b+x) + (a+2x)(b+2x) + \ldots + 2n$ *terms*

The excess of sum of last n terms over the sum of first n terms is the excess of the last term over the first as n^2 *to* $2n-1$.

SOLUTION Let S_1 be the sum of first n terms, S be the sum of $2n$ terms and S_2 be the sum of last n terms of the given series. Then,

$$S_1 = ab + (a+x)(b+x) + (a+2x)(b+2x) + \ldots + \{a+(n-1)x\}\{b+(n-1)x\}$$

$$= \sum_{r=1}^{n} \{a+(r-1)x\}\{b+(r-1)x\}$$

$$= \sum_{r=1}^{n} [ab + (a+b)(r-1)x + (r-1)^2 x^2]$$

$$= nab + (a+b)x \sum_{r=1}^{n} (r-1) + x^2 \sum_{r=1}^{n} (r-1)^2$$

$$= nab + (a+b)x\,\frac{n(n-1)}{2} + x^2\,\frac{n(n-1)(2n-1)}{6}$$

$$S = ab + (a+x)(b+x) + \ldots + \{a+(2n-1)x\}\{b+(2n-1)x\}$$

$$= \sum_{r=1}^{2n} \{a+(r-1)x\}\{b+(r-1)x\}$$

$$= \sum_{r=1}^{2n} [ab + (r-1)x(a+b) + (r-1)^2 x^2]$$

$$= 2n\,ab + (a+b)x \sum_{r=1}^{2n} (r-1) + x^2 \sum_{r=1}^{2n} (r-1)^2$$

$$= 2n\,ab + (a+b)x\,n(2n-1) + x^2\,\frac{n(2n-1)(4n-1)}{3}$$

$$\therefore \quad S_2 = S - S_1$$

$$= n\,ab + (a+b)\,\frac{n(3n-1)x}{2} + \frac{n}{6}(7n-1)(2n-1)x^2$$

$$\therefore \quad S_2 - S_1 = (a+b)\,x\,n^2 + x^2 n^2 (2n-1)$$

$$= n^2 x\,[(a+b) + x(2n-1)]$$

Now,

Last term — First term

$$= \{a+(2n-1)x\}\{b+(2n-1)x\} - ab$$

$$= (a+b)(2n-1)x + (2n-1)^2 x^2$$

$$= (2n-1)x\,[(a+b) + (2n-1)x]$$

Hence,

$$\text{Required ratio} = \frac{S_2 - S_1}{\text{Last term} - \text{First term}} = \frac{n^2}{2n-1}.$$

EXAMPLE 23 *The first and last term of a series of* $(n+1)$ *terms in A.P. are a and b. A new series of n terms is formed by multiplying each of the first n terms by the next consecutive term. Show that the sum of the new series is*

$$\frac{[(n^2-1)(a^2+b^2) + (n^2+2)\,ab]}{3n}$$

SOLUTION Let $a_1 + a_2 + a_3 + \ldots + a_n + a_{n+1}$ be the given series in A.P. It is given that

$$a_1 = a \text{ and } a_{n+1} = b$$

Let d be the common difference of the A.P. Then,

$$a_{n+1} = b \Rightarrow a + nd = b \Rightarrow d = \frac{b-a}{n}$$

Now,

$$a_1 a_2 + a_2 a_3 + \ldots + a_n a_{n+1}$$

$$= \sum_{r=1}^{n} a_r a_{r+1}$$

$$= \sum_{r=1}^{n} \{a + (r-1)d\}\{a + rd\}$$

$$= \sum_{r=1}^{n} \{a^2 + (2r-1)ad + (r^2 - r)d^2\}$$

$$= na^2 + ad \sum_{r=1}^{n} (2r-1) + d^2 \sum_{r=1}^{n} (r^2 - r)$$

$$= na^2 + n^2 ad + \left\{\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right\} d^2$$

$$= n a^2 + n^2 ad + \frac{n(n^2-1)}{3} d^2$$

$$= na^2 + n^2 a\left(\frac{b-a}{n}\right) + \frac{n(n^2-1)}{3}\left(\frac{b-a}{n}\right)^2$$

$$= na^2 + na(b-a) + \frac{n^2-1}{3n}(b-a)^2$$

$$= \frac{(n^2-1)(a^2+b^2) + (n^2+2)ab}{3n}$$

EXAMPLE 24 *If* $a_n = \sum_{r=1}^{n} \log_e a^r, a > 0$, *find* $\sum_{r=1}^{n} a_r$. *Also find how many terms of the series are to be taken so that the sum is greater than* $1335 \log_e a^n$.

SOLUTION We have,

$$a_n = \sum_{r=1}^{n} \log_e a^r$$

$$= \sum_{r=1}^{n} r \cdot \log_e a$$

$$= (\log_e a) \sum_{r=1}^{n} r$$

$$= (\log_e a) \frac{n(n+1)}{2}$$

$$\therefore \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} (\log_e a) \cdot \frac{r(r+1)}{2}$$

$$= \frac{1}{2} (\log_e a) \sum_{r=1}^{n} (r^2 + r)$$

$$= \frac{1}{2} (\log_e a) \left\{\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right\}$$

$$= \frac{1}{2} (\log_e a) \left\{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)(n+2)}{6} \log_e a$$

Let the sum of n terms of the series be greater than $1335 \log_e a^n$. Then,

$$\frac{n(n+1)(n+2)}{6} \log_e a > 1335 \log_e a^n$$

$$\Rightarrow \frac{n(n+1)(n+2)}{6} \log_e a > 1335\, n \log_e a$$

$$\Rightarrow (n+1)(n+2) > 8010$$

$$\Rightarrow n^2 + 3n - 8008 > 0$$

$$\Rightarrow (n+91)(n-88) > 0$$

$$\Rightarrow n - 88 > 0 \Rightarrow n > 88$$

Hence, the least number of terms is 89.

EXERCISE 21.2

1. Find the sum of n terms of the series :

$$\log a + \log \frac{a^2}{b} + \log \frac{a^3}{b^2} + \log \frac{a^4}{b^3} + \ldots.$$

2. Find the sum of first 16 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_4 + a_7 + \ldots + a_{16} = 147$. Also, find the sum $a_1 + a_6 + a_{11} + a_{16}$.

3. If the sum S_n of first n terms of a sequence is given by $S_n = An^2 + Bn$, then prove that it is an A.P. Also, prove that the common difference is 2A.

4. Prove that the square of any odd integer $(2n+1)$ is always equal to the sum of n terms of an A.P. incresed by unit.

5. Prove that the square of any even integer $2n$ is always equal to the sum of n terms of an A.P.

6. If the sums of pth, qth and rth terms of an A.P. be a, b and c respectively, prove that

$$a(q-r) + b(r-p) + c(p-q) = 0$$

7. Find the sum of first 100 numbers appearing in both the progressions : 17, 21, 25, ... and 16, 21, 26,.....

8. The sum of n terms of m A.P.'s are $S_1, S_2, S_3, \ldots S_m$. The first term and the common differences are $1, 2, 3, \ldots, m$ respectively. Prove that

$$S_1 + S_2 + S_3 + \ldots + S_m = \frac{1}{4} mn(m+1)(n+1).$$

9. If there are m A.P.'s beginning with unity whose common differences are 1, 2, 3, ..., m respectively, show that the sum of their nth terms is

$$\frac{1}{2} m (mn - m + n + 1)$$

10. If S_1 be the sum of $(2n + 1)$ terms of an A.P. and S_2 be the sum of its odd terms, then prove that

$$S_1 : S_2 = (2n + 1) : (n + 1)$$

11. Find an A.P. in which the sum of any number of terms is always three times the squared number of these terms.

12. The number of terms of an A.P. is even; the sum of odd terms is 24, of the even terms is 30, and the last term exceeds the first by 10 1/2, find the number of terms and the series.

13. If $S_n = n^2 p$ and $S_m = m^2 p$, $m \neq n$, in an A.P., prove that $S_p = p^3$.

14. The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that

$$\frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) = 0$$

15. The sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3 respectively. Prove that $S_3 = 3 (S_2 - S_1)$.

16. The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1 + S_3 = 2 S_2$.

17. Prove that the sum of the latter half of $2n$ terms of an A.P. is one third of the sum of $3n$ terms of the same A.P.

18. In an A.P. of which a is the first term, if the sum of first p terms is zero, show that the sum of the next q terms is $\frac{-a(p+q)q}{p-1}$.

19. The ratio between the sums of n terms of two A.P's is $\frac{2n+5}{7n+4}$, find the ratio between their nth terms.

20. Find the A.P. in which the ratio of the sum to n terms to the sum of succeeding n terms is independent of n.

ANSWERS

1. $\frac{n}{2} \log \left(\frac{a^{n+1}}{b^{n-1}}\right)$ 2. 392, 98 7. 101100 11. 3, 9, 15, 21, ... 12. 8 terms; $\frac{3}{2}$, 3, $\frac{9}{2}$, ... 19. $\frac{4n+1}{14n-3}$ 20. $a, 3a, 5a, \ldots$ for all a

HINTS TO SELECTED PROBLEMS

4. We have, $(2n + 1)^2 = 4n^2 + 4n + 1 = (4n^2 + 4n) + 1 = S_n + 1$, where S_n denotes the sum of n terms of an A.P. with common difference 8 and first term = 8.

5. We have, $(2n)^2 = 4n^2 = 4n^2 + 0n$. We know that a squence is an A.P. With common difference 2A iff the sum of its n terms is given by $S_n = An^2 + Bn$. Therefore, $4n^2 + 0n$ i.e. $(2n)^2$ is the sum of n terms of an A.P. with common difference 8 and first term 4.

11. Use : $S_n = 3n^2$ and $a_n = S_n - S_{n-1}$.

15. Replace n by $(2 \times 5 - 1) = 9$.

—❑❑❑—

21.6 INSERTION OF ARITHMETIC MEANS

If between two given quantities a and b we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots A_n, b$ form an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

Since 15, 11, 7, 3, –1, –5 are in A.P., it follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.

If a, A, b are in A.P., we say that A is the arithmetic mean of a and b.

INSERTION OF n ARITHMETIC MEANS BETWEEN a AND b.

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then,

$a, A_1, A_2, \ldots, A_n, b$ is an A.P.

Let d be the common difference of this A.P. Clearly, it contains $(n + 2)$ terms.

$\therefore$ b = $(n + 2)$th term

$\Rightarrow b = a + (n + 1) d$

$\Rightarrow d = \frac{b-a}{n+1}$.

Now,

$A_1 = a + d$

$\Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$

$$A_2 = a + 2d \Rightarrow A_2 = \left(a + \frac{2(b-a)}{n+1}\right), \ldots\ldots\ldots\ldots\ldots,$$

$$A_n = a + nd \Rightarrow A_n = \left(a + \frac{n(b-a)}{n+1}\right).$$

These are the required arithmetic means between a and b.

INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN a AND b

Let A be the arithmetic mean of a and b. Then, a, A, b are in A.P.

$$\Rightarrow A - a = b - A$$

$$\Rightarrow 2A = a + b$$

$$\Rightarrow A = \frac{a+b}{2}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b?*

SOLUTION Since A.M. of a and b is $\frac{a+b}{2}$. Therefore,

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$$\Rightarrow 2(a^{n+1}+b^{n+1}) = (a^n+b^n)(a+b)$$

$$\Rightarrow 2a^{n+1}+2b^{n+1} = a^{n+1}+a^n b+b^n a+b^{n+1}$$

$$\Rightarrow a^{n+1}+b^{n+1} = a^n b+b^n a$$

$$\Rightarrow a^n(a-b) = b^n(a-b)$$

$$\Rightarrow a^n = b^n$$

$$\Rightarrow \frac{a^n}{b^n} = 1$$

$$\Rightarrow \left(\frac{a}{b}\right)^n = 1$$

$$\Rightarrow \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow n = 0$$

EXAMPLE 2 *If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of n.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A. P. $20, A_1, A_2, \ldots A_n, 80$. Then,

$$d = \frac{80-20}{n+1} = \frac{60}{n+1} \qquad \left[\text{Using : } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_1 = 20 + d \Rightarrow A_1 = 20 + \frac{60}{n+1} = 20\left(\frac{n+4}{n+1}\right)$$

And,

$$A_n = 20 + nd \Rightarrow A_n = 20 + \frac{60n}{n+1} = 20\left(\frac{4n+1}{n+1}\right)$$

$$\therefore \frac{A_1}{A_n} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}} = \frac{1}{3}$$

$$\Rightarrow \frac{n+4}{4n+1} = \frac{1}{3}$$

$$\Rightarrow 4n+1 = 3n+12 \Rightarrow n = 11$$

EXAMPLE 3 *Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and $(m-1)$th means is 5 : 9. Find the value of m.*

SOLUTION Let $A_1, A_2, \ldots, A_m$ be m arithmetic means between 1 and 31. Then, $1, A_1, A_2, \ldots, A_m, 31$ is an A.P. with common difference d given by

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \qquad \left[\text{Using : } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_7 = 1 + 7d$$

$$\Rightarrow A_7 = 1 + \frac{7 \times 30}{m+1} = \frac{m+211}{m+1}$$

and,

$$A_{m-1} = 1 + (m-1)d$$

$$\Rightarrow A_{m-1} = 1 + \frac{30}{m+1}(m-1)$$

$$\Rightarrow A_{m-1} = \frac{31m-29}{m+1}$$

It is given that

$$\Rightarrow \frac{A_7}{A_{m-1}} = \frac{5}{9}$$

$$\Rightarrow \frac{m+211}{31m-29} = \frac{5}{9}$$

$\Rightarrow 9m + 1899 = 155m - 145$

$\Rightarrow 146m = 2044 \Rightarrow m = 14$

EXAMPLE 4 *Prove that the sum of n arithmetic means between two numbers is n times the single A.M. between them.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference d given by

$$d = \frac{b-a}{n+1}$$

Now,

$$A_1 + A_2 + \ldots + A_n$$

$$= \frac{n}{2}[A_1 + A_n]$$

$$= \frac{n}{2}[a+b] \qquad \left[\because a, A_1, A_2, \ldots, A_n, b \text{ is an A.P.} \Rightarrow a+b = A_1 + A_n\right]$$

$$= n\left(\frac{a+b}{2}\right)$$

$= n \times$ (A.M. between a and b).

EXAMPLE 5 *The sum of two numbers is $\frac{13}{6}$. An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted.*

SOLUTION Let a and b be two numbers such that

$$a + b = \frac{13}{6} \qquad \ldots(i)$$

Let $A_1, A_2, \ldots, A_{2n}$ be $2n$ arithmetic means between a and b. Then,

$$A_1 + A_2 + \ldots + A_{2n} = 2n\left(\frac{a+b}{2}\right) \qquad \text{[Using Example 4]}$$

$$= n(a+b) = \frac{13}{6}n \qquad \text{[Using (i)]}$$

It is given that

$$A_1 + A_2 + \ldots + A_{2n} = 2n + 1.$$

$$\therefore \frac{13}{6}n = 2n + 1 \Rightarrow n = 6$$

So, the number of means is 12.

EXAMPLE 6 *If the A.M. between pth and qth terms of an A.P. be equal to the A.M. between rth and sth terms of the A.P., then show that $p + q = r + s$.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then

$$a_p = p\text{th term} = a + (p-1)d;$$

$$a_q = q\text{th term} = a + (q-1)d$$

$$a_r = r\text{th term} = a + (r-1)d$$

and, $$a_s = s\text{th term} = a + (s-1)d$$

It is given that

A.M. between a_p and a_q = A.M. between a_r and a_s

$$\Rightarrow \frac{1}{2}(a_p + a_q) = \frac{1}{2}(a_r + a_s)$$

$$\Rightarrow a_p + a_q = a_r + a_s$$

$$\Rightarrow \{a + (p-1)d\} + \{a + (q-1)d\} = \{a + (r-1)d\} + \{a + (s-1)d\}$$

$$\Rightarrow (p + q - 2)d = (r + s - 2)d$$

$$\Rightarrow p + q = r + s$$

EXAMPLE 7 *Suppose x and y are two real numbers such that the rth mean between x and 2y is equal to the rth mean between 2x and y when n arithmetic means are inserted between them in both the cases. Show that*

$$\frac{n+1}{r} - \frac{y}{x} = 1$$

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between x and $2y$. Then,

$x, A_1, A_2, \ldots, A_n, 2y$ are in AP with common difference d_1 given by

$$d_1 = \frac{2y - x}{n+1}$$

$\therefore$ r*th* mean $= A_r$

$$= x + r\,d_1$$

$$= x + r\left(\frac{2y-x}{n+1}\right)$$

Let $A_1', A_2', \ldots, A_n'$ be n arithmetic means between $2x$ and y. Then, $2x, A_1', A_2', \ldots, A_n', y$ are in AP with common difference d_2 given by

$$d_2 = \frac{y - 2x}{n+1}$$

$\therefore$ rth mean $= A_r'$

$$= 2x + r\,d_2$$

$$= 2x + r\left(\frac{y-2x}{n+1}\right)$$

It is given that :

$$A_r = A_r'$$

$$\Rightarrow x + r\left(\frac{2y-x}{n+1}\right) = 2x + r\left(\frac{y-2x}{n+1}\right)$$

$$\Rightarrow (n+1)x + r(2y - x) = (n+1)2x + r(y - 2x)$$

$$\Rightarrow (n+1)x - ry = rx$$

$$\Rightarrow \frac{n+1}{r} - \frac{y}{x} = 1$$

21.7 GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviated as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words, a sequence, $a_1, a_2, a_3, ..., a_n, ...$ is called a geometric progression if

$$\frac{a_{n+1}}{a_n} = \text{constant for all } n \in N$$

ILLUSTRATION 1 *The sequence 4, 12, 36, 108, ... is a G.P., because* $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = ... = 3$, *which is constant.*

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

ILLUSTRATION 2 *The sequence* $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, ...$ *is a G.P. with first term* $\frac{1}{3}$ *and common ratio* $\left(-\frac{1}{2}\right) \Big/ \left(\frac{1}{3}\right) = -\frac{3}{2}$

ILLUSTRATION 3 *Show that the sequence given by* $a_n = 3(2^n)$, *for all* $n \in N$, *is a G.P. Also, find its common ratio.*

SOLUTION We have, $a_n = 3(2^n)$

$$\therefore \quad a_{n+1} = 3(2^{n+1})$$

Now, $$\frac{a_{n+1}}{a_n} = \frac{3(2^{n+1})}{3(2^n)} = 2$$

Clearly, $\frac{a_{n+1}}{a_n} = 2$ (constant), for all $n \in N$. So, the given sequence is an A.P. with common ratio 2.

GEOMETRIC SERIES *If $a_1, a_2, a_3, ... a_n, ...$ is a G.P., then the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called a geometric series.*

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

The nth term of a G.P. with first term 'a' and common ratio 'r' is given by

$$a_n = a\, r^{n-1}$$

Also, the G.P. can be written as

$$a, ar, ar^2, ar^3, ..., ar^{n-1}, ...$$

If a finite G.P. consists of m terms, then nth term from the end is $(m - n + 1)$ th term from the beginning. Therefore,

$$n^{th} \text{ term from the end } = a\, r^{m-n}$$

If l denotes the last term of a finite G.P., then nth term from the end is also given by $l\left(\frac{1}{r}\right)^{n-1}$.

21.9 SELECTION OF TERMS IN G.P.

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner :

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as $a, ar, ar^2, ar^3, ...$

21.10 PROPERTIES OF GEOMETRIC PROGRESSIONS

In this section, we shall discuss some properties of geometric progressions and geometric series.

PROPERTY I *If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.*

PROOF Let $a_1, a_2, a_3, ..., a_n, ...$ be a G.P. with common ratio r.

Then, $\frac{a_{n+1}}{a_n} = r$, for all $n \in N$ ...(i)

Let k be a non-zero constant. Multiplying all the terms of the given G.P. by k, we obtain the sequence

$$ka_1, ka_2, ka_3, ... ka_n, ...$$

Clearly, $\frac{ka_{n+1}}{k\,a_n} = \frac{a_{n+1}}{a_n} = r$ for all $n \in N$ [Using (i)]

Hence, the new sequence also forms a G.P. with common ratio r.

PROPERTY II *The reciprocals of the terms of a given G.P. form a G.P.*

PROOF Let $a_1, a_2, a_3, ..., a_n, ...$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad ...(i)$$

The sequence formed by the reciprocals of the terms of the given G.P. is

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ... \frac{1}{a_n}, ...$$

We have, $$\frac{1/a_{n+1}}{1/a_n} = \frac{a_n}{a_{n+1}} = \frac{1}{r}$$ [Using (i)]

So, the new sequence is a G.P. with common ratio $1/r$.

PROPERTY III *If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.*

PROOF Let $a_1, a_2, a_3, ... a_n, ...$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad \ldots(i)$$

Let k be a non-zero real number. Consider the sequence

$$a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$$

We have, $\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k$

$= r^k$ for all $n \in N$ [Using (i)]

Hence, $a_1^k, a_2^k, a_3^k, \ldots a_n^k, \ldots$ is a G.P. with common ratio r^k.

PROPERTY IV *In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be a finite G.P. with common ratio r. Then,

$$k\text{th term from the beginning} = a_k = a_1 r^{k-1}$$

kth term from the end = $(n-k+1)$th term from the beginning

$$= a_{n-k+1}$$

$$= a_1 r^{n-k}$$

$\therefore$ (kth term from the beginning) (kth term from the end)

$= a_k \, a_{n-k+1}$

$= a_1 r^{k-1} a_1 r^{n-k}$

$= a_1^2 r^{n-1} = a_1 . a_1 r^{n-1}$

$= a_1 a_n$ for all $k = 2, 3, \ldots, n-1$

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

PROPERTY V *Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$*

PROOF a, b, c are in G.P.

$\Leftrightarrow \frac{b}{a} = \frac{c}{b}$ = (common ratio)

$\Leftrightarrow b^2 = ac$

NOTE *When a, b, c are in G.P., then b is known as the Geometric mean of a and c.*

PROPERTY VI *If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.*

PROPERTY VII *If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. of non-zero non-negative terms, then $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. and vice-versa.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. of non-zero non-negative terms with common ratio r. Then,

$$a_n = a_1 r^{n-1} \text{ for all } n \in N$$

$\Rightarrow \log a_n = \log a_1 + (n-1) \log r$ for all $n \in N$

Let $b_n = \log a_n = \log a_1 + (n-1) \log r$ for all $n \in N$

Then, $b_{n+1} - b_n$

$= [\log a_1 + n \log r] - [\log a_1 + (n-1) \log r]$.

$= \log r$ for all $n \in N$

Clearly, $b_{n+1} - b_n = \log r$ = constant for all $n \in N$.

Hence, $b_1, b_2, \ldots, b_n, \ldots$ i.e. $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. with common difference $\log r$.

Conversely, let $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ be an A.P. with common difference d. Then,

$$\log a_{n+1} - \log a_n = d, \text{ for all } n \in N.$$

$$\Rightarrow \log\left(\frac{a_{n+1}}{a_n}\right) = d, \text{ for all } n \in N$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = e^d, \text{ for all } n \in N.$$

$\Rightarrow a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio e^d.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.*

SOLUTION Let r be the common ratio of the given G.P. Then,

$$b = nth \text{ term} = ar^{n-1}$$

$$\Rightarrow r^{n-1} = \frac{b}{a}$$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

and,

P = Product of the first n terms

$= a . ar . ar^2 \ldots ar^{n-1}$

$= a^n r^{1+2+3+\ldots+(n-1)}$

$= a^n r^{\frac{n(n-1)}{2}}$

$= a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}}$

$$= a^n\left(\frac{b}{a}\right)^{n/2} = a^{n/2}b^{n/2} = (ab)^{n/2}$$

$$\therefore \quad P^2 = [(ab)^{n/2}]^2 = (ab)^n$$

EXAMPLE 2 *The third term of a G.P. is 4. Find the product of its first five terms.*

SOLUTION Let a be the first term and r the common ratio. Then,

$$a_3 = 4 \Rightarrow ar^2 = 4 \qquad ...(i)$$

Now,

Product of first five terms

$$= a_1 a_2 a_3 a_4 a_5$$

$$= a\,(ar)\,(ar^2)\,(ar^3)\,(ar^4)$$

$$= a^5 r^{10} = (ar^2)^5 = 4^5 \qquad \text{[Using (i)]}$$

EXAMPLE 3 *The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{m/2n}$ respectively.*

SOLUTION Let a be the first term and r be the common ratio. Then,

$$a_{m+n} = p \text{ and } a_{m-n} = q$$

$$\Rightarrow ar^{m+n-1} = p \text{ and, } ar^{m-n-1} = q$$

$$\Rightarrow \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q}$$

$$\Rightarrow r^{2n} = \frac{p}{q}$$

$$\Rightarrow r = \left(\frac{p}{q}\right)^{1/2n}$$

$$\Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now,

$$a_m = ar^{m-1}$$

$$\Rightarrow a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n$$

$$\Rightarrow a_m = a_{m+n}\left(\frac{1}{r}\right)^n \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow a_m = p\left(\frac{q}{p}\right)^{n/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow a_m = p\left(\frac{q}{p}\right)^{1/2}$$

$$\Rightarrow a_m = \sqrt{pq}.$$

and,

$$a_n = ar^{n-1}$$

$$\Rightarrow a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow a_n = p\left(\frac{q}{p}\right)^{m/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

EXAMPLE 4 *Find all sequences which are simultaneously A.P. and G.P.*

SOLUTION Let $< a_n >$ be a sequence which is both an A.P. and a G.P. Let a_n, a_{n+1}, a_{n+2} be three consecutive terms of the A.P. Then

$$2a_{n+1} = a_n + a_{n+2}, \; n \geq 1 \qquad ...(i)$$

Let r be the common ratio of the sequence when it is considered a G.P. Then,

$$a_n = a_1 r^{n-1}, \; a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}$$

Putting these values in (i), we get

$$2a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1}$$

$$\Rightarrow 2r = 1 + r^2$$

$$\Rightarrow r^2 - 2r + 1 = 0$$

$$\Rightarrow (r-1)^2 = 0$$

$$\Rightarrow r = 1$$

So, $a_1 = a_2 = a_3 = \ldots$ i.e. the constant sequence is the only sequence which is both an A.P. and a G.P.

EXAMPLE 5 *Does there exist a G.P. containing 27, 8 and 12 as three of its terms ? If it exists, how many such progressions are possible ?*

SOLUTION If possible, let there be a G.P. with first term a and common ratio r such that

$$a_m = 27, \; a_n = 8 \qquad \text{and } a_p = 12$$

$$\Rightarrow ar^{m-1} = 27, \; ar^{n-1} = 8 \text{ and } ar^{p-1} = 12$$

$$\Rightarrow \frac{ar^{m-1}}{ar^{n-1}} = \frac{27}{8} \qquad \text{and} \qquad \frac{ar^{p-1}}{ar^{n-1}} = \frac{12}{8}$$

$$\Rightarrow r^{m-n} = \left(\frac{3}{2}\right)^3 \qquad \text{and} \qquad r^{p-n} = \left(\frac{3}{2}\right)$$

$$\Rightarrow r^{(m-n)} = [r^{(p-n)}]^3$$

$$\Rightarrow r^{(m-n)} = r^{3(p-n)}$$

$$\Rightarrow m - n = 3(p-n)$$

$$\Rightarrow m - 3p = -2n$$

$$\Rightarrow 3p - m = 2n$$

$$\Rightarrow m = 3p - 2n$$

Clearly, the equation $m = 3p - 2n$ has infinitely many solutions in N. For example, $p = 2, n = 1, m = 4$; $p = 3, n = 2, m = 5$ etc. Hence, there can be infinitely many geometric progressions whose three terms are 8, 12, 27 (not consecutive).

EXAMPLE 6 *Show that the numbers 10, 11, 12 cannot be the terms of a single G.P. with common ratio not equal to 1.*

SOLUTION If possible, let there be a G.P. with first term a and common ratio r such that

$$a_m = 10,\ a_n = 11 \quad \text{and} \quad a_p = 12$$

$$\Rightarrow ar^{m-1} = 10,\ ar^{n-1} = 11 \text{ and } ar^{p-1} = 12$$

$$\Rightarrow \frac{ar^{m-1}}{ar^{n-1}} = \frac{10}{11} \quad \text{and} \quad \frac{ar^{n-1}}{ar^{p-1}} = \frac{11}{12}$$

$$\Rightarrow r^{m-n} = \frac{10}{11} \quad \text{and} \quad r^{n-p} = \frac{11}{12}$$

$$\Rightarrow r = \left(\frac{10}{11}\right)^{\frac{1}{m-n}} \quad \text{and} \quad r = \left(\frac{11}{12}\right)^{\frac{1}{n-p}}$$

$$\Rightarrow \left(\frac{10}{11}\right)^{\frac{1}{m-n}} = \left(\frac{11}{12}\right)^{\frac{1}{n-p}}$$

$$\Rightarrow \left(\frac{10}{11}\right)^{n-p} = \left(\frac{11}{12}\right)^{m-n}$$

$$\Rightarrow (10)^{n-p}(12)^{m-n} = (11)^{m-p}$$

Clearly, LHS of the above equality is even whereas RHS is an odd number which is not possible. Hence, 10, 11, 12 cannot be the terms of a single G.P.

EXAMPLE 7 *If pth, qth and rth terms of both an A.P. and a G.P. be respectively a, b and c, then prove that* $a^{b-c}b^{c-a}c^{a-b} = 1$.

SOLUTION Let A be the first term and D the common difference of the A.P. Then,

$$a = pth \text{ term} \Rightarrow a = A + (p-1)D,$$
$$b = qth \text{ term} \Rightarrow b = A + (q-1)D$$
$$c = rth \text{ term} \Rightarrow c = A + (r-1)D$$

$$\therefore\ a - b = (p-q)D,\ b - c = (q-r)D \text{ and } c - a = (r-p)D \quad ...(i)$$

Again, let x be the first term and R be the common ratio of the G.P. Then,

$$a = pth \text{ term} \Rightarrow a = xR^{p-1},$$
$$b = qth \text{ term} \Rightarrow b = xR^{q-1}$$
$$c = rth \text{ term} \Rightarrow c = xR^{r-1}$$

$$\therefore\ a^{b-c}b^{c-a}c^{a-b}$$
$$= (xR^{p-1})^{b-c}(xR^{q-1})^{c-a}(xR^{r-1})^{a-b}$$
$$= x^{(b-c+c-a+a-b)}\ R^{(p-1)(b-c)+(q-1)(c-a)+(r-1)(a-b)}$$
$$= x^0\ R^{p(b-c)+q(c-a)+r(a-b)-(b-c)-(c-a)-(a-b)}$$
$$= R^{p(b-c)+q(c-a)+r(a-b)}$$
$$= R^{p(q-r)D+q(r-p)D+r(p-q)D} \qquad \text{[Using (i)]}$$
$$= R^0 = 1$$

EXAMPLE 8 *Given an A.P. and a G.P. with positive terms, with first and second terms of the corresponding progressions being equal, prove that any other term of the A.P. does not exceed the corresponding term of the G.P.*

SOLUTION Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ be the A.P. and G.P. respectively, with $a_1 = b_1$ and $a_2 = b_2$. Since all the terms of the A.P. are positive. Therefore, $a_1 > 0$ and the common difference d of the A.P. is also non-negative. It is also given that the terms of the G.P. are non-negative, therefore, the common ratio r is positive.

It is given that :

$$a_1 = b_1 \text{ and } a_2 = b_2$$

$$\Rightarrow a_1 + d = a_1 r$$

$$\Rightarrow r = 1 + \frac{d}{a_1} \geq 1 \quad ...(i) \qquad \left[\because \frac{d}{a_1} \geq 0\right]$$

Now, for $n = 2, 3, 4, \ldots$, we have

$$b_{n+1} - a_{n+1}$$
$$= a_1 r^n - (a_1 + nd)$$
$$= a_1 r^n - a_1 - nd$$
$$= a_1\left(1 + \frac{d}{a_1}\right)^n - a_1 - nd \qquad \text{[Using (i)]}$$
$$= a_1\left[{}^nC_0 + {}^nC_1\left(\frac{d}{d_1}\right) + {}^nC_2\left(\frac{d}{a_1}\right)^2 + \ldots + {}^nC_n\left(\frac{d}{a_1}\right)^n\right] - a_1 - nd$$
$$= {}^nC_2\left(\frac{d}{a_1}\right)^2 + {}^nC_3\left(\frac{d}{a_1}\right)^3 + \ldots + {}^nC_n\left(\frac{d}{a_1}\right)^n \geq 0$$

$$\Rightarrow b_{n+1} - a_{n+1} \geq 0 \text{ for } n = 2, 3, 4, \ldots$$

$$\Rightarrow b_{n+1} \geq a_{n+1} \text{ for } n = 2, 3, 4, \ldots.$$

Hence, the terms of the A.P. do not exceed the corresponding terms of the G.P.

EXAMPLE 9 *If a_1, a_2, a_3 are three consecutive terms of a G.P. with common ratio r. Find all the values of r for which the inequality $a_3 > 4a_2 - 3a_1$ is satisfied.*

SOLUTION We have,

$$a_2 = a_1 r \text{ and } a_3 = a_1 r^2$$

$$\therefore\ a_3 > 4a_2 - 3a_1$$

$$\Rightarrow a_1 r^2 > 4a_1 r - 3a_1$$

$$\Rightarrow a_1(r^2 - 4r + 3) > 0$$

Now, two cases arise :

CASE I *When* $a_1 > 0$:

In this case,

$a_1 (r^2 - 4r + 3) > 0$

$\Rightarrow \quad r^2 - 4r + 3 > 0$

$\Rightarrow \quad (r-1)(r-3) > 0$

$\Rightarrow \quad r < 1$ or $r > 3$

$\Rightarrow \quad r \in (-\infty, 1) \cup (3, \infty)$

CASE II When $a_1 < 0$:

In this case,

$a_1 (r^2 - 4r + 3) > 0$

$\Rightarrow \quad r^2 - 4r + 3 < 0$

$\Rightarrow \quad (r-1)(r-3) < 0$

$\Rightarrow \quad 1 < r < 3$

$\Rightarrow \quad r \in (1, 3)$

EXAMPLE 10 *Find three numbers a, b, c between 2 and 18 such that:*

(i) their sum is 25.

(ii) the numbers 2, a, b are in A.P.

(iii) the numbers b, c, 18 are in G.P.

SOLUTION It is given that :

$a + b + c = 25$ ...(i)

a, b are in A.P.

$\Rightarrow \quad 2a = 2 + b$ ...(ii)

And,

$b, c, 18$ are in G.P.

$\Rightarrow \quad c^2 = 18b$ ...(iii)

From (i) and (ii), we get

$\frac{2+b}{2} + b + c = 25$

$\Rightarrow \quad 2 + b + 2b + 2c = 50$

$\Rightarrow \quad 3b + 2c = 48$ ...(iv)

$\Rightarrow \quad 3\left(\frac{c^2}{18}\right) + 2c = 48$ [Using (iii)]

$\Rightarrow \quad c^2 + 12c - 288 = 0$

$\Rightarrow \quad (c + 24)(c - 12) = 0$

$\Rightarrow \quad c = 12, -24$

$\Rightarrow \quad c = 12$ [$\because$ a, b, c lie between 2 and 18]

Putting $c = 12$ in (iv), we get $b = 8$

Putting $b = 8$ in (ii), we get $a = 5$

Hence, $a = 5, b = 8$ and $c = 12$.

EXAMPLE 11 *The sum of the squares of three distinct real numbers, which are in GP is* S^2. *If their sum is* αS, *show that* $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$

SOLUTION Let the three distinct real numbers in G.P. be

a, ar, ar^2

Since the numbers are distinct. Therefore, $r \neq \pm 1$

It is given that :

$a^2 + a^2 r^2 + a^2 r^4 = S^2$

$\Rightarrow \quad a^2 (1 + r^2 + r^4) = S^2$ ...(i)

And,

$a + ar + ar^2 = \alpha S$

$\Rightarrow \quad a (1 + r + r^2) = \alpha S$ ...(ii)

$\Rightarrow \quad a^2 (1 + r + r^2)^2 = \alpha^2 S^2$ ...(iii)

From (i) and (iii), we get

$\frac{a^2 (1 + r + r^2)^2}{a^2 (1 + r^2 + r^4)} = \frac{\alpha^2 S^2}{S^2}$

$\Rightarrow \quad \frac{(1 + r + r^2)^2}{1 + r^2 + r^4} = \alpha^2$

$\Rightarrow \quad \frac{(1 + r + r^2)^2}{(r^2 + r + 1)(r^2 - r + 1)} = \alpha^2$

$\Rightarrow \quad \frac{1 + r + r^2}{r^2 - r + 1} = \alpha^2$ ...(iv)

$\Rightarrow \quad r^2 (\alpha^2 - 1) - r (\alpha^2 + 1) + (\alpha^2 - 1) = 0$

Since r is real. Therefore,

$(\alpha^2 + 1)^2 - 4 (\alpha^2 - 1)^2 \geq 0$ [Disc ≥ 0]

$\Rightarrow \quad (\alpha^2 + 1 + 2\alpha^2 - 2)(\alpha^2 + 1 - 2\alpha^2 + 2) \geq 0$

$\Rightarrow \quad -(3\alpha^2 - 1)(\alpha^2 - 3) \geq 0$

$\Rightarrow \quad (3\alpha^2 - 1)(\alpha^2 - 3) \leq 0$

$\Rightarrow \quad \frac{1}{3} \leq \alpha^2 \leq 3$ ...(v)

Now, $r = 1 \quad \Rightarrow \alpha^2 = 3$

and, $r = -1 \Rightarrow \alpha^2 = \frac{1}{3}$ [Putting $r = \pm 1$ in (iv)]

Also, $r = 0 \quad \Rightarrow \alpha^2 = 1$ [Putting $r = 0$ in (iv)]

But, $r \neq \pm 1, 0$. Therefore, $\alpha^2 \neq 3, \frac{1}{3}, 1$.

Thus, from (v), we get

$\alpha^2 \in (1/3, 1) \cup (1, 3)$

EXAMPLE 12 *Find the values of the common ratio for which three successive terms of a G.P. are the sides of a triangle.*

SOLUTION Let a, ar and ar^2 be three successive terms of a G.P. Clearly, these three terms must be positive. Therefore, $r > 0$ and $a > 0$.

Now, three cases arise.

CASE I *When* $r = 1$:

In this case, the three terms are each equal to a. Clearly these three terms form an equilateral triangle.

CASE II *When* $0 < r < 1$

In this case, given terms are a, ar, ar^2. These three terms will represent the sides of a triangle, if the sum of any two is greater than the third. That is

if $a + ar > ar^2, a + ar^2 > ar$ and $ar + ar^2 > a$.

Since $0 < r < 1$, therefore a is the largest term and ar^2 is the smallest term. Therefore, $a + ar > ar^2$ and $a + ar^2 > ar$ clearly hold good.

Thus, a, ar, ar^2 will form the sides of the triangle, if

$$ar + ar^2 > a$$

$$\Rightarrow r^2 + r - 1 > 0$$

$$\Rightarrow r^2 + r - 1 > 0$$

$$\Rightarrow \left(r - \frac{\sqrt{5}-1}{2}\right)\left(r + \frac{\sqrt{5}+1}{2}\right) > 0$$

$$\Rightarrow \left(r - \frac{\sqrt{5}-1}{2}\right) > 0 \qquad \left[\because r + \frac{\sqrt{5}+1}{2} > 0\right]$$

$$\Rightarrow r > \frac{\sqrt{5}-1}{2}$$

But, $0 < r < 1$. Therefore,

$$\frac{\sqrt{5}-1}{2} < r < 1 \qquad ...(i)$$

CASE III *When* $r > 1$

Three terms are a, ar, ar^2. These three terms will represent the sides of a triangle, if the sum of any two is greater than the third i.e.

if $a + ar > ar^2, a + ar^2 > ar$ and $ar + ar^2 > a$.

Since $r > 1$, therefore ar^2 is the largest term and a is the smallest term. Therefore, $a + ar^2 > ar$ and $ar + ar^2 > a$ clearly hold good.

Thus, a, ar, ar^2 will form the sides of a triangle, if

$$a + ar > ar^2$$

$$\Rightarrow 1 + r > r^2$$

$$\Rightarrow r^2 - r - 1 < 0$$

$$\Rightarrow \left(r - \frac{1-\sqrt{5}}{2}\right)\left(r - \frac{1+\sqrt{5}}{2}\right) < 0$$

$$\Rightarrow \frac{1-\sqrt{5}}{2} < r < \frac{1+\sqrt{5}}{2}$$

But, $r > 1$. Therefore,

$$1 < r < \frac{1+\sqrt{5}}{2} \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}$$

Hence, three successive terms of a GP will form the sides of a triangle, if the common ratio lies between $\frac{\sqrt{5}-1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

EXAMPLE 13 *Given a G.P. and an A.P. with positive terms* $a, a_1, a_2, \ldots, a_n, \ldots$ *and* $b, b_1, b_2, \ldots, b_n, \ldots$. *The common ratio of the G.P. is different from 1. Prove that there exists* $x \in R, x > 0$ *such that*

$$\log_x a_n - b_n = \log_x a - b \text{ for all } n \in N.$$

SOLUTION Since $a, a_1, a_2, \ldots, a_n$, are in GP. Therefore, $\log a_1, \log a_2, \ldots, \log a_n \ldots$ are in A.P. Let D be the common difference of this A.P. Then,

$$\log a_n = \log a + nD \qquad ...(i)$$

Let d be the common difference of the A.P. $b, b_1, b_2, \ldots$ Then,

$$b_n = b + nd \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{\log a_n - \log a}{b_n - b} = \frac{nD}{nd}$$

$$\Rightarrow \frac{\log a_n - \log a}{b_n - b} = \frac{D}{d}$$

$$\Rightarrow \frac{\log a_n - \log a}{b_n - b} = \log x, \text{ where } x = \frac{D}{d} \text{ is a fixed positive real number.}$$

$$\Rightarrow \frac{\log\left(\frac{a_n}{a}\right)}{\log x} = b_n - b$$

$$\Rightarrow \log_x (a_n/a) = b_n - b$$

$$\Rightarrow \log_x a_n - \log_x a = b_n - b$$

$$\Rightarrow \log_x a_n - b_n = \log_x a - b, \text{ where } \log x = \frac{D}{d} \text{ is a fixed positive real number.}$$

EXAMPLE 14 *If* $a_1, a_2, a_3, \ldots, a_{2n}$ *are* $2n$ *positive real numbers in GP, then prove that*

$$\sqrt{a_1 a_2} + \sqrt{a_3 a_4} + \sqrt{a_5 a_6} + \ldots + \sqrt{a_{2n-1} a_{2n}} = \sqrt{(a_1 + a_3 + \ldots + a_{2n-1})}\ \sqrt{(a_2 + a_4 + \ldots + a_{2n})}$$

SOLUTION Let r be the common ratio of the G.P. Then,

$$a_{n+1} = a_n r \text{ for } n = 1, 2, \ldots, 2n - 1.$$

$$\Rightarrow a_n a_{n+1} = a_n^2 r \text{ for } n = 1, 2, 3, \ldots, (2n - 1)$$

$$\Rightarrow \sqrt{a_n a_{n+1}} = a_n \sqrt{r} \text{ for } n = 1, 2, 3\ldots, (2n - 1)$$

$$\therefore \sqrt{a_1 a_2} + \sqrt{a_3 a_4} + \ldots + \sqrt{a_{2n-1} a_{2n}}$$

$$= a_1 \sqrt{r} + a_3 \sqrt{r} + \ldots + a_{2n-1} \sqrt{r}$$

$$= (a_1 + a_3 + \ldots + a_{2n-1}) \sqrt{r} \qquad ...(i)$$

Also,

$$\sqrt{a_1+a_3+\ldots+a_{2n-1}}\ \sqrt{a_2+a_4+\ldots+a_{2n}}$$

$$=\sqrt{a_1+a_3+\ldots+a_{2n-1}}\ \sqrt{a_1r+a_3r+\ldots+a_{2n-1}r}$$

$$=(a_1+a_3+\ldots+a_{2n-1})\sqrt{r} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\sqrt{a_1a_2}+\sqrt{a_3a_4}+\ldots+\sqrt{a_{2n-1}a_{2n}}$$

$$=\sqrt{a_1+a_3+\ldots+a_{2n-1}}\cdot\sqrt{a_2+a_4+\ldots+a_{2n}}$$

EXAMPLE 15 *If $\log_x y, \log_z x, \log_y z$ are in G.P. ; $xyz = 64$ and x^3, y^3, z^3 are in A.P., then find x, y and z.*

SOLUTION Since x^3, y^3, z^3 are in A.P. Therefore,

$$2y^3 = x^3 + z^3 \qquad \ldots(i)$$

It is given that $\log_x y, \log_z x$ and $\log_y z$ are in G.P. Therefore,

$$(\log_z x)^2 = (\log_x y)(\log_y z)$$

$$\Rightarrow \left(\frac{\log x}{\log z}\right)^2 = \left(\frac{\log y}{\log x}\right)\left(\frac{\log z}{\log y}\right)$$

$$\Rightarrow (\log x)^3 = (\log z)^3$$

$$\Rightarrow \log x = \log z$$

$$\Rightarrow x = z.$$

Putting $x = z$ in (i), we get

$$2y^3 = 2x^3$$

$$\Rightarrow y = x$$

$$\therefore x = y = z$$

But, $xyz = 64$.

Hence, $x = y = z = 4$.

EXAMPLE 16 *Find all the complex numbers x and y such that x, $x+2y, 2x+y$ are in A.P. and $(y+1)^2, xy+5, (x+1)^2$ are in G.P. Also, find the progression.*

SOLUTION Since $x, x+2y, 2x+y$ are in A.P. and $(y+1)^2, xy+5, (x+1)^2$ are G.P. Therefore,

$$2(x+2y) = x+(2x+y) \text{ and } (xy+5)^2 = (y+1)^2(x+1)^2$$

$$\Rightarrow x-3y = 0 \text{ and } (xy+5)^2 = |(x+1)(y+1)|^2$$

$$\Rightarrow x = 3y \quad \text{and } (xy+5)^2 = (xy+x+y+1)^2$$

$$\Rightarrow (3y^2+5)^2 = (3y^2+3y+y+1)^2 \quad \left[\begin{array}{l}\text{Putting } x=3y \\ \text{in the secondequation}\end{array}\right]$$

$$\Rightarrow (3y^2+5)^2 = (3y^2+4y+1)^2$$

$$\Rightarrow 3y^2+5 = \pm(3y^2+4y+1)$$

$$\Rightarrow 3y^2+5 = 3y^2+4y+1 \text{ or, } 3y^2+5 = -(3y^2+4y+1)$$

$$\Rightarrow 4y = 4 \text{ or } 6y^2+4y+6 = 0$$

$$\Rightarrow y = 1 \text{ or } 3y^2+2y+3 = 0$$

$$\Rightarrow y = 1 \text{ or } y = \frac{-1\pm 2\sqrt{2}\,i}{3}$$

Now,

$$y = 1 \Rightarrow x = 3 \qquad [\because x = 3y]$$

and,

$$y = \frac{-1\pm 2\sqrt{2}i}{3} \Rightarrow x = -1\pm 2\sqrt{2}\,i \qquad [\because x = 3y]$$

Thus, the possible values of x and y are :

$$x = 3, y = 1;\ x = -1\pm 2\sqrt{2}\,i, y = \frac{-1\pm 2\sqrt{2}i}{3}$$

EXAMPLE 17 *If a, b, c are in G.P. and $\log_c a, \log_b c, \log_a b$ are in A.P., then find the common difference of the A.P.*

SOLUTION Let $x = \log_c a, y = \log_b c$ and $z = \log_a b$. Then,

$$xyz = \log_c a \cdot \log_b c \cdot \log_a b$$

$$= \log_b a \cdot \log_a b$$

$$= \log_b a \cdot \frac{1}{\log_b a}$$

$$\Rightarrow xyz = 1 \qquad \ldots(i)$$

Also, $\log_c a, \log_b c, \log_a b$ are in A.P.

$$\Rightarrow x, y, z \text{ are in A.P.}$$

$$\Rightarrow 2y = x+z. \qquad \ldots(ii)$$

Now,

a, b, c are in G.P.

$$\Rightarrow b^2 = ac$$

$$\Rightarrow \log_b b^2 = \log_b(ac)$$

$$\Rightarrow 2\log_b b = \log_b a + \log_b c$$

$$\Rightarrow 2 = \frac{1}{\log_a b} + \log_b c$$

$$\Rightarrow 2 = \frac{1}{z} + y$$

$$\Rightarrow y = 2 - \frac{1}{z} \qquad \ldots(iii)$$

Eliminating y from (ii) and (iii), we get

$$2\left(2-\frac{1}{z}\right) = x+z$$

$$\Rightarrow x = 4 - \frac{2}{z} - z \qquad \ldots(iv)$$

Putting the values of y and x from (iii) and (iv) respectively, in (i), we get

$$\left(4-\frac{2}{z}-z\right)\left(2-\frac{1}{z}\right)z = 1$$

$\Rightarrow 2z^3 - 9z^2 + 9z - 2 = 0$

$\Rightarrow (z-1)(2z^2 - 7z + 2) = 0$

$\Rightarrow 2z^2 - 7z + 2 = 0$ ...(v) $[\because z \neq 1]$

Now, common difference of the A.P.

$= z - y$

$= z - \left(2 - \frac{1}{z}\right)$ [Using (iii)]

$= \frac{z^2 - 2z + 1}{z} = \frac{\frac{7z-2}{2} - 2z + 1}{z} = \frac{7z - 2 - 4z + 2}{2z} = \frac{3z}{2z} = \frac{3}{2}$

EXERCISE 21.3

1. If $a, 2a+2, 3a+3$ are first three terms of GP, find fifth term of the sequence.
2. If $(m+n)$ *th* term of a GP is a and $(m-n)$ *th* term is b, prove that mth term is $\sqrt{ab}$
3. If a is the first term and b is the nth term of a G.P., then prove that the product P of its first terms is given by $P^2 = (ab)^n$.
4. If a, b, c be respectively the *pth*, *qth* and *r*th terms of a GP, then prove that

$$(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$$

5. If *mth* term of a G.P. is m and nth term is n, then prove that its *r*th term is

$$\left(\frac{m^{r-n}}{n^{r-m}}\right)^{1/m-n}$$

6. If q *th*, r *th* terms of an AP are in G.P., prove that the common ratio of the GP is $\frac{q-r}{p-q}$ and $p-q, q-r, r-s$ are in G.P.
7. If a, b, c, d and x are distinct real numbers such that

$$(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + (b^2 + c^2 + d^2) \leq 0$$

then prove that a, b, c, d are in GP.

8. If a, b, c are three distinct real numbers in G.P such that $a + b + c = xb$, then find the values of x.
9. If a, b, c are three distinct real numbers in GP such that $4b - 3a < c$, then prove that the common ratio r of the G.P. lies in the interval $(-\infty, 1) \cup (3, \infty)$.
10. If $a_1, a_2, \ldots, a_n$ are non-zero real numbers such that

$$\left(\sum_{r=1}^{n-1} a_r^2\right)\left(\sum_{r=2}^{n} a_r^2\right) \leq \left(\sum\sum_{1 \leq r < s \leq n} a_r a_s\right)^2$$

then prove that $a_1, a_2, \ldots, a_n$ are in G.P.

11. If a, b, c, x are real numbers such that

$$(a^2 + b^2)x^2 - 2b(a+c)x + (b^2 + c^2) = 0$$

then prove that a, b, c are in G.P. with common ratio x.

12. If $2x^4 = y^4 + z^4$, $xyz = 8$ and $\log_y x, \log_z y, \log_x z$ are in G.P., then find the real values of x, y, z.
13. Does there exist a G.P. having 20, 21 and 22 as three terms in it ?
14. If $5x - y, 2x + y$ and $x + 2y$ form an A.P. and the numbers $(y+1)^2, xy + 1, (x-1)^2$ form a G.P., find x and y, given that $x \neq 0$.
15. Prove that no G.P. can have three of its terms (not necessarily consecutive) as three consecutive integers.
16. The sequence a_1, a_2, a_3 is a G.P. with $\frac{a_4}{a_6} = \frac{1}{4}$ and $a_2 + a_5 = 216$. Find a_1.

ANSWERS

1. $\frac{-81}{4}$ 8. $(-\infty, -1) \cup (3, \infty)$ 12. $x = y = z = 2$ 13. No 14. $\frac{1}{4}, \frac{1}{2}$ or $-2, -4$ 16. $a_1 = 12, \frac{108}{7}$

—❑❑❑—

21.11 SUM OF *n* TERMS OF A G.P.

THEOREM *To prove that the sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by*

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ or, } S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1.$$

PROOF Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \quad \ldots(i)$$

Multiplying both sides by r, we get

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \quad \ldots(ii)$$

On subtracting (ii) from (i), we get

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n(1 - r) = a(1 - r^n)$$

$$\Rightarrow S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

or, $S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

Hence, $S_n = a\left(\frac{1 - r^n}{1 - r}\right)$ or, $S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

REMARK 1 *If l is the last term of the G.P., then $l = a r^{n-1}$.*

$$\therefore \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right) = \frac{a - ar^n}{1 - r} = \frac{a - (a r^{n-1})\, r}{1 - r} = \frac{a - lr}{1 - r}$$

Thus, $S_n = \frac{a - lr}{1 - r}$ or, $S_n = \frac{lr - a}{r - 1}$, $r \neq 1$

REMARK 2 *If $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$. Therefore, the sum S of an infinite G.P. with common ratio r satisfying $|r| < 1$ is given by*

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} a\left(\frac{1 - r^n}{1 - r}\right)$$

$$\Rightarrow \quad S = \frac{a}{1 - r}$$

Thus, the sum S of an infinite G.P. with first term a and common ratio r $(-1 < r < 1)$ is given by

$$S = \frac{a}{1 - r}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the sum to n terms of the sequence*

$$\left(x + \frac{1}{x}\right)^2, \left(x^2 + \frac{1}{x^2}\right)^2, \left(x^3 + \frac{1}{x^3}\right)^2, \ldots$$

SOLUTION Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \ldots + \left(x^n + \frac{1}{x^n}\right)^2$$

$$\Rightarrow S_n = \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \ldots + \left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$$

$$\Rightarrow S_n = (x^2 + x^4 + x^6 + \ldots + x^{2n}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots + \frac{1}{x^{2n}}\right) + \underbrace{(2 + 2 + \ldots)}_{n \text{ times}}$$

$$\Rightarrow S_n = x^2\left|\frac{(x^2)^n - 1}{x^2 - 1}\right| + \frac{1}{x^2}\left|\frac{(1/x^2)^n - 1}{(1/x^2) - 1}\right| + 2n$$

$$\Rightarrow S_n = x^2\left(\frac{x^{2n} - 1}{x^2 - 1}\right) + \frac{1}{x^{2n}}\left(\frac{1 - x^{2n}}{1 - x^2}\right) + 2n$$

$$\Rightarrow S_n = x^2\left(\frac{x^{2n} - 1}{x^2 - 1}\right) + \frac{1}{x^{2n}}\left(\frac{x^{2n} - 1}{x^2 - 1}\right) + 2n$$

$$\Rightarrow S_n = \left(\frac{x^{2n} - 1}{x^2 - 1}\right)\left(x^2 + \frac{1}{x^{2n}}\right) + 2n$$

EXAMPLE 2 *Sum the series*

$$x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \ldots \text{ to } n \text{ terms}$$

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \ldots + x^n(x^n + y^n)$$

$$\Rightarrow S_n = (x^2 + x^4 + x^6 + \ldots + x^{2n}) + (xy + x^2y^2 + x^3y^3 + \ldots + x^n y^n)$$

$$\Rightarrow S_n = x^2\left\{\frac{(x^2)^n - 1}{x^2 - 1}\right\} + xy\left\{\frac{(xy)^n - 1}{xy - 1}\right\}$$

$$\Rightarrow S_n = x^2\left\{\frac{x^{2n} - 1}{x^2 - 1}\right\} + xy\left\{\frac{(xy)^n - 1}{xy - 1}\right\}$$

EXAMPLE 3 *Find the sum of the following series :*

(i) $5 + 55 + 555 + \ldots$ to n terms

(ii) $0.7 + 0.77 + 0.777 + \ldots$ to n terms

SOLUTION (i) We have,

$5 + 55 + 555 + \ldots$ to n terms

$= 5\,[1 + 11 + 111 + \ldots + \text{to } n \text{ terms}]$

$= \frac{5}{9}\,[9 + 99 + 999 + \ldots + \text{to } n \text{ terms}]$

$= \frac{5}{9}\,[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots + (10^n - 1)]$

$= \frac{5}{9}\,[(10 + 10^2 + 10^3 + \ldots + 10^n) - \underbrace{(1 + 1 + 1 + \ldots + 1)}_{n \text{ times}}]$

$= \frac{5}{9}\left[10 \times \frac{(10^n - 1)}{10 - 1} - n\right]$

$= \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

$= \frac{5}{81}\,[10^{n+1} - 10 - 9n]$

(ii) We have,

$0.7 + 0.77 + 0.777 + \ldots$ to n terms

$= 7 \times 0.1 + 7 \times 0.11 + 7 \times 0.111 + \ldots$ to n terms

$$= 7\,[0.1 + 0.11 + 0.111 + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\,[0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n \text{ terms}\right]$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \ldots \text{ to } n \text{ terms}\right]$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{7}{9}\left[n - \left(\frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^n}\right)\right]$$

$$= \frac{7}{9}\left[n - \frac{1}{10}\,\frac{\left\{1 - \left(\frac{1}{10}\right)^n\right\}}{\left(1 - \frac{1}{10}\right)}\right]$$

$$= \frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right] = \frac{7}{81}\left[9n - 1 + \frac{1}{10^n}\right]$$

EXAMPLE 4 *Prove that the sum to n terms of the series*

$$11 + 103 + 1005 + \ldots \text{ is } \frac{10}{9}(10^n - 1) + n^2.$$

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11 + 103 + 1005 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow S_n = (10 + 1) + (10^2 + 3) + (10^3 + 5) + \ldots + [10^n + (2n - 1)]$$

$$\Rightarrow S_n = (10 + 10^2 + \ldots + 10^n) + (1 + 3 + 5 + \ldots + (2n - 1)$$

$$\Rightarrow S_n = 10\left(\frac{10^n - 1}{10 - 1}\right) + \frac{n}{2}(1 + 2n - 1)$$

$$\Rightarrow S_n = \frac{10}{9}(10^n - 1) + n^2$$

EXAMPLE 5 *Find the natural number a for which*

$$\sum_{k=1}^{n} f(a + k) = 16\,(2^n - 1),$$

where the function f satisfies the relation $f(x + y) = f(x)\,f(y)$ for all $x, y \in N$ and further $f(1) = 2$.

[IIT 1992]

SOLUTION For any natural number x, we have

$$f(x) = f(\underbrace{1 + 1 + 1 + 1 + \ldots + 1}_{x \text{ times}})$$

$$= f(1)\,f(1)\,f(1) \ldots x \text{ times} \qquad [\because f(x + y) = f(x)\,f(y)]$$

$$= [f(1)]^x \qquad [\because f(1) = 2]$$

$$= 2^x$$

Thus, $f(x) = 2^x$ for all $x \in N$

$$\therefore \sum_{k=1}^{n} f(a + k) = 16\,(2^n - 1)$$

$$\Rightarrow \sum_{k=1}^{n} 2^{a+k} = 16\,(2^n - 1)$$

$$\Rightarrow \sum_{k=1}^{n} 2^a \cdot 2^k = 16\,(2^n - 1)$$

$$\Rightarrow 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16\,(2^n - 1)$$

$$\Rightarrow 2^a\,(2 + 2^2 + \ldots + 2^n) = 16\,(2^n - 1)$$

$$\Rightarrow 2^a\left\{2\left(\frac{2^n - 1}{2 - 1}\right)\right\} = 16\,(2^n - 1)$$

$$\Rightarrow 2^{a+1}\,(2^n - 1) = 16\,(2^n - 1)$$

$$\Rightarrow 2^{a+1} = 2^4$$

$$\Rightarrow a + 1 = 4 \Rightarrow a = 3$$

EXAMPLE 6 *If S_1, S_2 and S_3 be respectively the sum of n, 2n and 3n terms of a G.P., prove that*

$$S_1\,(S_3 - S_2) = (S_2 - S_1)^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S_1 = a\left(\frac{r^n - 1}{r - 1}\right), S_2 = a\left(\frac{r^{2n} - 1}{r - 1}\right) \text{ and } S_3 = a\left(\frac{r^{3n} - 1}{r - 1}\right)$$

Now,

$$S_1\,(S_3 - S_2)$$

$$= a\left(\frac{r^n - 1}{r - 1}\right)\left\{a\left(\frac{r^{3n} - 1}{r - 1}\right) - a\left(\frac{r^{2n} - 1}{r - 1}\right)\right\}$$

$$= \frac{a^2}{(r - 1)^2}\,(r^n - 1)\,[(r^{3n} - 1) - (r^{2n} - 1)]$$

$$= \frac{a^2}{(r - 1)^2}\,(r^n - 1)\,(r^{3n} - r^{2n})$$

$$= \frac{a^2}{(r - 1)^2}\,(r^n - 1)\,r^{2n}\,(r^n - 1)$$

$$= \left\{ar^n\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

and,

$$(S_2 - S_1)^2$$

$$= \left[a\left(\frac{r^{2n} - 1}{r - 1}\right) - a\left(\frac{r^n - 1}{r - 1}\right)\right]^2$$

$$= \frac{a^2}{(r-1)^2}\{(r^{2n}-1)-(r^n-1)\}^2$$

$$= \frac{a^2}{(r-1)^2}\{r^n(r^n-1)\}^2$$

$$= \left[a r^n\left(\frac{r^n-1}{r-1}\right)\right]^2$$

Hence, $S_1(S_3-S_1) = (S_2-S_1)^2$

EXAMPLE 7 *If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., prove that*

$$\left(\frac{S}{R}\right)^n = P^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$\Rightarrow \quad S = a\left(\frac{r^n-1}{r-1}\right) \qquad \ldots(i)$$

$$P = a \cdot ar\, ar^2 \ldots ar^{n-1}$$

$$\Rightarrow \quad P = a^n r^{1+2+3+\ldots+(n-1)}$$

$$\Rightarrow \quad P = a^n r^{\frac{n(n-1)}{2}} \qquad \ldots(ii)$$

and,

$$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{ar^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\frac{\{(1/r)^n-1\}}{\{(1/r)-1\}}$$

$$\Rightarrow \quad R = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\left(\frac{r^n-1}{r-1}\right)\frac{1}{r^{n-1}} \qquad \ldots(iii)$$

$$\therefore \quad \frac{S}{R} = a\left(\frac{r^n-1}{r-1}\right).a\left(\frac{r-1}{r^n-1}\right)r^{n-1} = a^2 r^{n-1}$$

$$\Rightarrow \quad \left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = \left\{a^n r^{\frac{n(n-1)}{2}}\right\}^2$$

$$\Rightarrow \quad \left(\frac{S}{R}\right)^n = P^2 \qquad \text{[Using (ii)]}$$

EXAMPLE 8 *If S_n represents the sum of n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that*

(i) $S_1 + S_2 + S_3 + \ldots + S_n = \dfrac{an}{1-r} - \dfrac{ar(1-r^n)}{(1-r)^2}$

(ii) $S_1 + S_3 + S_5 + \ldots + S_{2n-1} = \dfrac{an}{1-r} - \dfrac{ar(1-r^{2n})}{(1-r)^2(1+r)}$

SOLUTION We have, $S_n = a\left(\dfrac{1-r^n}{1-r}\right)$, where $n \in N$.

(i) $S_1 + S_2 + S_3 + \ldots + S_n$

$$= \sum_{k=1}^{n} S_k$$

$$= \sum_{k=1}^{n} a\left(\frac{1-r^k}{1-r}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{a}{1-r} - \frac{ar^k}{1-r}\right)$$

$$= \sum_{k=1}^{n}\frac{a}{1-r} - \sum_{k=1}^{n}\frac{ar^k}{1-r}$$

$$= \frac{an}{1-r} - \frac{a}{1-r}\sum_{k=1}^{n} r^k$$

$$= \frac{an}{1-r} - \frac{a}{1-r}(r + r^2 + \ldots + r^n)$$

$$= \frac{an}{1-r} - \frac{a}{1-r}r\cdot\left(\frac{1-r^n}{1-r}\right)$$

$$= \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$$

(ii) $S_1 + S_3 + S_5 + \ldots + S_{2n-1}$

$$= \sum_{k=1}^{n} S_{2k-1}$$

$$= \sum_{k=1}^{n} a\left(\frac{1-r^{2k-1}}{1-r}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{a}{1-r} - \frac{ar^{2k-1}}{1-r}\right)$$

$$= \sum_{k=1}^{n}\frac{a}{1-r} - \sum_{k=1}^{n}\frac{ar^{2k-1}}{1-r}$$

$$= \frac{an}{1-r} - \frac{a}{1-r}\sum_{k=1}^{n} r^{2k-1}$$

$$= \frac{an}{1-r} - \frac{a}{1-r}(r + r^3 + r^5 + \ldots + r^{2n-1})$$

$$= \frac{an}{1-r} - \frac{a}{1-r}r\left|\frac{1-(r^2)^n}{1-r^2}\right|$$

$$= \frac{an}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$$

EXAMPLE 9 *Let $S_1, S_2, \ldots, S_p$ denote the sum of an infinite G.P. with the first terms 1, 2, 3, ..., p and common ratio $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p+1}$ respectively. Show that $S_1 + S_2 + \ldots + S_p = \frac{1}{2}\ p\,(p+3)$.*

SOLUTION We have,

S_n = Sum of an infinite G.P. with first term n and common ratio $\frac{1}{n+1}$.

$$= \frac{n}{1-\frac{1}{n+1}}$$

$$= n+1$$

$$\therefore\ S_1 + S_2 + \ldots + S_p$$

$$= \sum_{n=1}^{p} S_n$$

$$= \sum_{n=1}^{p} (n+1)$$

$$= 2+3+\ldots+(p+1)$$

$$= \frac{p}{2}(2+p+1) = \frac{p}{2}(p+3)$$

EXAMPLE 10 *If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3, ..., n and common ratios are $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$ respectively, then prove that*

$$S_1^2 + S_2^2 + \ldots + S_{2n-1}^2 = \frac{1}{3}n\,(2n+1)\,(4n+1) - 1.$$

SOLUTION We have,

S_k = Sum of infinite G.P. with first term k and common ratio $\frac{1}{k+1}$

$$= \frac{k}{1-\frac{1}{k+1}}$$

$$= k+1, k = 1, 2, 3, \ldots, n$$

$$\therefore\ S_1^2 + S_2^2 + \ldots + S_{2n-1}^2$$

$$= 2^2 + 3^2 + \ldots + (2n-1+1)^2$$

$$= \{1^2 + 2^2 + \ldots + (2n)^2\} - 1$$

$$= \frac{2n\,(2n+1)\,(2\,(2n)+1)}{6} - 1$$

$$= \frac{2n\,(2n+1)\,(4n+1)}{6} - 1$$

$$= \frac{n}{3}\,(2n+1)\,(4n+1) - 1.$$

EXAMPLE 11 *Let S_n denote the sum of first n terms of a G.P. Show that*

$$S_n\,(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S_n = a\left(\frac{r^n-1}{r-1}\right),\ S_{2n} = a\left(\frac{r^{2n}-1}{r-1}\right) \text{and},\ S_{3n} = a\left(\frac{r^{3n}-1}{r-1}\right)$$

$$\therefore\ S_{3n} - S_{2n} = a\left(\frac{r^{3n}-1}{r-1}\right) - a\left(\frac{r^{2n}-1}{r-1}\right)$$

$$\Rightarrow\ S_{3n} - S_{2n} = a\left(\frac{r^{3n}-r^{2n}}{r-1}\right) = a\,r^{2n}\left(\frac{r^n-1}{r-1}\right)$$

$$\Rightarrow\ S_n\,(S_{3n} - S_{2n}) = a\left(\frac{r^n-1}{r-1}\right).ar^{2n}\left(\frac{r^n-1}{r-1}\right)$$

$$\Rightarrow\ S_n\,(S_{3n} - S_{2n}) = a^2\,r^{2n}\left(\frac{r^n-1}{r-1}\right)^2 \qquad \ldots(i)$$

Also,

$$S_{2n} - S_n = a\left(\frac{r^{2n}-1}{r-1}\right) - a\left(\frac{r^n-1}{r-1}\right)$$

$$= \frac{a}{r-1}\left\{(r^{2n}-1) - (r^n-1)\right\}$$

$$= \frac{a}{r-1}\,(r^{2n} - r^n)$$

$$= \frac{ar^n\,(r^n-1)}{r-1}$$

$$\therefore\ (S_{2n} - S_n)^2 = a^2\,r^{2n}\left(\frac{r^n-1}{r-1}\right)^2 \qquad \ldots(ii)$$

From (i) and (ii), we get

$$S_n\,(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2.$$

EXAMPLE 12 *If $S_1, S_2, \ldots, S_n$ are the sums of n terms of n G.P's whose first term is 1 in each and common ratios are 1, 2, 3, .., n respectively, then prove that*

$$S_1 + S_2 + 2S_3 + \ldots + (n-1)\,S_n = 1^n + 2^n + 3^n + \ldots + n^n$$

SOLUTION We have,

S_1 = Sum of n terms of a GP with first term 1 and common ratio $1 = n$.

and,

S_k = Sum of n terms of a GP with first term 1 and common ratio k.

$$\Rightarrow\ S_k = 1\left(\frac{k^n-1}{k-1}\right),\ k = 2, 3, \ldots, n$$

$$\Rightarrow\ (k-1)\,S_k = k^n - 1,\ k = 2, 3, \ldots, n$$

$$\Rightarrow \sum_{k=2}^{n}(k-1)S_k = \sum_{k=2}^{n}(k^n-1)$$

$$\Rightarrow S_2+2S_3+3S_4+\ldots(n-1)S_n$$
$$=(2^n-1)+(3^n-1)+\ldots+(n^n-1)$$

$$\Rightarrow S_1+S_2+2S_3+3S_4+\ldots+(n-1)S_n$$
$$=n+(2^n-1)+(3^n-1)+\ldots+(n^n-1)$$

$$\Rightarrow S_1+S_2+2S_3+3S_4+\ldots+(n-1)S_n = 1^n+2^n+3^n+\ldots+n^n.$$

EXAMPLE 13 *Let $a_1, a_2, a_3, \ldots, a_n$ be a geometric progression with first term a and common ratio r. Then the sum of the products of $a_1, a_2, \ldots, a_n$ taken two at a time is*

$$\frac{a^2 r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$$

SOLUTION Required sum

$$=\frac{1}{2}\left[(a_1+a_2+\ldots+a_n)^2-(a_1^2+a_2^2+\ldots+a_n^2)\right]$$

$$=\frac{1}{2}\left[\left\{\frac{a(1-r^n)}{1-r}\right\}^2-(a^2+a^2r^2+a^2r^4+\ldots+a^2r^{2n-2})\right]$$

$$=\frac{1}{2}\left[a^2\left(\frac{1-r^n}{1-r}\right)^2-a^2\left(\frac{1-(r^2)^n}{1-r^2}\right)\right]$$

$$=\frac{1}{2}\left[a^2\left(\frac{1-r^n}{1-r}\right)^2-a^2\left(\frac{1-r^n}{1-r}\right)\left(\frac{1+r^n}{1+r}\right)\right]$$

$$=\frac{1}{2}a^2\left(\frac{1-r^n}{1-r}\right)\left[\frac{1-r^n}{1-r}-\frac{1+r^n}{1+r}\right]$$

$$=\frac{1}{2}a^2\left(\frac{1-r^n}{1-r}\right)\frac{2r(1-r^{n-1})}{(1-r^2)}$$

$$=\frac{a^2r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$$

EXAMPLE 14 *Let $a_1, a_2, \ldots, a_n$ be a G.P. with first term a and common ratio r. Show that :*

(i) $\dfrac{1}{a_1^2-a_2^2}+\dfrac{1}{a_2^2-a_3^2}+\ldots+\dfrac{1}{a_{n-1}^2-a_n^2}=\dfrac{r^2(1-r^{2n-2})}{a^2r^{2n-2}(1-r^2)^2}$

(ii) $\dfrac{1}{a_1^m+a_2^m}+\dfrac{1}{a_2^m+a_3^m}+\ldots+\dfrac{1}{a_{n-1}^m+a_n^m}$

$$=\frac{r^{mn-m}-1}{a^m(1+r^m)(r^{mn-m}-r^{mn-2m})}$$

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n$ is a GP. with first term a and common ratio r. Therefore,

$$a_k = ar^{k-1},\ k=1,2,3,\ldots,n.$$

$$\therefore\ a_{k-1}^2-a_k^2 = a^2r^{2k-4}-a^2r^{2k-2} \quad \text{for } k=2,3,\ldots,n$$

$$\Rightarrow a_{k-1}^2-a_k^2 = a^2r^{2k-4}(1-r^2), \quad \text{for } k=2,3,\ldots,n$$

$$\therefore\ \frac{1}{a_1^2-a_2^2}+\frac{1}{a_2^2-a_3^2}+\ldots+\frac{1}{a_{n-1}^2-a_n^2}$$

$$=\sum_{k=2}^{n}\frac{1}{a_{k-1}^2-a_k^2}$$

$$=\sum_{k=2}^{n}\frac{1}{a^2r^{2k-4}(1-r^2)}$$

$$=\frac{1}{a^2(1-r^2)}\sum_{k=2}^{n}\frac{1}{r^{2k-4}}$$

$$=\frac{1}{a^2(1-r^2)}\left(1+\frac{1}{r^2}+\frac{1}{r^4}+\ldots+\frac{1}{r^{2n-4}}\right)$$

$$=\frac{1}{a^2(1-r^2)}\cdot\left(\frac{\left(\frac{1}{r^2}\right)^{n-1}-1}{\frac{1}{r^2}-1}\right)$$

$$=\frac{r^2}{a^2(1-r^2)}\left(\frac{1-r^{2n-2}}{1-r^2}\right)\cdot\frac{1}{r^{2n-2}}$$

$$=\frac{r^2(1-r^{2n-2})}{a^2r^{2n-2}(1-r^2)^2}$$

(ii) We have,

$$a_k = ar^{k-1},\ k=1,2,\ldots,n$$

$$\therefore\ a_{k-1}^m+a_k^m = (ar^{k-2})^m+(ar^{k-1})^m$$
$$=a^mr^{mk-2m}(1+r^m),\ k=2,3,\ldots,n$$

$$\Rightarrow \frac{1}{a_{k-1}^m+a_k^m}=\frac{1}{a^mr^{mk-2m}(1+r^m)}$$

$$\Rightarrow \sum_{k=2}^{n}\frac{1}{a_{k-1}^m+a_k^m}$$

$$=\sum_{k=2}^{n}\frac{1}{a^mr^{mk-2m}(1+r^m)}$$

$$=\frac{1}{a^m(1+r^m)}\sum_{k=2}^{n}\frac{1}{r^{mk-2m}}$$

$$=\frac{1}{a^m(1+r^m)}\left[1+\frac{1}{r^m}+\frac{1}{r^{2m}}+\ldots+\frac{1}{r^{nm-2m}}\right]$$

$$= \frac{1}{a^m(1+r^m)} \left\{ \frac{\left(\frac{1}{r^m}\right)^{n-1} - 1}{\frac{1}{r^m} - 1} \right\}$$

$$= \frac{1}{a^m(1+r^m)} \cdot \frac{(1-r^{mn-m})\, r^m}{r^{mn-m}(1-r^m)}$$

$$= \frac{(r^{mn-m}-1)}{a^m(1+r^m)(r^{mn-m}-r^{mn-2m})}$$

EXAMPLE 15 *Let $a_1, a_2, a_3, \ldots, a_n$, form an A.P. with common difference d and $b_1, b_2, \ldots, b_n, \ldots$ form a G.P. with common ratio r ($\neq 1$). Show that :*

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \frac{a_{n+1} b_{n+1} - a_1 b_1}{r-1} - d \left| \frac{b_{n+2} - b_2}{(r-1)^2} \right|$$

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. with common difference d. Therefore,

$a_k = a_1 + (k-1)\,d$ and $a_{k+1} = a_k + d$ for all $k \geq 1$.

Also, $b_1, b_2, b_3, \ldots, b_n$, are in G.P. with common ratio r.

Therefore,

$b_k = b_1 r^{k-1}$ and $b_{k+1} = b_k r$ for all $k \geq 1$.

Let $S = a_1 b_1 + a_2 b_2 + \ldots + a_{n-1} b_{n-1} + a_n b_n$ ...(i)

$\Rightarrow$ $rS = a_1 b_1 r + a_2 b_2 r + \ldots + a_{n-1} b_{n-1} r + a_n b_n r$

$\Rightarrow$ $rS = a_1 b_2 + a_2 b_3 + \ldots + a_{n-1} b_n + a_n b_n r$ ...(ii)

Subtracting (i) from (ii), we get

$$rS - S = -a_1 b_1 + (a_1 - a_2) b_2 + (a_2 - a_3) b_3 + \ldots + (a_{n-1} - a_n) b_n + a_n b_n r$$

$$\Rightarrow S(r-1) = -a_1 b_1 + (a_1 - a_2) b_2 + (a_2 - a_3) b_3 + \ldots + (a_{n-1} - a_n) b_n + a_n b_{n+1}$$

$$\Rightarrow S(r-1) = -a_1 b_1 - d(b_2 + b_3 + \ldots + b_n) + a_n b_{n+1}$$

$$\Rightarrow S(r-1) = -a_1 b_1 - d\left\{ b_2 \left(\frac{r^{n-1}-1}{r-1} \right) \right\} + a_n b_{n+1}$$

$$\Rightarrow S(r-1) = a_n b_{n+1} - a_1 b_1 - d\left(\frac{b_2 r^{n-1} - b_2}{r-1} \right)$$

$$\Rightarrow S(r-1) = a_n b_{n+1} - a_1 b_1 - d\left(\frac{b_1 r^n - b_2}{r-1} \right)$$

$$\Rightarrow S(r-1) = a_n b_{n+1} - a_1 b_1 - d\left(\frac{b_{n+1} - b_2}{r-1} \right)$$

$$\Rightarrow S = \frac{a_n b_{n+1} - a_1 b_1}{r-1} - d\left(\frac{b_{n+1} - b_2}{(r-1)^2} \right)$$

EXAMPLE 16 *An A.P. and a G.P. each has a as the first term and b as the second term ($0 < b < a$). If S denotes the sum to infinity of the G.P, prove that the sum of the first n terms of the A.P. can be written as*

$$na - \frac{n(n-1)a^2}{2S}$$

SOLUTION Let d be the common difference of the given A.P. and r be the common ratio of the given G.P. Then,

$d = b - a$ and $r = \frac{b}{a}$.

Since $0 < b < a$. Therefore, $0 < \frac{b}{a} < 1 \Rightarrow 0 < r < 1$.

Now,

$$S = \frac{a}{1-r}$$

$$\Rightarrow \quad S = \frac{a}{1 - \frac{b}{a}}$$

$$\Rightarrow \quad S = \frac{a^2}{a-b}$$

$$\Rightarrow \quad b - a = -\frac{a^2}{S}$$

$$\Rightarrow \quad d = -\frac{a^2}{S} \qquad [\because\ d = b - a]$$

Let S_n denote the sum of n terms of the given A.P. Then,

$$S_n = \frac{n}{2}[2n + (n-1)\,d]$$

$$\Rightarrow S_n = \frac{n}{2}\left[2a + (n-1)\left(\frac{-a^2}{S} \right) \right] = a_n - \frac{n(n-1)}{2S} a^2.$$

EXAMPLE 17 *Let S_n, $n = 1, 2, 3\ldots$ be the sum of an infinite geometric series whose first term is 2n and the common ratio is $\frac{1}{2n+1}$. Evaluate :*

$$\lim_{n \to \infty} \frac{S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \ldots + S_n S_1}{S_1^{\,2} + S_2^{\,2} + \ldots + S_n^{\,2}}.$$

SOLUTION We have,

$$S_k = \frac{2k}{1 - \frac{1}{2k+1}} = 2k + 1, k = 1, 2\ldots,\ldots$$

$$\therefore \quad S_1 S_n + S_2 S_{n-1} + \ldots + S_n S_1$$

$$= \sum_{k=1}^{n} S_k S_{n-(k-1)}$$

$$= \sum_{k=1}^{n} (2k+1)\,\{2(n-(k-1)) + 1\}$$

$$= \sum_{k=1}^{n} (2k+1)\ (2n-2k+3)$$

$$= \sum_{k=1}^{n} [(2n+3)+4\,(n+1)\,k-4k^2]$$

$$= \sum_{k=1}^{n} (2n+3)+\sum_{k=1}^{n} 4\,(n+1)\,k-\sum_{k=1}^{n} 4k^2$$

$$= (2n+3)\,n+\frac{4\,(n+1)\,n\,(n+1)}{2}-\frac{4\,n\,(n+1)\,(2n+1)}{6}$$

$$= (2n+3)\,n+2n\,(n+1)^2-\frac{2}{3}\,n\,(n+1)\,(2n+1)$$

and,

$$S_1^2+S_2^2+\ldots+S_n^2$$

$$= \sum_{k=1}^{n} S_k^2$$

$$= \sum_{k=1}^{n} (2k+1)^2$$

$$= \sum_{k=1}^{n} (4k^2+4k+1)$$

$$= 4\sum_{k=1}^{n} k^2+4\sum_{k=1}^{n} k+\sum_{k=1}^{n} 1$$

$$= \frac{4n\,(n+1)\,(2n+1)}{6}+\frac{4n\,(n+1)}{2}+n$$

$$= \frac{2}{3}\,n\,(n+1)\,(2n+1)+2n\,(n+1)+n$$

$$\therefore \lim_{n\to\infty} \frac{S_1\,S_n+S_2\,S_{n-1}+\ldots.+S_n\,S_1}{S_1^2+S_2^2+\ldots+S_n^2}$$

$$= \lim_{n\to\infty} \frac{(2n+3)\,n+2n\,(n+1)^2-\frac{2}{3}\,n\,(n+1)\,(2n+1)}{\frac{2}{3}\,n\,(n+1)\,(2n+1)+2n\,(n+1)+n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2}{n}+\frac{3}{n^2}\right)+2\left(1+\frac{1}{n}\right)^2-\frac{2}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\frac{2}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+2\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{1}{n^2}}$$

[Dividing N^r and D^r by n^3]

$$= \frac{2-\frac{4}{3}}{\frac{4}{3}} = \frac{1}{2}.$$

EXAMPLE 18 *An A.P. and a G.P. with positive terms have the same number of terms and their first terms as well as last terms are equal. Show that the sum of the A.P. is greater than or equal to the sum of the G.P.*

SOLUTION Let $a_1, a_2, \ldots, a_n$ be the given A.P. and $b_1, b_2, \ldots, b_n$ be the given G.P. It is given that $a_1 = b_1$ and $a_n = b_n$. Let r be the common ratio of the given G.P. Then,

$$b_n = b_1\,r^{n-1}$$

$$\Rightarrow a_n = a_1\,r^{n-1} \quad [\because a_1 = b_1 \text{ and } a_n = b_n] \qquad \ldots(i)$$

Let S_n be the sum of terms of the given A.P. and T_n the sum of n terms of the given G.P. Then,

$$S_n = \frac{n}{2}\,(a_1+a_n)$$

$$\Rightarrow S_n = \frac{n}{2}\,(a_1+a_1\,r^{n-1}) \qquad \text{[Using (i)]}$$

$$\Rightarrow S_n = \frac{n}{2}\,a_1\,(1+r^{n-1}).$$

and,

$$T_n = b_1\left(\frac{1-r^n}{1-r}\right)$$

$$\Rightarrow T_n = a_1\left(\frac{1-r^n}{1-r}\right) \qquad [\because a_1 = b_1]$$

Now,

$$S_n - T_n$$

$$= \frac{n}{2}\,a_1\,(1+r^{n-1})-a_1\left(\frac{1-r^n}{1-r}\right)$$

$$= a_1\left[\frac{n}{2}\,(1+r^{n-1})-\left(\frac{1-r^n}{1-r}\right)\right]$$

$$= \frac{a_1}{2}\,[n\,(1+r^{n-1})-2\,(1+r+r^2+\ldots+r^{n-1}]$$

$$= \frac{a_1}{2}\Big[n\,(1+r^{n-1})-\{(1+r^{n-1})+(r+r^{n-2})+(r^2+r^{n-3})+\ldots+(r^k+r^{n-k-1})+\ldots+(r^{n-1}+1)\}\Big]$$

$$= \frac{a_1}{2}\left[\sum_{k=0}^{n-1}\{1+r^{n-1}-(r^k+r^{n-k-1})\}\right]$$

$$= \frac{-a_1}{2}\left[\sum_{k=0}^{n-1}(r^k+r^{n-k-1}-r^{n-1}-1)\right]$$

$$= \frac{-a_1}{2}\sum_{k=0}^{n-1}\{(r^k-1)-r^{n-k-1}\,(r^k-1)\}$$

$$= \frac{-a_1}{2} \sum_{k=0}^{n-1} (r^k - 1)\ (1 - r^{n-k-1})$$

$$= \frac{a_1}{2} \sum_{k=0}^{n-1} (r^k - 1)\ (r^{n-k-1} - 1)$$

Now, three cases arise :

CASE I *When* $r > 1$:

In this case, we have

$r^k > 1$ and $r^{n-k-1} > 1$.

Therefore,

$$(r^k - 1)(r^{n-k-1} - 1) > 0$$

$$\Rightarrow \sum_{k=0}^{n-1} (r^k - 1)\ (r^{n-k-1} - 1) > 0$$

$$\Rightarrow \frac{a_1}{2} \sum_{k=0}^{n-1} (r^k - 1)\ (r^{n-k-1} - 1) > 0$$

$$\Rightarrow S_n - T_n > 0$$

$$\Rightarrow S_n > T_n.$$

CASE II When $0 < r < 1$:

In this case, we have

$r^k < 1$ and $r^{n-k-1} < 1$

$\Rightarrow r^k - 1 < 0$ and $r^{n-k-1} - 1 < 0$

$\Rightarrow (r^k - 1)\ (r^{n-k-1} - 1) > 0$

$$\Rightarrow \sum_{k=0}^{n-1} (r^k - 1)\ (r^{n-k-1} - 1) > 0$$

$$\Rightarrow \frac{a_1}{2} \sum_{k=0}^{n-1} (r^k - 1)\ (r^{n-k-1} - 1) > 0$$

$$\Rightarrow S_n - T_n > 0$$

$$\Rightarrow S_n > T_n.$$

CASE III In this case, we have

$r^k = 1$ and $r^{n-k-1} = 1$

$\therefore\ r^k - 1 = 0$ and $r^{n-k-1} = 0$

$$\Rightarrow S_n - T_n = \frac{a_1}{2} \sum_{k=1}^{n-1} (r^k - 1)(r^{n-k-1} - 1)$$

$$\Rightarrow S_n - T_n = 0$$

$$\Rightarrow S_n = T_n$$

Hence, in all the three cases, we have $S_n \geq T_n$.

EXAMPLE 19 *Two geometric series have the same first term 1. The common ratio of the first series is greater than that of the second series and both are positive. If* S_n and T_n *denote the sums of n terms of the two series respectively, then prove that*

$$\frac{S_n}{S_{n+1}} < \frac{T_n}{T_{n+1}}.$$

SOLUTION Let a and b be the common ratios of the two geometric series. Then, $a > b > 0$.

Clearly, $S_n, S_{n+1}, T_n, T_{n+1}$ are all positive.

Now,

$$S_n\ T_{n+1} - T_n\ S_{n+1}$$

$$= (1 + a + a^2 + \ldots + a^{n-1})\ (1 + b + b^2 + \ldots + b^n)$$

$$- (1 + b + b^2 + \ldots + b^{n-1})(1 + a + a^2 + \ldots + a^n)$$

$$= (1 + a + a^2 + \ldots + a^{n-1})\ \{(1 + b + b^2 + \ldots + b^{n-1}) + b^n\}$$

$$- (1 + b + b^2 + \ldots + b^{n-1})\ \{(1 + a + a^2 + \ldots + a^{n-1}) + a^n\}$$

$$= (1 + a + a^2 + \ldots + a^{n-1})\ (1 + b + b^2 + \ldots + b^{n-1})$$

$$+ b^n (1 + a + a^2 + \ldots + a^{n-1})$$

$$- (1 + b + b^2 + \ldots + b^{n-1})\ (1 + a + a^2 + \ldots + a^{n-1})$$

$$- a^n (1 + b + b^2 + \ldots + b^{n-1})$$

$$= b^n\ (1 + a + a^2 + \ldots + a^{n-1}) - a^n\ (1 + b + b^2 + \ldots + b^{n-1})$$

$$= (b^n - a^n) + (b^n a - a^n b) + (b^n a^2 - a^n b^2) + \ldots$$

$$+ (b^n a^{n-1} - a^n b^{n-1})$$

$$= (b^n - a^n) + ab\ (b^{n-1} - a^{n-1}) + a^2 b^2\ (b^{n-2} - a^{n-2}) + \ldots$$

$$+ a^{n-1} b^{n-1}\ (b - a)$$

< 0 [$\because\ 0 < b < a\ \therefore$ each term on RHS is negative]

$$\Rightarrow S_n\ T_{n+1} < T_n\ S_{n+1}$$

$$\Rightarrow \frac{S_n}{S_{n+1}} < \frac{T_n}{T_{n+1}}$$

EXAMPLE 20 *Find the sum of an infinitely decreasing G.P. whose first term is equal to* $b + 2$ *and the common ratio to* $2/c$, *where b is the least value of the product of the roots of the equation* $(m^2 + 1)\ x^2 - 3x + (m^2 + 1)^2 = 0$, *and c is the greatest value of the sum of its roots.*

SOLUTION We have,

$$(m^2 + 1)\ x^2 - 3x + (m^2 + 1)^2 = 0$$

$$\therefore \quad \text{Sum of the roots} = \frac{3}{m^2 + 1}$$

and,

$$\text{Product of the roots} = (m^2 + 1)$$

Now,

b = least value of the product of roots

= least value of (m^2+1)

= 1 $[\because\ m^2+1 \geq 1$ for all $m]$

c = greatest value of the sum of the roots

= greatest value of $\dfrac{3}{m^2+1}$

Clearly, $\dfrac{3}{m^2+1}$ is greatest when m^2+1 is least and the least value of m^2+1 is 1.

$\therefore \quad c = \dfrac{3}{1} = 3$

So, first term of the infinite G.P. is $b+2 = 1+2 = 3$ and the common ratio is $\dfrac{2}{c} = \dfrac{2}{3}$.

Hence, the sum of the infinite G.P. is given by

$$S = \frac{3}{1-\frac{2}{3}} = 9 \qquad \left[\text{Using}: S = \frac{a}{1-r}\right]$$

EXAMPLE 21 *For $0 < \theta,\ \phi < \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, then prove that $xz + yz - z = xy$.*

SOLUTION We have,

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta$$

$$\Rightarrow\ x = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty$$

$$\Rightarrow\ x = \frac{1}{1-\cos^2\theta}$$

$$\Rightarrow\ \sin^2\theta = \frac{1}{x}$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi$$

$$\Rightarrow\ y = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$$

$$\Rightarrow\ y = \frac{1}{1-\sin^2\phi}$$

$$\Rightarrow \cos^2\phi = \frac{1}{y},$$

and,

$$z = \sum_{n=0}^{\infty} \cos^{2n}\theta, \sin^{2n}\phi$$

$$= 1 + \cos^2\theta \sin^2\phi + \cos^4\theta \sin^4\phi + \ldots \infty$$

$$\Rightarrow\ z = \frac{1}{1-\cos^2\theta \sin^2\phi}$$

$$\Rightarrow\ z = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$

$$\Rightarrow\ z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)}$$

$$\Rightarrow\ z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}}$$

$$\Rightarrow\ z = \frac{xy}{x+y-1}$$

$$\Rightarrow\ xz + yz - z = xy$$

EXAMPLE 22 *If there are m quantities in G.P. with common ratio 'r' and S_m denotes the sum of the first m terms, show that the sum of the products of these m terms taken two by two together is $\dfrac{r}{r+1} S_m \cdot S_{m-1}$.*

SOLUTION Let a be the first term of the given G.P. Then

$$S_m = a\left(\frac{1-r^m}{1-r}\right) \qquad \ldots(i)$$

and,

$$S_{m-1} = a\left(\frac{1-r^{m-1}}{1-r}\right) \qquad \ldots(ii).$$

We know that

$(x_1 + x_2 + \ldots + x_n)^2$

$= (x_1^2 + x_2^2 + \ldots + x_n^2) + 2$ (Sum of the products taken two by two)

$\Rightarrow$ 2 [Sum of the products taken two by two]

= (Sum of m terms)^2 – (Sum of the squares of m terms)

$\Rightarrow$ 2 [Sum of the products taken two by two]

$$= (a + ar + ar^2 + \ldots + ar^{m-1})^2 - (a^2 + a^2r^2 + a^2r^4 + \ldots + a^2r^{2m-2})$$

$$= \left\{a\left(\frac{1-r^m}{1-r}\right)\right\}^2 - \left\{a^2\left(\frac{1-r^{2m}}{1-r^2}\right)\right\}$$

$$= a^2\frac{(1-r^m)^2}{(1-r)^2} - \frac{a^2(1-r^m)(1+r^m)}{(1-r)(1+r)}$$

$$= a^2\frac{(1-r^m)}{(1-r)}\left[\frac{1-r^m}{1-r} - \frac{1+r^m}{1+r}\right]$$

$$= a^2\frac{(1-r^m)}{(1-r)}\left[\frac{(1-r^m)(1+r)-(1+r^m)(1-r)}{(1-r)(1+r)}\right]$$

$$= a^2\left(\frac{1-r^m}{1-r}\right)\frac{(2r-2r^m)}{(1-r^2)}$$

$$= 2a^2 r\left(\frac{1-r^m}{1-r}\right)\left(\frac{1-r^{m-1}}{1-r^2}\right)$$

$$= 2\left\{a\left(\frac{1-r^m}{1-r}\right)\right\}\left\{a\left(\frac{1-r^{m-1}}{1-r}\right)\right\}\left(\frac{r}{r+1}\right) \quad \text{[Using (i) and (ii)]}$$

Hence,

$$= \frac{2r}{r+1} S_m S_{m-1}$$

Sum of the products taken two by two $= \frac{r}{r+1} S_m \cdot S_{m-1}$.

EXAMPLE 23 *Let S denote the sum of the terms of an infinite G.P. and σ^2 denote the sum of the squares of the terms. Show that the sum of the first n terms of this geometric progression is given by*

$$S\left[1-\left(\frac{S^2-\sigma^2}{S^2+\sigma^2}\right)^n\right]$$

SOLUTION Let a be the first term and r the common ratio of the G.P. such that $|r| < 1$. Then,

$$S = \frac{a}{1-r} \quad \text{...(i)}$$

and,

$$\sigma^2 = a^2 + a^2 r^2 + a^2 r^4 + \ldots.$$

$$\Rightarrow \quad \sigma^2 = \frac{a^2}{1-r^2}$$

Now,

$$\frac{S^2}{\sigma^2} = \frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2}$$

$$\Rightarrow \quad \frac{S^2}{\sigma^2} = \frac{1+r}{1-r}$$

$$\Rightarrow \quad \frac{S^2+\sigma^2}{S^2-\sigma^2} = \frac{2}{2r} \quad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad r = \frac{S^2-\sigma^2}{S^2+\sigma^2} \quad \text{...(ii)}$$

Let S_n denote the sum of n terms of the G.P. Then,

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$\Rightarrow \quad S_n = S\left[1-\left(\frac{S^2-\sigma^2}{S^2+\sigma^2}\right)^n\right] \quad \text{[Using (i) and (ii)]}$$

EXAMPLE 24 *If $|x| < 1$ and $|y| < 1$, find the sum to infinity of the series*

$$(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots$$

SOLUTION We know that

$$\frac{x^n-a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \ldots + a^{n-1}, n \in N$$

$$\therefore \quad (x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots \text{ to } \infty$$

$$= \frac{1}{x-y}\left[(x^2-y^2) + (x^3-y^3) + (x^4-y^4) + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{x-y}\left[(x^2+x^3+x^4+\ldots \text{ to } \infty) - (y^2+y^3+y^4+\ldots \text{ to } \infty)\right]$$

$$= \frac{1}{x-y}\left[\frac{x^2}{1-x} - \frac{y^2}{1-y}\right]$$

$$= \frac{1}{x-y}\frac{[x^2-x^2y-y^2+y^2x]}{(1-x)(1-y)}$$

$$= \frac{1}{(x-y)}\frac{[(x^2-y^2)-xy(x-y)]}{(1-x)(1-y)}$$

$$= \frac{x+y-xy}{(1-x)(1-y)}$$

EXAMPLE 25 *If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156, find the numbers.*

SOLUTION Let the three numbers be $a/r, a, ar$. Then,

Product = 216

$\Rightarrow a/r \,.\, a \,.\, ar = 216$

$\Rightarrow a^3 = 216 \Rightarrow a = 6.$

Sum of the products in pairs = 156

$$\Rightarrow \quad \frac{a}{r}.a + a.ar + \frac{a}{r}.ar = 156$$

$$\Rightarrow \quad a^2\left(\frac{1}{r}+r+1\right) = 156$$

$$\Rightarrow \quad 36\left(\frac{1+r^2+r}{r}\right) = 156$$

$$\Rightarrow \quad 3(r^2+r+1) = 13r$$

$$\Rightarrow \quad 3r^2-10r+3 = 0$$

$$\Rightarrow \quad (3r-1)(r-3) = 0$$

$$\Rightarrow \quad r = \frac{1}{3} \text{ or } r = 3$$

Hence, putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 26 *Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.*

SOLUTION Let the numbers be a, ar, ar^2. Then,

Sum = 13

$$\Rightarrow \quad a + ar + ar^2 = 13$$

$$\Rightarrow a(1+r+r^2) = 13 \quad \text{...(i)}$$

Sum of the squares = 91

$\Rightarrow a^2 + a^2r^2 + a^2r^4 = 91$

$\Rightarrow a^2(1 + r^2 + r^4) = 91$ ...(ii)

Now, $a(1 + r + r^2) = 13$

$\Rightarrow a^2(1 + r + r^2)^2 = 169$

$\Rightarrow a^2(1 + r^2 + r^4) + 2a^2r(1 + r + r^2) = 169$

$\Rightarrow 91 + 2ar\{a(1 + r + r^2)\} = 169$

$\Rightarrow 91 + 2ar \times 13 = 169$ [Using (ii)]

$\Rightarrow ar = 3$ [Using (i)]

$\Rightarrow a = \frac{3}{r}$ ...(iii)

Putting $a = \frac{3}{r}$ in (i), we get

$\frac{3}{r}(1 + r + r^2) = 13$

$\Rightarrow \frac{3}{r} + 3 + 3r = 13$

$\Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow (3r - 1)(r - 3) = 0$

$\Rightarrow r = 3$ or $r = \frac{1}{3}$

Putting $r = 3$ in (iii), we get $a = 1$. So, the numbers are 1, 3, 9.

Putting $r = \frac{1}{3}$ in (iii), we get $a = 9$. So, the numbers are 9, 3, 1.

Hence, the numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 27 *Find the geometric progression of real numbers such that the sum of its first four terms is equal to 30 and the sum of the squares of the first four terms is 340.*

SOLUTION Let a, ar, ar^2, ar^3 be the first four terms of the G.P. with first term a and common ratio r. It is given that

$$a + ar + ar^2 + ar^3 = 30 \quad ...(i)$$

and,

$$a^2 + a^2r^2 + a^2r^4 + a^2r^6 = 340,\ r \neq 1 \quad ...(ii)$$

$\Rightarrow \frac{(a + ar + ar^2 + ar^3)^2}{a^2 + a^2r^2 + a^2r^4 + a^2r^6} = \frac{30^2}{340}$

$\Rightarrow \frac{(1 + r + r^2 + r^3)^2}{(1 + r^2 + r^4 + r^6)} = \frac{900}{340}$

$\Rightarrow \left(\frac{1 - r^4}{1 - r}\right)^2 \times \left(\frac{1 - r^2}{1 - r^8}\right) = \frac{90}{34}$

$\Rightarrow \frac{1 - r^4}{1 - r} \times \frac{1 + r}{1 + r^4} = \frac{45}{17}$

$\Rightarrow \frac{(1 + r + r^2 + r^3)(1 + r)}{1 + r^4} = \frac{45}{17}$

$\Rightarrow \frac{r^4 + 2r^3 + 2r^2 + 2r + 1}{r^4 + 1} = \frac{45}{17}$

$\Rightarrow 17r^4 + 34r^3 + 34r^2 + 34r + 17 = 45r^4 + 45$

$\Rightarrow 14r^4 - 17r^3 - 17r^2 - 17r + 14 = 0$

$\Rightarrow 14\left(r^2 + \frac{1}{r^2}\right) - 17\left(r + \frac{1}{r}\right) - 17 = 0$ [Dividing by r^2]

$\Rightarrow 14\left\{\left(r + \frac{1}{r}\right)^2 - 2\right\} - 17\left(r + \frac{1}{r}\right) - 17 = 0$

$\Rightarrow 14(y^2 - 2) - 17y - 17 = 0$, where $y = r + \frac{1}{r}$

$\Rightarrow 14y^2 - 17y - 45 = 0$

$\Rightarrow y = \frac{17 \pm \sqrt{17^2 + 4 \times 45 \times 14}}{2 \times 14} = \frac{17 \pm 53}{28} = \frac{5}{2}, \frac{-9}{7}$

$\Rightarrow r + \frac{1}{r} = \frac{5}{2}, r + \frac{1}{r} = \frac{-9}{7}$

Now,

$r + \frac{1}{r} = \frac{5}{2}$

$\Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow (2r - 1)(r - 2) = 0$

$\Rightarrow r = 2, \frac{1}{2}$

$r + \frac{1}{r} = \frac{-9}{7}$ does not give real values of r.

Putting $r = 2$ in (i), we get

$15a = 30 \Rightarrow a = 2$.

Putting $r = 2$ in (i), we get

$a + \frac{a}{2} + \frac{a}{4} + \frac{a}{8} = 30$

$\Rightarrow \frac{8a + 4a + 2a + a}{8} = 30$

$\Rightarrow 15a = 30 \times 8$

$\Rightarrow a = 16$

Hence, the required progressions are :

2, 4, 8, 16, 32, ..., and 16, 8, 4, 2, 1/2,..

EXAMPLE 28 *Prove that* $\underset{91 \text{ times}}{111 \ldots 1}$ *is not a prime number.*

SOLUTION We have,

$\underset{91 \text{ times}}{111 \ldots 1} = 10^{90} + 10^{89} + \ldots + 10 + 1$

$$= \frac{10^{91}-1}{10-1}$$

$$= \frac{10^{91}-1}{10^7-1} \times \frac{10^7-1}{10-1}$$

$$= \left(\frac{(10^7)^{13}-1}{10^7-1}\right) \times \left(\frac{10^7-1}{10-1}\right)$$

$$= \{(10^7)^{12} + (10^7)^{11} + \ldots + 1\} \times (10^6 + 10^5 + \ldots + 1)$$

$$= (10^{84} + 10^{77} + \ldots + 1)(10^6 + 10^5 + \ldots + 1)$$

Hence, $\underbrace{111 \ldots 1}_{91 \text{ times}}$ is not a prime number..

EXAMPLE 29 *Consider the sequence 49, 4489 444889,... in which every term is obtained by inserting 48 in the middle of the previous term. Prove that each term of the sequence is the square of an integer.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n \ldots$ be the given sequence. Then,

$$a_1 = 49 = 4 \times 10 + 9$$

$$a_2 = 4489$$

$$= (4 \times 10^3 + 4 \times 10^2) + (8 \times 10) + 9$$

$$a_3 = 444889$$

$$= (4 \times 10^5 + 4 \times 10^4 + 4 \times 10^3) + (8 \times 10^2 + 8 \times 10^1) + 9$$

$$a_4 = 44448889$$

$$= (4 \times 10^7 + 4 \times 10^6 + 4 \times 10^5 + 4 \times 10^4) + (8 \times 10^3 + 8 \times 10^2 + 8 \times 10^1) + 9 \text{ etc.}$$

$$\therefore \quad a_n = (4 \times 10^{2n-1} + 4 \times 10^{2n-3} + \ldots + 4 \times 10^n) + (8 \times 10^{n-1} + 8 \times 10^{n-2} + \ldots + 8 \times 10^1) + 9$$

$$= 4 \times 10^n \left(\frac{10^n - 1}{10-1}\right) + 8 \times 10^1 \left(\frac{10^{n-1}-1}{10-1}\right) + 9$$

$$= \frac{4}{9} \times 10^n (10^n - 1) + \frac{8}{9} \times (10^n - 10) + 9$$

$$= \frac{1}{9} [4 \times 10^{2n} - 4 \times 10^n + 8 \times 10^n - 8 \times 10 + 81]$$

$$= \frac{1}{9} [4 \times 10^{2n} + 4 \times 10^n + 1]$$

$$= \left\{\frac{2 \times 10^n + 1}{3}\right\}^2$$

Since the sum of the digits in $2 \times 10^n + 1$ is 3. Therefore, $2 \times 10^n + 1$ is divisible by 3.

$\Rightarrow \frac{2 \times 10^n + 1}{3}$ is an integer

$\Rightarrow a_n$ is the square of an integer.

Hence, each term of the given sequence is the square of an integer n.

EXAMPLE 30 *Show that all the terms of the sequence* 10001, 100010001, 1000100010001, ... *are composite numbers.*

SOLUTION Let $a_1, a_2, a_3, a_4, \ldots$ be the given sequence. Then,

$$a_1 = 10001 = 10^4 + 1$$

$$a_2 = 100010001 = 10^8 + 10^4 + 1$$

$$a_3 = 1000100010001 = 10^{12} + 10^8 + 10^4 + 1 \text{ etc.}$$

In general,

$$a_n = 10^{4n} + 10^{4(n-1)} + \ldots + 10^8 + 10^4 + 1, n \in N.$$

Clearly, $a_1 = 10001 = 73 \times 137$ is a composite number.

Now,

$$a_n = 10^{4n} + 10^{4(n-1)} + \ldots + 10^8 + 10^4 + 1$$

$$= \frac{(10^4)^{n+1} - 1}{10^4 - 1}$$

$$= \frac{10^{4n+4} - 1}{10^4 - 1}$$

$$= \frac{(10^{2n+2})^2 - 1^2}{10^{2n+2} - 1} \times \frac{10^{2n+2} - 1}{10^4 - 1}$$

$$= \frac{10^{2n+2} + 1}{10^2 + 1} \times \frac{(10^{n+1})^2 - 1}{10^2 - 1}$$

$$= \frac{10^{2n+2} + 1}{101} \times \frac{(10^2)^{n+1} - 1}{10^2 - 1}$$

$$= \frac{10^{2n+2} + 1}{101} \times [1 + 10^2 + 10^4 + \ldots + (10^2)^n]$$

$$\Rightarrow \quad 101\, a_n = [1 + 10^2 + 10^4 + \ldots + (10^2)^n]\ (10^{2n+2} + 1) \qquad \ldots(i)$$

Since 101 is a prime number. Therefore, it follows from the above relation that either $1 + 10^2 + 10^4 + \ldots + (10^2)^n$ is divisible by 101 or $10^{2n+2} + 1$ is divisible by 101. That is,

either $1 + 10^2 + 10^4 + \ldots + (10^2)^n = 101\, b_n$

or, $10^{2n+2} + 1 = 101\, c_n$

$\Rightarrow a_n = (10^{2n+2} + 1)\, b_n$

or, $a_n = [1 + 10 + 10^2 + \ldots + (10^2)^n]\, c_n$ [Using (i)]

$\Rightarrow a_n$ is a composite number.

Hence, a_n is a composite number for all $n \in N$.

EXAMPLE 31 *Prove that each term of the sequence* 121, 12321, 1234321, 123454321, ... *is a perfect square of an odd integer.*

SOLUTION Let $a_1, a_2, a_3, \ldots$ be the given sequence. Then,

$$a_1 = 121 = 10^2 + 2 \times 10 + 1$$

$$a_2 = 12321 = 10^4 + 2\times10^3 + 3\times10^2 + 2\times10^1 + 1$$

$$a_3 = 1234321 = 10^6 + 2\times10^5 + 3\times10^4 + 4\times10^3 + 3\times10^2 + 2\times10 + 1$$

and so on.

In general,

$$a_n = 123\ldots.n\,(n+1)\,n\ldots4321$$

$$= 1\times10^{2n} + 2\times10^{2n-1} + 3\times10^{2n-2} + \ldots + n\times10^{n+1} + (n+1)\,10^n + n\times10^{n-1} + (n-1)\times10^{n-2} + \ldots + 3\times10^2 + 2\times10^1 + 1$$

$$= [1\times10^{2n} + 2\times10^{2n-1} + 3\times10^{2n-2} + \ldots + (n+1)\times10^n] + [n\times10^{n-1} + (n-1)\,10^{n-2} + \ldots + 3\times10^2 + 2\times10^1 + 1]$$

$$= S_1 + S_2$$

where $S_1 = 1\times10^{2n} + 2\times10^{2n-1} + 3\times10^{2n-2} + \ldots + (n+1)\,10^n$

and,

$$S_2 = n\times10^{n-1} + (n-1)\times10^{n-2} + \ldots + 2\times10^1 + 1$$

Now,

$$S_1 = 1\times10^{2n} + 2\times10^{2n-1} + 3\times10^{2n-2} + \ldots + (n+1)\times10^n$$

$$\Rightarrow \frac{S_1}{10} = 10^{2n-1} + 2\times10^{2n-2} + \ldots + n\,10^n + (n+1)\,10^{n-1}$$

Subtracting these two, we get

$$S_1 - \frac{S_1}{10} = 10^{2n} + 10^{2n-1} + 10^{2n-2} + \ldots + 10^n - (n+1)\,10^{n-1}$$

$$= 10^n\left(\frac{10^{n+1}-1}{10-1}\right) - (n+1)\,10^{n-1}$$

$$\frac{9}{10}S_1 = \frac{10^{2n+1}-10^n}{9} - (n+1)\,10^{n-1}$$

$$\Rightarrow S_1 = \frac{10^{2n+2}-10^{n+1}}{81} - \frac{(n+1)\,10^n}{9} \quad \ldots(i)$$

and,

$$S_2 = n\times10^{n-1} + (n-1)\times10^{n-2} + \ldots + 2\times10^1 + 1$$

$$\Rightarrow \frac{S_2}{10} = n\times10^{n-2} + (n-1)\,10^{n-3} + \ldots + 2 + \frac{1}{10}$$

On subtracting, we get

$$S_2 - \frac{S_2}{10} = n\times10^{n-1} - \left\{10^{n-2} + 10^{n-3} + \ldots + 1 + \frac{1}{10}\right\}$$

$$\Rightarrow \frac{9}{10}S_2 = n\times10^{n-1} - \frac{1}{10}\left(\frac{10^n-1}{10-1}\right)$$

$$\Rightarrow \frac{9}{10}S_2 = n\times10^{n-1} - \left(\frac{10^{n-1}-10^{-1}}{9}\right)$$

$$\Rightarrow S_2 = \frac{n}{9}\times10^n - \frac{1}{81}(10^n - 1) \quad \ldots(ii)$$

$$\therefore a_n = S_1 + S_2$$

$$\Rightarrow a_n = \frac{10^{2n+2}-10^{n+1}}{81} - \left(\frac{n+1}{9}\right)\times10^n + \frac{n\times10^n}{9} - \frac{(10^n-1)}{81}$$

$$\Rightarrow a_n = \frac{10^{2n+2}-10^{n+1}-10^n+1}{81} - \frac{10^n}{9}$$

$$\Rightarrow a_n = \frac{10^{2n+2}-10^{n+1}-10^n+1-9\times10^n}{81}$$

$$\Rightarrow a_n = \frac{10^{2n+2}-2\times10^{n+1}+1}{81}$$

$$\Rightarrow a_n = \left(\frac{10^{n+1}-1}{9}\right)^2$$

$$\Rightarrow a_n = \left(\frac{9999\ldots..9\ (n+1)\ \text{digits}}{9}\right)^2$$

$$\Rightarrow a_n = \left(\underset{(n+1)\ \text{digits}}{111\ldots..1}\right)^2$$

Hence, a_n is the perfect square of an odd integer.

EXAMPLE 32 *If for* $0 < x < \pi/2$,

$$exp\,[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots)\log_e 2]$$

satisfies the quadratic equation $x^2 - 9x + 8 = 0$ *find the value of*

$$\frac{\sin x - \cos x}{\sin x + \cos x}.$$

SOLUTION We have,

$$0 < x < \frac{\pi}{2}$$

$$\Rightarrow 0 < \sin^2 x < 1$$

$$\therefore \sin^2 x + \sin^4 x + \sin^6 x + \ldots. = \frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$$

$$\Rightarrow \text{exp. } [(\sin^2 x + \sin^4 x + \sin^6 x \ldots +)\log_e 2] = e^{\tan^2 x \log_e 2} = e^{\log_e 2^{\tan^2 x}} = 2^{\tan^2 x}$$

Now,

$$x^2 - 9x + 8 = 0$$

$$\Rightarrow x = 1, 8$$

It is given that $2^{\tan^2 x}$ satifies the quadratic equation $x^2 - 9x + 8 = 0$. Therefore,

$$2^{\tan^2 x} = 1, 2^{\tan^2 x} = 8$$

$$\Rightarrow \quad 2^{\tan^2 x} = 2^0, 2^{\tan^2 x} = 2^3$$

$$\Rightarrow \quad \tan^2 x = 0, \tan^2 x = 3$$

$$\Rightarrow \quad \tan x = 0, \tan x = \sqrt{3} \qquad [\because 0 < x < \pi/2]$$

$$\Rightarrow \quad x = 0, \frac{\pi}{3}$$

EXAMPLE 33 *Find the sum of the infinite G.P, if it is given that first term is the point where the function* $f(x) = 7 + 2x \log_e 25 - 5^{x-1} - 5^{2-x}$ *has the greatest value and the common ratio is equal to* $\lim_{x \to 0} \int_0^x \frac{t^2}{x^2 \tan(\pi + x)} dt$.

SOLUTION We have,

$$f(x) = 7 + 2x \log_e 25 - 5^{x-1} - 5^{2-x}$$

$$\Rightarrow f'(x) = 2 \log_e 25 - 5^{x-1} \log_e 5 + 5^{2-x} \log_e 5$$

$$\Rightarrow f'(x) = (\log_e 5)(4 - 5^{x-1} + 5^{2-x})$$

$$\Rightarrow f'(x) = -\frac{(\log_e 5)}{5^{x+1}} (5^{2x} - 20 \cdot 5^x - 125)$$

$$\Rightarrow f'(x) = -\frac{(\log_e 5)}{5^{x+1}} (5^x - 25)(5^x + 5)$$

For maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow 5^x - 25 = 0 \qquad [\because 5^x + 5 \neq 0]$$

$$\Rightarrow \quad 5^x = 5^2$$

$$\Rightarrow \quad x = 2$$

Clearly, $f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = 2$. So, $f(x)$ attains a local maximum at $x = 2$.

Thus, if a is the first term of the infinite G.P. Then, $a = 2$.

Let r be the common ratio of the G.P. Then,

$$r = \lim_{x \to 0} \int_0^x \frac{t^2}{x^2 \tan(\pi + x)} dt = \lim_{x \to 0} \frac{x^3}{3x^2 \tan x} = \frac{1}{3}$$

Hence, required sum $= \frac{a}{1-r} = \frac{2}{1 - \frac{1}{3}} = 3.$

EXAMPLE 34 *Prove that :*

$$(666.....6)^2 + (888.....8) = 444.... 4$$
$$n\text{-digits} \qquad n\text{-digits} \qquad 2n\text{-digits}$$

SOLUTION We have,

$$(\underset{n\text{-digits}}{666 .... 6}) = 6 \times 10^{n-1} + 6 \times 10^{n-2} + ... + 6 \times 10^1 + 6 \times 10^0$$

$$= 6 (1 + 10 + 10^2 + ... + 10^{n-1})$$

$$= 6 \left(\frac{10^n - 1}{10 - 1}\right)$$

Similarly, we have

$$(\underset{n\text{-digits}}{888 .... 8}) = 8 \left(\frac{10^n - 1}{10 - 1}\right)$$

$$\therefore \quad (\underset{n\text{-digits}}{666...6})^2 + (\underset{n\text{-digits}}{888....8})$$

$$= 36 \left(\frac{10^n - 1}{10 - 1}\right)^2 + 8 \left(\frac{10^n - 1}{10 - 1}\right)$$

$$= 4 \left(\frac{10^n - 1}{10 - 1}\right) \left\{9 \left(\frac{10^n - 1}{10 - 1}\right) + 2\right\}$$

$$= 4 \left(\frac{10^n - 1}{10 - 1}\right) (10^n + 1)$$

$$= 4 \left(\frac{10^{2n} - 1}{10 - 1}\right)$$

$$= 4 (1 + 10 + 10^2 + ... + 10^{2n-1})$$

$$= (\underset{2n\text{-digits}}{444.....4})$$

EXAMPLE 35 *Find the sum of the infinite series :*

$$1 + (1 + a) b + (1 + a + a^2) b^2 + (1 + a + a^2 + a^3) b^3 + ... \infty$$

b and a being fractions.

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = (1 + a + a^2 + ... + a^{n-1}) b^{n-1}$$

$$= \left(\frac{1 - a^n}{1 - a}\right) b^{n-1}$$

$$= \frac{b^{n-1}}{1 - a} - \frac{a^n b^{n-1}}{1 - a}$$

$$= \frac{1}{1 - a} (b^{n-1}) - \frac{1}{b(1 - a)} (ab)^n$$

$\therefore$ Sum of the series

$$= \sum_{n=1}^{\infty} T_n$$

$$= \sum_{n=1}^{\infty} \left[\frac{b^{n-1}}{1 - a} - \frac{1}{b(1 - a)} (ab)^n\right]$$

$$= \frac{(1 - ab) - a(1 - b)}{(1 - a)(1 - b)(1 - ab)} = \frac{1}{(1 - b)(1 - ab)}$$

$$= \frac{1}{1-a}\left(\sum_{n=1}^{\infty} b^{n-1}\right) - \frac{1}{b(1-a)}\sum_{n=1}^{\infty}(ab)^n$$

$$= \frac{1}{(1-a)} \times \frac{1}{1-b} - \frac{1}{b(1-a)}\left(\frac{ab}{1-ab}\right)$$

$$= \frac{1}{(1-a)(1-b)} - \frac{a}{(1-a)(1-ab)}$$

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21.12 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

GEOMETRIC MEANS : *Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (G.M.'s) between a and b.*

GEOMETRIC MEAN : *If a single geometric mean G is inserted between two given numbers a and b, then G is known as the geometric mean between a and b.*

Thus,

G is the G.M. between a and b.

$\Leftrightarrow$ a, G, b are in G.P.

$\Leftrightarrow$ $G^2 = ab$

$\Leftrightarrow$ $G = \sqrt{ab}$.

The G.M. between 4 and 9 is given by $G = \sqrt{4 \times 9} = 6$, whereas the G.M. between –9 and –4 is given by $G = \sqrt{-9 \times -4} = -6$.

NOTE *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

INSERTION OF *n* GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS *a* AND *b*.

Let $G_1, G_2, \ldots, G_n$ be n geometric means between two given numbers a and b. Then,

$a, G_1, G_2, \ldots, G_n, b$ is a G.P. consisting of $(n+2)$ terms. Let r be the common ratio of this G.P. Then,

$b = (n+2)$th term $= ar^{n+1}$

$\Rightarrow r^{n+1} = \frac{b}{a}$

$\Rightarrow r = (b/a)^{1/n+1}$

$\therefore G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$,

$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$, ...,

$G_n = ar^n = a\left(\frac{b}{a}\right)^{(n+1)}$

AN IMPORTANT PROPERTY OF GEOMETRIC MEANS

THEOREM *If n geom-etric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.*

PROOF Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between two quantities a and b. Then,

$a, G_1, G_2, \ldots, G_n, b$ is a G.P

Let r be the common ratio of this G.P.

Then, $r = (b/a)^{1/n+1}$ and

$G_1 = ar, G_2 = ar^2, G_3 = ar^3, \ldots, G_n = ar^n$.

Now,

$G_1, G_2, G_3, \ldots, G_n$

$= (ar)(ar^2)(ar^3)\ldots(ar^n)$

$= a^n r^{1+2+3+\ldots+n}$

$= a^n r^{\frac{n(n+1)}{2}}$

$= a^n\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$

$= a^n\left(\frac{b}{a}\right)^{n/2} = a^{n/2}b^{n/2}$

$= (\sqrt{ab})^n = G^n$, where $G = \sqrt{ab}$ is the single geometric mean between a and b.

21.13 SOME IMPORTANT PROPERTIES OF ARITHMETIC AND GEOMETRIC MEANS BETWEEN TWO GIVEN QUANTITIES

PROPERTY I *If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then A > G.*

PROOF We have,

$A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$\therefore A - G = \frac{a+b}{2} - \sqrt{ab}$

$= \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{1}{2}[(\sqrt{a}-\sqrt{b})^2] > 0$

$\Rightarrow A > G$.

PROPERTY II *If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.*

PROOF We have,

$A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

The equation having a, b as its roots is

$x^2 - x(a+b) + ab = 0$

or, $x^2 - 2Ax + G^2 = 0$ $\left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$

PROP. III *If A and G be the A.M. and G.M. between two positive numbers, then the numbers are* $A \pm \sqrt{A^2 - G^2}$

PROOF The equation having its roots as the given numbers is

$$x^2 - 2Ax + G^2 = 0$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$

$$\Rightarrow \quad x = A \pm \sqrt{A^2 - G^2}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the value of n so that* $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ *may be the geometric mean between a and b.*

SOLUTION It is given that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the G.M. between a and b.

$$\therefore \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$$

$$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{(n+1/2)}\, b^{1/2} + a^{1/2}\, b^{(n+1/2)}$$

$$\Leftrightarrow \quad a^{n+1} - a^{(n+1/2)}\, b^{1/2} = a^{1/2}\, b^{(n+1/2)} - b^{n+1}$$

$$\Leftrightarrow \quad a^{(n+1/2)}\,(a^{1/2} - b^{1/2}) = b^{(n+1/2)}\,(a^{1/2} - b^{1/2})$$

$$\Leftrightarrow \quad a^{(n+1/2)} = b^{(n+1/2)} \qquad [\because\ a^{1/2} - b^{1/2} \neq 0, \text{ as } a \neq b]$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{(n+1/2)} = 1$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{(n+1/2)} = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow \quad \frac{n+1}{2} = 0 \Rightarrow n = -\frac{1}{2}$$

EXAMPLE 2 *If the A.M. and G.M. between two numbers are in the ratio m : n, then prove that the numbers are in the ratio*

$$m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}.$$

SOLUTION Let the two numbers be a and b. Let A and G be respectively the arithmetic and geometric means between a and b. Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

$$\therefore \quad a + b = 2A \text{ and } G^2 = ab \qquad \ldots(i)$$

The equation having a, b as its roots is

$$x^2 - (a+b)\,x + ab = 0$$

$$\text{or,} \quad x^2 - 2Ax + G^2 = 0 \qquad [\text{Using (i)}]$$

$$\therefore \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$

$$\Rightarrow \quad x = A \pm \sqrt{A^2 - G^2}$$

So, the two numbers are

$$a = A + \sqrt{A^2 - G^2} \text{ and, } b = A - \sqrt{A^2 - G^2}.$$

It is given that $A : G = m : n$

$$\Rightarrow \quad A = \lambda m \text{ and } G = \lambda n \text{ for some } \lambda.$$

Substituting the values of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$

$$\Rightarrow \quad a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$$

EXAMPLE 3 *Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then, prove that*

$$\frac{y^3 + z^3}{xyz} = 2$$

SOLUTION Let a and b be two positive numbers. Then,

$$x = \text{A.M. of } a \text{ and } b$$

$$\Rightarrow \quad x = \frac{a+b}{2} \qquad \ldots(i)$$

It is given that y and z are two geometric means between a and b. Then a, y, z, b is a G.P. with common ratio $\left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3}$

$$\therefore \quad y = ar \Rightarrow y = a\left(\frac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3}\, a^{2/3}$$

$$z = ar^2 \Rightarrow z = a\left(\frac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3}\, a^{1/3}.$$

$$\Rightarrow \quad y^3 + z^3 = (b^{1/3}\, a^{2/3})^3 + (b^{2/3}\, a^{1/3})^3 = ba^2 + b^2 a = ab\,(a+b)$$

$$\text{and,} \quad yz = (b^{1/3}\, a^{2/3})\,(b^{2/3}\, a^{1/3}) = ab.$$

$$\therefore \quad y^3 + z^3 = yz\,(a+b)$$

$$\Rightarrow \quad y^3 + z^3 = yz\,(2x) \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad \frac{y^3 + z^3}{xyz} = 2.$$

EXAMPLE 4 *If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that* $G_1^3 + G_2^3 = 2abc$

SOLUTION It is given that a is the A.M. of b and c, therefore

$$a = \frac{b+c}{2}$$

$$\Rightarrow \quad b + c = 2a \qquad \ldots(i)$$

Since G_1 and G_2 are two geometric means between b and c. Therefore, b, G_1, G_2, c is a G.P. with common ratio r given by

$$r = \left(\frac{c}{b}\right)^{1/3}$$

$$\therefore \quad G_1 = br = b\left(\frac{c}{b}\right)^{1/3} = c^{1/3} b^{2/3}$$

and,

$$G_2 = br^2 = b\left(\frac{c}{b}\right)^{2/3} = b^{1/3} c^{2/3}$$

$$\Rightarrow \quad G_1^3 = b^2 c \text{ and } G_2^3 = bc^2$$

$$\Rightarrow \quad G_1^3 + G_2^3 = b^2 c + bc^2$$

$$\Rightarrow \quad G_1^3 + G_2^3 = bc\,(b + c)$$

$$\Rightarrow \quad G_1^3 + G_2^3 = 2abc \qquad \text{[Using (i)]}$$

EXAMPLE 5 *If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that* $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

SOLUTION Let a and b be two given quantities. Then, G is the geometric mean of a and b

$$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab \qquad \ldots(i)$$

It is given that A_1, A_2 are two arithmetic means between a and b. Then a, A_1, A_2, b is an A.P. with common difference d given by

$$d = \frac{b-a}{3}.$$

$$\therefore \quad A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3},$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$$

$$\Rightarrow \quad 2A_1 - A_2 = 2\left(\frac{2a+b}{3}\right) - \left(\frac{a+2b}{3}\right) = a$$

and,

$$2A_2 - A_1 = 2\left(\frac{a+2b}{3}\right) - \left(\frac{2a+b}{3}\right) = b$$

$$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = ab$$

$$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2 \qquad \text{[Using (i)]}$$

EXAMPLE 6 *The A.M. between m and n and the G.M. between a and b are each equal to* $\frac{ma+nb}{m+n}$. *Find m and n in terms of a and b.*

SOLUTION We have,

$$\frac{m+n}{2} = \frac{ma+nb}{m+n} \text{ and } \sqrt{ab} = \frac{ma+nb}{m+n}$$

$$\Rightarrow \quad \frac{m+n}{2} = \sqrt{ab} = \frac{ma+nb}{m+n}$$

$$\Rightarrow \quad m+n = 2\sqrt{ab} \text{ and } ma+nb = (m+n)\sqrt{ab}$$

$$\Rightarrow \quad m+n = 2\sqrt{ab} \text{ and } ma+nb = 2ab$$

Solving these two equations, we get

$$m = \frac{2\sqrt{ab^2}}{\sqrt{a}+\sqrt{b}} \text{ and } n = \frac{2\sqrt{a^2 b}}{\sqrt{a}+\sqrt{b}}$$

EXERCISE 21.4

1. If S_1, S_2, S_3 are the sums of $n, 2n, 3n$ terms repectively of a G.P., then prove that :
   (i) $S_1, S_2 - S_1, S_3 - S_2$ are in G.P.
   (ii) $S_1^2 + S_2^2 = S_1(S_2 + S_3)$
2. Find the sum of $2n$ terms of a G.P. whose every even term is 'a' times the term before it and every odd term is c times the term before it, the first term being unity.
3. Prove that in an infinite G.P. whose common ratio r has its absolute value less than 1, the ratio of any term to the sum of all succeeding terms is $\frac{1-r}{r}$.
4. Find an infinite G.P. if it is given that the sum of its terms is 3 and the sum of the squares of its terms is also 3.
5. The sum of an infinite G.P. is 2 and the sum of the cubes of the terms of the G.P. is 24. Find the series.
6. If $S_1, S_2, S_3, \ldots S_n$ are the sums of the terms of n infinite geometric progressions whose first terms are 1, 2, 3, ..., n respectively and their common ratios are $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$ respectively, then find
   (i) $\sum_{r=1}^{n} s_r$, where $s_r = S_1 + S_2 + \ldots + S_r$
   (ii) $\sum_{1 \le i < j \le n} S_i S_j$
7. Find the sum to n terms of the series :
   $5 + 55 + 555 + \ldots.$
8. If $x = 1 + a + a^2 + \ldots. \infty$ and $y = 1 + b + b^2 + \ldots \infty$ show that
   $1 + ab + a^2 b^2 + a^3 b^3 + \ldots \infty = \frac{xy}{x+y-1}$, where $0 < a < 1$ and $0 < b < 1$.

9. If $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$, calculate the least value of n such that $2 - S_n < \frac{1}{100}$.

10. If AM of two numbers is twice their G.M., show that the numbers are in the ratio $2+\sqrt{3} : 2-\sqrt{3}$.

11. Prove that the each term of the sequence

121, 12321, 1234321, ... is a perfect square of an odd integer.

12. The sum of an infinite G.P. is 162 and the sum of its first n terms is 160. If the inverse of its common ratio is an integer, find all possible values of the common ratio, n and the first term of the series.

13. Find the sum :

$$\left(a+\frac{1}{a}\right)^3 + \left(a^2+\frac{1}{a^2}\right)^3 + \ldots + \left(a^n+\frac{1}{a^n}\right)^3$$

14. Consider an A.P. and a G.P. with their nth terms as a_n and b_n respectively such that $a_n > 0$ and $b_n > 1$. Prove that $a_n < b_n$ for $n > 2$, if the second term of A.P. exceeds the first term of the G.P. by the positive difference of the first two terms of the G.P. and the natural logarithm of the second term of A.P. exceeds that the first term of the G.P. by the positive difference of the natural logarithm of the first two terms of the A.P.

15. If $S_1, S_2, S_3, \ldots$ are the sums of infinite geometric series whose first terms are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$ and common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ respectively. Prove that

$$S_1 + S_2 + S_3 + \ldots \infty = 1 + \log_e 2.$$

16. Solve the following equations for x and y :

$$\log_{10} x + \frac{1}{2}\log_{10} x + \frac{1}{4}\log_{10} x + \ldots = y$$

and, $\frac{1+3+5\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)} = \frac{20}{7\log_{10} x}$

17. If a, b, c are in A.P. and p, p' be the AM and GM between a and b; and q, q' be the A.M. and G.M. between b and c, prove that

$$p^2 - p'^2 = q^2 - q'^2$$

18. Calculate : $(0.0625)\log_{4\sqrt{2}}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right)$

ANSWERS

2. $(1+a)\left(\frac{c^n a^n - 1}{ca-1}\right)$ 4. $\frac{3}{2^n}$ 5. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\ldots$ 7. $\frac{50}{81}(10^n-1)-\frac{5n}{9}$ 9. 8

6. (i) $\frac{n(n+1)(n+5)}{6}$

(ii) $\frac{1}{2}\left\{\frac{n^2(n+3)^2}{4} - \frac{(n+1)(n+2)(2n+3)}{6} + 1\right\}$

12. $r = 3, 9, 81, n = 4, 2, 1, a = 108, 144, 160$

13. $\frac{1}{1-a^3}\left[a^3 - a^{3(n+1)} - 1 + \frac{1}{a^{3n}}\right] + \frac{3}{1-a}\left[a - a^{n+1} - 1 + \frac{1}{a^n}\right]$

16. $x = 10^5, y = 10$ 18. 2^{16}

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21.13 ARITHMETICO GEOMETRIC SEQUENCE

A sequence of the form

$$a, (a+d)r, (a+2d)r^2, \ldots, (a+(n-1)d)r^{n-1}, \ldots.$$

is called an arithmetico geometric sequence.

Clearly, nth term of the above arithmetico-geometric series is equal to the product of the nth terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$ and the geometric progression $1, r, r^2, \ldots, r^{n-1}, \ldots$.

ILLUSTRATION 1. *The sequence $1, 3x, 5x^2, 7x^3, \ldots$ is an arithmetico-geometric sequence whose corresponding A.P. and G.P. are $1, 3, 5, 7, \ldots$ and $1, x, x^2, x^3, \ldots$ respectively. The nth term of the corresponding A.P. is $1+(n-1)\times 2 = 2n-1$ and the nth term of the corresponding G.P. by taking first term as unity is x^{n-1}. So, the nth term of the given sequence is $(2n-1)x^{n-1}$*

ILLUSTRATION 2. *The sequence $2, -5x, 8x^2, -11x^3, \ldots$ is an arithmetico-geometric sequence whose corresponding A.P. and G.P. are $2, 5, 8, 11, \ldots$ and $1, -x, x^2, -x^3, \ldots$. The nth term of these two sequences are $2+(n-1)\times 3 = 3n-1$ and $(-x)^{n-1} = (-1)^{n-1}x^{n-1}$ respectively. So, the nth term of the given arithmetico-geometric sequence is $(3n-1)(-1)^{n-1}x^{n-1}$.*

ARITHMETICO-GEOMETRIC SERIES *Let $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots$ be an arithmetico-geometric sequence. Then, $a+(a+d)r+(a+2d)r^2+(a+3d)r^3+\ldots$ is an arithmetico-geometric series.*

ILLUSTRATION 3 *Find the nth term of the series*

$$1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots.$$

SOLUTION The given seriesis an arithmetico-geometric series whose corresponding A.P. and G.P. are $1, 2, 3, 4, \ldots$ and $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \cdots$ respectively.

The *nth* term of A.P. $= 1+(n-1)\times 1 = n$;

The *nth* term of G.P. $= 1\times\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3^{n-1}}$

So, the *nth* term of the given series $= n \cdot \left(\frac{1}{3^{n-1}}\right) = \frac{n}{3^{n-1}}$

21.14 SUM OF *n* TERMS OF AN ARITHMETICO-GEOMETRIC SEQUENCE

THEOREM *The sum of n terms of an arithmetico-geometric sequence* $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots$ *is given by*

$$S_n = \begin{cases} \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{1-r} - \frac{\{a+(n-1)d\}r^n}{1-r}, & \text{when } r \neq 1 \\ \frac{n}{2}[2a+(n-1)d], & \text{when } r = 1. \end{cases}$$

PROOF Let S_n denote the sum of n terms of the given sequence. Then,

$$S_n = a + (a+d)r + (a+2d)r^2 + \ldots + \{a+(n-1)d\}r^{n-1} \quad \ldots(i)$$

$$\Rightarrow \quad rS_n = ar + (a+d)r^2 + \ldots + \{a+(n-2)d\}r^{n-1} + \{a+(n-1)d\}r^n \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$S_n - rS_n = a + [dr + dr^2 + \ldots + dr^{n-1}] - \{a+(n-1)d\}r^n$$

$$\Rightarrow \quad S_n(1-r) = a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - \{a+(n-1)d\}r^n$$

$$\Rightarrow \quad S_n = \frac{a}{1-r} + dr\frac{1-r^{n-1}}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad \ldots(i)$$

SUM OF AN INFINITE ARITHMETICO-GEOMETRIC SEQUENCE

Let $|r| < 1$. Then $r^n, r^{n-1} \to 0$ as $n \to \infty$ and it can also be shown that $n \cdot r^n \to 0$ as $n \to \infty$. So, from (i), we obtain that

$$S_n \to \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \text{ as } n \to \infty.$$

In other words, when $|r| < 1$ the sum to infinity of an arithmetico-geometric series is

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

In order to find the sum of an orithmetic-geometric the following algorithm.

ALGORITHM

STEP I *Obtain the corresponding A.P. and G.P.*

STEP II *Write the nth terms of corresponding A.P. and G.P.*

STEP III *Multiply the nth terms obtained in step II to get the nth term of the given arithmetico-geometric sequence. Replace n by (n – 1) in it to get the (n – 1)th term.*

STEP IV *Put S_n equal to the sum of first n terms of the given sequence.*

STEP V *Multiply both sides of S_n in Step IV by the common ratio of the corresponding G.P.*

STEP VI *Subtract the expression obtained in step V from the expression in step IV and simplify to get S_n.*

The sum of an infinite arithmetics-geometric series can be computed by using the following algorithm.

ALGORITHM

STEP I *Obtain the common ratio of the corresponding G.P.*

STEP II *Equate the given series to S_∞.*

STEP III *Multiply both sides of the equality in step II by the common ratio of the corresponding G.P.*

STEP IV *Subtract the expression in step III from the expression in step II.*

STEP V *On RHS of the expression in step IV write the sum of an infinite G.P. and obtain S_∞.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the sum to n terms of the series*

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots\ldots\ldots$$

SOLUTION Clearly, the given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are respectively $1, 4, 7, 10, \ldots$ and $1, 1/5, 1/5^2, 1/5^3, \ldots$.

The *nth* term of A.P. is $[1+(n-1)\times 3] = 3n-2$

The *nth* term of G.P. is $\left[1 \times \left(\frac{1}{5}\right)^{n-1}\right] = \left(\frac{1}{5}\right)^{n-1}$.

So, the *nth* term of the given series is

$$(3n-2) \times \frac{1}{5^{n-1}} = \frac{3n-2}{5^{n-1}}.$$

Let $S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots + \frac{3n-5}{5^{n-2}} + \frac{3n-2}{5^{n-1}}$ ...(i)

Multiplying both sides by 1/5, we get

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots + \frac{(3n-5)}{5^{n-1}} + \frac{3n-2}{5^n} \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$S_n - \frac{1}{5}S_n = 1 + \left[\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots + \frac{3}{5^{n-1}}\right] - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad \frac{4}{5}S_n = 1 + \frac{3}{5}\frac{\left\{1 - \left(\frac{1}{5}\right)^{n-1}\right\}}{\left(1 - \frac{1}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \frac{4}{5}S_n = 1 + \frac{3}{5}\frac{\left\{1 - \frac{1}{5^{n-1}}\right\}}{\left(\frac{4}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \frac{4}{5}S_n = 1 + \frac{3}{4}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{5^n}$$

$$\Rightarrow S_n = \frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{4.5^{n-1}}.$$

EXAMPLE 2 *Find the sum to infinity of the series :*

$$1 - 3x + 5x^2 - 7x^3 + \ldots, \infty \text{ when } |x| < 1.$$

SOLUTION Clearly, the given series is an arithmetico-geometric series whose corresponding G.P. is $1, -x, x^2, -x^3, \ldots$. The common ratio of this G.P. is $-x$. Let S_∞ denote the sum of the given infinite series. Then,

$$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \ldots \infty \quad \ldots(i)$$

$$\Rightarrow (-x)S_\infty = -x + 3x^2 - 5x^3 + \ldots \infty \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$(1+x)S_\infty = 1 + [-2x + 2x^2 - 2x^3 + \ldots \infty]$$

$$= 1 + \left\{\frac{-2x}{1-(-x)}\right\}$$

$$= 1 - \frac{2x}{1+x}$$

$$= \frac{1-x}{1+x}$$

$$\therefore \quad S_\infty = \frac{1-x}{(1+x)^2}$$

EXAMPLE 3 *Find the sum to infinity of the series*

$$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots \infty$$

SOLUTION The given series is not an arithmetico-geometric series, because $1^2, 2^2, 3^2, 4^2, \ldots$ are not in A.P. However, their successive differences $(2^2 - 1^2), (3^2 - 2^2), (4^2 - 3^2), \ldots$ i.e. $3, 5, 7, \ldots$ form an A.P. So, the process of finding the sum to infinity of an arithmetico-geometric series will be repeated twice as given below :

Let

$$S_\infty = 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots \infty \quad \ldots(i)$$

$$\Rightarrow x S_\infty = 1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots \infty \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$(1-x)S_\infty = 1^2 + (2^2 - 1^2)x + (3^2 - 2^2)x^2 + (4^2 - 3^2)x^3 + \ldots$$

$$\Rightarrow (1-x)S_\infty = 1 + 3x + 5x^2 + 7x^3 + \ldots \quad \ldots(iii)$$

This is an arithmetico-geometric series in which the common ratio of the corresponding G.P. is x.

Multiplying (iii) by x, we get

$$x(1-x)S_\infty = x + 3x^2 + 5x^3 + \ldots \quad \ldots(iv)$$

Subtracting (iv) from (iii), we get

$$(1-x)S_\infty - x(1-x)S_\infty = 1 + 2x + 2x^2 + 2x^3 + \ldots$$

$$\Rightarrow (1-x)^2 S_\infty = 1 + \frac{2x}{1-x}.$$

$$\Rightarrow (1-x)^2 S_\infty = \frac{1+x}{(1-x)}$$

$$\Rightarrow \quad S_\infty = \frac{1+x}{(1-x)^3}$$

EXAMPLE 4 *Sum to infinity the following series*

$$1^2 + 5^2 x + 9^2 x^2 + 13^2 x^3 + \ldots \infty, \text{ where } |x| < 1.$$

SOLUTION The given series is not an arithmetico-geometric series, because $1^2, 5^2, 9^2, 13^2, \ldots$ is not a G.P. However, their successive differences $(5^2 - 1^2), (9^2 - 5^2), (13^2 - 9^2), \ldots\ldots$, *i.e.* $24, 56, 88, \ldots$ form an A.P. So, the process of obtaining the sum of an infinite arithmetico-geometric series will be repeated twice as given below :

Let

$$S_\infty = 1^2 + 5^2 x + 9^2 x^2 + 13^2 x^3 + \ldots \infty \quad \ldots(i)$$

Multiplying (i) by x, we get

$$x S_\infty = 1^2 x + 5^2 x^2 + 9^2 x^3 + \ldots \infty \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$(1-x)S_\infty = 1^2 + (5^2 - 1^2)x + (9^2 - 5^2)x^2 + (13^2 - 9^2)x^3 + \ldots \infty$$

$$(1-x)S_\infty = 1 + 24x + 56x^2 + 88x^3 + \ldots \infty \quad \ldots(iii)$$

This is an arithmetico-geometric series in which the common ratio of the corresponding G.P. is x. Multiplying (iii) by x, we get

$$x(1-x)S_\infty = x + 24x^2 + 56x^3 + \ldots \infty \quad \ldots(iv)$$

Subtracting (iv) from (iii), we get

$$(1-x)S_\infty - x(1-x)S_\infty = 1 + 23x + 32x^2 + 32x^3 + \ldots \infty$$

$$\Rightarrow (1-x)^2 S_\infty = 1 + 23x + \frac{32x^2}{1-x}$$

$$\Rightarrow S_\infty = \frac{1}{(1-x)^2} + \frac{23x}{(1-x)^2} + \frac{32x^2}{(1-x)^3}$$

$$\Rightarrow S_\infty = \frac{1 + 22x + 9x^2}{(1-x)^3}$$

EXAMPLE 5 *Let* α and β *be two distinct positive roots of* $x^2-2ax+ab=0$. *If* $S_n=1+2\left(\frac{b}{a}\right)+3\left(\frac{b}{a}\right)^2+\ldots+n\left(\frac{b}{a}\right)^{n-1}$, *show that* $S_n \not> \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$.

SOLUTION Since α, β are the roots of the equation $x^2-2ax+ab=0$

Therefore,

$\alpha+\beta = 2a$ and $\alpha\beta = ab$.

$$\therefore \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4 = \frac{(\alpha+\beta)^4}{[(\alpha+\beta)^2-4\alpha\beta]^2}$$

$$= \frac{16a^4}{(4a^2-4ab)^2}$$

$$= \frac{a^2}{(a-b)^2}$$

Since the roots are positive. Therefore, a, b are positive and Discriminant > 0

$\Rightarrow 4a^2-4ab>0$

$\Rightarrow a(a-b)>0$

$\Rightarrow a-b>0$

$\Rightarrow a>b$

$\Rightarrow \frac{b}{a}<1$

Now,

$$S_n = 1+2\left(\frac{b}{a}\right)+3\left(\frac{b}{a}\right)^2+\ldots+(n-1)\left(\frac{b}{a}\right)^{n-2}+n\left(\frac{b}{a}\right)^{n-1}$$

$$\frac{b}{a}S_n = \left(\frac{b}{a}\right)+2\left(\frac{b}{a}\right)^2+\ldots\ldots\ldots\ldots+(n-1)\left(\frac{b}{a}\right)^{n-1}+n\left(\frac{b}{a}\right)^n$$

On subtracting, we get

$$S_n-\frac{b}{a}S_n = 1+\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^2+\ldots+\left(\frac{b}{a}\right)^{n-1}-n\left(\frac{b}{a}\right)^n$$

$$\Rightarrow S_n\left(1-\frac{b}{a}\right) = \frac{1-\left(\frac{b}{a}\right)^n}{1-\frac{b}{a}}-n\left(\frac{b}{a}\right)^n$$

$$\Rightarrow S_n = \frac{a^n-b^n}{a^{n-2}(a-b)^2}-\frac{nb^n}{a^{n-1}(a-b)}$$

$$\therefore S_n-\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4 = \frac{a^n-b^n}{a^{n-2}(a-b)^2}-\frac{nb^n}{a^{n-1}(a-b)}-\frac{a^2}{(a-b)^2}$$

$$= \frac{a^n-b^n}{a^{n-2}(a-b)^2}-\frac{nb^n}{a^{n-1}(a-b)}-\frac{a^n}{a^{n-2}(a-b)^2}$$

$$= -\frac{b^n}{a^{n-2}(a-b)^2}-\frac{nb^n}{a^{n-1}(a-b)}<0 \qquad [\because a>b>0]$$

$$\therefore S_n < \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$$

Hence, $S_n \not> \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^4$

EXAMPLE 6 *Show that the sum of the series*

$1+3\left(\frac{2n+1}{2n-1}\right)+5\left(\frac{2n+1}{2n-1}\right)^2+\ldots$ *to n terms is an even or an odd number according as n is even or odd.*

SOLUTION Let

$$S = 1+3\left(\frac{2n+1}{2n-1}\right)+5\left(\frac{2n+1}{2n-1}\right)^2+7\left(\frac{2n+1}{2n-1}\right)^3+\ldots$$

$$+(2n-3)\left(\frac{2n+1}{2n-1}\right)^{n-2}+(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n-1}$$

$$\Rightarrow \frac{2n+1}{2n-1}S = \left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\ldots$$

$$+(2n-3)\left(\frac{2n+1}{2n-1}\right)^{n-1}+(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$

On subtracting, we get

$$\left(S-\frac{2n+1}{2n-1}S\right)$$

$$= 1+2\left(\frac{2n+1}{2n-1}\right)+2\left(\frac{2n+1}{2n-1}\right)^2+2\left(\frac{2n+1}{2n-1}\right)^2+\ldots$$

$$+2\left(\frac{2n+1}{2n-1}\right)^{n-1}-(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$

$$\Rightarrow S\left(\frac{-2}{2n-1}\right)$$

$$= 1+2\left(\frac{2n+1}{2n-1}\right)\left\{\frac{1-\left(\frac{2n+1}{2n-1}\right)^{n-1}}{1-\frac{2n+1}{2n-1}}\right\}-(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$

$$\Rightarrow S\left(\frac{-2}{2n-1}\right) = -2n$$

$\Rightarrow S = n(2n-1)$, which is odd or even according as n is odd or even.

EXAMPLE 7 *Let* $x = 1+3a+6a^2+10a^3+\ldots, \ |a|<1$

$$y = 1+4b+10b^2+20b^3+\ldots, \ |b|<1$$

Find $S = 1+3(ab)+5(ab)^2+\ldots$ *in terms of x and y.*

SOLUTION We have,

$x = 1 + 3a + 6a^2 + 10a^3 + ... \infty$

$\Rightarrow ax = a + 3a^2 + 6a^3 + ... \infty$

On subtracting, we get

$x - ax = 1 + 2a + 3a^2 + 4a^3 + ... \infty$

$\Rightarrow x(1-a) = \frac{1}{1-a} + \frac{a}{(1-a)^2}$ [Using the formule forthe sum of AGP]

$\Rightarrow x(1-a) = \frac{1}{(1-a)^2}$

$\Rightarrow x = \frac{1}{(1-a)^3}$

$\Rightarrow (1-a)^3 = x$

$\Rightarrow 1 - a = x^{-1/3}$

$\Rightarrow a = 1 - x^{-1/3}$ ...(i)

$y = 1 + 4b + 10b^2 + 20b^3 + ... \infty$

$\Rightarrow by = b + 4b^2 + 10b^3 + ... \infty$

On subtracting, we get

$y - by = 1 + 3b + 6b^2 + 10b^3 + ... \infty$

$\Rightarrow b(y - by) = b + 3b^2 + 6b^3 + ... \infty$ [Multiplying by b]

On subtracting, we get

$(y - by) - b(y - by) = 1 + 2b + 3b^2 + 4b^3 + ... \infty$

$\Rightarrow y(1-b)^2 = \frac{1}{1-b} + \frac{b}{(1-b)^2}$

$\Rightarrow y(1-b)^2 = \frac{1}{(1-b)^2}$

$\Rightarrow y = (1-b)^{-4}$

$\Rightarrow b = 1 - y^{-1/4}$ ...(ii)

Now,

$S = 1 + 3(ab) + 5(ab)^2 + ... \infty$

$\Rightarrow S(ab) = (ab) + 3(ab)^2 + ... \infty$

$\therefore S - S(ab) = 1 + 2(ab) + 2(ab)^2 + ... \infty$

$\Rightarrow S(1-ab) = 1 + \frac{2ab}{1-ab}$

$\Rightarrow S = \frac{1+ab}{(1-ab)^2}$

$\Rightarrow S = \frac{1 + (1 - x^{-1/3})(1 - y^{-1/4})}{[1 - (1 - x^{-1/3})(1 - y^{-1/4})]^2}$

$\Rightarrow S = \frac{2 - x^{-1/3} - y^{-1/4} + x^{-1/3} y^{-1/4}}{(x^{-1/3} + y^{-1/4} - x^{-1/3} y^{-1/4})^2}$

EXAMPLE 8 *If* $x = 1 + 4a + 10a^2 + 20a^3 + ...$ *to* ∞, $|a| < 1$

$y = 1 + 5b + 15b^2 + 35b^3 + ...$ *to* ∞, $|b| < 1$

Find $S = 3 + 5ab + 7(ab)^2 + ...$ *to* ∞.

SOLUTION We have,

$x = 1 + 4a + 10a^2 + 20a^3 + ...$

$\Rightarrow ax = a + 4a^2 + 10a^3 + ...$

On subtracting, we get

$x - ax = 1 + 3a + 6a^2 + 10a^3 + ...$

$\Rightarrow x(1-a)a = a + 3a^2 + 6a^3 + ...$

On subtracting, we get

$x(1-a) - x(1-a)a = 1 + 2a + 3a^2 + 4a^3 + ... \infty$

$\Rightarrow x(1-a)^2 = \frac{1}{1-a} + \frac{a}{(1-a)^2}$

$\Rightarrow x = \frac{1}{(1-a)^4}$

$\Rightarrow a = 1 - x^{-1/4}$.

Similarly, we have

$b = 1 - y^{-1/5}$.

Now,

$S = 3 + 5(ab) + 7(ab)^2 + ... \infty$

$\Rightarrow abS = 3ab + 5(ab)^2 + ... \infty$

$\therefore S - abS = 3 + 2ab + 2(ab)^2 + 2(ab)^3 + ... \infty$

$\Rightarrow S(1-ab) = 3 + \frac{2ab}{1-ab}$

$\Rightarrow S = \frac{3-ab}{(1-ab)^2}$

$\Rightarrow S = \frac{3 - (1 - x^{-1/4})(1 - y^{-1/5})}{[1 - (1 - x^{-1/4})(1 - y^{-1/5})]^2}$

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21.15 HARMONIC PROGRESSION

DEFINITION *A sequence* $a_1, a_2, ..., a_n, ...$ *of non-zero numbers is called a Harmonic progression, if the sequence* $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}, ...$ *is an Arithmetic progression.*

The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...$ is a HP, because the sequence, 1, 3, 5, 7, is an A.P.

nth TERM OF A HP

The nth term of a H.P. is the reciprocal of the nth term of the corresponding A.P. Thus, if $a_1, a_2, a_3, ..., a_n, ...$ is a HP and the common difference of the corresponding AP is d i.e.

$d = \frac{1}{a_{n+1}} - \frac{1}{a_n}$, then

$$a_n = \frac{1}{\frac{1}{a_1} + (n-1)\,d}$$

If a, b, c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP. Therefore,

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $a_1, a_2, a_3, \ldots, a_n$ are in HP, prove that*

$$\frac{a_1}{a_2 + a_3 + \ldots + a_n}, \frac{a_2}{a_1 + a_3 + \ldots + a_n}, \ldots, \frac{a_n}{a_1 + a_2 + \ldots + a_{n-1}}$$

are in HP.

SOLUTION It is given that

$a_1, a_2, a_3, \ldots, a_n$ are in HP

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

$$\Rightarrow \frac{a_1 + a_2 + \ldots + a_n}{a_1}, \frac{a_1 + a_2 + \ldots + a_n}{a_2}, \frac{a_1 + a_2 + \ldots + a_n}{a_3} \ldots$$

$$\ldots\ldots \frac{a_1 + a_2 + \ldots + a_n}{a_n} \text{ are in AP}$$

$$\Rightarrow 1 + \frac{a_2 + a_3 + \ldots + a_n}{a_1}, 1 + \frac{a_1 + a_3 + \ldots + a_n}{a_2},$$

$$1 + \frac{a_1 + a_2 + a_4 + \ldots + a_n}{a_3}, \ldots, 1 + \frac{a_1 + a_2 + \ldots + a_{n-1}}{a_n} \text{ are in AP.}$$

$$\Rightarrow \frac{a_2 + a_3 + \ldots + a_n}{a_1}, \frac{a_1 + a_3 + \ldots + a_n}{a_2}, \frac{a_1 + a_2 + a_4 + \ldots + a_n}{a_3},$$

$$\ldots\ldots\ldots\ldots, \frac{a_1 + a_2 + \ldots + a_{n-1}}{a_n} \text{ are in AP}$$

$$\Rightarrow \frac{a_1}{a_2 + a_3 + \ldots + a_n}, \frac{a_2}{a_1 + a_3 + \ldots + a_n}, \frac{a_3}{a_1 + a_2 + \ldots a_n}, \ldots\ldots,$$

$$\frac{a_n}{a_1 + a_2 + \ldots + a_{n-1}} \text{ are in HP.}$$

EXAMPLE 2 *If $a_1, a_2, a_3, \ldots, a_n$ are in HP, prove that*

$$a_1 a_2 + a_2 a_3 + \ldots + a_{n-1}\, a_n = (n-1)\, a_1 a_n$$

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n$ are in HP. Therefore,

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

Let d be the common difference of the A.P. Then,

$$\frac{1}{a_2} - \frac{1}{a_1} = d, \frac{1}{a_3} - \frac{1}{a_2} = d, \ldots, \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$$

$\Rightarrow a_1 - a_2 = d\,(a_1 a_2)$, $a_2 - a_3 = d\,(a_2 a_3)$,

$\ldots a_{n-1} - a_n = d\,(a_{n-1} a_n)$

$\Rightarrow (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-1} - a_n)$

$= d\,(a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n)$

$\Rightarrow a_1 - a_n = d\,(a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n)$ ...(i)

Now,

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP with common difference d.

$\therefore \frac{1}{a_n} = \frac{1}{a_1} + (n-1)\,d$

$\Rightarrow \frac{1}{a_n} - \frac{1}{a_1} = (n-1)\,d$

$\Rightarrow \frac{a_1 - a_n}{a_1 a_n} = (n-1)\,d$

$\Rightarrow a_1 - a_n = (n-1)\,d\,(a_1 a_n)$ ...(ii)

From (i) and (ii), we get

$(n-1)\,d\,(a_1 a_n) = d\,(a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n)$

$\Rightarrow (n-1)\, a_1 a_n = a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n$.

EXAMPLE 3 *An A.P. and a H.P. each having n number of terms also have the same first and last terms, viz. a and b. Prove that the product of $(r+1)^{th}$ term of one sequence and the $(n-r)$ th terms of the other sequence is independent of r.*

SOLUTION Let the A.P. be $a, a_1, a_2, \ldots, a_{n-2}, b$ and the H.P. be $a, h_1, h_2, \ldots, h_{n-2}, b$. Let d be the common difference of the A.P. Then,

$$b = a + (n-1)\,d \Rightarrow d = \frac{b-a}{n-1}$$

Now,

$(r+1)$ *th* term of the A.P.

$= a_r = a + rd$

$$= a + r\,\frac{(b-a)}{n-1}$$

$$= \frac{(n-1)\,a + (b-a)\,r}{n-1}$$

$$= \frac{(n-r-1)\,a + rb}{n-1}$$

Since $a, h_1, h_2, \ldots, h_{n-2}, b$ are in H.P. Therefore,

$\frac{1}{a}, \frac{1}{h_1}, \frac{1}{h_2}, ..., \frac{1}{h_{n-2}}, \frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$$\frac{1}{b} = \frac{1}{a} + (n-1)\,D$$

$$\Rightarrow D = \frac{a-b}{(n-1)\,ab}$$

Now, $(n-r)$ *th* term of the H.P. is h_{n-r-1}.

$$\therefore \frac{1}{h_{n-r-1}} = \frac{1}{a} + (n-r-1)\,D$$

$$\Rightarrow \frac{1}{h_{n-r-1}} = \frac{1}{a} + (n-r-1)\frac{(a-b)}{(n-1)\,ab}$$

$$\Rightarrow \frac{1}{h_{n-r-1}} = \frac{b\,(n-1) + (n-r-1)\,(a-b)}{(n-1)\,ab}$$

$$\Rightarrow \frac{1}{h_{n-r-1}} = \frac{(n-r-1)\,a + rb}{ab\,(n-1)}$$

$$\Rightarrow h_{n-r-1} = \frac{ab\,(n-1)}{(n-r-1)\,a + rb}$$

$$\therefore a_r\, h_{n-r-1} = \frac{(n-r-1)\,a + rb}{n-1} \times \frac{ab\,(n-1)}{(n-r-1)\,a + rb} = ab$$

which is independent of r.

Similarly, we can show that $a_{n-r-1}\, h_r$ is independent of r.

EXAMPLE 4 *Three sequences, in A.P., G.P. and H.P. respectively, have a and b (with $a \neq b$) as their first two terms. Show that if their $(n+2)$th terms are in G.P., then*

$$\frac{b^{2n+2} - a^{2n+2}}{ba\,(b^{2n} - a^{2n})} = \frac{n+1}{n}$$

SOLUTION The common difference of the sequence in A.P. is $d = b - a$. Let A be the $(n+2)$ *th* term of the A.P. Then,

$$A = a + (n+2-1)\,d$$

$$\Rightarrow A = a + (n+1)\,(b-a)$$

$$\Rightarrow A = (n+1)\,b - na \qquad ...(i)$$

The common ratio of the G.P. is $r = \frac{b}{a}$. Let G be the $(n+2)$ *th* term of the G.P. Then,

$$G = ar^{n+1}$$

$$\Rightarrow G = a\left(\frac{b}{a}\right)^{n+1} \qquad ...(ii)$$

The first two terms of the H.P. are a and b. Therefore, the two terms of the corresponding A.P. are $\frac{1}{a}$ and $\frac{1}{b}$. Let D be the common difference of the corresponding A.P. Then,

$$D = \frac{1}{b} - \frac{1}{a}$$

Let H be the $(n+2)$th terms of the H.P. Then,

$$\frac{1}{H} = \frac{1}{a} + (n+1)\,D$$

$$\Rightarrow \frac{1}{H} = \frac{1}{a} + (n+1)\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\Rightarrow \frac{1}{H} = \frac{n+1}{b} - \frac{n}{a}$$

$$\Rightarrow H = \frac{ab}{(n+1)\,a - nb} \qquad ...(iii)$$

It is given that A, G and H are in G.P. Therefore,

$$G^2 = AH$$

$$\Rightarrow a^2\left(\frac{b}{a}\right)^{2n+2} = \{(n+1)\,b - na\}\left(\frac{ab}{(n+1)\,a - nb}\right)$$

$$\Rightarrow \left(\frac{b}{a}\right)^{2n+1} = \frac{(n+1)\,b - na}{(n+1)\,a - nb}$$

$$\Rightarrow b^{2n+1}\,\{(n+1)\,a - nb\} = \{(n+1)\,b - na\}\,a^{2n+1}$$

$$\Rightarrow ab\,(n+1)\,(a^{2n} - b^{2n}) = n\,(a^{2n+2} - b^{2n+2})$$

$$\Rightarrow \frac{a^{2n+2} - b^{2n+2}}{ab\,(a^{2n} - b^{2n})} = \frac{n+1}{n}$$

$$\Rightarrow \frac{b^{2n+2} - a^{2n+2}}{ba\,(b^{2n} - a^{2n})} = \frac{n+1}{n}$$

EXAMPLE 5 *If S_1, S_2 and S_3 denote the sums up to n (>1) terms of three non-constant sequences in A.P., whose first terms are unity and common differences are in H.P., prove that*

$$n = \frac{2S_1\,S_3 - S_1\,S_2 - S_2\,S_3}{S_1 - 2S_2 + S_3}$$

SOLUTION Let d_1, d_2, d_3 be the common differences of the three A.P.'s. Then,

$$S_i = \frac{n}{2}\,[2 \times 1 + (n-1)\,d_i]\,;\, i = 1, 2, 3.$$

$$\Rightarrow S_i = n + \frac{n\,(n-1)}{2}\,d_i$$

$$\Rightarrow 2\,(S_i - n) = n\,(n-1)\,d_i$$

$$\Rightarrow d_i = \frac{2\,(S_i - n)}{n\,(n-1)}\,;\, i = 1, 2, 3.$$

$$\Rightarrow \frac{1}{d_i} = \frac{n\,(n-1)}{2\,(S_i - n)}\,;\, i = 1, 2, 3.$$

It is given that d_1, d_2, d_3 are in H.P. Therefore,

$\frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_3}$ are in A.P.

$$\Rightarrow \frac{1}{d_2} - \frac{1}{d_1} = \frac{1}{d_3} - \frac{1}{d_2}$$

$$\Rightarrow \frac{n(n-1)}{2(S_2-n)}-\frac{n(n-1)}{2(S_1-n)}=\frac{n(n-1)}{2(S_3-n)}-\frac{n(n-1)}{2(S_2-n)}$$

$$\Rightarrow \frac{1}{S_2-n}-\frac{1}{S_1-n}=\frac{1}{S_3-n}-\frac{1}{S_2-n}$$

$$\Rightarrow \frac{S_1-S_2}{S_1-n}=\frac{S_2-S_3}{S_3-n}$$

$$\Rightarrow (S_1-S_2)(S_3-n)=(S_2-S_3)(S_1-n)$$

$$\Rightarrow S_3(S_1-S_2)-n(S_1-S_2)=S_1(S_2-S_3)-n(S_2-S_3)$$

$$\Rightarrow S_3(S_1-S_2)-S_1(S_2-S_3)=n(S_1-S_2-S_2+S_3)$$

$$\Rightarrow n(S_1-2S_2+S_3)=2S_3S_1-S_2S_3-S_1S_2$$

$$\Rightarrow n=\frac{2S_1S_3-S_2S_3-S_1S_2}{S_1-2S_2+S_3}$$

EXAMPLE 6 *Find the largest positive term of the HP whose first two terms are* $\frac{2}{5}$ and $\frac{12}{23}$.

SOLUTION First two terms of the corresponding AP are $\frac{5}{2}$ and $\frac{23}{12}$.

Let d be the common difference of the corresponding A.P. Then,

$$d=\frac{23}{12}-\frac{5}{2}=\frac{-7}{12}.$$

Let a_n be the nth term of the given H.P. Then,

$$a_n=\frac{1}{\frac{5}{2}+(n-1)\left(\frac{-7}{12}\right)}=\frac{12}{30-7n+7}=\frac{12}{37-7n}$$

Clearly, a_n will be greatest, if $37-7n$ is least.

$37-7n$ is least for $n=5$.

Hence, $a_5=\frac{12}{37-35}=6$ is the largest positive term.

EXAMPLE 7 *If the* $(m+1)$*th,* $(n+1)$*th and* $(r+1)$*th terms of an A.P. are in G.P. and m, n, r are in H.P., show that the ratio of the common difference to the first term of the A.P. is* $\frac{-2}{n}$.

SOLUTION Let the first term and the common difference of the A.P. be a and d respectively. Then,

$$a_{m+1}=a+md,\ a_{n+1}=a+nd \text{ and } a_{r+1}=a+rd.$$

It is given that $a_{m+1}, a_{n+1}, a_{r+1}$ are in G.P. Therefore,

$$(a_{n+1})^2=a_{m+1}\cdot a_{r+1}$$

$$\Rightarrow (a+nd)^2=(a+md)(a+rd)$$

$$\Rightarrow 2and+n^2d^2=ad(m+r)+mrd^2$$

$$\Rightarrow ad(2n-m-r)=(mr-n^2)d^2$$

$$\Rightarrow \frac{d}{a}=\frac{2n-m-r}{mr-n^2}$$

$$\Rightarrow \frac{d}{a}=\frac{2n-\frac{2mr}{n}}{mr-n^2} \qquad \left[\because m, n, r \text{ are in H.P. } \therefore n=\frac{2mr}{m+r} \Rightarrow m+r=\frac{2mr}{n}\right]$$

$$\Rightarrow \frac{d}{a}=\frac{2}{n}\left(\frac{n^2-mr}{mr-n^2}\right)=\frac{-2}{n}$$

EXAMPLE 8 *If a, x, y, z, b are in A.P when* $x+y+z=15$ *and a,* α, β, γ, b *are in HP when* $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{5}{3}$, *find a, b.*

SOLUTION Since a, x, y, z, b are in A.P. Therefore, x, y, z are also in AP. Hence,

$$2y=x+z$$

$$\Rightarrow 3y=x+y+z$$

$$\Rightarrow 3y=15 \qquad [\because x+y+z=15 \text{ (given)}]$$

$$\Rightarrow y=5$$

$$\Rightarrow x+z=10 \qquad [\because 2y=x+z]$$

Again,

a, x, y, z, b are in A.P. Therefore,

$$a+b=x+z$$

$$\Rightarrow a+b=10 \qquad \text{...(i)}$$

Since $a, \alpha, \beta, \gamma, b$ are in H.P. Therefore,

$\frac{1}{a}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{b}$ are in AP.

$$\Rightarrow \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma} \text{ are in AP}$$

$$\Rightarrow \frac{2}{\beta}=\frac{1}{\alpha}+\frac{1}{\gamma}$$

$$\Rightarrow \frac{3}{\beta}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \qquad [\text{Adding } \tfrac{1}{\beta} \text{ on both sides}]$$

$$\Rightarrow \frac{3}{\beta}=\frac{5}{3}$$

$$\Rightarrow \beta=\frac{9}{5}$$

$$\Rightarrow \frac{1}{\alpha}+\frac{1}{\gamma}=\frac{10}{9} \qquad \left[\because \frac{2}{\beta}=\frac{1}{\alpha}+\frac{1}{\gamma}\right]$$

Again, $\frac{1}{a}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{b}$ are in A.P. Therefore,

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{\alpha}+\frac{1}{\gamma}$$

$$\Rightarrow \frac{a+b}{ab}=\frac{10}{9}$$

$$\Rightarrow \frac{10}{ab}=\frac{10}{9} \qquad [\text{Using (i)}]$$

$$\Rightarrow ab=9 \qquad \text{...(ii)}$$

The quadratic equation having a, b as its roots is

$$x^2 - x(a+b) + ab = 0$$

or, $x^2 - 10x + 9 = 0$

or, $(x-1)(x-9) = 0$

or, $x = 1, 9.$

Hence, either $a = 1, b = 9$ or, $a = 9, b = 1$.

EXAMPLE 9 *Three numbers form a HP. The sum of the numbers is 11 and the sum of their reciprocals is 1. Find the numbers.*

SOLUTION Let the numbers be $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$. Then,

$$\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 11 \quad \ldots(i)$$

$$a - d + a + a + d = 1 \quad \ldots(ii)$$

From (ii), we get $a = \frac{1}{3}$

Putting $a = \frac{1}{3}$ in (i), we get

$$\frac{3}{1-3d} + 3 + \frac{3}{1+3d} = 11$$

$$\Rightarrow \frac{6}{1-9d^2} = 8$$

$$\Rightarrow 3 = 4 - 36d^2$$

$$\Rightarrow d = \pm\frac{1}{6}.$$

When $a = \frac{1}{3}$ and $d = \frac{1}{6}$, then numbers are 6, 3, 2.

When $a = \frac{1}{3}$ and $d = -\frac{1}{6}$, the numbers are 2, 3, 6.

EXAMPLE 10 *The value of xyz is 55 or $\frac{343}{55}$ according as the series a, x, y, z, b is an AP or HP. Find the values of a and b given that they are positive.*

SOLUTION If a, x, y, z, b to are in AP, then the common difference d of the AP is given by

$$b = a + 4d \Rightarrow d = \frac{b-a}{4}$$

$$\therefore x = a + d = \frac{a+b-a}{4} = \frac{b+3a}{4}$$

$$y = a + 2d = \frac{a+b-a}{2} = \frac{a+b}{2}$$

$$z = a + 3d = a + 3\left(\frac{b-a}{4}\right) = \frac{a+3b}{4}$$

$$\therefore xyz = \frac{b+3a}{4} \times \frac{a+b}{2} \times \frac{a+3b}{4}$$

$$\Rightarrow 55 = \frac{(3a+b)(a+b)(a+3b)}{32}$$

$$\Rightarrow (3a+b)(a+b)(a+3b) = 55 \times 32 \quad \ldots(i)$$

When a, x, y, z, b are in HP. Then,

$\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in AP.

Let D be the common difference of this A.P. Then,

$$\frac{1}{b} = \frac{1}{a} + 4D \Rightarrow D = \frac{a-b}{4ab}$$

$$\therefore \frac{1}{x} = \frac{1}{a} + D = \frac{1}{a} + \frac{a-b}{4ab} = \frac{3b+a}{4ab}$$

$$\frac{1}{y} = \frac{1}{a} + 2D = \frac{1}{a} = \frac{a-b}{2ab} = \frac{a+b}{2ab}$$

$$\frac{1}{z} = \frac{1}{a} + 3D = \frac{1}{a} + \frac{3(a-b)}{4ab} = \frac{3a+b}{4ab}$$

$$\therefore \frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z} = \frac{(3b+a)(a+b)(3a+b)}{32a^3b^3}$$

$$\Rightarrow \frac{1}{xyz} = \frac{(3a+b)(a+b)(a+3b)}{32a^3b^3}$$

$$\Rightarrow \frac{55}{343} = \frac{55 \times 32}{32a^3b^3} \quad \text{[Using (i)]}$$

$$\Rightarrow (ab)^3 = 7^3$$

$$\Rightarrow ab = 7$$

$$\Rightarrow a = 1, b = 7 \text{ or } a = 7, b = 1.$$

EXAMPLE 11 *If $\log_{14} 5, \log_{(3^x-11)} 5, \log_{(3^x-61/7)} 5$ are in H.P., find the values of x.*

SOLUTION Since $\log_{14} 5, \log_{(3^x-11)} 5, \log_{(3^x-61/7)} 5$ are in H.P. Therefore,

$$\frac{1}{\log_{14} 5}, \frac{1}{\log_{(3^x-11)} 5}, \frac{1}{\log_{(3^x-61/7)} 5} \text{ are in A.P.}$$

$$\Rightarrow \log_5 14, \log_5 (3^x - 11), \log_5\left(3^x - \frac{61}{7}\right) \text{are in A.P.}$$

$$\Rightarrow 2\log_5 (3^x - 11) = \log_5 14 + \log_5\left(3^x - \frac{61}{7}\right)$$

$$\Rightarrow \log_5 (3^x - 11)^2 = \log_5\left(3^x - \frac{61}{7}\right) \times 14$$

$$\Rightarrow (3^x - 11)^2 = 14\left(3^x - \frac{61}{7}\right)$$

$$\Rightarrow (3^x - 11)^2 = (14 \cdot 3^x - 122)$$

$$\Rightarrow (y - 11)^2 = (14y - 122), \text{where } y = 3^x$$

$$\Rightarrow y^2 - 36y + 243 = 0$$

$$\Rightarrow (y-9)(y-27) = 0$$

$$\Rightarrow y = 9, 27$$

$$\Rightarrow 3^x = 9, 27$$

$\Rightarrow \quad x = 2, 3.$

But, $\log_{3^x-11} 5$ and $\log_{3^x-61/7} 5$ are defined, if

$3^x - 11 > 0$ and $3^x - \frac{61}{7} > 0$ i.e. if $3^x - 11 > 0$.

Clearly, $x = 2$ does not satisfy this condition.

Hence, $x = 3$.

EXAMPLE 12 *Prove that if a, b, c are distinct, the expression* $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ *is a perfect square iff a, b, c are in H.P.*

SOLUTION Clearly, $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ is a perfect square, if

$$b^2(c-a)^2 - 4ac(b-c)(a-b) = 0 \qquad [\text{Disc} = 0]$$

$$\Rightarrow \quad [a(b-c) + c(a-b)]^2 - 4ac(b-c)(a-b) = 0$$

$$\left[\because a(b-c) + b(c-a) + c(a-b) = 0\right]$$

$$\Rightarrow \quad [a(b-c) - c(a-b)]^2 = 0$$

$$\Rightarrow \quad a(b-c) - c(a-b) = 0$$

$$\Rightarrow \quad ab - ac - ac + bc = 0$$

$$\Rightarrow \quad b(a+c) = 2ac$$

$$\Rightarrow \quad b = \frac{2ac}{a+c}.$$

$\Rightarrow$ a, b, c are in H.P.

EXAMPLE 13 *If a, b, c are in HP. b, c, d are in G.P. and c, d, e are in A.P., show that* $e = \frac{ab^2}{(2a-b)^2}$.

SOLUTION It is given that :

a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ ...(i)

b, c, d are in G.P. $\Rightarrow c^2 = bd$ ...(ii)

c, d, e are in A.P. $\Rightarrow 2d = c + e$ ...(iii)

It is evident that we have to eliminate c and d from the above three relations.

From (i), we get $c = \frac{ab}{2a-b}$.

Substituting this value of c in (ii), we get

$$\left(\frac{ab}{2a-b}\right)^2 = bd \Rightarrow d = \frac{a^2b}{(2a-b)^2}$$

Substituting the values of c and d in (iii), we get

$$e = \frac{2a^2b}{(2a-b)^2} - \frac{ab}{2a-b} = \frac{ab^2}{(2a-b)^2}$$

EXAMPLE 14 *Three unequal numbers are in HP and their squares are in A.P. show that the numbers are in the ratio*

$$1-\sqrt{3} : -2 : 1+\sqrt{3} \quad \text{or} \quad 1+\sqrt{3} : -2 : 1-\sqrt{3}.$$

SOLUTION Let the number be a, b, c. It is given that a, b, c are in HP. Therefore,

$$b = \frac{2ac}{a+c}$$

$$\Rightarrow \quad \frac{b}{2} = \frac{ac}{a+c} = \lambda \text{ (say)}$$

$$\Rightarrow \quad b = 2\lambda \text{ and } ac = \lambda(a+c) \qquad ...(i)$$

It is also given that a^2, b^2, c^2 are in A.P. Therefore,

$$2b^2 = a^2 + c^2$$

$$\Rightarrow \quad 2b^2 = (a+c)^2 - 2ac$$

$$\Rightarrow \quad 2(2\lambda)^2 = (a+c)^2 - 2\lambda(a+c)$$

$$\Rightarrow \quad (a+c)^2 - 2\lambda(a+c) - 8\lambda^2 = 0$$

$$\Rightarrow \quad (a+c-4\lambda)(a+c+2\lambda) = 0$$

$$\Rightarrow \quad a+c = 4\lambda \text{ or } a+c = -2\lambda$$

If $a + c = 4\lambda$, then

$$ac = 4\lambda^2 \qquad [\text{From (i)}]$$

$$\therefore \quad (a-c)^2 = (a+c)^2 - 4ac = 16\lambda^2 - 16\lambda^2 = 0$$

$$\Rightarrow \quad a = c.$$

But, a, b, c are distinct numbers. Therefore, $a + c = 4\lambda$ is not true.

$$\therefore \quad a+c = -2\lambda$$

$$\Rightarrow \quad ac = -2\lambda^2 \qquad [\text{From (i)}]$$

Now,

$$(a-c)^2 = (a+c)^2 - 4ac$$

$$\Rightarrow \quad (a-c)^2 = 4\lambda^2 + 8\lambda^2$$

$$\Rightarrow \quad a - c = \pm 2\sqrt{3}\lambda$$

If $a - c = 2\sqrt{3}\lambda$, then

$a + c = -2\lambda$ and $a - c = 2\sqrt{3}\lambda$

$\Rightarrow$ $a = (\sqrt{3}-1)\lambda$ and $c = -(\sqrt{3}+1)\lambda$

If $a - c = -2\sqrt{3}\lambda$, then

$a + c = -2\lambda$ and $a - c = -2\sqrt{3}\lambda$

$\Rightarrow$ $a = -(1+\sqrt{3})\lambda$ and $c = (-1+\sqrt{3})\lambda$

Thus,

$$a : b : c = (\sqrt{3}-1)\lambda : 2\lambda : -(\sqrt{3}+1)\lambda$$
$$= \sqrt{3}-1) : 2 : -(\sqrt{3}+1)$$
$$= 1-\sqrt{3} : -2 : 1+\sqrt{3}$$

or, $a : b : c = -(1+\sqrt{3})\lambda : 2\lambda : (-1+\sqrt{3})\lambda$

$$= 1+\sqrt{3} : -2 : 1-\sqrt{3}.$$

EXAMPLE 15 *If a, b, c are in HP and a > c, show that*

$$\frac{1}{b-c} + \frac{1}{a-b} > \frac{4}{a-c}.$$

SOLUTION Since a, b, c are in H.P. Therefore,

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\Rightarrow \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}=\frac{1}{2}\left(\frac{1}{c}-\frac{1}{a}\right)$$

$$\Rightarrow \frac{a-b}{ab}=\frac{b-c}{bc}=\frac{1}{2}\left(\frac{a-c}{ac}\right)$$

$$\Rightarrow \frac{1}{a-b}=\frac{2c}{b(a-c)} \text{ and } \frac{1}{b-c}=\frac{2a}{b(a-c)}$$

$$\Rightarrow \frac{1}{a-b}+\frac{1}{b-c}=\frac{2c}{b(a-c)}+\frac{2a}{b(a-c)}$$

$$\Rightarrow \frac{1}{b-c}+\frac{1}{a-b}=\frac{2(c+a)}{b(a-c)}$$

$$\Rightarrow \frac{1}{b-c}+\frac{1}{a-b}=\frac{4ac}{b^2(a-c)} \qquad \left[\because a, b, c \text{ are in HP}, \therefore b=\frac{2ac}{a+c}\right]$$

$$\Rightarrow \frac{1}{b-c}+\frac{1}{a-b}>\frac{4ac}{ac(a-c)} \qquad \left[\because b \text{ is the HM of } a \text{ and } c, \sqrt{ac} \text{ is the GM of } a \text{ and } c \text{ and } GM>HM\right]$$

$$\Rightarrow \frac{1}{b-c}+\frac{1}{a-b}>\frac{4}{a-c}$$

EXAMPLE 16 *If $a, b, c, \alpha, \beta, \gamma$ are six positive real numbers such that a, b, c are in A.P., α, β, γ are in HP, and $a\alpha_1, b\beta_1, c\gamma$ are in GP. with common ratio not equal to 1, then prove that*

$$a:b:c=\frac{1}{\gamma}:\frac{1}{\beta}:\frac{1}{\gamma}.$$

SOLUTION It is given that

a, b, c are in AP $\Rightarrow 2b=a+c$ ...(i)

α, β, γ are in HP $\Rightarrow \beta=\frac{2\alpha\gamma}{\alpha+\gamma}$ ...(ii)

and, $a\alpha, b\beta, c\gamma$ are in GP

$\Rightarrow (b\beta)^2=(a\alpha)(c\gamma)$

$\Rightarrow b^2\beta^2=ac\,\alpha\gamma$ ...(iii)

$$\Rightarrow \left(\frac{a+c}{2}\right)^2\left(\frac{2\alpha\gamma}{\alpha+\gamma}\right)^2=ac\,\alpha\gamma \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \frac{(a+c)^2}{ac}=\frac{(\alpha+\gamma)^2}{\alpha\gamma}$$

$$\Rightarrow \frac{a^2+c^2+2ac}{ac}=\frac{\alpha^2+\gamma^2+2\alpha\gamma}{\alpha\gamma}$$

$$\Rightarrow \frac{a}{c}+\frac{c}{a}=\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}$$

$$\Rightarrow \left(\frac{a}{c}\right)^2+1-\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)\frac{a}{c}=0 \qquad \text{[Multiplying both sides by } a/c\text{]}$$

$$\Rightarrow \left(\frac{a}{c}-\frac{\alpha}{\gamma}\right)\left(\frac{a}{c}-\frac{\gamma}{\alpha}\right)=0$$

$$\Rightarrow \frac{a}{c}=\frac{\alpha}{\gamma} \text{ or } \frac{a}{c}=\frac{\gamma}{\alpha}$$

If $\frac{a}{c}=\frac{\gamma}{\alpha}$, then

$a\alpha=c\gamma$

Putting $c\gamma=a\alpha$ in (iii), we get

$\Rightarrow b^2\beta^2=a^2\alpha^2$

$\Rightarrow b\beta=a\alpha$

$\therefore a\alpha=b\beta=c\gamma$

This contradicts the hypothesis that $a\alpha, b\beta, c\gamma$ are in GP with common ratio not equal to one.

$\frac{a}{c}=\frac{\alpha}{\gamma} \Rightarrow a\gamma=c\alpha$

Putting $a\gamma=c\alpha$ in (iii), we get

$b^2\beta^2=c^2\alpha^2$

$\Rightarrow b\beta=c\alpha$

$$\Rightarrow \frac{b}{1/\beta}=\frac{c}{1/\alpha}$$

Also, $\frac{a}{c}=\frac{\alpha}{\gamma} \Rightarrow \frac{a}{1/\gamma}=\frac{c}{1/\alpha}$

$$\therefore \frac{a}{1/\gamma}=\frac{b}{1/\beta}=\frac{c}{1/\alpha}$$

$$\Rightarrow a:b:c=\frac{1}{\gamma}:\frac{1}{\beta}:\frac{1}{\alpha}.$$

EXAMPLE 17 *If a, b, c are in HP, prove that*

$$\log(a+c)+\log(a+c-2b)=2\log(a-c)$$

SOLUTION Since a, b, c are in H.P. Therefore,

$$b=\frac{2ac}{a+c}$$

$$\Rightarrow a+c-2b=a+c-\frac{4ac}{a+c}$$

$$\Rightarrow a+c-2b=\frac{(a+c)^2-4ac}{a+c}$$

$$\Rightarrow a+c-2b=\frac{(a-c)^2}{a+c}$$

$$\Rightarrow \log(a+c-2b)=\log\left\{\frac{(a-c)^2}{a+c}\right\}$$

$$\Rightarrow \log(a+c-2b)=2\log(a-c)-\log(a+c)$$

$$\Rightarrow \log(a+c)+\log(a+c-2b)=2\log(a-c).$$

EXAMPLE 18 *If $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ and p, q, r be in A.P., then prove that x, y, z are in H.P.*

SOLUTION We have,

$$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}=\lambda \text{ (say)}$$

$\Rightarrow p = \frac{a-x}{\lambda x}, q = \frac{a-y}{\lambda y}, r = \frac{a-z}{\lambda z}$

Now, p, q, r are in A.P.

$\Rightarrow \frac{a-x}{\lambda x}, \frac{a-y}{\lambda y}, \frac{a-z}{\lambda z}$ are in A.P.

$\Rightarrow \frac{a}{x}-1, \frac{a}{y}-1, \frac{a}{z}-1$ are in A.P.

$\Rightarrow \frac{a}{x}, \frac{a}{y}, \frac{a}{z}$ are in A.P.

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\Rightarrow x, y, z$ are in HP.

EXAMPLE 19 *Let a, b, c be positive real numbers. If a, A_1, A_2, b are in A.P., a, G_1, G_2, b are in G.P. and a, H_1, H_2, b are in H.P., show that*

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}$$ **[IIT 2002]**

SOLUTION Since, a, A_1, A_2, b are in A.P. and we know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

$\therefore a+b = A_1+A_2$ ...(i)

Since, a, H_1, H_2, b *are in* H.P. Therefore,

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{1}{H_1}+\frac{1}{H_2} = \frac{1}{a}+\frac{1}{b} \Rightarrow \frac{H_1+H_2}{H_1H_2} = \frac{a+b}{ab}$ ...(ii)

We know that the product of the terms equidistant from the beginning and the end in a G.P. is always same and is equal to the product of its first and last term.

It is given that a, G_1, G_2, b are in G.P. Therefore,

$G_1G_2 = ab$ ... (iii)

Now, $\frac{G_1G_2}{A_1+A_2} = \frac{ab}{a+b}$ [From (i) and (iii)]

and, $\frac{H_1H_2}{H_1+H_2} = \frac{ab}{a+b}$ [From (ii)]

$\therefore \frac{G_1G_2}{A_1+A_2} = \frac{H_1H_2}{H_1+H_2} \Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_2+H_2}$

Now, a, H_1, H_2, b are in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$\frac{1}{b} = \frac{1}{a}+3D$

$\Rightarrow D = \frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right)$

$\therefore \frac{1}{H_1} = \frac{1}{a}+D$ and $\frac{1}{H_2} = \frac{1}{a}+2D$

$\Rightarrow \frac{1}{H_1} = \frac{1}{a}+\frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right)$ and $\frac{1}{H_2} = \frac{1}{a}+\frac{2}{3}\left(\frac{1}{b}-\frac{1}{a}\right)$

$\Rightarrow \frac{1}{H_1} = \frac{a+2b}{3ab}$ and $\frac{1}{H_2} = \frac{2a+b}{3ab}$

$\Rightarrow H_1 = \frac{3ab}{a+2b}$ and $H_2 = \frac{3ab}{2a+b}$

Let r be the common ratio of the G.P. a, G_1, G_2, b. Then,

$b = a\,r^3 \Rightarrow r = \left(\frac{b}{a}\right)^{1/3}$

$\therefore G_1 = ar = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}b^{1/3}$

and, $G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}b^{2/3}$

$\therefore \frac{G_1G_2}{H_1H_2} = \frac{ab(a+2b)(2a+b)}{9a^2b^2} = \frac{(a+2b)(2a+b)}{9ab}$

Hence, $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}$

EXAMPLE 20 *If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ form a G.P.* **[IIT 2003]**

SOLUTION It is given that a, b, c are in A.P. and a^2, b^2, c^2 are in H.P.

$\therefore 2b = a+c$ ...(i)

and, $b^2 = \frac{2a^2c^2}{a^2+c^2}$ ...(ii)

Now, $2b = a+c$ and $b^2 = \frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow \left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow (a+c)^2(a^2+c^2) = 8a^2c^2$

$\Rightarrow (a^2+c^2+2ac)(a^2+c^2) = 8a^2c^2$

$\Rightarrow (a^2+c^2)^2+2ac(a^2+c^2) = 8a^2c^2$

$\Rightarrow (a^2+c^2)^2+2ac(a^2+c^2)+a^2c^2 = 9a^2c^2$

$\Rightarrow (a^2+c^2+ac)^2 = 9a^2c^2$

$\Rightarrow a^2+c^2+ac = \pm 3ac$

$\Rightarrow a^2+c^2 = -4ac$ or $a^2+c^2 = 2ac$

Now, $a^2+c^2 = -4ac$

$\Rightarrow a^2+c^2+2ac = -2ac$

$\Rightarrow (a+c)^2 = -2ac$

$\Rightarrow 4b^2 = -2ac$ [Using (i)]

$\Rightarrow b^2 = -\frac{ac}{2}$

$\Rightarrow a, b, -c/2$ are in G.P.

and,

$a^2 + c^2 = 2ac$

$\Rightarrow (a-c)^2 = 0$

$\Rightarrow a = c$

$\Rightarrow a = b = c$ [Using (i)]

EXERCISE 21.5

1. If x, y, z are in G.P. and $x-y, z-x$ and $y-z$ are in H.P., then prove that $x + 4y + z = 0$.

2. Let a, b, c be three distinct real numbers in A.P. such that $|a| < 1, |b| < 1$ and $|c| < 1$. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$ and, $z = \sum_{n=0}^{\infty} c^n$, then prove that x, y, z are in H.P.

3. If a, b, c are respectively the pth, qth rth terms of a H.P., prove that

$$bc(q-r) + ca(r-p) + ab(p-q) = 0$$

4. If pth and qth terms of a HP are qr and pr respectively, then prove that its rth term is pq.

5. If pth term of a HP is q and qth term is p, then prove that $(p+q)^{th}$ term is $\frac{pq}{p+q}$.

6. A G.P. and H.P. have the same pth, qth and rth terms as a, b, c respectively. Show that

$$a(b-c)\log a + b(c-a)\log b + c(a-b)\log c = 0$$

7. First three of the four numbers are in A.P. and the last three are in HP, prove that the four numbers are proportional.

[**Hint** : Prove that : First × Fourth = Second × Third]

8. If a, b, c, d, e are five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in HP, prove that

(i) a, c, e are in G.P. (ii) $e = \frac{(2b-a)^2}{a}$

9. If a, b, c are in GP and $b-c, c-a, a-b$ are in HP, then prove that $a + b + c = -3\sqrt{ac}$.

10. If m, n, p are three numbers in G.P. Prove that the first term of an A.P. whose mth, nth and rth terms are in HP is to the common difference as $(n+1) : 1$.

11. First two terms of an A.P. as well as a HP are a and b. If x be any term of the A.P. and y the corresponding term of HP, then prove that $\frac{x-a}{y-a} = \frac{b}{y}$.

12. An AP and a HP have the first term, the same last term, and the same number of terms, prove that the product of the rth term from the beginning in one series and the rth term from the end of the other is independent of r.

—❑❑❑—

21.16 INSERTION OF HARMONIC MEANS BETWEEN TWO GIVEN NUMBERS

Let a, b be two given numbers. If n numbers $H_1, H_2, \ldots, H_n$ are inserted between a and b such that the sequence $a, H_1, H_2, H_3, \ldots, H_n, b$ is an HP, then $H_1, H_2, \ldots, H_n$ are called n harmonic means between a and b.

Now, $a, H_1, H_2, \ldots, H_n, b$ are in HP.

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ are in AP.

Let D be the common difference of this A.P. Then,

$\frac{1}{b} = (n+2)th$ term

$\Rightarrow \frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow D = \frac{a-b}{(n+1)ab}$

Thus, if n harmonic means are inserted between two given numbers a and b, then the common difference of the corresponding AP is given by

$$D = \frac{a-b}{(n+1)ab}$$

Also, $\frac{1}{H_1} = \frac{1}{a} + D, \frac{1}{H_2} = \frac{1}{a} + 2D, \ldots, \frac{1}{H_n} = \frac{1}{a} + nD.$

On putting the value of D, we can obtain the values of $H_1, H_2, \ldots, H_n$.

HARMONIC MEAN OF TWO GIVEN NUMBERS

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the sequence a, H, b is a HP.

Now, a, H, b is a HP

$\Rightarrow \frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ is an AP

$\Rightarrow \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \Rightarrow H = \frac{2ab}{a+b}$

Thus, the harmonic mean H between two numbers a and b is given by

$$H = \frac{2ab}{a+b}$$

If $a_1, a_2, a_3, \ldots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}{n}$$

21.17 PROPERTIES OF ARITHMETIC, GEOMETRIC AND HARMONIC MEANS BETWEEN TWO GIVEN NUMBERS

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

These three means possess the following properties :

PROPERTY I $A \geq G \geq H$

PROOF We have,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$\Rightarrow \quad A - G = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

$$\Rightarrow \quad A \geq G \qquad \ldots(i)$$

$$G - H = \sqrt{ab} - \frac{2ab}{a+b}$$

$$\Rightarrow \quad G - H = \sqrt{ab}\left(\frac{a+b-2\sqrt{ab}}{a+b}\right)$$

$$\Rightarrow \quad G - H = \frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$$\Rightarrow \quad G \geq H \qquad \ldots(ii)$$

From (i) and (ii), we get

$$A \geq G \geq H.$$

Note that the equality holds only when $a = b$.

PROPERTY II *A, G, H form a GP i.e, $G^2 = AH$*

PROOF We have,

$$AH = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$\Rightarrow \quad AH = ab = (\sqrt{ab})^2 = G^2.$$

Hence, $G^2 = AH$.

PROPERTY III *The equation having a and b as its roots is*

$$x^2 - 2Ax + G^2 = 0$$

PROOF The equation having a and b as its roots is

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$$

PROPERTY IV *If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is*

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

PROOF We have,

$$A = \frac{a+b+c}{3}, G = (abc)^{1/3} \text{ and } \frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

$$\Rightarrow \quad a+b+c = 3A, \; abc = G^3 \text{ and } \frac{3G^3}{H} = ab+bc+ca$$

The equation having a, b, c as its roots is

$$x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$$

$$\Rightarrow \quad x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *The A.M. and H.M. between two numbers are 27 and 12 respectively, then find their G.M.*

SOLUTION Let A, G and H denote respectively the A.M., G.M. and H.M. between the two numbers. Then,

$$A = 27 \text{ and } H = 12$$

Since A, G, H are in G.P. Therefore,

$$G^2 = AH \Rightarrow G^2 = 27 \times 12 \Rightarrow G = 18.$$

EXAMPLE 2 *The harmonic mean between two numbers is 4, their A.M. A and G.M. G satisfy the relation $2A + G^2 = 27$. Find the two numbers.*

SOLUTION Let a and b be the two numbers and let. H be the harmonic mean between them

Then, $H = 4$. (given).

Since A, G, H are in G.P. Therefore,

$$G^2 = AH$$

$$\Rightarrow \quad G^2 = 4A.$$

But, $2A + G^2 = 27$ [Given]

$$\therefore \quad 6A = 27 \qquad [\because G^2 = 4A]$$

$$\Rightarrow \quad A = \frac{9}{2}$$

$$\Rightarrow \quad \frac{a+b}{2} = \frac{9}{2}$$

$$\Rightarrow \quad a+b = 9$$

Now, $G^2 = 4A$ and $A = \frac{9}{2}$

$\Rightarrow G^2 = 18$

$\Rightarrow ab = 18.$

The quadratic equation having a, b as its roots is

$$x^2 - (a+b)x + ab = 0$$

or, $x^2 - 9x + 18 = 0$

$\Rightarrow x = 3, 6.$

Hence, the two numbers are 3 and 6.

EXAMPLE 3 *If a, b, c, d are in HP, prove that*

(i) $a + d > b + c$ (ii) $ad > bc$, and (iii) $a^n + c^n > 2b^n$.

SOLUTION a, b, c, d are in HP

$\Rightarrow a, b, c$ are in HP

$\Rightarrow b$ is the harmonic mean of a and c.

But, the arithmetic mean of a and c is $\frac{a+c}{2}$

$\therefore$ AM > HM

$\Rightarrow \frac{a+c}{2} > b \Rightarrow a + c > 2b$ ...(i)

Similarly,

b, c, d are in HP $\Rightarrow$ c is HM of b and d.

But, AM of b and d is $\frac{b+d}{2}$. Therefore,

$$\text{AM} > \text{HM} \Rightarrow \frac{b+d}{2} > c$$

$$\Rightarrow b + d > 2c \quad \text{...(ii)}$$

From (i) and (ii), we get

$$a + c + b + d > 2b + 2c$$

$$\Rightarrow a + d > b + c$$

(ii) a, b, c, d are in HP

$\Rightarrow a, b, c$ are in HP

$\Rightarrow b$ is HM of a and c

But, geometric mean of a and c is $\sqrt{ac}$.

$\therefore$ GM > HM

$\Rightarrow \sqrt{ac} > b$ ...(iii)

Similarly, a, c, d are in HP

$\Rightarrow \sqrt{bd} > c$ ...(iv)

From (iii) and (iv), we get

$$\sqrt{ac}\sqrt{bd} > bc \Rightarrow abcd > b^2c^2 \Rightarrow ad > bc$$

(iii) From (i), we get $\frac{a+c}{2} > b$ ...(iv)

We know that $\frac{a^n + c^n}{2} > \left(\frac{a+c}{2}\right)^n$ ...(v)

From (iv) and (v), we get

$$\frac{a^n + c^n}{2} > b^n \Rightarrow a^n + c^n > 2b^n$$

EXAMPLE 4 *Let a, b, c be distinct positive real numbers and n be a natural number. If a, b, c are either in A.P. or in G.P. or in H.P., then prove that*

$$a^n + c^n > 2b^n.$$

SOLUTION Case I *When a, b, c are in A.P.*

If a, b, c are in A.P., then

$$a + c = 2b \quad \text{...(i)}$$

We know that :

$$\frac{a^n + c^n}{2} > \left(\frac{a+c}{2}\right)^n$$

$\Rightarrow \frac{a^n + c^n}{2} > b^n$ [Using (i)]

$\Rightarrow a^n + c^n > 2b^n$

CASE II *When a, b, c are in in G.P.*

If a, b, c are in G.P., then

$b^2 = ac$ ...(ii)

We know that :

A.M. > G.M.

$\Rightarrow \frac{a^n + c^n}{2} > \sqrt{a^n c^n} \Rightarrow \frac{a^n + c^n}{2} > (ac)^{n/2}$

$\Rightarrow \frac{a^n + c^n}{2} > b^n$ [Using (ii)]

$\Rightarrow a^n + c^n > 2b^n$

CASE III *When a, b, c are in H.P.*

If a, b, c are in H.P., then b is the HM of a and c. But the G.M. of a and c is $\sqrt{ac}$.

$\therefore$ GM > HM

$\Rightarrow \sqrt{ac} > b$ ...(iii)

Again, AM > GM

$\Rightarrow \frac{a^n + c^n}{2} > \sqrt{a^n c^n}$

$\Rightarrow \frac{a^n + c^n}{2} > b^n$ [Using (iii)]

$\Rightarrow a^n + c^n > 2b^n$

Hence, in all the cases, we have

$$a^n + c^n > 2b^n.$$

EXAMPLE 5 *If $2(y-a)$ is the harmonic mean of $y-x$ and $y-z$, show that $x-a, y-a$ and $z-a$ are in G.P.*

SOLUTION Since, $2(y-a)$ is the HM of $y-x$ and $y-z$. Therefore,

$y-x, 2(y-a), y-z$ are in HP

$\Rightarrow \frac{1}{y-x}, \frac{1}{2(y-a)}, \frac{1}{y-z}$ are in A.P.

$\Rightarrow \frac{1}{2(y-a)} - \frac{1}{y-x} = \frac{1}{y-z} - \frac{1}{2(y-a)}$

$\Rightarrow \frac{y-x-2y+2a}{2(y-a)(y-x)} = \frac{2y-2a-y+z}{(y-z)2(y-a)}$

$\Rightarrow \frac{2a-x-y}{y-x} = \frac{y+z-2a}{y-z}$

$\Rightarrow \frac{(y-a)+(x-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)}$

$\Rightarrow \frac{2(x-a)}{2(y-a)} = \frac{2(y-a)}{2(z-a)}$ [Applying componendo and dividendo]

$\Rightarrow (y-a)^2 = (x-a)(z-a)$

$\Rightarrow x-a, y-a, z-a$ are in G.P.

EXAMPLE 6 *If 9 harmonic means be inserted between 2 and 3, prove that* $A + \frac{6}{H} = 5$, *where A is any of the A.M.'s and H the corresponding H.M.*

SOLUTION Let $H_1, H_2, ..., H_9$ be 9 harmonic means between 2 and 3. Then,

$2, H_1, H_2, ..., H_9, 3$ are in HP.

$\therefore \frac{1}{2}, \frac{1}{H_1}, \frac{1}{H_2}, ..., \frac{1}{H_9}, \frac{1}{3}$ are in A.P. with common difference

$$D = \frac{2-3}{(9+1)\times 2\times 3} = -\frac{1}{60}$$

$\therefore \frac{1}{H_i} = \frac{1}{2} + iD \; ; \; i = 1, 2, 3, ..., 9$

$\Rightarrow \frac{1}{H_i} = \frac{1}{2} - \frac{i}{60} \Rightarrow \frac{6}{Hi} = 3 - \frac{i}{10}$ ...(i)

Let $A_1, A_2, ... A_9$ be 9 A.M.'s between 2 and 3. Then, $2, A_1, A_2, ..., A_9, 3$ are in A.P. with common difference

$$d = \frac{3-2}{9+1} = \frac{1}{10}$$

$\therefore A_i = 2 + id \; ; i = 1, 2, ..., 9$

$\Rightarrow A_i = 2 + \frac{i}{10}$ ...(ii)

From (i) and (ii), we have $A_i + \frac{6}{H_i} = 3 - \frac{i}{10} + 2 + \frac{i}{10}$

$= 5$ for $i = 1, 2, ..., 9$.

EXAMPLE 7 *If* $H_1, H_2, ..., H_n$ *be n harmonic means between a and b, show that*

$$\frac{H_1+a}{H_1-a} + \frac{H_n+b}{H_n-b} = 2n$$

SOLUTION Since $H_1, H_2, ..., H_n$ are n harmonic means between a and b. Therefore,

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, ..., \frac{1}{H_n}, \frac{1}{b}$ are in A.P. with common difference

$D = \frac{a-b}{(n+1)ab}$

$\therefore \frac{1}{H_1} = \frac{1}{a} + D$ and $\frac{1}{H_n} = \frac{1}{a} + nD$

$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab}$ and $\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$

$\Rightarrow \frac{1}{H_1} = \frac{nb+a}{(n+1)ab}$ and $\frac{1}{H_n} = \frac{na+b}{(n+1)ab}$

$\Rightarrow \frac{a}{H_1} = \frac{nb+b}{nb+a}$ and $\frac{b}{H_n} = \frac{na+b}{na+a}$

$\Rightarrow \frac{H_1}{a} = \frac{nb+b}{nb+a}$ and $\frac{H_n}{b} = \frac{na+a}{na+b}$

$\Rightarrow \frac{H_1+a}{H_1-a} = \frac{2nb+a+b}{b-a}$ and $\frac{H_n+b}{H_n-b} = \frac{2na+a+b}{a-b}$

[Applying componendo and dividendo]

$\Rightarrow \frac{H_1+a}{H_1-a} + \frac{H_n+b}{H_n-b} = \frac{2nb+a+b}{b-a} + \frac{2na+a+b}{a-b} = 2n.$

EXAMPLE 8 *If p be the first of n arithmetic means between two numbers and q be the first of n harmonic means between the same two numbers, prove that the value of q cannot be between p and* $\left(\frac{n+1}{n-1}\right)^2 p$. [IIT 1991]

SOLUTION Let the two positive numbers be a and b. Let $A_1, A_2, .., A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, ..., A_n, b$ is an AP with common difference $d = \frac{b-a}{n+1}$.

$\therefore p = A_1 = a + d = a + \frac{b-a}{n+1}$

$\Rightarrow p = \frac{na+b}{n+1}$ ...(i)

Let $H_1, H_2, ..., H_n$ be n harmonic means between a and b. Then, $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, ..., \frac{1}{H_n}, \frac{1}{b}$ is an A.P. with common difference

$D = \frac{a-b}{(n+1)ab}$

$\therefore \frac{1}{q} = \frac{1}{a} + D$

$\Rightarrow \frac{1}{q} = \frac{1}{a} + \frac{a-b}{(n+1)ab} \Rightarrow \frac{1}{q} = \frac{nb+a}{(n+1)ab}$

$\Rightarrow q = \frac{(n+1)ab}{nb+a}$ ...(ii)

From (i), we get

$$b = (n+1)p - na$$

Putting the value of b in (ii), we get

$$q\{n(n+1)p - n^2a + a\} = (n+1)a\{(n+1)p - na\}$$

$$\Rightarrow n(n+1)a^2 - \{(n+1)^2p + (n^2-1)q\}a + n(n+1)pq = 0$$

$$\Rightarrow na^2 - \{(n+1)p + (n-1)q\}a + npq = 0$$

Since a is real, therefore

$$\{(n+1)p + (n-1)q\}^2 - 4n^2pq \geq 0$$

$$\Rightarrow (n+1)^2p^2 + (n-1)^2q^2 + 2(n^2-1)pq - 4n^2pq \geq 0$$

$$\Rightarrow (n+1)^2p^2 + (n-1)^2q^2 - 2(n^2+1)pq \geq 0$$

$$\Rightarrow q^2 - \frac{2(n^2+1)}{(n-1)^2}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 \geq 0$$

$$\Rightarrow q^2 - \left\{1 + \left(\frac{n+1}{n-1}\right)^2\right\}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 \geq 0$$

$$\Rightarrow (q-p)\left\{q - \left(\frac{n+1}{n-1}\right)^2 p\right\} \geq 0$$

$$\Rightarrow q < p \text{ or } q > \left(\frac{n+1}{n-1}\right)^2 p \qquad \left[\because \left(\frac{n+1}{n-1}\right)^2 p > p\right]$$

Hence q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

EXAMPLE 9 *Between two quantities a and b are inserted n arithmetic means and also n harmonic means. If x and y be the rth terms of A.P. and H.P. respectively, then prove that*

$\frac{x}{a} + \frac{b}{y}$ is independent of r and n.

SOLUTION Proceeding as in Example 8, we have

$$x = A_{r-1} = a + (r-1)d$$

$$\Rightarrow x = a + (r-1)\frac{(b-a)}{n+1}$$

$$\Rightarrow x = \frac{na + r(b-a) - b + 2a}{n+1}$$

$$\Rightarrow \frac{x}{a} = \frac{na + r(b-a) - b + 2a}{(n+1)a} \qquad ...(i)$$

and,

$$\frac{1}{y} = \frac{1}{a} + (r-1)D$$

$$\Rightarrow \frac{1}{y} = \frac{1}{a} + (r-1)\cdot\frac{a-b}{(n+1)ab}$$

$$\Rightarrow \frac{1}{y} = \frac{(n+1)b + (r-1)(a-b)}{(n+1)ab}$$

$$\Rightarrow \frac{1}{y} = \frac{nb + r(a-b) - a + 2b}{(n+1)ab}$$

$$\Rightarrow \frac{b}{y} = \frac{nb + r(a-b) - a + 2b}{(n+1)a} \qquad ...(ii)$$

From (i) and (ii), we get

$$\frac{x}{a} + \frac{b}{y} = \frac{na + r(b-a) - b + 2a + nb + r(a-b) - a + 2b}{(n+1)a}$$

$$= \frac{n(a+b) + (a+b)}{(n+1)a}$$

$$= \frac{(n+1)(a+b)}{(n+1)a}$$

$$= \frac{a+b}{a}$$, which is independent of n and r.

EXAMPLE 10 *If $A_1, A_2, ..., A_n$ are n arithmetic means between a and b and $H_1, H_2, ..., H_n$ are n harmonic means between a and b, show that*

$$A_r H_{n-r+1} = ab \text{ for } 1 \leq r \leq n$$

SOLUTION Proceeding as in Example 8, we get

$$A_r = a + rd = a + r\left(\frac{b-a}{n+1}\right) = \frac{(n+1)a + r(b-a)}{n+1}$$

and, $$\frac{1}{H_{n-r+1}} = \frac{1}{a} + (n-r+1)D$$

$$\Rightarrow \frac{1}{H_{n-r+1}} = \frac{1}{a} + (n-r+1)\frac{a-b}{(n+1)ab}$$

$$\Rightarrow \frac{1}{H_{n-r+1}} = \frac{(n+1)b + (n-r+1)(a-b)}{(n+1)ab}$$

$$\Rightarrow \frac{1}{H_{n-r+1}} = \frac{(n+1)a - r(a-b)}{(n+1)ab}$$

$$\Rightarrow H_{n-r+1} = \frac{(n+1)ab}{(n+1)a - r(a-b)}$$

$$\therefore A_r \cdot H_{n-r+1} = \frac{(n+1)a + r(b-a)}{n+1} \times \frac{(n+1)ab}{(n+1)a + r(b-a)}$$

$$= ab \text{ for all } r \text{ satisfying } 1 \leq r \leq n.$$

EXAMPLE 11 *If the AM, GM and HM of first and last term of the sequence 25, 26, 27, ..., n–1, n are the terms of the sequence, find the value of n.*

SOLUTION Let A, G and H denote respectively the A.M, G.M. and H.M. of 25 and n. Then,

$$A = \frac{25+n}{2},\ G = \sqrt{25n} \text{ and } H = \frac{50n}{25+n}.$$

Let A, G, H be the pth, qth and rth terms of the given sequence. Then,

$$25 + (p-1)\times 1 = \frac{25+n}{2} \Rightarrow n = 2p + 23 \qquad ...(i)$$

$$25+(q-1)\times 1=\sqrt{25n}\Rightarrow \sqrt{n}=\frac{q+24}{5}$$

$$\Rightarrow n=\frac{(q+24)^2}{25} \quad ...(ii)$$

and,

$$25+(r-1)\times 1=\frac{50n}{25+n}\Rightarrow n=\frac{25(24+r)}{26-r} \quad ...(iii)$$

Since $p\in N$, therefore on putting $p=1,2,3...$ in (i), we get

$$n=25,27,29,31 \quad ...(iv)$$

Since n and q are positive integers. Therefore, we give those values to q for which n given by (ii) is a positive integer. Clearly, such values of q are :

$$1, 6, 11, 16, 21, ...$$

$$\therefore \quad n=25,36,49,64,81,100,.... \quad ...(v)$$

In (iii), we give those positive integral values to r which make n a positive integer. These values of r are :

$$1, 21, 25.$$

$$\therefore \quad n=25,225,1225 \quad ...(vi)$$

From (iv), (v) and (vi), we get $n=1225$.

EXAMPLE 12 *If a, b, c are in HP, prove that*

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4.$$

SOLUTION Since a, b, c are in H.P. Therefore,

$$b=\frac{2ac}{a+c}$$

Now,

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b}=\frac{a+\frac{2ac}{a+c}}{2a-\frac{2ac}{a+c}}+\frac{c+\frac{2ac}{a+c}}{2c-\frac{2ac}{a+c}}$$

$$=\frac{a+3c}{2a}+\frac{c+3a}{2c}$$

$$=\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{3}{2}\left(\frac{c}{a}+\frac{a}{c}\right)$$

$$\geq 1+\frac{3}{2}\times 2\sqrt{\frac{a}{c}\times\frac{c}{a}} \qquad [\because AM\geq GM]$$

$$\geq 1+\frac{3}{2}\times 2=4$$

EXAMPLE 13 *Prove that :*

$$1+\sqrt{2}+\sqrt{3}+....+\sqrt{n}\leq\frac{n^2x+nx+2n}{4\sqrt{x}}$$ *for all $x>0$ and for all $n\in N$*

SOLUTION We know that :

A.M. $\geq$ G.M.

$$\therefore \quad \frac{1+x}{2}\geq\sqrt{x}$$

$$\frac{1+2x}{2}\geq\sqrt{2x}$$

$$\frac{1+3x}{2}\geq\sqrt{3x}$$

$$\vdots \qquad \vdots$$

$$\frac{1+nx}{2}\geq\sqrt{nx}$$

Adding all these inequalities, we get

$$\frac{1}{2}\left\{n+x\frac{n(n+1)}{2}\right\}\geq\sqrt{x}\,(1+\sqrt{2}+\sqrt{3}+....+\sqrt{n})$$

$$\Rightarrow 1+\sqrt{2}+\sqrt{3}+...+\sqrt{n}\leq\frac{2n+n^2x+nx}{4\sqrt{x}}$$

EXAMPLE 14 *If a, b, c are positive real numbers, prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

SOLUTION We know that,

$$AM\geq GM$$

$$\therefore \quad \frac{a^2c+b^2a+c^2b}{3}\geq(a^2c\,.\,b^2a\,.\,c^2b)^{1/3}$$

$$\Rightarrow a^2c+b^2a+c^2b\geq 3abc$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3.$$

EXAMPLE 15 *Let $a_1, a_2, ...$ be positive real numbers in G.P. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, ..., a_n$. Find an expression for the geometric mean of $G_1, G_2, ..., G_n$ in terms of $A_1, A_2, ..., A_n, H_1, H_2, ..., H_n$.*

[IIT 2001]

SOLUTION Let a be the first term and r be the common roots of the G.P. $a_1, a_2, a_3, ...$ Then,

$$a_k=ar^{k-1} \text{ for } k=1,2,3,.....$$

It is given that $a_1, a_2, a_3, ...$ are positive real numbers. Therefore, $a>0$ and $r>0$.

Now, two cases are.

CASE I *When $r=1$*

In this case, We have

$$a_1=a_2=....=a_n=a$$

$$\therefore \quad A_n=\frac{1}{n}(a_1+a_2+...+a_n)=a$$

$$G_n=(a_1,a_2,...,a_n)^{1/n}=a$$

and,

$$\frac{1}{H_n} = \frac{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}{n} = \frac{1}{a}$$

$\therefore\ H_n = a$

Also, $A_n H_n = a^2 = G_n^2$ ...(i)

Now, let G be the geometric mean of $G_1, G_2, \ldots, G_n$. Then,

$$G = (G_1 G_2 G_3 \ldots G_n)^{1/n}$$

$$= (\sqrt{A_1H_1}, \sqrt{A_2H_2} \ldots \sqrt{A_nH_n})^{1/n} \quad \text{[Using (i)]}$$

$$= (A_1A_2 \ldots A_n . H_1 H_2 \ldots H_n)^{\frac{1}{2n}}$$

CASE II *When* $r \neq 1$.

We have,

$$A_n = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$$

$$= \frac{1}{n}(a + ar + \ldots + ar^{n-1})$$

$$= \frac{1}{n}\left\{a\left(\frac{1-r^n}{1-r}\right)\right\}$$

$$G_n = (a_1a_2 \ldots a_n)^{1/2}$$

$$= (a \,.\, ar \ldots ar^{n-1})^{1/n}$$

$$= \left\{a^n r^{\frac{n(n-1)}{2}}\right\}^{1/2}$$

$$= ar^{\frac{n-1}{2}}$$

and,

$$\frac{1}{H_n} = \frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

$$= \frac{1}{n}\left(\frac{1}{a}+\frac{1}{ar}+\ldots+\frac{1}{ar^{n-1}}\right)$$

$$= \frac{1}{n}\left[\frac{1}{a}\left\{\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}\right\}\right]$$

$$= \frac{1}{n}\left(\frac{1-r^n}{1-r}\right)\frac{1}{ar^{n-1}}$$

$$= \frac{1}{n}\left(\frac{1-r^n}{1-r}\right).\frac{1}{ar^{n-1}}$$

$$\Rightarrow\ H_n = \frac{n(1-r)ar^{n-1}}{1-r^n}$$

Thus,

$$A_nH_n = \frac{a(1-r^n)}{n(1-r)} \cdot \frac{n(1-r)ar^{n-1}}{1-r^n} = a^2 r^{n-1} = G_n^2 \quad \ldots\text{(ii)}$$

Let G be the geometric mean of $G_1, G_2, \ldots, G_n$. Then,

$$G = (G_1 G_2 \ldots G_n)^{\frac{1}{n}}$$

$$= (\sqrt{A_1H_1}\,\sqrt{A_2H_2} \ldots \sqrt{A_nH_n})^{\frac{1}{n}} \quad \text{[Using (ii)]}$$

$$= (A_1A_2 \ldots A_n . H_1H_2 \ldots H_n)^{\frac{1}{2n}}$$

EXERCISE 21.6

1. If the A.M. of two numbers exceeds their G.M. by 15 and HM by 27, find the numbers.

2. If the A.M between two numbers exceeds their GM by 2 and GM exceeds their HM by $\frac{8}{5}$, find the numbers.

3. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean of a and b?

4. If H is the harmonic mean between a and b, then prove that

(i) $\frac{1}{H-a}+\frac{1}{H-b} = \frac{1}{a}+\frac{1}{b}$ (ii) $\frac{a-A}{a-H}\times\frac{b-A}{b-H} = \frac{A}{H}$

5. If A_1, A_2 ; G_1, G_2 ; H_1, H_2 be two A.M.'s, G.M's and H.M.s between two numbers a and b, then prove that :

(i) $\frac{A_1+A_2}{H_1+H_2} = \frac{G_1G_2}{H_1H_2}$

(ii) $A_1H_2 = A_2H_1 = G_1G_2 = ab$

6. If $H_1, H_2, \ldots, H_n$ be n harmonic means between a and b and n is a root of the equation $x^2(1-ab) - x(a^2+b^2) - (1+ab) = 0$, then prove that $H_1 - H_n = ab(a-b)$.

7. If n harmonic means $H_1, H_2, \ldots, H_n$ be inserted between 1 and r, then prove that

$$\frac{H_1}{H_n} = \frac{n+r}{nr+1}$$

8. If A, G, H are the arithmetic, geometric and harmonic means of two positive real numbers a and b, and if $A = kH$,

prove that $A^2 = kG^2$. Find the ratio of a to b. For what value of k does this ratio exist ?

9. a, b, c are the first three terms of a geometric series. If the harmonic mean of a and b is 12 and that of b and c is 36, find the first five terms of the series.

10. If a be the A.M. of b and c, b be the G.M. of c and a, then prove that c is the H.M. of a and b.

ANSWERS

1. 120, 30 2. 16, 4 or 4, 16 3. $n = -1$ 8. $(2k-1) \pm 2\sqrt{k(k-1)}, k \geq 1$ 9. 8, 24, 72, 216, 648

—❑❑❑—

21.18 SUM TO n TERMS OF SOME SPECIAL SEQUENCES

In this section, we shall discuss some applications of the sum to n terms of the sequences of natural numbers, their squares and cubes.

I SUM OF FIRST n NATURAL NUMBERS

Consider the sequence 1, 2, 3, ..., n, of natural numbers. Let S_n denote the sum of its first n terms i.e.

$$S_n = 1+2+3+\ldots+n = \sum_{k=1}^{n} k$$

Clearly, it is an arithmetic series with first term $a = 1$, common difference $d = 1$ and last term $l = n$.

$$\therefore \quad S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}.$$

Hence, $$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

II SUM OF THE SQUARES OF FIRST n NATURAL NUMBERS

Consider the sequence of squares of the natural numbers, viz. $1^2, 2^2, 3^2, \ldots + n^2, (n+1)^2, \ldots$

Let S_n denote the sum to n terms of this sequence, i.e.

$$S_n = 1^2+2^2+\ldots+n^2 = \sum_{k=1}^{n} k^2$$

Consider the identity

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1$$

Putting $x = 1, 2, 3, \ldots, (n-1), n$ successively, we get :

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$
$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$
$$4^3 - 3^3 = 3.3^2 + 3.3 + 1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$n^3 - (n-1)^3 = 3.(n-1)^2 + 3.(n-1) + 1$$
$$(n+1)^3 - n^3 = 3.n^2 + 3.n + 1$$

Adding column wise, we obtain

$$(n+1)^3 - 1^3 = 3.(1^2+2^2+\ldots+n^2) + 3(1+2+3+\ldots+n) + \underbrace{(1+1+\ldots+1)}_{n \text{ terms}}$$

$$\Rightarrow \quad (n+1)^3 - 1^3 = 3\left(\sum_{k=1}^{n} k^2\right) + 3\left(\sum_{k=1}^{n} k\right) + n$$

$$\Rightarrow n^3 + 3n^2 + 3n = 3\left(\sum_{k=1}^{n} k^2\right) + 3\frac{n(n+1)}{2} + n$$

$$\Rightarrow \quad 3\left(\sum_{k=1}^{n} k^2\right) = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \quad 3\left(\sum_{k=1}^{n} k^2\right) = \frac{2n^3+3n^2+n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $$S_n = \sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

III SUM OF THE CUBES OF FIRST n NATURAL NUMBERS

Consider the sequence of cubes of the natural numbers, viz. $1^3, 2^3, 3^3, \ldots, n^3, (n+1)^3, \ldots$

Let S_n denote the sum to n terms of this sequence i.e.

$$S_n = 1^3+2^3+3^3+\ldots+n^3 = \sum_{k=1}^{n} k^3$$

Consider the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1 \quad \ldots(i)$$

Putting $x = 1, 2, 3, \ldots, (n-1), n$ successively, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$
$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$
$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$
$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

$$n^4-(n-1)^4 = 4(n-1)^3+6(n-1)^2+4(n-1)+1$$
$$(n+1)^4-n^4 = 4.n^3+6.n^2+4.n+1$$

Adding column wise, we get

$$(n+1)^4-1^4 = 4.(1^3+2^3+\ldots+n^3)+6(1^2+2^2+3^2+\ldots+n^2)+4(1+2+3+\ldots+n)+\underbrace{(1+1+\ldots+1)}_{n \text{ terms}}$$

$$\Rightarrow n^4+4n^3+6n^2+4n = 4\left(\sum_{k=1}^{n}k^3\right)+6\left(\sum_{k=1}^{n}k^2\right)+4\left(\sum_{k=1}^{n}k\right)+n$$

$$\Rightarrow n^4+4n^3+6n^2+4n = 4\left(\sum_{k=1}^{n}k^3\right)+6\left(\frac{n(n+1)(2n+1)}{6}\right)+4\left(\frac{n(n+1)}{2}\right)+n$$

$$\Rightarrow 4\left(\sum_{k=1}^{n}k^3\right) = n^4+4n^3+6n^2+4n-n(n+1)(2n+1)-2n(n+1)-n$$

$$\Rightarrow 4\left(\sum_{k=1}^{n}k^3\right) = n^4+2n^3+n^2 = n^2(n+1)^2$$

$$\Rightarrow \sum_{k=1}^{n}k^3 = \frac{n^2(n+1)^2}{4}$$

$$\Rightarrow \sum_{k=1}^{n}k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n}k\right)^2$$

Hence, $\sum_{k=1}^{n}k^3 = 1^3+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n}k\right)^2$

REMARK Proceeding on the same lines, we have

$$\sum_{k=1}^{n}k^4 = 1^4+2^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1. *Find the sum to n terms of the series* $1^2+3^2+5^2+\ldots$ to n terms.

SOLUTION Let T_n be the nth term of this series. Then,

$$T_n = [1+(n-1)\times 2]^2 = (2n-1)^2 = 4n^2-4n+1.$$

$\therefore$ $1^2+3^2+5^2+\ldots$ to n terms

$$= \sum_{k=1}^{n}T_k$$

$$= \sum_{k=1}^{n}(4k^2-4k+1)$$

$$= 4\left(\sum_{k=1}^{n}k^2\right)-4\left(\sum_{k=1}^{n}k\right)+\sum_{k=1}^{n}1$$

$$= 4\frac{n(n+1)(2n+1)}{6}-4\left(\frac{n(n+1)}{2}\right)+n$$

$$= \frac{n}{3}[2(n+1)(2n+1)-6(n+1)+3]$$

$$= \frac{n}{3}[4n^2+6n+2-6n-6+3]$$

$$= \frac{n}{3}(4n^2-1)$$

EXAMPLE 2. *Find the sum of the series* $2^2+4^2+6^2+\ldots+(2n)^2$

SOLUTION Let T_n be the *nth* term of this series. Then,

$T_n = (2n)^2 = 4n^2$

$\therefore$ $2^2+4^2+6^2+\ldots+(2n)^2$

$$= \sum_{k=1}^{n}T_k = \sum_{k=1}^{n}4k^2 = 4\sum_{k=1}^{n}k^2$$

$$= 4\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \frac{2}{3}n(n+1)(2n+1)$$

EXAMPLE 3. *Find the sum to n terms of the series* $1.2.3+2.3.4+3.4.5+\ldots$

SOLUTION Let T_n the *nth* term of the given series. Then,

$T_n =$ (*nth* term of the sequence formed by first digits in each term) $\times$ (*nth* term of the sequence of second digits in each term) $\times$ (*nth* term of the sequence of third digits in each term)

$=$ (*nth* term of 1, 2, 3, ...) $\times$ (*nth* term of 2, 3, 4, ...) $\times$ (*nth* term of 3, 4, 5, ...)

$= (1+(n-1)\times 1)\times(2+(n-1)\times 1)(3+(n-1)\times 1)$

$= n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n}T_k$$

$$= \sum_{k=1}^{n}k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}\{n^2 + n + 4n + 2 + 4\}$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

EXAMPLE 4. *Find the sum of n terms of the series* $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \ldots$

SOLUTION Let T_n be the *nth* term of the given series. Then,

T_n = (*nth* term of the sequence formed by first digits in each term) × (*nth* term of the sequence formed by second digits in each term)

= (*nth* term of $1, 2, 3, \ldots$) × (*nth* term of $2^2, 3^2, 4^2 \ldots$)

$= n(n+1)^2 = n^3 + 2n^2 + n$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^3 + 2k^2 + k)$$

$$= \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right\}$$

$$= \frac{n(n+1)}{2}\{3n^2 + 3n + 8n + 4 + 6\}$$

$$= \frac{n(n+1)}{2}\{3n^2 + 11n + 10\}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}$$

EXAMPLE 5. *Sum the series* $3 \cdot 8 + 6 \cdot 11 + 9 \cdot 14 + \ldots$ *to n terms.*

SOLUTION Let T_n be the *nth* term of the given series. Then,

T_n = (*nth* term of $3, 6, 9, \ldots$) × (*nth* term of $8, 11, 14, \ldots$)

$= [3 + (n-1) \times 3] \times [8 + (n-1) \times 3]$

$= 3n(3n+5) = 9n^2 + 15n$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (9k^2 + 15k)$$

$$= 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 15\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{3}{2}n(n+1)[2n+1+5]$$

$$= 3n(n+1)(n+3)$$

EXAMPLE 6 *Find the sum of n terms of the series whose nth term is*

(i) $2n^2 - 3n + 5$ *(ii)* $n^2 + 2^n$

SOLUTION (i) We have $T_n = 2n^2 - 3n + 5$. Let S_n denote the sum of n terms of the series whose *nth* term is T_n. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (2k^2 - 3k + 5)$$

$$= 2\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 5$$

$$= 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} - 3\left\{\frac{n(n+1)}{2}\right\} + 5n$$

$$= \frac{n}{6}[2(n+1)(2n+1) - 9(n+1) + 30]$$

$$= \frac{n}{6}[4n^2 + 6n + 2 - 9n - 9 + 30]$$

$$= \frac{n}{6}(4n^2 - 3n + 23)$$

(ii) We have $T_n = n^2 + 2^n$. Let S_n denote the sum of n terms of the series having T_n as its *nth* term. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^2 + 2^k)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$= \frac{n(n+1)(2n+1)}{6} + (2^1 + 2^2 + 2^3 + \ldots + 2^n)$$

$$= \frac{n(n+1)(2n+1)}{6} + 2\left(\frac{2^n - 1}{2-1}\right)$$

$$= \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

EXAMPLE 7 *Find the sum of the following series to n terms :*

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots$$

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = \frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots+(2n-1)}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}(1+(2n-1))}$$

$$= \frac{(n+1)^2}{4} = \frac{1}{4}(n^2+2n+1)$$

Let S_n denote the sum of n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} \frac{1}{4}(k^2+2k+1)$$

$$= \frac{1}{4}\left[\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + 2\left(\frac{n(n+1)}{2}\right) + n\right]$$

$$= \frac{n}{24}(2n^2+9n+13)$$

EXAMPLE 8 *Show that*

$$\frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)} = \frac{3n+5}{3n+1}$$

SOLUTION Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

T_n = nth term of the series : $1\times2^2+2\times3^2+\ldots+n\times(n+1)^2$

$= n(n+1)^2 = n^3+2n^2+n$

and,

T_n' = nth term of the series : $1^2\times2+2^2\times3+\ldots+n^2\times(n+1)$

$= n^2(n+1)$

$= n^3+n^2$

$$\therefore \quad \text{LHS} = \frac{\sum_{k=1}^{n} T_k}{\sum_{k=1}^{n} T_k'}$$

$$= \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)}$$

$$= \frac{\sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k}{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$= \frac{\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}}{\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$

$$= \frac{\frac{3n^2+3n+8n+4+6}{6}}{\frac{3n^2+3n+4n+2}{6}}$$

$$= \frac{3n^2+11n+10}{3n^2+7n+2}$$

$$= \frac{(3n+5)(n+2)}{(3n+1)(n+2)}$$

$$= \frac{3n+5}{3n+1}.$$

EXAMPLE 9 *If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that*

$$9S_2^2 = S_3(1+8S_1).$$

SOLUTION We have,

$$S_1 = \sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2},$$

$$S_2 = \sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

and, $$S_3 = \sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

$$\therefore \quad 9S_2^2 = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2$$

$$= \frac{9}{36}[n(n+1)(2n+1)]^2$$

$$= \frac{1}{4}[n(n+1)(2n+1)]^2.$$

and,

$$S_3(1+8S_1) = \left\{\frac{n(n+1)}{2}\right\}^2\left\{1+8.\frac{n(n+1)}{2}\right\}$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2\{4n^2+4n+1\}$$

$$= \frac{n^2(n+1)^2(2n+1)^2}{4} = \frac{1}{4}[n(n+1)(2n+1)]^2.$$

Hence, $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE 10 *Find the sum to n terms of the series :*

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$$

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = 1^2+2^2+3^2+\ldots+n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}(2n^3+3n^2+n)$$

Let S_n be the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}\frac{1}{6}(2k^3+3k^2+k)$$

$$= \frac{2}{6}\left(\sum_{k=1}^{n}k^3\right)+\frac{3}{6}\sum_{k=1}^{n}k^2+\frac{1}{6}\sum_{k=1}^{n}k$$

$$= \frac{1}{3}\left(\sum_{k=1}^{n}k^3\right)+\frac{1}{2}\left(\sum_{k=1}^{n}k^2\right)+\frac{1}{6}\left(\sum_{k=1}^{n}k\right)$$

$$= \frac{1}{3}\left(\frac{n(n+1)}{2}\right)^2+\frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{1}{6}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{12}[n(n+1)+(2n+1)+1]$$

$$= \frac{n(n+1)}{12}(n^2+3n+2)$$

$$= \frac{n}{12}(n+1)^2(n+2).$$

EXAMPLE 11 *The sequence N of natural numbers is divided into classes as follows :*

1 2
3 4 5 6
7 8 9 10 11 12
.....
.....

Show that the sum of the numbers in nth row is $n(2n^2+1)$.

SOLUTION Since the first row consists of 2 natural numbers, second row 4 natural numbers, third row 6 natural numbers and so on. So, the total number of natural numbers in nth row is $2n$. Now,

Total number of natural numbers upto the end of *nth* row

$$= 2+4+6+\ldots+2n$$

$$= 2(1+2+\ldots+n)$$

$$= \frac{2n(n+1)}{2}$$

$$= n(n+1).$$

$\therefore$ Total number of natural numbers upto the end of $(n-1)$th row

$$= (n-1)(n-1+1)$$

$$= n(n-1).$$

Let S_n denote the sum of first n natural numbers. Then, sum of the natural numbers in *nth* row

= Sum of the natural numbers upto the end of nth row − Sum of the natural numbers upto the end of $(n-1)$th row

$$= S_{n(n+1)} - S_{n(n-1)}$$

$= S_m - S_p$, where $m = n(n+1)$, $p = n(n-1)$.

$$= \frac{m(m+1)}{2} - \frac{p(p+1)}{2} \qquad \left[\text{Using : } S_n = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{n(n-1)\{n(n-1)+1\}}{2}$$

$$= \frac{n}{2}\{(n+1)(n^2+n+1)-(n-1)(n^2-n+1)\}$$

$$= \frac{n}{2}(4n^2+2) = n(2n^2+1).$$

EXAMPLE 12 *Sum to n terms the series*

$$1^2-2^2+3^2-4^2+5^2-6^2+\ldots$$

SOLUTION Clearly, nth term of the given series is negative or positive according as n is even or odd respectively.

CASE I *When n is even :* In this case, the given series is

$$1^2-2^2+3^2-4^2+5^2-6^2+\ldots+(n-1)^2-n^2$$

$$= (1^2-2^2)+(3^2-4^2)+(5^2-6^2)+\ldots+((n-1)^2-n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + \{(n-1)-(n)\}(n-1+n)$$

$$= -(1+2+3+4+\ldots+(n-1)+n)$$

$$= -\frac{n(n+1)}{2}.$$

CASE II *When n is odd* : In this case, the given series is

$$(1^2-2^2)+(3^2-4^2)+\ldots+\{(n-2)^2-(n-1)^2\}+n^2$$

$$= (1-2)(1+2)+(3-4)(3+4)+\ldots + \{(n-2)-(n-1)\}\{(n-2)+(n-1)\}+n^2$$

$$= -\{1+2+3+4+\ldots+(n-2)+(n-1)\}+n^2$$

$$= -\frac{(n-1)(n-1+1)}{2}+n^2 = \frac{n(n+1)}{2}.$$

EXAMPLE 13 *Find the sum of all possible products of the first n natural numbers taken two by two.*

SOLUTION We have,

$$(x_1+x_2+\ldots+x_n)^2 = (x_1^2+x_2^2+\ldots+x_n^2) + 2\sum\sum_{1\le i<j\le n} x_i x_j$$

$$\text{or, } \left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right) + 2\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} x_i x_j\right)$$

$$\Rightarrow \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} x_i x_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{j=1}^{n} x_i^2\right)\right]$$

$\therefore$ Required sum

$$= \frac{1}{2}\left[\left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n} k^2\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - \frac{2n+1}{3}\right\}\right]$$

$$= \frac{n(n+1)}{4}\left[\frac{3n^2+3n-4n-2}{6}\right]$$

$$= \frac{n(n+1)}{24}(3n^2-n-2)$$

$$= \frac{n(n+1)}{24}(n-1)(3n+2)$$

EXAMPLE 14 *Show that the sum of the cubes of any number of consecutive integers is divisible by the sum of those integers.*

SOLUTION Let $m+1, m+2, m+3, \ldots, m+n$ be n consecutive integers. Then, we have to show that

$$a = (m+1)^3 + (m+2)^3 + \ldots + (m+n)^3$$

is divisible by

$$b = (m+1) + (m+2) + \ldots + (m+n)$$

Now,

$$a = (m+1)^3 + (m+2)^3 + \ldots + (m+n)^3$$

$$= [1^3+2^3+\ldots+m^3+(m+1)^3+(m+2)^3+\ldots+(m+n)^3] - [1^3+2^3+\ldots+m^3]$$

$$= \left\{\frac{(m+n)(m+n+1)}{2}\right\}^2 - \left\{\frac{m(m+1)}{2}\right\}^2$$

$$= \left[\frac{(m+n)(m+n+1)}{2} + \frac{m(m+1)}{2}\right] \times \left[\frac{(m+n)(m+n+1)}{2} - \frac{m(m+1)}{2}\right] \quad \ldots(i)$$

and, $b = (m+1)+(m+2)+\ldots+(m+n)$

$$= \{1+2+\ldots+m+(m+1)+(m+2)+\ldots+(m+n)\} - \{1+2+\ldots+m\}$$

$$= \frac{(m+n)(m+n+1)}{2} - \frac{m(m+1)}{2} \quad \ldots(ii)$$

From (i) and (ii), we get

$$a = \left[\frac{(m+n)(m+n+1)}{2} + \frac{m(m+1)}{2}\right] b \quad \ldots(iii)$$

Since the product of any two consecutive natural numbers is even, therefore

$(m+n)(m+n+1)$ and $m(m+1)$ are even natural numbers.

$\Rightarrow$ $\frac{(m+n)(m+n+1)}{2}$ and $\frac{m(m+1)}{2}$ are natural numbers.

$\Rightarrow$ $\frac{(m+n)(m+n+1)}{2} + \frac{m(m+1)}{2}$ is a natural number, say k.

Therefore, from (iii), we get

$$a = bk$$

Hence, b divides a.

EXAMPLE 15 *Show that the sum of the first n terms of the series :*

$1.5.9. + 2.6.10 + 3.7.11 + \ldots$ is $\frac{1}{4}n(n+1)(n+8)(n+9)$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = r(r+4)(r+8),\ r = 1, 2, \ldots$$

$$= r^3 + 12r^2 + 32r$$

$\therefore$ Required sum

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n} (r^3 + 12r^2 + 32r)$$

$$= \sum_{r=1}^{n} r^3 + 12 \sum_{r=1}^{n} r^2 + 32 \sum_{r=1}^{n} r$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 12 \left\{\frac{n(n+1)(2n+1)}{6}\right\} + 32 \left\{\frac{n(n+1)}{2}\right\}$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2n(n+1)(2n+1) + 16n(n+1)$$

$$= \frac{1}{4}[n^2(n+1)^2 + 8n(n+1)(2n+1) + 64n(n+1)]$$

$$= \frac{n}{4}(n+1)[n(n+1) + 8(2n+1) + 64]$$

$$= \frac{n(n+1)}{4}(n^2 + 17n + 72)$$

$$= \frac{n(n+1)(n+8)(n+9)}{4}$$

EXAMPLE 16 *Find the sum of the series*

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \ldots + (n-1) \cdot 2 + n \cdot 1$$

and hence find the coefficient of x^{n-1} *in the expansion of* $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$.

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = r(n - r + 1)$$

$$= (n+1)r - r^2$$

$\therefore$ Required sum

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n} [(n+1)r - r^2]$$

$$= (n+1)\left(\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

Now,

$$(1 + 2x + 3x^2 + .. + nx^{n-1})^2$$

$$= (1 + 2x + 3x^2 + \ldots + nx^{n-1}) \times (nx^{n-1} + (n-1)x^{n-2} + \ldots + 2x + 1)$$

$\therefore$ Coefficient of x^{n-1} in $(1 + 2x + 3x^2 + \ldots + nx^{n-1})^2$

$$= 1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \ldots + (n-1) \cdot 2 + n.$$

$$= \frac{n(n+1)(n+2)}{6}$$

EXAMPLE 17 *Find the sum of n terms of the series :*

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \ldots$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = 1 + 2 + 3 + \ldots + r, \quad r = 1, 2, 3....$$

$$\Rightarrow T_r = \frac{r(r+1)}{2}, \quad r = 1, 2, 3...$$

$\therefore$ Sum of the series

$$= \sum_{r=1}^{n} T_r$$

$$= \frac{1}{2} \sum_{r=1}^{n} r(r+1)$$

$$= \frac{1}{2} \sum_{r=1}^{n} (r^2 + r)$$

$$= \frac{1}{2}\left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

EXAMPLE 18 *Sum to n terms :*

$$1 + 2^2 - 3^3 + 4 + 5^2 - 6^3 + 7 + 8^2 - 9^3 + \ldots$$

SOLUTION We have,

$$S = 1 + 2^2 - 3^3 + 4 + 5^2 - 6^3 + 7 + 8^2 - 9^3 + ...$$

$$= (1 + 4 + 7 + \ldots) + (2^2 + 5^2 + 8^2 + \ldots) - (3^3 + 6^3 + 9^3 + ...)$$

$$= S_1 + S_2 - S_3, \quad \text{where}$$

$$S_1 = 1 + 4 + 7 + \ldots,$$

$$S_2 = 2^2 + 5^2 + 8^2 + \ldots$$

and,

$$S_3 = 3^3 + 6^3 + 9^3 + \ldots$$

Clearly, the number of terms in S_1, S_2, S_3 depend on the value of n. So, we have the following cases :

CASE I *When n is a multiple of 3 i.e.* $n = 3m, m \in N$:

In this case, each of S_1, S_2, S_3 contains m terms :

$\therefore$ $S_1 = 1 + 4 + 7 + \ldots$ to m terms

$$= \frac{m}{2}[2 \times 1 + (m-1) \times 3]$$

$$= \frac{m(3m-1)}{2}$$

$$= \frac{n}{6}(n-1) \qquad [\because n = 3m]$$

$S_2 = 2^2 + 5^2 + 8^2 + \ldots$ to m terms

$$= \sum_{r=1}^{m} [2 + (r-1)\times 3]^2$$

$$= \sum_{r=1}^{m} (3r-1)^2$$

$$= \sum_{r=1}^{m} (9r^2 - 6r + 1)$$

$$= 9\sum_{r=1}^{m} r^2 - 6\sum_{r=1}^{m} r + \sum_{r=1}^{m} 1$$

$$= 9\frac{m(m+1)(2m+1)}{6} - 6\times\frac{m(m+1)}{2} + m$$

$$= \frac{3m(m+1)(2m-1)}{2} + m$$

$$= \frac{n\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}-1\right)}{2} + \frac{n}{3}$$

$$= \frac{n(n+3)(2n-3)}{18} + \frac{n}{3}$$

$S_3 = 3^3 + 6^3 + 9^3 + \ldots$ to m terms

$$= \sum_{r=1}^{m} (3r)^3$$

$$= 27\left(\sum_{r=1}^{m} r^3\right)$$

$$= 27\left\{\frac{m(m+1)}{2}\right\}^2$$

$$= \frac{27}{4}\left\{\frac{n}{3}\left(\frac{n}{3}+1\right)\right\}^2$$

$$= \frac{n^2(n+3)^2}{12}$$

$\therefore\ S = S_1 + S_2 - S_3$

$$= \frac{n(n-1)}{6} + \frac{n(n+3)(2n-3)}{18} + \frac{n}{3} - \frac{n^2(n+3)^2}{12}$$

CASE II *When* $n = 3m+1, m \in N$.

In this case, S_1 will have $m+1$ terms while others will have m terms each.

$\therefore\ S_1 = 1 + 4 + 7 + \ldots +$ to $(m+1)$ terms

$$= \frac{m+1}{2}[2\times 1 + (m+1-1)\times 3]$$

$$= \frac{(m+1)(3m+2)}{2}$$

$$= \frac{1}{2}\left(\frac{n-1}{3}+1\right)(n-1+2) \qquad [\because n = 3m+1]$$

$$= \frac{(n+2)(n+1)}{6}$$

$S_2 = 2^2 + 5^2 + 8^2 + \ldots$ to m terms

$$= \frac{3m(m+1)(2m-1)}{2} + m \qquad \text{[See case I]}$$

$$= \frac{(n-1)\left(\frac{n-1}{3}+1\right)\left(\frac{2n-2}{3}-1\right)}{2} + \frac{n-1}{3}$$

$$= \frac{(n-1)(n+2)(2n-5)}{18} + \frac{n-1}{3}$$

and,

$S_3 = 3^3 + 6^3 + 9^3 + \ldots$ to m terms

$$= \frac{27}{4}[m(m+1)]^2$$

$$= \frac{27}{4}\left\{\left(\frac{n-1}{3}\right)\left(\frac{n-1}{3}+1\right)\right\}^2$$

$$= \frac{(n-1)^2(n+2)^2}{12}$$

$\therefore\ S = S_1 + S_2 - S_3$

$$= \frac{(n+2)(n+1)}{6} + \frac{(n-1)(n+2)(2n-5)}{18} + \frac{n-1}{3} - \frac{(n-1)^2(n+2)^2}{12}$$

CASE III *When* $n = 3m+2, m \in N$:

In this case, S_1 and S_2 will have $(m+1)$ terms while S_3 will have m terms.

$\therefore\ S_1 = 1 + 4 + 7 + \ldots$ to $(m+1)$ terms

$$= \frac{(m+1)(3m+2)}{2} \qquad \text{[See case II]}$$

$$= \frac{1}{2}\left(\frac{n-2}{3}+1\right)(n-2+2)$$

$$= \frac{n(n+1)}{6}$$

$S_2 = 2^2 + 5^2 + 8^2 + \ldots$ to $(m+1)$ terms

$$= \sum_{r=1}^{m+1} (3r-1)^2$$

$$= \sum_{r=1}^{m+1} (9r^2 - 6r + 1)$$

$$= 9\left(\sum_{r=1}^{m+1} r^2\right) - 6\left(\sum_{r=1}^{m+1} r\right) + \sum_{r=1}^{m+1} 1$$

$$= 9\frac{(m+1)(m+2)(2m+3)}{6} - 6\frac{(m+1)(m+2)}{2} + m + 1$$

$$= \frac{3}{2}\left(\frac{n+1}{3}\right)\left(\frac{n+4}{3}\right)\left(\frac{2n+5}{3}\right) - 3\left(\frac{n+1}{3}\right)\left(\frac{n+4}{3}\right) + \frac{n+1}{3}$$

$$= \frac{(n+1)(n+4)(2n+5)}{18} - \frac{(n+1)(n+4)}{3} + \frac{n+1}{3}$$

$$= \frac{(n+1)(n+4)(2n-1)}{18} + \frac{n+1}{3}$$

and,

$S_3 = 3^3 + 6^3 + 9^3 + \ldots$ to m terms

$$= \frac{27}{4}[m(m+1)]^2 \qquad \text{[See case II]}$$

$$= \frac{[(n-2)(n+1)]^2}{12}$$

$\therefore \quad S = S_1 + S_2 - S_3$

$$= \frac{n(n+1)}{6} + \frac{(n+1)(n+4)(2n-1)}{18} + \frac{n+1}{3} - \frac{[(n+1)(n-2)]^2}{12}$$

EXAMPLE 19 *Find the sum to n terms of the series :*

$$1^3 + 3 \cdot 2^2 + 3^3 + 3 \cdot 4^2 + 5^3 + 3 \cdot 6^2 + \ldots.$$

SOLUTION We have,

$$S = 1^3 + 3 \cdot 2^2 + 3^3 + 3 \cdot 4^2 + 5^3 + 3 \cdot 6^2 + \ldots..$$

$$S = (1^3 + 3^3 + 5^3 + \ldots) + 3 \cdot (2^2 + 4^2 + 6^2 + \ldots)$$

$$S = (1^3 + 3^3 + 5^3 + \ldots) + 12(1^2 + 2^2 + 3^2 + \ldots.)$$

$$S = S_1 + 12 S_2 \qquad \ldots(i)$$

where $S_1 = 1^3 + 3^3 + 5^3 + \ldots.$ and $S_2 = 1^2 + 2^2 + 3^2 + \ldots$

Now two cases arise :

CASE I *Wehn n is even, say* $n = 2m\ m \in N$:

In this case, S_1 and S_2 both contain m terms.

$\therefore \quad S_1 = 1^3 + 3^3 + 5^3 + \ldots + (2m-1)^3$

$$= \sum_{r=1}^{m} (2r-1)^3$$

$$= \sum_{r=1}^{m} (8r^3 - 12r^2 + 6r - 1)$$

$$= 8\sum_{r=1}^{m} r^3 - 12\sum_{r=1}^{m} r^2 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 8\left\{\frac{m(m+1)}{2}\right\}^2 - 12\left\{\frac{m(m+1)(2m+1)}{6}\right\} + \frac{6m(m+1)}{2} - m$$

$$= 8\left\{\frac{n(n+2)}{8}\right\}^2 - \frac{12}{6}\left\{\frac{n}{2}\left(\frac{n+2}{2}\right)(n+1)\right\} + 3\frac{n}{2}\left(\frac{n+2}{3}\right) - \frac{n}{2}$$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + 3\frac{n(n+2)}{4} - \frac{n}{2}$$

$S_2 = 1^2 + 2^2 + 3^2 + \ldots + m^2$

$$= \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{n(n+2)(n+1)}{24}$$

$\therefore \quad S = S_1 + 12 S_2$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + \frac{3}{4}n(n+2) - \frac{n}{2} + \frac{n(n+1)(n+2)}{2}$$

$$= \frac{n^2(n+2)^2}{8} + \frac{3}{4}n(n+2) - \frac{n}{2}$$

$$= \frac{n}{8}(n^3 + 4n^2 + 10n + 8)$$

CASE II *When n is odd i.e.* $n = 2m + 1$.

In this case, S_1 contains $m + 1$ terms and S_2 contains m terms.

Therefore,

$S_1 = 1^3 + 3^3 + 5^3 + \ldots + (2m+1)^3$

$$= \sum_{r=1}^{m+1} (2r-1)^3$$

$$= 8\left\{\frac{(m+1)(m+2)}{2}\right\}^2 - \frac{12}{6}(m+1)(m+2)(2m+3) + 6\frac{(m+1)(m+2)}{2} - (m+1)$$

$$= 8\left\{\frac{(n+1)(n+3)}{8}\right\}^2 - 2\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)(n+2) + 3\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right) - \frac{n+1}{2}$$

$$= \frac{[(n+1)(n+3)]^2}{8} - \frac{(n+1)(n+2)(n+3)}{2} + \frac{3}{4}(n+1)(n+3) - \frac{n+1}{2}$$

and,

$S_2 = 1^2 + 2^2 + \ldots + m^2$

$$= \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{n(n-1)(n+1)}{24}$$

$$\therefore \quad S = S_1 + 12\,S_2$$

$$= \frac{(n+1)^2 + (n+3)^2}{8} - \frac{(n+1)(n+2)(n+3)}{2} + \frac{3}{4}(n+1)(n+3) - \frac{n+1}{2} + \frac{n(n-1)(n+1)}{2}$$

$$= \frac{(n+1)}{8}\left\{(n+1)(n+3)^2 - 4(n+2)(n+3) + 6(n+3) - 4 + 4n(n-1)\right\}$$

$$= \frac{(n+1)}{8}(n^3 + 7n^2 - 3n - 1)$$

EXAMPLE 20 *If* $S_m = \dfrac{1+2+3+\ldots+m}{m}$, *find the value of* $S_1^2 + S_2^2 + \ldots + S_n^2$.

SOLUTION We have,

$$S_m = \frac{1+2+3+\ldots+m}{m}$$

$$= \frac{m(m+1)}{2m}$$

$$= \frac{m+1}{2}$$

$$\therefore \quad S_1^2 + S_2^2 + \ldots + S_n^2$$

$$= \sum_{m=1}^{n} S_m^2$$

$$= \sum_{m=1}^{n} \left(\frac{m+1}{2}\right)^2$$

$$= \frac{1}{4}\sum_{m=1}^{n} (m+1)^2$$

$$= \frac{1}{4}[2^2 + 3^2 + \ldots + (n+1)^2]$$

$$= \frac{1}{4}\left[\frac{(n+1)(n+2)(2n+3)}{6} - 1\right]$$

$$= \frac{1}{24}[(n+1)(n+2)(2n+3) - 6]$$

$$= \frac{n}{24}(2n^2 + 9n + 13)$$

EXAMPLE 21 *Find the sum to 2n terms of the series*

$$5^3 + 4 \cdot 6^3 + 7^3 + 4 \cdot 8^3 + 9^3 + 4 \cdot 10^3 + \ldots.$$

SOLUTION We have,

$$5^3 + 4 \cdot 6^3 + 7^3 + 4 \cdot 8^3 + 9^3 + 4 \cdot 10^3 + \ldots \text{ to } 2n \text{ terms}$$

$$= (5^3 + 7^3 + 9^3 + \ldots n \text{ terms}) + 4(6^3 + 8^3 + 10^3 + \ldots n \text{ terms})$$

$$= \sum_{r=1}^{n} (2r+3)^3 + 4\sum_{r=1}^{n} (2r+4)^3$$

$$= \sum_{r=1}^{n} (8r^3 + 36r^2 + 54r + 27) + 4\sum_{r=1}^{n} (8r^3 + 48r^2 + 96r + 64)$$

$$= 40\sum_{r=1}^{n} r^3 + 228\sum_{r=1}^{n} r^2 + 438\sum_{r=1}^{n} r + 283\sum_{r=1}^{n} 1$$

$$= 40\left\{\frac{n(n+1)}{2}\right\}^2 + 228 \times \frac{n(n+1)(2n+1)}{6} + 438 \times \frac{n(n+1)}{2} + 283n$$

$$= n(10n^3 + 96n^2 + 343n + 540).$$

EXAMPLE 22 *The natural numbers are divided into groups 1; 2, 3, 4, 5; 6, 7, 8, 9, 10, 11, 12, 13, 14; and so on. Find the sum of the numbers in the nth group.*

SOLUTION We observe that the first group has 1 number, second group has $2^2 = 4$ numbers, third group has $3^2 = 9$ numbers and so on. So, the number of numbers in nth group is n^2.

The total number of numbers in the first $(n-1)$ groups

$$= \sum_{r=1}^{n} r^2 = \frac{n(n-1)(2n-1)}{6}$$

The nth group consists of n^2 numbers starting with $\dfrac{n(n-1)(2n-1)}{6} + 1$ and ending with $\dfrac{n(n-1)(2n-1)}{6} + n^2$.

$\therefore$ Sum of the numbers in the nth group

$$= \frac{n^2}{2}\left[\frac{n(n-1)(2n-1)}{6} + 1 + \frac{n(n-1)(2n-1)}{6} + n^2\right]$$

$$= \frac{n^2}{6}[n(n-1)(2n-1) + 3(n^2+1)]$$

$$= \frac{n^2}{6}(2n^3 + n + 3)$$

EXAMPLE 23 *The sequence of odd positive integers are written as under :*

1

3 5 7 9

11 13 15 17 19 21 23 25 27

.....

.....

Find the sum of the numbers in the nth row.

SOLUTION We observe that the number of terms in

first row is 1

second row is $4 = 2^2$

third row is $9 = 3^2$

and so on.

So, the number of terms in nth row $= n^2$.

Total number of terms in first $(n-1)$ rows

$$= \sum_{r=1}^{n-1} r^2 = \frac{n(n-1)(2n-1)}{6}$$

$\therefore$ First term of nth row

$$= 1 + \left\{\frac{n(n-1)(2n-1)}{6} + 1 - 1\right\} + 2 \left[\because \begin{matrix}\text{Terms are in A.P. with} \\ \text{common difference 2}\end{matrix}\right]$$

$$= 1 + \frac{n(n-1)(2n-1)}{3}$$

Sum of the terms in nth row

$$= \frac{n^2}{2}\left[2\left\{\frac{n(n-1)(2n-1)}{6} + 1\right\} + (n^2-1)\times 2\right]$$

$$= n^2\left[\frac{n(n-1)(2n-1)}{6} + n^2\right]$$

$$= \frac{n^3}{6}(4n^2+2)$$

21.19 METHOD OF DIFFERENCES

Sometimes the nth term of a sequence or a series cannot be determined by the methods discussed in the earlier sections. In such cases, we use the following steps to find the nth term T_n of the given sequence.

STEP I *Obtain the terms of the sequence and compute the differences between the successive terms of the given sequence. If these differences are in A.P, then take $T_n = an^2 + bn + c$, where a, b, c are constants. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.*

STEP II *If the successive differences computed in step I are in G.P. with common ratio r, then take $T_n = ar^{n-1} + bn + c$.*

STEP III *If the differences of the differences computed in step I are in A.P., then take*

$$T_n = an^3 + bn^2 + cn + d.$$

and find the values of constants a, b, c, d.

STEP IV *If the differences of the differecnes computed in step I are in G.P. with common ratio r, then take*

$$T_n = ar^{n-1} + bn^2 + cn + d$$

The following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Sum to n terms the series*

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$$

SOLUTION Clearly, *nth* term of the given series is negative or positive according as n is even or odd respectively.

CASE I *When n is even :*

In this case, the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + \{(n-1)^2 - n^2\}$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + \{(n-1) - (n)\}(n-1+n)$$

$$= -(1+2+3+4+\ldots+(n-1)+n)$$

$$= -\frac{n(n+1)}{2}.$$

CASE II *When n is odd :*

In this case, the given series is

$$(1^2 - 2^2) + (3^2 - 4^2) + \ldots + \{(n-2)^2 - (n-1)^2\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \ldots + \{(n-2) - (n-1)\}\{(n-2) + (n-1)\} + n^2$$

$$= -(1+2+3+4+\ldots+(n-2)+(n-1)) + n^2$$

$$= -\frac{(n-1)(n-1+1)}{2} + n^2$$

$$= \frac{n(n+1)}{2}.$$

EXAMPLE 2 *Sum the following series to n terms :*

$$5 + 7 + 13 + 31 + 85 + \ldots.$$

SOLUTION The sequence of differences between successive terms is 2, 6, 18, 54,... Clearly, it is a G.P. with common ratio 3. So, let the nth term T_n of the given series be

$$T_n = a \cdot 3^{n-1} + bn + c \quad \ldots(i)$$

Putting $n = 1, 2, 3$, we get

$$5 = a + b + c \quad \ldots(ii)$$

$$7 = 3a + 2b + c \quad \ldots(iii)$$

$$13 = 9a + 3b + c \quad \ldots(iv)$$

Solving these equations, we get

$$a = 1, b = 0, c = 4.$$

Putting the values of a, b, c in (i), we get

$$T_n = 3^{n-1} + 4, \; n = 1, 2, 3, \ldots.$$

Hence, sum of the series

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n} (3^{r-1} + 4)$$

$$= \sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n} 4$$

$$= \left(\frac{3^n - 1}{3 - 1}\right) + 4n$$

$$= \frac{3^n - 1}{2} + 4n$$

$$= \frac{1}{2}[3^n + 8n - 1].$$

EXAMPLE 3 *Find the nth term and the sum to n terms of the series:*

$$2 + 5 + 12 + 31 + 86 + \ldots.$$

SOLUTION The sequence of differences between successive terms is 3, 7, 19, 55, ... The sequence of the second order differences is 4, 12, 36, ... Clearly, it is a G.P. with common ratio 3.

So, let the nth term T_n of the given series be

$$T_n = a \cdot 3^{n-1} + bn^2 + cn + d \qquad \ldots(i)$$

Putting $n = 1, 2, 3, 4$, we get

$$a + b + c + d = 2$$

$$3a + 4b + 2c + d = 5$$

$$9a + 9b + 3c + d = 12$$

and $$27a + 16b + 4c + d = 31$$

Solving these equations, we get

$$a = 1, b = 0, c = 1, d = 0$$

Putting the values of a, b, c, d in (i), we get

$$T_n = 3^{n-1} + n$$

$\therefore$ Sum of the series

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n} (3^{r-1} + r)$$

$$= \sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n} r$$

$$= \left(\frac{3^n - 1}{3 - 1}\right) + n$$

$$= \frac{1}{2}[3^n + 2n - 1]$$

EXAMPLE 4 *Find the sum* $1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots$ *to n terms.*

SOLUTION Consider the sequence 1, 3, 6, 10, 15, ... We observe that the sequence of successive differences is 2, 3, 4, 5, ... Clearly, it is an A.P. Let T_n be the nth term of the sequence 1, 3, 6, 10, 15, ... Then,

$$T_n = an^2 + bn + c \qquad \ldots(i)$$

Putting $n = 1, 2, 3$, we get

$$a + b + c = 1$$

$$4a + 2b + c = 3$$

$$9a + 3b + c = 6$$

Solving these equations, we get $a = \frac{1}{2}, b = \frac{1}{2}, c = 0$

$$\therefore \quad T_n = \frac{1}{2}(n^2 + n)$$

Thus, nth term of the series $1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots$ is

$$T_n^2 = \left\{\frac{1}{2}(n^2 + n)\right\}^2 = \frac{(n^2 + n)^2}{4}$$

Hence,

Sum of the series

$$= \sum_{r=1}^{n} T_r^2$$

$$= \frac{1}{4} \sum_{r=1}^{n} (r^2 + r)^2$$

$$= \frac{1}{4} \sum_{r=1}^{n} (r^4 + 2r^3 + r^2)$$

$$= \frac{1}{4}\left[\sum_{r=1}^{n} r^4 + 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} + 2\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{6}\right]$$

$$\left[\because \sum_{r=1}^{n} r^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}\right]$$

$$= \frac{1}{120}[n(n+1)(6n^3 + 9n^2 + n - 1) + 15n^2(n+1)^2 + 5n(n+1)2n+1)]$$

$$= \frac{n(n+1)}{120}\left[6n^3+9n^2+n-1+15n(n+1)+5(2n+1)\right]$$

$$= \frac{n(n+1)(n+2)(3n^2+6n+1)}{60}$$

EXAMPLE 5 *Find the sum to n terms of the series :*

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n.(n+1)}$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{r(r+1)}, \quad r = 1, 2, \ldots, n$$

$$= \frac{1}{r} - \frac{1}{r+1}$$

$\therefore$ Required sum

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1}$$

EXAMPLE 6 *Find the sum to n terms of the series :*

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)}, \quad r = 1, 2, 3 \ldots n$$

$$= \frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right) \quad \because \text{ [Using partical fractions]}$$

$\therefore$ Required sum of the sereis

$$= \sum_{r=1}^{n} T_r$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left\{\frac{1}{2r-1}-\frac{1}{2r+1}\right\}$$

$$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

$$= \frac{n}{2n+1}.$$

EXAMPLE 7 *Find the sum :* $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$.

SOLUTION $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$

$$= \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b}-\frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b}-\frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\left[\left(\frac{1}{a+b}-\frac{1}{2a+b}\right)+\left(\frac{1}{2a+b}-\frac{1}{3a+b}\right)+\ldots+\left(\frac{1}{na+b}-\frac{1}{(n+1)a+b}\right)\right]$$

$$= \frac{1}{a}\left(\frac{1}{a+b}-\frac{1}{(n+1)a+b}\right)$$

$$= \frac{n}{(a+b)\{(n+1)a+b\}}$$

EXAMPLE 8 *Show that :*

$$\frac{1^4}{1.3}+\frac{2^4}{3.5}+\frac{3^4}{5.7}+\ldots+\frac{n^4}{(2n-1)(2n+1)} = \frac{n(4n^2+6n+5)}{48}+\frac{n}{16(2n+1)}$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{r^4}{(2r-1)(2r+1)}$$

$$= \frac{r^4}{4r^2-1}$$

$$= \frac{1}{16}\cdot\left|\frac{16r^4}{4r^2-1}\right|$$

$$= \frac{1}{16}\left|\frac{16r^4-1+1}{4r^2-1}\right|$$

$$= \frac{1}{16}\left\{4r^2+1+\frac{1}{4r^2-1}\right\}$$

$$= \frac{1}{4}r^2+\frac{1}{16}+\frac{1}{16(2r-1)(2r+1)}$$

$$= \frac{1}{4}r^2+\frac{1}{16}+\frac{1}{32}\left\{\frac{1}{2r-1}-\frac{1}{2r+1}\right\}$$

$\therefore$ Required sum

$$= \sum_{r=1}^{n} T_r$$

$$= \frac{1}{4}\sum_{r=1}^{n} r^2 + \frac{1}{16}n + \frac{1}{32}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{n(n+1)(2n+1)}{24} + \frac{n}{16} + \frac{1}{32}\cdot\frac{2n}{2n+1} \qquad \text{[See Ex. 6]}$$

$$= \frac{n}{48}(4n^2+6n+5) + \frac{n}{16(2n+1)}$$

EXAMPLE 9 *If*

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

and, $H_n' = \frac{n+1}{2} - \left[\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + \frac{3}{(n-2)(n-3} + \ldots + \frac{n-2}{2.3}\right]$

Show that $H_n = H_n'$.

SOLUTION We have,

$$H_n' = \frac{n+1}{2} - \sum_{r=1}^{n-2}\frac{r}{(n-r+1)(n-r)}$$

$$= \frac{n+1}{2} + \sum_{r=1}^{n-2}\left(\frac{n+1}{n-r+1} - \frac{n}{n-r}\right) \qquad \text{[Using partial fractions]}$$

$$= \frac{n+1}{2} + (n+1)\left(\sum_{r=1}^{n-2}\frac{1}{n-r+1}\right) - n\left(\sum_{r=1}^{n-2}\frac{1}{n-r}\right)$$

$$= \frac{n+1}{2} + (n+1)\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\right) - n\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)$$

$$= \frac{n+1}{2} + (n+1)\left(H_n - 1 - \frac{1}{2}\right) - n\left(H_n - 1 - \frac{1}{n}\right)$$

$$= \frac{n+1}{2} + H_n - (n+1) - \frac{1}{2}(n+1) + n + 1$$

$$= H_n$$

EXAMPLE 10 *Find the sum of first n terms of the series whose nth term is*

$$\frac{1}{n(n+1)}(n^4+2n^3+2n^2+n-1)$$

SOLUTION We have,

$$T_n = \frac{n^4+2n^3+2n^2+n-1}{n^2+n}$$

$$= \frac{(n^2+n)^2+n^2+n-1}{n^2+n}$$

$$= n^2+n+1-\frac{1}{n^2+n}$$

$$= n^2+n+1-\frac{1}{n(n+1)}$$

$$= n^2+n+1-\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$\therefore$ Required sum

$$= \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n}\left[r^2+r+1-\left(\frac{1}{r}-\frac{1}{r+1}\right)\right]$$

$$= \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 - \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n - \left(1 - \frac{1}{n+1}\right)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + \frac{n^2-1+1}{n+1}$$

$$= \frac{n(n+1)(n+2)}{3} + \frac{n^2}{n+1}.$$

EXAMPLE 11 *Find the sum to n terms of the series :*

$$\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots.$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{r(r+1)(r+2)(r+3)}$$

$$\Rightarrow T_{r+1} = \frac{1}{(r+1)(r+2)(r+3)(r+4)}$$

$$\therefore r\,T_r = (r+4)\,T_{r+1}$$

$$\Rightarrow r\,T_r - (r+1)\,T_{r+1} = 3\,T_{r+1}$$

$$\Rightarrow \sum_{r=1}^{n-1}\left\{rT_r - (r+1)\,T_{r+1}\right\} = 3\sum_{r=1}^{n-1} T_{r+1}$$

$$\Rightarrow T_1 - n\,T_n = 3(T_2+T_3+\ldots+T_n)$$

$$\Rightarrow 4T_1 - nT_n = 3(T_1+T_2+\ldots+T_n)$$

$$\Rightarrow T_1+T_2+\ldots+T_n$$

$$= \frac{4}{3}T_1 - \frac{n}{3}T_n$$

$$= \frac{4}{3}\cdot\frac{1}{1.2.3.4} - \frac{n}{3}\cdot\frac{1}{n(n+1)(n+2)(n+3)}$$

$$\Rightarrow T_1+T_2+\ldots+T_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

EXAMPLE 12 *If* $\sum_{r=1}^{n} a_r = \frac{n(n+1)(n+2)}{6}$ *for all* $n \geq 1$, *then prove that*

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{a_r}=2.$$

SOLUTION We have,

$$a_n=\sum_{r=1}^{n}a_r-\sum_{r=1}^{n-1}a_r$$

$$=\frac{n(n+1)(n+2)}{6}-\frac{(n-1)n(n+1)}{6}$$

$$=\frac{n(n+1)}{2}$$

$$\therefore\ \frac{1}{a_n}=\frac{2}{n(n+1)}$$

$$\Rightarrow\ \frac{1}{a_r}=\frac{2}{r(r+1)}$$

$$\Rightarrow\ \frac{1}{a_r}=2\left[\frac{1}{r}-\frac{1}{r+1}\right]$$

$$\Rightarrow\ \sum_{r=1}^{n}\frac{1}{a_r}=2\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$=2\left(\frac{1}{1}-\frac{1}{n+1}\right)$$

$$=\frac{2n}{n+1}.$$

$$\therefore\ \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{a_r}=\lim_{n\to\infty}\frac{2n}{n+1}=2.$$

EXAMPLE 13 *Find the sum to n terms of the series :*

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r=\frac{(2r+1)}{r^2.(r+1)^2}$$

$$=\left[\frac{1}{r^2}-\frac{1}{(r+1)^2}\right]$$

$\therefore$ Required sum

$$=\sum_{r=1}^{n}T_r$$

$$=\sum_{r=1}^{n}\left[\frac{1}{r^2}-\frac{1}{(r+1)^2}\right]$$

$$=1-\frac{1}{(n+1)^2}$$

$$=\frac{2n+n^2}{(n+1)^2}$$

EXAMPLE 14 *Find the sum to n terms of the series :*

$$\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r=\frac{n-r+1}{r(r+1)(r+2)}$$

$$=\frac{n+1}{r(r+1)(r+2)}-\frac{1}{(r+1)(r+2)}$$

$$=(n+1)\left[\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right]-\left[\frac{1}{r+1}-\frac{1}{r+2}\right]$$

$$=(n+1)\left[\frac{1}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{2}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]-\left[\frac{1}{r+1}-\frac{1}{r+2}\right]$$

$$=\frac{(n+1)}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\left(\frac{n+3}{2}\right)\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$\therefore\ \sum_{r=1}^{n}T_r=\left(\frac{n+1}{2}\right)\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\left(\frac{n+3}{2}\right)\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$=\left(\frac{n+1}{2}\right)\left(1-\frac{1}{n+1}\right)-\left(\frac{n+3}{2}\right)\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$=\frac{n}{2}-\frac{n+3}{4}+\frac{n+3}{2(n+2)}$$

$$=\frac{n(n+1)}{4(n+2)}$$

Hence, the sum of n terms of the given series is $\frac{n(n+1)}{4(n+2)}$.

EXAMPLE 15 *If* $\sum_{r=1}^{n}T_r=\frac{n(n+1)(n+2)(n+3)}{12}$, *where* T_r *denotes the rth term of the series. Find,* $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r}$.

SOLUTION We have,

$$T_n=\sum_{r=1}^{n}T_r-\sum_{r=1}^{n-1}T_r$$

$$=\frac{n(n+1)(n+2)(n+3)}{12}-\frac{n(n-1)(n+1)(n+2)}{12}$$

$$=\frac{n}{12}(n+1).(n+2)[(n+3)-(n-1)]$$

$$=\frac{n(n+1)(n+2)}{3}$$

$$\therefore\ \frac{1}{T_n}=\frac{3}{n(n+1)(n+2)}$$

$$\Rightarrow \frac{1}{T_n} = 3\left[\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}\right] \quad \text{[Using Partial fractions]}$$

$$\Rightarrow \frac{1}{T_n} = \frac{3}{2}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{T_r} = \frac{3}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)\right]$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)\right]$$

$$= \frac{3}{2}\left[\frac{n}{n+1} - \frac{n}{2(n+2)}\right]$$

$$= \frac{3n(n+3)}{4(n+1)(n+2)}$$

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{T_r} = \frac{3}{4} \lim_{n\to\infty} \frac{n^2+3n}{n^2+3n+2} = \frac{3}{4}.$$

EXAMPLE 16 *Find the sum to n terms of the series :*

$$\frac{1}{3.5} + \frac{16}{3^2.5^2} + \frac{1}{5.7} + \frac{24}{5^2.7^2} + \frac{1}{7.9} + \frac{32}{7^2.9^2} + \ldots$$

SOLUTION We have,

$$S = \frac{1}{3.5} + \frac{16}{3^2.5^2} + \frac{1}{5.7} + \frac{24}{5^2.7^2} + \frac{1}{7.9} + \frac{32}{7^2.9^2} + \ldots$$

$$= \left(\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots\right) + \left(\frac{16}{3^2.5^2} + \frac{24}{5^2.7^2} + \frac{32}{7^2.9^2} + \ldots\right)$$

$$= S_1 + S_2,$$

where $S_1 = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots$

and, $S_2 = \frac{16}{3^2.5^2} + \frac{24}{5^2.7^2} + \frac{32}{7^2.9^2} + \ldots$

Now two cases arise :

CASE I *When n is even i.e. $n = 2m$, $m \in N$:*

In this case, each of S_1 and S_2 contains m terms.

$$\therefore S_1 = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2m+1)(2m+3)}$$

$$= \sum_{r=1}^{m} \frac{1}{(2r+1)(2r+3)}$$

$$= \frac{1}{2}\sum_{r=1}^{m}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \ldots + \left(\frac{1}{2m+1} - \frac{1}{2m+3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2m+3}\right]$$

$$= \frac{m}{6(2m+3)}$$

$$= \frac{n}{6(n+3)} \qquad [\because n = 2m]$$

$$S_2 = \frac{16}{3^2.5^2} + \frac{24}{5^2.7^2} + \frac{32}{7^2.9^2} + \ldots + , r \text{ terms}$$

$$= \left(\frac{1}{3^2} - \frac{1}{5^2}\right) + \left(\frac{1}{5^2} - \frac{1}{7^2}\right) + \ldots + \left(\frac{1}{(2m+1)^2} - \frac{1}{(2m+3)^2}\right)$$

$$= \frac{1}{3^2} - \frac{1}{(2m+3)^2}$$

$$= \frac{1}{9} - \frac{1}{(n+3)^2} \qquad [\because n = 2m]$$

$$= \frac{n^2+6n}{9(n+3)^2}$$

$$\therefore S = S_1 + S_2$$

$$= \frac{n}{6(n+3)} + \frac{n^2+6n}{9(n+3)^2}$$

$$= \frac{n(5n+21)}{18(n+3)^2}$$

CASE II *When n is odd i.e. $n = (2m+1)$:*

In this case, S_1 contains $(n+1)$ terms and S_2 contains m terms.

$$\therefore S_1 = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2m+3)(2m+5)}$$

$$= \sum_{r=1}^{m+1} \frac{1}{(2r+1)(2r+3)}$$

$$= \frac{1}{2}\sum_{r=1}^{m+1}\left(\frac{1}{2r+1,} - \frac{1}{2r+3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2m+5}\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{n+4}\right)$$

$$= \frac{n+1}{6(n+4)}$$

and,

$$S_2 = \frac{1}{3^2} - \frac{1}{(2m+3)^2} \qquad \text{[See case I]}$$

$$= \frac{1}{9} - \frac{1}{(n+2)^2}$$

$$= \frac{n^2+4n-5}{9(n+2)^2}$$

$$\therefore \quad S = S_1 + S_2$$

$$= \frac{n+1}{6(n+4)} + \frac{n^2+4n-5}{9(n+2)^2}$$

$$= \frac{5n^3+31n^2+46n-28}{18(n+4)(n+2)^2}$$

EXAMPLE 17 *Find the sum to n terms of the series :*

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots\ldots$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, r = 1, 2, 3, \ldots, n$$

$$= \frac{r}{(r^2+r+1)(r^2-r+1)}$$

$$= \frac{1}{2}\left(\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right)$$

$\therefore$ Sum of the series

$$= \sum_{r=1}^{n} T_r$$

$$= \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right)\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\ldots+\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^2+n+1}\right]$$

$$= \frac{n^2+n}{2(n^2+n+1)}$$

EXAMPLE 18 *Find the general term and the sum to n terms of the series :*

$$2.2 + 4.2^2 + 7.2^3 + 11.2^4 + 16.2^5 + \ldots.$$

SOLUTION The given series is

$$2.2 + 4.2^2 + 7.2^3 + 11.2^4 + 16.2^5 + \ldots$$

The sequence of the first factors in each term of the above series is 2, 4, 7, 11, 16,...

The sequence of successive differences is 2, 3, 4, 5... which is clearly an A.P. So, let the general term be

$$t_r = ar^2 + br + c.$$

Putting $r = 1, 2, 3$, we get

$$a + b + c = 2$$

$$4a + 2b + c = 4$$

$$9a + 3b + c = 7$$

Solving these equations, we get

$$a = 1, b = \frac{1}{2} = c.$$

$$\therefore \quad t_r = r^2 + \frac{r}{2} + \frac{1}{2}$$

Let T_r be rth term of the given series. Then,

$$T_r = t_r . 2^4$$

$$\Rightarrow \quad T_r = \left(r^2 + \frac{r}{2} + \frac{1}{2}\right)2^r, r = 1, 2, 3, .., n$$

Let

$$\left(r^2 + \frac{r}{2} + \frac{1}{2}\right)2^r$$

$$= (Ar^2 + Br + C)2^r - [A(r-1)^2 + B(r-1) + C]2^{r-1}.$$

Equating the coefficients of r^2, r and the constant terms, we get

$$A = 1, B = -1, C = 4$$

$$\therefore \quad T_r = (r^2 - r + 4)2^r - [(r-1)^2 - (r-1) + 4]2^{r-1}.$$

$$\therefore \quad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left[(r^2 - r + 4)2^r - \left\{(r-1)^2 - (r-1) + 4\right\}2^{r-1}\right]$$

$$= (n^2 - n + 4)2^n - 4.$$

Hence, the sum of n terms of the given series is $(n^2 - n + 4)2^n - 4$.

EXERCISE 21.7

1. Sum to n terms :

$$1.3^2 + 3.5^2 + 5.7^2 + \ldots$$

2. Sum to n terms :

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots$$

3. Find the sum to n terms of the series :

$$1^3 + 3.2^2 + 3^3 + 3.4^2 + 5^3 + 3.6^2 + \ldots.$$

4. If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

5. Show that :

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\ldots+\frac{2^n}{x^{2^n}+1}=\frac{1}{x-1}-\frac{2^{n+1}}{x^{2^{n+1}}-1}$$

6. Show that :

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\ldots\left(1+\frac{1}{3^{2n}}\right)=\frac{3}{2}\left(1-\frac{1}{3^{2n-1}}\right)$$

7. Find the sum of the series :

$$399\,.\,1+397\,.\,3+395\,.\,5+393\,.\,7+\ldots$$
$$+5\,.\,395+3\,.\,397+1\,.\,399$$

8. Let S_n denote the sum to n terms of the series $1.2+2.3+3.4+\ldots$ and σ_{n-1} that to $(n-1)$ terms of the series

$$\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.8}+\ldots$$

Then, show that $18\,S_n\,\sigma_{n-1}-S_n=-2$

9. Find the nth term and sum to n term sof the series :

$$1+5+19+49+101+181+295+\ldots$$

10. Sum the series to n terms :

$$\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\ldots$$

11. Show that :

$$\left(\frac{n-1}{n+1}\right)+3\left(\frac{n-1}{n+1}\right)^2+5\left(\frac{n-1}{n+1}\right)^3+\ldots\text{ to }\infty=\sum_{r=1}^{n}r.$$

12. Sum the following series :

$$(2n-1)\,.\,1+(2n-3)\,.\,3+(2n-5)\,.\,5+\ldots+1\,.\,(2n-1)$$

13. The sequence of natural numbers is divided into groups 1; 2,3,4;5,6,7,8,9;... and so on. Show that the sum of the numbers in nth group is $(n-1)^3+n^3$.

14. The sequence of natural numbers is divided into groups 1; 2, 3, 4; 3, 4, 5,6, 7 ; 4, 5, 6, 7, 8, 9, 10 ; and so on. Find the sum of the numbers in nth group.

15. The sequence of natural numbers is divided into groups 1; 3, 5 ; 7, 9, 11; ... and so on. Show that the sum of the numbers in nth group is n^3.

ANSWERS

1. $\frac{1}{3}n(6n^3+16n^2+9n-4)$

2. $\frac{1}{20}n^5+\frac{1}{4}n^2(n+1)^2-\frac{1}{24}n(n+1)(2n+1)$
$+\frac{1}{8}n(n+1)-\frac{1}{20}n$

3. $\frac{n^2(n+1)^2}{4}-\frac{n^2(n+2)^2}{8}+\frac{n(n+1)(n+2)}{8}$, when n is even
$\frac{n^2(n+1)^2}{4}-\frac{n^2-1}{8}+\frac{n(n^2-1)}{2}$, when n is odd

4. $\frac{n^2+3n}{2(n+1)(n+2)}$

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9. $n^3-n^2+1,\ \frac{n(n^2-1)(3n+2)}{12}+n$

10. $\frac{n}{(1+x)[1+(n+1)x]}$

12. $\frac{n}{3}(2n^2+1)$

14. $(2n-1)^2$

CHAPTER

22

PERMUTATIONS AND COMBINATIONS

22.1 FACTORIAL

In this section, we shall introduce the term and notation of factorial which will be very useful to present various concepts and results in permutations and combinations in compact form.

FACTORIAL *The continued product of first n natural numbers is called the "n factorial" and is denoted by $\lfloor n$ or $n!$ i.e.*

$$n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$$

For example,

$$2! = 1 \times 2$$

$$3! = 1 \times 2 \times 3$$

$$4! = 1 \times 2 \times 3 \times 4$$

$$5! = 1 \times 2 \times 3 \times 4 \times 5 \text{ and so on.}$$

We have,

$$n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$$

and,

$$(n-1)! = 1 \times 2 \times 3 \times 4 \ldots \times (n-2) \times (n-1)$$

Clearly, $n! = (n-1)!\, n$

Thus, $n! = n\,\{(n-1)!\}$

For example,

$$8! = 8 \times 7!$$

$$= 8 \times 7 \times 6!$$

$$= 8 \times 7 \times 6 \times 5! \text{ etc.}$$

REMARK 1 *As we will require zero factorial in the later sections of this chapter and it does not make any sense to define it as the product of the integers from 1 to 0. So, we define* $0! = 1$.

REMARK 2 *Factorials of proper fractions or negative integers are not defined.*

In order to illustrate the use of the above notation i.e. factorial, let us discuss the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Ccompute :*

(i) $\dfrac{8!}{5!}$ (ii) $\dfrac{10!}{7!\,3!}$

SOLUTION (i) We have,

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 8 \times 7 \times 6 = 336$$

(ii) We have,

$$\frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8 \times 7!}{7! \times (1 \times 2 \times 3)} = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$$

EXAMPLE 2 *Convert the following products into factorials :*

(i) $7 \times 8 \times 9$ (ii) $2 \times 4 \times 6 \times 8$

(iii) $2 \times 4 \times 6 \times 8 \times 10 \times \ldots \times (2n-2) \times 2n$

(iv) $1 \times 3 \times 5 \times 7 \times 9 \times \ldots \times (2n-1)$

SOLUTION (i) We have,

$$7 \times 8 \times 9 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$$

$$= \frac{9!}{6!}$$

(ii) We have,

$$2 \times 4 \times 6 \times 8 = (2 \times 1) \times (2 \times 2) \times (2 \times 3) \times (2 \times 4)$$

$$= 2^4 (1 \times 2 \times 3 \times 4)$$

$$= 2^4 (4!)$$

(iii) We have,

$$2 \times 4 \times 6 \times 8 \times 10 \times \ldots \times (2n-2) \times 2n$$

$= (2\times1)\times(2\times2)\times(2\times3)\times\ldots\times(2\times(n-1))\times(2\times n)$

$= 2^n\ (1\times2\times3\times4\times\ldots\times(n-1)\times n)$

$= 2^n\, n!$

(iv) We have,

$1\times3\times5\times7\times\ldots\times(2n-1)$

$= \dfrac{1\times2\times3\times4\times5\times6\times7\times8\ldots\times(2n-2)\times(2n-1)\times2n}{2\times4\times6\times8\times\ldots\times(2n-2)\times2n}$

$= \dfrac{2n!}{2^n\, n!}$

EXAMPLE 3 *Prove that* : $\dfrac{(2n)!}{n!} = \{1\cdot3\cdot5\ldots.(2n-1)\}\,2^n$.

SOLUTION We have,

$$\frac{(2n)!}{n!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\ldots(2n-2)(2n-1)(2n)}{n!}$$

$$= \frac{\{1\cdot3\cdot5\cdot7\ldots(2n-1)\}\cdot\{2\cdot4\cdot6\cdot8\ldots(2n-2)(2n)\}}{n!}$$

$$= \frac{\{1\cdot3\cdot5\cdot7\ldots(2n-1)\}\,2^n\{1\cdot2\cdot3\cdot4\ldots(n-1)n\}}{n!}$$

$$= \frac{\{1\cdot3\cdot5\cdot7\ldots(2n-1)\}\cdot2^n\cdot n!}{n!}$$

$$= \{1\cdot3\cdot5\cdot7\ldots(2n-1)\}\,2^n$$

EXAMPLE 4 *If* $(n+2)! = 60\,\{(n-1)!\}$, *find the value of* n.

SOLUTION We have,

$$(n+2)! = 60\,\{(n-1)!\}$$

$$\Rightarrow (n+2)(n+1)n\,\{(n-1)!\} = 60\,\{(n-1)!\}$$

$$\Rightarrow (n+2)(n+1)n = 5\times4\times3$$

$$\Rightarrow n = 3 \qquad \text{[By comparison]}$$

EXAMPLE 5 *Prove that* $(n!+1)$ *is not divisible by any natural number between 2 and* n.

SOLUTION Let m be divisible by k and r be any natural number between 1 and k. We observe that when $m+r$ is divided by k, then we obtain r as the remainder.

Since $n! = 1\cdot2\cdot3\cdot4\ldots(n-1)\cdot n$, it follows that $n!$ is divisible by every natural number between 2 and n. So, $(n!+1)$, when divided by any natural number between 2 and n, leaves 1 as the remainder. Hence, $(n!+1)$ is not divisible by any natural number between 2 and n.

EXAMPLE 6 *Prove that* $S_n = 1!+2!+3!+4!+\ldots+n!$ *cannot be the square of a natural number except for* $n = 1, 3$.

SOLUTION For $n = 1$, we have

$S_1 = 1! = 1$, which is a perfect square.

For $n = 2$, we have

$S_2 = 1!+2! = 1+2 = 3$, which is not a perfect square.

For $n = 3$, we have

$S_3 = 1!+2!+3! = 1+2+6 = 9$, which is a perfect square.

For $n = 4$, we have

$S_4 = 1!+2!+3!+4! = 1+2+6+24 = 33$,

which is not a perfect square.

For $n \geq 5$, we find that the digits at units place in $n!$ is 0 and $S_4 = 1!+2!+3!+4!$ has 3 as the digit at units place. Therefore, for $n \geq 5$, S_n has 3 at units place. But the square of any natural number does not have 3 at units place. Therefore, S_n is not a perfect square for $n \geq 5$.

Hence, $S_n = 1!+2!+3!+\ldots+n!$ is not a perfect square of a natural number except for $n = 1, 3$.

EXAMPLE 7 *Prove that the equation*

$1!+2!+3!+\ldots+x! = y^3$ *has just one solution in the set of natural numbers.*

SOLUTION Clearly, $x = 1, y = 1$ is a solution of the given equation.

We have,

$1!+2! = 3$

and,

$3 \mid x! \qquad$ for $x \geq 3$

$\therefore\ 3 \mid 1!+2!+3!+\ldots+x! \quad$ for $x \geq 2$

$\Rightarrow 3 \mid y^3$

$\Rightarrow 3 \mid y$

$\Rightarrow 27 \mid y^3$

Thus, the equation $1!+2!+3!+\ldots+x! = y^3$ will have a solution other than $x = 1, y = 1$ in the set of natural numbers, if

$27 \mid 1!+2!+3!+\ldots+x!$

We observe that

$27 \mid x!$ for $x \geq 9$

and none of the following

$1!+2! = 3$

$1!+2!+3! = 9$

$1!+2!+3!+4! = 33$

$1!+2!+3!+4!+5! = 153$

$1!+2!+3!+4!+5!+6! = 873$

$1!+2!+3!+4!+5!+6!+7! = 5913$

and,

$1!+2!+3!+4!+5!+6!+7!+8! = 46233$

is not divisible by 27.

Therefore,

$1!+2!+3!+4!+\ldots+x!$ is not divisible by 27.

Thus, if $x \geq 2$, the equation

$1!+2!+3!+\ldots+x! = y^3$

is not solvable in N.

Hence, the equation

$1! + 2! + 3! + ... + x! = y^3$

has just one solution $x = 1, y = 1$ in the set of natural numbers.

EXAMPLE 8 *Find ones and tens digit in* $1! + 2! + 3! + ... + 49!$.

SOLUTION We know that $n!$ is divisible by 100 for $n \geq 10$ and $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! = 373113$.

Therefore,

$1! + 2! + 3! + ... + 49! = 373113 + a$ multiple of 100

Hence, one's and ten's digit in $1! + 2! + ... + 49!$ are 3 and 1 repsectively.

22.2 EXPONENT OF PRIME p IN $n!$

Let p be a prime number and n be a positive integer. Then, the last integer amongst $1, 2, 3, ..., (n-1), n$ which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}$.

For example, $\left[\frac{10}{3}\right] = 3, \left[\frac{12}{5}\right] = 2, \left[\frac{15}{3}\right] = 5$ etc.

Let $E_p(n)$ denote the exponent of the prime p in the positive integer n. Then,

$E_p(n!)$

$= E_p(1 \cdot 2 \cdot 3 \ldots (n-1)\, n)$

$= E_p\left(p \cdot 2p \cdot 3p \ldots \left[\frac{n}{p}\right]p\right)$ $\left[\because \text{Remaining integers between 1 and } n \text{ are not divisible by } p\right]$

$= \left[\frac{n}{p}\right] + E_p\left(1 \cdot 2 \cdot 3 \ldots \left[\frac{n}{p}\right]\right)$

Now, the last integer amongst $1, 2, 3, ..., \left[\frac{n}{p}\right]$ which is divisible by p is

$$\left[\frac{n/p}{p}\right]p = \left[\frac{n}{p^2}\right]p.$$

$\therefore\ E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p \cdot 2p \cdot 3p \ldots \left[\frac{n}{p^2}\right]p\right)$

$\left[\because \text{Remaining integers between 1 and } \left[\frac{n}{p}\right] \text{ are not divisible by } p\right]$

$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1 \cdot 2 \cdot 3 \ldots \left[\frac{n}{p^2}\right]\right)$

Continuing in this manner, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^s}\right],$$

where s is the largest positive integer such that $p^s \leq n < p^{s+1}$

ILLUSTRATIVE EXMAPLES

EXAMPLE 1 *Find the exponent of 3 in 100!.*

SOLUTION Let $E_p(n)$ denote the exponent of p in n.

Then, $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^s}\right],$

where s is the largest positive integer such that $p^s \leq n < p^{s+1}$

Here, $n = 100,\ p = 3$

$\because\ 3^4 < 100 < 3^5$

$\therefore\ s = 4.$

So, $E_3(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$

$= 33 + 11 + 3 + 1$

$= 48$

Hence, the exponent of 3 in 100! is 48.

EXAMPLE 2 *Prove that 33! is divisible by* 2^{15}. *What is the largest integer n such that 33! is divisible by* 2^n?

SOLUTION Let $E_p(n)$ denote the exponent of prime p in n. Then, we know that

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^s}\right],$$

where s is the largest integer such that $p^s \leq n < p^{s+1}$

Here, $n = 33, p = 2.$

$\because\ 2^5 < 33 < 2^6\ \ \therefore\ s = 5$

So, $E_2(33!) = \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right]$

$= 16 + 8 + 4 + 2 + 1$

$= 31$

Hence, the exponent of 2 in 33! is 31 *i.e.* 33! is divisible by 2^{31}. But, 2^{31} is divisible by 2^{15} also.

Hence, 33! is divisible by 2^{15} and the largest integer n such that 33! is divisible by 2^n is 31.

EXAMPLE 3 *Find the exponent of 15 in 100!*

SOLUTION We have, $15 = 3 \times 5$

Now,

$E_3(100!) = 48$ [See Ex. 1]

and,

$$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

$\therefore$ Exponent of 15 in 100! = min (24, 48) = 24

EXAMPLE 4 *Find the number of zeros at the end of 100!.*

SOLUTION In terms of prime factors 100! can be written as $2^a 3^b 5^c, 7^d$ .....

Now,

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and,

$$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

Therefore,

$$100! = 2^{97} \times 3^b \times 5^{24} \times 7^d \times \ldots$$

$$= 2^{73} \times (2 \times 5)^{24} \times 7^d \times \ldots$$

$$= 10^{24} \times 2^{73} \times 3^b \times 7^d \times \ldots$$

Thus, the number of zeros at the end of 100! is 24.

EXAMPLE 5 *Find the exponent of 6 in 33!*

SOLUTION We have,

$E_2(33!) = 31$ [See Ex 2]

and,

$$E_3(33!) = \left[\frac{33}{3}\right] + \left[\frac{33}{9}\right] + \left[\frac{33}{27}\right] = 11 + 3 + 1 = 15$$

Hence, exponent of 6 in 33! = min (31, 15) = 15

EXAMPLE 6 *The exponent of 12 in 100!.*

SOLUTION We have

$12 = 2^2 \times 3 = 2 \times 2 \times 3$ and $100! = 2^a \cdot 3^b \cdot 5^c$ .....

Now,

$$a = E_2(100!)$$

$$= \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and,

$$b = E_3(100!)$$

$$= \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$$

$$= 33 + 11 + 3 + 1 = 48$$

$\therefore$ $100! = 2^{97} \times 3^{48} \times 5^c \times$ .....

$= (2^2 \times 3)^{48} \times 2 \times 5^c \times$ .....

$= 12^{48} \times 2 \times 5^c \times$ .....

Hence, $E_{12}(100!) = 48$

EXERICSE 22.1

1. If $\frac{(2n)!}{3!\,(2n-3)!}$ and $\frac{n!}{2!\,(n-2)!}$ are in the ratio 44 : 3, find n.
2. Prove that
   (i) $\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))$
   (ii) $\frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} = \frac{(n+1)!}{r!\,(n-r+1)!}$
3. Prove that $\frac{(2n+1)!}{n!} = 2^n [1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1)]$
4. Find the number of zeros at the end of 70!.
5. Find the largest positive integer n for which 35! is divisible by 3^n.

ANSWERS

1. 6 5. 15

—❑❑❑—

22.3 SOME USEFUL SYMBOLS

In this section, we shall introduce you to two symbols viz. ${}^nP_r$ or, $P(n, r)$ and ${}^nC_r$ or, $C(n, r)$. These two symbols will be used in the later part of this chapter. We shall also discuss some properties of these two symbols.

22.3.1 SYMBOL ${}^nP_r$ OR, $P(n, r)$:

If n is a natural number and r is a positive integer satisfying $0 \le r \le n$, then the natural number $\frac{n!}{(n-r)!}$ is denoted by the symbol ${}^nP_r$ or, $P(n, r)$ i.e.

$${}^nP_r = P(n,r) = \frac{n!}{(n-r)!}$$

For example,

$${}^5P_2 = P(5,2)$$

$$= \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3!}{3!} = 120$$

$${}^7P_6 = P(7,6) = \frac{7!}{(7-6)!} = \frac{7!}{1!} = 7! = 840,$$

$${}^4P_4 = P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} - 4! = 24.$$

Clearly,

$${}^nP_r = P(n,r) = \frac{n!}{(n-r)!}$$

$$= n(n-1)(n-2)\ldots(n-(r-1))$$

Also,

$${}^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

and,

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Following examples will illustrate some more applications of the above symbol.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* $2.P(5,3) = P(n,4)$, *find* n.

SOLUTION We have,

$$2.P(5,3) = P(n,4)$$

$$\Rightarrow P(n,4) = 2.P(5,3)$$

$$\Rightarrow \frac{n!}{(n-4)!} = 2\left(\frac{5!}{(5-3)!}\right)$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = \frac{2(5!)}{2!}$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 5!$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 5\times4\times3\times2\times1$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 5\cdot(5-1)\cdot(5-2)(5-3)$$

$$\Rightarrow n = 5 \quad \text{[By comparing two sides]}$$

EXAMPLE 2 *If* $P(n,4) = 20\times P(n,2)$, *find* n.

SOLUTION We have,

$$P(n,4) = 20\times P(n,2)$$

$$\Rightarrow \frac{n!}{(n-4)!} = 20\times\frac{n!}{(n-2)!}$$

$$\Rightarrow (n-2)! = 20\times(n-4)!$$

$$\Rightarrow (n-2)(n-3)(n-4)! = 20\times(n-4)!$$

$$\Rightarrow (n-2)(n-3) = 20$$

$$\Rightarrow n^2-5n+6 = 20$$

$$\Rightarrow n^2-5n-14 = 0$$

$$\Rightarrow (n-7)(n+2) = 0$$

$$\Rightarrow n = 7 \quad [\because n\in N]$$

EXAMPLE 3 *If* $P(5,r) = 2.P(6,r-1)$, *find* r.

SOLUTION We have,

$$P(5,r) = 2.P(6,r-1)$$

$$\Rightarrow \frac{5!}{(5-r)!} = 2.\frac{6!}{(6-(r-1))!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{2\times6\times5!}{(7-r)!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{12\times5!}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow 1 = \frac{12}{(7-r)(6-r)}$$

$$\Rightarrow (7-r)(6-r) = 12$$

$$\Rightarrow 42-13r+r^2 = 12$$

$$\Rightarrow r^2-13r+30 = 0$$

$$\Rightarrow (r-10)(r-3) = 0$$

$$\Rightarrow r = 10 \text{ or, } r = 3.$$

$$\Rightarrow r = 3 \quad [\because P(5,r) \text{ is meaningful for } r\le5]$$

EXAMPLE 4 *If* ${}^{10}P_r = 5040$, *find the value of* r.

SOLUTION We have,

$${}^{10}P_r = 5040$$

$$\Rightarrow \frac{10!}{(10-r)!} = 10\times504$$

$$\Rightarrow \frac{10!}{(10-r)!} = 10\times9\times8\times7$$

$$\Rightarrow \frac{10!}{(10-r)!} = \frac{10\times9\times8\times7\times6!}{6!}$$

$$\Rightarrow \frac{10!}{(10-r)!} = \frac{10!}{6!}$$

$$\Rightarrow (10-r)! = 6!$$

$$\Rightarrow 10-r = 6$$

$$\Rightarrow r = 4.$$

EXAMPLE 5 *If* $P(n-1,3):P(n,4)=1:9$, find n.

SOLUTION We have,

$$P(n-1,3):P(n,4)=1:9$$

$$\Rightarrow \frac{P(n-1,3)}{P(n,4)}=\frac{1}{9}$$

$$\Rightarrow \frac{\frac{(n-1)!}{(n-1-3)!}}{\frac{n!}{(n-4)!}}=\frac{1}{9}$$

$$\Rightarrow \frac{(n-1)!}{(n-4)!}\times\frac{(n-4)!}{n!}=\frac{1}{9}$$

$$\Rightarrow \frac{(n-1)!}{n!}=\frac{1}{9}$$

$$\Rightarrow \frac{(n-1)!}{n\cdot(n-1)!}=\frac{1}{9}$$

$$\Rightarrow n=9$$

EXAMPLE 6 *If* ${}^{9}P_5+5\cdot{}^{9}P_4={}^{10}P_r$, *find* r.

SOLUTION We have,

$${}^{9}P_5+5\cdot{}^{9}P_4={}^{10}P_r$$

$$\Rightarrow \frac{9!}{(9-5)!}+5\cdot\frac{9!}{(9-4)!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{9!}{4!}+5\cdot\frac{9!}{5!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{9!}{4!}+\frac{9!}{4!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow 2\times\frac{9!}{4!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{5\times2\times9!}{5\times4!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{10\times9!}{5!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow \frac{10!}{5!}=\frac{10!}{(10-r)!}$$

$$\Rightarrow (10-r)!=5!$$

$$\Rightarrow 10-r=5$$

$$\Rightarrow r=5$$

EXAMPLE 7 *If* ${}^{56}P_{r+6}:{}^{54}P_{r+3}=30800:1$, *find* r.

SOLUTION We have,

$${}^{56}P_{r+6}:{}^{54}P_{r+3}=30800:1$$

$$\Rightarrow \frac{56!}{(56-r-6)!}:\frac{54!}{(54-r-3)!}=\frac{30800}{1}$$

$$\Rightarrow \frac{56!}{(50-r)!}:\frac{54!}{(51-r)!}=30800:1$$

$$\Rightarrow \frac{56!}{(50-r)!}\times\frac{(51-r)!}{54!}=\frac{30800}{1}$$

$$\Rightarrow \frac{56\times55\times54!}{(50-r)!}\times\frac{(51-r)\times(50-r)!}{54!}=\frac{30800}{1}$$

$$\Rightarrow 56\times55\times(51-r)=30800$$

$$\Rightarrow (51-r)=10$$

$$\Rightarrow r=41.$$

EXAMPLE 8 *If* ${}^{2n+1}P_{n-1}:{}^{2n-1}P_n=3:5$, *find* n.

SOLUTION We have,

$${}^{2n+1}P_{n-1}:{}^{2n-1}P_n=3:5$$

$$\Rightarrow \frac{{}^{2n+1}P_{n-1}}{{}^{2n-1}P_n}=\frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)!}{(n+2)!}\times\frac{(n-1)!}{(2n-1)!}=\frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)n(n-1)!}\times\frac{(n-1)!}{(2n-1)!}=\frac{3}{5}$$

$$\Rightarrow \frac{2(2n+1)}{(n+2)(n+1)}=\frac{3}{5}$$

$$\Rightarrow 10(2n+1)=3(n+2)(n+1)$$

$$\Rightarrow 3n^2+9n+6=20n+10$$

$$\Rightarrow 3n^2-11n-4=0$$

$$\Rightarrow (n-4)(3n+1)=0$$

$$\Rightarrow n=4 \qquad [\because n\neq -1/3]$$

EXAMPLE 9 *If* ${}^{22}P_{r+1}:{}^{20}P_{r+2}=11:52$, *find* r.

SOLUTION We have,

$${}^{22}P_{r+1}:{}^{20}P_{r+2}=11:52$$

$$\Rightarrow \frac{22!}{(21-r)!}:\frac{20!}{(18-r)!}=11:52$$

$$\Rightarrow \frac{22!}{(21-r)!}\times\frac{(18-r)!}{20!}=\frac{11}{52}$$

$$\Rightarrow \frac{22\times21\times20!}{(21-r)(20-r)(19-r)\cdot(18-r)!}\times\frac{(18-r)!}{20!}=\frac{11}{52}$$

$$\Rightarrow \frac{22\times21}{(21-r)(20-r)(19-r)}=\frac{11}{52}$$

$$\Rightarrow (21-r)(20-r)(19-r)=2\times21\times52$$

$$\Rightarrow (21-r)(20-r)(19-r)=2\times3\times7\times4\times13$$

$$\Rightarrow (21-r)(20-r)(19-r)=12\times13\times14$$

$$\Rightarrow (21-r)(20-r)(19-r)=(21-7)(20-7)(19-7)$$

$$\Rightarrow r=7$$

EXAMPLE 10 *Prove the following :*

(i) $P(n, n) = 2\,P(n, n-2)$

(ii) $P(n, n) = P(n, n-1)$

(iii) $P(n, r) = P(n-1, r) + r\,.\,P(n-1, r-1)$

(iv) $P(n, r) = n \cdot P(n-1, r-1)$

SOLUTION (i) We have,

$2P(n, n-2)$

$= 2\dfrac{n!}{(n-(n-2))!}$

$= 2\left(\dfrac{n!}{2!}\right)$

$= n! = P(n, n)$

(ii) We have,

$P(n, n-1)$

$= \dfrac{n!}{(n-(n-1))!}$

$= \dfrac{n!}{1!} = n! = P(n, n)$

(iii) We have,

$P(n-1, r) + r \cdot P(n-1, r-1)$

$= \dfrac{(n-1)!}{(n-1-r)!} + r \cdot \dfrac{(n-1)!}{((n-1)-(r-1))!}$

$= \dfrac{(n-1)!}{(n-r-1)!} + r\dfrac{(n-1)!}{(n-r)!}$

$= \dfrac{(n-1)!}{(n-r-1)!} + r\dfrac{(n-1)!}{(n-r)(n-r-1)!}$

$= \dfrac{(n-1)!}{(n-r-1)!}\left\{1 + \dfrac{r}{n-r}\right\}$

$= \dfrac{(n-1)!}{(n-r-1)!}\left\{\dfrac{n-r+r}{n-r}\right\}$

$= \dfrac{(n-1)!}{(n-r-1)!} \cdot \dfrac{n}{n-r}$

$= \dfrac{n!}{(n-r)!} = P(n, r)$

(iv) We have,

$n \cdot P(n-1, r-1)$

$= n\dfrac{(n-1)!}{((n-1)-(r-1))!}$

$= \dfrac{n!}{(n-r)!} = P(n, r)$

EXAMPLE 11 *Prove that if $r \le s \le n$, then $P(n, s)$ is divisible by $P(n, r)$.*

SOLUTION Let $s = r + k$ where $0 \le k \le s - r$. Then,

$P(n, s) = \dfrac{n!}{(n-s)!}$

$= n(n-1)(n-2)\ldots(n-(s-1))$

$= n(n-1)(n-2)\ldots(n-(r+k-1))$

$= n(n-1)(n-2)\ldots\{n-(r-1)\}(n-r)\{n-(r+1)\}\ldots$

$\ldots\ldots\ldots\ldots\{n-(r+k-1)\}$

$= [n(n-1)(n-2)\ldots\{n-(r-1)\}](n-r)\{n-(r+1)\}\ldots$

$\ldots\ldots\ldots\ldots\{n-(r+k-1)\}$

$= P(n, r) \cdot [(n-r)\{n-(r+1)\}\ldots\{n-(r+k-1)\}]$

$\Rightarrow \quad P(n, s) = P(n, r) \cdot [(n-r)\{n-(r+1)\}\ldots\{n-(r+k-1)\}]$

$\Rightarrow \quad P(n, s)$ is divisible by $P(n, r)$.

EXAMPLE 12 *If P_m stands for ${}^mP_m$, then prove that*

$$1 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + \ldots + n \cdot P_n = (n+1)!$$

SOLUTION We have,

$P_m = {}^mP_m = m!.$

$\therefore \quad 1 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + \ldots + n \cdot P_n$

$= 1 + 1 + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + \ldots + n \cdot n!$

$= 1 + \sum_{r=1}^{n} r \cdot r!$

$= 1 + \sum_{r=1}^{n} \{(r+1) - 1\}\, r!$

$= 1 + \sum_{r=1}^{n} [(r+1)\, r! - r!]$

$= 1 + \sum_{r=1}^{n} [(r+1)! - r!]$

$= 1 + [(2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)]$

$= 1 + [(n+1)! - 1!]$

$= (n+1)!$

22.3.2. SYMBOL ${}^nC_r$ OR , $C(n, r)$

If n is a natural number and r is a positive integer satisfying $0 \le r \le n$, then the natural number $\dfrac{n!}{(n-r)!\,r!}$ is denoted by the symbol ${}^nC_r$ or, $C(n, r)$. Thus,

$${}^nC_r = C(n, r) = \frac{n!}{(n-r)!\,r!}$$

For example,

$${}^7C_5 = C(7, 5) = \frac{7!}{(7-5)!\,5!} = \frac{7!}{2!\,5!} = \frac{7 \times 6 \times 5!}{2! \times 5!} = 21,$$

$${}^9C_1 = C(9, 1) = \frac{9!}{(9-1)!\,1!} = \frac{9!}{8!} = \frac{9 \times 8!}{8!} = 9,$$

$${}^8C_8 = C(8, 8) = \frac{8!}{(8-8)!\,8!} = \frac{8!}{0!\,8!} = \frac{8!}{8!} = 1.$$

Clearly,

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(r-1))(n-r)!}{(n-r)!\,r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}$$

Also,

$${}^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1,$$

$${}^nC_1 = \frac{n!}{(n-1)!\,1!} = \frac{n!}{(n-1)!} = n,$$

$${}^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2!} \quad \text{etc.}$$

and,

$${}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = \frac{n!}{n!} = 1$$

RELATION BETWEEN ${}^nC_r$ and ${}^nP_r$

If n is a natural number and r is a positive integer such that $0 \le r \le n$, then

$${}^nP_r = \frac{n!}{(n-r)!} \text{ and } {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Now,

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow {}^nC_r = \left(\frac{n!}{(n-r)!}\right)\cdot\frac{1}{r!}$$

$$\Rightarrow {}^nC_r = \frac{{}^nP_r}{r!}$$

Thus,

$${}^nC_r = \frac{{}^nP_r}{r!} \qquad \text{or, } {}^nP_r = {}^nC_r \times r!$$

We shall now discuss some useful properties of ${}^nC_r$ or, $C(n, r)$.

22.3.3 PROPERTIES OF ${}^nC_r$ or, $C(n, r)$

We shall now discuss some important properties of ${}^nC_r$.

PROPERTY I ${}^nC_r = {}^nC_{n-r}$ *for* $0 \le r \le n$.

PROOF We have,

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

REMARK 1 *The use of this property simplifies the calculation of* ${}^nC_r$ *when* r *is large.*

For example, if we want to calculate ${}^{20}C_{19}$, *by using this property, we have*

$${}^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20.$$

REMARK 2 *The above property can be restated as follows :*

If x *and* y *are non-negative integers such that* $x + y = n$, *then*

$${}^nC_x = {}^nC_y.$$

ILLUSTRATION 1 *If* ${}^nC_7 = {}^nC_4$, *find the value of* n.

SOLUTION We have,

$${}^nC_x = {}^nC_y \Leftrightarrow x + y = n \text{ or, } x = y.$$

$$\therefore \quad {}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11.$$

PROPERTY II *Let* n *and* r *be non-negative integers such that* $r \le n$. *Then,*

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

PROOF We have,

$${}^nC_r + {}^nC_{r-1}$$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{[n-(r-1)]!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r(r-1)!} + \frac{n!}{(n-r+1)(n-r)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$$

$$= \frac{n!\,(n+1)}{(n-r)!\,(r-1)!\,r(n-r+1)}$$

$$= \frac{(n+1)\,n!}{(n-r+1)(n-r)!\,r(r-1)!}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!} = \frac{(n+1)!}{((n+1)-r)!\,r!}$$

$$= {}^{n+1}C_r.$$

ILLUSTRATION 2 *Find the value of the expression*

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3.$$

SOLUTION We have,

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$

$$= {}^{47}C_3 + {}^{47}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \quad [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= ({}^{48}C_3 + {}^{48}C_4) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \quad [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= ({}^{49}C_3 + {}^{49}C_4) + {}^{50}C_3 + {}^{51}C_3 \quad [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_4$

$= ({}^{50}C_3 + {}^{50}C_4) + {}^{51}C_3 \quad [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= {}^{51}C_4 + {}^{51}C_3$

$= {}^{52}C_4 \quad [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

PROPERTY III *Let n and r be non-negative integers such that $1 \le r \le n$. Then, ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$*

PROOF We have,

$${}^nC_r = \frac{n!}{(n-r)!\, r!}$$

$$= n \frac{(n-1)!}{\{(n-1)-(r-1)\}!\, r\,(r-1)!}$$

$$= \frac{n}{r} \frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!}$$

$$= \frac{n}{r}\, {}^{n-1}C_{r-1}$$

REMARK *This property is very useful to find the value of ${}^nC_r$. For example,*

$${}^{10}C_3 = \frac{10}{3} \times {}^9C_2 = \frac{10}{3} \times \frac{9}{2} \times {}^8C_1 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times {}^7C_0$$

$$= \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120$$

PROPERTY IV *If $1 \le r \le n$, then*

$$n \cdot {}^{n-1}C_{r-1} = (n-r+1)\, {}^nC_{r-1}.$$

PROOF We have,

$n \cdot {}^{n-1}C_{r-1}$

$= n \cdot \frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!}$

$= \frac{n!}{(n-r)!\,(r-1)!}$

$= \frac{(n-r+1) \cdot n!}{(n-r+1)\,(n-r)!\,(r-1)!}$

$= (n-r+1)\left[\frac{n!}{(n-r+1)!\,(r-1)!}\right]$

$= (n-r+1)\left[\frac{n!}{\{n-(r-1)\}!\,(r-1)!}\right]$

$= (n-r+1)\, {}^nC_{r-1}$

PROPERTY V ${}^nC_x = {}^nC_y \Rightarrow x = y$ or, $x + y = n$.

PROOF We have,

$${}^nC_x = {}^nC_y$$

$\Rightarrow {}^nC_x = {}^nC_y = {}^nC_{n-y} \quad [\because {}^nC_y = {}^nC_{n-y}]$

$\Rightarrow \quad x = y \text{ or, } x = n - y$

$\Rightarrow \quad x = y \text{ or, } x + y = n$

REMARK *If ${}^nC_x = {}^nC_y$ and $x \ne y$, then $x + y = n$.*

PROPERTY VI *If n is even, then the greatest value of ${}^nC_r\, (0 \le r \le n)$ is ${}^nC_{n/2}$.*

PROOF We have,

$$\frac{{}^nC_r}{{}^nC_{r+1}}$$

$$= \frac{n!}{(n-r)!\, r!} \times \frac{(r+1)!\,(n-r-1)!}{n!}$$

$$= \frac{r+1}{n-r} \quad \text{...(i)}$$

Now,

$$0 \le r < \frac{n}{2}$$

$$\Rightarrow 0 < r+1 \le \frac{n}{2} \text{ and } \frac{n}{2} < n - r \le n$$

$$\Rightarrow \frac{r+1}{n-r} < 1$$

$$\Rightarrow \frac{{}^nC_r}{{}^nC_{r+1}} < 1 \quad \text{[Using (i)]}$$

$$\Rightarrow {}^nC_r < {}^nC_{r+1}$$

Putting $r = 0, 1, 2, \ldots, \frac{n}{2} - 1$, we get

$${}^nC_0 < {}^nC_1,\ {}^nC_1 < {}^nC_2,\ {}^nC_2 < {}^nC_3 \ldots, {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}}$$

$$\Rightarrow \quad {}^nC_0 < {}^nC_1 < {}^nC_2 < \ldots < {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}} \quad \text{...(ii)}$$

Since, ${}^nC_{n-r} = {}^nC_r$

$\therefore \quad {}^nC_0 = {}^nC_n\ {}^nC_1 = {}^nC_{n-1}$

${}^nC_2 = {}^nC_{n-2}, \ldots, {}^nC_{\frac{n}{2}-1} = {}^nC_{\frac{n}{2}+1}$,

Substituting these values in (ii), we get

$${}^nC_n < {}^nC_{n-1} < {}^nC_{n-2} < \ldots < {}^nC_{\frac{n}{2}+1} < {}^nC_{\frac{n}{2}} \quad \text{...(iii)}$$

From (ii) and (iii), we obtain that the maximum value of ${}^nC_r$ is ${}^nC_{n/2}$.

PROPERTY VII *If n is odd, then the greatest value of* ${}^nC_r$ $(0 \le r \le n)$ *is*

$${}^nC_{\frac{n+1}{2}} \text{ or, } {}^nC_{\frac{n-1}{2}}$$

PROOF From property VI, we have,

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r} \quad \ldots(i)$$

Now,

$$0 \le r < \frac{n-1}{2}$$

$$\Rightarrow 0 < r+1 < \frac{n+1}{2}$$

and,

$$\frac{n+1}{2} \le n-r \le n$$

$$\Rightarrow \frac{r+1}{n-r} < 1$$

$$\Rightarrow \frac{{}^nC_r}{{}^nC_{r+1}} < 1 \qquad \text{[Using (i)]}$$

$$\Rightarrow {}^nC_r < {}^nC_{r+1}$$

Putting $r = 0, 1, 2, \ldots, \left(\frac{n-3}{2}\right)$, we get

$${}^nC_0 < {}^nC_1, {}^nC_1 < {}^nC_2, {}^nC_2 < {}^nC_3, \ldots, {}^nC_{\frac{n-3}{2}} < {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$$

$$\Rightarrow {}^nC_0 < {}^nC_1 < {}^nC_2 < {}^nC_3 < \ldots < {}^nC_{\frac{n-3}{2}} < {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}} \quad \ldots(ii)$$

Since, ${}^nC_{n-r} = {}^nC_r$

$$\therefore {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}, \ldots, {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$$

Substituting these values in (ii), we get

$${}^nC_n < {}^nC_{n-1} < {}^nC_{n-2} < \ldots < {}^nC_{\frac{n+3}{2}} < {}^nC_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} \quad \ldots(iii)$$

From (ii) and (iii), it follows that the maximum value of

$${}^nC_r \text{ is } {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$$

ILLUSTRATION 3 *If* ${}^nC_{15} = {}^nC_8$, *find the value of* ${}^nC_{21}$.

SOLUTION We have,

$${}^nC_{15} = {}^nC_8$$

$$\Rightarrow n = (15+8) = 23 \qquad [{}^nC_x = {}^nC_y \Rightarrow x+y=n]$$

$$\therefore {}^nC_{21} = {}^{23}C_{21}$$

$$= {}^{23}C_{23-21} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{23}C_2$$

$$= \frac{23}{2} \times \frac{22}{1} \times {}^{21}C_0 \qquad \left[\because {}^nC_r = \frac{n}{r} \times \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}\right]$$

$$= \frac{23}{2} \times \frac{22}{1} \times 1 \qquad [\because {}^nC_0 = 1]$$

$$= 23 \times 11 = 253$$

ILLUSTRATION 4 *If* ${}^{10}C_x = {}^{10}C_{x+4}$, *find the value of x.*

SOLUTION We have,

$${}^{10}C_x = {}^{10}C_{x+4}$$

$$\Rightarrow x + x + 4 = 10$$

$$\Rightarrow 2x = 6 \Rightarrow x = 3$$

Following examples will illustrate the applications of the above properties in solving problems.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If* ${}^nC_8 = {}^nC_6$, *find* ${}^nC_2$

SOLUTION If ${}^nC_x = {}^nC_y$ and $x \ne y$, then we have $x + y = n$.

$$\therefore {}^nC_8 = {}^nC_6$$

$$\Rightarrow n = (8+6) = 14.$$

Now, ${}^nC_2 = {}^{14}C_2$

$$= \frac{14}{2} \times \frac{13}{1} \times {}^{12}C_0 \qquad [\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}]$$

$$= \frac{14}{2} \times \frac{13}{1} \times 1 \qquad [\because {}^nC_0 = 1]$$

$$= 91$$

EXAMPLE 2 *If* ${}^nP_r = 720$ and ${}^nC_r = 120$, find r.

SOLUTION We know that

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\Rightarrow 120 = \frac{720}{r!}$$

$$\Rightarrow r! = 6$$

$$\Rightarrow r! = 3! \Rightarrow r = 3.$$

EXAMPLE 3 *If the ratio* ${}^{2n}C_3 : {}^nC_3$ *is equal to 11 : 1, find n.*

SOLUTION We have,

$$^{2n}C_3 : {}^nC_3 = 11 : 1$$

$$\Rightarrow \quad \frac{^{2n}C_3}{{}^nC_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\frac{(2n)!}{(2n-3)!\,3!}}{\frac{n!}{(n-3)!\,(3!)}} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)\,(2n-1)\,(2n-2)\,(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n\,(n-1)\,(n-2)\,(n-3)!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)\,(2n-1)\,(2n-2)}{n\,(n-1)\,(n-2)} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{4\,(2n-1)}{n-2} = \frac{11}{1}$$

$$\Rightarrow \quad 8n - 4 = 11n - 22$$

$$\Rightarrow \quad 3n = 18 \Rightarrow n = 6.$$

EXAMPLE 4 *Prove that the product of r consecutive positive integers is divisible by r!.*

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), (n+3), \ldots, (n+r)$. Then,

Their product

$$= (n+1)\,(n+2)\,(n+3) \ldots (n+r)$$

$$= \frac{n!\,(n+1)\,(n+2)\,(n+3) \ldots (n+r)}{n!}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot \ldots n \cdot (n+1)\,(n+2) \ldots (n+r)}{n!}$$

$$= \frac{(n+r)!}{n!}$$

$$= \frac{(n+r)!}{r!\,[(n+r)-r]!}\,(r!)$$

$= (^{n+r}C_r)\,r!$, which is divisible by $r!$ $\quad [\because {}^{n+r}C_r$ is an integer$]$

EXAMPLE 5 *Prove that:* $^{2n}C_n = \dfrac{2^n\,[1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)]}{n!}$.

SOLUTION We have,

$$^{2n}C_n$$

$$= \frac{2n!}{(2n-n)!\,n!}$$

$$= \frac{(2n)!}{n!\,n!}$$

$$= \frac{(2n)\,(2n-1)\,(2n-2) \ldots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)]\,[2 \cdot 4 \cdot 6 \cdot \ldots 2n]}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)] \times 2^n\,[1 \cdot 2 \cdot 3 \cdot \ldots n]}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)] \times 2^n \times n!}{n!\,n!}$$

$$= 2^n\,\frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)]}{n!}$$

EXAMPLE 6 *If* $^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, find n.

SOLUTION We have,

$$^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$$

$$\Rightarrow \quad \frac{^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)\,(n+1)\,n\,(n-1)\,(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad (n+2)\,(n+1)\,n\,(n-1) = \frac{57}{16} \times 8! = 143640$$

$$\Rightarrow \quad (n+2)\,(n+1)\,(n-1)\,n = 143640$$

$$\Rightarrow \quad (n+2)\,(n-1)\,(n+1)\,n = 143640$$

$$\Rightarrow \quad (n^2+n-2)\,(n^2+n) = 143640$$

$$\Rightarrow \quad (n^2+n)^2 - 2\,(n^2+n) = 143640$$

$$\Rightarrow \quad (n^2+n)^2 - 2\,(n^2+n) + 1 = 143641$$

$$\Rightarrow \quad (n^2+n)^2 - 2\,(n^2+n) + 1 = (379)^2$$

$$\Rightarrow \quad (n^2+n-1)^2 = (379)^2$$

$$\Rightarrow \quad n^2+n-1 = 379 \qquad [\because n^2+n-1 \geq 0 \text{ for all } n \in N]$$

$$\Rightarrow \quad n^2+n-380 = 0$$

$$\Rightarrow \quad (n-19)\,(n+20) = 0$$

$$\Rightarrow \quad n = 19 \qquad [\because n \in N]$$

EXAMPLE 7 *Show that for* $n \geq r$

$$^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^nC_r = {}^{n+1}C_{r+1}$$

i.e., $\quad \sum_{k=r}^{n} {}^kC_r = {}^{n+1}C_{r+1}$

SOLUTION We have,

$$\sum_{k=r}^{n} {}^kC_r$$

$= {}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{n}C_r$

$= {}^{r+1}C_{r+1} + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{n}C_r \quad [\because {}^{r}C_r = {}^{r+1}C_{r+1} = 1]$

$= ({}^{r+1}C_{r+1} + {}^{r+1}C_r) + {}^{r+2}C_r + \ldots + {}^{n}C_r$

$= {}^{r+2}C_{r+1} + {}^{r+2}C_r + \ldots + {}^{n}C_r \quad [\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r]$

$= ({}^{r+2}C_{r+1} + {}^{r+2}C_r) + \ldots + {}^{n}C_r$

$= {}^{r+3}C_{r+1} + \ldots + {}^{n}C_r$

.....

$= {}^{n}C_{r+1} + {}^{n}C_r$

$= {}^{n+1}C_{r+1}$

EXAMPLE 8 *Prove that* $\sum_{k=m}^{n} {}^{k}C_r = {}^{n+1}C_{r+1} - {}^{m}C_{r+1}$

SOLUTION We have,

$\sum_{k=m}^{n} {}^{k}C_r$

$= {}^{m}C_r + {}^{m+1}C_r + {}^{m+2}C_r + \ldots + {}^{n}C_r$

$= ({}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{m-1}C_r + {}^{m}C_r + {}^{m+1}C_r \ldots + {}^{n}C_r) - ({}^{r}C_r + {}^{r+1}C_r + \ldots + {}^{m-1}C_r)$

$= \sum_{k=r}^{n} {}^{k}C_r - \sum_{k=r}^{m-1} {}^{k}C_r$

$= {}^{n+1}C_{r+1} - {}^{(m-1)+1}C_{r+1} \quad [\text{From Ex. 6, } \sum_{k=r}^{n} {}^{k}C_r = {}^{n+1}C_{r+1}]$

$= {}^{n+1}C_{r+1} - {}^{m}C_{r+1}$

EXAMPLE 9 *If* ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, *then find* ${}^{r}C_2$.

SOLUTION We know that,

$$\frac{{}^{n}C_r}{{}^{n}C_{r+1}} = \frac{r+1}{n-r}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3}$$

$$\Rightarrow \quad 2n - 5r = 3 \qquad \ldots(i)$$

Replacing r by $(r-1)$ in $\frac{{}^{n}C_r}{{}^{n}C_{r+1}}$, we get

$$\frac{{}^{n}C_{r-1}}{{}^{n}C_r} = \frac{r}{n-(r-1)}$$

$$\therefore \quad \frac{{}^{n}C_{r-1}}{{}^{n}C_r} = \frac{36}{84}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{36}{84}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow \quad 3n - 10r = -3 \qquad \ldots(ii)$$

Solving (i) and (ii), we get r = 3.

$$\therefore \quad {}^{r}C_2 = {}^{3}C_2 = \frac{3!}{(3-2)!\,2!} = 3.$$

EXAMPLE 10 *If* ${}^{n}P_r = {}^{n}P_{r+1}$ and ${}^{n}C_r = {}^{n}C_{r-1}$, *find the values of n and r.*

SOLUTION: We have,

$${}^{n}P_r = {}^{n}P_{r+1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow \quad n - r = 1 \qquad \ldots(i)$$

And,

$${}^{n}C_r = {}^{n}C_{r-1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r\,(r-1)!} = \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{n-r+1}$$

$$\Rightarrow \quad n - r + 1 = r$$

$$\Rightarrow \quad n - 2r = -1 \qquad \ldots(ii)$$

Solving (i) and (ii), we obtain $n = 3$ and $r = 2$.

EXAMPLE 11 *If* ${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 11 : 6 : 3$, *find the values of n and r.*

SOLUTION We have,

$$\frac{{}^{n+1}C_{r+1}}{{}^{n}C_r} = \frac{11}{6} \text{ and, } \frac{{}^{n}C_r}{{}^{n-1}C_{r-1}} = \frac{6}{3}$$

$$\Rightarrow \quad \frac{\frac{n+1}{r+1} \cdot {}^{n}C_r}{{}^{n}C_r} = \frac{11}{6}$$

$$\text{and, } \frac{n}{r} \cdot \frac{{}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3} \qquad \left[\because {}^{n}C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad \frac{n+1}{r+1} = \frac{11}{6} \quad \text{and,} \quad \frac{n}{r} = \frac{6}{3}$$

$$\Rightarrow \quad 6n - 11r = 5 \quad \text{and,} \quad n = 2r$$

$$\Rightarrow \quad n = 10 \quad \text{and,} \quad r = 5$$

EXAMPLE 12 *If* ${}^nP_{r+1} : {}^nP_r : {}^nP_{r-1} = a : b : c$, *prove that* $b^2 = c(a+b)$

SOLUTION We have,

$$\frac{{}^nP_{r+1}}{{}^nP_r} = \frac{a}{b} \text{ and, } \frac{{}^nP_r}{{}^nP_{r-1}} = \frac{b}{c}$$

$$\Rightarrow n - r = \frac{a}{b} \text{ and, } n - r + 1 = \frac{b}{c} \qquad \left[\because \frac{{}^nP_r}{{}^nP_{r-1}} = n - r + 1\right]$$

$$\Rightarrow \frac{a}{b} + 1 = \frac{b}{c}$$

$$\Rightarrow (a+b)\,c = b^2$$

$$\Rightarrow b^2 = c(a+b)$$

EXAMPLE 13 *If*

${}^nC_{n-r} + 3 \cdot {}^nC_{n-r+1} + 3 \cdot {}^nC_{n-r+2} + {}^nC_{n-r+3} = {}^xC_r$, *find the value of* x.

SOLUTION We have,

$${}^nC_{n-r} + 3 \cdot {}^nC_{n-r+1} + 3 \cdot {}^nC_{n-r+2} + {}^nC_{n-r+3} = {}^xC_r$$

$$\Rightarrow \left({}^nC_{n-r} + {}^nC_{n-r+1}\right) + 2\left({}^nC_{n-r+1} + {}^nC_{n-r+2}\right) + \left({}^nC_{n-r+2} + {}^nC_{n-r+3}\right) = {}^xC_r$$

$$\Rightarrow {}^{n+1}C_{n-r+1} + 2\,{}^{n+1}C_{n-r+2} + {}^{n+1}C_{n-r+3} = {}^xC_r \qquad \left[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\right]$$

$$\Rightarrow \left\{{}^{n+1}C_{n-r+1} + {}^{n+1}C_{n-r+2}\right\} + \left\{{}^{n+1}C_{n-r+2} + {}^{n+1}C_{n-r+3}\right\} = {}^xC_r$$

$$\Rightarrow {}^{n+2}C_{n-r+2} + {}^{n+2}C_{n-r+3} = {}^xC_r$$

$$\Rightarrow {}^{n+3}C_{n-r+3} = {}^xC_r$$

$$\Rightarrow {}^{n+3}C_r = {}^xC_r \qquad \left[\because {}^{n+3}C_{n-r+3} = {}^{n+3}C_r\right]$$

$$\Rightarrow \quad x = n+3$$

EXAMPLE 14 *Evaluate :*

$$(1^2+1)\,1! + (2^2+1)\,2! + (3^2+1)\,3! + \ldots + (n^2+1)\,n!$$

SOLUTION We have,

$$(1^2+1)\,1! + (2^2+1)\,2! + (3^2+1)\,3! + \ldots + (n^2+1)\,n!$$

$$= \sum_{r=1}^{n} (r^2+1)\,r!$$

$$= \sum_{r=1}^{n} \{(r+2)(r+1) - 3(r+1) + 2\}\,r! \qquad [\because r^2 + 1 = (r+2)(r+1) - 3(r+1) + 2]$$

$$= \sum_{r=1}^{n} \{(r+2)(r+1)\,r! - 3(r+1)\,r! + 2r!\}$$

$$= \sum_{r=1}^{n} \{(r+2)! - 3(r+1)! + 2r!\}$$

$$= \sum_{r=1}^{n} \{(r+2)! - (r+1)!\} - 2\sum_{r=1}^{n} \{(r+1)! - r!\}$$

$$= [(3! - 2!) + (4! - 3!) + (5! - 4!) + \ldots + \{(n+2)! - (n+1)!\}] - 2\,[(2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)]$$

$$= [(n+2)! - 2!] - 2\,[(n+1)! - 1!]$$

$$= (n+2)! - 2(n+1)!$$

$$= (n+2)(n+1)! - 2(n+1)!$$

$$= (n+2-2)(n+1)!$$

$$= n(n+1)!$$

EXAMPLE 15 *Solve :*

$${}^{18}C_r + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_{r-2} > {}^{20}C_{13}$$

SOLUTION We have,

$${}^{18}C_r + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_{r-2} > {}^{20}C_{13}$$

$$\Rightarrow ({}^{18}C_r + {}^{18}C_{r-1}) + ({}^{18}C_{r-1} + {}^{18}C_{r-2}) > {}^{20}C_{13}$$

$$\Rightarrow {}^{19}C_r + {}^{19}C_{r-1} > {}^{20}C_{13}$$

$$\Rightarrow {}^{20}C_r > {}^{20}C_{13}$$

But, ${}^{20}C_{13} = {}^{20}C_7$. Therefore,

$${}^{20}C_r > {}^{20}C_{13} \text{ and } {}^{20}C_r > {}^{20}C_7$$

We know that ${}^{20}C_r$ for varying values of r, is greatest for $r = 10$.

Also, ${}^{20}C_r = {}^{20}C_{20-r}$. Therefore,

$${}^{20}C_r > {}^{20}C_{13} \text{ and } {}^{20}C_r > {}^{20}C_7$$

$\Rightarrow r = 8, 9, 10, 11, 12$

EXAMPLE 16 *Show that* $(kn)!$ *is divisible by* $(n!)^k$.

SOLUTION We have,

$(kn)!$

$= 1 \cdot 2 \cdot 3 \ldots\ldots (kn)$

$= (1 \cdot 2 \cdot 3 \ldots\ldots n)$

$\times \{(n+1)(n+2) \ldots (n+n)\}$

$\times \{(2n+1)(2n+2) \ldots (2n+n)\}$

.....

$\times \{(k-1)n+1\}\{(k-1)n+2\}$

$\times \{(k-1)n+3\}\{(k-1)n+n\}]$

$$= \prod_{r=0}^{k-1} (rn+1)(rn+2)(rn+3) \ldots (rn+n)$$

$$= \prod_{r=0}^{k-1} \frac{(rn)!\,(rn+1)(rn+2) \ldots (rn+n)}{(rn)!}$$

$$= \prod_{r=0}^{k-1} \frac{(rn+n)!}{(rn)!}$$

$$= \prod_{r=0}^{k-1} \frac{(rn+n)!}{(rn)!\,n!}\, n!$$

$$= (n!)^k \prod_{r=0}^{k-1} {}^{rn+n}C_n$$

$= (n!)^k \times$ a natural number

Hence, $(kn)!$ is divisible by $(n!)^k$.

EXAMPLE 17 *Solve :*

$${}^{n-1}C_4 - {}^{n-1}C_3 - \frac{5}{4}\cdot {}^{n-2}P_2 < 0, \text{ where } n \in N$$

SOLUTION We have,

$${}^{n-1}C_4 - {}^{n-1}C_3 - \frac{5}{4}\, {}^{n-2}P_2 < 0$$

$$\Rightarrow \frac{(n-1)(n-2)(n-3)(n-4)}{4!} - \frac{(n-1)(n-2)(n-3)}{3!} - \frac{5}{4}(n-2)(n-3) < 0$$

$$\Rightarrow \frac{(n-2)(n-3)}{24}\{(n-1)(n-4) - 4(n-1) - 30\} < 0$$

$$\Rightarrow (n-2)(n-3)(n^2-9n-22) < 0$$

$$\Rightarrow (n-2)(n-3)(n-11)(n+2) < 0$$

$$\Rightarrow (n-2)(n-3)(n-11) < 0 \qquad [\because\ n+2>0 \text{ for } n \in N]$$

$$\Rightarrow n \in (-\infty, 2) \cup (3, 11)$$



Fig. 22.1

$$\Rightarrow n \in (0, 2) \cup (3, 11) \qquad [\because\ n \in N]$$

$$\Rightarrow n = 1, 4, 5, 6, 7, 8, 9, 10$$

But, ${}^{n-1}C_4$ and ${}^{n-2}P_2$ both are meaningful for $n \geq 5$.

Hence, $n = 5, 6, 7, 8, 9, 10$.

EXAMPLE 18 *Show that two positive integers n and r cannot be found such that* ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}, {}^nC_{r+3}$ *are in A.P.*

SOLUTION If possible, let there be positive integers n and r such that ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}$ and ${}^nC_{r+3}$ are in A.P.

Now,

${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}, {}^nC_{r+3}$ are in A.P.

$$\Rightarrow 2\,{}^nC_{r+1} = {}^nC_r + {}^nC_{r+2} \qquad ...(i)$$

and,

$$2\,{}^nC_{r+2} = {}^nC_{r+1} + {}^nC_{r+3} \qquad ...(ii)$$

Now,

$$2\,{}^nC_{r+1} = {}^nC_r + {}^nC_{r+2}$$

$$\Rightarrow 2\frac{n!}{(n-r-1)!\,(r+1)!} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r-2)!\,(r+2)!}$$

$$\Rightarrow \frac{2}{(n-r-1)} = \frac{1}{(n-r)(n-r-1)} + \frac{1}{(r+2)(r+1)}$$

$$\Rightarrow 2(n-r)(r+2) = (r+2)(r+1) + (n-r-1)(n-r)$$

$$\Rightarrow (n-r)^2 - (2r+5)(n-r) + (r^2+3r+2) = 0 \qquad ...(iii)$$

and,

$$2\,.\,{}^nC_{r+2} = {}^nC_{r+1} + {}^nC_{r+3}$$

$$\Rightarrow (n-r-1)^2 - (2r+7)(n-r-1) + \{(r+1)^2 - 3(r+1) + 2\} = 0$$

[Replacing r by $r+1$ in (iii)]

$$\Rightarrow (n-r)^2 - (2r+9)(n-r) + (r^2+7r+14) = 0 \qquad ...(iv)$$

If the values of n and r exist, then (iii) and (iv) must be identical quadratic equations in $(n-r)$. Therefore,

$$2r+5 = 2r+9 \text{ and } r^2+3r+2 = r^2+7r+14$$

These two equations are inconsistent. Therefore, equations (i) and (ii) cannot hold good simultaneously.

Hence, ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}, {}^nC_{r+3}$ cannot be in A.P.

EXAMPLE 19 *If* ${}^nP_r = {}^nP_{r+1}$ *and* ${}^nC_r = {}^nC_{r-1}$, *find the values of* n and r.

SOLUTION We have,

$${}^nP_r = {}^nP_{r+1}$$

$$\Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow (n-r)! = (n-r-1)!$$

$$\Rightarrow (n-r)(n-r-1)! = (n-r-1)!$$

$$\Rightarrow (n-r-1)(n-r-1)! = 0$$

$$\Rightarrow n-r-1 = 0$$

$$\Rightarrow n = r+1 \qquad ...(i)$$

and,

$${}^nC_r = {}^nC_{r-1}$$

$$\Rightarrow r+r-1 = n \qquad [\because\ {}^nC_x = {}^nC_y \Rightarrow x+y=n]$$

$$\Rightarrow n = 2r-1 \qquad ...(ii)$$

Solving (i) and (ii), we have

$$n = 3 \text{ and } r = 2.$$

EXAMPLE 20 *If* ${}^{m+n}P_2 = 90$ *and* ${}^{m-n}P_2 = 30$, *find the values of* m *and* n.

SOLUTION We have,

$${}^{m+n}P_2 = 90$$

$$\Rightarrow \frac{(m+n)!}{(m+n-2)!} = 90$$

$$\Rightarrow \frac{(m+n)(m+n-1)(m+n-2)!}{(m+n-2)!} = 90$$

$$\Rightarrow (m+n)(m+n-1) = 90$$

$$\Rightarrow (m+n)^2 - (m+n) - 90 = 0$$

$$\Rightarrow (m+n-10)(m+n+9) = 0$$

$$\Rightarrow m+n = 10 \qquad \text{...(i)} \quad [\because m, n \in N \therefore m+n+9 \neq 0]$$

and,

$${}^{m-n}P_2 = 30$$

$$\Rightarrow \frac{(m-n)!}{(m-n-2)!} = 30$$

$$\Rightarrow (m-n)(m-n-1) = 30$$

$$\Rightarrow (m-n)^2 - (m-n) - 30 = 0$$

$$\Rightarrow (m-n-6)(m-n+5) = 0$$

$$\Rightarrow m-n = 6 \qquad \text{...(ii)} \quad [\because m-n>0 \therefore m-n+5 \neq 0]$$

Solving (i) and (ii), we get

$$m = 8 \text{ and } n = 2.$$

EXAMPLE 21 *Find the values of natural number n for which*

$$4^{n+3}P_3 \, {}^nP_n - 195^{n+1}P_{n+1} < 0.$$

SOLUTION We have,

$$4^{n+3}P_3 \, {}^nP_n - 195^{n+1}P_{n+1} < 0$$

$$\Rightarrow 4 \times \frac{(n+3)!}{n!} \times \frac{n!}{0!} - 195 \frac{(n+1)!}{0!} < 0$$

$$\Rightarrow 4(n+3)(n+2)(n+1)! - 195(n+1)! < 0$$

$$\Rightarrow \{4(n+3)(n+2) - 195\}(n+1)! < 0$$

$$\Rightarrow 4n^2 + 20n - 171 < 0$$

Clearly, this inequality holds true for $n = 1, 2, 3, 4$.

EXAMPLE 22 *Evaluate* $\frac{{}^{20}C_n}{{}^{25}C_m}$ *when both numerator and denominator have their greatest values.*

SOLUTION We know that

$$\text{Greatest value of } {}^nC_r = \begin{cases} {}^nC_{n/2}, & \text{if } n \text{ is even} \\ {}^nC_{(n+1)/2}, & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ ${}^{20}C_n$ is greatest for $n = \frac{20}{2} = 10$

and,

${}^{25}C_m$ is greatest for $m = \frac{25+1}{2} = 13$

For $n = 10$ and $m = 13$, we have

$$\frac{{}^{20}C_n}{{}^{25}C_m} = \frac{{}^{20}C_{10}}{{}^{25}C_{13}} = \frac{20!}{10! \times 10!} \times \frac{12! \times 13!}{25!} = \frac{143}{4025}$$

EXAMPLE 23 *Prove that the product of 2n consecutive negative integers is divisible by (2n)!.*

SOLUTION Let r be a natural number. Then, $-r, -r-1, -r-2, \ldots, -r-(2n-1)$ are $2n$ consecutive negative integers. Let P be their product. Then,

$$P = (-r)(-r-1)(-r-2)\ldots(-r-(2n-1))$$

$$= (-1)^{2n} r(r+1)(r+2)\ldots(r+2n-1)$$

$$= r(r+1)(r+2)\ldots(r+2n-1)$$

$$= \frac{(r-1)!\, r(r+1)(r+2)\ldots(r+2n-1)}{(r-1)!}$$

$$= \frac{(r+2n-1)!}{(r-1)!}$$

$$= \left\{\frac{(2n+r-1)!}{(2n)!\,(r-1)!}\right\}(2n)!$$

$$= (2n)!\, {}^{2n+r-1}C_{2n}$$

$$= (2n)! \text{ (a natural number)}$$

Hence, P is divisible by $(2n)!$.

EXAMPLE 24 *Evaluate :*

$${}^{35}C_8 + \sum_{r=1}^{7} {}^{42-r}C_7 + \sum_{s=1}^{5} {}^{47-s}C_{40-s}$$

SOLUTION We have,

$${}^{35}C_8 + \sum_{r=1}^{7} {}^{42-r}C_7 + \sum_{s=1}^{5} {}^{47-s}C_{40-s}$$

$$= {}^{35}C_8 + \left\{{}^{41}C_7 + {}^{40}C_7 + {}^{39}C_7 + {}^{38}C_7 + \ldots + {}^{35}C_7\right\}$$

$$+ \left\{{}^{46}C_{39} + {}^{45}C_{38} + \ldots + {}^{42}C_{35}\right\}$$

$$= {}^{35}C_8 + \left\{{}^{35}C_7 + {}^{36}C_7 + \ldots + {}^{41}C_7\right\}$$

$$+ \left\{{}^{42}C_7 + {}^{43}C_7 + \ldots + {}^{46}C_7\right\} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= \left({}^{35}C_8 + {}^{35}C_7\right) + \left({}^{36}C_7 + \ldots + {}^{41}C_7 + \ldots + {}^{46}C_7\right)$$

$$= \left({}^{36}C_8 + {}^{36}C_7\right) + {}^{37}C_7 + \ldots + {}^{46}C_7$$

$$= {}^{37}C_8 + {}^{37}C_7 + {}^{38}C_7 + \ldots + {}^{46}C_7$$

$$= \ldots\ldots\ldots\ldots$$

$$= {}^{46}C_8 + {}^{46}C_7 = {}^{47}C_8$$

EXAMPLE 25 *Evaluate :*

$$\left\{{}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{n-1} + {}^{2n+1}C_n\right\}$$

$$- \left\{{}^{2n+1}C_{2n+1} + {}^{2n+1}C_{2n} + \ldots + {}^{2n+1}C_{n+1}\right\}$$

SOLUTION We have,

$$\left\{{}^{2n+1}C_1+{}^{2n+1}C_2+\ldots+{}^{2n+1}C_n\right\}$$
$$-\left\{{}^{2n+1}C_{2n+1}+{}^{2n+1}C_{2n}+\ldots+{}^{2n+1}C_{n+1}\right\}$$
$$=\left\{{}^{2n+1}C_1+{}^{2n+1}C_2+\ldots+{}^{2n+1}C_n\right\}$$
$$-\left\{{}^{2n+1}C_0+{}^{2n+1}C_1+\ldots+{}^{2n+1}C_n\right\} \qquad \left[\because {}^nC_r={}^nC_{n-r}\right]$$
$$=-{}^{2n+1}C_0$$
$$=-1.$$

EXAMPLE 26 *Prove that* ${}^{2n}C_n < \frac{4^n}{\sqrt{2n+1}}$ *for all* $n \in N$.

SOLUTION We have,

$${}^{2n}C_n$$
$$=\frac{(2n)!}{n!\,n!}$$
$$=\frac{2^n\{1.3.5\ldots(2n-1)\}}{n!}$$
$$=\frac{2^{2n}\{1.3.5\ldots(2n-1)\}}{2.4.6\ldots2n} \qquad \ldots(i)$$

Let $P=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$

Then,

$$P=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}$$

$$\Rightarrow P<\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n}{2n+1}$$

$$\left[\because \frac{n}{n+1}<\frac{n+1}{n+2} \text{ for all } n\in N\right]$$

$$\Rightarrow P<\left\{\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\right\}\frac{1}{2n+1}$$

$$\Rightarrow P<\frac{1}{P}\cdot\frac{1}{2n+1}$$

$$\Rightarrow P^2<\frac{1}{2n+1}$$

$$\Rightarrow P<\frac{1}{\sqrt{2n+1}}$$

$$\Rightarrow \frac{1.3.5\ldots2n-1}{2.4.6\ldots2n}<\frac{1}{\sqrt{2n+1}} \qquad \ldots(ii)$$

From (i) and (ii), we obtain

$${}^{2n}C_n<\frac{4^n}{\sqrt{2n+1}}$$

EXAMPLE 27 *Show that there cannot exist two positive integers n and r for which* ${}^nC_r, {}^nC_{r+1}$ *and* ${}^nC_{r+2}$ *are in H.P.*

SOLUTION If possible, let there be two positive integers n and r such that ${}^nC_r, {}^nC_{r+1}$ and ${}^nC_{r+2}$ are in H.P. Then,

$$\frac{1}{{}^nC_r}, \frac{1}{{}^nC_{r+1}}, \frac{1}{{}^nC_{r+2}} \text{ are in A.P.}$$

$$\Rightarrow \frac{2}{{}^nC_{r+1}}=\frac{1}{{}^nC_r}+\frac{1}{{}^nC_{r+2}}$$

$$\Rightarrow 2=\frac{{}^nC_{r+1}}{{}^nC_r}+\frac{{}^nC_{r+1}}{{}^nC_{r+2}}$$

$$\Rightarrow 2=\frac{n-r}{r+1}+\frac{r+2}{n-r-1} \qquad \left[\because \frac{{}^nC_{r+1}}{{}^nC_r}=\frac{n-r}{r+1}\right]$$

$$\Rightarrow 2(r+1)(n-r-1)=(n-r)(n-r-1)+(r+1)(r+2)$$

$$\Rightarrow n^2-4nr+4r^2+8r-3n+4=0$$

$$\Rightarrow (n-2r-2)^2+n=0$$

But, this is impossible as $(n-2r-2)^2\geq 0$ and $n>0$.

Hence, ${}^nC_r, {}^nC_{r+1}$ and ${}^nC_{r+2}$ cannot be in H.P. for any $n, r \in N$.

EXAMPLE 28 *Prove that* $(n!)!$ *is divisible by* $(n!)^{(n-1)!}$.

SOLUTION Clearly, $(n!)!$ is the product of natural numbers from 1 to $n!$.

$$\therefore (n!)! = \{1\times2\times3\ldots\times n\}$$
$$\times\{(n+1)(n+2)\ldots(2n)\}$$
$$\times\{(2n+1)(2n+2)\ldots(3n)\}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\times\{(n!-n+1)(n!-n+2)(n!-n+3)\ldots n!\} \qquad \ldots(i)$$

We observe that :

Last term of first bracket on RHS of (i) is n

Last term of second bracket on RHS of (i) is $2\,.\,n$

Last term of third bracket on RHS of (i) is $3\,.\,n$

and so on.

Last term of the last bracket on RHS of (i) is $(n-1)!\,n$

It is clear from this that there are $(n-1)!$ brackets on the RHS of (i). and in each bracket there is product of n consecutive natural numbers.

From (i), we have

$$(n!)!$$
$$=\prod_{r=1}^{(n-1)!}[\{(r-1)n+1\}\{(r-1)n+2\}\{(r-1)n+3\}\ldots\{r-1)n+n\}]$$

We know that the product of n consecutive natural numbers is divisible by $n!$. So, let

$\{(r-1)n+1\}\{(r-1)n+2\}\ldots\{(r-1)n+n\} = n!\,M_r,$

$r = 1, 2, \ldots (n-1)!$

$\therefore\ (n!)! = \prod_{r=1}^{(n-1)!} n!\,M_r$

$= (n!)^{(n-1)!} \cdot \prod_{r=1}^{(n-1)!} M_r$

$= (n!)^{(n-1)!} \times a$ natural number

Hence, $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

EXERCISE 22.2

1. If $P(5, r) = P(6, r-1)$, find r.
2. If $5\,P(4, n) = 6.\,P(5, n-1)$, find n.
3. If $P(n, 5) = 20\,.\,P(n, 3)$, find n
4. If ${}^nP_4 = 360$, find the value of n.
5. If $P(9, r) = 3024$, find r.
6. If $P(11, r) = P(12, r-1)$ find r.
7. If $P(n, 4) = 12\,.\,P(n, 2)$, find n.
8. If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n.
9. If $P(2n-1, n) : P(2n+1, n-1) = 22 : 7$ find n.
10. If $P(n, 5) : P(n, 3) = 2 : 1$, find n.
11. Prove that

$1\,.\,P(1, 1) + 2\,.\,P(2, 2) + 3\,.\,P(3, 3) + \ldots + n\,.\,P(n, n)$

$= P(n+1, n+1) - 1.$

12. If $P(15, r-1) : P(16, r-2) = 3 : 4$, find r.
13. If ${}^{n+5}P_{n+1} = \frac{11(n-1)}{2}\,{}^{n+3}P_n$, find n.
14. If ${}^nC_{12} = {}^nC_5$, find the value of n.
15. If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.
16. If ${}^nC_{10} = {}^nC_{12}$, find ${}^{23}C_n$.
17. If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.
18. If ${}^{18}C_x = {}^{18}C_{x+2}$, find x.
19. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
20. If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.
21. If ${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$, find r.
22. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, find n.
23. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.
24. Prove that

${}^{4n}C_{2n} : {}^{2n}C_n = [1 \cdot 3 \cdot 5 \ldots (4n-1)] : [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2$

25. If ${}^{2n}C_3 : {}^nC_2 = 44 : 3$, find n
26. If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.
27. Evaluate ${}^{20}C_5 + \sum_{r=2}^{5} {}^{25-r}C_4$.
28. Prove that the product of $2n$ consecutive negative integers is divisible by $(2n)!$
29. Prove that the greatest value of ${}^{2n}C_r$ for $0 \le r \le 2n$ is ${}^{2n}C_n$.
30. Prove that

${}^{r-1}C_{r-1} + {}^rC_{r-1} + {}^{r+1}C_{r-1} + \ldots + {}^{n+r-2}C_{r-1} = {}^{n+r-1}C_r.$

31. If n is even, then prove that ${}^nC_0 < {}^nC_1 < {}^nC_2 < \ldots < {}^nC_{n/2}$.
32. For all positive integers n, show that

$${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}({}^{2n+2}C_{n+1}).$$

33. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then find n.
34. If $\alpha = {}^mC_2$, then find the value of ${}^\alpha C_2$.
35. Let r and n be positive integers such that $1 \le r \le n$. Then, prove the following :

(i) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ (ii) $n \cdot {}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$

(iii) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$ (iv) ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r.$

36. Prove that : $\sum_{k=m}^{n} {}^kC_r = {}^{n+1}C_{r+1} - {}^mC_{r+1}$.
37. Evaluate : ${}^nC_k + \sum_{r=0}^{m} {}^{n+r}C_{k-1}$.
38. Prove that : ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$, if $n > 7$
39. Prove that : ${}^{2n}P_n = 2^n(1 \cdot 3 \cdot 5 \ldots (2n-1))$.
40. Prove that there cannot exist two positive integers n and r such that ${}^nC_r$, ${}^nC_{r+1}$, ${}^nC_{r+2}$ are in G.P.
41. If $n\,(>1)$ and r are positive integers such that ${}^{3n}C_r = {}^{3n}C_{n+r}$, then prove that each of them is equal to

$$\frac{(3n)(3n-1)(3n-2)\ldots(2n+1)}{n!}$$

42. Prove that the greatest value of ${}^{2n}C_r$ is double the greatest value of ${}^{2n-1}C_r$

43. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, find ${}^rC_2$.

44. Show that $(kn)!$ is divisible by $(n!)^k$.

ANSWERS

1. 4	2. 3	3. 8	4. 6	22. 19	23. 7	25. 6
5. 4	6. 9	7. 6	8. 9			26. 35
9. 10	10. 5	12. 4	13. 6, 7	27. 42504	33. 14, 7	34. $\frac{(m+1)(m)(m-1)(m-2)}{8}$
14. 17	15. 66	16. 23	17. 7	37. ${}^{n+m+1}C_k$		43. 3
18. 8	19 3	20. 3	21. 5			

HINTS TO SELECTED PROBLEMS

28. Let $(-r), (-r-1), (-r-2), \ldots, (-r-2n+1)$ be $2n$ consecutive negative integers. Then,

$$\text{Their product} = (-1)^{2n} r(r+1)(r+2)\ldots(r+2n-1)$$
$$= \frac{(r-1)!\,(r)(r+1)\ldots(r+2n-1)}{(r-1)!}$$
$$= \frac{(r+2n-1)!}{(r-1)!} = \frac{(r+2n-1)!}{(r-1)!\,(2n)!}(2n)!$$
$$= \{{}^{r+2n-1}C_{2n}\}(2n)!$$

—❑❑❑—

22.3 FUNDAMENTAL PRINCIPLES OF COUNTING

In this section, we shall discuss two fundamental principles viz. principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations. In fact these two principles are the building blocks of permutations and combinations.

FUNDAMENTAL PRINCIPLE OF MULTIPLICATION

If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

ILLUSTRATION 1 *In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection ?*

SOLUTION Here, the teacher is to perform two jobs:

(i) selecting a boy among 10 boys,

and, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

REMARK *The above principle can be extended for any finite number of jobs as stated below :*

If there are n jobs $J_1, J_2, \ldots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, \ldots, n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.

FUNDAMENTAL PRINCIPLE OF ADDITION

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

ILLUSTRATION 2 *In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection ?*

SOLUTION Here, the teacher is to perform either of the following two jobs :

(i) selecting a boy among 10 boys.

or, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in $(10 + 8) = 18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

Difference between the two principles : As we have discussed in the principle of multiplication a job is divided or decomposed into a number of sub-jobs which are uncorrected to each other and the job is said to be performed if each sub-job is performed. While in the principle of addition there are a number of independent jobs and we have to perform one of them. So, the total number of ways of completing any one of the sub-jobs is the sum of the number of ways of completing each sub-job.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural science scholarship.*

(i) In how many ways can these scholarships be awarded ?

(ii) In how many ways one of these scholarships be awarded ?

SOLUTION Clearly, Classical scholarship can be awarded to any one of the three candidates. So, there are 3 ways of awarding the Classical scholarship.

Similarly, Mathematical and Natural science scholarships can be awarded in 5 and 4 ways respectively. So,

Number of ways of awarding three scholarships

$= 3 \times 5 \times 4$

$= 60$ [By Fundamental Principle of multiplication]

And,

Number of way of awarding one of the three scholarships

$= 3 + 5 + 4 = 12$

EXAMPLE 2 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin (i) at any one of the 7 floors (ii) at different floors.*

SOLUTION Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is

$$7 \times 6 \times 5 \times 4 \times 3 = 2520.$$

EXAMPLE 3 *Find the number of ways in which n distinct balls can be put into three boxes.*

SOLUTION Here, the job of putting n differnt balls in three boxes is divided into n sub-jobs as given below :

J_1 : Putting first ball in one of the three boxes.

J_2 : Putting second ball in one of the three boxes.

⋮ ⋮ ⋮ ⋮

J_n : Putting nth ball in one of the three boxes.

We observe that each sub-job can be completed in three ways. So, by the fundamental principle of counting the total number of ways of putting n balls in three different boxes is

$$\underbrace{3 \times 3 \times 3 \times \ldots \times 3}_{n\text{-times}} = 3^n$$

EXAMPLE 4 *Find the number of ways in which n distinct balls can be put into three boxes so that no two boxes remain empty.*

SOLUTION In the above example, we have seen that n distinct balls can be put into three boxes in 3^n ways. Out of these 3^n ways, there are three ways

(i) When all balls are put in first box

(ii) When all balls are put in second box

(iii) When all balls are put in third box.

Thus, there are $3^n - 3$ ways in which no two of the three boxes are empty.

EXAMPLE 5 *Find the total number of ways of answering 5 objective type questions, each question having 4 choices.*

SOLUTION Since each question can be answered in 4 ways. So, the total number of ways of answering 5 questions

$$= 4 \times 4 \times 4 \times 4 \times 4 = 4^5.$$

EXAMPLE 6 *For a set of five true/false questions, no student has written all correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible ?*

SOLUTION Since a true/false type question can be answered in 2 ways either by marking it true or false. So, there are 2 ways of answering each of the 5 questions. So,

Total number of different sequences of answers

$$= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

Out of these 32 sequences of answers there is only one sequence of answering all the five questions correctly. But no student has written all the correct answers and different students have given different sequences of answers. So,

Maximum number of students in the class

= Number of sequences except one sequence in which all answers are correct

$= 32 - 1 = 31$

EXAMPLE 7 *In how many ways one can post 4 letters in 5 letter boxes ?*

SOLUTION Let L_1, L_2, L_3 and L_4 be four letters and B_1, B_2, B_3, B_4 and B_5 be five letter boxes. The job of posting 4 letters in five letter boxes can be divided into following four sub-jobs.

J_1 : Posting letter L_1 in a letter box.

J_2 : Posting letter L_2 in a letter box.

J_3 : Posting letter L_3 in a letter box.

J_4 : Posting letter L_4 in a letter box.

Since letter L_1 can be posted in any one of the five letter boxes. Therefore, job J_1 can be done in five ways. Similarly, each job can be done in 5 ways.

So, required number of ways $= 5 \times 5 \times 5 \times 5 = 5^4$.

EXAMPLE 8 *How many numbers of 4-digits can be made with the non-zero digits in which no two consecutive digits are the same ?*

SOLUTION We have nine non-zero digits, viz. 1, 2, 3, ..., 9. In order to construct a 4-digit number we have to fill 4 places by using these digits. In other words, the job of constructing a four digit number is divided into four sub-jobs, say, J_1, J_2, J_3, J_4.

J_1 : Filling unit's place.

J_2 : Filling ten's place.

J_3 : Filling hundred's place.

J_4 : Filling thousand's place.

Since there are 9 non-zero digits. So, job J_1 can be completed in 9 ways. As no two consecutive digits are to be the same i.e. a digit used in a place cannot be used in the next place but it can be used again in the place coming after the next place. So, job J_2 can be performed in 8 ways. In order to fill up hundred's place we cannot use the digit used to fill up ten's place but the digit used to fill up one's place can be used. So, job J_3 can be completed in 8 ways. Similarly, Job J_4 can be completed in 8 ways.

Hence, required number of 4 digit numbers

$$= 9 \times 8 \times 8 \times 8$$
$$= 4608$$

EXAMPLE 9 *A letter lock consists of 4 rings each marked with 15 different letters. In how many ways it is possible to make an unsuccessful attempt to open the lock ?*

SOLUTION Since each ring is marked with 15 different letters. Therefore, each ring has 15 positions. Thus, the total number of attempts that can be made to open the lock is $15 \times 15 \times 15 \times 15 = 15^4$. Out of these, there is just one attempt in which the lock will open. Therefore,

Total number of unsuccessful attempts $= 15^4 - 1$

EXAMPLE 10 *Four dice are rolled. Find the total number of possible outcomes. Also, find the number of ways in which*

(i) *no die exhibits 3.*

(ii) *at least one die shows 3.*

SOLUTION Since each die can result in 6 ways. So, the total number of possible outcomes is $6 \times 6 \times 6 \times 6 = 6^4$.

(i) If 3 does not occur on any die, then each die can result in 5 ways. Therefore,

Total number of outcomes in which no die shows 3

$$= 5 \times 5 \times 5 \times 5 = 5^4.$$

(ii) Total number of possible outcomes in which at least one die shows 3

= Total number of possible outcomes

– Number of outcomes in which no die shows 3

$= 6^4 - 5^4$.

EXAMPLE 11 *A question paper is split into two parts — Part A and Part B. Part A contains 5 questions and Part B has 4 questions. Each question in Part A has an alternative. A student has to attempt at least one question from each part. Find the number of ways in which the student can attempt the question paper.*

SOLUTION Here the job of answering the question paper is broadly divided into two subjobs.

J_1 : Answering Part *A*.

J_2 : Answering Part *B*.

Part *A* contains 5 questions and each question has an alternative. Therefore, for each question in part A the student has following three choices :

I The student does not attempt the question ;

II The student attempts the first part of the question;

III The student attempts the alternative part of the question.

Therefore, the total number of choices of answering a question is 3. But, there are 5 questions in part *A*. So, the total number of ways of answering Part *A* is 3^5. But, this includes a way in which the student does not attempt any question in Part *A*. Thus, the total number of ways of answering Part *A* is $3^5 - 1 = 243 - 1 = 242$.

Part *B* has 4 questions. For each question in Part *B* the student has two choices :

I The student attempts the question.

II The student does not attempt the question.

Therefore, the total number of choices for one question is 2. Since there are 4 questions in Part *B*. So, there are 2^4 ways of answering Part *B* which also include a way in which no question of Part *B* is answered. Therefore, there are $2^4 - 1 = 15$ ways of answering Part *B*.

Hence, the number of ways in which the student can attempt the question paper is $242 \times 15 = 3630$.

EXAMPLE 12 *In how many ways 5 prizes can be given away to 9 atheletes, if each athelete is eligible for each prize ?*

SOLUTION Since a prize can be given to any one of the 9 atheletes. Therefore, each prize can be given in 9 ways. There are 5 prizes. So, the total number of ways to give away 5 prizes is $9 \times 9 \times 9 \times 9 \times 9 = 9^5$.

EXAMPLE 13 *Let $a = a_1 a_2 a_3$ and $b = b_1 b_2 b_3$ be two three digit numbers. How many pairs of a and b can be formed so that a can be subtracted from b without borrowing.*

SOLUTION Clearly, a_1 and b_1 can take integral values from 1 to 9 whereas each of a_2, a_3, b_2 and b_3 can take integral values from 0 to 9.

We observe that the number a can be subtracted from the integer b without borrowing if $b_3 \geq a_3$, $b_2 \geq a_2$ and $b_1 \geq a_1$.

Let $a_1 = r$, where $r = 1, 2, 3, \ldots, 9$. Then, b_1 can take values $r, r+1, \ldots, 9$. Thus, corresponding to one choice of a_1, there are $(10 - r)$ choices for b_1. So, a_1 and b_1 can be chosen in

$$\sum_{r=1}^{9} (10 - r) \text{ ways.}$$

Similarly, let $a_2 = s$, where $s = 0, 1, 2, \ldots, 9$. Then, b_2 can take values $s, s+1, \ldots, 9$. Therefore, corresponding to one choice of a_2, there are $(10 - s)$ choices for b_2. So, a_2 and b_2 can be chosen in

$$\sum_{s=0}^{9} (10 - s) \text{ ways.}$$

Similarly, a_3 and b_3 can be chosen in $\sum_{s=0}^{n} (10 - s)$ ways.

Hence, by the fundamental principle of counting, the total number of pairs of a and b is

$$\sum_{r=1}^{9} (10 - r) \times \sum_{s=0}^{n} (10 - s) \times \sum_{s=0}^{9} (10 - s)$$

$$= 45 \times 55 \times 55 = 45 \times (55)^2.$$

EXAMPLE 14 *How many three digit numbers are of the form abc, with $a, c < b$ and $a \neq 0$?*

SOLUTION Since the digit at hundred's place cannot be zero. Therefore, we must have

$$a \geq 1$$

But, it is given that $a, c < b$. Therefore,

$$b \geq 2$$

$$\Rightarrow \quad b = 2, 3, 4, 5, \ldots, 9$$

Let $b = r$, where $r = 2, 3, \ldots, 9$. Then, a can take $(r - 1)$ values $1, 2, \ldots, r - 1$ and b can take r values $0, 1, 2, \ldots, r - 1$. Thus, for each value of r, abc can take $r(r - 1)$ values. But, r can take values from 2 to 9. Therefore, by the fundamental principle of addition

Required number of numbers

$$= \sum_{r=2}^{9} r(r - 1)$$

$$= \sum_{r=2}^{9} r^2 - \sum_{r=2}^{9} r$$

$$= \sum_{r=1}^{9} r^2 - \sum_{r=1}^{9} r$$

$$= \frac{9(9+1)(9\times 2+1)}{6} - \frac{9(9+1)}{2}$$

$$= 285 - 45 = 245$$

EXAMPLE 15 *A sequence is a ternary sequence, if it contains digits 0, 1 and 2. How many ternary sequences of length 9 are there which either begin with 210 or end with 210 ?*

SOLUTION Since each digit in a ternary sequence of length 9 can be filled in 3 ways. Therefore, the number of nine-digit ternary sequence beginning with 210 is 3^6 and the number of nine-digit ternary sequence ending with 210 is also 3^6. The number of ternary sequence of 9 digits which begin and end with 210 is 3^3.

Thus, the total number of sequences which either begin with 210 or end with 210 is

$$3^6 + 3^6 - 3^3 = 729 + 729 - 27 = 1431.$$

EXAMPLE 16 *A valid identifier in a programming language consists of a string of one to six alpha numeric characters (the 36 characters A, B, C, ..., x, y, z, 0, 1, 2, ..., 9) beginning with a letter. How may valid identifiers are there ?*

SOLUTION Since a valid identifier begins with a letter. Therefore, there are 26 ways to choose an identifier of one character. An identifier of 2 characters can be chosen in 26×36 ways. Similarly, an identifier of 3 characters can be chosen in $26 \times 36 \times 36$ ways.

The number of ways of choosing an identifier of 4, 5 and 6 characters are $26 \times 36 \times 36 \times 36$, $26 \times 36 \times 36 \times 36 \times 36$ and $26 \times 36 \times 36 \times 36 \times 36 \times 36$ respectively.

Hence, the total number of valid identifiers

$$= 26 + 26 \times 36 + 26 \times 36^2 + \ldots + 26 \times 36^5$$

$$= 26\left(\frac{36^6 - 1}{36 - 1}\right) = \frac{26}{35}(36^6 - 1).$$

EXAMPLE 17 *Find the number of ways in which n distinct objects can be put into two identical boxes so that no box remains empty.*

SOLUTION Let us first label the boxes B_1 and B_2. Now each object can be put either in B_1 or in B_2. So, there are two ways to deal with each of the n objects. Consequently, n objects can be dealt with 2^n ways. Out of these 2^n ways, there are two ways (i) when all objects are put in box B_1 (ii) when all objects are put in box B_2. Thus, there are $2^n - 2$ ways in which neither box is empty. If we now remove the labels from the boxes so that they become identical, this number must be divided by 2 to get the required number of ways.

Hence, the required number of ways $= \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$.

EXAMPLE 18 *How many numbers are there between 100 and 1000 in which all the digits are distinct ?*

SOLUTION A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits.

We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3, ..., 9. So, there are 9 ways of filling the hundred's place.

Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the total number of required numbers

$$= 9 \times 9 \times 8 = 648.$$

EXAMPLE 19 *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?*

SOLUTION Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three digit numbers such that every digit is either 2 or 9.

Clearly, each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$

EXAMPLE 20 *How many numbers are there between 100 and 1000 such that 7 is in the unit's place.*

SOLUTION Every number between 100 and 1000 is a three digit number. So, we have to form 3-digit numbers with 7 at the unit's place by using the digits 0, 1, 2, ..., 9. Clearly, repetition of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways.

Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, unit's place can be filled in only one way.

Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

EXAMPLE 21 *How many numbers are there between 100 and 1000 such that at least one of their digits is 7 ?*

SOLUTION Clearly, a number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7

= (Total number of three-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

Total number of three-digit numbers : We have to form three-digit numbers by using the digits 0, 1, 2, 3, ..., 9.

Clearly, hundred's place can be filled in 9 ways and each of the ten's and one's place can be filled in 10 ways.

So, total number of 3-digit number $= 9 \times 10 \times 10 = 900$.

Total number of three-digit number in which 7 does not appear at all : Here we have to form three-digit numbers by using the digits 0 to 9, except 7.

So, hundred's place can be filled in 8 ways and each of the ten's and one's place can be filled in 9 ways. So, total number of three-digit numbers in which 7 does not appear at all $= 8 \times 9 \times 9$.

Hence, total number of 3-digit numbers having at least one of their digits as 7

$$= 9 \times 10 \times 10 - 8 \times 9 \times 9 = 252.$$

EXAMPLE 22 *How many numbers are there between 100 and 1000 which have exactly one of their digits as 7 ?*

SOLUTION A number between 100 and 1000 contains 3-digits. So, we have to form 3-digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types:

(I) Those numbers that have 7 in the unit's place but not in any other place.

(II) Those numbers that have 7 in the ten's place but not in any other place.

(III) Those numbers that have 7 in the hundred's place but not in any other place.

Required number of numbers is the total number of these three types of numbers.

We shall now count these three types of numbers separately.

(I) *Those three-digit numbers that have 7 in the unit's place but not in any other place.*

The hundred's place can have any one of the digits from 0 to 9 except 0 and 7. So, hundred's place can be filled in 8 ways. The ten's place can have any one of the digits from 0 to 9 except 7. So, the number of ways the ten's place can be filled is 9. The unit's place has 7. So, it can be filled in only one way.

Thus, there are $8 \times 9 \times 1 = 72$ numbers of the first kind.

(II) *Those three-digit numbers that have 7 in the ten's place but not in any other place.*

The number of ways to fill the hundred's place = 8

(by any one of the digits from 1, 2, 3, 4, 5, 6, 8, 9)

The number of ways to fill the ten's place = 1 (by 7 only)

The number of ways to fill the one's place = 9

(by any one of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9)

Thus, there are $8 \times 1 \times 9 = 72$ numbers of the second kind.

(III) *Those three-digit numbers that have 7 in the hundred's place but not at any other place.*

In this case, the hundred's place can be filled only in one way and each of the ten's and one's place can be filled in 9 ways.

So, there are $1 \times 9 \times 9 = 81$ numbers of the third kind.

Hence, the total number of required type of numbers

$$= 72 + 72 + 81 = 225$$

EXAMPLE 23 *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

SOLUTION Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card

to the first friend = 3. Similarly, invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways

$$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729.$$

EXAMPLE 24 *How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed ?*

SOLUTION (i) In a four-digit number 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. (viz. 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways.

Now, any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining three digits in 3 ways.

Hence, the required number of numbers

$$= 5 \times 5 \times 4 \times 3 = 300.$$

(ii) For a four-digit number we have to fill up four places and 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the remaining three places viz. hundred's, ten's and one's can be filled in 6 ways.

Hence, the required number of numbers

$$= 5 \times 6 \times 6 \times 6 = 1080.$$

EXAMPLE 25 *How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if : (i) repetition of digits is allowed ? (ii) repetition of digits is not allowed ?*

SOLUTION (i) Every number between 1000 and 4000 is a four digit number. In thousand's place we can put either 1 or 2 or 3 but not 4. So, thousand's place can be filled in 3 ways. Since repetition of digits is allowed, so each of the hundred's, ten's and one's place can be filled in 5 ways. So, total number of numbers between 1000 and 4000, including 1000 and excluding 4000 is $3 \times 5 \times 5 \times 5 = 375$. But we have to find the total number of numbers greater than 1000 but not greater than 4000.

Hence, required number of numbers

$$= 375 + 1 \text{ (for 4000)} - 1 \text{ (for 1000)} = 375.$$

(ii) As discussed above thousand's place can be filled in 3 ways. Since repetition of digits is not allowed, so, hundred's place can be filled from the remaining digits in 4 ways. Now, three digits are left, so ten's place can be filled in 3 ways. One's place can be filled in 2 ways.

Hence, required number of numbers $= 3 \times 4 \times 3 \times 2 = 72$.

EXAMPLE 26 *How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if :*

(i) *the repetition of digits is not allowed ?*

(ii) *the repetition of digits is allowed ?*

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers $= 3 \times 5 \times 4 = 60$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 27 *There are unlimited number of identical balls of four different colours. How many arrangements of at most 8 balls in a row can be made by using them ?*

SOLUTION Since there are balls of four different colours. Therefore,

Number of arrangements of one ball $= 4$

Number of arrangements of two balls $= 4 \times 4 = 4^2$

Number of arrangements of three balls $= 4 \times 4 \times 4 = 4^3$ etc.

Hence,

Required number of arrangements $= 4 + 4^2 + 4^3 + \ldots + 4^8$

$$= 4\left(\frac{4^8 - 1}{4 - 1}\right)$$

$$= \frac{4}{3}(4^8 - 1)$$

$$= \frac{4}{3} \times 65535$$

$$= 87380.$$

EXAMPLE 28 *There are two sets of parallel lines, their equations being* $x \cos \alpha + y \sin \alpha = p$; $p = 1, 2, 3, \ldots m$ *and* $y \cos \alpha - x \sin \alpha = q$; $q = 1, 2, 3, \ldots, n$ $(n > m)$ *where* α *is a given constant. Show that the lines form* $\frac{1}{6} m (m-1)(3n - m - 1)$ *squares.*

SOLUTION The equation

$x \cos \alpha + y \sin \alpha = p$; $p = 1, 2, 3, \ldots, m$ represents m parallel lines such that the distance between two consecutive lines is one unit.

Similarly, the equation

$y \cos \alpha - x \sin \alpha = q$; $q = 1, 2, 3, \ldots, n$ represents n parallel lines such that the distance between any two consecutive lines is one unit.

We observe that the slope of each line of first set is $m_1 = -\cot \alpha$ and the slope of each line of the second set is $m_2 = \tan \alpha$.

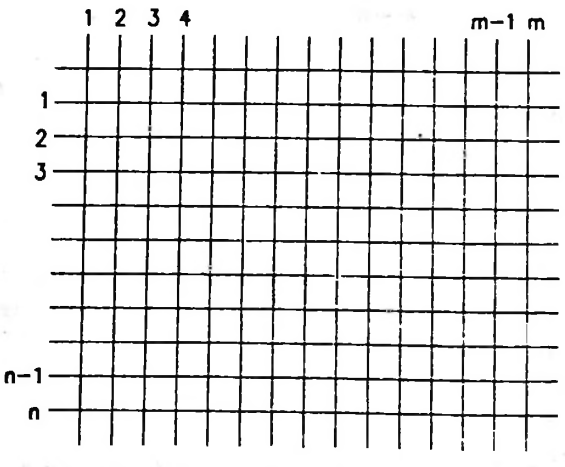


Fig. 22.2

Clearly, $m_1 m_2 = -1$.

Therefore, every line of first set is perpendicular to every line of second set.

We observe that four lines consisting of two lines of the first set and two lines of the second set will form a square, if the distance between two parallel lines of first set is same as the distance between two parallel lines of second set.

Since $m < n$, therefore the length of the side of the largest square formed by the two sets of lines is $(m-1)$ units and the length of the side of the smallest square is 1 unit.

Clearly, two lines at a unit distance from the set of m parallel lines can be chosen in $(m-1)$ ways, namely $(1, 2)$, $(2, 3)$, $(3, 4)$ ..., $(m-1, m)$ and two lines at a unit distance from the set of n parallel lines can be chosen in $(n-1)$ ways, namely $(1, 2)$, $(2, 3)$, ..., $(n-1, n)$. Therefore, number of squares whose sides are of length 1 unit $= (m-1)(n-1)$.

Similarly, two lines at a distance of 2 units from the set of m parallel lines can be chosen in $(m-2)$ ways, namely $(1, 3)$ $(2, 4)$, ..., $(m-2, m)$ and two lines at a distance of 2 units from the set of n parallel lines can be chosen in $(n-2)$ ways, namely $(1, 3)$, $(2, 4)$ ...., $(n-2, n)$. Therefore,

Number of squares whose sides are of length 2 units

$$= (m-2)(n-2)$$

Continuing in this manner, we find that

the number of squares whose sides are of length $(m-1)$ units

$$= (m-(m-1))\ (n-(m-1))$$

Hence,

Total number of squares

$$= (m-1)(n-1) + (m-2)(n-2) + \ldots\ldots\ldots\ldots\ldots\ldots + (m-(m-1))(n-(m-1))$$

$$= \sum_{r=1}^{m-1} (m-r)(n-r)$$

$$= \sum_{r=1}^{m-1} \{mn - r(m+n) + r^2\}$$

$$= mn(m-1) - (m+n)\sum_{r=1}^{m-1} r + \sum_{r=1}^{m-1} r^2$$

$$= mn(m-1) - (m+n)\frac{m(m-1)}{2} + \frac{(m-1)m(2m-1)}{6}$$

$$= \frac{m(m-1)}{6}\{6n - 3(m+n) + (2m-1)\}$$

$$= \frac{m(m-1)}{6}\{6n - 3m - 3n + 2m - 1\}$$

$$= \frac{m(m-1)(3n-m-1)}{6}$$

EXAMPLE 29 *From a well-shuffled pack of 52 cards, a player is then 5 cards. If the cards have consecutive values the its is said that the player has a run and if they are also from the same suit it is said that the player has a running flash. In how many different ways can a player hold a (i) running flash ? (ii) run ?*

SOLUTION There are four suits of cards in a pack of 52 cards and each suit has 13 cards. The values of cards in a suit are

2, 3, 4, 5, ..., 10, 11 (= Jack), 12 (= queen), 13 (= King) and 14 (= Ace)

(i) From a suit five consecutively numbered cards can be drawn in one of the following ways :

(2, 3, 4, 5, 6) (3, 4, 5, 6, 7) ..... (10, 11, 12, 13, 14)

Therefore, there are 9 ways of getting a running flash from a suit.

Since there are four suits of cards. So,

Total number of ways of getting a running flash

$$= 9 + 9 + 9 + 9 = 36.$$

(ii) From a pack of 52 cards, five consecutively numbered cards can be drawn in one of the following ways :

(2, 3, 4, 5, 6), (3, 4, 5, 6, 7)..., (10, 11, 12, 13, 14)

But, in each place we have 4 choices of suits. Therefore,

Total number of runs $= 9 \times (4 \times 4 \times 4 \times 4 \times 4)$

$$= 9 \times 4^5$$

EXAMPLE 30 *What can be the maximum population of a country in which no two persons have an identical set of teeth ? (Disregard the shape and size of the teeth. Take only the positioning of the teeth in consideration. Also assume that there is no persons without teeth and no person has more than 32 teeth)*

SOLUTION There are 32 places for the teeth in the mouth. For each places, we have two choices, either there is a tooth or there is no tooth at the place. Therefore, the number of ways to fill up 32 places is 2^{32}. It is given that there is no person without a tooth. Therefore, the maximum population of the countary in which no two persons have identical set of teeth is $2^{32} - 1$.

EXERCISE 22.3

1. A man has 3 jackets, 10 shirts and 5 pairs of slacks. If an outfit consists of a jacket, a shirt and a pair of slacks, how many different outfits can the man make ?
2. A student has a choice of 5 foreign languages and 4 sciences. In how many ways can he choose 1 language and 1 science ?
3. In how many ways can 5 letters be mailed if there are 3 mailboxes available ?
4. There are 4 candidates for president of a club, 6 for vice-president and 2 for secretary. In how many ways can these three positions be filled ?
5. In how many ways can a club consisting of 12 members choose a president, a secretary, and a treasurer ?

6. Three six faced die, marked with numbers 1 to 6 on six faces of each die, are thrown and outcomes are recorded. How many possible outcomes are there ?

7. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection ?

8. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles ?

9. From Goa to Mumbai there are two roots; air, and sea. From Mumbai to Delhi there are three routes; air, rail and road. From Goa to Delhi via Mumbai, how many kinds of routes are there ?

10. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of calendars should it prepare to serve for all the possibilities in future years ?

11. There are four parcels and five post-offices. In how many different ways can the parcels be sent by registered post ?

12. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there ?

13. In how many ways can an examinee answer a set of ten true/false type questions? [No. of ways = 2^{10}]

14. A letter lock consists of three rings each marked with 10 different letters. In how many ways is it possible to make an unsuccessful attempt to open the lock ?

15. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each ?

16. There are 5 books on Mathematics and 6 books on Physics in a book shop. In how many ways can a student buy : (i) a Mathematics book and a Physics book (ii) either a Mathematics book or a Physics book ?

17. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other ?

18. A team consists of 6 boys and 4 girls and other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl ?

19. Twelve students compete in a race. In how many ways first three prizes be given ?

20. How many A.P.'s with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5} ?

21. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-incharge are to be appointed. In how many ways can this be done ?

22. How many three-digit numbers are there with no digit repeated ?

23. How many three-digit numbers are there ?

24. How many three-digit odd numbers are there ?

25. How many different five-digit number licence plates can be made if

(i) first digit cannot be zero and the repetition of digits is not allowed,

(ii) the first-digit cannot be zero, but the repetition of digits is allowed ?

26. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 7000, if repetition of digits is not allowed ?

27. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 800, if repetition of digits is not allowed ?

28. In how many ways can six persons be seated in a row ?

29. How many 9-digit numbers of different digits can be formed ?

30. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed ?

31. How many 3-digit numbers are there, with distinct digits, with each digit odd ?

32. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed ?

33. How many different numbers of six digits can be formed from the digits 3, 1, 7, 0, 9, 5 when repetition of digits is not allowed ?

34. How many different numbers, greater than 5000 can be formed with the digits 1, 2, 5, 9, 0 when repetition of digits is not allowed ?

35. In how many ways can the following prizes be given to a class of 40 students ?

(i) First and second prizes for Mathematics.

(ii) First and second prizes for Physics.

(iii) First prize for English.

(iv) First prize for Indian Classical Music.

36. In a particular programming language, a valid variable name consists of a sequence of one to eight alphanumeric characters *A, B, C, ..., Z*, 0, 1, 2, ... 9 beginning with a letter. Find the total number of valid variable names.

37. A straight is a five-card hand containing consecutive values. How many different straights are there ? How many straights are there if the cards are not all from the same suit ?

ANSWERS

1. 150	2. 20	3. 243	4. 48	13. 1024	14. 999	15. 512	16. (i) 30 (ii) 11
5. 1320	6. 6^3	7. 378	8. 600	17. 42	18. 42	19. 1320	20. 15
9. 6	10. 14	11. 625	12. 32	21. 42840	22. 648	23. 900	24. 450

25. (i) 27216 (ii) 90000 **26.** 72 **27.** 48
28. 720 **29.** 9(9!) **30.** 21 **31.** 60
32. 720 **33.** 600 **34.** 48 **35.** $(40)^4 \times (39)^2$
36. $\frac{36}{35}(36^8 - 1)$ **37.** $9(4^5), 9(4^5 - 1)$

—❑❑❑—

22.4 PERMUTATIONS AND COMBINATIONS

In this section, we shall introduce the notion of combinations and permutations. Normally we discuss permutations (arrangements) prior to combinations (selections). However, practically first we make selections from a given group of distinct items and then selected items are arranged or permutated in each group. This means that in the process of arranging a group of distinct items by taking some of them at a time, the job of selection occurs prior to the job of arranging the selected items. Keeping this thing in mind, we shall develop the notion of permutations and combinations by introducing combinations (selections) prior to permutations (arrangements).

COMBINATIONS *Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination.*

PERMUTATIONS *Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.*

Consider the set $\{a, b, c\}$ of first three letters of the English alphabet. Various combinations (selections) of these three letters taken two at a time are as follows :

$$ab \qquad bc \qquad ca$$

Here selection ab is same as ba. Similarly, selections bc and ca are same as cb and ac respectively.

Now, if we want to list all possible arrangements of these three letters taking two at a time, then letters in each of the above selections will be arranged and we will obtain following arrangements :

ab	bc	ca
ba	cb	ac

Let us now consider the set $\{a, b, c, d\}$ of first four letters of the English alphabet. The list of all possible selections of three letters from these four letters is as given below

$$abc, \quad abd, \quad bcd, \quad acd$$

It is to mention here that selection abc is same as bca or cab or bca etc.

Now, if we wish to arrange letters a, b, c, d by taking 3 at a time, then in each of the above selections, letters are arranged and we obtain the following arrangements.

abc	abd	bcd	acd
acb	adb	bdc	adc
bca	bda	cbd	cda
bac	bad	cdb	cad
cab	dba	dcb	dca
cba	dab	dbc	dac

THEOREM 1 *Let r and n be positive integers such that $1 \le r \le n$. Then, prove that the number of all permutations of n distinct items or objects taken r at a time is*

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$

PROOF The number of permutations of n distinct things, taken r at a time, is same as the number of ways in which we can fill up r-places when we have n different things at our disposal.

The first place can be filled in n ways, for any one of the n things can be used to fill up the first place. Having filled it, there are $(n-1)$ things left and any one of these $(n-1)$ things can be used to fill up the second place. So, the second place can be filled in $(n-1)$ ways. Hence, by the fundamental principle of counting, the first two places can be filled in $n(n-1)$ ways. When the first two places are filled, there are $(n-2)$ places left, so that the third place can be filled from the remaining $(n-2)$ things in $(n-2)$ ways. Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this manner, we find that the first $(r-1)$ places can be filled in $n(n-1)(n-2)\ldots(n-(r-2))$ ways. After filling up first $(r-1)$ places, exactly $n-(r-1) = n-r+1$ things are left. So, the rth place can be filled in $(n-(r-1))$ ways. Hence, the r places can be filled in $n(n-1)(n-2)\ldots(n-(r-1))$ ways.

Hence, the total number of permutations of n distinct things, taken r at a time is $n(n-1)(n-2)(n-3)\ldots(n(r-1))$.

REMARK 1 We have,

$$n(n-1)(n-2)\ldots(n-(r-1))$$

$$= \frac{n(n-1)(n-2)\ldots(n-(r-1))(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^nP_r.$$

So, the total number of arrangements (permutations) of n-distinct items, taking r at a time is ${}^nP_r$ or $P(n, r)$.

THEOREM 2 *The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.*

PROOF The number of all permutations of n distinct objects, taken all at a time, is same as the number of ways of filling n-places when we have n distinct things at our disposal. Proceeding as in the above theorem, we find that the required number of ways is

$$n(n-1)(n-2)\ldots 3.2.1 = n!$$

Hence, the number of all permutations of n distinct objects taken all at a time is n !.

ALITER By using theorem 1, we find that n distinct items can be arranged in a row in ${}^nP_n = \frac{n!}{0!} = n!$ ways.

THEOREM 3 *The number of ways of selecting r items or objects from a group of n distinct items or objects is*

$$\frac{n!}{(n-r)!\,r!} = {}^nC_r$$

PROOF Let the total number of ways of selecting r items from a group of n distinct items be x. This means that x groups of r items each can be prepared from a group of n distinct items. Consider one of these x groups. We find that there are r-distinct items in this group. These r items can be arranged in $r!$ ways. Thus, each of the x-groups gives rise to $r!$ arrangements. So, x groups will give rise to $x \times r!$ arrangements. Consequently, the number or permutations of n items, taken r at a time is $x \times r!$. But, this number is also equal to

$n(n-1)(n-2)\ldots(n-(r-1))$ [See Theorem 1]

$\therefore \quad x \times r! = n(n-1)(n-2)\ldots(n-(r-1))$

$$\Rightarrow \quad x = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}$$

$$\Rightarrow \quad x = \frac{n(n-1)(n-2)\ldots(n-(r-1))(n-r)!}{r!\,(n-r)!}$$

$$\Rightarrow \quad x = \frac{n!}{r!\,(n-r)!}$$

$$\Rightarrow \quad x = {}^nC_r$$

REMARK 2 *Let there be n-distinct objects which are to be arranged in a row by taking r-at a time. The job of arranging n items by taking r at a time can be divided into two ordered subjobs J_1 and J_2 given by*

J_1 : Selecting r items from n-distinct items.

J_2 : Arranging r selected items.

We find that the sub-job J_1 can be completed in ${}^nC_r$ ways and sub-job J_2 can be completed in $r!$ ways. Therefore,

Number of ways of arranging n-distinct items, taking r at a time

$$= {}^nC_r \times r!$$

Clearly, it is consistent with the result stated in Theorem 1, because ${}^nC_r \times r! = n(n-1)(n-2)\ldots(n-(r-1)) = {}^nP_r$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *How many 5-letter words, with or without meaning, can be formed out of the letters of the word 'EQUATIONS' if repetition of letters is not allowed ?*

SOLUTION There are 9 letters in the word 'EQUATIONS'. In order to construct 5-letter words from these 9 letters, we first select 5 letters out of these 9 letters and then the selected letters are arranged. So, the job of construction of 5 letter words is divided into two sub-jobs.

J_1 : Selecting 5 letters from 9 letters of the word "EQUATIONS".

J_2 : Arranging 5 selected letters to form 5-letter words.

We find that job J_1 can be completed in ${}^9C_5$ ways and job J_2 in 5! ways.

So,

Total number of 5 letter words $= {}^9C_5 \times 5!$

$$= \frac{9!}{4!\,5!} \times 5!$$

$$= 9 \times 8 \times 7 \times 6 \times 5 = 15120.$$

EXAMPLE 2 *How many four digit numbers are there with distinct digits ?*

SOLUTION There are 10 digits viz. 0, 1, 2, ..., 9. The total number of arrangements of these 10 digits taking 4 at a time is ${}^{10}C_4 \times 4!$. But these arrangements also include those numbers which have 0 at thousand's place. Such numbers are three digit numbers. When 0 is fixed at thousand's, we have to arrange remaining 9 digits by taking three at a time. The number of such arrangements is ${}^9C_3 \times 3!$. So, the total number of numbers having 0 at thousand's place is ${}^9C_3 \times 3!$

Hence, the total number of four digit numbers with distinct digits $= {}^{10}C_4 \times 4! - {}^9C_3 \times 3! = 5040 - 504 = 4536$.

EXAMPLE 3 *It is required to seat 7 men and 3 women in a row so that women occupy the even places. How many such arrangements are possible ?*

SOLUTION In all 10 persons are to be seated in a row and in the row of 10 positions there are exactly 5 even places viz. second, fourth, sixth, eighth and tenth. It is given that 3 of these five even places are to be occupied by three women. This can be done in ${}^5C_3 \times 3!$ ways. The remaining 7 positions can be filled by the 7 men in ${}^7C_7 \times 7!$ ways.

Hence, the total number of seating arrangements

$$= ({}^5C_3 \times 3!) \times ({}^7C_7 \times 7!)$$

$$= 30240.$$

EXAMPLE 4 *How many signals can be made by 5 flags from 8 flags of different colours ?*

SOLUTION In order to generate signals of 5 flags from 8 flags of different colours, we first select 5 flags from 8 flags. This can be done in ${}^8C_5$ ways. Now we arrange selected 5 flags in 5! ways.

Hence, the total number of signals $= {}^8C_5 \times 5! = 6720$.

EXAMPLE 5 *How many different signals can be given using any number of flags from 6 flags of different colours ?*

SOLUTION Proceeding as in Example 3, the total number of signals of r flags from 6 flags of different colours is ${}^6C_r \times r!$. But, can vary from 1 to 6. Therefore, by the fundamental principle of addition

$$\text{Required number of signals} = \sum_{r=1}^{6} {}^6C_r \times r!$$

$$= \sum_{r=1}^{6} \frac{6!}{(6-r)!}$$

$$= \frac{6!}{5!} + \frac{6!}{4!} + \frac{6!}{3!} + \frac{6!}{2!} + \frac{6!}{1!} + \frac{6!}{0!}$$

$$= 6 + 30 + 120 + 360 + 720 + 720$$

$$= 1956$$

EXAMPLE 6 *Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated.*

SOLUTION The total number of 5-letter words which can be formed from 10 letters when one or more of its letters is repeated

$$= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$$

The total number of 5-letter words when none of their letters is repeated

$$= {}^{10}C_5 \times 5! = 30240$$

Hence, the number of 5-letter words which have at least one of their letters repeated

$$= 10^5 - 30240 = 69760$$

EXAMPLE 7 *How many different automobile licence plates can be constructed if the licences contain three letters of the English alphabet followed by a three digit number if (i) repetitions are allowed ? (ii) repetitions are not allowed ?*

SOLUTION (i) When repetitions are allowed, three letters from the English alphabet can be chosen in $26 \times 26 \times 26 = 26^3$ ways, and a three digit number can be chosen in $(10 \times 10 \times 10 - 1) = 999$ ways. Thus, by the fundamental principle of counting the total number of licence plates is $26^3 \times 999$.

(ii) When repetitions are not allowed, three letter words with distinct letters can be made in ${}^{26}C_3 \times 3!$ ways. Corresponding to each such way a number with three distinct digits can be constructed in ${}^{10}C_3 \times 3!$ ways. Thus, the number of licence plates in this case is $({}^{26}C_3 \times 3!) \times ({}^{10}C_3 \times 3!)$.

EXAMPLE 8 *Find the total number of times the digit 3 will be written when listing integers from 1 to 1000.*

SOLUTION Since 3 does not occur in 1000. So, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form abc where $0 \le a, b, c \le 9$.

Clearly,

Number of times 3 occurs

= (No. of numbers in which 3 occurs exactly at one place)
+ 2 (No. of numbers in which 3 occurs exactly at two places)
+ 3 (No. of numbers in which 3 occurs exactly at three places)

Now,

No. of numbers in which 3 occurs exactly at one place:

Since 3 can occur at one place in ${}^3C_1$ ways and each of the remaining two places can be filled in 9 ways. So, number of numbers in which 3 occurs exactly at one place $= {}^3C_1 \times 9 \times 9$.

No. of numbers in which 3 occurs exactly at two places :

Since 3 can occur exactly at two places in ${}^3C_2$ ways and the remaining place can be filled in 9 ways.

So, number of numbers in which 3 occurs exactly at two places $= {}^3C_2 \times 9$.

No. of numbers in which 3 occurs at all the three places :

Since 3 can occur in all the three digits in one way only. So, number of numbers in which 3 occurs at all the three places is one.

Hence, Numbers of times 3 occurs

$$= {}^3C_1 \times 9 \times 9 + 2({}^3C_2 \times 9) + 3 \times 1 = 300$$

EXAMPLE 9 *There are n intermediate stations between two junctions J_1 and J_2. A train starts from junction J_1 and stops at junction J_2. m persons $(m < n)$ enter the train during the journey with m different tickets of the same class. How many different set of tickets they could have had ?*

SOLUTION Let $S_1, S_2, \ldots, S_n$ be n intermediate stations between junctions J_1 and J_2.

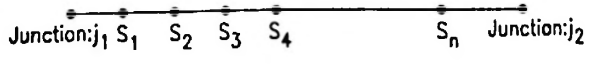


Fig. 22.3

We observe that a person taking the train from ith station (S_i) can get down at any one of the next $(n+1)-i$ stations. Therefore,

Total number of possible tickets

$$= \sum_{i=0}^{n} (n+1-i)$$

$$= (n+1)+n+(n-1)+(n-2)+\ldots+1$$

$$= \frac{(n+1)(n+2)}{2} = N \text{ (say)}$$

Clearly, m different tickets are out of these N tickets. Therefore, required number of ways

$$= {}^{N}C_{m}, \text{ where } N = \frac{(n+1)(n+2)}{2}$$

EXAMPLE 10 *Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the seating arrangement can be made.*

SOLUTION Since four particular guests want to sit on a particular side A (say) and three others on the other side B (say). So, we are left with 11 guests out of which we choose 5 for side A in ${}^{11}C_5$ ways and the remaining 6 for side B in ${}^{6}C_6$ ways. Hence, the number of selections for the two sides is ${}^{11}C_5 \times {}^{6}C_6$.

Now 9 persons on each side of the table can be arranged among themselves in 9! ways.

Hence, the total number of arrangements

$$= {}^{11}C_5 \times {}^{6}C_6 \times 9! \times 9! = \frac{11!}{6!\,5!} \times 9! \times 9!$$

EXAMPLE 11 *How many different words, each containing 2 vowels and 3 consonants can be formed with 5 vowels and 17 consonants ?*

SOLUTION There are 5 vowels and 17 consonants. Two vowels out of 5 vowels and 3 consonants out of 17 consonants can be chosen in ${}^{5}C_2 \times {}^{17}C_3$ ways. Now we have 5 letters which can be arranged in 5! ways.

Hence, the total number of words

$$= {}^{5}C_2 \times {}^{17}C_3 \times 5! = 816000$$

EXAMPLE 12 *A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated ?*

SOLUTION It is given that 4 persons wish to sit on one side, say, A and two on the other side, say B. Therefore, 10 persons are left, out of which 4 persons for side A can be selected in ${}^{10}C_4$ ways and 6 persons for side B from the remaining 6 persons can be chosen in ${}^{6}C_6$ ways. Hence, the total number of ways of selecting persons for two sides of the table is ${}^{10}C_4 \times {}^{6}C_6$. Now, 8 persons on each side can be arranged amongst themseleves in 8! ways. Hence, the total number of seating arrangements is

$${}^{10}C_4 \times {}^{6}C_6 \times 8! \times 8! = {}^{10}C_4\,(8!)^2.$$

EXAMPLE 13 *A tea party is arranged for 2m persons along two sides of a long table with m chairs on each side. r persons wish to sit on one particular side and s on the other. In how many ways can they be seated.*

SOLUTION Let us mark the sides as A and B. It is given that r persons are willing to sit on one side, say A, and s persons on the other side B. Out of the remaining $(2m-r-s)$ persons we first select $(m-r)$ persons for side A and $m-s$ persons for side B. This can be done in ${}^{2m-r-s}C_{m-r} \times {}^{m-s}C_{m-s}$ ways. Now, m persons on each side can be arranged amongst themselves in $m!$ ways.

Hence, the total number of arrangements

$$= {}^{2m-r-s}C_{m-r} \times {}^{m-s}C_{m-s} \times m! \times m!$$

$$= {}^{2m-r-s}C_{m-r} \times (m!)^2$$

EXAMPLE 14 *Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.*

SOLUTION The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time = Number of arrangements of 4 digits, taken all at a time = ${}^{4}P_4 = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers. Each of the digits 2, 3, 4, 5 occurs in 3! (= 6) times in the unit's place.

So, total for the digits in the unit's place in all the numbers

$$= (2+3+4+5) \times 3! = 84.$$

Since each of the digits 2, 3, 4, 5 occurs 3! times in any one of the remaining places. So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers

$$= (2+3+4+5) \times 3! = 84.$$

Hence, the sum of all the numbers

$$= 84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$$

REMARK *It follows from the above example that sum of the numbers formed by n non-zero digits*

$$= (\text{Sum of the digits})\,(n-1)!\left(\frac{10^n - 1}{10 - 1}\right)$$

EXAMPLE 15 *Find the sum of all the numbers greater than 10000 formed by using digits 0, 2, 4, 6, 8, no digit being repeated in any number.*

SOLUTION The total number of numbers formed by arranging the digits 0, 2, 4, 6, 8 taken all at a time is 5! = 120. Some of these numbers are less than 10000 and some are greater than 10000. In fact all those numbers having 0 at ten thousand's place are four digit numbers. Therefore,

Required sum

= Sum of the numbers formed by using digits 0, 2, 4, 6, 8

– Sum of the numbers formed by using digits 2, 4, 6, 8

$$= (0+2+4+6+8)(5-1)!\left(\frac{10^5-1}{10-1}\right)$$

$$-(2+4+6+8)(4-1)!\left(\frac{10^4-1}{10-1}\right)$$

$$= 480\left(\frac{10^5-1}{10-1}\right)-120\left(\frac{10^4-1}{10-1}\right)$$

$$= 5333280 - 133320$$

$$= 5199960$$

EXAMPLE 16 *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party ?*

SOLUTION It is to note here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons

Hence, required number of ways

$$= {}^{12}C_2 = \frac{12!}{10! \times 2!} = 66.$$

EXAMPLE 17 *A question paper has two parts, Part A and Part B, each containing 10 questions. If the student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in ${}^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in ${}^{10}C_5$ ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= {}^{10}C_8 \times {}^{10}C_5 = \frac{10!}{8!\,2!} \times \frac{10!}{5! \times 5!} = 11340$$

EXAMPLE 18 *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?*

SOLUTION There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing :

(I) 5 women and 7 men

(II) 6 women and 6 men

(III) 7 women and 5 men

(IV) 8 women and 4 men

(V) 9 women and 3 men

∴ Total number of ways of forming the committee

$$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$$

$$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062.$$

Clearly, women are in majority in (III), (IV) and (V) cases as discussed above.

So, total number of committees in which women are in majority

$$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$$

$$= 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$$

Clearly, men are in majority in only (I) case as discussed above.

So, total number of committees in which men are in majority

$$= {}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008.$$

EXAMPLE 19 *A man has 7 relatives, 4 women and 3 men. His wife also has 7 relatives, 3 women and 4 men. In how many ways can they invite 3 women and 3 men so that 3 of them are the men's relatives and 3 his wife's ?*

SOLUTION Three women and three men can be chosen from 7 relatives of the husband and 7 relatives of his wife in the following ways :

	I		II		III		IV	
	H	W	H	W	H	W	H	W
M	0	3	1	2	2	1	3	0
F	3	0	2	1	1	2	0	3

Hence the rquired number of ways

$$= ({}^4C_3 \times {}^4C_3) + ({}^4C_2 \times {}^3C_1) \times ({}^3C_1 \times {}^4C_2) + ({}^4C_1 \times {}^3C_2) \times ({}^3C_2 \times {}^4C_1) + ({}^3C_3 \times {}^3C_3)$$

$$= 16 + 324 + 144 + 1 = 485$$

EXAMPLE 20 *There are 3 sections in a question paper each containing 5 questions. A candidate has to solve only 5 questions, choosing at least one question from each section. In how many ways can he make his choice ?*

SOLUTION We observe that :

Total number of ways of choosing at least one question from each section

= Total number of ways of choosing 5 questions out of 15 questions

– Number of ways of selecting 5 questions from 2 sections only

– Number of ways of choosing 5 questions from one section only.

$$= {}^{15}C_5 - {}^3C_2\left({}^{10}C_5 - {}^2C_1 \times {}^5C_5\right) - {}^3C_1 \times {}^5C_5$$

$$= {}^{15}C_5 - 3 \times {}^{10}C_5 + 6 - 3$$

$$= 2250.$$

EXAMPLE 21 *A person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. How many friends should he invite ?*

SOLUTION Suppose he invites r friends at a time. Then, the total number of parties is ${}^{20}C_r$. We have to find the maximum

value of ${}^{20}C_r$ which is for $r = 10$ (if n is even, then ${}^nC_r$ is maximum for $r = n/2$). Hence, he should invite 10 friends at a time in order to form the maximum number of parties.

EXAMPLE 22 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included ?*

SOLUTION The committee can be formed in the following ways :

(I) By selecting 2 men and 1 woman.

(II) By selecting 1 man and 2 women

Now, 2 men out of 5 men and 1 woman out of 2 women can be chosen in ${}^5C_2 \times {}^2C_1$ ways.

And, 1 man out of 5 men and 2 women out of 2 women can be chosen in ${}^5C_1 \times {}^2C_2$ ways.

$\therefore$ Total number of ways of forming the committee

$$= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25.$$

EXAMPLE 23 *A firm of Chartered Accountants in Delhi has to send 10 clerks to 5 concerns, two to each concern. Two of the concerns are in Delhi and the others are outside. Two of the clerks prefer to work in Delhi while three others prefer to work outside. In how many ways can the assignment be made if the preferences are to be satisfied ?*

SOLUTION It is given that out of 5 concerns two are in Delhi and three outside Delhi. This means that 4 clerks are to be selected for Delhi and 6 clerks for outside Delhi. But two clerks prefer to work in Delhi and 3 prefer to work outside Delhi. This means that out of the remaining 5 clerks we have to select 2 clerks for Delhi and 3 for outside Delhi. This can be done in ${}^5C_3 \times {}^2C_2$ ways. Now, 4 clerks in Delhi are to work in two concerns, two to each concern. This can be done in ${}^4C_2 \times {}^2C_2$ ways. 6 clerks for outside Delhi are to work in three concerns, two to each concern. This can be done in ${}^6C_2 \times {}^4C_2 \times {}^2C_2$ ways.

Hence, the total number of ways in which assignements can be made

$$= ({}^5C_3 \times {}^2C_2) \times ({}^4C_2 \times {}^2C_2) \times ({}^6C_2 \times {}^4C_2 \times {}^2C_2)$$

$$= 5400.$$

EXAMPLE 24 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicket keeper.*

SOLUTION The selection of team is divided into four phases :

I Selection of 5 batsmen out of 10. This can be done in ${}^{10}C_5$ ways.

II Selection of 3 all-rounders out of 5. This can be done in ${}^5C_3$ ways.

III Selection of 2 bowlers out of 8. This can be done in ${}^8C_2$ ways.

IV Selection of one wicket keeper out of 2. This can be done in ${}^2C_1$ ways.

The selection of team is completed by completing all the four phases.

$\therefore$ The team can be selected in

$${}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1 = 141120 \text{ ways.}$$

EXAMPLE 25 *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when*

(i) at least two ladies are included ;

(ii) at most two ladies are included ?

SOLUTION (i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways :

I Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

II Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

III Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, therefore, by the fundamental principle of addition, the total number of ways of forming the committee

$$= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1 = 120 + 60 + 6 = 186$$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways :

I Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

II Selecting 4 gents only out of 6 and one lady out of 4. This can be done in ${}^6C_4 \times {}^4C_1$ ways.

III Selecting 3 gents only out of 6 and two ladies out of 4. This can be done is ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so, the total number of ways of forming the committee

$$= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186.$$

EXAMPLE 26 *On how many nights may a watch of 4 men be arranged by a security agency from a crew of 16 so that no two watches are identical ? On how many of these nights would a particular person be off-duty ?*

SOLUTION The required number of nights

= Number of ways of selecting 4 men out of 16

$$= {}^{16}C_4$$

$$= 1820$$

The number of nights a particular person is on duty

$$= {}^{15}C_3$$

$$= 455$$

$\therefore$ The number of nights a particular person is off-duty

$= 1820 - 455$

$= 1365$

EXAMPLE 27 *A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw ?*

SOLUTION There are 9 balls in the box out of which three balls can be drawn in ${}^9C_3$ ways. But these ways also include those ways in which no black ball is drawn. The total number of such ways is ${}^6C_3$.

Hence, the number of ways in which at least one black ball is included

= Total number of ways of drawing 3 balls from 9 balls

– Number of ways of drawing no black ball from 9 balls

$= {}^9C_3 - {}^6C_3 = 84 - 20 = 64.$

EXAMPLE 28 *How many diagonals are there in a polygon with n sides ?*

SOLUTION A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time

= Number of ways of selecting 2 out of n

$= {}^nC_2 = \frac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \frac{n(n-3)}{2}$

$= \frac{n(n-1)}{2} - n$

EXAMPLE 29 *A polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let there be n sides of the polygon. We know that the number of diagonals of n sided polygon is $\frac{n(n-3)}{2}$. Therefore,

$$\frac{n(n-3)}{2} = 44$$

$\Rightarrow \quad n^2 - 3n - 88 = 0$

$\Rightarrow \quad (n-11)(n+8) = 0$

$\Rightarrow \quad n = 11 \qquad [\because n > 0]$

Hence, there are 11 sides of the polygon.

EXAMPLE 30 *There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points; (ii) number of triangles that can be formed with the vertices at these points.*

SOLUTION (i) Number of straight lines formed joining the 10 points, taking 2 at a time

$= {}^{10}C_2 = \frac{10!}{2!\,8!} = 45.$

Number of straight lines formed by joining the four points, taking 2 at a time

$= {}^4C_2 = \frac{4!}{2!\,2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines $= 45 - 6 + 1 = 40.$

(ii) Number of triangles formed by joining the points, taking 3 at a time

$= {}^{10}C_3 = \frac{10!}{3!\,7!} = 120$

Number of triangles formed by joining the 4 points, taken 3 at a time

$= {}^4C_3 = {}^4C_1 = 4.$

But, 4 collinear points cannot form a triangle when taken 3 at a time. So,

Required number of triangles $= 120 - 4 = 116$

EXAMPLE 31 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

SOLUTION The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But, 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore, instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535.$

EXAMPLE 32 *There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices ? How many can be formed if point A is also included ?*

SOLUTION A triangle can be constructed in the following ways:

(i) *By taking two points on AB and one point on AC,*

(ii) *By taking two points on AC and one point on AB.*

Therefore,

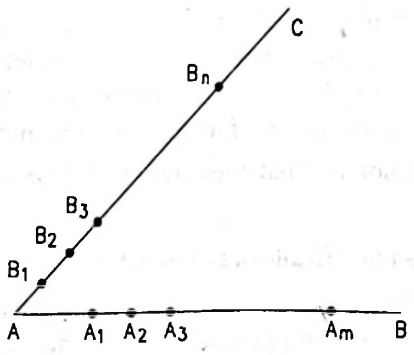


Fig. 22.4

Total number of triangles

$$= {}^{m}C_2 \times {}^{n}C_1 + {}^{m}C_1 \times {}^{n}C_2$$

$$= \frac{m(m-1)}{2} \times n + m \times \frac{n(n-1)}{2}$$

$$= \frac{mn}{2}(m+n-2)$$

If point A is included, then total number of triangles

$$= {}^{m+1}C_2 \times {}^{n}C_1 + {}^{m}C_1 \times {}^{n}C_2$$

$$= \frac{(m+1)\,mn}{2} + \frac{mn(n-1)}{2}$$

$$= \frac{mn}{2}(m+n)$$

EXAMPLE 33 *There are n straight lines in a plane, no two of which are parallel, and no three pass through the same point. If their points of intersection are joined, find the number of fresh lines thus obtained.*

SOLUTION Since no two lines are parallel and no three are concurrent, therefore n straight lines intersect at ${}^{n}C_2 = N$ (say) points. These N points are joined to generate some new lines as well as old lines. Since two points are required to determine a line, therefore total number of lines obtained by joining N-points is ${}^{N}C_2$. In those ${}^{N}C_2$ lines each old line has been counted ${}^{n-1}C_2$ times because on each old line there will be $(n-1)$ points of intersection made by the remaining $(n-1)$ lines.

Hence, the total number of fresh lines.

$$= {}^{N}C_2 - n \,.\, {}^{n-1}C_2$$

$$= \frac{N(N-1)}{2} - \frac{n(n-1)(n-2)}{2}$$

$$= \frac{n(n-1)}{4}\left\{\frac{n(n-1)}{2} - 1\right\} - \frac{n(n-1)(n-2)}{2}$$

$$= \frac{n(n-1)(n^2-n-2)}{8} - \frac{n(n-1)(n-2)}{2}$$

$$= \frac{n(n-1)}{8}(n^2-5n+6)$$

$$= \frac{n(n-1)(n-2)(n-3)}{8}$$

EXAMPLE 34 *If m parallel lines in plane are intersected by a family of n parallel lines, find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in ${}^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^{n}C_2$ ways. Hence, the number of parallelograms formed

$$= {}^{m}C_2 \times {}^{n}C_2 = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2}$$

$$= \frac{mn(m-1)(n-1)}{4}$$

EXAMPLE 35 *There are n points in space, no four of which are in the same plane with the exception of m, which are all in the same plane, find how many planes are there each contaning three of the points.*

SOLUTION Total number of planes containing three points only is ${}^{n}C_3$. But it is given that m points of the given n points are in the same plane. So, the plane containing these m points is counted ${}^{m}C_3$ times in computing ${}^{n}C_3$.

Hence, required number of planes

$$= {}^{n}C_3 - {}^{m}C_3 + 1$$

$$= \frac{n(n-1)(n-2)}{3!} - \frac{m(m-1)(m-2)}{3!} + 1$$

$$= \frac{n(n-1)(n-2)}{6} - \frac{m(m-1)(m-2)}{6} + 1$$

EXAMPLE 36 *Suppose a city has m parallel roads running East-West and n parallel roads running North-South. How many rectangles are formed with their sides along these roads ? If the distance between every consecutive pair of parallel roads is the same, how many shortest possible routes are there to go from one corner of the city to its diagonally opposite corner ?*

SOLUTION A rectangle is formed by two points of parallel lines such that two lines in one pair are perpendicular to the lines in another pair. Therefore to form a rectangle we require two parallel roads from East to West and two parallel roads from North to South. Two roads from East to West can be chosen in ${}^{m}C_2$ ways and two roads from North to South in ${}^{n}C_2$ ways.

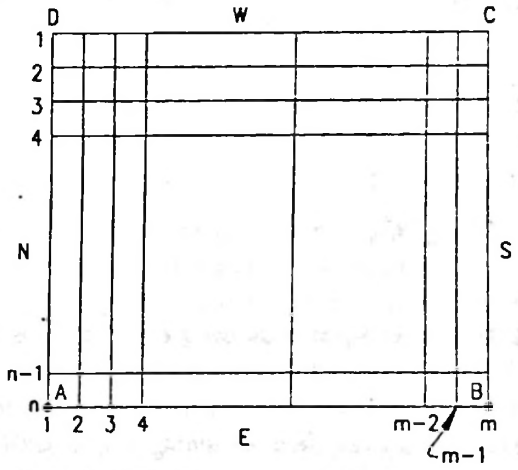


Fig. 22.5

Therefore, the total number of rectangles formed

$$= {}^{m}C_2 \times {}^{n}C_2 = \frac{mn(m-1)(n-1)}{4}$$

In order to go from one corner, say A, of the city to its diagonally opposite corner C a man has to cross $(m-1)$ roads from East to West and $(n-1)$ roads from South to North. This

means that out of total $(m+n-2)$ cuts he has to select $m-1$ cuts from East to West and $(n-1)$ cuts from South to North. This can be done in

$$^{m+n-2}C_{m-1} \times {}^{n-1}C_{n-1} \text{ ways.}$$

Hence, total number of possible routs = $^{m+n-2}C_{m-1}$.

EXAMPLE 37 *There are n straight lines in a plane such that n_1 of them are parallel in one direction, n_2 are parallel in different direction and so on, n_k are parallel in another direction such that $n_1 + n_2 + \ldots + n_k = n$. Also no three of the given lines meet at a point. Prove that the total number of points of intersection is*

$$\frac{1}{2}\left[n^2 - \sum_{r=1}^{k} n_r^2\right]$$

SOLUTION If no two of n given lines are parallel and no three of them meet at a point, then the total number of points of intersection is $^nC_2$. But it is given that there are k sets of $n_1, n_2, n_3, \ldots, n_k$ parallel lines such that no line in one set is parallel to a line in any other set. Also, lines of one set do not intersect with each other. Therefore, lines of one set do not provide any point of intersection. Hence,

Total number of points of intersection

$$= {}^nC_2 - \left({}^{n_1}C_2 + {}^{n_2}C_2 + \ldots + {}^{n_k}C_2\right)$$

$$= \frac{n(n-1)}{2} - \left\{\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \ldots + \frac{n_k(n_k-1)}{2}\right\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\left\{(n_1^2 + n_2^2 + \ldots + n_k^2) - (n_1 + n_2 + \ldots + n_k)\right\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\left\{(n_1^2 + n_2^2 + \ldots + n_k^2) - n\right\}$$

$$= \frac{n_2}{2} - \frac{1}{2}(n_1^2 + n_2^2 + \ldots + n_k^2)$$

$$= \frac{1}{2}\left[n^2 - \sum_{r=1}^{k} n_r^2\right]$$

EXAMPLE 38 *Haneef Mohammad has x children by his first wife and Mrs Fatima has $x+1$ children by her first husband. If Haneef and Fatima marry with each other and have children of their own. The whole family has 10 children. Assuming that two children of the same parents do not fight, find the maximum number of fights that can take place among children.*

SOLUTION We have,

Number of children of Haneef Mohammad and Fatima

$$= 10 - (x + x + 1)$$

$$= 9 - 2x$$

It is given that two children of the same parents do not fight. Therefore,

Number of fights between the children

$$= {}^{10}C_2 - \left({}^xC_2 + {}^{x+1}C_2 + {}^{9-2x}C_2\right)$$

$$= \frac{10 \times 9}{2} - \left\{\frac{x(x-1)}{2} + \frac{x(x+1)}{2} + \frac{(9-2x)(8-2x)}{2}\right\}$$

$$= 45 - \frac{1}{2}\{x^2 - x + x^2 + x + 72 - 34x + 4x^2\}$$

$$= 45 - \{3x^2 - 17x + 36\}$$

$$= -3x^2 + 17x + 9$$

$$= -3\left(x^2 - \frac{17}{3}x - 3\right)$$

$$= -3\left\{\left(x - \frac{17}{6}\right)^2 - \frac{397}{36}\right\}$$

$$= \frac{397}{12} - 3\left(x - \frac{17}{6}\right)^2$$

Clearly, this will be maximum if $\frac{x-17}{6} = 0$ i.e. $x = \frac{17}{6}$. But x cannot be a fraction. Therefore, $x = 3$.

Thus, the number of fights will be maximum when $x = 3$ and in that case, we have

$$\text{Maximum number of fights} = \frac{397}{12} - 3\left(3 - \frac{17}{6}\right)^2$$

$$= \frac{397}{12} - \frac{1}{12} = \frac{396}{12} = 33.$$

EXAMPLE 39 *In how many ways three different rings can be worn in four fingers with at most one ring in each finger?*

SOLUTION In order to put 3 rings in four finger we first select 3 fingers out of 4 and then arrange the rings in the selected fingers. Hence, the required number of ways is

$$^4C_3 \times 3! = \frac{4!}{(4-3)!} = 24.$$

EXAMPLE 40 *Find the number of ways of putting five different rings on four fingers of the left hand. (ignoring the difference in the size of rings and fingers).*

SOLUTION Let R_1, R_2, R_3, R_4 and R_5 be five rings. Since these five rings are to be put on four fingers. This is possible only when more than one ring is put on a finger. Consider ring R_1. It can be placed on any of the four fingers. So, there are four ways to place the ring R_1. After placing ring R_1, we have 5 choices for ring R_2, since for the finger which holds R_1 we have two choices to place R_2, either below R_2 or above R_2. Similarly, ring R_3 can be placed in 6 ways, ring R_4 in 7 ways and ring R_5 in 8 ways. Thus, the total number of ways to arrange five distinct rings on four fingers of the left hand is

$$4 \times 5 \times 6 \times 7 \times 8 = \frac{8!}{3!} = 6720.$$

EXAMPLE 41 *In how many ways five different rings can be worn in four fingers with at least one ring in each finger ?*

SOLUTION Since at least one ring is to be worn in each finger. So, we first select four rings out of 5 given rings and then they are arranged in fingers. This can be done in ${}^5C_4 \times 4!$ ways. Now, one ring is left which can be worn in any one of the four fingers in 4 ways.

Hence, required number of ways $= {}^5C_4 \times 4! \times 4$

$= 480.$

EXAMPLE 42 *In how many ways five different rings can be put on four fingers with at least one ring on each finger ?*

SOLUTION Since at least one ring is to be put in each finger. So, we select four rings out of 5 and then they are arranged in fingers. This can be done in ${}^5C_4 \times 4!$ ways. Now, one ring is left out which can be put on finger in two ways, either above the ring already placed on it or below it.

So, the required number of ways $= {}^5C_4 \times 4! \times 4 \times 2$

$= 960$

EXAMPLE 43 *There are r distinct objects and n distinct boxes such that $n > r$. Find the number of ways of*

(i) *putting r objects in n boxes if each box is sufficient to contain r objects.*

(ii) *arranging r objects in n boxes so that each box contains only one object.*

(iii) *arranging r objects in n boxes if, a box may contain any number of objects.*

SOLUTION Let $O_1, O_2, O_3, \ldots, O_r$ be r objects and $B_1, B_2, \ldots, B_n$ be n boxes.

(i) In this case, an object can be placed in any one of the n boxes. So, each object can be placed in n ways. Since there are r objects. So,

Total number of ways $= n \times n \times n \times \ldots \times n$

$= n^r.$

(ii) In this case, we first select r boxes out of n boxes and then we arrange r objects in the chosen boxes. So, required number of ways

$= {}^nC_r \times r!$

(iii) In this case, object O_1 can be placed in any one of n boxes. So, O_1 can be placed in n ways. After placing O_1 in a box, we have $(n+1)$ choices for the objects O_2, since for the box that contains O_1 we have two choices to place O_2 either before O_1 or after O_1. Similarly, object O_3 has $(n+2)$ choices, object O_4 has $(n+3)$ choices and so on. Finally, object O_r has $(n+r-1)$ choices.

Hence,

The number of ways of arranging r distinct objects in n-distinct boxes is

$$n(n+1)(n+2)\ldots(n+r-1)$$
$$= \frac{(n+r-1)!}{(n-1)!} = {}^{n+r-1}P_r.$$

EXAMPLE 44 *Let A and B be two sets consisting of m and n elements respectively. Determine the number of functions from A to B. Also, find the number of one-one functions from A to B.*

SOLUTION Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the given sets. In order to define a function from A to B, we have to associate each element of set A to a unique element of B. Consider a_1. This can be associated to any one of $b_1, b_2, \ldots, b_n$. So, a_1 can be associated in n ways. Similarly, $a_2, a_3, \ldots, a_m$ each can be associated in n ways. So, total number of functions from A to B

$$= \underbrace{n \times n \times \ldots \times n}_{\text{m-times}} = n^m.$$

In order to define a one-one function from A to B, we associate different elements of set A to distinct elements in set B. This is possible only when $n \geq m$. To determine the total number of one-one functions from A to B, we first select m elements from B and then elements in A are associated to elements in B. The first part can be done in ${}^nC_m$ ways and the second part in $m!$ ways.

Hence, total number of one-one functions $= {}^nC_m \times m!$

EXAMPLE 45 *In how many ways can a pack of 52 cards be divided equally among four players in order ?*

SOLUTION We have to divide 52 playing cards equally among 4 players. This means that each player must get 13 cards. For the first player there are ${}^{52}C_{13}$ ways. Now, 13 cards are selected from the remaining 39 cards for the second player. This can be done in ${}^{39}C_{13}$ ways. Now, 26 cards are left, out of which 13 cards are to be given to the third player. This can be done in ${}^{26}C_{13}$ ways. We have now 13 cards which can be given to the fourth player in ${}^{13}C_{13}$ ways.

Hence, the total number of ways

$$= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$$
$$= \frac{52!}{(13!)^4}$$

EXAMPLE 46 *There are two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that men played between themselves exceeds by 66 to the number of games the men played with women. How many participants were there ? How many games were played ?*

SOLUTION Let there be m men participating in the tournament.

Since each participant played two games with the other. Therefore,

Number of games played between men themselves

$$= 2({}^nC_2) = n(n-1)$$

Number of games played between men and women

$$= 2(n+n) = 4n$$

It is given that the number of games that men played between themselves exceeds by 66 the number of games the men played with women. Therefore,

$$n(n-1) = 66 + 4n$$

$$\Rightarrow n^2 - 5n - 66 = 0$$

$$\Rightarrow (n-11)(n+6) = 0$$

$$\Rightarrow n = 11 \qquad [\because n \in N]$$

Hence,

Total number of participants $= 11 + 2 = 13.$

and,

Total number of games played $= 2\,(^{13}C_2) = 156.$

EXAMPLE 47 *Two different packs of cards are shuffled together. Cards are dealt equally among 4 players, each getting 13 cards. In how many ways can a player get his cards if no two cards are from the same suit with the same denomination.*

SOLUTION Here, we have 52 cards, each card being 2 in number. It is given that no two cards are to be of the same suit with the same denomination. So, we first draw 13 cards from 52 cards. This can be done in $^{52}C_{13}$ ways. Now, each of 13 cards selected can be chosen in 2 ways either from first pack or from 2nd pack.

Hence, required number of ways $= {}^{52}C_{13} \times 2^{13}$.

EXAMPLE 48 *There are n bags which are marked by n consecutive positive integers starting with the positive integer r. Each bag contains as many different flowers as the number levelled against the bag. A boy has to pickup r flowers from any one of the bags. In how many different ways can this be done ?*

SOLUTION It is given that the bags are marked with the numbers $r, r+1, r+2, \ldots, r+(n-1)$ and each bag contains as many different flowers as the number levelled against it. Therefore,

The number of ways of selecting r flowers from the bag marked with the number $r+(k-1)$ is $^{r+k-1}C_r$, where $k = 1, 2, 3, \ldots, n$.

Hence, the required number of ways

$$= \sum_{k=1}^{n} {}^{r+k-1}C_r$$

$$= {}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{r+n-1}C_r$$

$$= \left\{{}^{r+1}C_{r+1} + {}^{r+1}C_r\right\} + {}^{r+2}C_r + {}^{r+3}C_r + \ldots + {}^{r+n-1}C_r$$

$$\left[\because {}^{r}C_r = {}^{r+1}C_{r+1}\right]$$

$$= \left\{{}^{r+2}C_{r+1} + {}^{r+2}C_r\right\} + {}^{r+3}C_r + \ldots + {}^{r+n-1}C_r$$

$$= {}^{r+3}C_{r+1} + {}^{r+3}C_r + \ldots + {}^{r+n-1}C_r$$

$$= \ldots\ldots\ldots\ldots$$

$$= {}^{r+n-1}C_{r+1} + {}^{r+n-1}C_r$$

$$= {}^{r+n}C_{r+1}.$$

EXAMPLE 49 *A student is allowed to select at most n books from a collection of (2n+ 1) books. If the total number of ways in which he can select a book is 63, find the value of n.*

SOLUTION The total number of ways of selecting at most n books out of $(2n + 1)$ books is

$$^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$$

By hypothesis, we have

$$^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 63$$

$$\Rightarrow 2\left\{{}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n\right\} = 126$$

$$\Rightarrow 2\,.\,{}^{2n+1}C_1 + 2\,.\,{}^{2n+1}C_2 + 2\,.\,{}^{2n+1}C_3 + \ldots + 2\,.\,{}^{2n+1}C_n = 126$$

$$\Rightarrow \left\{{}^{2n+1}C_1 + {}^{2n+1}C_{2n}\right\} + \left\{{}^{2n+1}C_2 + {}^{2n+1}C_{2n-1}\right\}$$

$$+ \left\{{}^{2n+1}C_3 + {}^{2n+1}C_{2n-2}\right\} + \ldots + \left\{{}^{2n+1}C_n + {}^{2n+1}C_{n+1}\right\} = 126$$

$$\left[\because {}^{2n+1}C_r = {}^{2n+1}C_{(2n+1)-r},\ r = 1, 2, 3, \ldots, n\right]$$

$$\Rightarrow {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots$$

$$+ {}^{2n+1}C_{2n} = 126$$

$$\Rightarrow {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots$$

$$+ {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1} = 126 + {}^{2n+1}C_0 + {}^{2n+1}C_{2n+1}$$

$$\Rightarrow 2^{2n+1} = 126 + 1 + 1 \qquad \left[\because \sum_{r=0}^{2n+1} {}^{2n+1}C_r = 2^{2n+1}\right]$$

$$\Rightarrow 2^{2n+1} = 128$$

$$\Rightarrow 2^{2n+1} = 2^7$$

$$\Rightarrow 2n + 1 = 7 \Rightarrow n = 3.$$

EXAMPLE 50 *Six X's have to be placed in the squares of the diagram shown in Fig.22.6 such that each row contains at least one X. In how many ways can this be done ?*

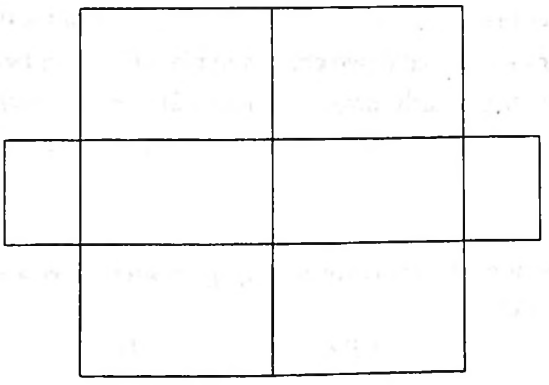


Fig. 22.6

SOLUTION Six X's can be placed in 8 squares shown in Fig. 22.6 such that each row contains at least one X in the following mutually exclusive ways :

	Ist row	2nd row	3rd row
I	2	3	1
II	2	2	2
III	1	4	1
IV	1	3	2

Hence, the required number of ways

$= {}^2C_2 \times {}^4C_3 \times {}^2C_1 + {}^2C_2 \times {}^4C_2 \times {}^2C_2$

$+ {}^2C_1 \times {}^4C_4 \times {}^2C_1 + {}^2C_1 \times {}^4C_3 \times {}^2C_2$

$= 8 + 6 + 4 + 8$

$= 26.$

EXAMPLE 51 *Two packs of 52 playing cards are shuffled together the number of ways in which a man can deal with 26 cards so that he does not get two cards of the same suit and same denomination.*

SOLUTION The number of ways of choosing 26 cards out of 52 cards is ${}^{52}C_2$. Now there are two ways in which each card can be dealt, because a card can be either from the first pack or from the second pack. Hence, the number of ways in which the cards can be dealt is ${}^{52}C_{26} \times 2^{26}$.

EXERCISE 22.4

1. Seven athletes are participating in a race. In how many ways can the first three prizes be won ?
2. How many different signals can be made by 5 flags from 8 flags of different colours ?
3. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
4. Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them ?
5. How many different signals can be given using any number of flags from 5 flags of different colours ?
6. In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated ?
7. In how many ways 7 pictures can be hung from 5 picture rais on a wall ?
8. Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.
9. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.
10. How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed ?
11. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done ?
12. Find the number of different 4-letter words, with or without meanings, that can be formed from the letters of the word 'NUMBER'.
13. How many three-digit numbers are there, with distinct digits, with each digit odd ?
14. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once ?
15. How many words, with or without meaning, can be formed by using the letters of the word 'TRIANGLE' ?
16. There are two works each of 3 volumes and two works each of 2 volumes. In how many ways can the 10 books be placed on a shelf so that the volumes of the same work are not separated ?
17. There are 6 items in column *A* and 6 items in column *B*. A student is asked to match each item in column *A* with an item in column *B*. How many possible, correct or incorrect, answers are there to this question ?
18. How mnay three-digit numbers are there, with no digit repeated?
19. How many 6-digit telephone numbers can be constructed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once ?
20. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them ?
21. If a denotes the number of permutations of $(x + 2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x - 11$ things taken all at a time such that $a = 182\,bc$, find the value of x.
22. find the sum of all the numbers greater than 10000 formed by using the digits 1, 3, 5, 7, 9 no digit being repeated in any number.
23. An eight -oared boat is to be manned by a crew chosen from 11 men, of whom 3 can steer but cannot row and the rest can row but cannot steer. In how many ways can the crew be arranged, if two of the men only row on bow side ?
24. A question paper has two parts, Part *A* and Part *B*, each containing 10 questions. If the student has to choose 8 from Part *A* and 5 from Part *B*, in how many ways can he choose the questions ?
25. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?
26. A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when
    (i) at least two ladies are included ;
    (ii) at most two ladies are included ?

27. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour ?

28. How many triangles can be formed by joining the vertices of a hexagon ?

29. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition) ?

30 From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition; at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made ?

31. How many different selections of 4 books can be made from 10 different books, if

(i) there is no restriction;
(ii) two particular books are always selected;
(iii) two particular books are never selected ?

32. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer ?

33. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted ?

34. A student has to answer 10 questions, choosing at least 4 from each of part *A* and part *B*. If there are 6 questions in part *A* and 7 in part *B*, in how many ways can the student choose 10 questions ?

35. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.

36. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions ?

37. There are 10 points in a plane of which 4 are collinear. How many different straight lines can be drawn by joining these points.

38. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.

39. How many triangles can be obtained by joining 12 points, five of which are collinear ?

40. In how many ways can a committee of 5 persons be formed out of 6 men and 4 women when at least one woman has to be necessarily selected ?

41. In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made ?

42. A parallelogram is cut by two sets of m lines parallel to its sides. Find the number of parallelograms thus formed.

43. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them ?

44. Determine the number of 5 cards combinations out of a deck of 52 cards if at lest one of the 5 cards has to be a king ?

45. In how many ways can a committee of 5 women and 6 men be chosen from 10 women and 8 men if Mr *A* refuses to serve on the committee if Ms *B* is a member ?

ANSWERS

1. 210	**2.** 6720	**3.** 2880	**4.** 172800
5. 325	**6.** $2(16! \times 16!)$		**7.** 2520
8. 5274	**9.** 8!	**10.** 5040	**11.** 24
12. 360	**13.** 60	**14.** 120	**15.** 8!
16. 3456	**17.** 720	**18.** 648	**19.** 1680
20. 86400	**21.** 12	**22.** 6666600	**23.** 25920
24. 11340	**25.** 1008	**26.** 186	**27.** 425
28. 20	**29.** 11	**30.** 104874	
31. (i) 210		(ii) 28	(iii) 70
32. (i) 224		(ii) 896	**33.** $2({}^{20}C_5 \times {}^{20}C_6)$
34. 266		**35.** 3	**36.** 780
37. 40		**38.** (i) 9	(ii) 104
39. 210		**40.** 246	
41. ${}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20} \times {}^{35}C_0$			
42. $({}^{m+2}C_2)^2$		**43.** (i) 144	(ii) 806.
44. 886656		**45.** 4410	

HINTS TO SELECTED PROBLEMS

11. No. of ways = No. of arrangements of 4 books, taken all at a time

$= {}^4P_4$

12. Total no. of words = No. of arrangements of 6 letters, taken 4 at a time

$= {}^6P_4$

13. Required number of numbers = No. of arrangements of digits 1, 3, 5, 7, 9 by taking 3 at a time $= {}^5P_3$

16. Let $\frac{W_{11}, W_{12}, W_{13}}{W_1}; \frac{W_{21}, W_{22}, W_{23}}{W_2}; \frac{W_{31}, W_{32}}{W_3}; \frac{W_{41}, W_{42}}{W_4}$

be 4 works. These 4 works can be arranged in 4! ways. Now, volumes of each work can be arranged in the following ways : $W_1 \to 3!$ ways ; $W_2 \to 3!$ ways, $W_3 \to 2!$ ways, $W_4 \to 2!$ ways.

Hence, total no. of ways to arrange all books

$= 4!\,(3! \times 3! \times 2! \times 2!) = 3456$

17. Each answer to the given question is an arrangement of the 6 items of column *B* keeping the order of items in column *A*

fixed. Hence, the total number of answers = no. of arrangements of 6 items in column $B = {}^6P_6 = 6!$

18. Total number of three digit numbers with distinct digits
$$= {}^{10}P_3 - {}^9P_2$$

19. Required number of telephone numbers $= {}^8P_4$

20. Five girls can sit on chairs in a row in ${}^5P_5 = 5!$ ways. Also, 6 boys can stand behind them in a row in ${}^6P_6 = 6!$ ways.
Hence, the total number of ways $= 5! \times 6!$

21. Total number of products
= No. of ways of selecting 2 or 3 or all out of 4 numbers 3, 5, 7, 11
$= {}^4C_2 + {}^4C_3 + {}^4C_4 = 6 + 4 + 1 = 11.$

30. Since two girls who won the prizes last year are to be included in every selection. So, we have to select 8 students out of 12 boys and 8 girls, choosing at least 4 boys and at least two girls. This can be done in
$${}^{12}C_6 \times {}^8C_2 + {}^{12}C_5 \times {}^8C_3 + {}^{12}C_4 \times {}^8C_4 = 104874 \text{ ways.}$$

32. (i) Required number of ways $= {}^4C_1 \times {}^8C_5$
(ii) Required no. of ways
= Total no. of ways – No. of ways of selecting no officer
$= {}^{12}C_6 - {}^8C_6$

33. Required no. of ways $= {}^{20}C_5 \times {}^{20}C_6 + {}^{20}C_6 \times {}^{20}C_5$.

34. The various possibilities are : (i) 4 from part A and 6 from part B (ii) 5 from part A and 5 from part B (iii) 6 from part A and 4 from part B.
So, the required no. of ways
$$= {}^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4 = 266.$$

35. Required number of ways $= {}^3C_2$.

36. Required no. of ways
$$= {}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5 = 780.$$

37. Number of straight lines $= {}^{10}C_2 - {}^4C_2 + 1$.

39. Number of triangles $= {}^{12}C_3 - {}^5C_3$.

41. 52 families have at most 2 children, while 35 families have more than 2 children. The selection of 20 families of which at least 18 families must have at most 2 children can be made as under :
(I) 18 families out of 52 and 2 families out of 3
or, (II) 19 families out of 52 and 1 family out of 35
or, (III) 20 families out of 52.

42. Each set of parallel lines consists of $(m + 2)$ lines and each parallelogram is formed by choosing two lines from the first set and two straight lines from the second set. Hence, the total number of parallelograms $= {}^{m+2}C_2 \times {}^{m+2}C_2$.

—❑❑❑—

22.5 PERMUTATIONS UNDER CERTAIN CONDITIONS

In the previous section, we have discussed permutations of a given nubmer of distinct objects by taking some of them at a time. In this section, we shall discuss permutations of distinct objects under certain conditions like a certain item (object) is never taken in each arrangement or, a certain number of items occur together in every arrangement or, a certain object is always included in each arrangement etc.

In order to illustrate the method, let us discuss the following theorem.

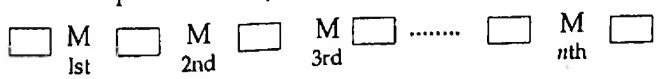
THEOREM *The number of all permutations (arrangements) of n different objects taken r at a time,*

(i) *when a particular object is to be always included in each arrangement is* ${}^{n-1}C_{r-1} \times r!$

(ii) *when a particular object is never taken in each arrangement is* ${}^{n-1}C_r \times r!$

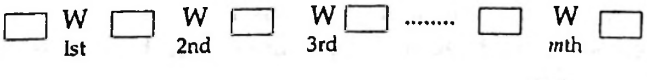
(iii) *when two specified objects always occur together is* ${}^{n-2}C_{r-2} \times (r-1)! \times 2!$

PROOF (i) In order to arrange n distinct items by taking r at a time when a particular object is to be always included in each arrangement, let us first put aside the specified item and select $(r-1)$ items from the remaining $(n-1)$ items. This can be done in ${}^{n-1}C_{r-1}$ ways. Now, we have r items, namely, one specified item and $(r-1)$ selected items. These r items can be arranged in $r!$ ways. Hence, the required number of arrangements is ${}^{n-1}C_{r-1} \times r!$



(ii) If a particular object is never taken in each arrangement, then we first select r objects from the remaining $(n-1)$ objects and then they are arranged. So, required number of arrangements is
$${}^{n-1}C_r \times r!$$

(iii) In order to arrange n distinct items by taking r at a time when two specified objects always occur together, we first select $(r-2)$ objects from the remaining $(n-2)$ objects. This can be done in ${}^{n-2}C_{r-2}$ ways. Now, we consider two specified objects temporarily as a single object and mix it with $(r-2)$ selected objects. Now, these $(r-1)$ objects can be arranged in $(r-1)!$ ways. But, two specified objects can be put together in 2! ways. Hence, required number of arrangements is
$${}^{n-2}C_{r-2} \times (r-1)! \times 2!$$



ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *m men and n women are to be seated in a row so that no two women sit together. If m > n, show that the number of ways in which they can be seated is*

$$\frac{m!\,(m+1)!}{(m-n+1)!}.$$

SOLUTION Since no two women should sit together and $m > n$. So, we first arrange m men in a row in $m!$ ways. Now, $(m+1)$ places are there for n women to sit. This can be done in

$$^{m+1}C_n \times n! \text{ ways.}$$

Hence, by the fundamental principle of counting the total number of seating arrangements is

$$m! \times {}^{m+1}C_n \times n!$$

$$= m! \times \frac{(m+1)!}{(m+1-n)!\,n!} \times n!$$

$$= \frac{m!\,(m+1)!}{(m-n+1)!}$$

EXAMPLE 2 *Find the number of ways of arranging m white and n black balls in a row $(m > n)$ so that no two black balls are placed together.*

SOLUTION Since $m > n$ and we have to find the total number of arrangements of $m + n$ balls so that no two black balls are together. So, we first arrange m white balls in a row. This can be done in only one way as all white balls are identical.

Now, we have $(m+1)$ places between m white balls and we want that no two black balls are together. So, we select n places out of $(m+1)$ and then arrange them in these n places. This can be done in

$$^{m+1}C_n \times 1 = {}^{m+1}C_n \text{ ways.}$$

Hence, the required number of ways $= {}^{m+1}C_n$.

EXAMPLE 3 *In how many ways can a committee of m women and n men be chosen from M women and N men if Mr. A refuses to serve on the committee if Ms. B is a member ?*

SOLUTION The committee can be constituted in the following ways :

I *When Ms. B is a member of the committee.*

II *When Ms B is not a member of the committee.*

CASE I *When Ms. B is a member of the committee.*

In this case Mr. A is not a member of the commitee. Therefore, to form the committee we have to select $(m-1)$ women from $(M-1)$ remaining women and n men from $(N-1)$ men. This can be done in

$$^{M-1}C_{m-1} \times {}^{N-1}C_n \text{ ways.}$$

CASE II *When Ms. B is not a member of the committee :*

In this case, Mr. A may or may not be a member of the committee. So to form the committee, we have to select m women out of $(M-1)$ women and n men from N men. This can be done in

$$^{M-1}C_m \times {}^{N}C_n \text{ ways.}$$

Hence, the total number of ways in which the committee can be formed is

$$^{M-1}C_{m-1} \times {}^{N-1}C_n + {}^{M-1}C_m \times {}^{N}C_n.$$

EXAMPLE 4 *A family consists of a grandfather, m sons and daughters and 2n grand children. They are to be seated in a row for dinner. The grand children wish to occupy the n seats at each end and the grandfather refuses to have a grand child on either side of him. In how many ways can the family be made to sit ?*

SOLUTION There are $(m + 2n + 1)$ members in the family. So, they can be seated in a row on one side of a $(m + 2n + 1)$ seater table. The grand children can occupy n seats at each end of the table in $({}^{2n}C_n \times n! \times n!)$ ways. Since the grandfather does not want a gradchild on either side of him. So he can occupy any one of the remaining $(n-1)$ seats which can be done in $^{m-1}C_1$ ways. The remaining m seate can be occupied by m sons and daughters in $^{m}C_m \times m!$ ways.

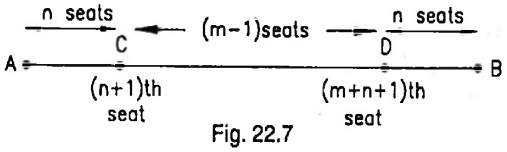


Fig. 22.7

Hence, by the fundamental principle of counting the required number of ways in which the family can be made to sit is

$$({}^{2n}C_n \times n! \times n!)\,({}^{m-1}C_1)\,({}^{m}C_m \times m!) \text{ ways}$$

$$= (2n!)\,(m-1)\,m! \text{ ways}$$

EXAMPLE 5 *Find the number of ways in which 5 boys and 5 girls be seated in a row so that*

(i) No two girls may sit together.

(ii) All the girls sit togetner and all the boys sit together

(iii) All the girls are never together.

SOLUTION (i) 5 boys can be seated in a row in $= 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in $^{6}C_5 \times 5!$ ways.

Hence, the number of ways in which no two girls sit together

$$= 5! \times ({}^{6}C_5 \times 5!) = 6! \times 5!$$

(ii) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements

$$= 2!\,(5! \times 5!) = 2\,(5!)^2.$$

(iii) The total number of ways in which all the girls are never together

= Total number of arrangements
– Total number of arrangements in which all the girls are always together

$= 10! - 5! \times 6!$

EXAMPLE 6 *Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.*

SOLUTION 5 boys can be arranged in a line in 5! ways. Since the boys and girls are alternating. So, corresponding each of the 5! ways of arrangements of 5 boys we obtain 5 places marked by cross as shown below :

(i) $B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$

(ii) $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5$.

Clearly, 5 girls can be arranged in 5 places marked by cross in (5! + 5!) ways.

Hence, the total number of ways of making the line

$$= 5! \times (5! + 5!) = 2\,(5!)^2$$

EXAMPLE 7 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats ?*

SOLUTION We have,

Total number of persons = 3 girls + 9 boys = 12.

Total number of numbered seats = $2 \times 3 + 4 \times 2 = 14$

So, total number of ways in which 12 persons can be seated on 14 seats = Number of arrangements or 14 seats by taking 12 at a time

$= {}^{14}C_{12} \times 12!$

Three girls can be seated together in a back row on adjacent seats in the following ways :

1, 2, 3 or 2, 3, 4 of first van

and, 1, 2, 3 or 2, 3, 4 of second-one.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats = $4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}C_9 \times 9!$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}C_9 \times 4 \times 3!$.

EXAMPLE 8 *How many natural numbers are there lying between 20,000 and 60,000, the sum of digits being even ?*

SOLUTION The numbers lying between 20,000 and 60,000 will be five digit numbers having 2, 3, 4 or 5 at ten thousands place. So, the ten thousands place can be filled in 4 ways.

T Th	Th	H	T	0

Each of thousands, hundreds and tens places can be filled in 10 ways. So, the first four places can be filled in

$4 \times 10 \times 10 \times 10$ ways.

After filling these the sum of the digits at these four places is either even or odd. Therefore, the units place can be filled by one of the digits 0, 2, 4, 6, 8 when the sum is even and if the sum is odd, the units place can be filled by one of the digits 1, 3, 5, 7, 9. In either case, the units place is filled in 5 ways.

Hence, the required number of numbers

$$= 4 \times 10 \times 10 \times 10 \times 5 = 20{,}000.$$

EXAMPLE 9 *How many numbers of five digits can be made with at least one repeated digit ?*

SOLUTION We have,

Total number of five digit numbers $= 9 \times 10 \times 10 \times 10 \times 10$

$= 90000.$

and,

Total number of five digit numbers with distinct digits

$= ({}^{10}C_5 \times 5! - {}^{9}C_4 \times 4!)$

$= (10 \times 9 \times 8 \times 7 \times 6 - 9 \times 8 \times 7 \times 6)$

$= 27216$

Hence,

The required number of numbers having at least one repeated digit

$= 90000 - 27216$

$= 62784.$

EXAMPLE 10 *How many four digit numbers with distinct digits can be made with the digits 0, 1, 2, 3, 4, 5 which are divisible by 3 ? How many of these will be divisible by 6 ?*

SOLUTION We know that a number is divisible by 3 if the sum of the digits in the number is divisible by 3.

We obseve that 0 + 1 + 2 + 3 + 4 + 5 = 15 which is divisible by 3. Therefore, to construct a four digit number divisible by 3, we have to omitt two digits out of the given digits whose sum is a multiple of 3. Thus, we have following options for four digits :

1, 2, 4, 5;

0, 3, 4, 5;

0, 1, 3, 5;

0, 2, 3, 4;

0, 1, 2, 3.

CASE I *When four digits of the number are 1, 2, 4, 5:*

In this case, the number of 4 digit numbers is 4!.

CASE II *When four digits of the number are 0, 3, 4, 5 or 0, 1, 3, 5 or 0, 2, 3, 4 or 0, 1, 2, 3 :*

In this case, the number of 4 digit numbers that can be made by the digits in any one of the four groups is 4! – 3!. Therefore, the number of four digit numbers formed by all the four groups

$= 4\,(4! - 3!)$

Hence, the required number of four-digit numbers divisible by 3

$= 4! + 4(4! - 3!)$

$= 24 + 4(24 - 6)$

$= 96.$

. A number is divisible by 6, if it is even as well as divisible by 3. So, let us now find the even numbers generated by the four digits in each of the five groups discussed above.

The number of 4 digit numbers divisible by 6 that can be made with the digits 1, 2, 4, 5 is equal to the number of four digit even numbers that can be mde with the digits 1, 2, 4, 5.

$= 2 \times 3!$ [$\because$ Units place can be filled in 2 ways and the remaining 3 places in 3! ways]

$= 12$

Similarly,

The number of four digit even numbers that can be made with the digits 0, 3, 4, 5

$= 3! + (3! - 2!) = 10$

[$\because$ The number should have 0 or 4 at units place. If 0 is at units place, then the number of 4 digit numbers is 3 ! . If 4 is at units place, then number of 4 digit numbers = (3! – 2!)]

The number of four digit even numbers that can be made with the digits 0, 1, 3, 5 .

$= 3! = 6$

The number of four digit even numbers that can be made with the digits 0, 2, 3, 4

$= 3! + (3! - 2!) + (3! - 2!)$

$= 14$

The number of four digit even numbers that can be made with the digits 0, 1, 2, 3

$= 3! + (3! - 2!)$

$= 10$

Hence, the number of 4-digit numbers divisible by 6

$= 12 + 10 + 6 + 14 + 10$

$= 52.$

EXAMPLE 11 *Find the sum of the four digit numbers that can be made with the digits 0, 3, 4, 5. Also find the sum of the four digit even numbers that can be made with the given digits.*

SOLUTION We know that the sum of the numbers formed by n digits is

$$(n-1)!\,(\text{Sum of the digits})\left(\frac{10^n - 1}{10 - 1}\right)$$

We obseve that the digits 0, 3, 4, 5 when taken all at a time generate 4 digit numbers as well as 3 digit numbers.

$\therefore$ Sum of the four digit numbers

= Sum of the numbers made with the digits 0, 3, 4, 5

– Sum of the numbers made with 0, 3, 4, 5 and having 0 at thousands place

$= (4-1)!\,(0+3+4+5)\left(\frac{10^4 - 1}{10 - 1}\right)$

$-(3-1)!\,(3+4+5)\left(\frac{10^3 - 1}{10 - 1}\right)$

$= 72\,(10^3 + 10^2 + 10 + 1) - 24\,(10^2 + 10 + 1)$

$= 72 \times 1111 - 24 \times 111$

$= 79992 - 2664$

$= 77328.$

In order to find the sum of the four digit even numbers that can be made with the digits 0, 3, 4, 5, we will first find the sum of the digits at ones, tens, hundreds and thousands places of all the even numbers formed with the given digits.

The even numbers have 0 or 4 in units place. Therefore,

S_1 = Sum of the digits in units place for all the numbers

= 0 × (Number of even numbers having 0 in units place) + 4 (Number of even numbers having 4 in units place)

$= 0 \times 3! + 4 \times (3! - 2!) = 16$

S_2 = Sum of the digits in tens place

= 0 × (Number of numbers with 0 in tens place) + 3 × (Number of even numbers with 3 in tens place) + 4 × (Number of even numbers with 4 in tens place) + 5 × (Number of even numbers with 5 in tens place)

$= 0 \times 2! + 3 \times (2! + 1) + 4 \times 2! + 5\,(2! + 1)$

$= 0 + 9 + 8 + 15$

$= 32.$

S_3 = Sum of the digits in hundreds place

= 0 × (Number of even numbers with 0 in hundreds place) + 3 × (Number of even numbers with 3 in hundred place) + 4 × (Number of even numbers with 4 in hundreds place) + 5 × (Number of even numbers with 5 in hundreds place)

$= 0 \times 2! + 3\,(2! + 1) + 4\,(2!) + 5\,(2! + 1)$

$= 0 + 9 + 8 + 15$

$= 32.$

S_4 = Sum of the digits in thousands place

= 3 × (Number of even numbers with 3 in thousands place)

+ 4 × (Number of even numbers with 4 in thousands place)

+ 5 × (Number of even numbers with 5 in thousands place)

$= 3 \times (1 \times 2! \times 2) + 4\,(2!) + 5\,(1 \times 2! \times 2)$

$= 12 + 8 + 20$

$= 40.$

Hence, required sum of even numbers

$= S_1 \times 10^0 + S_2 \times 10^1 + S_3 \times 10^2 + S_4 + \times 10^3$

$= 16 + 32 \times 10 + 32 \times 100 + 40 \times 1000$

$= 43536.$

EXAMPLE 12 *Seven men and 3 women are to be seated in a row. If three women sit together and two of the men refuse to take consecutive seats, find the total number of seating arrangements.*

SOLUTION We have,

Required number of seating arrangements

= Number of seating arrangements of 10 persons when three women sit together

– Number of seating arrangements of 10 persons when three women sit together and two men (who refused to sit together).

$= 8! \times 3! - 7! \times 3! \times 2!$

$= (8 - 2) \times 7! \times 3!$

$= 6 \times 7! \times 3! = 181440$

EXAMPLE 13 *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together ?*

SOLUTION In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below :

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here, E denotes the position of an English book and × that of a Hindi book.

Since there are 21 books on English, the number places mark × are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways $= 1 \times {}^{22}C_{19} \times 1 = 1540.$

EXAMPLE 14 *A regular polygon of 10 sides is constructed. In how many ways can 3 vertices be selected so that no two vertices are consecutive ?*

SOLUTION The required number of vertices

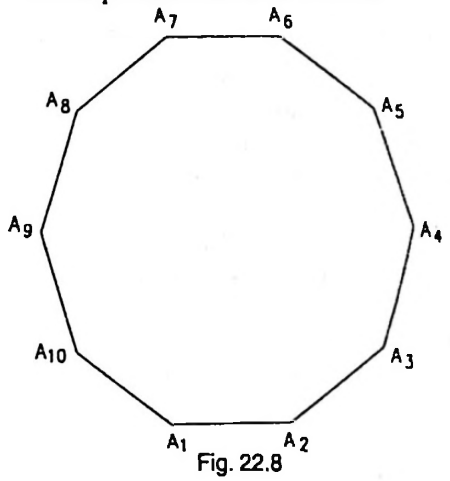


Fig. 22.8

= The total number of ways of selecting 3 vertices out of 10

– Number of ways of selecting three consecutive vertics

– Number of ways of selecting two consecutive vertices

Now,

Total number of ways of selecting 3 vertices out of 10 is ${}^{10}C_3$

Number of ways of selecting three consecutive vertices

$= 10$ [∵ Three consecutive vertices can be $A_1 A_2 A_3, A_2 A_3 A_4, \ldots, A_{10} A_1 A_2$]

Number of ways of selecting 2 consecutive vertices

$= 10 \times {}^6C_1$

[∵ Two consecutive vertices can be $A_1 A_2, A_2 A_3, \ldots, A_{10} A_1$
After selecting these two we select one vertex from the remaining 6 vertices]

Hence,

The required number of ways $= {}^{10}C_3 - 10 - 10 \times {}^6C_1 = 50.$

EXERCISE 22.5

1. In how many ways can the letters of the word PENCIL be arranged so that (i) N is always next to E ? (ii) N and E are always together ?

2. How many different words can be formed with the letters of the word EQUATION so that

(i) the words begin with E ?

(ii) the words begin with E and end with N! ?

(iii) the words begin and end with a consonant ?

3. How many words can be formed from the letters of the word, 'TRIANGLE' ? How many of these will begin with T and end with E ?

4. How many words can be formed with the letters of the word 'ORDINATE' so that vowels occupy odd places ?

5. In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places ?

6. How many words can be formed from the letters of the word 'DAUGHTER' so that

(i) the vowels always come together ?

(ii) the vowels never come together ?

7. In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together ?

8. In how many ways can 5 children be arranged in a row such that

(i) two of them, Ram and Shyam, are always together ?

(ii) two of them, Ram and Shyam, are never together ?

9. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible ?

10. How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit ?

11. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular and two on the other side. In how many ways can they be seated ?

12. How many different car licence plates can be constructed if the licences contain three letters of the English alphabet followed by a three digit number

(i) if repetitions are allowed ?

(ii) if repetitions are not allowed ?

13. A family consists of a grandfather, 5 sons and daughters and 8 grand children. They are to be seated in a row for dinner. The grand children wish to occupy the 4 seats at each end and the grandfather refuses to have a grand child on either side of him. In how many ways can the family be made to sit ?

14. There are 10 candidates for an examination out of which 4 are appearing in mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together ?

15. Four digit number are formed with the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. Find

(i) total number of numbers that can be formed.

(ii) total number of even numbers that can be formed.

(iii) total number of numbers which are divisible by 4.

(iv) total number of numbers which are divisible by 25.

16. Ten persons are to be seated in a row. Three of them are to be seated together, while two of the remaining seven do not wish to sit side by side. Find the number of ways in which this can be done.

ANSWERS

1. (i) 120 (ii) 240 **2.** (i) 7! (ii) 6! (iii) 4320 **3.** 40320, 720 **4.** 576 **5.** 12 **6.** (i) 4320 (ii) 36000 **7.** 282240 **8.** (i) 48 (ii) 72 **9.** $18! \times 1440$ **10.** 365 **11.** ${}^{8}P_4 \times {}^{5}P_2 \times 10!$ **12.** (i) $999 \times (26)^3$ (ii) ${}^{26}C_3 \times 3! \times {}^{10}C_3 \times 3!$ **13.** 11520 **14.** 604800 **15.** (i) 840 (ii) 360 (iii) 200 (iv) 40 **16.** 181440

—❑❑❑—

22.6 PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

So far we have discussed permutations of distinct objects (things) by taking some or all at a time. In this section, we intend to discuss the permutations of a given number of objects when objects are not all different. For example, the number of arrangements of the letters of the word MISSISSIPPI, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

THEOREM *The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is*

$$\frac{n!}{p!\,q!}$$

PROOF Let the required number of permutations be x. Consider one of these x permutations.

Replace p alike things in this permutation by p distinct things which are also different from others. These p different things may be permuted among themselves in $p!$ ways without changing the positions of other things. Similarly, if we replace q alike things by q distinct things, which are also different from others, then they can be permuted among themselves in $q!$ ways.

Thus, if both the replacements are done simultaneously, then we find that each one of the x permutations give rise to $p! \times q!$ permutations. Therefore, x permutations give rise to $x \times p! \times q!$ permutations. Now, each of these $x \times p! \times q!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore x \times p! \times q!$ = Number of permutations of n different things taken all at a time

$$= n!$$

Hence, $x = \dfrac{n!}{p!\,q!}$

REMARK 1 *The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; p_r are alike of rth kind such that $p_1 + p_2 + \ldots + p_r = n$, is*

$$\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$$

REMARK 2 *The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is*

$$\frac{n!}{p!\,q!}$$

REMARK 3 *Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is*

$$\frac{(p_1+p_2+\ldots+p_r)!}{p_1!\,p_2!\,p_3!\ldots p_r!}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *How many different words can be formed with the letters of the word 'MISSISSIPPI' ?*

SOLUTION There are 11 letters in the given word, of which 4 are *S*'s, 4 are *I*'s and 2 are *P*'s. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e.

$$\frac{11!}{4!\,4!\,2!}$$

Hence, the total number of words $= \frac{11!}{4!\,4!\,2!} = 34650$

EXAMPLE 2 *How many permutations of the letters of the word 'APPLE' are there ?*

SOLUTION Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. So, the required number of permutations is

$$\frac{5!}{2!\,1!\,1!\,1!} = \frac{120}{2} = 60$$

EXAMPLE 3 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice ?*

SOLUTION We are given 8 letters viz. A, A, A, B, B, C, C, C. Clearly, there are 8 letters of which three are of one kind, two are of second kind and three are of third kind. So, the total number of permutations is $\frac{8!}{3!\,2!\,3!} = 560$.

Hence, the requisite number of words = 560.

EXAMPLE 4 *Find the number of different permutations of the letters of the word BANANA.*

SOLUTION Clearly, there are six letters in the word 'BANANA' of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind.

$\therefore$ Total number of their permutations $= \frac{6!}{3!\,2!\,1!} = 60$.

Hence, the requisite number of words = 60

EXAMPLE 5 *(i) How many different words can be formed with the letters of the word HARYANA ?*

(ii) How many of these begin with H and end with N ?

(iii) In how many of these H and N are together ?

SOLUTION (i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \frac{7!}{3!\,1!\,1!\,1!\,1!} = \frac{7!}{3!} = 840$

(ii) After fixing *H* in first place and *N* in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words $= \frac{5!}{3!} = 20$

(iii) Considering *H* and *N* together we have 7 – 2 + 1 = 6 letters out of which three are alike i.e. *A*'s and others are each of its own kind. These six letters can be arranged in $\frac{6!}{3!}$ ways. But *H* and *N* can be arranged amongst themselves in 2! ways.

Hence, the requisite number of words

$$= \frac{6!}{3!} \times 2! = 120 \times 2 = 240$$

EXAMPLE 6 *How many different words can be formed by using all the letters of the word 'ALLAHABAD' ?*

(i) In how many of them vowels occupy the even positions ?

(ii) In how many of them both L do not come together ?

SOLUTION There are 9 letters in the word 'ALLAHABAD' out of which 4 are *A*'s, 2 are *L*'s and the rest are all distinct.

So, the requisite number of words $= \frac{9!}{4!\,2!} = 7560$

(i) There are 4 vowels and all are alike i.e. 4 *A*'s. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in $\frac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 *L*'s) and other distinct, can be arranged in $\frac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places $= \frac{5!}{2!} \times \frac{4!}{4!} = \frac{5!}{2!} = 60$.

(ii) Considering both *L* together and treating them as one letter we have 8 letters out of which *A* repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So, the number of words in which both *L* come together

$$= \frac{8!}{4!} = 1680.$$

Hence, the number of words in which both L do not come together

= Total no. of words

– No. of words in which both L come together

= 7560 – 1680 = 5880

EXAMPLE 7 *How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together ?*

SOLUTION There are 11 letters in the word 'MATHEMATICS' of which two are *M*'s, two are *A*'s, two are *T*'s and all other are distinct. So,

$$\text{Required number of arrangements} = \frac{11!}{2! \times 2! \times 2!} = 4989600$$

There are 4 vowels viz. *A, E, A, I*. Considering these four vowels as one letter we have 8 letters (*M, T, H, M, T, C, S* and one letter obtained by combining all vowels), out of which *M* occurs twice, *T* occurs twice and the rest all different. These 8 letters can be arranged in $\frac{8!}{2! \times 2!}$ ways.

But, the four vowels (*A, E, A, I*) can be put together in $\frac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together $= \frac{8!}{2! \times 2!} \times \frac{4!}{2!} = 10080 \times 12 = 120960$

EXAMPLE 8 *If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word ?*

SOLUTION In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter *A*, and arranging the other four letters GAIN, we obtain 4! = 24 words.

Thus, there are 24 words which start with *A*. These are the first 24 words.

Then, starting with *G*, and arranging the other four letters *A, A, I, N* in different ways, we obtain $\frac{4!}{2!} = \frac{24}{2} = 12$ words.

Thus, there are 12 words, which start with *G*.

Now, we start with *I*. The remaining 4 letters *A, G, A, N* can be arranged in $\frac{4!}{2!} = 12$ ways. So, there are 12 words, which start with *I*.

Thus, we have so far constructed 48 words.

The 49th word is NAAGI and hence the 50th word is NAAIG.

EXAMPLE 9 *The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with *A, D, M, N, O, R* in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. Similarly, *D, M, N, O* will occur in the first place the same number of times.

∴ Number of words starting with *A* = 5! = 120

Number of words starting with *D* = 5! = 120

Number of words starting with *M* = 5! = 120

Number of words starting with *N* = 5! = 120

Number of words starting with *O* = 5! = 120

Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM.

No. of words starting with RAD = 3! = 6

No. of words starting with RAM = 3! = 6

Now, the words beginning with 'RAN' must follow.

There are 3! words beginning with RAN. One of these words is the word RANDOM itself.

The first word beginning with RAN is the word RANDMO and the next word is RANDOM.

∴ Rank of RANDOM = 5 × 120 + 2 × 6 + 2 = 614.

EXAMPLE 10 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

SOLUTION Any number of greater than a million will contain all the seven digits.

Now, we have to arrange these seven digits, out of which 2 occurs twice, 3 occurs twice and the rest are distinct.

$$\text{The number of such arrangements} = \frac{7!}{2! \times 3!} = 420$$

These arrangements also include those numbers which contain 0 at the million's place.

Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

These 6 digits can be arranged in $\frac{6!}{2! \times 3!} = 60$ ways

Hence, the number of required numbers = 420 – 60 = 360.

EXAMPLE 11 *If the different permutations of the word, 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E ?*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we have to find the total number of words starting with *A*, because the very next word will start with *E*.

For finding the number of words starting with *A*, we have to find the number of arrangements of the remaining 10 letters, EXMINATION, of which there are 2 *I*'s, 2 *N*'s and the others each of its own kind.

$$\text{The number of such arrangements} = \frac{10!}{2! \times 2!} = 907200$$

Hence, the required number of items = 907200.

EXAMPLE 12 *How many four-letter words can be formed using the letters of the word 'FAILURE', so that*

(i) F is included in each word ?

(ii) F is not included in any word ?

SOLUTION There are 7 letters in the word 'FAILURE'.

(i) To include F in every 4 letter word, we first select four letters from the 7 letters of the word 'FAILURE' such that F is included in every selection. This can be done by selecting three letters from the remaining 6 letters i.e. A, I, L, U, R, E in ${}^6C_3$ ways. Now, there are 4 letters in each of ${}^6C_3$ selections.

Consider one of these ${}^6C_3$ selections. This selection contains 4 letters which can be arranged in 4! ways.

Thus, each of ${}^6C_3$ selections provides 4! words. Hence, the total number of words $= {}^6C_3 \times 4! = 480$.

(ii) If F is not to be included in any word, then we first select 4 letters from the remaining 6 letters. This can be done in ${}^6C_4$ ways. Now, every selection has 4 letters which can be arranged in a row in 4! ways.

Hence, the total number of words $= {}^6C_4 \times 4! = 360$.

EXAMPLE 13 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together ?*

SOLUTION There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in ${}^5C_3 \times {}^3C_2$ ways. So, there are ${}^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways.

Hence, the required number of words

$$= ({}^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440.$$

EXAMPLE 14 *How many numbers of five digits can be made with the digits 1, 2, 3 each of which can be used at most thrice in a number ?*

SOLUTION A five digit number can be made with the digits in the following ways :

I *By taking three identical digits and 2 distinct digits like 1, 1, 1, 2, 3, etc.*

The total number of such numbers $= \left({}^3C_1 \times {}^2C_2\right) \times \frac{5!}{3!}$

$= 60.$

II *By taking three identical digits of one type and 2 identical digits of second type like 1, 1, 1, 2, 2 etc.*

The total number of such numbers $= \left({}^3C_1 \times {}^2C_1\right) \times \frac{5!}{3!\,2!}$

$= 60.$

III *By taking two identical digits of one type, two identical digits of second type and one different digit like 1, 1, 2, 2, 3.*

The total number of such numbers $= ({}^3C_2 \times {}^1C_1) \times \frac{5!}{2!\,2!}$

$= 90.$

Hence, required number of numbers $= 60 + 60 + 90$

$= 210.$

EXAMPLE 15 *How many words can be formed by taking 4 letters at a time out of the letters of the word 'MATHEMATICS'.*

SOLUTION There are 11 letters viz. MM, AA, TT, H, E, I, C, S. All these letters are not distinct, so we cannot use ${}^nP_r$. We can choose 4 letters from the following ways :

(I) *All the four distinct letters* : There are 8 distinct letters viz. M, A, T, H, E, I, C, S out of which 4 can be chosen in ${}^8C_4$ ways.

So, the total number of groups of 4 letters $= {}^8C_4$.

Each such group has 4 letters which can be arranged in 4! ways. Hence,

the total number of words $= {}^8C_4 \times 4! = {}^8P_4 = 1680$.

(II) *Two distinct and two alike letters* : There are 3 pairs of alike letters viz MM, AA, TT, out of which one pair can be chosen in ${}^3C_1$ ways. Now we have to choose two letters out of the remaining 7 different types of letters which can be done in ${}^7C_2$ ways. So, the total number of groups of 4 letters in which two are different and 2 are alike is ${}^3C_1 \times {}^7C_2$. Each such group has 4 letters of which 2 are alike and remaining two distinct and they can be arranged in $\frac{4!}{2!}$ ways. Hence, the total number of words in which two letters are alike $= {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$.

(III) *Two alike of one kind and two alike of other kind* : There are 3 pairs of 2 alike letters out of which 2 pairs can be chosen in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups of 4 letters each. In each group there are 4 letters of which 2 are alike of one kind and two alike of other kind. These 4 letters can be arranged in $\frac{4!}{2!\,2!}$ ways. Hence, the total number of words in which two letters are alike of one kind and two alike of other kind $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18$.

From (i), (ii) and (iii)

The total number of 4 letter words

$$= 1680 + 756 + 18 = 2454.$$

EXAMPLE 16 *How many four-letter words can be formed using the letter of the word 'INEFFECTIVE' ?*

SOLUTION There are 11 letters in the word 'INEFFECTIVE'. viz. EEE, FF, II, C, T, N, V.

The four-letter words may consist of :

(I) 3 alike letters and 1 distinct letter

(II) 2 alike letters of one kind and 2 alike letters of the second kind

(III) 2 alike letters and 2 distinct letters

(IV) all different letters

Now we shall discuss these four cases one by one :

(I) *3 alike letters and 1 distinct letter* : There is one set of three alike letters viz. *EEE*. So, three alike letters can be selected in one way. Out of the 6 different letters *F, I, T, N, V, C* one letter can be selected in ${}^6C_1$ ways. Thus, three alike and one different letter can be selected in $1 \times {}^6C_1 = {}^6C_1$ ways. So, there are ${}^6C_1$ groups each of which contains 3 alike letters and one different letter. These 4 letters can be arranged in $\frac{4!}{3!\,1!}$ ways.

Hence, the total number of words consisting of three alike and one distinct letters $= {}^6C_1 \times \frac{4!}{3!\,1!} = {}^6C_1 \times 4 = 24.$

(II) *2 alike letters of one kind and 2 alike letters of second kind* : There are three sets of two alike letters viz *EE, FF, II*. Out of these three sets two can be selected in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups each of which contains 4 letters out of which 2 are alike of one type and two are alike of second type. Now, 4 letters in each group can be arranged in $\frac{4!}{2!\,2!}$ ways.

Hence, the total number of words consisting of two alike letters of one type and 2 alike letters of second type

$$= {}^3C_2 \times \frac{4!}{2!\,2!} = 18$$

(III) *2 alike and 2 different letters* : Out of 3 sets of two alike letters one set can be chosen in ${}^3C_1$ ways. Now, from the remaining 6 distinct letters, 2 letters can be chosen in ${}^6C_2$ ways. Thus, 2 alike letters and 2 distinct letters can be selected in $({}^3C_1 \times {}^6C_2)$ ways. So, there are $({}^3C_1 \times {}^6C_2)$ groups of 4 letters each. Now, letters of each group can be arranged among themselves in $\frac{4!}{2!}$ ways.

Hence, the total number of words consisting of two alike letters and 2 distinct $= {}^3C_1 \times {}^6C_2 \times \frac{4!}{2!} = 540$

(IV) *All different letters* : There are 7 distinct letters *E, F, I, T, N, V, C* out of which 4 can be selected in ${}^7C_4$ ways. So, there are ${}^7C_4$ groups of 4 letters each. The letters in each of ${}^7C_4$ groups can be arranged in 4! ways.

So, the total number of 4 letter words in which all letters are distinct $= {}^7C_4 \times 4! = 840.$

Hence, the total number of 4-letter words

$$= 24 + 18 + 540 + 840 = 1422.$$

EXAMPLE 17 *How many numbers of five digits can be made having exactly two identical digits ?*

SOLUTION We have 10 digits, namely 0, 1, 2, 3, ..., 8, 9. Let us select one of these digits which occurs twice in the five digit number. This can be done in ${}^{10}C_1$ ways. Now, we select 3 more digits from the remaining 9 digits in ${}^9C_3$ ways. We have now 5 digits like 2, 2, 1, 3, 4; 2, 3, 5, 0, 0 etc.

Total number of numbers formed by these digits is

$${}^{10}C_1 \times {}^9C_3 \times \frac{5!}{2!} = 50400$$

All these numbers are not five digit numbers. Some of these numbers are 4-digit numbers and some are 3-digit numbers. Let us now find out the number of such numbers.

Number of numbers having 0 at ten-thousands place and 0 as the repeating digit $= {}^1C_1 \times {}^9C_3 \times 4! = 2016$

Number of numbers having 0 at ten-thousands place and having a non-zero repeating digit

$$= {}^1C_1 \times {}^9C_1 \times {}^8C_2 \times \frac{4!}{2!} = 3024$$

Hence, the number of five digit numbers having exactly two identical digits

$$= 50400 - 2016 - 3024$$
$$= 45360.$$

EXAMPLE 18 *How many numbers of five digits can be made having exactly two repeated digits at consecutive places ?*

SOLUTION In order to construct a five digit number having exactly two consectively repeating digits, let us first select a digit out of 10 digits 0, 1, 2, ... , 9 which repeats and then select three more digits from the remaining 9 digits. This can be done in ${}^{10}C_1 \times {}^9C_3$ ways. Considering repeating digits like 00, 11, 22, ... , 99 as one object and three other digits, we now arrange them in 4! ways. The total number of numbers obtained in this manner is

$${}^{10}C_1 \times {}^9C_3 \times 4! = 20160$$

These numbers also include 4-digit numbers having 0 at ten thousands place and 3 digit numbers having 0 at ten thousands and thousands place. The total number of such numbers is

$${}^1C_1 \times {}^9C_1 \times {}^8C_2 \times 3! + {}^9C_3 \times 3!$$
$$= 1512 + 504 = 2016$$

Hence, the number of five digit numbers having exactly two consecutive repeated digits

$$= 20160 - 2016 = 18144.$$

EXAMPLE 19 *Find the total number of selections of 5 letters from five A's, four B's, three C's and two D's. Also, find the total number of 5-letter words that can be made.*

SOLUTION Various possible selections and arrangements of five letters from 5 *A*'s, 4 *B*'s, 3 *C*'s and 2 *D*'s are as follows :

Possibilities	*Number of selections*	*Number of arrangements*
All 5 letters are identical i.e. All 5 *A's* are selected	${}^1C_1 = 1$	${}^1C_1 \times \frac{5!}{5!} = 1$
4 identical of one type and 1 from the remaining	${}^2C_1 \times {}^3C_1 = 6$	${}^2C_1 \times {}^3C_1 \times \frac{5!}{4!} = 30$

3 identical of one type and 2 identical of second type	${}^3C_1 \times {}^3C_1 = 9$	${}^3C_1 \times {}^3C_1 \times \frac{5!}{3!\,2!} = 90$
2 identical of one type and 2 distinct from the remaining	${}^3C_1 \times {}^3C_2 = 9$	${}^3C_1 \times {}^3C_2 \times \frac{5!}{3!} = 180$
2 identical of one type 2 identical of second type and one from the remaining	${}^4C_2 \times {}^2C_1 = 12$	${}^4C_2 \times {}^2C_1 \times \frac{5!}{2!\,2!} = 360$
2 identical of one type and remaining three distinct	${}^4C_1 \times {}^3C_3 = 4$	${}^4C_1 \times {}^3C_3 \times \frac{5!}{2!} = 240$

Thus, the total number of selections of 5 letters

$$= 1+6+9+9+12+4 = 41.$$

The total number of arrangements of the given letters taken 5 at a time

$$= 1+30+90+180+360+240 = 901.$$

ALITER See Example 6 on page. 22.74

EXAMPLE 20 *In how many ways can 7 plus (+) signs and 5 minus (−) signs be arranged in a row so that no two minus signs are together ?*

SOLUTION We want to separate minus signs. So, let us first arragne 7 plus signs in a row. This can be done in $\frac{7!}{7!} = 1$ way as shown below.

	+		+		+		+		+		+		+	

A blank box in the above arrangement shows available space for minus signs. Clearly, there are eight blank boxes in which we have to arrange minus signs. This can be done in

$${}^8C_5 \times \frac{5!}{5!} = {}^8C_5 \text{ ways}$$

Hence, the number of all possible arrangements

$$= 1 \times {}^8C_5 = {}^8C_5 = 56$$

EXAMPLE 21 *Six dice are thrown simultaneously. Find the number of ways in which*

(i) *all of them show the same face.*

(ii) *all of them show different faces.*

(iii) *exactly three show the same face and the remaining show distinct faces.*

(iv) *at least four show the same face and the remaining show distinct faces.*

SOLUTION When six dice are thrown simultaneously there are 6^6 possible outcomes of the form (1, 2, 3, 3, 4, 5), (2, 2, 2, 2, 2, 2), (2, 3, 4, 5, 6, 3) etc.

(i) All six dice show the same face in the following ways: (1, 1, 1, 1, 1, 1), (2, 2, 2, 2, 2, 2), (3, 3, 3, 3, 3, 3), (4, 4, 4, 4, 4, 4), (5, 5, 5, 5, 5, 5) and (6, 6, 6, 6, 6, 6).

Thus, the total number of ways is 6.

(ii) The total number of ways in whcih all six die show different faces is the number of arrangements of six numbers 1, 2, 3, 4, 5, 6 taken all at a time i.e. 6!.

(iii) Select a number which occurs on three dice out of six numbers 1, 2, 3, 4, 5, 6. This can be done in ${}^6C_1$ ways. Now select three numbers from the remaining 5 numbers. This can be done in ${}^5C_3$ ways. We now have six numbers out of which three are identical and three distinct like 1, 1, 1, 2, 3, 4; 2, 2, 2, 4, 5, 6 etc. These six numbers can be arranged in $\frac{6!}{3!}$ ways.

Hence, the number of ways of getting three identical and three distinct faces is

$${}^6C_1 \times {}^5C_3 \times \frac{6!}{3!} = 6 \times 10 \times 120 = 7200$$

(iv) Proceeding as in (iii), the total number of ways in which r dice show the same face is

$${}^6C_1 \times {}^5C_{6-r} \times \frac{6!}{r!};\ r = 2, 3, 4, 5, 6$$

$\therefore$ Total number of ways in which at least 4 dice show the same face

$$= \sum_{r=4}^{6} {}^6C_1 \times {}^5C_{6-r} \times \frac{6!}{r!}$$

EXAMPLE 22 *Find the number of ways in which four S come consecutively in the word MISSISSIPPI.*

SOLUTION There are 11 letters viz. 4S's, 4I's, 2P's and M in the word MISISSIPPI. Considering 4S's as one letter, we have 8 letters which can be arragned in $\frac{8!}{4!\,2!}$ ways. But, 4S's can be put together in $\frac{4!}{4!} = 1$ way.

Hence, the total number of words

$$= \frac{8!}{4!\,2!} \times \frac{4!}{4!} = \frac{8!}{4!\,2!} = 840.$$

EXAMPLE 23 *Find the number of ways in which five different letters can be put in the correspondingly five addressed envelopes so that no letter is put in the correct envelope.*

SOLUTION We observe that two letters can be put in two addressed envelopes in 2! ways out of which there is only one way in which no letter is put in the correct envelope. Thus,

Number of ways to put 2 letters in 2 addressed envelopes so that no letter is put in the correct envelope = 1.

Now,

Number of ways to put 3 letters in 3 addressed envelopes so that no letter is put in correct envelope

= Number of ways without restriction

− Number of ways in which one is incorrect envelope and two wrong envelopes

– Number of ways in which all come in correct envelopes

$= 3! - {}^3C_1 \times 1 - 1$

$= 6 - 3 - 1 = 2.$

The number of ways to put 4 letters in 4 addressed envelopes so that no letter is in correct envelope

= Number of ways without restriction

– Number of ways in which 1 is in correct envelope and 3 in wrong envelopes

– Number of ways in which 2 are in correct envelopes and 2 in wrong envelopes

– Number of ways in which all are in correct envelopes

$= 4! - {}^4C \times 2 - {}^4C_2 \times 1 - 1$

$= 24 - 8 - 6 - 1$

$= 9$

Similarly,

The number of ways to put 5 letters in 5 addressed envelopes so that no letter is in correct envelope

$= 5! - {}^5C_1 \times 9 - {}^5C_2 \times 2 - {}^5C_3 \times 1 - 1$

$= 120 - 45 - 20 - 10 - 1$

$= 44.$

EXAMPLE 24 *Find the total number of ways of selecting 5 letters from the letters of the word "INDEPENDENT".*

SOLUTION There are 11 letters in the word "INDEPENDENT" viz.

$$NNN, EEE, DD, I, P, T$$

Five letters can be selected from these 11 letters in the following ways :

(i) *When all five letters are distinct* : There are 6 distinct letters *N, E, D, I, P* and *T* out of which five letters can be selected in ${}^6C_5$ ways.

(ii) *When two letters are alike and remaining three distinct* : There are three pairs of alike letters viz. *NN, EE* and DD, out of these three pairs one pair can be chosen in ${}^3C_1$ ways. Now we have to select three letters from the remaining 5 distinct letters which can be done in ${}^5C_3$ ways. Therefore, the number of ways of selecting 5 letters in this case is

$${}^3C_1 \times {}^5C_3$$

(iii) *When three letters are alike and remaining two are distinct* : There are two sets of three alike letters out of which one can be chosen in ${}^2C_1$ ways. Now we have 5 distinct letters from which 2 letters can be chosen in ${}^5C_2$ ways. So,

the number of ways of selecting 5 letters in this case

$$= {}^2C_1 \times {}^5C_2$$

(iv) *When three letters are alike of one type and two letters are alike of another type* : There are two sets of three alike letters out of which one set can be selected in ${}^2C_1$ ways. Now we have two sets one containing three alike letters and other containing two alike letters out of which one set of two letters can be chosen in ${}^2C_1$ ways. So, the number of ways of selecting 5 letters in this case

$$= {}^2C_1 \times {}^2C_1$$

(v) *When two letters are alike of one type, two letters alike of second type and one from the remaining* : In this case, the number of ways of selecting 5 letters is

$${}^3C_2 \times {}^4C_1$$

Hence, the total number of selections

$= {}^6C_5 + {}^3C_1 \times {}^5C_3 + {}^2C_1 \times {}^5C_2 + {}^2C_1 \times {}^2C_1 + {}^3C_2 \times {}^4C_1$

$= 6 + 30 + 20 + 4 + 12$

$= 72.$

ALITER See section 22.14.

EXAMPLE 25 *In how many ways can 5 identical black balls, 7 identical red balls and 6 identical green balls be arranged in a row so that at least one ball is separated from the balls of the same colour ?*

SOLUTION We have,

Required number of ways

= Total number of arrangements of 18 balls without any restriction

– Number of ways in which balls of the same colour are together

Now,

Total number of arrangements of 18 balls without any restriction

$$= \frac{18!}{5!\,7!\,6!}$$

In order to find the number of ways in which all balls of the same colour are put together, we consider 5 identical black balls as one object, 7 identical red balls as second object and 6 identical green balls as third object. These three objects can be arranged in a row in 3! ways. Since balls of the same colour are identical, therefore there is no arrangement between balls of the same colour.

Thus,

Number of ways in which all balls of the same colour are together = 3!.

Hence,

$$\text{Required number of ways} = \frac{18!}{5!\,7!\,6!} - 3!$$

EXAMPLE 26 *Prove that the number of ways of selecting n objects out of 3n objects of which n are identical and the rest are distinct, is*

$$2^{2n-1} + \frac{1}{2}\left\{\frac{2n!}{(n!)^2}\right\}$$

SOLUTION Suppose r objects are selected from n identical objects and the remaining $(n - r)$ are selected from the remaining $2n$ distinct objects. The number of such ways is

$$1 \times {}^{2n}C_{n-r} = {}^{2n}C_{n-r},\ r = 0, 1, 2, \ldots, n$$

Hence, required number of ways

$$= \sum_{r=0}^{n} {}^{2n}C_{n-r}$$

$$= \frac{1}{2} \sum_{r=0}^{n} ({}^{2n}C_{n-r} + {}^{2n}C_{n-r})$$

$$= \frac{1}{2} \sum_{r=0}^{n} ({}^{2n}C_{n-r} + {}^{2n}C_{n+r}) \qquad [\because {}^{2n}C_{n-r} = {}^{2n}C_{n+r}]$$

$$= \frac{1}{2} [({}^{2n}C_n + {}^{2n}C_n) + ({}^{2n}C_{n-1} + {}^{2n}C_{n+1}) + \ldots + ({}^{2n}C_{n-2} + {}^{2n}C_{n+2}) + \ldots + ({}^{2n}C_1 + {}^{2n}C_{2n-1}) + ({}^{2n}C_0 + {}^{2n}C_{2n})]$$

$$= \frac{1}{2} [({}^{2n}C_0 + {}^{2n}C_1 + \ldots + {}^{2n}C_{n-1} + {}^{2n}C_n + {}^{2n}C_{n+1} + \ldots\ldots + \ldots\ldots\ldots\ldots + {}^{2n}C_{2n}) + {}^{2n}C_n]$$

$$= \frac{1}{2} \left[2^{2n} + \frac{(2n)!}{n!\, n!}\right] = 2^{2n-1} + \frac{1}{2} \left\{\frac{2n!}{n!\, n!}\right\}$$

EXAMPLE 27 *A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q is again chosen at random. Find the number of ways of selecting P and Q so that*

(i) *P and Q are disjoint sets.*

(ii) $P = Q$

(iii) $P \cup Q = A$

(iv) $P \cap Q$ *contains just one element.*

(v) $P \cup Q$ *contains just one element.*

(vi) $P \cap Q$ *contains just two elements.*

SOLUTION The set A has n elements. So, it has 2^n subsets. A subset of set A out of these 2^n subsets can be chosen in ${}^{2^n}C_1 = 2^n$ ways. Therefore, each of the subsets P and Q can be chosen in 2^n ways.

Thus, P and Q both can be chosen in $2^n \times 2^n = 4^n$ ways.

(i) Suppose P is constructed by selecting r elements out of n elements of set A. This can be done in ${}^nC_r$ ways. For Q to be disjoint from P, we must construct Q by selecting any number of elements from the remaining $(n - r)$ elements. This can be done in

$${}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r} \text{ ways.}$$

Thus, P and Q both can be chosen in

$${}^nC_r \times \left({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r}\right) \text{ ways}$$

But, r can vary from 0 to n. Therefore, total number of ways of selecting P and Q such that they are disjoint is

$$= \sum_{r=0}^{n} {}^nC_r \left({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r}\right)$$

$$= \sum_{r=0}^{n} {}^nC_r\ 2^{n-r}$$

$$= (1 + 2)^n$$

$$= 3^n.$$

(ii) If P contains r elements, then the number of ways of selecting P is ${}^nC_r$. Since $P = Q$. Therefore, number of ways of selecting P and Q both when P contains r elements is ${}^nC_r$. But, r can vary from 0 to n. Therefore, total number of ways of selecting P and Q such that $P = Q$ is

$$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(iii) Let P be constructed by selecting r elements from n elements of set A. This can be done in ${}^nC_r$ ways. In order to construct Q such that $P \cup Q = A$, we must include remaining $(n - r)$ elements of set A in Q and any number of elements from r elements selected for P. This can be done in

$$1 \times \left({}^rC_0 + {}^rC_1 + \ldots + {}^rC_r\right)$$

Therefore, number of ways of selecting P and Q such that $P \cup Q = A$ and P contains r elements is

$${}^nC_r \times 1 \times ({}^rC_0 + {}^rC_1 + \ldots + {}^rC_r) = {}^nC_r\, 2^r.$$

But, r can vary from 0 to n. Therefore, the number of ways of selecting P and Q such that $P \cup Q = A$ is

$$\sum_{r=0}^{n} {}^nC_r\ 2^r = (1 + 2)^n = 3^n$$

(iv) Let P be constructed by selecting r elements from n elements of set A. This can be done in ${}^nC_r$ ways. In order to construct Q such that $P \cap Q$ contains just one element, we must select one element from the r elements selected for P and any number of elements from the remaining $(n - r)$ elements in A. This can be done in ${}^rC_1 \times ({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r})$ ways. Therefore, the number of ways of selecting P and Q such that P contains r elements and $P \cap Q$ contains just one element is

$${}^nC_r \times {}^rC_1 \times ({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r})$$

$$= {}^nC_r \times r \times 2^{n-r}$$

$$= \frac{n}{r}\ {}^{n-1}C_{r-1} \cdot r \cdot 2^{n-r}$$

$$= n \cdot {}^{n-1}C_{r-1}\, 2^{n-r}$$

But, r can vary from 1 to n. Therefore, the number of ways of selecting P and Q such that $P \cap Q$ contains just one element is

$$\sum_{r=1}^{n} n \cdot {}^{n-1}C_{r-1}\, 2^{n-r}$$

$$= n \sum_{r=1}^{n} {}^{n-1}C_{r-1}\, 2^{(n-1)-(r-1)}$$

$$= n\,(1+2)^{n-1}$$

$$= n\,3^{n-1}.$$

(v) $P \cup Q$ contains exactly one element in one of the following ways :

(i) P and Q each has the same element

(ii) P has no element and Q has just one element

(iii) P has just one element and Q has no element.

Therefore, required number of ways

$$= {}^{n}C_1 + {}^{n}C_0 \times {}^{n}C_1 + {}^{n}C_1 \times {}^{n}C_0$$

$$= n + n + n = 3n.$$

(vi) Suppose P is constructed by selecting r elements from n elements of set A. This can be done in ${}^{n}C_r$ ways. In order to construct Q such that $P \cap Q$ contains just two elements, we select two elements from the r elements chosen in P and any number of elements from the remaining $(n-r)$ elements in A. This can be chosen in

$${}^{r}C_2\ ({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots\ldots + {}^{n-r}C_{n-r}) \text{ ways}$$

Therefore, the number of ways of selecting P and Q such that P contains r elements and $P \cap Q$ contains just two elements is

$${}^{n}C_r \times {}^{r}C_2\ ({}^{n-r}C_0 + {}^{n-r}C_1 + \ldots + {}^{n-r}C_{n-r})$$

$$= {}^{n}C_r \times \frac{r(r-1)}{2} \times 2^{n-r}$$

$$= \frac{n}{r} \cdot \frac{n-1}{r-1}\, {}^{n-2}C_{r-2} \times \frac{r(r-1)}{2} \times 2^{n-r}$$

$$= \frac{n(n-1)}{2}\, {}^{n-2}C_{r-2}\, 2^{n-r}$$

But, r can take values from 2 to n. Therefore, the number of ways of selecting P and Q such that $P \cap Q$ contains just two elements is

$$\sum_{r=2}^{n} \frac{n(n-1)}{2}\, {}^{n-2}C_{r-2}\, 2^{n-r}$$

$$= \frac{n(n-1)}{2} \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, 2^{(n-2)-(r-2)}$$

$$= \frac{n(n-1)}{2}\,(1+2)^{n-2}$$

$$= \frac{n(n-1)}{2}\,3^{n-2}$$

EXAMPLE 28 *In how many ways can two distinct subsets of the set A consisting of n $(n \geq 3)$ elements be selected so that they have exactly two common elements ?*

SOLUTION Let P and Q be two distinct subsets of set A. Let set P be constructed by selecting r elements out of n elements of set A. Since P and Q have exactly two common elements. So, select 2 elements out of r elements in P in ${}^{r}C_2$ ways. The set Q must contain these two elements and any number of elements from the remaining $(n-r)$ elements of set A. Therefore, set Q can be constructed in

$${}^{r}C_2 \times \left({}^{n-r}C_0 + {}^{n-r}C_1 + {}^{n-r}C_2 + \ldots. + {}^{n-r}C_{n-r}\right)$$

$$= {}^{r}C_2\, 2^{n-r} \text{ ways.}$$

Thus, first set P consisting of r elements and then set Q consisting of any number of elements from the remaining $n-r$ elements of set A and exactly two elements from P can be chosen in

$${}^{n}C_r \times {}^{r}C_2\, 2^{n-r} \text{ ways}$$

$$= \frac{n}{r} \cdot \frac{n-1}{r-1}\, {}^{n-2}C_{r-2} \cdot \frac{r(r-1)}{2}\, 2^{n-r} \text{ ways}$$

$$= \frac{n(n-1)}{2}\, {}^{n-2}C_{r-2}\, 2^{n-r} \text{ ways}$$

But, r can vary from 2 to n. Therefore, the total number of ways of selecting first P containing r elements and then Q such that P and Q can have exactly two common elements is

$$\sum_{r=2}^{n} \frac{n(n-1)}{2}\, {}^{n-2}C_{r-2}\, 2^{n-r}$$

$$= \frac{n(n-1)}{2}\,(1+2)^{n-2}$$

$$= \frac{n(n-1)}{2}\,3^{n-2}$$

In this way there is a selection in which P and Q have only two elements which are common to them.

Therefore, the number of ways of selecting first P and then Q such that P and Q are disjoint is

$$\frac{n(n-1)}{2}\,3^{n-2} - 1$$

But we can interchange P and Q also i.e. first we construct Q and then P. Therefore, the required number of ways

$$\frac{1}{2}\left[\frac{n(n-1)}{2} \cdot 3^{n-2} - 1\right]$$

EXAMPLE 29 *Three integers are selected from $3n$ consecutive integers. In how many ways can these integers be selected such that their sum is divisible by 3.*

SOLUTION Let $m, m+1, m+2, \ldots, m+(3n-1)$ be $3n$ consecutive integers, where m is an integer.

Let us divide these $3n$ consecutive integers into three groups G_1, G_2 and G_3 as follows :

$G_1 : m, m+3, m+6, \ldots, m+(3n-3)$

$G_2 : m+1, m+4, m+7, \ldots, m+(3n-2)$

$G_3 : m+2, m+5, m+8, \ldots, m+(3n-1)$

The sum of 3 integers chosen from the given $3n$ integers will be divisible by 3 in each of the following cases :

CASE I *When all the three integers are chosen from the same group:*

In this case, the number of ways $= {}^nC_3 + {}^nC_3 + {}^nC_3$

$= 3 \cdot {}^nC_3$

$= \frac{n(n-1)(n-2)}{2}$

CASE II *When one integer is chosen from each group :*

In this case, the number of ways $= {}^nC_1 \times {}^nC_1 \times {}^nC_1 = n^3$.

Hence, the number of ways in which the sum is divisible by 3 is

$$\frac{n(n-1)(n-2)}{2} + n^3 = \frac{n}{2}(3n^2 - 3n + 2)$$

EXAMPLE 30 *Two numbers a and b are chosen from the set $\{1, 2, ..., 3n\}$. In how many ways can these integers be selected such that*

(i) *$a^2 - b^2$ is divisible by 3?*

(ii) *$a^3 + b^3$ is divisible by 3?*

SOLUTION Let us divide the given set of first $3n$ natural numbers in three groups G_1, G_2 and G_3 as given below :

$G_1 : 3, 6, 9, \ldots. 3n$

$G_2 : 1, 4, 7, \ldots., (3n-1)$

$G_3 : 2, 5, 8, \ldots., (3n-2)$

(i) We have,

$$a^2 - b^2 = (a-b)(a+b).$$

Therefore, $a^2 - b^2$ will be divisible by 3, if either $a-b$ is divisible by 3 or $a+b$ is divisible by 3 or both are divisible by 3.

Clearly, this is possible if either a and b are chosen from the same group or one of them is chosen from G_2 and the other from G_3. This can be done in

${}^nC_2 + {}^nC_2 + {}^nC_2 + {}^nC_1 \times {}^nC_1$ ways

Hence, the required number of ways $= 3 \cdot {}^nC_2 + ({}^nC_1)^2$

$= 3\frac{n(n-1)}{2} + n^2$

$= \frac{5n^2 - 3n}{2}$

(ii) We have,

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

Therefore, $a^3 + b^3$ will be divisible by 3, if $a+b$ is divisible by 3. Clearly, $a+b$ will be divisible by 3 in each of the following cases :

CASE I *When a and b are chosen from groups G_1 :*

This can be done in ${}^nC_2$ ways.

CASE II *When one of the two numbers is chosen from group G_1 and the other from group G_3 :*

This can be done in ${}^nC_1 \times {}^nC_1$ ways.

Thus, the number of ways of selecting a and b such that $a^3 + b^3$ is divisible by 3 is

$${}^nC_2 + {}^nC_1 \times {}^nC_1 = \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2}.$$

EXAMPLE 31 *Find the number of ways of selecting two integers a and b from the set $\{1, 2, ..., 5n\}$, $n \in N$ so that $a^4 - b^4$ is divisible by 5.*

SOLUTION Let us divide the given set of $5n$ integers in 5 groups as follows :

$G_1 : 1, 6, 11, \ldots., 5n-4$

$G_2 : 2, 7, 12, \ldots, 5n-3$

$G_3 : 3, 8, 13, \ldots, 5n-2$

$G_4 : 4, 9, 14, \ldots, 5n-1$

$G_5 : 5, 10, 15, \ldots, 5n$

We have,

$$a^4 - b^4 = (a-b)(a+b)(a^2+b^2) \quad \ldots(i)$$

We observe that the RHS of (i) will be divisible by 5 if a and b are taken from the last group G_5. This can be done in ${}^nC_2$ ways. Also, if a and b are chosen from $G_1 \cup G_2 \cup G_3 \cup G_4$, then at least one of the factors on RHS of (i) will be divisible by 5. This can be done in ${}^{4n}C_2$ ways. Hence, the number of ways of selecting a and b so that $a^4 - b^4$ is divisible by 5 is

$${}^nC_2 + {}^{4n}C_2 = \frac{n(n-1)}{2} + \frac{4n(4n-1)}{2}$$

$$= \frac{17n^2 - 5n}{2}$$

EXERCISE 22.6

1. Find the number of words formed by permuting all the letters of the following words :

(i) INDEPENDENCE (ii) INTERMEDIATE

(iii) ARRANGE (iv) INDIA

(v) PAKISTAN (vi) RUSSIA

(vii) SERIES (viii) EXERCISES

(ix) CONSTANTINOPLE

2. In how many ways can the letters of the word 'ALGEBRA' be arranged without changing the relative order of the vowels and consonants ?

3. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remaining together ?

4. Find the total number of arrangements of the letters in the expression $a^3 b^2 c^4$ when written at full length.

5. How many words can be formed with the letters of the word 'PARALLEL' so that all *L*'s do not come together ?

6. How many words can be formed by arranging the letters of the word 'MUMBAI' so that all *M*'s come together ?

7. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word LATE.

8. If the letters of the word 'MOTHER' are written in all possible orders and these words are written out as in a dictionary, find the rank of the word 'MOTHER'.

9. If the permutations of *a, b, c, d, e* taken all together be written down in alphabetical order as in dictionary and numbered, find the rank of the permutation *debac*.

10. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places ?

11. Find the total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together.

12. How many different signals can be made from 4 red, 2 white and 3 green flags by arranging all of them vertically on a flagstaff ?

13. How many number of four digits can be formed with the digits 1, 3, 3, 0 ?

14. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two *R*'s are never together ?

15. How many different numbers, greater than 50000 can be formed with the digits 0, 1, 1, 5, 9.

16. How many words can be formed from the letters of the word 'SERIES' which start with *S* and end with *S* ?

17. How many permutations of the letters of the word 'MADHUBANI' do not begin with *M* but end with *I* ?

18. Find the number of numbers, greater than a million, that can be formed with the digits 2, 3, 0, 3, 4, 2, 3.

19. In how many ways can the letters of the word "INTERMEDIATE" be arranged so that

(i) the vowels always occupy even places ?

(ii) the relative order of vowels and consonants do not alter ?

(iii) the order of vowels do not change ?

20. There are three copies each of 4 different books. In how many ways can they be arranged in a shelf ?

21. The letters of the word 'ZENITH' are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word 'ZENITH' ?

22. How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'. How many of them begin with *C* ? How many of them begin with *T* ?

23. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.

24. Five letter words are to be formed out of the letters of the word 'INFINITESIMAL'. What is their number.

25. Find the total number of four letter words that can be formed from the letters of the word INEFFECTIVE.

26. A five letter word is to be formed such that letters appearing in the odd positions are taken from the unrepeated letters of the word MATHEMATICS whereas the letters which occupy even places are taken from amongst the repeated letters. Find the total number of such words.

27. A dictionary is printed consisting of 7-lettered words that can be made with the letters of the word CRICKET. If the words are printed in the alphabetic order, as in an ordinary dictionary, find the position of the word CRICKET in that dictionary.

28. How many words can be made with letters of the word INTERMEDIATE if

(i) the words neither begin with *I* nor end with *E*.

(ii) the vowels and consonants alternate in the words.

(iii) the vowels are always consecutive.

(iv) no vowel is between two consonants i.e. all consonants are consecutive.

(v) the relative order of vowels and consonants does not change.

(vi) the order of vowels does not change i.e. no two vowels interchange their places.

29. Three integers are selected from the integers 1, 2, ..., 1000. In how many ways can these integers be selected such that their sum is divisible by 4 ?

30. Three integers are selected from the integers 1, 2, 3, ..., 4*n*. In how many ways can these integers be selected such that their sum is divisible by 4?

ANSWERS

1. (i) 1663200 (ii) 19958400 (iii) 1260 (iv) 60 (v) 20160 (vi) 360 (vii) 180 (viii) 30240 (ix) $\frac{14!}{24}$

2. 72 **3.** 60480 **4.** 1260 **5.** 3000 **6.** 120 **7.** 14 **8.** 309 **9.** 93 **10.** 18 **11.** 35 **12.** 1260 **13.** 9 **14.** 900 **15.** 24 **16.** 12

17. 17640 **18.** 360 **19.** (i) 21600 (ii) 21600 (iii) 332640 **20.** $12!/(3!)^4$ **21.** 616 **22.** $\frac{11!}{2!\,2!\,2!}, \frac{10!}{2!\,2!\,2!}, \frac{10!}{2!\,2!}$ **23.** 236 **24.** 22660 **25.** 1422 **26.** 540 **27.** 531th place **28.** (i) $\frac{83 \times 10!}{24}$ (ii) 43200 (iii) 151200

(iv) 151200 (v) 21600 (vi) $\frac{12!}{6!\,2!}$ 29. 41541750 30. $\frac{n}{3}(8n^2-6n+1)$

HINTS TO SELECTED PROBLEMS

2. The consonants can be arranged among themselves in 4! ways and the vowels among themselves in $\frac{3!}{2!}$ ways.

Hence, the required number of arrangements $= 4! \times \frac{3!}{2!} = 72$.

4. There are 3 *a's*, 2 *b's* and 4 *c's*. So the total number of arrangements is $\frac{9!}{3!\,2!\,4!} = 1260$.

10. There are 4 odd digits 1, 1, 3, 3 and 4 odd places. So, odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways. The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways.

Hence, the requisite number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$.

11. Six '+' signs can be arranged in a row in $\frac{6!}{6!} = 1$ way. Now, we are left with seven places in which four different things can be arranged in ${}^7C_4 \times 4!$ ways but as all the four '–' signs are identical, therefore, four '–' signs can be arranged in $\frac{{}^7C_4 \times 4!}{4!} = 35$ ways. Hence, the required number of ways $= 1 \times 35 = 35$.

12. We have to arrange 9 flags, out of which 4 are of one kind, 2 are of another kind and 3 are of third kind.

So, total number of signals $= \frac{9!}{4!\,2!\,3!}$.

13. Required number of numbers $= \frac{4!}{2!} - \frac{3!}{2!}$.

15. Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits.

Number of numbers with digit 5 at first place $= \frac{4!}{2!}$

Number of numbers with digit 9 at first place $= \frac{4!}{2!}$

Hence, the required number of numbers $= \frac{4!}{2!} + \frac{4!}{2!} = 24$.

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22.7 CIRCULAR PERMUTATIONS

So far we have discussed permutations of objects (or things) in a row. This type of permutations are generally known as linear permutations. If we arrange the objects along a closed curve viz. a circle, the permutations are known as *circular permutations*. As we have seen in the earlier sections of this chapter that every linear arrangement has a beginning and an end, but there is nothing like beginning or end in a circular permutation. Thus, in a circular permutation, we consider one object as fixed and the remaining objects are arranged as in case of linear arrangements.

THEOREM *The number of circular permutations of n distinct objects is (n – 1)!*

PROOF Let $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ be n distinct objects. Let the total number of circular permutations be x. Consider one of these x permutations as shown in Fig. 22.9.

Clearly, this circular permutation provides n linear permutations as given below.

$a_1, a_2, a_3, \ldots, a_{n-1}, a_n$

$a_2, a_3, a_4, \ldots, a_n, a_1$

$a_3, a_4, a_5, \ldots, a_n, a_1, a_2$

$a_4, a_5, a_6, \ldots, a_n, a_1, a_2, a_3$

... ...

$a_n, a_1, a_2, a_3, \ldots, a_{n-1}$

Thus, each circular permutation gives n linear permutations. But there are x circular permutations. So, total number of linear permutations is xn. But the number of linear permutations of n distinct objects is $n!$

$\therefore \quad xn = n! \Rightarrow x = \frac{n!}{n} = (n-1)!$

Q.E.D.

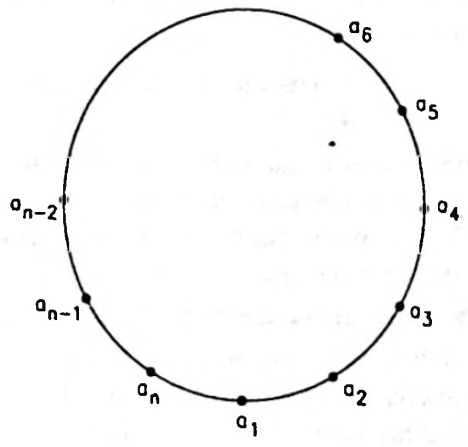


Fig. 22.9

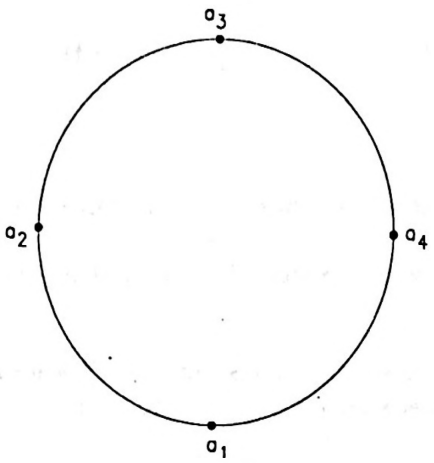


Fig. 22.10

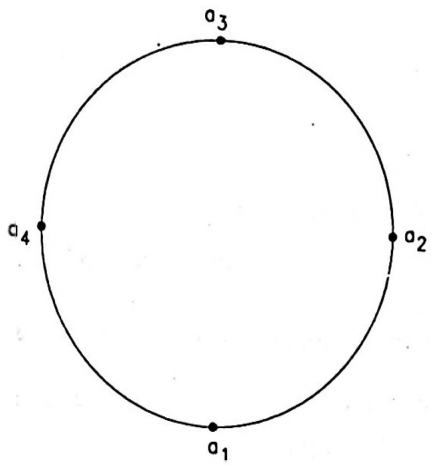


Fig. 22.11

NOTE *In the above theorem anti-clockwise and clockwise order of arrangements are considered as distinct permutations.*

DIFFERENCE BETWEEN CLOCKWISE AND ANTI-CLOCKWISE ARRANGEMENTS

Consider the following circular permutations. We observe that in both, the order of the circular arrangement is a_1, a_2, a_3, a_4. In Fig. 22.10 the order is anti-clockwise whereas in Fig. 22.11 the order is clockwise. Thus, the number of circular permutation of n things in which clockwise and anti-clockwise arrangements give rise to different permutations is $(n-1)!$. For example, the number of permutations of 5 persons seated around a table is $(5-1)! = 4!$. Because with respect to the table, the clockwise and anti-clockwise arrangements are distinct.

If anti-clockwise and clockwise order of arrangements are not distinct e.g. arrangements of beads in a necklace, arrangements of flowers in a garland etc., then the number of circular permutations of n distinct items is $\frac{1}{2}\{(n-1)!\}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In how many ways can 8 students be seated in a (i) line (ii) circle ?*

SOLUTION (i) The number of ways in which 8 students can be seated in a line $= {}^8P_8 = 8! = 40320$.

(ii) The number of ways in which 8 students can be seated in a circle $= (8-1)! = 7! = 5040$.

EXAMPLE 2 *(i) In how many ways can 5 persons be seated around a circular table ?*

(ii) In how many of these arrangements will two particular persons be next to each other ?

SOLUTION (i) Five persons can be seated around a circular table in $(5-1)! = 4! = 24$ ways.

(ii) Considering two particular persons as one person, we have 4 persons in all. These 4 persons can be seated around a circular table in $(4-1)! = 3!$ ways. But two particular persons can be arranged among themselves in 2! ways.

Hence, the total number of arrangements $= 3! \times 2! = 12$

EXAMPLE 3 *If 20 persons were invited for a party, in how many ways can they and the host be seated at a circular table ? In how many of these ways will two particular persons be seated on either side of the host ?*

SOLUTION (i) Clearly, there are 21 persons, including the host, to be seated round a circular table. These 21 persons can be seated round a circular table in $(21-1)! = 20!$ ways.

(ii) Let P_1, P_2 be two particular persons and H be the host. These two particular persons can be seated on either side of the host in the following two ways : (i) $P_1\, H\, P_2$ and (ii) $P_2\, H\, P_1$.

Consider the two particular persons and the host as one person, we have 18 persons in all. These 19 persons can be seated round a circular table in $(19-1)! = 18!$ ways. But two particular persons can be seated on either side of the host in 2 ways. So, the number of ways of seating 21 persons at a circular table with two particular persons on either side of the host is $18! \times 2$.

EXAMPLE 4 *There are 20 persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.*

SOLUTION Let B_1 and B_2 be two brothers among 20 persons and let M be a person. Clearly, person M can be chosen from 18 persons (excluding B_1 and B_2) in 18 ways. Considering the two brothers B_1 and B_2 and person M as one person, we have 18 persons in all. These 18 persons can be arranged around a circle in $(18-1)! = 17!$ ways.

But B_1 and B_2 can be arranged among themselves in 2! ways (B_1MB_2 and B_2MB_1).

Hence, the total number of ways = $18 \times 17! \times 2! = 2 \times 18!$.

EXAMPLE 5 *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent ?*

SOLUTION The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men in $(4-1)! = 3!$ ways. Now, 4 vacant seats can be occupied by 4 women in 4! ways.

Hence, the required number of seating arrangements

$$= 3! \times 4! = 144.$$

EXAMPLE 6 *A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates are (i) always together? (ii) never together ?*

SOLUTION (i) Let D_1 and D_2 be two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. These 19 delegates can be seated round a circular table in $(19-1)! = 18!$ ways. But two particular delegates can arrange among themselves in 2! ways (D_1D_2 and D_2D_1). Hence, the total number of ways = $18! \times 2! = 2\,(18!)$

(ii) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of seating arrangements of 20 persons. Clearly 20 persons can be seated round a circular table in $(20-1)! = 19!$ ways.

Hence, the required number of seating arrangements

$$= 19! - 2 \times 18! = 17\,(18!).$$

EXAMPLE 7 *There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together ?*

SOLUTION Five gentlemen can be seated at a round table in $(5-1)! = 4!$ ways. Now, 5 places are created in which 4 ladies can be arranged in ${}^5P_4$ ways.

Hence, the total number of ways in which no two ladies sit together = $4! \times {}^5P_4 = 4! \times 5! = 2880$.

EXAMPLE 8 *In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements ?*

SOLUTION Clearly, 7 persons can sit at a round table in $(7-1)! = 6!$ ways. But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways = $\frac{1}{2}(6!) = 360$.

EXAMPLE 9 *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible ?*

SOLUTION Let B_1, B_2 and X be three boys and G_1, G_2 and Y be three girls. Since the boy X does not want any girl neighbour. Therefore boy X will have his neighbours as boys B_1 and B_2 as shown in Fig. 22.12. Similarly, girl Y has her neighbours as girls G_1 and G_2 as shown in Fig. 22.12. But the boys B_1 and B_2 can be arranged among themselves in 2! ways and the girls G_1 and G_2 can be arranged among themselves in 2! ways.

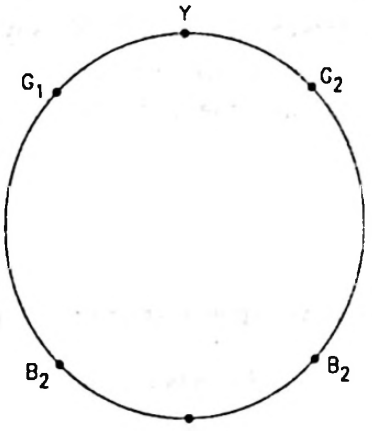


Fig. 22.12

Hence, the required number of arrangements = $2! \times 2! = 4$.

EXAMPLE 10 *Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.*

SOLUTION Considering 4 particular flowers as one flower, we have five flowers which can be strung to form a garland in 4! ways. But 4 particular flowers can be arranged in 4! ways. Thus, the required number of ways = $4! \times 4! = 576$.

EXAMPLE 11 *Find the number of ways in which 10 different beads can be arranged to form a necklace.*

SOLUTION Ten different beads can be arranged in circular form in $(10-1)! = 9!$ ways. Since there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements = $\frac{1}{2}(9!)$

EXAMPLE 12 *If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these objects so that no two of them are consecutive is* $\frac{n(n-4)(n-5)}{6}$.

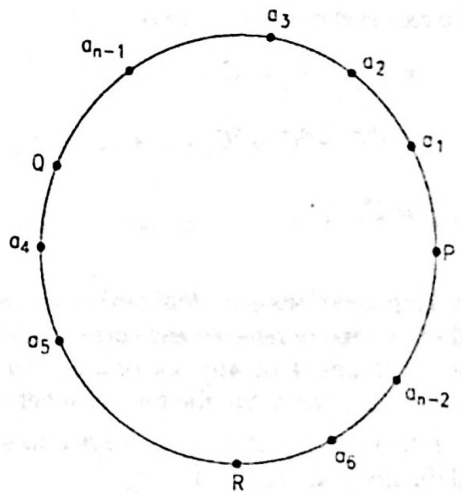


Fig. 22.13

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n$ be n distinct objects arranged in a circle. Choose an object P out of these n objects and call it the first object. This can be done in ${}^nC_1$ ways. Since no two of the three selected objects are consecutive. Therefore, the remaining two objects, say Q and R, are to be chosen from the remaining $n-3$ objects obtained by leaving P and two more objects one on each side of P. This can be done in ${}^{n-3}C_2$ ways. Out of these there are $(n-4)$ ways in which Q and R are consecutive. Therefore, the number of ways of selecting Q and R so that they are not consecutive is

$$ {}^{n-3}C_2 - (n-4) = \frac{(n-4)(n-5)}{2} $$

But Q and R can be arranged in 2! ways. Therefore, the number of ways of selecting 3 objects (in order first, second and third) is

$$ {}^nC_1 \times \frac{1}{2}(n-4)(n-5) \times 2! = n(n-4)(n-5) $$

Since the order in which the objects are taken is immaterial, therefore, the number of ways of selecting three objects

$$ = \frac{1}{3!} \times n(n-4)(n-5) = \frac{n\,n-4)(n-5)}{6} $$

EXERICSE 22.7

1. In how many ways 10 persons may be arranged in a (i) line (ii) circle ?
2. In how many ways 8 beads can be arranged to form a necklace ?
3. In how many ways can 7 persons form a ring ? In how many ways can 7 Indian and 7 Americans sit down at a round table, no two Americans being together ?
4. In how many ways can 11 members of a committee sit at a round table so that the secretary and the joint secretary are always the neighbours of the president ?
5. A gentleman invites 6 of his friends to a party. In how many different arrangements they, along with the wife of the gentleman can sit at a round table for a dinner if the host and his wife always sit side by side ?
6. In how many ways can a garland of 20 different flowers be made ?
7. In how many ways can n men be seated around a round table when in no two ways a man has the same neighbours ?
8. In how many ways can 6 Hindus and 6 Muslims sit around a round table so that two Muslims may never sit together ?

ANSWERS

1. (i) 10! (ii) 9! 2. $\frac{1}{2}(7!)$ 3. 720, 3628800 4. 80640 5. 1440 6. $\frac{1}{2}(19!)$ 7. $\frac{1}{2}(n-1)!$ 8. 86400

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22.8 SELECTION OF ONE OR MORE OBJECTS

SELECTION FROM DIFFERENT ITEMS

THEOREM 1 *The number of ways of selecting one or more items from a group of n distinct items is* $2^n - 1$.

PROOF Out of n items, 1 item can be selected in ${}^nC_1$ ways; 2 items can be selected in ${}^nC_2$ ways; three items can be selected in ${}^nC_3$ ways and so on.

Hence, the required number of ways

$$ = {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n $$

$$ = ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) - {}^nC_0 $$

$$ = 2^n - 1 $$

Q.E.D.

ALITER In making selections each item can be dealt with in two ways; it is either selected or rejected and corresponding to each way of dealing with one item, any one of the other items can also be dealt with in 2 ways. So, the total number of ways of dealing with n items is 2^n. But these 2^n ways also include the case when all the items are rejected.

Hence, required number of ways = $2^n - 1$.

ILLUSTRATION *A man has 6 friends, in how many ways may he invite one or more of them to dinner ?*

SOLUTION The man has to select some or all of his 6 friends. So, required number of ways = $2^6 - 1 = 63$.

SELECTION FROM IDENTICAL OBJECTS

RESULT I *The number of ways of selecting r items out of n identical items is 1.*

RESULT II *The total number of ways of selecting zero or more i.e. at least one item from a group of n identical items is* $(n+1)$.

RESULT III *The total number of selections of some or all out of* $p+q+r$ *items where p are alike of one kind, q are alike of second kind and rest are alike of third kind is* $[(p+1)(q+1)(r+1)]-1$.

SELECTION FROM IDENTICAL AND DISTINCT OBJECTS

THEOREM 2 *The total number of ways of selecting one or more items from p identical items of one kind; q identical items of second kind; r identical items of third kind and n different items is*

$$ (p+1)(q+1)(r+1)2^n - 1 $$

PROOF Since we may select 0, 1, 2, 3, ... p items from p identical items, therefore p identical items may be dealt with in $(p+1)$ ways. Similarly, q identical items may be dealt with in $(q+1)$ ways and r identical items in $(r+1)$ ways. Now, each of the n distinct items can be disposed of in two ways, either by select-

ing it or by rejecing it. Therefore, the total number of ways of dealing with all the items is

$$(p+1)(q+1)(r+1)\times\underbrace{2\times2\times2\times\ldots\times2}_{n\text{ - times}}$$

But, this number also includes a case in which none of the items is taken, therefore the required number of ways of selecting at least one item is $(p+1)(q+1)(r+1)2^n-1$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1. *Find the total number of factors (excluding 1) of 2160.*

SOLUTION We have,

$$2160=2^4\times3^3\times5^1.$$

The total number of divisors is same as the number of ways of selecting some or all out of four 2's, three 3's and one 5's. The number of such ways is

$$(4+1)(3+1)(1+1)-1=39.$$

Hence, the total number of factors = 39.

EXAMPLE 2. *Find the total number of proper factors of 7875.*

SOLUTION We have ,

$$7875=3^2\times5^3\times7^1$$

The total number of ways of selcting some or all out of two 3's, three 5's and one 7's is

$$(2+1)(3+1)(1+1)-1=23$$

But, this includes the given number itself. Therefore, the required number of proper factors is 22.

EXAMPLE 3. *Find the number of factors (excluding 1 and the expression itself) of the product of $a^7\,b^4\,c^3\,def$, where a, b, c, d, e, f are all prime numbers.*

SOLUTION The total number of factors of the product $a^7\,b^4\,c^3\,def$ is equal to the number of ways of selecting at least one from seven a's, four b's, three c's, one d's, one e's and one f's. The number of such ways is

$$(7+1)(4+1)(3+1)(2)(2)(2)-1=1279$$

But, this includes the given product. Hence, the required number of factors

$$=1279-1=1278.$$

EXAMPLE 4. *Find the number of ways of selecting one or more letters from the letters AAAA BBB CDE.*

SOLUTION The given letters can be divided into following five groups :

$$(AAAA),(BBB),C,D,E$$

The required number of ways is equal to the number of ways of selecting at least one letter from four A's, three B's, one C's, one D's and one E's i.e.

$$=(4+1)(3+1)\times2\times2\times2-1=159$$

EXAMPLE 5 *An unlimited number of coupons bearing the letters A, B and C are available. Find the number of ways of choosing m of these coupons so that they cannot be used to spell BAC.*

SOLUTION BAC cannot be spelt in the following cases :

CASE I *When all the selected coupons bear the same letter.*

There are three letters out of which one letter can be selected in ${}^3C_1$ ways. Therefore, the total number of ways of choosing m coupons bearing the same letter is ${}^3C_1\times1=3$.

CASE II *When the selected coupons bear two letters only :*

There are three letters A, B, C out of which two can be chosen in ${}^3C_2$ ways. Now corresponding to each way there are two choices for each coupon. Therefore, there are 2^m choices for all m coupons. But these 2^m choices include two choices in which all the coupons bear the same letter. So, the number of choices in which all coupons do not bear the same letter out of two selected letters in (2^m-2).

Thus, the number of ways in which selected coupons bear two letters only $={}^3C_2\times(2^m-2)$.

Hence, the required number of ways $=3+{}^3C_2\,(2^m-2)$

$$=3+3(2^m-2)$$

$$=3(2^m-1)$$

EXAMPLE 6 *There are 5 mangoes and 4 apples. In how many different ways can a selection of fruits be made if*

(i) *fruits of the same kind are different ?*

(ii) *fruits of the same kind are identical ?*

SOLUTION (i) The total number of ways of selecting any number of fruits $=(2^9-1)$.

(ii) The required number of ways $=(5+1)(4+1)-1=29$.

—❑❑❑—

22.9 NUMBER OF DIVISORS AND THE SUM OF THE DIVISORS OF A GIVEN NATURAL NUMBER

The concept of selecting one or more objects from a group consisting of some identical objects and some objects distinct from each other is very useful to find the number of divisors of a given natural number greater than 1. By, fundamental theorem of algebra any natural number n can be expressed as a product of primes. So, let

$$n = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k} \quad \dots(i)$$

where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $n_1, n_2, \dots, n_k$ are positive integers.

Clearly, any divisor of n is of the form

$$d = p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \dots p_k^{m_k} \quad \dots(ii)$$

where $m_1, m_2, \dots m_k$ are natural numbers such that $0 \le m_i \le n_i ; i = 1, 2, 3. \dots, k$

Therefore, the total number of divisors of (i) will be equal to the number of ways of selecting at least one from n_1 identical primes p_1, n_2 identical primes p_2, and so on, finally n_k identical primes p_k. The number of such ways is

$$(n_1 + 1)(n_2 + 1) \dots (n_k + 1)$$

Hence, the total numbers of divisors of

$$n = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k} \text{ is}$$

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1).$$

This includes 1 and n as divisors. Therefore, number of divisors other than 1 and n is

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1) - 2.$$

The sum of all divisors of (i) is given by

$$\sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \sum_{r_3=0}^{n_3} \dots \sum_{r_k=0}^{n_k} p_1^{r_1} \cdot p_2^{r_2} \cdot p_3^{r_3} \dots p_k^{r_k}$$

$$= \left(\frac{p_1^{n_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{n_2+1} - 1}{p_2 - 1}\right)\left(\frac{p_3^{n_3+1} - 1}{p_3 - 1}\right) \dots \left(\frac{p_k^{n_k+1} - 1}{p_k - 1}\right)$$

ILLUSTRATION 1 *If $n = 10800$, then find*

(i) *The total number of divisors of n.*

(ii) *The number of even divisors of n.*

(iii) *The number of divisors of the form $4m + 2$.*

(iv) *The number of divisors which are multiples of 15.*

SOLUTION We have,

$$n = 10800 = 2^4 \times 3^3 \times 5^2$$

(i) Total number of divisors of $n = (4 + 1)(3 + 1)(2 + 1)$

$= 60.$

(ii) For a divisor to be even, we must have at least one 2 out of four 2's any number of 3's and 5's.

So, total number of even divisors $= 4(3 + 1)(2 + 1) = 48.$

(iii) We have,

$$4m + 2 = 2(2m + 1)$$

Therefore, in any divisor of the form $4m + 2$, 2 must occur exatly once. Hence, the number of divisors of the form $4m + 2$ is

$$1 \times (3 + 1)(2 + 1) = 12$$

(iv) A divisor of n will be a multiple of 15, if it contains at least one 3 out of three 3's and at least one 5 out of two 5's.

Hence, the number of divisors which are multiples of 15 is equal to $(4 + 1)(3)(2) = 30$.

22.10 DIVISION OF OBJECTS INTO GROUPS

DIVISION OF ITEMS INTO GROUPS OF UNEQUAL SIZE

THEOREM 1 *Number of ways in which $(m + n)$ distinct items can be divided into two unequal groups containing m and n items is*

$$\frac{(m+n)!}{m!\, n!}$$

PROOF The number of ways in which $(m + n)$ items are divided into two groups containing m and n items is same as the number of combinations of $(m + n)$ things taken m at a time. Thus, the required number $= {}^{m+n}C_m = \frac{(m+n)!}{m!\, n!}$

Q.E.D.

REMARK 1 *The number of ways in which $(m + n + p)$ items can be divided into unequal groups contaning m, n, p items is*

$${}^{m+n+p}C_m \cdot {}^{n+p}C_n = \frac{(m+n+p)!}{m!\, n!\, p!}$$

REMARK 2 *The number of ways to distribute $(m + n + p)$ items among 3 persons in the groups containing m, n and p items is*

= (No. of ways to divide) × (No. of groups)!

$$= \frac{(m+n+p)!}{m!\, n!\, p!} \times 3!$$

DIVISION OF OBJECTS INTO GROUPS OF EQUAL SIZE

The number of ways in which mn different objects can be divided equally into m groups, each containing n objects and the order of the groups is not important, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}$$

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of groups is important, is

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!}\right) m! = \frac{(mn)!}{(n!)^m}$$

ILLUSTRATION 1 *In how many ways can a pack of 52 cards be divided equally among four players in order?*

SOLUTION Here 52 cards are to be divided into four equal groups and the order of the groups is important.

So, required number of ways

$$= \left(\frac{52!}{(13!)^4\, 4!}\right) 4! = \frac{52!}{(13!)^4}$$

ALITER For the first player we have ${}^{52}C_{13}$ choices, for the second player ${}^{39}C_{13}$ choices, for the third player ${}^{26}C_{13}$ choices and for the last player we have ${}^{13}C_{13}$ choices.

Hence, the total number of way

$$= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} = \frac{52!}{(13!)^4}$$

ILLUSTRATION 2 *In how many ways can a pack of 52 cards be divided equally into four groups ?*

SOLUTION The total number of ways is $\frac{52!}{(13!)^4 4!}$

ILLUSTRATION 3 *In how many ways can a pack of 52 cards be divided into 4 sets, three of them having 17 cards each and the fourth just one card ?*

SOLUTION First we divide 52 cards into two groups of 1 card and 51 cards. This can be done in $\frac{52!}{51!1!}$ ways. Now, every group of 51 cards can be divided into 3 groups of 17 cards each in $\frac{51!}{(17!)^3 3!}$.

Hence, the required number of ways

$$= \frac{52!}{51!1!} \times \frac{51!}{(17!)^3 3!}$$

$$= \frac{52!}{(17!)^3 3!}$$

ALITER For the first set we have ${}^{52}C_{17}$ choices, for the second set ${}^{35}C_{17}$ choices, for the third set ${}^{18}C_{17}$ choices and for the last set we have ${}^{1}C_{1}$ choices. But the first three sets can be interchanged in 3! ways.

Hence, the total number of ways for the required distribution

$$= {}^{52}C_{17} \times {}^{35}C_{17} \times {}^{18}C_{17} \times {}^{1}C_{1} \times \frac{1}{3!}$$

$$= \frac{52!}{(17!)^3 3!}$$

ILLUSTRATION 4 *In how many ways can 2n people be divided into n couples?*

SOLUTION Here the order of the couples is not important.

So, the required numbers of ways is

$$\frac{(2n)!}{(2!)^n n!}$$

ILLUSTRATION 5 *Find the number of ways of arranging 2m white and 2n red counters in a straight line so that each arrangement is symmetrical to a central mark.*

SOLUTION Since each arrangement is symmetrical about the central mark, so, we have to arrange m white and n red counters on either side of the central mark. This can be done in $\frac{(m+n)!}{m!\,n!}$ ways.

DIVISION OF IDENTICAL OBJECTS INTO GROUPS

THEOREM 2 *The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2, or more items* $(\le n)$ *is* ${}^{n+r-1}C_{r-1}$

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

PROOF Consider r brackets corresponding to r persons. In each bracket, take an expression given by $x^0 + x^1 + x^2 + \ldots + x^n$. Here the various powers of x viz; 0, 1, 2, ..., n correspond to the number of items each person can have in the distribution.

Since the total number of items is n. So, the required number of ways is the coefficient of x^n in the product

$$(x^0 + x^1 + x^2 + \ldots + x^n)(x^0 + x^1 + \ldots + x^n) \ldots\ldots\ldots$$
$$\ldots (x^0 + x^1 + x^2 + \ldots + x^n)$$
$$(r - \text{brackets})$$

Thus, the required number of ways

$$= \text{Coeff of } x^n (x^0 + x^1 + x^2 \ldots + x^n)^r$$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1 - x^{n+1}}{1 - x}\right)^r$$

$$= \text{Coefficient of } x^n \text{ in } (1 - x^{n+1})^r (1 - x)^{-r}$$

$$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$$

$$= \frac{(r+1)(r+2)(r+3)\ldots(r+n-1)}{1.2.3\ldots r} = {}^{n+r-1}C_{r-1}$$

Q.E.D.

THEOREM 3 *The total number of ways of dividing n identical items among r persons, each one of whom, receives at least one item is* ${}^{n-1}C_{r-1}$

OR

The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$

PROOF The required number of ways is the coeff. of x^n in $(x + x^2 + \ldots + x^n)^r$

Now,

$$\text{Coeff. of } x^n \text{ in } (x + x^2 + x^3 + \ldots + x^n)^r$$

$$= \text{Coeff. of } x^{n-r} \text{ in } (1 + x^2 + \ldots + x^{n-1})^r$$

$$= \text{Coeff. of } x^{n-r} \text{ in } \left(\frac{1 - x^n}{1 - x}\right)^r$$

$$= \text{Coeff. of } x^{n-r} \text{ in } (1 - x)^{-r}$$

$$= {}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$$

Q.E.D.

THEOREM 4 *The number of ways in which n identical items can be divided into r groups so that no group contains less than m items and more than k* $(m < k)$ is

Coefficient of x^n in the expansion of $(x^m + x^{m+1} + \ldots + x^k)^r$

ILLUSTRATION 6 *The total number of ways in which 30 mangoes can be distributed among 5 persons is* $^{30+5-1}C_{5-1} = {}^{34}C_4$.

ILLUSTRATION 7 *Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls.*

SOLUTION The required number of ways is the number of ways of distributing 5 items among 3 persons so that a person receives at least one item = $^{5-1}C_{3-1} = {}^4C_2 = 6$.

ILLUSTRATION 8 *In how many ways can 20 identical toys be distributed among 4 children so that each one gets at least 3 toys ?*

SOLUTION Required number of ways

$= $ Coefficient of x^{20} in $(x^3 + x^4 + \ldots + x^{20})^4$

$= $ Coefficientof x^8 in $(1 + x + \ldots + x^{17})^4$

$= $ Coefficient of x^8 in $\left(\frac{1 - x^{18}}{1 - x}\right)^4$

$= $ Coefficient of x^8in $(1 - x)^{-4}$

$= {}^{8\times4-1}C_{4-1}$ $[\because$ Coeff. of x^n in $(1-x)^{-r} = {}^{n+r-1}C_{r-1}]$

$= {}^{11}C_3$

$= 165.$

22.11 NUMBER OF INTEGRAL SOLUTIONS OF LINEAR EQUATIONS AND INEQUATIONS

The concepts introduced in the previous sections are very useful to determine the total number of integral solutions of linear equations and inequations.

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n \qquad \ldots(i)$$

where $x_1, x_2, x_3, x_4, \ldots x_r$ and n are non-negative integers.

This equation may be interpreted as that n identical objects are to be divided into r groups where a group may contain any number of objects. Therefore,

Total number of solutions of equation (i)

$= $ Coefficient of x^n in $(x^0 + x^1 + \ldots + x^n)^r$

$= {}^{n+r-1}C_{r-1}$ [See section 22.10]

Now, let us consider the equation

$$x_1 + x_2 + x_3 + \ldots + x_m = n \qquad \ldots(ii)$$

where $x_1, x_2, \ldots, x_m$ are integers such that

$a_i \le x_i \le b_i\ ; i = 1, 2, \ldots., m$

Proceeding as above, we find that

Total number of solutions of equation (ii)

$= $ Coefficient of x^n in $\{(x^{a_1} + x^{a_1+1} + x^{a_1+2} + \ldots + x^{b_1}) \cdot (x^{a_2} + x^{a_2+1} + \ldots + x^{b_2}) \ldots. (x^{a_m} + x^{a_m+1} + \ldots. + x^{b_m})\}$

In the above discussion, the coefficients of variables in the equations were unity. Now, let us consider the equation

$$ax_1 + bx_2 + cx_3 = n \qquad \ldots(iii)$$

where x_1, x_2, x_3 are integers such that

$$a_i \le x_i \le b_i\ ;\ i = 1, 2, 3.$$

The total number of solutions of this equation is equal to

Coefficient of x^n in

$$\left[\{(x^a)^{a_1} + (x^a)^{a_1+1} + \ldots. + (x^a)^{b_1}\}\ \{(x^b)^{a_2} + (x^b)^{a_2+1} + \ldots + (x^b)^{b_2}\}\ \{(x^c)^{a_3} + (x^c)^{a_3+1} + \ldots + (x^c)^{b_3}\}\right]$$

REMARK 1 *The total number of non-negative integral solutions of the equation* $x_1 + x_2 + \ldots + x_r = n$ *is* $^{n+r-1}C_{r-1}$ *and the total number of solutions of the same equation in the set N of natural numbers is* $^{n-1}C_{r-1}$.

REMARK 2 *If the upper limit of a variable in solving an equation of the form* $x_1 + x_2 + \ldots + x_m = n$ *subject to the conditions* $a_i \le x_i \le b_i\ ; i = 1, 2, \ldots, m$ *is more than or equal to the sum required and lower limit of all the variables are non-negative, then upper limit of that variable can be taken as infinite.*

REMARK 3 *In order to solve inequations of the form*

$$x_1 + x_2 + \ldots + x_m \le n \qquad \ldots(i)$$

we introduce a dummy (artificial) variable x_{m+1} such that

$x_1 + x_2 + \ldots + x_m + x_{m+1} = n$, where $x_{m+1} \ge 0$

The number of solutions of this equation are same as the number of solutions of inequation (i).

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Determine the total number of non-negative integral solutions of* $x_1 + x_2 + x_3 + x_4 = 100$.

SOLUTION Here the upper limit for the variables x_1, x_2, x_3 is more than 100 and the lower limit is non-negative. Therefore, required number of solutions

$= $ Coefficient of x^{100} in $(1 + x + x^2 + x^3 + \ldots)^4$

$= $ Coefficient of x^{100} in $(1 - x)^{-4}$

$= {}^{100+4-1}C_{4-1}$

$= {}^{103}C_3$.

ALITER Total number of non-negative integral solutions of the given equation is same as the number of ways of distributing 100 items among 4 persons such that each person can receive any number of items.

Hence, total number of solutions = $^{100+4-1}C_{4-1} = {}^{103}C_3$

EXAMPLE 2 *How many integral solutions are there to the equation* $x+y+z+t=29$ *when* $x\geq 1, y\geq 2, z\geq 3$ and $t\geq 0$ *?*

SOLUTION Here upper limit of all the variables is more than 29 and the lower limit is non-negative. So, the upper limit of each variable can be taken as infinitey.

Hence, required number of solutions

$$= \text{Coefficient of } x^{29} \text{ in } (x^1+x^2+x^3+\ldots)(x^2+x^3+\ldots.)(x^3+x^4+\ldots.)(x^0+x+x^2+\ldots.)$$

$$= \text{Coefficient of } x^{29} \text{ in } x^6(1+x+x^2+\ldots)^4$$

$$= \text{Coefficient of } x^{23} \text{ in } (1-x)^{-4}$$

$$= {}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$$

ALITER We have,

$x\geq 1,\ y\geq 2,\ z\geq 3$ and $t\geq 0$

Let $u=x-1,\ v=y-2,\ w=z-3$. Then,

$$x\geq 1 \Rightarrow u\geq 0$$
$$y\geq 2 \Rightarrow v\geq 0$$
$$z\geq 3 \Rightarrow w\geq 0$$

Thus, we have

$u+1+v+2+w+3+t=29$, where $u\geq 0, v\geq 0, w\geq 0, t\geq 0$

$$u+v+w+t = 23$$

The total number of solutions of this equation is

$${}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$$

EXAMPLE 3 *Find the number of ways in which 16 sovereigns can be distributed among three applicants such that each applicant does not receive less than 3 sovereigns.*

SOLUTION Let x, y, z be the number of sovereigns received by the three applicants. Then,

$x\geq 3, y\geq 3, z\geq 3$ and $x+y+z = 16$.

Thus, the required number of ways is equal to the number of solutions of the equation $x+y+z=16$, where $x\geq 3, y\geq 3$, $z\geq 3$.

Clearly, when x and y take their minimum values, z will assume its maximum value. Therefore, the maximum value of z is given by

$$3+3+z = 16 \Rightarrow z = 10$$

Similarly, the maximum values of x and y are each equal to 10.

Thus, the required number of ways is same as the number of solutions of the equation

$x+y+z = 16$, where $3\leq x, y, z\leq 10$.

Hence,

required number of ways

$$= \text{Coefficient of } x^{16} \text{ in } (x^3+x^4+\ldots+x^{10})^3$$

$$= \text{Coefficient of } x^7 \text{ in } (1+x+\ldots+x^7)^3$$

$$= \text{Coefficient of } x^7 \text{ in } \left(\frac{1-x^8}{1-x}\right)^{-3}$$

$$= \text{Coefficient of } x^7 \text{ in } (1-x)^{-3}$$

$$= {}^{7+3-1}C_{3-1}$$

$$= {}^9C_2 = 36.$$

ALITER Let x, y, z be the number of sovereigns received by the three applicants. Then,

$$x\geq 3, y\geq 3, z\geq 3 \text{ and } x+y+z=16$$

Let $u=x-3, v=y-3$ an $w=z-3$. Then, $u\geq 0, v\geq 0, w\geq 0$ and,

$$u+3+v+3+w+3=16$$
$$\Rightarrow \quad u+v+w=7$$

The total number of solutions of this equation is

$${}^{7+3-1}C_{3-1} = {}^9C_2 = 36.$$

EXAMPLE 4 *How many integral solutions are there to the system of equations* $x_1+x_2+x_3+x_4+x_5=20$ *and* $x_1+x_2+x_3=5$ *when* $x_k\geq 0$ *?*

SOLUTION We have, $x_1+x_2+x_3+x_4+x_5 = 20$

and, $x_1+x_2+x_3 = 5$

These two equations reduce to

$$x_4+x_5 = 15 \qquad \ldots(i)$$

and, $x_1+x_2+x_3 = 5$ ...(ii

Since corresponding to each solution of (i) there are solutions of equation (ii). So, total number of solutions of the given system of equations

= No. of solutions of (i) × No. of solutions of (ii)

$$= {}^{15+2-1}C_{2-1} \times {}^{5+3-1}C_{3-1}$$

$$= {}^{16}C_1 \times {}^7C_2 = 336.$$

EXAMPLE 5 *In how many ways can four persons, each throwing a die once, make a sum of 6?*

SOLUTION Let x_1, x_2, x_3, x_4 be the numbers obtained by the four persons on the upper faces of their respective dice. Then, the required number of ways will be equal to the number of solutions of the equation

$x_1+x_2+x_3+x_4 = 6$, where $1\leq x_1, x_2, x_3, x_4\leq 6$.

Since the upper limit is six which is equal to the sum required. So, the upper limit for each variable can be taken as infinite.

Hence, required number of ways

$$= \text{Coefficient of } x^6 \text{ in } (1+x+x^2+\ldots.)^4$$

$$= \text{Coefficient of } x^6 \text{ in } (1-x)^{-4}$$

$$= {}^{6+4-1}C_{4-1} = {}^9C_3 = 84.$$

EXAMPLE 6 *In how many ways can four persons, each throwing a die once, make a sum of 13 ?*

SOLUTION Let x_1, x_2, x_3 and x_4 be the numbers on the upper faces of the four dice. Then, the required number of ways is equal to the number of solutions of

$x_1 + x_2 + x_3 + x_4 = 13$, where $1 \le x_1, x_2, x_3, x_4 \le 6$

The number of solutions of this equation is equal to the

Coefficient of x^{13} in $(x^1 + x^2 + ... + x^6)^4$

$= $ Coefficient of x^{13} in $x^4\ (1 + x + x^2 + ... + x^5)^4$

$= $ Coefficient of x^9 in $\left(\frac{1-x^6}{1-x}\right)^4$

$= $ Coefficient of x^9 in $(1-x^6)^4 (1-x)^{-4}$

$= $ Coefficient of x^9 in $({}^4C_0 - {}^4C_1 x^6 + {}^4C_2 x^{12} + ...)(1-x)^{-4}$

$= $ Coefficient of x^9 in ${}^4C_0 (1-x)^{-4}$

$-$ Coefficient of x^9 in ${}^4C_1 x^6 (1-x)^{-4}$

$= {}^4C_0 \times$ (Coefficient of x^9 in $(1-x)^{-4}$

$- {}^4C_1 \times$ Coefficient of x^3 in $(1-x)^{-4}$

$= {}^4C_0 \times {}^{9+4-1}C_{4-1} - {}^4C_1 \times {}^{3+4-1}C_{4-1}$

$= {}^{12}C_3 - 4 \times {}^6C_3$

$= 220 - 80$

$= 140.$

EXAMPLE 7 *Three six faced die are thrown together. Find the total number of ways of getting k as the sum if*

(i) $3 \le k \le 8$ (ii) $9 \le k \le 14$

SOLUTION Let x_1, x_2 and x_3 be the numbers on the upper faces of three dice. Then, the required number of ways is equal to the number of solutions of the equation

$x_1 + x_2 + x_3 = k$, where $1 \le x_1, x_2, x_3 \le 6$.

The number of solutions of this equation is equal to

Coefficient of x^k in $(x + x^2 + ... + x^6)^3$

$= $ Coefficient of x^{k-3} in $(1 + x + ... + x^5)^3$

$= $ Coefficient of x^{k-3} in $\left(\frac{1-x^6}{1-x}\right)^3$

$= $ Coefficient of x^{k-3} in $(1-x^6)^3 (1-x)^{-3}$

(i) If $3 \le k \le 8$, then $0 \le k-3 \le 5$.

$\therefore$ Coefficient of x^{k-3} in $(1-x^6)^3 (1-x)^{-3}$

$= $ Coefficient of x^{k-3} in $(1-x)^{-3}$

$= {}^{k-3+3-1}C_{3-1}$

$= {}^{k-1}C_2.$

Hence, required number of ways $= {}^{k-1}C_2$.

(ii) If $9 \le k \le 14$, then $6 \le k-3 \le 11$.

$\therefore$ Coefficient of x^{k-3} in $(1-x^6)^3 (1-x)^{-3}$

$= $ Coefficient of x^{k-3} in $({}^3C_0 - {}^3C_1 x^6 + {}^3C_2 x^{12} - {}^3C_3 x^{18})(1-x)^{-3}$

$= {}^3C_0$ Coefficient of x^{k-3} in $(1-x)^{-3}$

$- {}^3C_1$ Coefficient of x^{k-9} in $(1-x)^{-3}$

$= {}^3C_0 \times {}^{k-3+3-1}C_{3-1} - {}^3C_1 \times {}^{k-9+3-1}C_{3-1}$

$= {}^{k-1}C_2 - 3 \times {}^{k-7}C_2$

$= 21k - k^2 - 83.$

EXAMPLE 8 *How many integers between 1 and 10^6 have the sum of their digits equal to 18 ?*

SOLUTION Let $n = x_6 x_5 x_4 x_3 x_2 x_1$ be a number between 1 and 10^6, where $x_1, x_2, x_3, ..., x_6$ are digits at one's, tens, hundreds .... places. Clearly $0 \le x_i \le 9, i = 1, 2, ..., 6$.

The total number of values of n is the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $0 \le x_i \le 9\ ; i = 1, 2, ..., 6$

The number of solutions of the above equation is

Coefficient of x^{18} in $(x^0 + x^1 + x^2 + ... + x^9)^6$

$= $ Coefficient of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

$= $ Coefficient of x^{18} in $(1-x^{10})^6 (1-x)^{-6}$

$= $ Coefficient of x^{18} in $(1 - {}^6C_1 x^{10} + {}^6C_2 x^{20} + ...)(1-x)^{-6}$

$= $ Coefficient of x^{18} in $(1-x)^{-6}$

$- {}^6C_1 \times$ Coefficient of x^8 in $(1-x)^{-6}$

$= {}^{18+6-1}C_{6-1} - {}^6C_1 \times {}^{8+6-1}C_{6-1}$

$= {}^{23}C_5 - 6 \times {}^{13}C_5$

$= 33649 - 7722$

$= 25927.$

Hence, the required number of integers is 25927.

EXAMPLE 9 *An examination consists of four papers. Each paper has a maximum of m marks. Show that the number of ways in which a student can get 2m marks in the examination is*

$$\frac{m+1}{3}(2m^2 + 4m + 3).$$

SOLUTION Suppose the student scores x_i marks in ith paper, $i = 1, 2, 3, 4$. Then, $0 \le x_i \le m\ ; i = 1, 2, 3, 4$.

We have to find the total number of non-negative integral solutions of the equation

$x_1 + x_2 + x_3 + x_4 = 2m$, where $0 \le x_i \le m\ ; i = 1, 2, 3, 4$.

Hence, required number of ways

$= \text{Coefficient of } x^{2m} \text{ in } (1+x+x^2+\ldots+x^m)^4$

$= \text{Coefficient of } x^{2m} \text{ in } \left(\frac{1-x^{m+1}}{1-x}\right)^4$

$= \text{Coefficient of } x^{2m} \text{ in } (1-x^{m+1})^4\ (1-x)^{-4}$

$= \text{Coefficient of } x^{2m} \text{ in } \{({}^4C_0 - {}^4C_1 x^{m+1} + {}^4C_2 x^{2m+2} \ldots) \times (1-x)^{-4}\}$

$= {}^4C_0 \times \text{Coefficient of } x^{2m} \text{ in } (1-x)^{-4} - {}^4C_1 \times \text{Coefficient of } x^{m-1} \text{ in } (1-x)^{-4}$

$= {}^4C_0 \times {}^{2m+4-1}C_{4-1} - {}^4C_1 \times {}^{m-1+4-1}C_{4-1}$

$= {}^{2m+3}C_3 - 4 \times {}^{m+2}C_3$

$= \frac{(2m+3)(2m+2)(2m+1)}{6} - 4 \times \frac{(m+2)(m+1)m}{6}$

$= \frac{1}{3}(m+1)(2m^2+4m+3)$

EXAMPLE 10 *In how many ways can 15 identical objects be distributed among six persons such that every one gets at least one object and two particular persons get equal number of objects and another three particular persons get equal number of objects.*

SOLUTION Suppose each of the three particular persons gets x objects, each of the two particular persons gets y objects and the sixth person gets z objects. Then,

$x+x+x+y+y+z = 15$, where $x, y, z \geq 1$.

$\Rightarrow 3x+2y+z=15$ ...(i)

Thus, the total number of ways of distributing 15 objects is equal to the number of solutions of equation (i) subject to the condition $x, y, z \geq 1$.

Now,

Number of solutions of equation (i)

$= \text{Coefficient of } x^{15} \text{ in } \{(x^3+x^6+x^9+\ldots+x^{18}) \times (x^2+x^4+x^6+\ldots+x^{18})(x+x^2+\ldots+x^{18})\}$

$= \text{Coefficient of } x^9 \text{ in } \{(1+x^3+x^6+\ldots+x^{15})(1+x^2+x^4+\ldots+x^{16})(1+x+\ldots+x^{17})\}$

$= \text{Coefficient of } x^9 \text{ in } \left|\left(\frac{1-x^{18}}{1-x^3}\right)\left(\frac{1-x^{18}}{1-x^2}\right)\left(\frac{1-x^{18}}{1-x}\right)\right|$

$= \text{Coefficient of } x^9 \text{ in } \{(1-x^3)^{-1}\ (1-x^2)^{-1}\ (1-x)^{-1}\}$

$= \text{Coefficient of } x^9 \text{ in } \{(1+x^3+x^6+x^9+\ldots) \times (1+x^2+x^4+x^6+x^8+\ldots)(1+x+x^2+\ldots+x^9+\ldots.)\}$

$= \text{Coefficient of } x^9 \text{ in } \{(1+x^2+x^4+x^6+x^8+x^3+x^5+x^7+x^9+x^6+x^8+x^9)(1+x+x^2+\ldots+x^9)\}$

$= \text{Coefficient of } x^9 \text{ in } \{(1+x^2+x^3+x^4+x^5+2x^6+x^7+2x^8+2x^9)(1+x+x^2+\ldots+x^9)\}$

$= 1+1+1+1+1+2+1+2+2$

$= 12.$

EXAMPLE 11 *Find the number of non-negative inetegral solutions of the equation* $x_1+x_2+x_3+4x_4=20$.

SOLUTION The total number of non-negative integral solutions of the given equation

$= \text{Coefficient of } x^{20} \text{ in } (x^0+x^1+x^2+\ldots)^3\ (x^0+x^4+x^8+x^{16}+\ldots.)$

$= \text{Coefficient of } x^{20} \text{ in } \left(\frac{1}{1-x}\right)^3 \left(\frac{1}{1-x^4}\right)$

$= \text{Coefficient of } x^{20} \text{ in } [(1-x)^{-3}\ (1-x^4)^{-1}]$

$= \text{Coefficient of } x^{20} \text{ in } [(1-x)^{-3}\ (1+x^4+x^8+x^{12}+x^{16}+x^{20}+\ldots)]$

$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-3} + \text{Coeff. } x^{16} \text{ in } (1-x)^{-3} + \text{Coeff. of } x^{12} \text{ in } (1-x)^{-3} + \ldots. + \text{Coeff. of } x^0 \text{ in } (1-x)^{-3}$

$= {}^{20+3-1}C_{3-1} + {}^{16+3-1}C_{3-1} + {}^{12+3-1}C_{3-1} + {}^{8+3-1}C_{3-1} + {}^{4+3-1}C_{3-1} + 1$

$= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^6C_2 + 1$

$= 231+153+91+45+15+1$

$= 536.$

ALITER Let $x_4 = k$. Then,

$x_1+x_2+x_3+4x_4=20$

$\Rightarrow x_1+x_2+x_3 = 20-4k$ ...(i)

Since x_1, x_2, x_3 and x_4 are non-negative integers. Therefore, $20 \geq 20-4k \geq 0 \Rightarrow 0 \leq k \leq 5$.

For a given value of k, the total number of integral solutions of (i) is

$${}^{20-4k+3-1}C_{3-1} = {}^{22-4k}C_2.$$

But k varies from 0 to 5. So, the total number of integral solutions of the given equation is

$$\sum_{k=0}^{5} {}^{22-4k}C_2$$

$$= \sum_{k=0}^{5} (8k^2-86k+231)$$

$$= 8\sum_{k=0}^{5} k^2 - 86\sum_{k=0}^{5} k + 231 \times 6$$

$= 8 \times 55 - 86 \times 15 + 231 \times 6 = 536$

EXAMPLE 12 *Find the number of non-negative integral solutions of the equation* $3x + y + z = 24$.

SOLUTION Required number of solutions

$= \text{Coefficient of } x^{24} \text{ in } (x^0 + x^3 + x^6 + \ldots + x^{24})(x^0 + x^1 + \ldots + x^{24})^2$

$= \text{Coefficient of } x^{24} \text{ in } \left(\frac{1-x^{27}}{1-x^3}\right)\left(\frac{1-x^{25}}{1-x}\right)^2$

$= \text{Coefficient of } x^{24} \text{ in } (1-x^3)^{-1}(1-x)^{-2}$

$= \text{Coefficient of } x^{24} \text{ in } (1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18} + x^{21} + x^{24} + \ldots.)(1-x)^{-2}$

$= \text{Coefficient of } x^{24} \text{ in } \left(\sum_{n=0}^{8} x^{3n}\right)(1-x)^{-2}$ [Neglecting higher powers]

$= \sum_{n=0}^{8} \text{Coefficient of } x^{24-3n} \text{ in } (1-x)^{-2}$

$= \sum_{n=0}^{8} {}^{24-3n+2-1}C_{2-1}$

$= \sum_{n=0}^{8} {}^{25-3n}C_1$

$= \sum_{n=0}^{8} (25-3n)$

$= 25 \times 9 - \frac{3 \times 8 \times 9}{2} = 225 - 108 = 117$

ALITER We have, $3x + y + z = 24$ where $x \geq 0, y \geq 0, z \geq 0$.

Let $x = k$. Then,

$3x + y + z = 24$

$\Rightarrow 3k + y + z = 24$

$\Rightarrow y + z = 24 - 3k$ ...(i)

$\Rightarrow 24 - 3k \geq 0$ [$\because y + z \geq 0$]

$\Rightarrow k \leq 8$

$\Rightarrow 0 \leq k \leq 8$ [$\because x \geq 0 \Rightarrow k \geq 0$]

For a fixed value of k the number of solutions of (i) is

${}^{24-3k+2-1}C_{2-1}$

$= {}^{25-3k}C_1$

$= 25 - 3k$

But, k varies from 0 to 8.

Hence, the total number of solutions of the given equation is

$$\sum_{k=0}^{8} (25-3k) = 25 \times 9 - \frac{3 \times 8 \times 9}{2} = 225 - 108 = 117$$

EXAMPLE 13 *Find the total number of positive unequal integral solutions of the equation* $x + y + z + w = 20$.

SOLUTION The given equation is

$$x + y + z + w = 20 \quad \ldots(i)$$

We have to find the values of x, y, z such that $x \neq y \neq z \neq w$ and $x, y, z, w \geq 1$.

Let us assume that

$x < y < z < w$

and, $x = x_1, y - x = x_2, z - y = x_3$ and $w - z = x_4$.

Then, $x = x_1, y = x_1 + x_2, z = x_1 + x_2 + x_3$

and $w = x_1 + x_2 + x_3 + x_4$.

Also, $x_1, x_2, x_3, x_4 \geq 1$.

Substituting these values in (i), we get

$4x_1 + 3x_2 + 2x_3 + x_4 = 20$, ...(ii)

where $x_1, x_2, x_3, x_4 \geq 1$.

Now,

Number of solutions of equation (ii)

$= \text{Coefficient of } x^{20} \text{ in } (x^4 + x^8 + \ldots)(x^3 + x^6 + \ldots) \times (x^2 + x^4 + \ldots)(x + x^2 + \ldots)$

$= \text{Coefficient of } x^{10} \text{ in } (1 + x^4 + x^8 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x + x^2 + \ldots)$

$= \text{Coefficient of } x^{10} \text{ in } (1 + x^4 + x^8)(1 + x^3 + x^6 + x^9) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + \ldots + x^{10})$ [Neglecting higher powers]

$= \text{Coefficient of } x^{10} \text{ in } (1 + x^3 + x^4 + x^6 + x^7 + x^8 + x^9 + x^{10}) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + \ldots. + x^{10})$

$= \text{Coefficient of } x^{10} \text{ in } (1 + x^2 + x^3 + 2x^4 + x^5 + 3x^6 + 2x^7 + 4x^8 + 3x^9 + 5x^{10})(1 + x + x^2 + \ldots + x^{10})$

$= 1 + 1 + 1 + 2 + 1 + 3 + 2 + 4 + 3 + 5$

$= 23.$

But, x, y, z and w can be arranged in 4! ways and corresponding to each way there are 23 solutions. Hence, required number of solutions $= 23 \times 24 = 552$.

EXAMPLE 14 *In how many ways can 14 identical toys be distributed among three boys so that each one gets at least one toy and no two boys get equal number of toys.*

SOLUTION Let the three boys get x, y and z toys respectively. Then,

$$x + y + z = 14, \quad \ldots(i)$$

where $x, y, z \geq 1$ and $x \neq y \neq z$.

Let us assume that $x < y < z$ and $x = x_1$, $y - x = x_2$ and $z - y = x_3$. Then,

$x = x_1, y = x_1 + x_2$ and $z = x_1 + x_2 + x_3$.

Substituting these values in (i), we get

$$3x_1 + 2x_2 + x_3 = 14 \quad ...(iii)$$

where $x_1, x_2, x_3 \geq 1$.

The number of solutions of (ii)

$= $ Coefficient of x^{14} in $(x^3 + x^6 + x^9 + ...)$

$\times (x^2 + x^4 + ...)(x + x^2 + ...)$

$= $ Coefficient of x^8 in $(1 + x^3 + x^6 + ...)(1 + x^2 + x^4 + ...)$

$(1 + x + x^2 + ...)$

$= $ Coefficient of x^8 in $(1 + x^3 + x^6)(1 + x^2 + x^4 + x^6 + x^8)$

$\times (1 + x + x^2 + ... + x^8)$ [Neglecting higher powers]

$= $ Coefficient of x^8 in $(1 + x^2 + x^3 + x^4 + x^5 + 2x^6 + x^7 + 2x^8)$

$\times (1 + x + x^2 + ... + x^8)$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10$

Since x, y, z can be assigned to three boys in 3! ways. So, corresponding to each solution, we have six ways of distribution.

Hence, the total number of ways $= 10 \times 6 = 60$.

EXAMPLE 15 *Between two junction stations A and B there are 12 intermediate stations. Find the total number of ways in which a train can be made to stop at 4 of these stations so that no two of these halting stations are consecutive.*

SOLUTION Let x_1 be the number of stations before the first halting station, x_2 between first and second, x_3 between second and third, x_4 between third and fourth and x_5 on the right of the 4th stations. Then,

$x_1 \geq 0, x_5 \geq 0, x_2, x_3, x_4 \geq 1$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8 \quad ...(i)$$

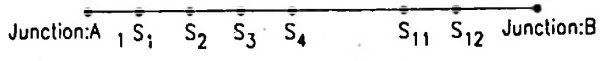


Fig. 22.14

The total number of ways is the number of solutions of the above equation.

Let $y_2 = x_2 - 1, y_3 = x_3 - 1, y_4 = x_4 - 1$. Then, (i) reduces to

$x_1 + y_2 + y_3 + y_4 + x_5 = 5$, where $y_2, y_3, y_4 \geq 0$.

The number of solutions of this equation is

$$^{5+5-1}C_{5-1} = {}^9C_4.$$

EXAMPLE 16 *If n objects are arranged in a row, then determine the number of ways of selecting three of these objects so that no two of them are next to each other.*

SOLUTION Let x_1 be the number of objects to the left of first object, x_2 the number of objects between the first and second, x_3 the number objects between second and third and x_4 the number of objects to the right of the third object. Then,

$x_1 \geq 0, x_2 > 1, x_3 \geq 1$ and $x_4 \geq 0$ such that

$$x_1 + x_2 + x_3 + x_4 = n - 3 \quad ....(i)$$

Clearly, the total number of ways is the total number of solutions of (i).

Let $y_2 = x_2 - 1$ and $y_3 = x_3 - 1$ Then, (i) reduces to

$$x_1 + y_2 + y_3 + x_4 = n - 5, \text{ where } x_1, x_4 \geq 0, y_2, y_3 \geq 0$$

The number of solutions of this equation is $^{n-5+4-1}C_{4-1}$ i.e. $^{n-2}C_3$.

EXAMPLE 17 *If $n \in N$, then find the total number of non-negative integeral solutions of the inequation $x + y + z \leq n$.*

SOLUTION The given inequation is

$$x + y + z \leq n \quad ...(i)$$

Let t be a variable such that

$$x + y + z + t = n \quad ...(ii)$$

Clearly, $t \geq 0$.

The total number of solutions of this equation is same as the number of solutions of inequation (i).

The number of solutions of (ii) is

$$^{n+4-1}C_{4-1} = {}^{n+3}C_2.$$

Hence, the total number of solutions of (i) is $^{n+3}C_2$.

EXAMPLE 18 *In how many ways can we get a sum of at most 17 by throwing six ditinct dice ?*

SOLUTION Let x_1, x_2, x_3, x_4, x_5 and x_6 be the numbers appearing on six dice. Then, the required number of ways is same as the number of solutions of the inequations

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 17,$$

when $1 \leq x_i \leq 6$; $i = 1, 2, ..., 6$

Let x_7 be *a* variable such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 17 \quad ...(i)$$

Clearly, $x_7 \geq 0$.

Now,

Required number of ways

$= $ Number of solutions of equation (i)

$= $ Coefficient of x^{17} in $(x^1 + ... + x^6)^6 (1 + x + x^2 + ...)$

$= $ Coefficient of x^{11} in $\left(\frac{1 - x^6}{1 - x}\right)^6 \left(\frac{1}{1 - x}\right)$

$= $ Coefficient of x^{11} in $(1 - x^6)^6 (1 - x)^{-7}$

$= $ Coefficient of x^{11} in $(1 - {}^6C_1 x^6 + {}^6C_2 x^{12} ...)(1 - x)^{-7}$

$= $ Coefficient of x^{11} in $(1 - x)^{-7}$

$- {}^6C_1 \times$ Coefficient of x^5 in $(1 - x)^{-7}$

$= {}^{11+7-1}C_{7-1} - {}^{6}C_1 \times {}^{7+5-1}C_{7-1}$

$= {}^{17}C_6 - 6 \times {}^{11}C_6$

$= 9604.$

EXAMPLE 19 *Show that the total number of non-negative integral solutions of 4-tuples* (x_1, x_2, x_3, x_4) *is* ${}^{104}C_4$ *such that* $x_1 + x_2 + x_3 + x_4 \le 100$.

SOLUTION We have,

$x_1 + x_2 + x_3 + x_4 \le 100$, where $x_i \ge 0$ for $i = 1, 2, 3, 4$.

Let x_5 be a dummy variable such that

$x_1 + x_2 + x_3 + x_4 + x_5 = 100$, where $x_5 \ge 0$.

The total number of non-negative integral solutions of this equation is ${}^{100+5-1}C_{5-1} = {}^{104}C_4$.

Hence, the total number of non-negative integral 4-tuples satisfying the given inequation is ${}^{104}C_4$.

EXAMPLE 20 *A condolence meeting is bieng held in a big hall which has 7 doors by which the mourners enter the hall. One can use any one of the 7 doors to enter and can come at any time during meeting. At each door a register is kept in which a mourner has to put down his signature while enetering the hall. If 200 people attend the meeting, how many different sets of 7 lists of signatures can arise ?*

SOLUTION Suppose x_i persons enter into the hall through i^{th} door, where $i = 1, 2, \ldots, 6, 7$. Then,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 200, \quad \ldots(i)$$

where $x_i \ge 0$; $i = 1, 2, \ldots, 7$

The total number of non-negative integral solutions of equation (i) is

$${}^{200+7-1}C_{7-1} = {}^{206}C_6$$

Thus, there are ${}^{206}C_6$ ways in which 200 persons can enter into the hall through 7 doors. But corresponding to each way of entering into the hall the signatures of 200 persons can be arranged in 200! ways.

Hence, the required number of lists

$$= {}^{206}C_6 \times 200! = \frac{206!}{6!}$$

EXAMPLE 21 *In how many ways two ditinct natural numbers can be selected which are less than or equal to 100 and differ by at most 10 ?*

SOLUTION Let x and y be two required natural numbers. If we plot the natural numbers from 1 to 100 on number line, we find that x and y divide the remaining 98 natural numbers into three parts. First we will find the number of ways of selecting x and y such that the difference is at least 11. Let x_1 be the total number of natural numbers less than x, x_2 be the number of natural numbers between x and y and x_3 be the number of natural numbers greater than y. Then,

$$x_1 + x_2 + x_3 = 98 \quad \ldots(i)$$

where $x_1 \ge 0, x_2 \ge 10$ and $x_3 \ge 0$



Fig. 22.15

The number of solutions of equation (i)

$= $ Coefficient of x^{98} in $(x^0 + x^1 + x^2 + \ldots + x^{100})^2 \times (x^{10} + x^{11} + \ldots + x^{100})$

$= $ Coefficient of x^{88} in $\left(\frac{1-x^{101}}{1-x}\right)^2 \left(\frac{1-x^{91}}{1-x}\right)$

$= $ Coefficient of x^{88} in $(1-x)^{-3}$

$= {}^{88+3-1}C_{3-1}$

$= {}^{90}C_2$

Thus, the number of ways of selecting x and y such that the difference is at least 11 is ${}^{90}C_2$.

Hence, the required number of ways

$$= {}^{100}C_2 - {}^{90}C_2 = 945.$$

EXAMPLE 22 *Let there be n identical toys and n children. Find the number of ways of distributing the toys to children so that exactly two of them do not get any toy.*

SOLUTION Out of n boys two boys can be chosen in ${}^{n}C_2$ ways. Now, distribute n toys to the remaining $n-2$ boys such that each boy gets at least one toy. This can be done in

$${}^{n-1}C_{n-2-1} = {}^{n-1}C_{n-3} \text{ ways.}$$

Hence, the required number of ways $= {}^{n}C_2 \times {}^{n-1}C_{n-3}$

$= {}^{n}C_2 \times {}^{n-1}C_2$

$= \frac{n(n-1)^2(n-2)}{4}$

EXAMPLE 23 *In how many ways can 25 identical objects be distributed among five persons, if*

(i) *each gets odd number of objects.*

(ii) *each gets at least one object but not more than eleven.*

SOLUTION (i) Required number of ways

$= $ Coefficient of x^{25} in $(x + x^3 + x^5 + x^7 + \ldots + x^{25})^5$

$= $ Coefficient of x^{20} in $(1 + x^2 + x^4 + x^6 + \ldots + x^{24})^5$

$= $ Coefficient of x^{20} in $\left|\frac{1-(x^2)^{13}}{1-x^2}\right|^5$

$= $ Coefficient of x^{20} in $(1-x^2)^{-5}$

$= $ Coefficient of $(x^2)^{10}$ in $(1-x^2)^{-5}$

$= {}^{10+5-1}C_{5-1}$

$= {}^{14}C_4$

$= 1001$

(ii) Required number of ways

$=$ Coefficient of x^{25} in $(x + x^2 + \ldots + x^{11})^5$

$=$ Coefficient of x^{20} in $(1 + x + x^2 + \ldots + x^{10})^5$

$= $ Coefficient of x^{20} in $\left(\frac{1-x^{11}}{1-x}\right)^5$

$=$ Coefficient of x^{20} in $(1-x^{11})^5\,(1-x)^{-5}$

$=$ Coefficient of x^{20} in $({}^5C_0 - {}^5C_1 x^{11} + \ldots)(1-x)^{-5}$

$= {}^5C_0 \times$ Coefficient of x^{20} in $(1-x)^{-5}$

$- {}^5C_1 \times$ Coefficient of x^9 in $(1-x)^{-5}$

$= {}^5C_0 \times {}^{20+5-1}C_{5-1} - {}^5C_1 \times {}^{9+5-1}C_{5-1}$

$= {}^{24}C_4 - 5 \times {}^{13}C_4$

$= 10626 - 3575$

$= 7051$

EXAMPLE 24 *In how many ways can 10 persons take seats in a row of 24 fixed seates so that no two persons take consecutive seats ?*

SOLUTION Let x_0 be the number of vacant seats before the first person, x_1 be the number of vacant seats between first and second person, .., x_9 be the number of vacant seats between 9th and 10th person and x_{10} be the number of vacant seats on the right side of 10th person. Then,



Fig. 22.16

$x_0 + x_1 + \ldots + x_{10} = 24 - 10 = 14,$

where $x_0 \geq 0, x_1, x_2, \ldots, x_9 \geq 1$ and $x_{10} \geq 0$.

Let $y_0 = x_0, y_{10} = x_{10}$ and $y_i = x_i - 1$ for $i = 1, 2, \ldots, 9$

Then, the above equation reduces to

$$y_0 + y_1 + \ldots + y_{10} = 5,$$

where $y_i \geq 0$; $i = 1, 2, \ldots, 10$.

The total number of solutions of this equation is

$${}^{11+5-1}C_{11-1} = {}^{15}C_{10}$$

But, 10 persons can arrange among themselves in 10! ways.

Hence, the total number of seating arrangements

$$= {}^{15}C_{10} \times 10!$$

EXAMPLE 25 *There are 15 seats in a row numbered as 1 to 15. In how many ways can 4 persons sit in such a way that seat number 6 is always occupied and no two persons sit in adjacent seats.*

SOLUTION Since seat number '6' is always occupied and no two persons can occupy adjacent seats. Therefore, at most two persons can sit on the left side of sixth seat. Thus, we have the following cases for the selection of seats :

CASE I *When two seats are selected on the left side of sixth seat and one seat on its right side:*

Since no two adjacent seats are selected. So, we can select either 1st and 3rd or 2nd and 4th or 1st and 4th seats. So, there are 3 ways to select 2 seats on the left side of sixth seat. On the right side of sixth seat there are 9 seats. Therefore, one seat (excluding 7th seat) on the right side of sixth seat can be chosen in 8 ways.

Thus, two seats on the left side of 6th seat and one seat on its right side can be chosen in $3 \times 8 = 24$ ways.

CASE II *When two seats are selected on the right side of sixth seat and one seat on its left side.*

In this case, one seat on the left side of sixth seat can be chosen in 4 ways (any one of the first four seats) and the number of ways of selecting two seats on the right side of sixth seat is same as the numberof non-negative integral solutions of the equation

$$x_1 + x_2 + x_3 = 7, \text{ where } x_1 \geq 1, x_2 \geq 1 \text{ and } x_3 \geq 0$$

Here, x_1 is the number of vacant seats between sixth seat and the fist seat selected on the right side of six th seat, x_2 is the number of vacant seats between first and second seat selected on the right side of sixth seat and x_3 is the number of vacant seats on the right side of the second selected seat.

Let $y_1 = x_1 - 1, y_2 = x_2 - 1$ and $y_3 = x_3$. Then,

$$x_1 + x_2 + x_3 = 7$$

$$\Rightarrow \quad y_1 + y_2 + y_3 = 5, \quad \text{where } y_1, y_2, y_3 \geq 0$$

Total number of integral solutions of this equation is

$${}^{5+3-1}C_{3-1} = {}^7C_2 = 21.$$

Thus, the number of ways in which two seats can be chosen on the right side of sixth seat = 21.

Hence, total number of ways of selection of seats in this case

$$= 4 \times 21 = 84.$$

CASE III *When all the three persons sit on the right side of sixth seat:*

Let x_1 be the number of vacant seats between 6th seat and first seat selected on the right of 6th seat, x_2 be the number of vacant seats between first selected seat and the second selected seat, x_3 be the number of seats between second and third selected seats and x_4 be the number of vacant seats on the right-side of fourth selected seat. Then, the number of ways of selecting 3 seats on the right side of sixth seat is equal to the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 6, \text{where } x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_4 \geq 0$$

Let $z_1 = x_1 - 1, z_2 = x_2 - 1, z_3 = x_3 - 1$ and $z_4 = x_4$. Then, we have

$$z_1 + z_2 + z_3 + z_4 = 3, \text{ where } z_i \geq 0\ ;\ i = 1, 2, 3, 4.$$

Total number of solutions of this equation is

$$^{3+4-1}C_{4-1} = {}^{6}C_3 = 20.$$

Thus, the number of ways of selecting 3 seats on the right side of 6th seat = 20.

Hence, total number of ways of selection of 4 seats

$$= 24 + 84 + 20 = 128.$$

But, corresponding to each way of selection of 4 seats there are 4! arrangements of 4 persons. Hence,

Total number of seating arrangement $= 128 \times 4! = 3072.$

EXAMPLE 26 *Find the number of ways of distribution of 12 identical balls in 3 identical boxes.*

SOLUTION Let us first find the number of ways of distribution of 12 identical balls in three distinct boxes. Suppose first box contains x balls, second box contains y balls and third box contains z balls. Then, the total number of distributions is equal to the number of non-negative integral solutions of the equation

$$x + y + z = 12 \qquad ...(i)$$

The number of solutions of this equation is

$$^{12+3-1}C_{3-1} = {}^{14}C_2 = 91.$$

Since the boxes are identical, so the number of ways of distribution will not be equal to the number of solutions of the above equation because $x = 2, y = 3, z = 7$ and $x = 3, y = 7, z = 2$ are two different solutions but they give the same distribution.

We observe that all solutions of (i) can be divided in the following groups :

(i) Solutions in which all variables are same.

(ii) Solutions in which two variables are same.

(iii) Solutions in which all variables are distinct.

CASE I *When all variables are equal :*

In this case, we have exactly one solution viz. $x = y = z = 4$. Corresponding to this solution there is exactly one distribution.

CASE II *When two variables are equal :*

Let $x = y \neq z$. Then, equation (i) reduces to

$$2x + z = 12$$

This equation has 6 solutions, namely

$x = 6, z = 0\ ;\ x = 5, z = 2\ ;\ x = 3, z = 6;\ x = 2, z = 8;\ x = 1, z = 10$ and $x = 0, z = 12$.

Since out of 3 variables, two variables can be chosen in $^3C_2$ ways. So, the total number of solutions in which exactly two variables are equal is

$$^3C_2 \times 6 = 18$$

Corresponding to these 18 solutions, number of distributions

$$= \frac{18}{\left(\frac{3!}{2!}\right)} = 6. \qquad \left[\because \text{Three objects in which 2 are identical can be arranged in } \frac{3!}{2!} \text{ ways}\right]$$

CASE III When variable assume distinct values in this case,

Number of solutions $= 91 - (18 + 1) = 72$

$\therefore$ Number of distributions $= \frac{27}{3!} = 12.$

Hence, total number of distributions $= 1 + 6 + 12$

$= 19.$

EXAMPLE 27 *There are 6n flowers of one kind and 3 another kind. Show that the number of different garlands made using all these flowers is $3n^2 + 3n + 1$.*

SOLUTION In a garland consisting of $6n$ flowers of one kind and 3 another kind, we observe that 3 flowers of another kind divide the $6n$ flowers of one kind in three parts. Let there be x_1, x_2, x_3 flowers in these parts. Then,

$$x_1 + x_2 + x_3 = 6n \qquad ...(i)$$

Clearly, $x_1, x_2, x_3 \geq 0$.

Number of solutions of equation (i)

$$= {}^{6n+3-1}C_{3-1}$$

$$= {}^{6n+2}C_2$$

$$= \frac{(6n+2)(6n+1)}{2}$$

$$= 18n^2 + 9n + 1.$$

We observe that the total number of garlands is not same as the number of solutions of equation (i) because $x_1 = n, x_2 = 2n, x_3 = 3n$ and $x_1 = 2n, x_2 = 3n, x_3 = n$ are two different solutions of (i) but these solutions will give the same garland. Similarly, $x_1 = n, x_2 = n, x_3 = 4n$; $x_1 = 4n, x_2 = n, x_3 = n$ etc give the same garland. So, let us divide all solutions of equation (i) in the following groups :

(i) Solutions in which $x_1 = x_2 = x_3$ i.e. all variables are equal.

(ii) solutions in which two variables are equal i.e. $x_1 = x_2 \neq x_3, x_1 \neq x_2 = x_3$ and $x_1 = x_3 \neq x_2$.

(iii) Solutions in which all variables are distinct i.e. $x_1 \neq x_2 \neq x_3$.

CASE I *When all variables are equal :*

In this case, we have exactly one solution viz. $x_1 = x_2 = x_3 = 2n$. Corresponding to this solution there is only one garland.

CASE II *When two variables are equal :*

Let us consider the case when

$$x_1 = x_2 \neq x_3.$$

In this case, equation (i) reduces to

$$2x_1 + x_3 = 6n$$

As x_1 can take values $0, 1, 2, ..., 2n-1, 2n+1, ..., 3n$. So, this equation has $3n$ solutions.

Since out of 3 variables, two variables can be chosen in ${}^3C_2$ ways. So, the total number of solutions of (i) in which exactly two variables are equal is

$${}^3C_2 \times 3n = 9n.$$

Thus, there are $9n$ solutions in which exactly two x_i's are same, so number of garlands corresponding to them is

$$\frac{9n}{3} = 3n.$$

CASE III *When variables are distinct i.e.* $x_1 \neq x_2 \neq x_3$.

In this case, number of solutions of (i) is

$$18n^2 + 9n + 1 - (9n + 1) = 18n^2$$

$\therefore$ Number of garlands $= \dfrac{18n^2}{3!} = 3n^2$.

Hence, total number of garlands $= 3n^2 + 3n + 1$.

EXAMPLE 28 *In how many ways can* $(2n+1)$ *identical balls be placed in distinct boxes so that any two boxes together will contain more balls than the third ?*

SOLUTION Required number of ways

= (Total number of ways in which $(2n+1)$ identical balls be placed in 3 distinct boxes) – (Total number of ways in which one of the three boxes contains more balls than the other two).

Now, the total number of ways to place $(2n+1)$ identical balls in three distinct boxes

$$= {}^{2n+1+3-1}C_{3-1}$$

$$= {}^{2n+3}C_2.$$

In order to find the number of ways in which one of the three boxes contains more balls than the other two boxes, we first select a box out of three distinct boxes and place $n+1$ balls in it and then divide the rest of n balls in three boxes arbitrarily. Therefore,

Total number of ways in which one of the three boxes contains more balls than the other two

$$= {}^3C_1 \times {}^{n+3-1}C_{3-1} = 3 \times {}^{n+2}C_2$$

Hence, required number of ways

$$= {}^{2n+3}C_2 - 3 \times {}^{n+2}C_2$$

$$= \frac{(2n+3)(2n+2)}{2} - 3\,\frac{(n+2)(n+1)}{2}$$

$$= \frac{n(n+1)}{2}$$

—❑❑❑—

22.12 SELECTIONS OF OBJECTS FROM TWO OR MORE GROUPS

In this section, we shall discuss various methods of determining the number of ways of selecting objects from two or more groups such that oibjects in a group are identical but they differe from objects in other groups. The following theorems determine the same.

THEOREM 1 If there are n_1 objects of one kind, n_2 objects of second kind, ..., n_k objects of kth kind, then the number of ways of selecting r objects out of these objects is equal to

Coefficient of x^r in $\left\{(1 + x + x^2 + ... + x^{n_1}) \times (1 + x + x^2 + ... + x^{n_2}) ..... (1 + x + x^2 + ... + x^{n_k})\right\}$

THEOREM 2 If there are n_1 objects of one kind, n_2 objects of second kind, ..., n_k objects of kth kind, then the number of ways of selecting r objects out of these objects such that each selection contains at least one object of each kind, is

Coefficient of x^r in $\left\{(x + x^2 + ... + x^{n_1})(x + x^2 + ... + x^{n_2}) ..... (x + x^2 + ... + x^{n_k})\right\}$

THEOREM 3 If there are n_1 objects of one kind, n_2 objects of second kind, ..., n_k objects of kth kind, then the number of arrangements of these objects taken r at a time is

Coefficient of x^r in $\left\{r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n_1}}{n_1!}\right) \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n_2}}{n_2!}\right) .... \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n_k}}{n_k!}\right)\right\}$

22.13 DERANGEMENT

Let $a_1, a_2, a_3, ..., a_n$ be n distinct objects such that their positions are fixed in a row. If we now re-arrange $a_1, a_2, ..., a_n$ in such a way that no one occupies its original place, then such an arrangement is called a derangement.

In other words, a derangement of $a_1, a_2, ..., a_n$ is a bijection

$f: \{a_1, a_2, ..., a_n\} \to \{a_1, a_2, ..., a_n\}$ such that

$f(a_i) \neq a_i ; i = 1, 2, ..., n$

If n distinct objects are arranged in a row, then the number of ways in which they can be deranged so that none of them occupies its original place is

$$n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - .... + (-1)^n \frac{1}{n!}\right\}$$

and it is denoted by $D(n)$.

If r $(0 \leq r \leq n)$ objects occupy the places assigned to them i.e. their original places and none of the remaining $(n-r)$ objects occupies its original places, then the number of such ways is

$$D(n-r) = {}^{n}C_r \cdot D(n-r)$$

$$= {}^{n}C_r \cdot (n-r)! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right\}$$

The proof of this is given in chapter 40.

ILLUSTRATION 1 *In how many ways can 10 letters be placed in 10 marked envelopes, so that no letter is in the right encelope ?*

SOLUTION The required number of ways is equal to the number of derangements of 10 objects.

Hence, required number of ways

$$= 10! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!}\right\}$$

ILLUSTRATION 2 *Ravish writes letters to his five friends and addresses the correspondig envelopes. In how many ways can the letters be placed in the envelopes so that at least two of them are in the wrong envelopes ?*

SOLUTION Required number of ways

$$= \sum_{r=2}^{5} {}^{5}C_{5-r}\, D(r)$$

$$= \sum_{r=2}^{5} \frac{5!}{r!\,(5-r)!}\, r! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^r}{r!}\right\}$$

$$= \sum_{r=2}^{5} \frac{5!}{(5-r)!} \left\{\frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^r}{r!}\right\}$$

$$= \frac{5!}{3!}\left(\frac{1}{2!}\right) + \frac{5!}{2!}\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{5!}{1!}\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) + \frac{5!}{0!}\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$= 10 + 20 + (60 - 20 + 5) + (60 - 20 + 5 - 1)$$

$$= 10 + 20 + 45 + 44$$

$$= 119.$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In an examination the maximum marks for each of three papers is n and that for fourth paper is 2n. Prove that the number of ways in which a candidate can get 3n marks is*

$$\frac{1}{6}(n+1)(5n^2 + 10n + 6).$$

SOLUTION The total number of ways of getting $3n$ marks

= Coefficient of x^{3n} in $(x^0 + x^1 + x^2 + \ldots + x^n)^3 (x^0 + x^1 + \ldots + x^{2n})$

$$= \text{Coefficient of } x^{3n} \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^3 \left(\frac{1-x^{2n+1}}{1-x}\right)$$

$$= \text{Coefficient of } x^{3n} \text{ in } (1-x^{n+1})^3 (1-x^{2n+1})(1-x)^{-4}$$

$$= \text{Coefficient of } x^{3n} \text{ in } [(1 - 3x^{n+1} + 3x^{2n+2} - 3x^{3n+3})(1-x^{2n+1})(1-x)^{-4}]$$

$$= \text{Coefficient of } x^{3n} \text{ in } [(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2})(1-x)^{-4}]$$

$$= \text{Coefficient of } x^{3n} \text{ in } (1-x)^{-4}$$

$$- 3 \text{ Coefficient of } x^{2n-1} \text{ in } (1-x)^{-4}$$

$$- \text{ Coefficient of } x^{n-1} \text{ in } (1-x)^{-4}$$

$$+ 3 \text{ Coefficient of } x^{n-2} \text{ in } (1-x)^{-4}$$

$$= {}^{3n+4-1}C_{4-1} - 3 \times {}^{2n-1+4-1}C_{4-1} - {}^{n-1+4-1}C_{4-1} + 3 \times {}^{n-2+4-1}C_{4-1}$$

$$= {}^{3n+3}C_3 - 3 \times {}^{2n+2}C_3 - {}^{n+2}C_3 + 3 \times {}^{n+1}C_3$$

$$= \frac{(3n+3)(3n+2)(3n+1)}{6} - 3\,\frac{(2n+2)(2n+1)(2n)}{6} - \frac{(n+2)(n+1)(n)}{6} + 3\,\frac{(n+1)(n)(n-1)}{6}$$

$$= \frac{1}{6}(n+1)(5n^2 + 10n + 6)$$

EXAMPLE 2 *Show that the number of ways of selecting n objects out of 3n objects, of which n are alike and rest are different is*

$$2^{2n-1} + \frac{(2n-1)!}{n!\,(n-1)!}.$$

SOLUTION The required number of ways

= Coefficient of x^n in $(x^0 + x^1 + x^2 + \ldots + x^n)(x^0 + x)^{2n}$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)(1+x)^{2n}$$

$$= \text{Coefficient of } x^n \text{ in } (1-x)^{-1}(1+x)^{2n}$$

$$= \text{Coefficient of } x^n \text{ in } \left(\sum_{r=0}^{\infty} x^r\right)(1+x)^{2n}$$

$$= \text{Coefficient of } x^n \text{ in } \left(\sum_{r=0}^{n} x^r\right)(1+x)^{2n} \quad [\text{Neglecting higher powers}]$$

$$= \text{Coefficient of } x^n \text{ in } \sum_{r=0}^{n} x^r (1+x)^{2n}$$

$$= \sum_{r=0}^{n} \text{Coefficient of } x^{n-r} \text{ in } (1+x)^{2n}$$

$$= \sum_{r=0}^{n} {}^{2n}C_{n-r}$$

$$= {}^{2n}C_n + {}^{2n}C_{n+1} + \ldots + {}^{2n}C_1 + {}^{2n}C_0$$

$$= \frac{1}{2}\left[{}^{2n}C_n + \left\{{}^{2n}C_0 + {}^{2n}C_1 + \ldots + {}^{2n}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n}C_{2n}\right\}\right]$$

$$[\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$= \frac{1}{2}\left[{}^{2n}C_n + 2^{2n}\right]$$

$$= \frac{1}{2}\left[\frac{2n!}{n!\,n!} + 2^{2n}\right]$$

$$= 2^{2n-1} + \frac{(2n-1)!}{n!\,(n-1)!}$$

EXAMPLE 3 *There are three papers of 100 marks each in an examination. In how many ways can a student get 150 marks such that he gets at least 60% in two parpers?*

SOLUTION Suppose the student gets at least 60% marks in first two papers, then he must get at most 30 marks in the third paper to make a total of 150 marks. So,

The required number of ways

$$= \text{Coefficient of } x^{150} \text{ in } \left\{(x^{60} + x^{61} + \ldots + x^{100})^2 (1 + x + x^2 + \ldots + x^{30})\right\}$$

$$= \text{Coefficient of } x^{30} \text{ in } \left\{(1 + x + .. + x^{40})^2 (1 + x + \ldots + x^{30})\right\}$$

$$= \text{Coefficient of } x^{30} \text{ in } \left(\frac{1-x^{41}}{1-x}\right)^2 \left(\frac{1-x^{31}}{1-x}\right)$$

$$= \text{Coefficient of } x^{30} \text{ in } (1-x)^{-3}$$

$$= {}^{30+3-1}C_{3-1}$$

$$= {}^{32}C_2.$$

Thus, the student gets at least 60% marks in first two papers to get 150 marks as total in ${}^{32}C_2$ ways. But the two papers, of at least 60% marks, can be chosen out of 3 papers in ${}^{3}C_2$ ways.

Hence, the required number of ways $= {}^{3}C_2 \times {}^{32}C_2$

$$= 3 \times 31 \times 16$$

$$= 1488.$$

EXAMPLE 4 *In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper are 100. Find the number of ways in which the candidate can score 60% marks in the aggregate.*

SOLUTION We have,

Aggregate marks of four papers $= 50 \times 3 + 100 = 250$

$\therefore$ 60% of aggregate marks $= \frac{60}{100} \times 250 = 150$

Number of ways of getting 150 marks in aggregate

$$= \text{Coefficient of } x^{150} \text{ in } \left\{(x^0 + x + x^2 + \ldots + x^{50})^3 (x^0 + x + x^2 + \ldots + x^{100})\right\}$$

$$= \text{Coefficient of } x^{150} \text{ in } \left(\frac{1-x^{51}}{1-x}\right)^3 \left(\frac{1-x^{101}}{1-x}\right)$$

$$= \text{Coefficient of } x^{150} \text{ in } (1-x^{51})^3 (1-x^{101})(1-x)^{-4}$$

$$= \text{Coefficient of } x^{150} \text{ in } (1 - 3x^{51} + 3x^{102} - x^{153})(1-x^{101})(1-x)^{-4}$$

$$= \text{Coefficient of } x^{150} \text{ in } (1 - 3x^{51} - x^{101} + 3x^{102})(1-x)^{-4}$$

$$= \text{Coefficient of } x^{150} \text{ in } (1-x)^{-4} - 3 \text{ Coefficient of } x^{99} \text{ in } (1-x)^{-4} - \text{Coefficient of } x^{49} \text{ in } (1-x)^{-4} + 3 \text{ Coefficient of } x^{48} \text{ in } (1-x)^{-4}$$

$$= {}^{150+4-1}C_{4-1} - 3 \times {}^{99+4-1}C_{4-1} - {}^{49+4-1}C_{4-1} + 3 \times {}^{48+4-1}C_{4-1}$$

$$= {}^{153}C_3 - 3 \times {}^{102}C_3 - {}^{52}C_3 + 3 \times {}^{51}C_3$$

$$= 110556.$$

EXAMPLE 5 *Find the number of ways of selecting 4 letters from the word 'EXAMINATION'.*

SOLUTION There are 11 letters in the word EXAMINATION viz. two *A*'s, 2*I*'s, 2*N*'s, *E*, *X*, *M*, *T*, *O*.

Therefore,

Number of ways of selecting 4 letters from these letters

$$= \text{Coefficient of } x^4 \text{ in } (x^0 + x^1 + x^2)^3 (1+x)^5$$

$$= \text{Coefficient of } x^4 \text{ in } \left(\frac{1-x^3}{1-x}\right)^3 (1+x)^5$$

$$= \text{Coefficient of } x^4 \text{ in } (1-x^3)^3 (1-x)^{-3} (1+x)^5$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - 3x^3 + 3x^6 - x^9)(1+x)^5 (1-x)^{-3}$$

$$= \text{Coefficient of } x^4 \text{ in } (1-3x^3)(1 + {}^{5}C_1\, x + {}^{5}C_2\, x^2 + {}^{5}C_3\, x^3 + {}^{5}C_4\, x^4)(1-x)^{-3}$$

$$= \text{Coefficient of } x^4 \text{ in } (1 + 5x + 10x^2 + 10x^3 + 5x^4 - 3x^3 - 15x^4)(1-x)^{-3}$$

$$= \text{Coefficient of } x^4 \text{ in } (1 + 5x + 10x^2 + 7x^3 - 10x^4)(1-x)^{-3}$$

$$= {}^{4+3-1}C_{3-1} + 5 \times {}^{3+3-1}C_{3-1} + 10 \times {}^{2+3-1}C_{3-1} + 7 \times {}^{1+3-1}C_{3-1} - 10$$

$$= {}^{6}C_2 + 5 \times {}^{5}C_2 + 10 \times {}^{4}C_2 + 7 \times {}^{3}C_2 - 10$$

$$= 15 + 50 + 60 + 21 - 10 = 136$$

ALITER See Example.11 given on page.22.46.

EXAMPLE 6 *Find the total number of selections of 5 letters from five A's four B's, three C's and two D's. Also, find the total number of 5 letter words that can be made.*

SOLUTION The total number of ways of selecting 4 letters from 5 A's 4B's, 3C's and 2D's is equal to

Coefficient of x^5 in $(1+x+x^2+x^3+x^4+x^5)\ (1+x+x^2+x^3+x^4)\ (1+x+x^2+x^3)\ (1+x+x^2)$

$= \text{Coefficient of } x^5 \text{ in } \left(\frac{1-x^6}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)$

$= \text{Coefficient of } x^5 \text{ in } (1-x^5)\,(1-x^4)\,(1-x^3)\,(1-x)^{-4}$

$= \text{Coefficient of } x^5 \text{ in } (1-x^5)\,(1-x^3-x^4)\,(1-x)^{-4}$

$= \text{Coefficient of } x^5 \text{ in } (1-x^3-x^4-x^5)\,(1-x)^{-4}$

$= \text{Coefficient of } x^5 \text{ in } (1-x)^{-4} - \text{Coefficient of } x^2 \text{ in } (1-x)^{-4}$
$- \text{Coefficient of } x \text{ in } (1-x)^{-4} - \text{Coefficient of } x^0 \text{ in } (1-x)^{-4}$

$= {}^{5+4-1}C_{4-1} - {}^{2+4-1}C_{4-1} - {}^{1+4-1}C_{4-1} - 1$

$= {}^8C_3 - {}^5C_3 - {}^4C_3 - 1$

$= 56 - 10 - 4 - 1$

$= 41.$

Total number of 5-letter words

$= 5! \times \text{Coefficient of } x^5 \text{ in } \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right)$
$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)$

$= 5! \times \text{Coefficient of } x^5 \text{ in } \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right)$
$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(1+2x+2x^2+\frac{7}{6}x^3+\frac{5}{12}x^4+\frac{x^5}{12}\right)$

$= 5! \times \text{Coefficient of } x^5 \text{ in } \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right)$
$\times \left(1+3x+\frac{9x^2}{2}+\frac{13x^3}{3}+\frac{71x^4}{24}+\frac{3x^5}{2}\right)$

$= 5!\left(\frac{1}{5!}+\frac{3}{4!}+\frac{9}{2\times 3!}+\frac{13}{6}+\frac{71}{24}+\frac{3}{2}\right)$

$= 1+15+90+260+355+180 = 901$

EXAMPLE 7 *Find the total number of 4 letter words formed from the letters of the word "EXAMINATION".*

SOLUTION In the word 'EXAMINATION' there are 11 letters viz. 2 A's, 2 I's 2 N's, 1E, 1X, 1M, 1T and one O. So,

Required number of arrangements

$= 4! \times \text{Coefficient of } x^4 \text{ in } \left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)^3\left(1+\frac{x}{1!}\right)^5$

$= 4! \times \text{Coefficient of } x^4 \text{ in } \left[\left\{(1+x)^3+3(1+x)^2\frac{x^2}{2!}+3(1+x)\frac{x^4}{4}+\frac{x^6}{8}\right\}\times(1+x)^5\right]$

$= 4! \times \text{Coefficient of } x^4 \text{ in } \left[(1+x)^8+\frac{3x^2}{2}(1+x)^7+\frac{3x^4}{4}(1+x)^6\right]$

$= 4!\left[{}^8C_4+\frac{3}{2}\times{}^7C_2+\frac{3}{4}\right]$

$= 4!\left[\frac{8\times7\times6\times5}{4\times3\times2\times1}+\frac{3}{2}\times\frac{7\times6}{2\times1}+\frac{3}{4}\right]$

$= 8\times7\times6\times5+6\times3\times7\times6+6\times3$

$= 2454.$

ALITER See Example 15 given on page 22.47.

EXAMPLE 8 *From a given number of 4n books, there are three sets of n identical books on Physics, Chemistry and Mathematics. The remaining n books are distinct books on other subjects. Find the number of ways of choosing n out of the 4n books.*

SOLUTION Required number of ways

$= \text{Coefficient of } x^n \text{ in } (x^0+x^1+x^2+\ldots.+x^n)^3\ (1+x)^n$

$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^3(1+x)^n$

$= \text{Coefficient of } x^n \text{ in } (1-x)^{-3}\ (1+x)^n$

$= \text{Coefficient of } x^n \text{ in } \left(\sum_{r=0}^{n} {}^nC_r\ x^r\right)(1-x)^{-3}$

$= \text{Coefficient of } x^n \text{ in } \sum_{r=0}^{n} {}^nC_r\ x^r\ (1-x)^{-3}$

$= \text{Coefficient of } x^{n-r} \text{ in } \sum_{r=0}^{n} {}^nC_r\ (1-x)^{-3}$

$= \sum_{r=0}^{n} {}^nC_r \times \{\text{Coefficient of } x^{n-r} \text{ in } (1-x)^{-3}\}$

$= \sum_{r=0}^{n} {}^nC_r\ {}^{n-r+3-1}C_{3-1}$

$= \sum_{r=0}^{n} {}^nC_r\ {}^{n-r+2}C_2$

$= \sum_{r=0}^{n} {}^nC_r\ \frac{(n-r+2)\ (n-r+1)}{2}$

$= \frac{1}{2}\sum_{r=0}^{n} {}^nC_r\ \{(n+2)\,(n+1)-(2n+3)\,r+r^2\}$

$$= \frac{1}{2}\sum_{r=0}^{n}\{(n+1)(n+2)\,{}^nC_r-(2n+3)\,r\,{}^nC_r+r^2\,{}^nC_r\}$$

$$= \frac{1}{2}(n+1)(n+2)\left(\sum_{r=0}^{n}{}^nC_r\right)-\frac{1}{2}(2n+3)\sum_{r=0}^{n}r\,.\,{}^nC_r$$

$$+\frac{1}{2}\sum_{r=0}^{n}\{r(r-1)+r\}\,{}^nC_r$$

$$= \frac{1}{2}(n+1)(n+2)\,2^n-\frac{1}{2}(2n+3)\sum_{r=0}^{n}r\,.\,{}^nC_r$$

$$+\frac{1}{2}\sum_{r=0}^{n}r(r-1)\,.\,{}^nC_r+\frac{1}{2}\sum_{r=0}^{n}r\,.\,{}^nC_r$$

$$= \frac{1}{2}(n+1)(n+2)\,2^n-\frac{1}{2}(2n+3)\sum_{r=0}^{n}r\,.\,\frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$+\frac{1}{2}\sum_{r=0}^{n}r(r-1)\,.\,\frac{n}{r}\,.\,\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}+\frac{1}{2}\sum_{r=0}^{n}r\,.\,\frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$= (n+1)(n+2)\,2^{n-1}-\frac{n}{2}(2n+3)\,2^{n-1}+\frac{1}{2}n(n-1)\,2^{n-2}$$

$$+\frac{1}{2}n\,.\,2^{n-1}$$

$$= (n+1)(n+2)\,2^{n-1}-n(2n+3)\,2^{n-2}+n(n-1)\,2^{n-3}+n\,2^{n-2}$$

$$= [4(n+1)(n+2)-2n(2n+3)+n(n-1)+2n]\,2^{n-3}$$

$$= [4n^2+12n+8-4n^2-6n+n^2-n+2n]\,2^{n-3}$$

$$= (n^2+7n+8)\,2^{n-3}.$$

EXAMPLE 9 *If there are n A's, n B's and n C's making 3n letters show that there are as many ways of selecting r letters out of them as 3n − r letters. If n < r < 2n + 1, show that the number of ways of selecting r letters is given by*

$$\frac{1}{2}(n+1)(n+2)+(r-n)(2n-r)$$

and that the maximum number of such selections is

$$\frac{1}{4}\{3(n+1)^2+1\} \text{ or } \frac{3}{4}(n+1)^2$$

according as n is even or odd.

SOLUTION The total number of ways of selecting r letters from n A's n B's and n C's is equal to

Coefficient of x^r in $(1+x+x^2+\ldots+x^n)^3$

$$= \text{Coefficient of } x^r \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^3$$

$$= \text{Coefficient of } x^r \text{ in } (1-x^{n+1})^3\,(1-x)^{-3}$$

$$= \text{Coefficient of } x^r \text{ in } (1-3x^{n+1}+3x^{2n+2}-x^{3n+3})\,(1-x)^{-3}$$

$$= \text{Coefficient of } x^r \text{ in } (1-x)^{-3}$$

$$-3\times\text{Coefficient of } x^{r-n-1} \text{ in } (1-x)^{-3}$$

$$+3\times\text{Coefficient of } x^{r-2n-2} \text{ in } (1-x)^3$$

$$= \begin{cases} {}^{r+3-1}C_{3-1} & \text{, if } 1\le r\le n \\ {}^{r+3-1}C_{3-1}-3\times{}^{r-n-1+3-1}C_{3-1} & \text{, if } n+1\le r\le 2n+1 \\ {}^{r+3-1}C_{3-1}-3\times{}^{r-n-1+3-1}C_{3-1} & \\ \quad +3\times{}^{r-2n-2+3-1}C_{3-1} & \text{, if } 2n+2\le r\le 3n \end{cases}$$

$$= \begin{cases} {}^{r+2}C_2 \text{, if } 1\le r\le n & \\ {}^{r+2}C_2-3\,.\,{}^{r-n+1}C_2 & \text{, if } n+1\le r\le 2n+1 \ldots\text{(i)} \\ {}^{r+2}C_2-3\,.\,{}^{r-n+1}C_2+3\,.\,{}^{r-2n}C_2 & \text{, if } 2n+2\le r\le 3n \end{cases}$$

If $0\le r\le n-2$, then from (i) the number of ways of selecting r letters is ${}^{r+2}C_2$.

Now, $0\le r\le n-2 \Rightarrow 2n+2\le 3n-r\le 3n$

Therefore, from (i) the number of ways of selecting $3n-r$ letters is

$$= {}^{3n-r+2}C_2-3\times{}^{3n-r-n+1}C_2+3\times{}^{3n-r-2n}C_2$$

[Replacing r by $3n-r$ in the third part of (i)]

$$= {}^{3n-r+2}C_2-3\times{}^{2n-r+1}C_2+3\times{}^{n-r}C_2$$

$$= \frac{(3n-r+2)(3n-r+1)}{2}-3\,\frac{(2n-r+1)(2n-r)}{2}$$

$$+3\,\frac{(n-r)(n-r-1)}{2}$$

$$= \frac{1}{2}\left[(3n+2)(3n+1)-(6n+3)\,r+r^2-3\{2n(2n+1)\right.$$

$$\left.-(4n+1)\,r+r^2\}+3\{n(n-1)-(2n-1)\,r+r^2\}\right]$$

$$= \frac{1}{2}\left[9n^2+9n+2-12n^2-6n+3n^2-3n\right.$$

$$\left.-(6n+3-12n-3-6n-3)\,r+r^2\right]$$

$$= \frac{1}{2}[r^2+3r+2]$$

$$= \frac{1}{2}(r+2)(r+1)$$

$$= {}^{r+1}C_2$$

Thus, if $0\le r\le n-2$, we have

Number of ways of selecting r letters

= Number of ways of selecting $(3n-r)$ letters.

Similarly, we can prove the result for the other cases.

If $n<r<2n+1$, then from (i) the number of ways of selecting r letters is given by

$$N = {}^{r+2}C_2-3\times{}^{r-n+1}C_2$$

$$= \frac{(r+2)(r+1)}{2}-3\,\frac{(r-n+1)(r-n)}{2}$$

$$= \frac{1}{2}\left[(r^2+3r+2)-3\left\{r^2-r(2n-1)+n(n-1)\right\}\right]$$

$$= \frac{1}{2}\left[r^2+3r+2-3r^2+3r(2n-1)-3n(n-1)\right]$$

$$= \frac{1}{2}\left[-2r^2+6nr+2-3n^2+3n\right]$$

$$= \frac{1}{2}\left[-2(r^2-3nr+2n^2)+n^2+3n+2\right]$$

$$= \frac{1}{2}(n+1)(n+2)-(r-2n)(r-n)$$

$$= \frac{1}{2}(n+1)(n+2)+(r-n)(2n-r)$$

In order to obtain the maximum value of N, let us re-write N as follows :

$$N = \frac{1}{2}(n+1)(n+2)-(r^2-3nr+2n^2)$$

$$= \frac{1}{2}(n+1)(n+2)-\left[\left(r-\frac{3n}{2}\right)^2-\frac{n^2}{4}\right]$$

$$= \left[\frac{1}{2}(n+1)(n+2)+\frac{n^2}{4}\right]-\left(r-\frac{3n}{2}\right)^2 \qquad ...(ii)$$

CASE I *When n is even :*

If n is even, then N is maximum when $r-\frac{3n}{2}=0$ i.e. $r=\frac{3n}{2}$.

The maximum value of N is given by

$$N_{max} = \frac{1}{2}(n+1)(n+2)+\frac{n^2}{4}$$

$$= \frac{1}{4}\left[3n^2+6n+4\right]$$

$$= \frac{1}{4}\left[3(n+1)^2+1\right]$$

CASE II *When n is odd :*

In this case, we have $r-\frac{3n}{2} \neq 0$.

Let $n = 2m+1$. Then,

$$\left(r-\frac{3n}{2}\right)^2 = \left(r-3m-\frac{3}{2}\right)^2$$

Clearly, it is least for $r = 3m+1$ or $r = 3m+2$. The least value of $\left(r-\frac{3n}{2}\right)^2$ is $\frac{1}{4}$.

Putting $\left(r-\frac{3n}{2}\right)^2 = \frac{1}{4}$ in (ii), the maximum value of N in this case is given by

$$N_{max} = \frac{1}{2}(n+1)(n+2)+\frac{n^2}{4}-\frac{1}{4}$$

$$= \frac{1}{4}\left[2n^2+6n+4+n^2-1\right]$$

$$= \frac{1}{4}\left[3n^2+6n+3\right]$$

$$= \frac{3}{4}(n+1)^2$$

EXERCISE 22.8

1. In how many ways can 20 identical books on Mathematics be distributed among 5 students.

2. In how many ways can three persons, each throwing a single die once, make a sum of 15?

3. In how many ways can an examiner assign 30 marks to 8 questions, giving not less than 2 marks to any question ?

4. Find the number of non-negative integral solutions of $2x+y+z=21$.

5. If n and m are positive integers such that $n \geq \frac{m(m+1)}{2}$. Find the number of solutions $(x_1, x_2, ..., x_m)$; $x_1 \geq 1, x_2 \geq 2, ..., x_m \geq m$, all integers satisfying the condition $x_1+x_2+...+x_m = n$.

6. Find the number of integral solutions of the equation

$$x_1+x_2+...+x_m = 4m^2-2m+1$$

where $x_k \geq (2k-1)$, $k = 1, 2, 3, ..., m$

7. In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

8. How many natural numbers below 1 million have the sum of digits equal to 18?

9. Suppose we have rupee 2, 3 and 5 notes. In how many ways can we get a sum of rupees 83 such that note of each type is present and the number of two rupee note (s) is less than the number of 3 rupee note (s) which is again less than the number of 5 rupee note (s).

10. How many words of 4 letters can be formed, starting with a vowel, using letters from the word 'EXAMINATION' ?

11. In how many ways some apples from 45 apples can be distributed among 5 persons so that none gets more than 9 apples and the total number of apples given is a multiple of 3 ?

12. Find the number of integral solutions of the equation $x+y+z=24$ subjected to the condition that $1 \leq x \leq 5$, $12 \leq y \leq 18$ and $z \geq -1$.

13. In how many ways can 5 letters be selected from 5 A's , 4 B's, 3 C's, 2 D's and one E.

14. Find the number of ways of selecting n objects, out of $3n$ objects of which n are of one kind and alike and n are of a second kind and alike and rest are unlike.

15. Find the number of non-negative integral solutions of $2x + y + z = 20$.

16. Show that the number of ways in which $2n$ things of one kind, $2n$ of another kind and $2n$ of a third kind can be divided between two persons, given $3n$ things to each, is $3n^2 + 3n + 1$.

17. In how many ways can we select 12 balls from a bag containing 23 balls in which 7 are identical and remaining distinct ?

18. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a_1, a_2, a_3, a_4, a_5\}$. How many one-one mappings $f : A \to B$ can be defined so that $f(i) \neq a_i$ for all $i = 1, 2, 3, 4, 5$.

ANSWERS

1. 10626 **2.** 10 **3.** $^{21}C_7$ **14.** $(n+2)\,2^{n-1}$ **15.** 121 **17.** $^{18}C_6 + {}^{18}C_8$

4. 132 **5.** $^{r+m-1}C_{m-1}$, where $r = n - \dfrac{m(m+1)}{2}$ **18.** $5!\left\{\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!}\right\}$

6. $^{m(3m-1)}C_{m-1}$ **7.** 2940 **8.** 25927 **13.** 71

HINTS TO SELECTED PROBLEMS

9. Let x, y, and z be the numbers of 2, 3 and 5 rupee notes respectively. Then, the required number of ways is equal to the total number of solutions of the equation $2x + 3y + 5z = 83$ subjected to the condition that $x < y < z$ and $x \geq 1, y \geq 1, z \geq 1$

12. Let $t = z + 1$. Then, the given equation reduces to $x + y + t = 25$, where $1 \leq x \leq 5, 12 \leq y \leq 18$ and $t \geq 0$.

$\therefore$ Required number of solutions

= Coefficient of x^{25} in

$[(x + x^2 + ... + x^5)(x^{12} + x^{13} + ... + x^{18})(1 + x + x^2 + ...)]$

14. Required number of ways

= Coefficient of x^n in $\{(1 + x + x^2 + ... + x^n)^2 (1 + x)^n\}$

16. Required number of ways

= Coefficient of x^{3n} in $(1 + x + x^2 + ... + x^{2n})^3$

—❑❑❑—

MATHEMATICS FOR
IIT-JEE

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R.D. SHARMA

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Volume-2	Chapters 23-41	Binomial Theorem, Matrices and Determinants, Inequalities, Co-ordinate Geometry, Trigonometry, Logarithms, Probability, Three Dimensional Geometry.

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