

Mathematical Methods in Physics

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Partial Differential Equations,
Fourier Series, and Special Functions

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To Bruce Adams,
a great friend

Contents

Introduction	xi
1 Fourier Series	1
1.1 Periodic Processes and Periodic Functions	1
1.2 Fourier Formulas	3
1.3 Orthogonal Systems of Functions	7
1.4 Convergence of Fourier Series	9
1.5 Fourier Series for Nonperiodic Functions	16
1.6 Fourier Expansions on Intervals of Arbitrary Length . . .	16
1.7 Fourier Series in Cosine or Sine Functions	18
1.8 The Complex Form of the Fourier Series	30
1.9 Complex Generalized Fourier Series	33
1.10 Fourier Series for Functions of Several Variables	35
1.11 Uniform Convergence of Fourier Series	36
1.12 The Gibbs Phenomenon	41
1.13 Completeness of a System of Trigonometric Functions .	42
1.14 General Systems of Functions: Parseval's Equality and Completeness	44
1.15 Approximation of Functions in the Mean	45
1.16 Fourier Series of Functions Given at Discrete Points . . .	49
1.17 Solution of Differential Equations by Using Fourier Series	51
1.18 Fourier Transforms	55
1.19 The Fourier Integral	60
Problems	68

2	Sturm-Liouville Theory	79
2.1	The Sturm-Liouville Problem	79
2.2	Mixed Boundary Conditions	90
2.3	Examples of Sturm-Liouville Problems	95
	Problems	107
3	One-Dimensional Hyperbolic Equations	113
3.1	Derivation of the Basic Equations	113
3.2	Boundary and Initial Conditions	116
3.3	Other Boundary Value Problems: Longitudinal Vibrations of a Thin Rod	121
3.4	Torsional Oscillations of an Elastic Cylinder	124
3.5	Acoustic Waves	127
3.6	Waves in a Shallow Channel	132
3.7	Electrical Oscillations in a Circuit	135
3.8	Traveling Waves: D'Alembert Method	138
3.9	Semi-infinite String Oscillations and the Use of Symmetry Properties	157
3.10	Finite Intervals: The Fourier Method for One-Dimensional Wave Equations	180
3.11	Generalized Fourier Solutions	211
3.12	Energy of the String	213
	Problems	222
4	Two-Dimensional Hyperbolic Equations	241
4.1	Derivation of the Equations of Motion	242
4.2	Oscillations of a Rectangular Membrane	247
4.3	The Fourier Method Applied to Small Transverse Oscillations of a Circular Membrane	277
	Problems	317
5	One-Dimensional Parabolic Equations	327
5.1	Physical Problems Described by Parabolic Equations: Boundary Value Problems	327
5.2	The Principle of the Maximum, Correctness, and the Generalized Solution	340

5.3	The Fourier Method of Separation of Variables for the Heat Conduction Equation	344
5.4	Heat Conduction in an Infinite Bar	383
5.5	Heat Equation for a Semi-infinite Bar	394
	Problems	409
6	Parabolic Equations for Higher-Dimensional Problems	419
6.1	Heat Conduction in More than One Dimension	419
6.2	Heat Conduction within a Finite Rectangular Domain . .	430
6.3	Heat Conduction within a Circular Domain	460
	Problems	482
7	Elliptic Equations	489
7.1	Elliptic Partial Differential Equations and Related Physical Problems	489
7.2	The Dirichlet Boundary Value Problem for Laplace's Equation in a Rectangular Domain	508
7.3	Laplace's and Poisson's Equations for Two-Dimensional Domains with Circular Symmetry	527
7.4	Laplace's Equation in Cylindrical Coordinates	561
	Problems	590
8	Bessel Functions	613
8.1	Boundary Value Problems Leading to Bessel Functions .	613
8.2	Bessel Functions of the First Kind	620
8.3	Properties of Bessel Functions of the First Kind: $J_n(x)$.	622
8.4	Bessel Functions of the Second Kind	627
8.5	Bessel Functions of the Third Kind	629
8.6	Modified Bessel Functions	630
8.7	The Effect of Boundaries on Bessel Functions	632
8.8	Orthogonality and Normalization of Bessel Functions . .	634
8.9	The Fourier-Bessel Series	638
8.10	Further Examples of Fourier-Bessel Series Expansions .	655
8.11	Spherical Bessel Functions	663
8.12	The Gamma Function	667
	Problems	670

9	Legendre Functions	677
9.1	Boundary Value Problems Leading to Legendre Polynomials	677
9.2	Generating Function for Legendre Polynomials	686
9.3	Recurrence Relations	687
9.4	Orthogonality of Legendre Polynomials	689
9.5	The Multipole Expansion in Electrostatics	692
9.6	Associated Legendre Functions $P_n^m(x)$	696
9.7	Orthogonality and the Norm of Associated Legendre Functions	701
9.8	Fourier-Legendre Series in Legendre Polynomials	702
9.9	Fourier-Legendre Series in Associated Legendre Functions	709
9.10	Laplace's Equation in Spherical Coordinates and Spherical Functions	713
	Problems	731
A	Eigenvalues and Eigenfunctions of the Sturm-Liouville Problem	743
B	Auxiliary Functions for Different Types of Boundary Conditions	747
C	The Sturm-Liouville Problem and the Laplace Equation	751
D	Vector Calculus	757
E	How to Use the Software Associated with this Book	769
E.1	Program Overview	770
E.2	Examples Using the Program TrigSeries	773
E.3	Examples Using the Program Waves	781
E.4	Examples Using the Program Heat	797
E.5	Examples Using the Program Laplace	811
E.6	Examples Using the Program FourierSeries	822
	Bibliography	831
	Index	833

Introduction

The topics of this book are partial differential equations (PDEs) of mathematical physics and boundary value problems, Fourier series, and special functions. This is the core content of many courses in the fields of engineering, physics, mathematics, and applied mathematics.

The book, along with the companion software, represents an innovative teaching and learning project that does not exist in the current literature in partial differential equations of mathematical physics. The book is significantly more detailed than the typical introduction to PDE. It contains many examples that show how to set up physical problems as mathematical ones; how to solve partial differential equations under different types of boundary conditions; how to work with special functions; and how to carry out a Fourier analysis using these functions. The topics discussed in this book are presented in full—not merely with brief explanations and solutions of basic examples. The text also contains many physical applications, which are important not only because of their significance in physics but also because they demonstrate how the same mathematical approaches may be used for different physical problems. This feature provides the reader with ways of extending these methods to other problems in the real world. The authors have many years of experience in teaching both physics and mathematics and believe that the text is a reasonable combination of a stringent mathematical approach and physical intuition.

The features of the book allow the presentation of a large amount of material of very significant depth during a one-semester undergraduate or graduate course for physicists, mathematicians, or engineers. The

book can also be used to teach a sequence of undergraduate and graduate courses. The text is substantially broader than most mathematical physics textbooks in PDEs, an aspect that will greatly aid instructors in their selection of topics for classes of different capabilities. Numerous problems are presented, from fairly simple to interestingly complex—an attribute that allows a lecturer to suggest problems of different levels to match the abilities of the students. Solutions to some of the more difficult problems are included; others are left for the reader to solve.

The detailed explanations and abundant examples, along with the companion software, also make this book useful for self-study, where it is important to illustrate all the steps of a solution and provide a tool for the extension of mathematical methods to more realistic and sophisticated problems. In keeping with practices currently recommended by physics education research, a number of *Reading Exercises* are scattered throughout the text. These exercises are designed to keep the reader engaged in the flow of arguments presented in the text and are particularly useful for self-study and self-evaluation. The reader is strongly encouraged to read the text with pencil and paper at hand, filling in the steps of the exercises while working through the text. The software provides a laboratory environment that allows the user to generate and model different physical situations and learn by experimentation. From this standpoint, the book along with the software can also be used as a reference for students and professionals alike on PDEs, Fourier series, and special functions.

The companion problem-solving software is a very important and intrinsic feature of the book and represents an approach to mathematical physics that integrates text, computational environment, and visualization. In typical undergraduate and even graduate courses in mathematical physics, students are limited to a small number of simple problems due to time constraints, which prohibit the study of a vast number of interesting, but more complex, applications. The software accompanying this text not only provides visualization and animation of various classes of mathematical problems, but also guides the reader and shows the sequence of all the steps needed to solve the problem. Once the problem is solved, the software allows a deeper investigation of the problem—an investigation of the dependence of the solution on the parameters, the accuracy of the solution, the speed of a series convergence, and related or similar questions. It also allows the reader to experiment with a much larger number of problems than are typically treated in a standard course.

The software is based on the same mathematical methods that are presented in the text. It also contains explanations of the theory, rich graphical capabilities, and a library of problems with hints or detailed solutions. As such, it is a platform for learning and investigating all the topics of the book; it is an inherent part of the learning experience rather than an interesting auxiliary. The software allows the presentation in full of almost any possible problem that an instructor might suggest to students for class and homework, as well as for independent study. The software is an open system and the reader can generate new examples to increase the existing library of sample cases. This feature also allows the instructor to build specific, novel problems tailored to a particular class objective.

The companion software does not require students to learn a programming language, as does general computer algebra systems software, such as Maple or Mathematica (although these are fine systems that are very useful for many purposes). Most computer algebra systems (CAS) require a significant investment in time spent learning commands, conventions, and other features—time that would be better spent learning the mathematical and physics content of a course in mathematical physics. In their evaluations, student users have commented on the short learning curve, the ease of use, and the flexibility of the software associated with this text as compared to more general computer systems for doing math. The software uses a simple, interactive interface, is user-friendly, and is intended to be an uncomplicated tool that will not take valuable class time away from the problems being studied. Extensive help sections that contain guided analytical solutions as well as detailed explanations of all the input menus are included within the software.

As an example of the advantages of the software in comparison with the solution of PDEs with more generic computer analysis software, the reader is encouraged to first use the included software to solve, for example, the heat equation in its general form, given by Equation (5.34) in Chapter 5, with nonhomogeneous mixed boundary conditions given by Equation (5.36), by using coefficients and other parameters of the reader's choice. Then, as a comparison, the reader should attempt to solve the same PDE with a generic CAS. Using the software included with the book, the solution can be obtained in only a few minutes with no programming effort at all. The solution, obtained with the Fourier method of separation of variables, is practically analytical; the only numeric calculations the software performs is the evaluation of the integrals for the coefficients of the series

expansion. Solving the same problem with a CAS requires a significant amount of programming even in the simplest case, and in more general cases, such as the one suggested above, a CAS does not help to reduce the problem to a transparent and manageable level.

Technical Overview

As an overview of the structure of the book, we begin with a brief discussion of what PDEs are from a mathematical point of view, with a few examples of several basic equations of mathematical physics, noting various phenomena they describe.

PDEs are differential equations for functions depending on several variables, the simplest example of which might be functions of location and time: $u(x, t)$. The unknown function can also depend on more than two variables, for instance in the three-dimensional case $u = u(\vec{r}, t)$.

A PDE can contain partial derivatives of different orders so that a general form of a second-order PDE for the function $u(x_1, x_2, \dots, x_n)$ is

$$F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_n^2}\right) = 0, \quad (1)$$

where F is a specified function.

To find the function u describing a particular process, the equation itself is not enough to determine the solution; additional information, which is contained in initial and boundary conditions, is needed. Thus we arrive at the so-called boundary value problem for PDEs.

Mathematical (theoretical) physics is largely based on PDEs so that knowledge of PDEs provides a powerful tool for investigating many different phenomena, as illustrated in the following examples of PDEs in physics.

Electromagnetic phenomena, from radio and computer communication to medical imaging, probably have the largest impact on modern life. Maxwell's equations for electric, $\vec{E}(\vec{r}, t)$, and magnetic, $\vec{H}(\vec{r}, t)$, fields, which describe all electromagnetic behavior, can be reduced to the D'Alembert equation

$$\nabla^2 \vec{A}(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \vec{A}(\vec{r}, t)}{\partial t^2} = 0 \quad (2)$$

for a vector potential $\vec{A}(\vec{r}, t)$, through which electric and magnetic fields can be obtained. Equation (2) is written for an electromagnetic wave in a vacuum, where ∇^2 is the Laplace operator and c is the speed of light.

Equation (2) is a particular case of the *wave equation*, which describes wave propagation. Indeed, it was Maxwell's derivation of the wave equation for electricity and magnetism that led to our present understanding of light and radio waves as electromagnetic phenomena. Similar equations describe other periodic behavior, such as sound waves, waves propagating in strings, membranes, and many other types of waves found in everyday life. The one-dimensional version of this equation is

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2}. \quad (3)$$

For an electrostatic field, when a magnetic field is absent (or constant), the equation for an electric potential $\varphi(\vec{r})$ is

$$\nabla^2 \varphi(\vec{r}) = 0. \quad (4)$$

Equation (4) is the *Laplace equation*; it too can be derived from Maxwell's equations and describes an electrostatic potential distribution in empty space. If electric charges are not absent, the equation for $\varphi(\vec{r})$ becomes

$$\nabla^2 \varphi(\vec{r}) = -4\pi \rho(\vec{r}), \quad (5)$$

where $\rho(\vec{r})$ is charge density distribution. Equation (5) is called *Poisson's equation*.

The one-dimensional *heat equation* for the function $u = u(x, t)$,

$$\frac{\partial u(x, t)}{\partial t} = a^2 \frac{\partial^2 u(x, t)}{\partial x^2} \quad (6)$$

(where a^2 is a constant), describes a process with dissipation, such as heat flow and diffusion. The solution of Equation (6) decays exponentially with time, which is the main difference compared to the periodic solutions of Equation (3).

The *Helmholtz equation*,

$$\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = 0, \quad (7)$$

represents the spatial (time-independent) part of either the wave or heat equations. The *Schrödinger equation* for a time-independent wave function also reduces to the Helmholtz equation.

All of the above equations are classified as either *hyperbolic*, *parabolic* or *elliptic*. These names come from a classification based on conic sections. The second-order algebraic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (8)$$

describes (for nontrivial cases) hyperbolas, parabolas, or ellipses, depending on the sign of the value of $b^2 - 4ac$. Similarly, considering functions of two variables, $u(x, y)$ for simplicity, the *second-order linear PDE* in its general form,

$$\begin{aligned} au''_{xx}(x, y) + bu''_{xy}(x, y) + cu''_{yy}(x, y) \\ + du'_x(x, y) + eu'_y(x, y) \\ + fu = g(x, y) \end{aligned} \quad (9)$$

is called hyperbolic if $b^2 - 4ac > 0$, elliptic if $b^2 - 4ac < 0$, and parabolic if $b^2 - 4ac = 0$. This classification is useful and important because, as already mentioned, the solutions of these different types of equations exhibit very different physical behaviors, but within a given class the solutions have many similar features. For instance, the solutions of all hyperbolic equations for very different physical phenomena are oscillatory. Obtaining the sign of the combination $b^2 - 4ac$ for the equations mentioned above, we can conclude that the wave equation is of the hyperbolic class, the heat equation is parabolic, and the Laplace and Helmholtz equations are elliptic.

The solution of PDEs involves the Fourier series expansion of the unknown function u and other functions, such as the initial conditions. Thus, the study of PDEs very naturally involves the study of the Fourier series (in addition to the fact that Fourier series are tremendously important in many other areas, not only for PDEs). The solutions of the PDEs of mathematical physics in the cases of cylindrical and spherical symmetry are the Bessel and Legendre functions, which also have a vast number of applications in many areas. Thus the scope of the book is naturally self-consistent: PDEs and boundary value problems, Fourier series and special functions.

This traditional set of topics is reflected in many books on PDEs (see the references below) and corresponds to a one-semester course on PDEs in many North American universities (in physics, mathematics, and engineering departments). In many European countries, the course on PDEs

is typically a separate, one- or two-semester course. Topics such as linear algebra (linear spaces), probability, and ordinary differential equations, for example, are usually taken by students as separate courses. In many North American schools, courses often use books on mathematical physics that cover practically all areas of mathematics. We do not think that a single course attempting to cover the entire spectrum of mathematical physics works well. Consequently, we have purposely focused on PDEs and related topics, which can be carefully considered in a normal semester-long course. Even for a more general course in mathematical physics PDEs, Fourier series and special functions will necessarily constitute a significant portion of class time. For such a course in which the instructor is trying to squeeze in other topics, our combined text-software approach is ideal because, due to the amount of detail provided, a significant amount of material can be assigned for study outside of the classroom.

The bibliography at the end of the book contains a short list of books the reader might consult as references (despite the self-consistency of the present text, it is always useful to consult other points of view). The first of these are the famous book by Tychonov and Samarski [1] and a much simpler and very clearly written text by Churchill and Brown [2]; the contents of both are very similar to this book. A book with very good physical emphasis and at the same time a rather high level of mathematics is Butkov [3]. The classical book on heat processes by Carslaw and Jaeger [4] contains many interesting problems. Landau and Lifshitz's book [5], as always, is very useful. The reference book by Korn and Korn [6] contains all the formulas necessary for practical applications. We also recommend the classic book by Courant and Hilbert [7] and, of course, the comprehensive text by Morse and Feshbach [8]. A good book on special functions is that of Lebedev [9].

In an attempt to raise the mood of the reader at the beginning of a challenging course, we quote two of our former students. When Tim Allen, a computer expert, heard about our troubles in extending the software to other computer platforms, he said, "What for? When I buy any book with software, first thing I do is I throw it away. Everybody does that." And trying to pass an exam on PDEs for the n th time, Sergei Chepelenko, a physics student and cheerful basketball player, answered a question about what the elliptic functions are by writing, "Well, there are three types of functions: the hyperbolic functions are waves, the parabolic functions are heat, the elliptic functions—that is even worse." Thank you, guys.

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1

Fourier Series

1.1 Periodic Processes and Periodic Functions

Very often in the sciences and in technology, we encounter periodic phenomena. These may be defined as physical processes that repeat after some time interval T , called the period. Alternating electric current, an object in circular motion, and wave phenomena are three examples of periodic physical phenomena. Such processes can be associated with periodic mathematical functions in time, t , which have the property

$$\varphi(t + T) = \varphi(t).$$

In the real world, this indicates that some physical quantity returns to its previous value after time intervals of one period.

The simplest periodic function is the sine (or cosine) function, $A \sin(\omega t + \alpha)$ (or $A \cos(\omega t + \alpha)$), where the angular frequency ω is related to the period by the relation

$$\omega = \frac{2\pi}{T}. \quad (1.1)$$

With these simple periodic functions, more complex periodic functions can be constructed, as was noted by the French mathematician Joseph Fourier.

For example, if we add the functions

$$\begin{aligned} y_0 &= A_0, \\ y_1 &= A_1 \sin(\omega t + \alpha_1), \\ y_2 &= A_2 \sin(2\omega t + \alpha_2), \\ y_3 &= A_3 \sin(3\omega t + \alpha_3), \dots, \end{aligned} \tag{1.2}$$

with multiple frequencies $\omega, 2\omega, 3\omega, \dots$ (i.e., with periods $T, T/2, T/3, \dots$), we obtain a periodic function (with period T), which, when graphed, has an appearance very distinct from the graphs of any of the functions in Equation (1.2). Almost any periodic function can be constructed in this fashion by using a combination of sine and cosine functions.

It is natural to also investigate the reverse problem. Is it possible to resolve a given arbitrary periodic function, $\varphi(t)$, with period T into a sum of simple functions such as those in Equation (1.2)? As we shall see, for a very wide class of functions, the answer to this question is positive, but to do so may require an infinite sequence of the functions in Equation (1.2). In these cases, the periodic function $\varphi(t)$ can be resolved into the infinite *trigonometric series*

$$\begin{aligned} \varphi(t) &= A_0 + A_1 \sin(\omega t + \alpha_1) + A_2 \sin(2\omega t + \alpha_2) + \dots \\ &= A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \alpha_n), \end{aligned} \tag{1.3}$$

where A_n and α_n are constants, and $\omega = 2\pi/T$. Each term in Equation (1.3) is called a *harmonic*, and the decomposition of periodic functions into harmonics is called *harmonic analysis*.

In many cases, it is useful to introduce the variable

$$x = \omega t = \frac{2\pi t}{T}$$

and to work with the functions

$$f(x) = \varphi\left(\frac{x}{\omega}\right),$$

which are also periodic but with the *standard period* 2π . Using this shorthand, Equation (1.3) becomes

$$\begin{aligned} f(x) &= A_0 + A_1 \sin(x + \alpha_1) + A_2 \sin(2x + \alpha_2) + \dots \\ &= A_0 + \sum_{n=1}^{\infty} A_n \sin(nx + \alpha_n). \end{aligned} \tag{1.4}$$

With the trigonometric identity $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ and the notation

$$A_0 = 2a_0, \quad A_n \sin \alpha_n = a_n, \quad A_n \cos \alpha_n = b_n, \quad (n = 1, 2, 3, \dots),$$

we obtain a standardized form for the harmonic analysis of a periodic function $f(x)$ as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \end{aligned} \quad (1.5)$$

which is referred to as the *trigonometric Fourier expansion*.

1.2 Fourier Formulas

To determine the limits of validity for the representation in Equation (1.5) of a given function $f(x)$ with period 2π and to find the coefficients a_n and b_n , we follow the approach that was originally elaborated by Fourier. We first assume that the function $f(x)$ can be integrated over the interval $[-\pi, \pi]$. If $f(x)$ is discontinuous at any point, we assume that the integral of $f(x)$ converges, and in this case we also assume that the integral of the absolute value of the function, $|f(x)|$, converges. A function with these properties is said to be *absolutely integrable*. Integrating Equation (1.5) term by term, we obtain

$$\int_{-\pi}^{\pi} f(x) dx = \pi a_0 + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right].$$

Since

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx dx &= \left. \frac{\sin nx}{n} \right|_{-\pi}^{\pi} = 0 \text{ and} \\ \int_{-\pi}^{\pi} \sin nx dx &= - \left. \frac{\cos nx}{n} \right|_{-\pi}^{\pi} = 0, \end{aligned} \quad (1.6)$$

all the terms in the sum are zero, and we obtain

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx. \quad (1.7)$$

To find coefficients a_n , we multiply Equation (1.5) by $\cos mx$ and then integrate term by term over the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \right].$$

The first term is zero, as was noted in Equation (1.6). For any n and m , we also have

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2} \int_{-\pi}^{\pi} [\sin(n+m)x + \sin(n-m)x] dx = 0, \quad (1.8)$$

and if $n \neq m$, we obtain

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x] dx = 0. \quad (1.9)$$

Using these formulas along with the identity

$$\int_{-\pi}^{\pi} \cos^2 mx dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} dx = \pi, \quad (1.10)$$

we see that all the integrals in the sum are zero except the one with the coefficient a_m . We thus have

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \quad (m = 0, 1, 2, \dots). \quad (1.11)$$

The usefulness of introducing the factor $1/2$ in the first term in Equation (1.5) is now apparent since it allows the same formulas to be used for all a_n , including $n = 0$.

Similarly, multiplying Equation (1.5) by $\sin mx$ and using, along with Equations (1.6) and (1.8), two other simple integrals

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0, \quad (1.12)$$

if $n \neq m$, and

$$\int_{-\pi}^{\pi} \sin^2 mx dx = \pi, \quad (1.13)$$

we obtain the second coefficient,

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \quad (m = 1, 2, 3, \dots). \quad (1.14)$$

Reading Exercise. Obtain the same result as in Equations (1.8), (1.9), and (1.12) by using the Euler expression

$$e^{imx} = \cos mx + i \sin mx.$$

Equations (1.7), (1.11), and (1.14) are known as the *Fourier coefficients*, and the series (1.5) with these definitions is called the *Fourier series*. Equation (1.5) is also referred to as the *Fourier expansion* of the function $f(x)$.

Notice that for the function $f(x)$ having period 2π , the integral

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx$$

does not depend on the value of α . As a result, we may also use the following expressions for the Fourier coefficients:

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx dx \quad \text{and} \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx. \quad (1.15)$$

It is important to realize that to obtain the results above we used a term-by-term integration of the series, which is justified only if the series converges

uniformly. Until we know for sure that the series converges we can only say that the series represented by Equation (1.5) *corresponds* to the function $f(x)$, which usually is denoted as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

At this point, we should remind the reader of the meaning of *uniform convergence*. The series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges to the sum $S(x)$ uniformly on the interval $[a, b]$ if, for any arbitrarily small $\varepsilon > 0$, we can find a number N such that for all $n \geq N$ the remainder of the series

$$\left| \sum_{n=N}^{\infty} f_n(x) \right| \leq \varepsilon$$

for all $x \in [a, b]$. This indicates that the series approaches its sum uniformly with respect to x . The most important features of a uniformly converging series are:

1. If $f_n(x)$ for any n is a continuous function, then $S(x)$ is also a continuous function.
2. The equality

$$\sum_{n=1}^{\infty} f_n(x) = S(x)$$

can be integrated term by term along any interval within the interval $[a, b]$.

3. If the series

$$\sum_{n=1}^{\infty} f'_n(x)$$

converges uniformly, then its sum is equal to $S'(x)$ (i.e., the formula

$$\sum_{n=1}^{\infty} f_n(x) = S(x)$$

can be differentiated term by term).

There is a simple and very practical criterion for convergence established by Karl Weierstrass that says that if $|f_n(x)| < c_n$ for each term $f_n(x)$ in the series defined on the interval $x \in [a, b]$ (i.e., $f_n(x)$ is limited by c_n), where

$$\sum_{n=1}^{\infty} c_n$$

is a converging numeric series, then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[a, b]$. For example, the numeric series

$$\sum_{n=1}^{\infty} 1/n^2$$

is known to converge, so any trigonometric series with terms such as $\sin nx/n^2$ or similar will converge uniformly for all x because

$$|\sin nx/n^2| \leq 1/n^2.$$

1.3 Orthogonal Systems of Functions

Equations (1.8), (1.9), and (1.12) also indicate that the system of functions that consists of all $\sin mx$ and $\cos nx$ is orthogonal on the interval $[-\pi, \pi]$. Recall that the system of real functions $\{\varphi_n(x)\}$ ($n = 1, 2, 3, \dots$) is defined to be *orthogonal* on the interval $[a, b]$ if

$$\int_a^b \varphi_n(x) \varphi_m(x) dx = 0 \quad (n, m = 1, 2, 3, \dots; n \neq m). \quad (1.16)$$

These integrals, as well as the integral

$$\int_a^b \varphi_n^2(x) dx \equiv \lambda_n > 0, \quad (1.17)$$

are assumed to exist. If all $\lambda_n = 1$, the system is said to be *orthonormal*. If the system is not orthonormal but satisfies Equation (1.17), we can construct one that is orthonormal by considering the system of functions

$$\left\{ \frac{\varphi_n(x)}{\sqrt{\lambda_n}} \right\}$$

in place of the original functions, $\varphi_n(x)$. Using these definitions, we see that the system of functions $\{\varphi_n(x)\}$ consisting of

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots \quad (1.18)$$

is orthogonal on $[-\pi, \pi]$, as was claimed previously.

It is important to notice that the above system is not orthogonal on the reduced interval $[0, \pi]$ because for n and m with different parity (i.e., one odd and the other even) we have

$$\int_0^\pi \sin nx \cos mx dx \neq 0.$$

However the system consisting of cosine functions only

$$1, \cos x, \cos 2x, \dots, \cos nx, \dots \quad (1.19)$$

is orthogonal on $[0, \pi]$, and the same is true for

$$\sin x, \sin 2x, \dots, \sin nx, \dots \quad (1.20)$$

A second observation, which we will need later, is that on an interval $[0, l]$ of arbitrary length l , both systems of functions

$$1, \cos \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \dots, \cos \frac{n\pi x}{l}, \dots \quad (1.21)$$

and

$$\sin \frac{\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \sin \frac{n\pi x}{l}, \dots \quad (1.22)$$

are orthogonal.

Reading Exercise. Prove statements (1.19) to (1.22).

Let us expand some function $f(x)$ defined on the interval $[a, b]$ into a series using the orthogonal system $\{\varphi_n(x)\}$ given in Equation (1.18):

$$f(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_n\varphi_n(x) + \dots \quad (1.23)$$

Multiplying this expression by $\varphi_m(x)$ and taking the integral over the interval $[a, b]$, we obtain

$$\int_a^b f(x)\varphi_m(x)dx = \sum_{n=0}^{\infty} c_n \int_a^b \varphi_n(x)\varphi_m(x)dx.$$

All the integrals on the right are zero except for the case $m = n$, which leads to a formula for the coefficients given by

$$c_m = \frac{1}{\lambda_m} \int_a^b f(x)\varphi_m(x)dx \quad (m = 1, 2, 3, \dots). \quad (1.24)$$

Again, it should be remembered that Equations (1.23) and (1.24) are valid only when the series given by Equation (1.24) converges uniformly on the interval $[a, b]$.

1.4 Convergence of Fourier Series

In this section, we study the range of validity of Equation (1.5) with Fourier coefficients given by Equations (1.11) and (1.14). To start, it is clear that if the function $f(x)$ is finite on $[-\pi, \pi]$, then the Fourier coefficients are bounded. This is easily verified, for instance for a_n :

$$|a_n| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} f(x) \cos nx dx \right| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)| \cdot |\cos nx| dx \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)| dx. \quad (1.25)$$

The same result is valid for cases in which $f(x)$ is not finite but is absolutely integrable; that is, the integral of its absolute value converges:

$$\int_{-\pi}^{\pi} |f(x)| dx < \infty. \quad (1.26)$$

The necessary condition that any series converges is that its terms tend to zero as $n \rightarrow \infty$. Because the absolute values of the sine and cosine functions are bounded by +1 and -1, the necessary condition that the trigonometric series in Equation (1.5) converges is that coefficients of expansion a_n and b_n tend to zero as $n \rightarrow \infty$. This condition is valid for functions that are integrable (or absolutely integrable in the case of functions that are not finite), which is clear from the following lemma.

Lemma 1.1 (Riemann's Lemma). *If the function $f(t)$ is absolutely integrable on $[a, b]$, then*

$$\lim_{\alpha \rightarrow \infty} \int_a^b f(t) \sin \alpha t dt = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} \int_a^b f(t) \cos \alpha t dt = 0. \quad (1.27)$$

We will not prove Riemann's lemma rigorously but its sense should be obvious. In the case of very fast oscillations, the sine and cosine functions change their sign very quickly as $\alpha \rightarrow \infty$. Thus these integrals vanish for "reasonable" (i.e., absolutely integrable) functions, $f(t)$ because they do not change substantially as the sine and cosine alternate with opposite signs in their semiperiods.

Thus, for absolutely integrating functions the necessary condition of convergence of Fourier series is satisfied. Before we discuss the problem of convergence of Fourier series in more detail, let us notice that practically any interesting function that has applications (periodic or given on some limited interval $[a, b]$) can be expanded in a converging Fourier series. It is interesting to note that continuity is not the key criterion, contrary to the power series expansion of the function $f(x)$, which, as well as all its derivatives, should be continuous in order for the power series to be converging. In fact, the continuity of $f(x)$ is not sufficient to guarantee the convergence of its Fourier expansion—it is possible to construct continuous functions with non-finite variation on a finite interval, for which the Fourier series do not converge. Such functions are quite uncommon in practical applications and we will not consider them. But contrary to power series, functions that are not necessarily continuous can be expanded in converging Fourier series—this makes Fourier series a very powerful tool. Most functions we encounter in science are piecewise continuous and piecewise monotonic on any finite interval, and can be expanded in converging Fourier series (see Section 1.4).

It is important to know how quickly the terms in Equation (1.5) decrease as $n \rightarrow \infty$. If they decrease rapidly, the series converges rapidly. In this case, by using very few terms, we have a good trigonometric approximation for $f(x)$, and the partial sum of the series, $S_n(x)$, is a good approximation to the sum, $S(x) = f(x)$. If the series converges more slowly, a larger number of terms is needed to have a sufficiently accurate approximation.

Assuming that the series of Equation (1.5) converges, the speed of its convergence to $f(x)$ depends on the behavior of $f(x)$ over its period, or, in the case of nonperiodic functions, on the way it is extended from the interval $[a, b]$ to the entire axis x , as we discuss next. Convergence is most rapid for very smooth functions (functions that have continuous derivatives of higher order). Discontinuities in the derivative of the function $f'(x)$ substantially reduce the rate of convergence, whereas discontinuities in $f(x)$ reduce the convergence rate even more, with the result that many terms in the Fourier series must be used to approximate the function $f(x)$ with the necessary precision. This should be fairly obvious, since the “smoothness” of $f(x)$ determines the rate of convergence of the coefficients a_n and b_n . As we will see in Lemma 1.2, the Fourier coefficients decrease (1) faster than $1/n^2$ (for example, $1/n^3$) when $f(x)$ and $f'(x)$ are continuous but $f''(x)$ has a discontinuity; (2) at about the same rate as $1/n^2$ when $f(x)$ is continuous but $f'(x)$ has discontinuities; and (3) at a rate similar to $1/n$ if $f(x)$ is not continuous. It is important to note that in the first two cases the series converges uniformly, which follows from Weierstrass’s criterion, because each term of Equation (1.5) is bounded by the corresponding term in the converging numeric series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

The following statement establishes the relationship between the differential properties of the function $f(x)$ and the speed of convergence of its Fourier series:

Lemma 1.2. *If the function, $f(x)$, and its derivatives, $f'(x), \dots, f^{(m)}(x)$ (with $m \geq 0$) are continuous on an interval $[-l, l]$ and have equal values on the ends of the interval, and $f^{(m+1)}(x)$ is piecewise continuous on $[-l, l]$, then the Fourier coefficients a_n and b_n of the expansion of the*

function $f(x)$ are of order

$$a_n = O\left(\frac{1}{n^{m+1}}\right), \quad b_n = O\left(\frac{1}{n^{m+1}}\right). \quad (1.28)$$

In other words, under these conditions, as $n \rightarrow \infty$, the Fourier coefficients decrease faster than

$$\left(\frac{1}{n^{m+1}}\right).$$

We now elaborate on some of the statements in the previous two paragraphs and give examples. First, we show that if a finite function $f(x)$ has a discontinuity, its Fourier coefficients decrease as $1/n$. Let a discontinuity be located at some point x_0 , in which case

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{x_0} f(x) \cos nx dx + \frac{1}{\pi} \int_{x_0}^{\pi} f(x) \cos nx dx.$$

Integrating each term by parts, we obtain

$$\begin{aligned} a_n &= \frac{1}{n\pi} [f(x_0 - 0) \sin nx_0 - f(-\pi) \sin(-n\pi) - \int_{-\pi}^{x_0} f'(x) \sin nx dx] \\ &\quad + \frac{1}{n\pi} [f(\pi) \sin n\pi - f(x_0 + 0) \sin(nx_0) - \int_{x_0}^{\pi} f'(x) \sin nx dx] \\ &= \frac{1}{n\pi} [f(x_0 - 0) - f(x_0 + 0)] \sin nx_0 - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx. \end{aligned}$$

On each interval (α, β) where the function $f(x)$ is continuous and monotone (we do not consider functions with an infinite number of maxima and minima, such as $\sin(1/x)$), the derivative $f'(x)$ does not change its sign; thus,

$$\int_{\alpha}^{\beta} |f'(x)| dx = \left| \int_{\alpha}^{\beta} f'(x) dx \right| = |f(\beta - 0) - f(\alpha + 0)| < \infty$$

and

$$\int_{-\pi}^{\pi} |f'(x)| dx < \infty.$$

We see that the last integral in the expression for a_n is finite and these coefficients decrease as $1/n$, a result that also holds for the coefficients b_n .

Now suppose the function $f(x)$ is continuous and its first derivative is finite but has discontinuities. Calculating as above, we obtain

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx.$$

As we just found, the integral on the right decreases as $1/n$, thus a_n decreases as $1/n^2$. Similarly, when $f(x)$ and $f'(x)$ are continuous, but $f''(x)$ has discontinuities, we must integrate by parts twice, which results in Fourier coefficients that decrease as $1/n^3$.

Having discussed the speed of convergence of Fourier coefficients (remember that $a_n \rightarrow 0$ and $b_n \rightarrow 0$ are only necessary conditions of convergence), let us discuss the convergence of the Fourier series given in Equation (1.5) at a point x_0 where $f(x)$ is continuous or where it may have a discontinuity. Notice the important fact that when $f(x)$ has a discontinuity at x_0 , the Fourier series cannot converge uniformly at this point because its terms are continuous functions and in the case of uniform convergence the sum would also be a continuous function.

Let us consider the partial sum of the series in Equation (1.5) at some point x_0 given by

$$S_n(x_0) = \frac{a_0}{2} + \sum_{m=1}^n (a_m \cos mx_0 + b_m \sin mx_0). \quad (1.29)$$

Substituting Equations (1.11) and (1.14) for a_m and b_m gives

$$\begin{aligned} S_n(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) du + \sum_{m=1}^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) [\cos mu \cos mx_0 + \sin mu \sin mx_0] du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \left\{ \frac{1}{2} + \sum_{m=1}^n \cos m(u - x_0) \right\} du. \end{aligned}$$

Using the trigonometric identity

$$\frac{1}{2} + \sum_{m=1}^n \cos m(u - x_0) = \frac{\sin(2n+1)\frac{u-x_0}{2}}{2 \sin \frac{u-x_0}{2}}, \quad (1.30)$$

we obtain

$$S_n(x_0) = \frac{1}{\pi} \int_{x_0-\pi}^{x_0+\pi} f(u) \frac{\sin(2n+1)\frac{u-x_0}{2}}{2 \sin \frac{u-x_0}{2}} du, \quad (1.31)$$

where we also have replaced the interval $[-\pi, \pi]$ by $[x_0 - \pi, x_0 + \pi]$ of the same length 2π .

Reading Exercise. Prove identity (1.30).

Hint. Multiply the left side by $2 \sin(u-x_0)/2$, then replace each of the terms $2 \cos m(u-x_0) \sin(u-x_0)/2$ by

$$\sin\left[\left(m + \frac{(u-x_0)}{2}\right)\right] - \sin\left[\left(m - \frac{(u-x_0)}{2}\right)\right].$$

Equation (1.31) is called the *Dirichlet integral*. The substitution $t = u - x_0$ brings this equation to a standard form given by

$$S_n(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0 + t) \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2}t} dt.$$

We may split the interval $[-\pi, \pi]$ into the two intervals $[-\pi, 0]$ and $[0, \pi]$. A change of sign of t in the first converts the interval to $[0, \pi]$; thus we may write

$$S_n(x_0) = \frac{1}{\pi} \int_0^{\pi} \frac{[f(x_0 + t) + f(x_0 - t)]}{2} \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2}t} dt. \quad (1.32)$$

We cannot obtain the limit of $S_n(x_0)$ directly from this formula because the integrand in Equation (1.32) does not exist when $n \rightarrow \infty$. We can, however, apply further arguments to yield interesting results.

Equation (1.32) is valid for any function $f(x)$. For instance, for $f(x) \equiv 1$ the formula should yield $S_n(x) \equiv 1$. Following this idea, we obtain

$$1 = \frac{1}{\pi} \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2}t} dt. \quad (1.33)$$

Equation (1.33) can be used to consider two important cases:

1. If the function $f(x)$ is continuous at x_0 , the integral in Equation (1.32) converges to $S_0 = f(x_0)$.
2. If the function $f(x)$ has a discontinuity of the first type (a finite jump discontinuity) at x_0 (i.e., both limits $f(x_0 + 0)$ and $f(x_0 - 0)$ exist), it is clear that the integral in Equation (1.32) converges to

$$S(x_0) = \frac{f(x_0 + 0) + f(x_0 - 0)}{2}.$$

Thus, we have arrived at the following theorem.

Theorem 1.3 (Dirichlet Theorem). *If the function $f(x)$ with period 2π is piecewise continuous in $[-\pi, \pi]$ and has a finite number of points of discontinuity in this interval, then its Fourier series converges to $f(x_0)$ when x_0 is a continuity point, and to*

$$S(x_0) = \frac{f(x_0 + 0) + f(x_0 - 0)}{2}$$

if x_0 is a point of discontinuity.

At the ends of the interval $[-\pi, \pi]$, the Fourier series converges to

$$\frac{f(-\pi + 0) + f(\pi - 0)}{2}.$$

Remember that a function $f(x)$ defined on $[a, b]$ is called *piecewise continuous* if:

1. It is continuous on $[a, b]$ except perhaps at a finite number of points;
2. If x_0 is one such point, then the left and right limits of $f(x)$ at x_0 exist and are finite;
3. Both the limit from the right of $f(x)$ at a and the limit from the left of $f(x)$ at b exist and are finite.

Stated more briefly, for the Fourier series of a function $f(x)$ to converge, this function should be piecewise continuous with a finite number of discontinuities. To be on the safe side, we also avoid functions with an infinite number of maxima and minima on a finite interval—thus we consider piecewise monotonic functions.

1.5 Fourier Series for Nonperiodic Functions

Thus far, we have assumed that the function $f(x)$ is defined on the entire x -axis and has period 2π . But very often we need to deal with nonperiodic functions defined only on the interval $[-\pi, \pi]$. The Dirichlet theory can still be used if we extend $f(x)$ periodically from $(-\pi, \pi)$ to all x . In other words, we assign the same values of $f(x)$ to all the intervals $(\pi, 3\pi)$, $(3\pi, 5\pi)$, $\dots$, $(-3\pi, -\pi)$, $(-5\pi, -3\pi)$, $\dots$ and then use Equations (1.11) and (1.14) for the Fourier coefficients of this new function, which is periodic. Many examples of such extensions are given in this chapter. If $f(-\pi) = f(\pi)$, we can include the endpoints, $x = \pm\pi$, and the Fourier series converges to $f(x)$ everywhere on $[-\pi, \pi]$. Over the entire axis the expansion gives a periodic extension of the function $f(x)$ given originally on $[-\pi, \pi]$. In many cases, $f(-\pi) \neq f(\pi)$, and the Fourier series at the ends of the interval $[-\pi, \pi]$ converges to

$$\frac{f(-\pi) + f(\pi)}{2},$$

which differs from both $f(-\pi)$ and $f(\pi)$.

The rate of convergence of the Fourier series depends on the discontinuities of the function and the derivatives of the function after its extension to the entire axis. Some extensions do not increase the number of discontinuities of the original function whereas others do increase this number. In the latter case, the rate of convergence is reduced. Among the examples given later in this chapter, two are of the Fourier series of $f(x) = x$ on the interval $[0, \pi]$. In the first expansion, the function is extended to the entire axis as an even function and remains continuous so that the coefficients of the Fourier series decrease as $1/n^2$. In the second example, this function is extended as an odd function and has discontinuities at $x = k\pi$ (integer k), in which case the coefficients decrease slower, as $1/n$.

1.6 Fourier Expansions on Intervals of Arbitrary Length

Suppose that a function $f(x)$ is defined on some interval $[-l, l]$ of arbitrary length $2l$ (where $l > 0$). Using the substitution

$$x = \frac{ly}{\pi} \quad (-\pi \leq y \leq \pi),$$

we obtain the function $f(y/\pi)$ of the variable y on the interval $[-\pi, \pi]$, which can be expanded by using the standard equations (1.5), (1.11), and (1.14) as

$$f\left(\frac{yl}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny),$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right) \cos ny dy \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right) \sin ny dy.$$

Returning to the variable x we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \quad (1.34)$$

with

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots \end{aligned} \quad (1.35)$$

If the function is given, but not on the interval $[-l, l]$, and instead on an arbitrary interval of length $2l$, for instance $[0, 2l]$, the formulas for the coefficients of the Fourier series (1.34) become

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx \quad \text{and} \quad b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx. \quad (1.36)$$

In both cases, the series in Equation (1.34) gives a periodic function with period $T = 2l$.

If the function $f(x)$ is given on an interval $[a, b]$ (where a and b may have the same or opposite signs; that is, the interval $[a, b]$ can include or exclude the point $x = 0$), different periodic continuations onto the entire x -axis may be made (see [Figure 1.1](#)). As an example, consider the periodic continuation $F(x)$ of the function $f(x)$, defined by the condition

$$F(x + n(b - a)) = f(x), \quad n = 0, \pm 1, \pm 2, \dots \text{ for all } x.$$

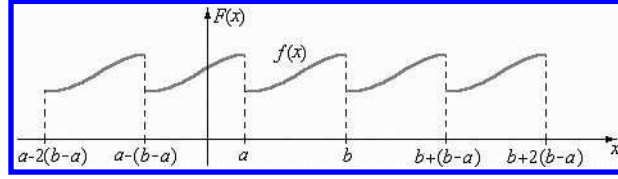


Figure 1.1. Arbitrary function $f(x)$ defined on the interval $[a, b]$ extended to the x -axis as the function $F(x)$.

In this case, the Fourier series is given by Equation (1.34), where $2l = b - a$. Clearly, instead of Equations (1.35), the following formulas for the Fourier coefficients should be used:

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a} dx, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a} dx. \end{aligned} \quad (1.37)$$

The series in Equation (1.34) gives a periodic function with period $T = 2l = b - a$; however, the original function was defined only on the interval $[a, b]$ and is not periodic in general.

1.7 Fourier Series in Cosine or Sine Functions

Suppose that $f(x)$ is an *even* function on $[-\pi, \pi]$ so that $f(x) \sin nx$ is odd. For this case,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0,$$

since the integral of an odd function over a symmetric interval equals zero. Coefficients a_n can be written as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad (1.38)$$

since the integrand is even. Thus, for an even function $f(x)$, we may write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad (1.39)$$

Similarly, if $f(x)$ is an *odd* function, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0 \quad \text{and} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \quad (1.40)$$

in which case we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \quad (1.41)$$

Any function can be presented as a sum of even and odd functions,

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_2(x) = \frac{f(x) - f(-x)}{2},$$

in which case $f_1(x)$ can be expanded into a cosine Fourier series and $f_2(x)$ into a sine Fourier series.

If the function $f(x)$ is defined only on the interval $[0, \pi]$, we can extend it to the interval $[-\pi, 0)$. This extension may be made in different ways corresponding to different Fourier series. In particular, such an extension can make $f(x)$ even or odd on $[-\pi, \pi]$, which leads to cosine or sine series with period 2π . In the first case, on the interval $[-\pi, 0)$ we have

$$f(-x) = f(x), \quad (1.42)$$

and in the second case

$$f(-x) = -f(x). \quad (1.43)$$

The points $x = 0$ and $x = \pi$ need special consideration because the sine and cosine series behave differently at these points. If $f(x)$ is continuous

at these points, because of Equations (1.42) and (1.43) the cosine series converges to $f(0)$ at $x = 0$ and to $f(\pi)$ at $x = \pi$. The situation is different for the sine series, however. At $x = 0$ and $x = \pi$ the sum of the sine series in Equation (1.41) is zero; thus, the series is equal to the functions $f(0)$ and $f(\pi)$, respectively, only when these values are zero.

If $f(x)$ is given on the interval $[0, l]$ (where $l > 0$), the cosine and sine series are

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad (1.44)$$

and

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad (1.45)$$

with the coefficients

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx, \quad (1.46)$$

or

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx. \quad (1.47)$$

If we then wish to extend these expansions to the entire axis, $-\infty < x < +\infty$, it is necessary in both cases to consider the interval $(-l, l)$, which can be expanded to the entire x -axis with period $T = 2l$. For example, suppose we have the function $f(x) = 2x + 1$ on the interval $[0, 1]$. Figure 1.2 presents different schemes of extension of this function to the x -axis.

To perform an even or odd continuation of the function $f(x)$ defined on the interval $[a, b]$ where a and b have the same sign, we can extend $f(x)$ on an interval containing the point $x = 0$. This continuation can be done in an arbitrary way; for example, we can choose $f(x) = 0$ at $x = 0$. Then the function $F(x)$ defined by

$$F(x) = \begin{cases} 0, & -a \leq x < a, \\ f(x), & a \leq x \leq b, \\ f(-x), & -b \leq x < -a \end{cases}$$

will be even and can be continued onto the entire axis with period $2l = 2b$.

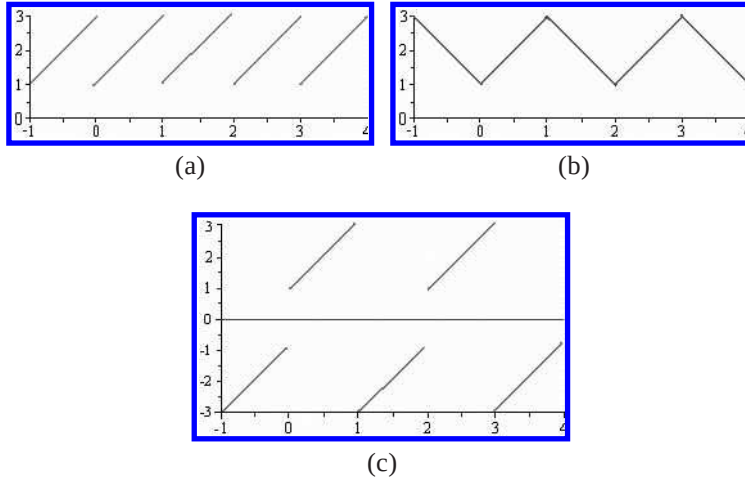


Figure 1.2. Three different ways of extending the function $f(x) = 2x + 1$ defined on the interval $[0, 1]$ to the function $F(x)$ on the x -axis. (a) General method. (b) Even terms method (cosines only). (c) Odd terms method (sines only).

To avoid the discontinuities at the points $x = \pm a$, we can set the function $F(x)$ to be equal to the value $f(a)$ for $-a \leq x \leq a$, and in this case the Fourier series will converge faster than the previous case. These two ways of even continuation are labeled as solid and dashed lines in Figure 1.3.

The Fourier series expansion is now given by Equations (1.44) and (1.46), where $f(x)$ is replaced by $F(x)$ and $l = b$.

Similarly, an odd continuation can be performed. If we choose $f(x) = 0$ at $x = 0$, then the function $F(x)$ defined by

$$F(x) = \begin{cases} 0, & -a \leq x < a, \\ f(x), & a \leq x \leq b, \\ -f(-x), & -b \leq x < -a \end{cases}$$

will be odd and can be continued onto the entire axis with period $2l = 2b$. This variant is shown with a solid line in Figure 1.4. One of the other possibilities is marked by a dashed line. The Fourier series expansion is now given by Equations (1.45) and (1.47) where $f(x)$ is replaced by $F(x)$ and $l = b$.

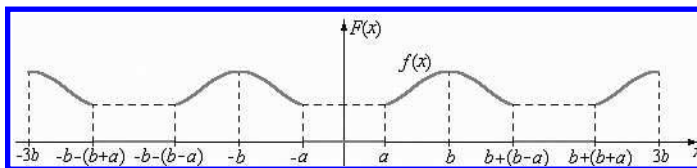


Figure 1.3. Two ways (solid and dashed lines) to extend function $f(x)$ to the x -axis as an even function, $F(x)$.

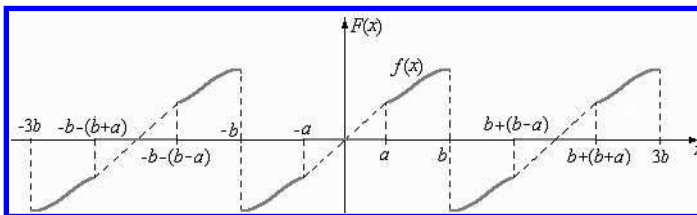


Figure 1.4. Two ways (solid and dashed curves) to extend function $f(x)$ to the x -axis as an odd function, $F(x)$.

Reading Exercise. Let $f(x)$ and $g(x)$ both have period T . Prove the following simple properties of periodic functions.

1. The function $af(x) + bg(x)$ has the same period for any constants a and b .
2. The function $f(ax)$ has period T/a .
3. The derivative $f'(x)$ has period T .

To summarize the preceding discussion, we see that the Fourier series provides a way to obtain an *analytic formula* for functions defined by different formulas on different intervals by combining these intervals into a larger one. Such analytic formulas replace a discontinuous function by a continuous Fourier series expansion, which is often more convenient in a given application. As we have seen, there are often many different choices of how to extend the original function, defined initially on an interval, to the entire axis. The specific choice of extension depends on the application to which the expansion is to be used. Examples demonstrating these points are presented at the end of this section and some of the following sections, as well as in the problems at the end of this chapter.

All the functions given below are differentiable or piecewise differentiable and can be represented by Fourier series. The expansions are given, but details of the calculation are left to the reader as an exercise. An explanation of how to use the program TrigSeries to solve the examples is given in Appendix E. Notice that the program uses the same formulas that the reader is directed to obtain and use when solving a problem analytically. The only numeric calculation the program performs is the evaluation of the coefficients of Fourier series (with Gauss's method and its modifications) and partial sums.

Example 1.1. Expand $f(x) = e^{2x}$ as a sine series and a cosine series on the interval $[0,1]$.

Solution. The coefficients are

$$b_n = 2 \int_0^1 e^{2x} \sin n\pi x dx = \frac{2n\pi}{4 + n^2\pi^2} [1 - (-1)^n e^2] \text{ for the sine series,}$$

$$a_n = 2 \int_0^1 e^{2x} \cos n\pi x dx = \frac{4}{4 + n^2\pi^2} [e^2(-1)^n - 1] \text{ for the cosine series.}$$

Notice that both series converge to e^{2x} for $0 < x < 1$ and that at $x = 0$ and $x = 1$ the sine series converges to zero whereas the cosine series converges to 1 for $x = 0$ and to e^2 for $x = 1$.

Example 1.2. Find the Fourier series for $f(x) = e^{ax}$ on the interval $(-\pi, \pi)$ where $(a = \text{const}, a \neq 0)$.

Solution. The coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = 2 \frac{\sinh a\pi}{a\pi},$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx dx = \frac{1}{\pi} \frac{a \cos nx + n \sin nx}{a^2 + n^2} e^{ax} \Big|_{-\pi}^{\pi} = (-1)^n \frac{1}{\pi} \frac{2a}{a^2 + n^2} \sinh a\pi,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx dx = \frac{1}{\pi} \frac{a \sin nx - n \cos nx}{a^2 + n^2} e^{ax} \Big|_{-\pi}^{\pi} = (-1)^{n-1} \frac{1}{\pi} \frac{2n}{a^2 + n^2} \sinh a\pi.$$

Thus, for $-\pi < x < \pi$, we have

$$e^{ax} = \frac{2}{\pi} \left\{ \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} [a \cos nx - n \sin nx] \right\} \sinh a\pi.$$

Reading Exercise. Find the series for the same function on the interval $(0, 2\pi)$. The Fourier coefficients will differ from the ones obtained above. Use the program TrigSeries to plot graphs of partial sums for both cases.

Example 1.3. Find the Fourier series for $f(x) = \frac{\pi-x}{2}$ on the interval $(0, 2\pi)$.

Solution. The coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} dx = \frac{1}{2\pi} \left(\pi x - \frac{1}{2}x^2 \right) \Big|_0^{2\pi} = 0, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \cos nx dx = \frac{1}{2\pi} (\pi-x) \frac{\sin nx}{n} \Big|_0^{2\pi} - \frac{1}{2n\pi} \int_0^{2\pi} \sin nx dx = 0, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \sin nx dx = -\frac{1}{2\pi} (\pi-x) \frac{\cos nx}{n} \Big|_0^{2\pi} - \frac{1}{2n\pi} \int_0^{2\pi} \cos nx dx = \frac{1}{n}. \end{aligned}$$

This contains the interesting result

$$\frac{\pi-x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi). \quad (1.48)$$

Equation (1.48) is not valid at $x = 0$ and $x = 2\pi$ because the sum of the series equals zero. The equality is also violated beyond $(0, 2\pi)$, as is obvious from the plot for the partial sums that we may obtain with the program TrigSeries.

Notice that the series in Example 1.3 converges more slowly than that in Example 1.2; thus, we need more terms to obtain the same deviation from the original function. Also, this series does not converge uniformly. (To understand why, attempt to differentiate it term by term and note what happens.)

For $x = \frac{\pi}{2}$, we have another interesting result, which was obtained by Leibniz by other means:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad (1.49)$$

Example 1.4. Find the cosine series for $f(x) = x^2$ on the interval $[-\pi, \pi]$.

Solution. The coefficients are

$$\frac{1}{2}a_0 = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi} x^2 \frac{\sin nx}{n} \Big|_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} x \sin nx dx \\ &= \frac{4}{n\pi} x \frac{\cos nx}{n} \Big|_0^{\pi} - \frac{4}{n^2\pi} \int_0^{\pi} \cos nx dx = (-1)^n \frac{4}{n^2}. \end{aligned}$$

Thus

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2} \quad (-\pi \leq x \leq \pi). \quad (1.50)$$

In the case where $x = \pi$, we obtain a famous expansion,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (1.51)$$

Reading Exercise. Using the program TrigSeries, plot graphs of partial sums, build the histogram of the squares of amplitudes A_n^2 , and study the rate of convergence.

Example 1.5. Let the function $f(x) = x$ on the interval $[0, \pi]$. Find the Fourier cosine series.

Solution. Figure 1.5 gives an even periodic continuation of $f(x) = x$ from $[0, \pi]$ onto the entire axis. It also represents the sum of the series shown in Equation (1.52). For coefficients, we have

$$\begin{aligned} \frac{1}{2}a_0 &= \frac{1}{\pi} \int_0^{\pi} x dx = \frac{\pi}{2}, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi} x \frac{\sin nx}{n} \Big|_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \sin nx dx \\ &= 2 \frac{\cos n\pi - 1}{n^2\pi} = 2 \frac{(-1)^n - 1}{n^2\pi} \quad (n > 0); \end{aligned}$$

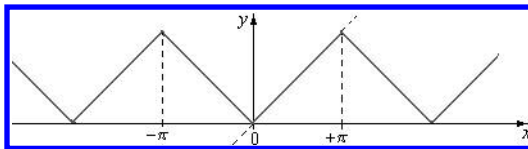


Figure 1.5. The function $f(x) = x$ extended to the x -axis.

that is,

$$a_{2k} = 0, \quad a_{2k-1} = -\frac{4}{(2k-1)^2\pi} \quad (k = 1, 2, 3, \dots),$$

and thus,

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} \quad (0 \leq x \leq \pi). \quad (1.52)$$

Figure 1.6 shows the graph of the partial sum

$$y = S_5(x) = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x \right)$$

together with the graph of the extended function.

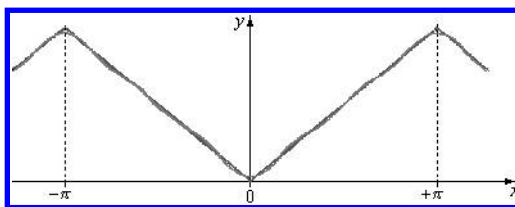


Figure 1.6. Original function extended to the x -axis plotted together with the partial sum of the first five terms.

Reading Exercise. Find the sine series of the same function on the interval $(-\pi, \pi)$.

The answer is

$$x = 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n}, \quad (1.53)$$

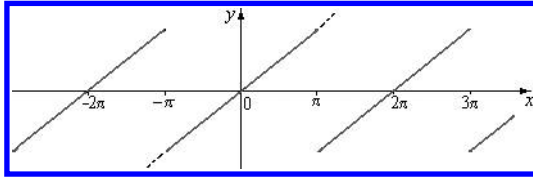


Figure 1.7. The function $f(x) = x$ with an extension to the x -axis alternate to Figure 1.5.

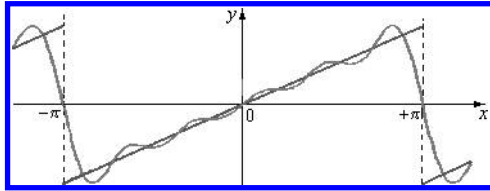


Figure 1.8. Original function with alternate extension to the x -axis plotted with the partial sum of the first five terms.

which is different than the cosine series for the same function. The graphs for the series sum and the partial sum

$$y = S_5(x) = 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \frac{1}{5} \sin 5x \right)$$

are presented in Figures 1.7 and 1.8, respectively.

Example 1.6. Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x & \text{if } 0 \leq x \leq \pi. \end{cases}$$

Solution. The coefficients are

$$\frac{1}{2}a_0 = \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{\pi}{4},$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{\pi} x \frac{\sin nx}{n} \Big|_0^{\pi} - \frac{1}{n\pi} \int_0^{\pi} \sin nx dx = \frac{\cos n\pi - 1}{n^2\pi};$$

that is,

$$a_{2k} = 0, \quad a_{2k-1} = -\frac{2}{(2k-1)^2\pi}.$$

Similarly,

$$b_n = -\frac{\cos n\pi}{n} = (-1)^{n-1} \frac{1}{n}.$$

The resulting expansion is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \cos x + \sin x - \frac{1}{2} \sin 2x - \frac{2}{9\pi} \cos 3x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots$$

Example 1.7. Find the Fourier series for the function

$$f(x) = \begin{cases} 1, & -l < x < -l + \alpha, \\ 0, & -l + \alpha < x < 0, \\ 1, & 0 < x < \alpha, \\ 0, & \alpha < x < l, \end{cases}$$

with $0 < \alpha < l$. Evaluate the resulting expression for the case when $\alpha = l/2$.

Solution. First, find the coefficients

$$\begin{aligned} a_0 &= \frac{1}{2l} \int_{-l}^l f(x) dx = \frac{1}{2l} \left(\int_{-l}^{-l+\alpha} + \int_{-l+\alpha}^0 + \int_0^{\alpha} + \int_{\alpha}^l \right) f(x) dx \\ &= \frac{1}{2l} \left(\int_{-l}^{-l+\alpha} 1 \cdot dx + \int_{-l+\alpha}^0 0 \cdot dx + \int_0^{\alpha} 1 \cdot dx + \int_{\alpha}^l 0 \cdot dx \right) = \frac{1}{2l} (\alpha + 0 + \alpha + 0) = \frac{\alpha}{l}, \\ a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos \frac{k\pi x}{l} dx = \frac{1}{l} \left(\int_{-l}^{-l+\alpha} \cos \frac{k\pi x}{l} dx + \int_0^{\alpha} \cos \frac{k\pi x}{l} dx \right) \\ &= \frac{1}{n\pi} \left[\sin \frac{n\pi(-l+\alpha)}{l} + \sin \frac{n\pi\alpha}{l} \right] \\ &= \frac{1}{n\pi} \left(-\sin \frac{n\pi l}{l} \cos \frac{n\pi\alpha}{l} + \cos \frac{n\pi l}{l} \sin \frac{n\pi\alpha}{l} + \sin \frac{n\pi\alpha}{l} \right) \\ &= \frac{1}{n\pi} [(-1)^n + 1] \sin \frac{n\pi\alpha}{l}, \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{l} \left(\int_{-l}^{-l+\alpha} \sin \frac{k\pi x}{l} dx + \int_0^\alpha \sin \frac{k\pi x}{l} dx \right) \\
&= -\frac{1}{n\pi} \left[\cos \frac{n\pi(-l+\alpha)}{l} - \cos(-n\pi) + \cos \frac{n\pi\alpha}{l} - 1 \right] \\
&= -\frac{1}{n\pi} \left(\cos \frac{n\pi l}{l} \cos \frac{n\pi\alpha}{l} + \sin \frac{n\pi l}{l} \sin \frac{n\pi\alpha}{l} - (-1)^n + \cos \frac{n\pi\alpha}{l} - 1 \right) \\
&= -\frac{1}{n\pi} \left\{ [(-1)^n + 1] \cos \frac{n\pi\alpha}{l} - (-1)^n - 1 \right\} = \frac{1}{n\pi} [(-1)^n + 1] \left(1 - \cos \frac{n\pi\alpha}{l} \right).
\end{aligned}$$

Thus, for odd n , all $a_n = b_n = 0$, and for even n ,

$$a_{2k} = \frac{1}{k\pi} \sin \frac{2k\pi\alpha}{l}, \quad b_{2k} = \frac{1}{k\pi} \left(1 - \cos \frac{2k\pi\alpha}{l} \right), \quad (k = 1, 2, 3, \dots).$$

When $\alpha = l/2$

$$a_{2k} = \frac{1}{k\pi} \sin k\pi = 0, \quad b_{2k} = \frac{1}{k\pi} (1 - \cos k\pi) = \frac{1}{k\pi} [1 - (-1)^n],$$

and the series becomes

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left(\frac{1}{1} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{6\pi x}{l} + \frac{1}{5} \sin \frac{10\pi x}{l} + \dots \right).$$

This series gives an analytical formula for the function given by 0 and 1 on alternating intervals.

Example 1.8. Suppose a function is defined on its half-period, $[0, 2]$, by the equation

$$f(x) = x - x^2/2.$$

Find a Fourier series that represents this function.

Solution. There are several different ways to expand this function into a Fourier series. Here, we consider two distinct possibilities.

- (a) Extend $f(x)$ onto $[-2, 0]$ as an even function with $l = 2$, which gives

$$a_n = \int_0^2 (x - x^2/2) \cos(n\pi x/2) dx, \quad b_n = 0.$$

(b) Extend $f(x)$ onto $[-2, 0]$ using the sine function, which results in

$$b_n = \int_0^2 (x - x^2/2) \sin(n\pi x/2) dx, \quad a_n = 0.$$

We leave it as a reading exercise to obtain the final formulas and to draw the graphs of the partial sums. There are other ways to extend the function beyond the given interval. For example, different periods for the extended function may be used in addition to the values 2 or 4.

1.8 The Complex Form of the Fourier Series

It is often useful to present Fourier series in complex form. Let us start with the Fourier expansion of the function $f(x)$ with period 2π ,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (1.54)$$

with

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (n = 0, 1, 2, \dots), \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (n = 1, 2, 3, \dots). \end{aligned} \quad (1.55)$$

From Euler's formula we have

$$\begin{aligned} \cos nx &= \frac{1}{2} (e^{inx} + e^{-inx}), \\ \sin nx &= \frac{1}{2i} (e^{inx} - e^{-inx}) = \frac{i}{2} (e^{-inx} - e^{inx}), \end{aligned}$$

from which we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2} (a_n - b_n i) e^{inx} + \frac{1}{2} (a_n + b_n i) e^{-inx} \right].$$

Using the notations

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - b_n i), \quad c_{-n} = \frac{1}{2}(a_n + b_n i),$$

we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \quad (1.56)$$

With the Fourier equations for a_n and b_n (1.55), it is easy to see that the coefficients c_n can be written as

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad (n = 0, \pm 1, \pm 2, \dots). \quad (1.57)$$

It is clear that for functions with period $2l$, Equations (1.56) and (1.57) have the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \quad (1.58)$$

and

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx \quad (n = 0, \pm 1, \pm 2, \dots). \quad (1.59)$$

For periodic functions in time t and processes with a period T , the same formulas can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2in\pi t}{T}} \quad (1.60)$$

and

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-\frac{2in\pi t}{T}} dt \quad (n = 0, \pm 1, \pm 2, \dots). \quad (1.61)$$

Several useful properties of these results can be easily verified:

1. Because $f(x)$ is real, c_n and c_{-n} are complex conjugates, and we have $c_{-n} = c_n^*$ (where the $*$ denotes complex conjugate);
2. If $f(x)$ is even, all c_n are real;

3. If $f(x)$ is odd, $c_0 = 0$ and all c_n are pure imaginary.

If the function $f(x)$ of the real variable x is complex, we have

$$f(x) = f_1(x) + if_2(x),$$

where $f_1(x)$, $f_2(x)$ are real functions, in which case the Fourier series for $f(x)$ is the sum of Fourier series for $f_1(x)$ and $f_2(x)$ where the second series is multiplied by the imaginary number i . Equations (1.56) and (1.57) remain unchanged but the above three properties of the coefficients are not valid (in particular, coefficients c_n and c_{-n} are not complex conjugates). Instead of the above three properties, in this case we have:

1. If $f(x)$ is even, then $c_{-n} = c_n$;
2. If $f(x)$ is odd, then $c_{-n} = -c_n$.

Example 1.9. Represent the function

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0, \\ 1, & 0 < x \leq \pi \end{cases}$$

by a complex Fourier series.

Solution. The coefficients are

$$c_0 = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}, \quad (1.62)$$

and

$$c_n = \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx = \frac{1 - e^{-in\pi}}{2\pi ni} = \begin{cases} 0 & \text{if } n = \text{even,} \\ \frac{1}{\pi ni} & \text{if } n = \text{odd.} \end{cases} \quad (1.63)$$

Thus,

$$f(x) = \frac{1}{2} + \frac{1}{\pi i} \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{1}{n} e^{inx}.$$

Reading Exercise. Using Euler's formula, check that from this expression it follows that

$$\operatorname{Im} f(x) = 0 \quad \text{and} \quad \operatorname{Re} f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,\dots}^{\infty} \frac{\sin nx}{n}.$$

The same result can be obtained by using the real Fourier series for the given function.

Example 1.10. Find the Fourier series of the function $f(x) = e^{-x}$ on the interval $(-\pi, \pi)$.

Solution. First, use the complex Fourier series with coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1+in)x} dx = \frac{e^{\pi} e^{in\pi} - e^{-\pi} e^{-in\pi}}{2\pi(1+in)}.$$

Then, with $e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n$, we have $c_n = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{2\pi(1+in)}$ and

$$e^{-x} = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1+in}.$$

In the interval $(-\pi, \pi)$ this series converges to e^{-x} , and at points $x = \pm\pi$ its sum is $(e^{\pi} + e^{-\pi})/2$.

Reading Exercise. Apply Euler's formula and check that this series in real form becomes

$$e^{-x} = \frac{e^{\pi} - e^{-\pi}}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (\cos nx + n \sin nx) \right].$$

The same result can be obtained if we apply the real form of the Fourier series from the beginning.

1.9 Complex Generalized Fourier Series

We do not have to use sine and cosine functions in a Fourier series; any set of orthogonal functions will work as a basis for the expansion. Recall that

two complex functions of a real variable x , $\varphi(x)$ and $\psi(x)$, are said to be orthogonal on the interval $[a, b]$ (which can be an infinite interval) if

$$\int_a^b \varphi(x)\psi^*(x)dx = 0, \quad (1.64)$$

where $\psi^*(x)$ is the complex conjugate of $\psi(x)$.

Let us expand an arbitrary function $f(x)$ into a complex set of orthogonal functions $\{\varphi_n(x)\}$:

$$f(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_n\varphi_n(x) + \dots$$

Multiplying by $\varphi_n(x)$, then integrating and using the orthogonality condition, we obtain the coefficients

$$c_n = \frac{\int_a^b f(x)\varphi_n^*(x)dx}{\int_a^b \varphi_n(x)\varphi_n^*(x)dx} = \frac{1}{\lambda_n} \int_a^b f(x)\varphi_n^*(x)dx, \quad (1.65)$$

where

$$\lambda_n = \int_a^b |\varphi_n(x)|^2 dx$$

are real numbers.

If the set $\{\varphi_n(x)\}$ is normalized, $\lambda_n = 1$ and the previous formula becomes

$$c_n = \int_a^b f(x)\varphi_n^*(x)dx. \quad (1.66)$$

Using for $\{\varphi_n(x)\}$ the set of exponential functions

$$\dots, e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots, \quad (1.67)$$

which are orthogonal on $[-\pi, \pi]$, we have the expansion (1.56). Similarly, the set of functions orthogonal on $[-l, l]$

$$\dots, e^{-\frac{i2\pi x}{l}}, e^{-\frac{i\pi x}{l}}, 1, e^{\frac{i\pi x}{l}}, e^{\frac{i2\pi x}{l}}, \dots \quad (1.68)$$

corresponds to the expansion (1.58). Notice that for this set $\lambda_n = b - a$ and the functions $\{\varphi_n(x)/\lambda_n\}$ are normalized, that is, have norms equal to 1.

1.10 Fourier Series for Functions of Several Variables

In this section we extend the previous ideas to generate the Fourier series for functions of two variables, $f(x, y)$, which have period 2π in both the variables x and y . Analogous to the development of Equation (1.56), we write a double Fourier series for the function $f(x, y)$ as

$$f(x, y) = \sum_{n,m=-\infty}^{+\infty} \alpha_{nm} e^{i(n x + m y)} \quad (1.69)$$

in the domain $(D) = (-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$.

The coefficients α_{nm} can be obtained by multiplying Equation (1.69) by $e^{-i(n x + m y)}$ and integrating over the domain (D) , performing this integration for the series term by term. Because the e^{inx} form a complete set of orthogonal functions on $[-\pi, \pi]$ (and the same for e^{imy}), we obtain

$$\alpha_{nm} = \frac{1}{4\pi^2} \iint_{(D)} f(x, y) e^{-i(n x + m y)} dx dy \quad (n, m = 0, \pm 1, \pm 2, \dots). \quad (1.70)$$

Equations (1.69) and (1.70) give the Fourier series for $f(x, y)$ in complex form. For the real Fourier series, instead of Equation (1.69), we have

$$f(x, y) = \sum_{n,m=0}^{+\infty} [a_{nm} \cos nx \cos my + b_{nm} \cos nx \sin my + c_{nm} \sin nx \cos my + d_{nm} \sin nx \sin my], \quad (1.71)$$

where

$$\begin{aligned} a_{00} &= \frac{1}{4\pi^2} \iint_{(D)} f(x, y) dx dy, & a_{n0} &= \frac{1}{2\pi^2} \iint_{(D)} f(x, y) \cos nx dx dy, \\ a_{0m} &= \frac{1}{2\pi^2} \iint_{(D)} f(x, y) \cos my dx dy, & a_{nm} &= \frac{1}{\pi^2} \iint_{(D)} f(x, y) \cos nx \cos my dx dy, \\ b_{0m} &= \frac{1}{2\pi^2} \iint_{(D)} f(x, y) \sin my dx dy, & b_{nm} &= \frac{1}{\pi^2} \iint_{(D)} f(x, y) \cos nx \sin my dx dy, \\ c_{n0} &= \frac{1}{2\pi^2} \iint_{(D)} f(x, y) \sin nx dx dy, & c_{nm} &= \frac{1}{\pi^2} \iint_{(D)} f(x, y) \sin nx \cos my dx dy, \end{aligned} \quad (1.72)$$

and

$$d_{nm} = \frac{1}{\pi^2} \iint_{(D)} f(x, y) \sin nx \sin my dx dy \quad \text{for } n, m = 1, 2, 3, \dots$$

The double Fourier series for $f(x, y)$ converges at the point (x_0, y_0) if two conditions are valid:

1. The partial derivatives f'_x and f'_y exist and are finite in (D) ;
2. The second derivative, f''_{xy} , exists and is continuous in the vicinity of (x_0, y_0) .

1.11 Uniform Convergence of Fourier Series

Uniformly converging series have many important features distinguishing them from other series. For this reason, it is helpful to know whether the Fourier series of some given function $f(x)$ converges uniformly and what class of functions can be expanded in uniformly converging Fourier series. Theorem 1.4 provides practical criteria for determining uniform convergence.

Theorem 1.4. *The Fourier series expansion of $f(x)$ converges uniformly on $[-\pi, \pi]$ for functions $f(x)$ given on the interval $[-\pi, \pi]$ that satisfy the following conditions:*

1. *The functions are periodic (i.e., $f(-\pi) = f(\pi)$);*
2. *The function is continuous on $[-\pi, \pi]$;*
3. *The interval $[-\pi, \pi]$ can be resolved into a finite number of intervals such that $f(x)$ is monotone on each interval.*

The importance of the periodicity condition (1) and continuity (2) should be obvious. Condition (3) may be replaced by the requirement that $f'(x)$ be piecewise continuous on the interval $[-\pi, \pi]$. Here, as before, the standard interval $[-\pi, \pi]$ can be replaced by arbitrary intervals.

Proof: Denote the Fourier coefficients of $f'(x)$ as A_n and B_n . In this case we have for the coefficient A_0 ,

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) dx = f(\pi) - f(-\pi) = 0.$$

In other words, $A_0 = 0$ is a direct result of the periodicity of $f(x)$.

For the coefficients A_n , we may integrate by parts to obtain

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \frac{\sin nx}{n} \Big|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = -\frac{b_n}{n} (n = 1, 2, \dots).$$

Similarly, $B_n = \frac{a_n}{n}$ (here a_n and b_n are Fourier coefficients of function $f(x)$).

Using the identities

$$0 \leq \left(|A_n|^2 - \frac{1}{n} \right)^2 = A_n^2 - \frac{2}{n} |A_n| + \frac{1}{n^2}$$

and

$$0 \leq \left(|B_n|^2 - \frac{1}{n} \right)^2 = B_n^2 - \frac{2}{n} |B_n| + \frac{1}{n^2},$$

we can establish that

$$\frac{1}{n} |A_n| + \frac{1}{n} |B_n| \leq \frac{1}{n^2} + \frac{1}{2} (A_n^2 + B_n^2),$$

and hence

$$|a_n| + |b_n| \leq \frac{1}{n^2} + \frac{1}{2} (A_n^2 + B_n^2).$$

It is known that the numeric series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. The convergence of the series

$$\sum_{n=1}^{\infty} (A_n^2 + B_n^2)$$

can be established, for instance, from Bessel's inequality (defined later in Equation (1.92)) for the Fourier coefficients of $f'(x)$. Therefore,

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)$$

converges also. Next, notice that for all $-\pi \leq x \leq \pi$ we have

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n|.$$

From here, using Weierstrass's criterion (Section 1.2), the Fourier series of $f(x)$ converges uniformly on $[-\pi, \pi]$. $\square$

Very often the problem of term-by-term differentiation and integration of Fourier series arises. It is easy to give examples directly showing that such a differentiation can lead to incorrect results. For example, the Fourier series of $f(x) = x$ defined on $[-\pi, \pi]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

and converges to x on $(-\pi, \pi)$. The derivative of the given function is $f'(x) = 1$ on the interval $(-\pi, \pi)$, but term-by-term differentiation of the series gives

$$\sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx,$$

which does *not* converge.

The above problem does not occur for uniformly converging series because, as we know, they can be differentiated term by term. As an example, we consider $f(x) = x^2$ on the interval $[-\pi, \pi]$. From Theorem 1.4, we conclude that the corresponding Fourier series converges uniformly. The main difference with the previous example is that now the periodicity condition, $f(-\pi) = f(\pi)$, holds.

Reading Exercise. Find the Fourier series of x^2 on $[-\pi, \pi]$, differentiate the series term by term, and verify that the result is $f'(x) = 2x$.

Reading Exercise. Repeat the previous reading exercise for $f(x) = |x|$ on $[-1, 1]$.

In contrast to differentiation, Fourier series can be integrated term by term without strong restrictions on the convergence of the series.

Theorem 1.5. *The integral along any interval $[x_1, x_2]$, where $a \leq x_1 < x_2 \leq b$, of any function piecewise continuous on $[a, b]$ equals the term-by-term integration of the Fourier series of this function.*

Proof: Let us prove this theorem for the case of $f(x)$ defined on the interval $[-\pi, \pi]$, and $x_1 = 0, x_2 = x$. We assume $f(x)$ to be absolutely integrable on the interval $[-\pi, \pi]$; thus, its Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (1.73)$$

We next introduce the continuous function

$$F(x) = \int_0^x (f(x) - a_0/2)dx, \quad (1.74)$$

for which

$$F(\pi) = F(-\pi) = \int_{-\pi}^{\pi} f(x)dx - \pi a_0 = 0.$$

Further, $F'(x) = f(x) - a_0/2$ at every point where $f(x)$ is continuous, thus $F'(x)$ is piecewise continuous on $[-\pi, \pi]$. We conclude that $F(x)$ can be resolved into an absolutely converging Fourier series

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

When we proved Theorem 1.4 about absolute convergence, we found that

$$A_n = -\frac{b_n}{n} \quad \text{and} \quad B_n = \frac{a_n}{n}.$$

To find A_0 , set $x = 0$ in the previous Fourier series. As seen from Equation (1.74), $F(0) = 0$, which gives

$$\frac{A_0}{2} = -\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

With these values of A_0 , A_n , and B_n , the Fourier series for $F(x)$ may be written as

$$F(x) = \sum_{n=1}^{\infty} [a_n \sin nx + b_n(1 - \cos nx)]/n.$$

By using Equation (1.74), we have the result

$$\int_0^x f(x)dx = \int_0^x \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_0^x [a_n \cos nx + b_n \sin nx]dx. \quad (1.75)$$

A similar formula is valid for the interval $[x_1, x_2]$:

$$\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_{x_1}^{x_2} [a_n \cos nx + b_n \sin nx]dx. \quad (1.76) \quad \square$$

Notice that we have proved the possibility of term-by-term integration without the requirement that the Fourier series of $f(x)$ converges to $f(x)$. Thus, the resulting Equations (1.75) and (1.76) hold even if the Fourier series does not converge to $f(x)$ at this particular x (for example, in the case of a finite discontinuity at x).

Example 1.11. Find an expansion for the cosine series: $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Solution. Integrate the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad (0 < x < 2\pi),$$

term by term to obtain

$$\sum_{n=1}^{\infty} \frac{1 - \cos nx}{n^2} = \frac{\pi}{2}x - \frac{1}{4}x^2;$$

therefore,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = -\frac{\pi}{2}x + \frac{1}{4}x^2 + c \quad (0 \leq x \leq 2\pi),$$

where the constant c is defined as the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (1.77)$$

1.12 The Gibbs Phenomenon

In this section, we take a closer look at the behavior of the Fourier series of a function $f(x)$ near a point of discontinuity (a finite jump) of the function. At these points, the series cannot converge uniformly, and, in fact, partial sums exhibit defects in such cases.

Let us begin with an example. The Fourier series for the function

$$f(x) = \begin{cases} -\pi/2 & \text{if } -\pi < x < 0, \\ 0 & \text{if } x = 0, \pm\pi, \\ \pi/2 & \text{if } 0 < x < \pi \end{cases} \quad (1.78)$$

is

$$2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = 2 \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right].$$

This expansion gives (an odd) continuation of the function $f(x)$ from the interval $(-\pi/2, \pi/2)$ to the entire x -axis. Because of the periodicity, we can restrict the analysis to the interval $(0, \pi/2)$. We suggest that the reader plot partial sums with the program TrigSeries, as shown in Figure 12.1 (Instructions for using the program FourierSeries are given in Appendix E.) These sums, like the original function $f(x)$, have jumps at points $x = 0$ and $x = \pi$.

We may isolate these discontinuity points within infinitely small regions $[0, \varepsilon)$ and $(\pi - \varepsilon, \pi]$ so that on the rest of the interval, $[\varepsilon, \pi - \varepsilon]$, this series converges uniformly. In Figure 1.9, this corresponds to the fact that the graphs of the partial sums, for large enough n , are very close to the line $y = \pi/2$ along the interval $[\varepsilon, \pi - \varepsilon]$. Close to the points $x = 0$ and

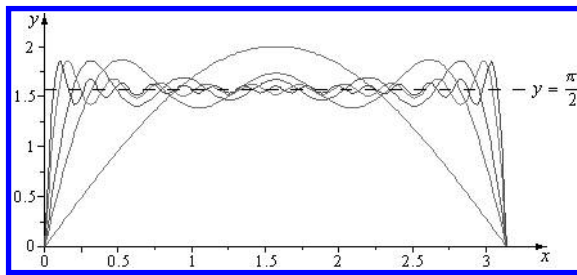


Figure 1.9. The first five partial sums of the expansion of Equation (1.78) demonstrating the Gibbs phenomenon.

$x = \pi$, it is clear that the uniformity of the approximation of $f(x)$ with partial sums is violated because of the jump discontinuity in $f(x)$.

Next, we point out another phenomenon that can be observed near the points $x = 0$ and $x = \pi$. Near $x = 0$, approaching the origin from the right, the graphs of the partial sums (shown in Figure 1.9) oscillate about the line $y = \pi/2$. The significant thing to note is that the amplitudes of these oscillations do not diminish to zero as $n \rightarrow \infty$. On the contrary, the height of the first bump (closest to $x = 0$) approaches the value of $\delta = 0.281$ above the $y = \pi/2$ line. This corresponds to an additional $\delta : (\pi/2) = 18\%$ of the height of the partial sum above the “expected” value. The situation is similar when x approaches the value π from the left. In general, if the function $f(x)$ has a finite jump $|D|$ at some point x , the maximum elevation of the partial sum value near x when $n \rightarrow \infty$ is bigger than $|D|$ by

$$2|D|\frac{\delta}{\pi}$$

that is, by about 18%. Such a defect of the convergence was first found by J. Gibbs and is known as the *Gibbs phenomenon*.

1.13 Completeness of a System of Trigonometric Functions

A simple question can be raised as to whether the system of orthogonal functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx \dots \quad (1.79)$$

is *complete* on the interval $[-\pi, \pi]$. A similar question can be asked about a system of sines or a system of cosines on the interval $[0, \pi]$. Rather than the standard interval $[-\pi, \pi]$, we may also wish to consider any interval of length 2π , or the interval $[-l, l]$, where, instead of the argument x in Equation (1.79), we have $\frac{n\pi x}{l}$, etc. The answer to these questions is positive. The system is complete, which means that when the function $f(x)$ is expanded into a trigonometric Fourier series, this series contains only the set of functions in Equation (1.79); no other functions need be included.

The following useful theorem is easy to prove for continuous functions.

Theorem 1.6. *If the function $f(x)$ is continuous on $[-\pi, \pi]$ and has Fourier coefficients all equal to zero, then this function is identically equal to zero.*

Proof: Using Theorem 1.5, we have

$$\int_0^x f(x)dx = \int_0^x \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_0^x [a_n \cos nx + b_n \sin nx]dx,$$

and from here,

$$\int_0^x f(x)dx = 0$$

for all x . Differentiating by x and recalling that $f(x)$ is continuous, we obtain the identity $f(x) = 0$. $\square$

A more interesting statement of the previous result is that, aside from a function that identically equals zero, no other continuous function exists on the interval $[-\pi, \pi]$ that can be orthogonal to all the functions in Equation (1.79). Lemma 1.7 states the previous conclusion more concisely.

Lemma 1.7. *The trigonometric system of functions is complete in the class of continuous functions.*

If two continuous functions have the same Fourier coefficients they are identically equal because their difference would have Fourier coefficients all equal to zero. Thus, a *continuous function is completely defined by its Fourier coefficients*. This is a second formulation of the completeness property.

If the function $f(x)$ is not equal to zero in a limited number of points, it is still orthogonal to all the functions in Equation (1.79) so that its coefficients are zero. Such a function, $f(x)$, can be said to be equivalent to zero. It can be shown that, besides functions that are equivalent to zero, no absolutely integrable function exists that is orthogonal to all functions in Equation (1.79) on the interval $[-\pi, \pi]$. It follows from this that an absolutely integrable function is completely defined by its Fourier coefficients.

1.14 General Systems of Functions: Parseval's Equality and Completeness

Consider a generalized Fourier series in which some function $f(x)$ is resolved into a complete set of orthogonal functions:

$$f(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_n\varphi_n(x) + \dots = \sum_{n=1}^{\infty} c_n\varphi_n(x). \quad (1.80)$$

Suppose that all functions $f(x)$ and $\varphi_n(x)$ are real and defined on the interval $[a, b]$ or on $(-\infty, \infty)$. We may square Equation (1.80) and integrate over the interval $[a, b]$ (or the entire axis). On the left side we have

$$\int_a^b f^2(x)dx,$$

and we assume this integral to be finite (converging). On the right side, due to the orthogonality of the functions $\{\varphi_n(x)\}$, the cross-integrals vanish and we have

$$\int_a^b f^2(x)dx = \sum_{n=1}^{\infty} c_n^2 \int_a^b \varphi_n^2(x)dx = \sum_{n=1}^{\infty} c_n^2 \lambda_n, \quad (1.81)$$

where λ_n are the *norms* of the functions,

$$\lambda_n = \int_a^b \varphi_n^2(x)dx.$$

Equation (1.81) is known as the *completeness equation*, or *Parseval's equality*. If this equation is satisfied for the integral of the square of any function $f(x)$, the set of functions $\{\varphi_n(x)\}$ is complete. Equation (1.81) is an extension of the Pythagorean Theorem to a space with an infinite number of dimensions: the square of the diagonal of an (infinite dimensional) parallelepiped is equal to the sum of the squares of all of its sides.

When the norms of the functions equal unity (i.e., $\lambda_n = 1$), Equation (1.81) has its simplest form:

$$\int_a^b f^2(x)dx = \sum_{n=1}^{\infty} c_n^2. \quad (1.82)$$

Note that it follows from these formulas that $c_n \rightarrow 0$ as $n \rightarrow \infty$.

For the trigonometric Fourier series on $[-\pi, \pi]$, Equation (1.81) becomes

$$\int_{-\pi}^{\pi} f^2(x) dx = \frac{1}{2} \pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2), \quad (1.83)$$

and for a series on the interval $[-l, l]$ we have

$$\int_{-l}^l f^2(x) dx = \frac{1}{2} l a_0^2 + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \quad (1.84)$$

The relationship between Equation (1.83) and the meaning of the completeness of the set of functions (1.80) should be clear from the discussion given in Sections 1.13 and 1.14.

1.15 Approximation of Functions in the Mean

Suppose we want to approximate a function $f(x)$ on the interval $[a, b]$ with a linear combination of a set of functions $\{\varphi_n(x)\}$ orthogonal on the interval $[a, b]$. The squares of the functions $\{\varphi_n(x)\}$, as well as the function $f(x)$, are assumed to be integrable. For an approximation, we can consider the mean deviation,

$$\delta = \frac{1}{b-a} \int_a^b |r(x)| dx, \quad (1.85)$$

or more commonly (because it is closer to the method of least squares), the total square deviation

$$\Delta = \int_a^b r^2(x) dx. \quad (1.86)$$

Here, $r(x) = f(x) - \sigma_n(x)$, and

$$\sigma_n(x) = \gamma_1 \varphi_1(x) + \gamma_2 \varphi_2(x) + \dots + \gamma_n \varphi_n(x) = \sum_{k=1}^n \gamma_k \varphi_k(x) \quad (1.87)$$

is a linear combination the first n functions from the infinite set $\{\varphi_n(x)\}$ (usually the contribution of the states decreases when $n \rightarrow \infty$). The goal is to find the combination in Equation (1.87) that gives the best total square deviation; that is, we want to minimize the value

$$\Delta_n = \int_a^b [f(x) - \sigma_n(x)]^2 dx. \quad (1.88)$$

With Equation (1.87) we have

$$\Delta_n = \int_a^b f^2(x) dx - 2 \sum_{m=0}^n \gamma_m \int_a^b f(x) \varphi_m(x) dx + \sum_{m=0}^n \gamma_m^2 \int_a^b \varphi_m^2(x) dx + 2 \sum_{k < m} \gamma_k \gamma_m \int_a^b \varphi_k(x) \varphi_m(x) dx.$$

The last term equals zero due to the orthogonality of the functions $\{\varphi_n(x)\}$.

By using the Fourier coefficients

$$c_m = \frac{1}{\lambda_m} \int_a^b f(x) \varphi_m(x) dx \quad \text{and} \quad \lambda_m = \int_a^b \varphi_m^2(x) dx,$$

we have

$$\Delta_n = \int_a^b f^2(x) dx - 2 \sum_{m=1}^n \lambda_m c_m \gamma_m + \sum_{m=1}^n \lambda_m \gamma_m^2.$$

Adding and subtracting $\lambda_m c_m^2$, we finally obtain

$$\Delta_n = \int_a^b f^2(x) dx - \sum_{m=1}^n \lambda_m c_m^2 + \sum_{m=1}^n \lambda_m (\gamma_m - c_m)^2.$$

It is clear that Δ_n takes its minimal value when the last sum becomes zero. This will occur for

$$\gamma_1 = c_1, \quad \gamma_2 = c_2, \quad \dots, \quad \gamma_n = c_n.$$

Thus, among all the possible combinations in Equation (1.87), a partial sum of a (generalized) Fourier series given by Equation (1.80) can be written as

$$S_n(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \dots + c_n \varphi_n(x) \quad (1.89)$$

and gives the smallest possible value of Δ_n defined as

$$\delta_n = \int_a^b [f(x) - S_n(x)]^2 dx = \int_a^b f^2(x) dx - \sum_{m=1}^n \lambda_m c_m^2. \quad (1.90)$$

Thus, the coefficients, c_n , of the Fourier series result in the optimal approximation in the mean.

From Equation (1.90), it follows that

$$\sum_{m=1}^n \lambda_m c_m^2 \leq \int_a^b f^2(x) dx. \quad (1.91)$$

When $n \rightarrow \infty$, we obtain *Bessel's inequality*

$$\sum_{m=1}^{\infty} \lambda_m c_m^2 \leq \int_a^b f^2(x) dx. \quad (1.92)$$

As n increases, the values of δ_n decrease because additional negative terms appear in Equation (1.90). It is clear that the equation

$$\sum_{m=1}^{\infty} \lambda_m c_m^2 = \int_a^b f^2(x) dx \quad (1.93)$$

embodies the criterion that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. We also recognize in Equation (1.93) a statement of Parseval's formula (or the completeness equation) given in equation (1.81).

Reading Exercise. We leave it to the reader to repeat the discussion presented in this section for a complete set of complex orthogonal functions and to prove that Equation (1.93) in this case becomes

$$\int_a^b |f(x)|^2 dx = \sum_{m=1}^{\infty} |c_m|^2 \int_a^b |\varphi_m|^2(x) dx. \quad (1.94)$$

Now, as an example, let us apply these considerations to the trigonometric system $\{\cos mx, \sin mx\}$ on $[-\pi, \pi]$. The partial sum of the Fourier series is

$$S_n(x) = \frac{a_0}{2} + \sum_{m=1}^n a_m \cos mx + b_m \sin mx \quad (1.95)$$

and

$$\delta_n = \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi} f^2(x) dx - \pi \left\{ \frac{a_0^2}{2} + \sum_{m=1}^n (a_m^2 + b_m^2) \right\}. \quad (1.96)$$

Bessel's inequality now is

$$\frac{a_0^2}{2} + \sum_{m=1}^n (a_m^2 + b_m^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx. \quad (1.97)$$

The following theorem was proved rigorously by A. M. Lyapunov:

Theorem 1.8. *For trigonometric sets of functions, the condition*

$$\lim_{n \rightarrow \infty} \delta_n = 0 \quad (1.98)$$

for the integral of the square of the difference of the partial sum and the function represented by the partial sum leads to the completeness equation

$$\frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx. \quad (1.99)$$

Versions of Parseval's equality occur frequently in physics and engineering. If $f(x)$ is a wave function of a quantum system, then $|c_n|^2$ in Equation (1.93) gives the probability of finding the system in the n th state. Thus the total probability

$$\int_a^b |f(x)|^2 dx$$

is equal to a sum of probabilities for all n states. In a second example, if some wave field (for example electromagnetic waves) is expanded into harmonics, Parseval's equality implies that the total energy of the wave is

equal to the sum of the energies associated with the individual harmonics. As a further example, it follows from Parseval's equation for a current, $I(t)$, expanded in a Fourier series that the heat dissipated in a resistor, $I^2 R$, is the sum of the energies (heat) dissipated in each harmonic.

Example 1.12. Apply Parseval's formula to find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution. First, notice that this series can be obtained by expanding the function $f(x) = x$ on the interval $(-1, 1)$. Let us use the complex form of the Fourier series. We leave it to the reader to check that Equations (1.58) and (1.59) give

$$f(x) = -\frac{i}{\pi} \left[(e^{i\pi x} - e^{-i\pi x}) - \frac{1}{2}(e^{2i\pi x} - e^{-2i\pi x}) + \frac{1}{3}(e^{3i\pi x} - e^{-3i\pi x}) - \dots \right].$$

An application of Parseval's formula, Equation (1.93), with $\varphi_n(x) = e^{in\pi x/l}$, $l = 1$, $\lambda_n = 2$, yields

$$\int_{-1}^1 x^2 dx = \frac{2}{3} = \sum_{n=-\infty}^{\infty} \lambda_n |c_n|^2 = \frac{2}{\pi^2} \left(1 + 1 + \frac{1}{4} + \frac{1}{4} + \dots \right) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2};$$

thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

1.16 Fourier Series of Functions Given at Discrete Points

Very often we do not have an analytic expression for a function. Instead, we might have, for example, a function given by a table of values or by a graph from which we can get the values of the function at a collection of points. Sometimes in these cases a partial sum of a Fourier series may

$x_1 = 0$	$x_2 = 1/3$	$x_3 = 2/3$	$x_4 = 3/4$	$x_5 = 1$
$f(x_1) = 0$	$f(x_2) = 1/3$	$f(x_3) = 2/3$	$f(x_4) = 3/4$	$f(x_5) = 1$

Table 1.1.

be used to obtain an approximate analytical formula for the function defined on the discrete set of points. This approach is the basis behind many rendering algorithms used in computer graphics.

Suppose the values $f(x_1), f(x_2), \dots, f(x_m)$ of a function $f(x)$ are given at m points $x_1, x_2, \dots, x_m$. We will write down the Fourier series for this function on the interval (x_1, x_m) and this decomposition will have period $T = (x_m - x_1)$. For a linear interpolation between nearby nodes (using the method of trapezoids for unequally separated points x_i), we find the following formulas for the coefficients of the series, which approximate the function $f(x)$ given at m discrete points:

$$\begin{aligned}
 A_0 &= \frac{2}{T} \cdot \frac{1}{2} \sum_{i=1}^{m-1} (x_{i+1} - x_i) [f(x_i) + f(x_{i+1})], \\
 A_n &= \frac{2}{T} \cdot \frac{1}{2} \sum_{i=1}^{m-1} (x_{i+1} - x_i) \left[f(x_i) \cos \frac{2\pi n x_i}{T} + f(x_{i+1}) \cos \frac{2\pi n x_{i+1}}{T} \right], \\
 B_n &= \frac{2}{T} \cdot \frac{1}{2} \sum_{i=1}^{m-1} (x_{i+1} - x_i) \left[f(x_i) \sin \frac{2\pi n x_i}{T} + f(x_{i+1}) \sin \frac{2\pi n x_{i+1}}{T} \right].
 \end{aligned} \tag{1.100}$$

Reading Exercise. Compare the expansion of the function defined by $f(x) = x$ on the interval $[0, 1]$ with $T = 1$ and the same function defined by the values shown in Table 1.1.

This function is defined on the interval $[0, 1]$ with $T = 1$.

Reading Exercise. Suppose that the partial sum

$$S_m(x) = \frac{A_0}{2} + \sum_{n=1}^m A_n \cos \frac{2\pi n x}{T} + B_n \sin \frac{2\pi n x}{T} \tag{1.101}$$

of the series poorly describes the function defined by the values in the table. Try to improve the description by increasing the number of points x_i in the table.

Note that the approximate equality $f(x) \approx S_m(x)$ does not improve with a larger number of points, m . In fact, the results become worse,

even when the difference between coefficients of the analytical function $f(x) = x$ and the function defined in the table is very small. This is related to the difficulty that the summation of a Fourier series with approximate coefficients does not work well when small changes in the coefficients cause large variations in the partial sum. Often we encounter such a difficulty as the result of limited accuracy of the measurement of the values $f(x_i)$. Also, it is clear that when the number of points is not large, using a large number of harmonics does not improve the approximation.

1.17 Solution of Differential Equations by Using Fourier Series

Fourier series expansions provide a means for solving many types of differential equations. Here we consider periodic solutions of an arbitrary linear differential equation with derivatives to order m given by

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + a_2 y^{(n-2)}(x) + \dots + a_n y(x) = g(x), \quad (1.102)$$

where the $a_i = \text{const}$, and $g(x + 2\pi) = g(x)$ is a given periodic function with period 2π .

We may search for the solution to this equation in the form of the Fourier series given by

$$y(x) = \sum_{k=-\infty}^{\infty} y_k e^{ikx}. \quad (1.103)$$

The periodic function $g(x)$ can also be represented by its Fourier series,

$$g(x) = \sum_{k=-\infty}^{\infty} g_k e^{ikx}. \quad (1.104)$$

First we substitute Equations (1.103) and (1.104) into Equation (1.102) to obtain

$$\sum_{k=-\infty}^{\infty} [a_0 (ik)^n + a_1 (ik)^{n-1} + \dots + a_{n-1} ik] y_k e^{ikx} = \sum_{k=-\infty}^{\infty} g_k e^{ikx} \quad (k = 0, \pm 1, \pm 2, \dots).$$

From here we obtain for the coefficients y_k

$$y_k = \frac{g_k}{a_0(ik)^n + a_1(ik)^{n-1} + \dots + a_{n-1}ik};$$

thus, for $y(x)$ we have the Fourier series

$$y(x) = \sum_{k=-\infty}^{\infty} \frac{g_k}{a_0(ik)^n + a_1(ik)^{n-1} + \dots + a_{n-1}ik} e^{ikx}. \quad (1.105)$$

From this, we see that the solution to the differential equation (1.102) is represented as a sum of harmonics with periods $2\pi/k$.

Similarly, we can obtain the solution using a real Fourier series for $y(x)$,

$$y(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$

and for $g(x)$,

$$g(x) = \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} \tilde{a}_k \cos kx + \tilde{b}_k \sin kx.$$

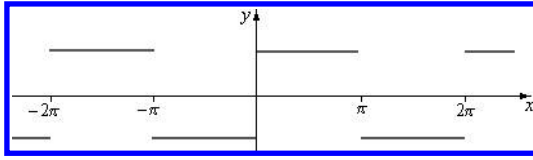
Reading Exercise. Following a line of reasoning similar to the one presented above, develop periodic solutions of Equation (1.102) when the function $g(x)$: (1) has an arbitrary period $2l$; (2) is defined on $[0, l]$ and extended to the entire axis as an even function; and (3) is defined on $[0, l]$ and extended to the entire axis as an odd function.

Example 1.13. Solve the equation $y'' + 4y' = f(x)$ with $f(x) = \operatorname{sgn} x$, $x \in (-\pi, \pi)$.

Solution. The extension of the function $\operatorname{sgn} x$ given initially on $(-\pi, \pi)$ to the x -axis makes it a periodic function with period $T = 2\pi$, defined as

$$\operatorname{sgn} x = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1},$$

as shown in Figure 1.10.

Figure 1.10. Periodic function $f(x) = \operatorname{sgn} x$, $x \in (-\pi, \pi)$.

Using this result, the differential equation becomes

$$y'' + 4y' = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Let us search for the solution, $y(x)$, as a real Fourier series. Substituting the series for $y(x)$ into the expression $y'' + 4y'$, we have

$$\begin{aligned} y'' + 4y' &= \sum_{k=1}^{\infty} [-k^2 a_k \cos kx - k^2 b_k \sin kx + 4b_k \cos kx - 4a_k k \sin kx] \\ &= \sum_{k=1}^{\infty} [(-k^2 a_k + 4k b_k) \cos kx + (-k^2 b_k + 4k a_k) \sin kx]. \end{aligned}$$

Because the expansion is unique we can equate the coefficients of sines and cosines in this equation with the coefficients for the expansion of $\operatorname{sgn} x$, which yields

$$-k^2 a_k + 4k b_k = 0,$$

from which we have $a_k = \frac{4b_k}{k}$, and

$$-k^2 b_k + 4k a_k = \begin{cases} 0, & k = 2q, \\ \frac{4}{\pi(2p-1)}, & k = 2p-1, p = 1, 2, \dots \end{cases}$$

Using the expression $a_k = \frac{4b_k}{k}$, we have for $k = 2p-1$

$$-k^2 b_k - 16b_k = \frac{4}{\pi(2p-1)}.$$

Thus,

$$\begin{aligned} b_k &= \begin{cases} -\frac{4}{\pi(2p-1)(k^2+16)}, & k = 2p-1, \\ 0, & k = 2p, \end{cases} \\ a_{2k-1} &= -\frac{16}{\pi(2k-1)^2 [(2k-1) + 16]}. \end{aligned}$$

Using these results, the solution to the differential equation becomes

$$\begin{aligned} y &= -\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{4 \cos(2k-1)x + (2k-1) \sin(2k-1)x}{(2k-1)^2 [(2k-1)^2 + 16]} \\ &= -\frac{4}{\pi} \left[\frac{4 \cos x}{17} + \frac{\sin x}{17} + \frac{4 \cos 3x}{9 \cdot 25} + \frac{\sin 3x}{3 \cdot 25} + \dots \right]. \end{aligned}$$

Let us consider the application of the Fourier series method to the common problem of forced oscillations (physical examples might include mechanical systems such as a body attached to a string, electrical circuits driven by an oscillating voltage, etc.). The linear differential equation

$$L(y) \equiv y'' + 2\lambda y' + \omega_0^2 y = f(t) \quad (1.106)$$

describes such forced oscillations, damped due to some form of dissipation (for instance, by viscous friction in a mechanical system). In this equation, $f(t)$ is a periodic external force with period T , ω_0 is the frequency of free oscillations obeying the equation $y'' + \omega_0^2 y = 0$, and λ is the damping coefficient. The values $y(0)$ and $y'(0)$ define the initial position and velocity of the system (if we are talking about a mechanical system), respectively.

The solution to this problem can be expressed as the sum of functions

$$y(t) = \bar{y}(t) + C_1 V_1(t) + C_2 V_2(t), \quad (1.107)$$

where $\bar{y}(t)$ is a particular solution to the nonhomogenous problem and $V_1(t)$, $V_2(t)$ are the fundamental functions for the respective homogeneous equation $L(y) \equiv 0$. The initial conditions $y(0)$ and $y'(0)$ determine the constants C_1 and C_2 .

It is easy to check that:

(a) If $\lambda < \omega_0$, then

$$V_1(t) = e^{-\lambda t} \cos \tilde{\omega} t, \quad V_2(t) = e^{-\lambda t} \sin \tilde{\omega} t.$$

(b) If $\lambda = \omega_0$, then

$$V_1(t) = e^{-\lambda t}, \quad V_2(t) = t \cdot e^{-\lambda t}. \quad (1.108)$$

(c) If $\lambda > \omega_0$, then

$$V_1(t) = e^{k_1 t}, \quad V_2(t) = e^{k_2 t}.$$

Here,

$$\begin{aligned}\tilde{\omega} &= \sqrt{\omega_0^2 - \lambda^2}, \\ k_1 &= -\lambda - \sqrt{\lambda^2 - \omega_0^2}, \\ k_2 &= -\lambda + \sqrt{\lambda^2 - \omega_0^2}.\end{aligned}$$

Reading Exercise. Expand the external force, $f(t)$, in a Fourier series with period T to get

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t] = \sum_{n=0}^{\infty} c_n \cos(n\omega t + \alpha_n), \quad (1.109)$$

where $\omega = 2\pi/T$, $c_0 = a_0/2$, $c_n = \sqrt{a_n^2 + b_n^2}$, $\alpha_0 = 0$, and $\alpha_n = -\arctan(b_n/a_n)$, and prove that the particular solution of the nonhomogeneous equation is

$$\bar{y}(t) = \sum_{n=0}^{\infty} y_n(t) = \sum_{n=0}^{\infty} d_n \cos(\omega_n t + \delta_n), \quad (1.110)$$

where

$$d_n = \frac{\sqrt{a_n^2 + b_n^2}}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + (2\lambda\omega_n)^2}}, \quad \delta_n = \arctan\left(\frac{2\lambda\omega_n}{\omega_n^2 - \omega_0^2}\right), \quad \omega_n = n\omega.$$

1.18 Fourier Transforms

A Fourier series is a representation of a function that uses a *discrete* system of orthogonal functions. This idea may be expanded to a *continuous* set of orthogonal functions. The corresponding expansion in this case is referred to as a *Fourier transform*. We start with the complex form of the Fourier series for function $f(x)$ on the interval $[-l, l]$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}, \quad (1.111)$$

with coefficients

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx \quad (n = 0, \pm 1, \pm 2, \dots). \quad (1.112)$$

In physics terminology, Equation (1.111) resolves function $f(x)$ in a discrete spectrum with wave numbers $k_n = \frac{n\pi}{l}$. Here, $c_n e^{ik_n x}$ is a harmonic with complex amplitude c_n defined by Equation (1.112), or

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-ik_n x} dx. \quad (1.113)$$

Suppose now that l is very large; thus, the distance between two neighboring wave numbers, $\Delta k = \frac{\pi}{l}$, is very small. Using the notation

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad (1.114)$$

we may write Equation (1.113) in the form

$$c_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ik_n x} dx \cdot \frac{\pi}{l} = \hat{f}(k_n) \Delta k. \quad (1.115)$$

By using this definition, Equation (1.111) can be written as

$$f(x) = \sum_n c_n e^{ik_n x} = \frac{1}{2\pi} \sum_n \hat{f}(k_n) e^{ik_n x} \Delta k \quad (-l < x < l). \quad (1.116)$$

In the limit $l \rightarrow \infty$, this becomes the integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \quad (-\infty < x < \infty). \quad (1.117)$$

In this limit, the wave number takes all the values from $-\infty$ to ∞ (i.e., when $l \rightarrow \infty$, the spectrum is continuous). The amplitudes are distributed

continuously and for each infinitesimal interval from k to $k + dk$, there is an infinitesimal amplitude

$$dc = \frac{1}{2\pi} \hat{f}(k) dk. \quad (1.118)$$

With this equation as a definition, $\hat{f}(k)$ is called the *spectral density* of $f(x)$.

Equations (1.114) and (1.117) define the *Fourier transform*. Equation (1.114) is called the direct Fourier transform, and Equation (1.117) is referred to as the inverse Fourier transform. These formulas are valid if the function $f(x)$ is absolutely integrable on $(-\infty, \infty)$, or

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \quad (1.119)$$

Note that there are different ways to deal with the factor $1/2\pi$ in the formulas for direct and inverse transforms. Often, this factor is placed in the direct transform formula, whereas other authors split this factor into two identical factors, $1/\sqrt{2\pi}$, one in each equation. Using the definition given by Equations (1.114) and (1.117) has the advantage that the Fourier transform of the Dirac delta function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \quad (1.120)$$

equals 1, as can be seen by comparing Equations (1.117) and (1.120). Here we remind the reader that the most useful property of the delta function is

$$\int_{-\infty}^{\infty} f(x') \delta(x - x') dx' = f(x). \quad (1.121)$$

The delta function defined with the coefficient in Equation (1.120) obeys the normalization condition

$$\int_{-\infty}^{\infty} \delta(x - x') dx' = 1. \quad (1.122)$$

The step or Heaviside function, defined as

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

is related to the delta function by the relation

$$\frac{d}{dx}H(x) = \delta(x). \quad (1.123)$$

Reading Exercise. Prove the following two properties of delta function:

$$\delta(-x) = \delta(x) \quad \text{and} \quad \delta(ax) = \frac{1}{|a|}\delta(x). \quad (1.124)$$

For many practical applications, it is useful to present Fourier transform formulas for another pair of physical variables, time and frequency. Using Equations (1.114) and (1.117), we may write the direct and inverse transforms, respectively, as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \quad \text{and} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega. \quad (1.125)$$

Fourier transform equations are easy to generalize to cases of higher dimensions. For instance, for an application with spatial variables represented as vectors, Equations (1.114) and (1.117) become, respectively,

$$\hat{f}(\vec{k}) = \int_{-\infty}^{\infty} f(\vec{x})e^{-i\vec{k}\vec{x}} d\vec{x} \quad \text{and} \quad f(\vec{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\vec{k})e^{i\vec{k}\vec{x}} d\vec{k}. \quad (1.126)$$

Next, we briefly discuss *Fourier transforms of even or odd functions*. If the function $f(x)$ is even, we have

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \cos kx dx - i \int_{-\infty}^{\infty} f(x) \sin kx dx = 2 \int_0^{\infty} f(x) \cos kx dx.$$

From here we see that $\hat{f}(k)$ is also even, and with Equation (1.117) we obtain

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \hat{f}(k) \cos kx dk. \quad (1.127)$$

These formulas give what is known as the *Fourier cosine transform*. Similarly, if $f(x)$ is odd we obtain the *Fourier sine transform*,

$$i\hat{f}(k) = 2 \int_0^{\infty} f(x) \sin kx dx, \quad f(x) = \frac{1}{\pi} \int_0^{\infty} i\hat{f}(k) \sin kx dk. \quad (1.128)$$

In this case, usually $i\hat{f}(k)$ (rather than $\hat{f}(k)$) is called the Fourier transform. We leave it to the reader to obtain Equations (1.128) as a reading exercise.

If the function $f(x)$ is given on the interval $0 < x < \infty$, it can be extended to $-\infty < x < 0$ in either an even or odd way and we may use either sine or cosine transforms.

Example 1.14. Let $f(x) = 1$ on $-1 < x < 1$ and 0 outside this interval. This is an even function; thus, with the cosine Fourier transform we have

$$\hat{f}(k) = 2 \left(\int_0^1 1 \cdot \cos kx dx + \int_1^{\infty} 0 \cdot \cos kx dx \right) = \frac{2 \sin k}{k}.$$

The inverse transform gives

$$f(x) = 2 \int_0^{\infty} \frac{\sin k}{\pi k} \cos kx dk. \quad (1.129)$$

As in the case for the regular Fourier series, if we substitute some value, x_0 , into the formula for the inverse transform, we obtain the value $f(x_0)$ at the point where this function is continuous. The equation

$$[f(x_0 + 0) + f(x_0 - 0)]/2$$

gives the value at a point where it has a finite discontinuity. For instance, substituting $x = 0$ in (1.129) gives

$$1 = 2 \int_0^{\infty} \frac{\sin k}{\pi k} dk,$$

from which we obtain the interesting result

$$\int_0^{\infty} \frac{\sin k}{k} dk = \frac{\pi}{2}. \quad (1.130)$$

1.19 The Fourier Integral

In this section, we briefly discuss the famous Fourier integral formula. Substituting Equation (1.114) into (1.117), we obtain the representation of the function $f(x)$ as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} dz \int_{-\infty}^{\infty} f(u) e^{izu} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} f(u) e^{iz(u-x)} du, \quad (1.131)$$

which is called the *Fourier integral* of the function $f(x)$.

There are other ways to write the Fourier integral. The real part of this formula can be written in the form

$$f(x) = \frac{1}{\pi} \int_0^{\infty} dz \int_{-\infty}^{\infty} f(u) \cos z(u-x) du. \quad (1.132)$$

Using the identity for the cosine of a difference, we may write Equation (1.132) as

$$f(x) = \int_0^{\infty} [a(z) \cos zx + b(z) \sin zx] dz, \quad (1.133)$$

where

$$a(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos zu du \quad \text{and} \quad b(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin zu du. \quad (1.134)$$

Equations (1.133) and (1.134) are very similar to formulas for the Fourier series. In the present case, the discrete number n is replaced by the continuous parameter z , and the series is replaced by the integral in Equation (1.133). Equation (1.133) also gives the representation of the function $f(x)$ using the Fourier integral. In contrast to the Fourier series, which gives a function written as a discrete spectrum with frequencies $n\pi/l$ depending on the integer number n (harmonic number), the Fourier integral represents a function as a continuum of harmonics with frequencies distributed continuously from zero to infinity.

For an even function, where $f(-x) = f(x)$, the coefficient $b(z) = 0$, and Equations (1.133) and (1.134) may be written as

$$f(x) = \int_0^{\infty} a(z) \cos zx dz, \quad a(z) = \frac{2}{\pi} \int_0^{\infty} f(u) \cos zu du,$$

which gives

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos zx dz \int_0^{\infty} f(u) \cos zu du. \quad (1.135)$$

For an odd function, where $f(-x) = -f(x)$, the coefficient $a(z) = 0$, and Equations (1.133) and (1.134) yield

$$f(x) = \int_0^{\infty} b(z) \sin zx dz, \quad b(z) = \frac{2}{\pi} \int_0^{\infty} f(u) \sin zu du,$$

which gives

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin zx dz \int_0^{\infty} f(u) \sin zu du. \quad (1.136)$$

As was the case for the Fourier series, the Fourier integral of the function $f(x)$ converges to this function everywhere except at points of discontinuity where, as for the Fourier series, it equals

$$\frac{f(x_0 + 0) + f(x_0 - 0)}{2}.$$

Even or odd functions as well as functions given on a semiaxis can be subjects of *Fourier cosine or sine transforms*. We already discussed them in Section 1.18 and present them below in a slightly different form, using the formulas of the current section. Most commonly, cosine and sine transforms are written in the form

$$\left. \begin{aligned} \hat{f}_c(z) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos zu du \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(z) \cos xz dz \end{aligned} \right\} \quad (1.137)$$

$$\left. \begin{aligned} \hat{f}_s(z) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin zu du \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(z) \sin xz dz \end{aligned} \right\} \quad (1.138)$$

In Equations (1.137) and (1.138), we use the most common choice of coefficients, $\sqrt{2/\pi}$. For sine and cosine Fourier transforms, this corresponds to the factor $1/\sqrt{2\pi}$ in both $f(x)$ and $\hat{f}(z)$. Also, we have used the variable z in place of k in the equations of this section. In Equations (1.137) and (1.138), the variable x is commonly used instead of u .

For *even* functions $f(x)$, we have

$$f(z) = \hat{f}_c(z)$$

(for $z < 0$, function $\hat{f}_c(z)$ extends as *even*). For *odd* $f(x)$, we have

$$f(z) = i\hat{f}_s(z)$$

(for $z < 0$, function $\hat{f}_s(z)$ extends as *odd*).

In general, instead of $f(x)$ we can work with even or odd functions defined by

$$g(x) = \frac{f(x) + f(-x)}{2}, \quad h(x) = \frac{f(x) - f(-x)}{2},$$

respectively, and obtain the Fourier transform of $f(x)$ from the Fourier transforms of $g(x)$ and $f(x)$:

$$\hat{f}(z) = \hat{g}_c(z) + i\hat{h}_s(z).$$

Example 1.15. Find the cosine and sine Fourier transforms of $f(x) = e^{-ax}$ ($a > 0, x \geq 0$).

Solution. The coefficients are

$$\begin{aligned} \hat{f}_c(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-az} \cos xz dz = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2}, \\ \hat{f}_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-az} \sin xz dz = \sqrt{\frac{2}{\pi}} \frac{x}{a^2 + x^2}. \end{aligned}$$

The inverse transforms are

$$\frac{2a}{\pi} \int_0^{\infty} \frac{\cos zx}{a^2 + z^2} dz = e^{-ax} \quad (x \geq 0),$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{z \sin zx}{a^2 + z^2} dz = e^{-ax} \quad (x > 0).$$

From here, we obtain two well-known integrals (Laplace's integrals):

$$\int_0^{\infty} \frac{\cos xz}{a^2 + z^2} dz = \frac{\pi}{2a} e^{-ax} \quad \text{and} \quad \int_0^{\infty} \frac{z \sin xz}{a^2 + z^2} dz = \frac{\pi}{2} e^{-ax}.$$

Example 1.16. Find the cosine transform of the function $f(x) = e^{-x^2/2}$.

Solution.

$$\hat{f}_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-z^2/2} \cos zx dz = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} e^{-x^2/2} = e^{-x^2/2},$$

thus, the cosine transform of the function $e^{-x^2/2}$ is equal to the original function.

Reading Exercise. Find the sine transform of $f(x)$.

Differentiating the formula for $\hat{f}_c(x)$ with respect to x we find that the sine transform of $f(x) = xe^{-x^2/2}$ is equal to $f(x)$.

Example 1.17. Investigate the connection between a Gaussian function and its Fourier transform.

Solution. The Fourier transform of a Gaussian, $f(x) = e^{-ax^2}$, $a > 0$, is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx.$$

Since

$$\hat{f}'(k) = \int_{-\infty}^{\infty} (-ix)e^{-ax^2} e^{-ikx} dx = \frac{i}{2a} \int_{-\infty}^{\infty} \frac{d}{dx}(e^{-ax^2}) e^{-ikx} dx = -\frac{k}{2a} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = -\frac{k}{2a} \hat{f}(k),$$

$\hat{f}(k)$ can be obtained as a solution of a simple differential equation (by separation of variables):

$$\hat{f}(k) = \hat{f}(0)e^{-k^2/4a}.$$

Here,

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-ax^2} dx.$$

With the substitution $z = x\sqrt{a}$, we obtain

$$\hat{f}(0) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-z^2} dz.$$

Because it is well known that

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi},$$

we have that $\hat{f}(0) = \sqrt{\frac{\pi}{a}}$ and $\hat{f}(k) = \sqrt{\frac{\pi}{a}} e^{-k^2/4a}$. Thus we have obtained a remarkable result: the Fourier transform of a Gaussian is also a Gaussian. Both functions are bell-shaped and their widths are determined by the value of a . If a is small, then $f(x)$ is a broadly spread Gaussian and its Fourier transform is sharply peaked near $k = 0$. Conversely, if $f(x)$ is a narrowly peaked Gaussian function corresponding to a being large, its Fourier transform is broadly spread. An application of this result is the uncertainty principle in quantum mechanics, in which momentum and position probabilities may be represented as Fourier transforms of each other with the result that a narrow uncertainty in the probable location of an object results in a broad uncertainty in the momentum of the object and vice versa.

Example 1.18. Represent the function

$$f(x) = \begin{cases} -e^x & \text{if } x \leq 0, \\ e^{-x} & \text{if } x > 0, \end{cases}$$

as a Fourier integral.

Solution. This function is odd, and we can use a Fourier representation in complex form, Equation (1.131), or in real form, Equation (1.136). Let us use the real form given in Equation (1.136). Performing the inner integral by parts, we obtain

$$\int_0^{\infty} e^{-u} \sin zu du = \frac{u}{1+u^2}, \text{ so that } f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{u \sin xu}{1+u^2} du.$$

In this Fourier integral, $x \neq 0$ because at $x = 0$ the result must be one-half of the limits of the given function when $x \rightarrow 0$ from both sides (i.e., $f(x) = 0$).

Example 1.19. Represent the function

$$f(x) = \begin{cases} 2 & \text{if } 0 < x < 3, \\ 1 & \text{if } x = 3, \\ 0 & \text{if } x > 3 \end{cases}$$

as a Fourier integral.

Solution. The function is given on the interval $(0, \infty)$ and can be extended to the interval $(-\infty, 0)$ as an even or odd function. In the first case, we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos zx dz \int_0^3 2 \cos zu du = \frac{4}{\pi} \int_0^{\infty} \frac{\cos zx \sin 3z}{z} dz.$$

In the second case, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin zx dz \int_0^3 2 \sin zu du = \frac{4}{\pi} \int_0^{\infty} \frac{(1 - \cos 3z) \sin zx}{z} dz.$$

Both Fourier integrals represent the function everywhere, including the discontinuity point $x = 3$, because at this point the given function takes the value $f(3) = 1$. This is the same as the half sum of the limits of the Fourier integral when $x \rightarrow 3$ from the left and right.

Example 1.20. Represent the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \pi x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x > 1 \end{cases}$$

as a Fourier integral.

Solution. One approach is to use Equations (1.133) and (1.134):

$$\begin{aligned} a(z) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos zu \, du = \int_0^1 u \cos zu \, du = \frac{z \sin z + \cos z - 1}{z^2}, \\ b(z) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin zu \, du = \int_0^1 u \sin zu \, du = \frac{\sin z - z \cos z}{z^2}, \end{aligned}$$

which gives the Fourier integral

$$f(x) = \int_0^{\infty} \frac{(z \sin z + \cos z - 1) \cos zx + (\sin z - z \cos z) \sin zx}{z^2} dz.$$

This integral converges to the original $f(x)$ for all x except $x = 1$, where it is equal to $\pi/2$.

The solution is shorter if we use the complex Fourier integral given in Equation (1.131):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} dz \int_{-\infty}^{\infty} f(u) e^{izu} du = \frac{1}{2} \int_{-\infty}^{\infty} e^{-izx} dz \int_0^1 u e^{izu} du = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iz}(1 - iz) - 1}{z^2} e^{-izx} dz.$$

This solution, if necessary, can be converted to the previous one with Euler formulas.

Reading Exercise. Here are several important properties of the Fourier transform, proofs of which we leave to the reader.

1. Prove that the Fourier transform of $f(-x)$ is equal to $\hat{f}(-k)$.
2. Prove that the Fourier transform of $\hat{f}(k)$ is equal to $2\pi f(-k)$.
3. Prove that the Fourier transform of $f'(x)$ is equal to $ik\hat{f}(k)$.

Hint. The Fourier transform of $f'(x)$ is

$$\int_{-\infty}^{\infty} f'(x)e^{-ikx} dx;$$

differentiate it by parts and take into account that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

4. Prove that the Fourier transform of $f(x - x_0)$ (a shift of origin) is equal to $e^{-ikx_0}\hat{f}(k)$.
5. Prove that the Fourier transform of $f(\alpha x)$ (where α is constant) is equal to $\frac{1}{\alpha}\hat{f}(\frac{k}{\alpha})$. This property shows that if we stretch the size of an “object” along the x -axis, then the size of Fourier “image” compresses by the same factor. This means that it is not possible to localize a function in both “ x - and k -spaces,” which is a mathematical expression representing the *uncertainty principle* in quantum mechanics.
6. Prove Parseval’s equality for the Fourier transform

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk.$$

Hint. Apply the formula

$$\int_{-l}^l |f(x)|^2 dx = \sum_{n=1}^{\infty} |c_n|^2 \int_{-l}^l |\varphi_n|^2(x) dx$$

for $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{l}}$, which gives

$$\int_{-l}^l |f(x)|^2 dx = 2l \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Then, by using Equation (1.115), obtain

$$\int_{-l}^l |f(x)|^2 dx = 2l \sum_{n=-\infty}^{\infty} |\hat{f}(k_n)|^2 (\Delta k)^2 = 2\pi \sum_n |\hat{f}(k_n)|^2 \Delta k.$$

Taking the limit as $l \rightarrow \infty$, we obtain the above result.

Problems

1.1. Let the function $f(x) = x$ be defined on the interval $[0, l]$. Expand the function $f(x) = x$ in a trigonometric Fourier series by using

1. the general series;
2. the cosine series;
3. the sine series.

Study the behavior of individual harmonics of these series and their partial sums with the help option "Choose Terms to Be Included" in the program TrigSeries. Using the program TrigSeries, build the histogram of the squares of the amplitudes A_n^2 , and graph the periodic continuation on the interval $[-1, 3]$.

Answer.

1. Expansion using the general series yields

$$y = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{2\pi n}{l} x$$

$$a_0 = \frac{1}{2}, \quad a_n = 0, \quad b_n = -\frac{1}{n\pi} \quad (n = 1, 2, \dots).$$

2. The cosine series expansion gives

$$y = \frac{1}{2} - \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi}{l} x$$

$$a_0 = \frac{1}{2}, \quad a_n = -\frac{2l}{(n\pi)^2} [1 - (-1)^n], \quad b_n = 0 \quad (n = 1, 2, \dots).$$

3. Expansion in a series of sine functions yields

$$y = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{l} x$$

$$a_n = 0, \quad b_n = (-1)^{n+1} \frac{2l}{n\pi} \quad (n = 1, 2, \dots).$$

1.2. Find the Fourier sine series for the function $f(x) = x^2$ on the interval $(0, \pi)$.

Answer.

$$x^2 = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_{2k} = -\frac{\pi}{k}, \quad b_{2k-1} = \frac{2\pi}{2k-1} - \frac{8}{\pi(2k-1)^3}.$$

Plot the graphs of partial sums on different intervals.

1.3. Find the general Fourier series for the function $f(x) = x^2$ given on the interval $[-1, 1]$.

1.4. Expand $f(x) = e^{ax}$ into; (1) a cosine on $[0, \pi]$; and (2) a sine Fourier series on the interval $(0, \pi)$.

Answer.

$$1. \quad e^{ax} = \frac{e^{a\pi} - 1}{a\pi} + \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{a\pi} - 1}{a^2 + n^2} \cos nx \quad (0 \leq x \leq \pi).$$

$$2. \quad e^{ax} = \frac{2}{\pi} \sum_{n=1}^{\infty} [1 - (-1)^n e^{a\pi}] \frac{n}{a^2 + n^2} \sin nx \quad (0 < x < \pi).$$

1.5. Find the Fourier series with period $T = 2\pi$ for the function $f(x) = x^3$ defined on the interval $[-\pi, \pi]$.

1.6. Find the Fourier series with period $T = 2\pi$ for the function $f(x) = \pi - 2x$ defined on the interval $[0, \pi]$.

1.7. Find the Fourier series with period $T = 2$ for the function $f(x) = x$ defined on the interval $[0, 1]$. Use an extension beyond the original interval.

1.8. Find the Fourier series with period $T = 2\pi$ for the function

$$f(x) = \begin{cases} -h, & -\pi \leq x \leq 0, \\ h, & 0 \leq x \leq \pi. \end{cases}$$

1.9. Find the Fourier series with period $T = 2\pi$ for the function

$$f(x) = \begin{cases} -3x, & -\pi \leq x \leq 0, \\ 4x, & 0 \leq x \leq \pi. \end{cases}$$

1.10. Find the Fourier series with period $T = 2\pi$ for the function

$$f(x) = \begin{cases} -x, & -\pi \leq x \leq 0, \\ 0, & 0 \leq x \leq \pi. \end{cases}$$

1.11. Find the Fourier series with period $T = 2$ for the function $f(x) = x$ defined on the interval $[0, 1]$. Use an extension beyond the original interval.

1.12. Find the Fourier series with period $T = 4$ for the function $f(x) = e^x$ defined on the interval $[-2, 2]$.

1.13. Extend the function $f(x) = 1 - x/2$ defined on the interval $[1, 4]$ to the entire x -axis by using sine and cosine Fourier series.

1.14. Find the Fourier series with period $T = 2$ of the function $f(x) = |x|$ defined on the interval $(-1, 1]$.

1.15. Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -3 < x \leq 0, \\ x, & 0 < x < 3. \end{cases}$$

Using the series obtained, find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} + \dots$$

1.16. Expand the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq h, \\ 0 & \text{if } h < x \leq \pi \end{cases}$$

in a cosine series on the interval $[0, \pi]$.

Answer.

$$f(x) = \frac{24}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin nh}{nh} \cos nx \right\}.$$

1.17. Expand the function

$$f(x) = \begin{cases} 1 - \frac{x}{2h} & \text{if } 0 \leq x \leq 2h, \\ 0 & \text{if } 2h < x \leq \pi \end{cases}$$

in a cosine series on the interval $[0, \pi]$.

Answer.

$$f(x) = \frac{24}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin nh}{nh} \right)^2 \cos nx \right\}.$$

1.18. Expand the function

$$f(x) = \begin{cases} \cos x & \text{if } 0 \leq x \leq \frac{\pi}{2}, \\ -\cos x & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$

in cosines on the interval $[0, \pi]$.

Answer.

$$f(x) = \frac{4}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos 2nx}{4n^2 - 1} \right\}.$$

1.19. Expand the function

$$f(x) = \begin{cases} -\frac{\pi}{2} & \text{if } -\pi < x < 0, \\ \frac{\pi}{2} & \text{if } 0 < x \leq \pi \end{cases}$$

in a Fourier series on the interval $[-\pi, \pi]$.

Answer.

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

1.20. Expand the function $f(x) = \operatorname{sgn} x$ in a Fourier series on the interval $[-\pi, \pi]$. (The $\operatorname{sgn} x$ function returns $+1$ for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$.)

Answer.

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

1.21. Let the function $f(x)$ be defined on the interval $[0, \pi]$.

1. If $f(\pi - x) = f(x)$, then for the cosine series show that $a_{2m-1} = 0$, and for the sine series that $b_{2m} = 0$.
2. If $f(\pi - x) = -f(x)$, then for the cosine series show that $a_{2m} = 0$ and for the sine series that $b_{2m-1} = 0$.

1.22. Let the function $f(x)$ be defined on the interval $[-\pi, \pi]$.

1. If $f(\pi + x) = f(x)$, show that $a_{2m-1} = b_{2m-1} = 0$.
2. If $f(x + \pi) = -f(x)$, show that $a_{2m} = 0$ and $b_{2m} = 0$.

1.23. Find the Fourier series of the function

$$f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0, \\ \pi - x, & 0 \leq x \leq \pi. \end{cases}$$

1.24. Find the Fourier cosine series of the function $f(x) = x$ on $[0, \pi]$. Repeat for the sine series. Compare and discuss the properties of these two series (see Example 1.5 in Section 1.7).

1.25. Let the function $g(x) = x$ be given on the interval $[a, b]$. Define some function $h(x)$ on the interval $[c, a]$, where $c = 2a - b$ (choose your own function $h(x)$ and the values of a , b , and c) and expand the composed function

$$f(x) = \begin{cases} h(x) & \text{if } c \leq x < a, \\ g(x) & \text{if } a \leq x < b \end{cases}$$

in a general trigonometric Fourier series. Repeat this problem for different functions.

1.26. Solve Problem 1.25 for several specific functions, $h(x)$, obeying the following conditions. Define the function $h(x)$ on $[c, a]$ so that $f(x)$ has continuous derivatives up to order $(m - 1)$ on interval $[c, b]$, the m th derivative is piecewise continuous, and $h(x)$ satisfies one of the following conditions at the ends of the interval $[c, a]$:

1. $h(a) \neq g(a)$ (i.e., a is a point of discontinuity of the function $f(x)$);
2. $h(a) = g(a)$, $h(c) \neq g(b)$;
3. $h(a) = g(a)$, $h(c) = g(b)$;

4. $h(a) = g(a)$, $h'(a) = g'(a)$, $\dots$, $h^{(m-1)}(a) = g^{(m-1)}(a)$, $h(c) = g(b)$, $h'(c) = g'(b)$, $\dots$, $h^{(m-1)}(c) = g^{(m-1)}(b)$ (in this case the Fourier coefficients of $f(x)$, a_k and b_k , converge more slowly than k^{-m}).

As a concrete example, let $a = 2$, $b = 3$, $c = 1$. Consider the following functions:

1. $h(x) = x + 1$;
2. $h(x) = x^2 - 2$;
3. $h(x) = (10 - x^2)/3$;
4. $h(x) = -8 + 25x - 18x^2 + 4x^3$ (in this case $m = 2$ and the Fourier coefficients a_k and b_k decrease more slowly than k^{-2}).

Expand these functions on the interval $[1,3]$ in a general trigonometric Fourier series. Using the program TrigSeries, build the histograms of the squares of amplitudes A_n^2 and graph the periodic continuations on the interval $[1,11]$. Tabulate this series on the interval $[2,3]$ and compare speeds of convergence. Try to define an $h(x)$ so that the coefficients a_k and b_k decrease as k^{-3} , k^{-4} , $\dots$ etc.

1.27. Perform a Fourier analysis of the periodic pulse function $f(x)$ defined on interval $[0, T]$. (A physical example might be where $f(x)$ corresponds to the sound pressure on the ear drum as a result of a series of hand claps.)

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \Delta x, \\ 0 & \text{if } \Delta x < x < T. \end{cases}$$

Here Δx is the duration of clap and $\omega_1 = 1/T$ is its frequency. Take $T = 1$, $\Delta x = 0.1$, $n = 30$ and use the general Fourier series expansion. Find coefficients of the series on the interval $[0, T]$.

For $\Delta x \ll T$, the main pitch with frequency ω_1 and the first several harmonics have nearly the same amplitudes, A_n . Check to see what happens as Δx increases. Plot a graph of the dependence of A_n on $\omega_n = n\omega_1$ up to the value n such that A_n has passed through 0 two times. Verify that the frequencies for which A_n has significant values occupy a band, $\Delta\omega$, in the region surrounding the main frequency ω_1 up to a frequency $(\Delta x)^{-1}$: $\omega = 0, \omega_1, 2\omega_1, 3\omega_1, \dots, \omega_{\max} = (\Delta x)^{-1}$; that is, $\Delta\omega \cdot \Delta x \approx 1$. This result is correct for any periodic function if the duration of a pulse satisfies the condition $\Delta x \ll T$.

1.28. Perform a Fourier analysis of a modulated signal, $f(t) = f_0(t) \cos \omega t$, with the amplitude $f_0(t)$ having a frequency $\Omega \ll \omega$. For example, $f_0(t) = \cos \Omega t$, $t \in [-\pi/2, \pi/2]$, $\omega = 50$, $\Omega = 1$. Plot the graph of the periodic continuation to the interval $t \in [-2\pi, 2\pi]$ and plot the histograms of the squares of amplitudes, A_n^2 .

Explain the appearance of the histograms. Try different forms of high frequency filling instead of $\cos \omega t$; for example, pulses in the form of a rectangle using the program TrigSeries and the pulse function Imp(a,b,x) defined in the program.

1.29. Let the function $f(x) = (\pi - x)/2$ be defined on the interval $[0, 2\pi]$. The Fourier series for this function was obtained earlier in Equation (1.48):

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

When $x = 0$ and $x = 2\pi$, the sum of the series equals 0 and the expansion fails.

Using the program TrigSeries, expand the function $f(x)$ in a Fourier series using $n = 10, 20$, and 30 terms. Check that in the vicinity of the ends of the interval, the Gibbs phenomenon occurs. Try to estimate the height of the first peak on the interval $[0, 1]$ of the partial sum $S_n(x)$ with help of the cursor. (It should be the case that $|S_n(x)| \rightarrow 1.85 \dots$ when $x \rightarrow 0$ and $n \rightarrow \infty$.)

1.30. Find sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

using Parseval's equality for the Fourier series of the function $f(x) = x^2$ on $-1/2 < x < 1/2$.

1.31. Find the sum of the series

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

using Parseval's equality for the Fourier series of the function $f(x) = |x|$ on $-\pi/2 < x < \pi/2$.

1.32. Suppose we have a function given by

$$f(x) = \begin{cases} \frac{1}{l^2}(x+l)^2, & x \leq 0, \\ \frac{1}{l^2}(x-l)^2, & x > 0. \end{cases}$$

Choose, for instance, $x_k = kl/m$, $l = 1$, $k = -6, -5, -4, \dots, 0, 1, \dots, 6$, $m = 6$ and let $y_k = f(x_k)$. Perturb y in the following way: $\hat{y}_k = y_k + \Delta y_k$, where $\Delta y_k = \sin k/2m$. Find the Fourier series for $\hat{y}_k$ and compare it with the Fourier series for unperturbed function $f(x)$ given analytically and by a table of values.

1.33. Expand the analytical function $f(x) = (x - \pi)^2/\pi^2$ given on the interval $[0, 2\pi]$ in a trigonometric Fourier series with $n = 5, 10, 20$ using the option "Choose Terms to Be Included" in the program TrigSeries.

1.34. Expand the corresponding discrete function given by the formulas

$$x_k = \frac{(k-1)\pi}{6}, \quad y_k = \frac{(x_k - \pi)^2}{\pi^2} = \frac{(k-7)^2}{36}, \quad k = 1, \dots, 13$$

on the interval $[0, 2\pi]$ in a trigonometric Fourier series with $n = 5, 10, 20$ terms. You can also visualize the expansion with the program TrigSeries.

1.35. Compare results of the expansions in Problems 1.33 and 1.34. Verify that the addition of terms in the series with numbers greater than $k_{max}/2$ only makes the precision of the approximation worse.

1.36. Use the TrigSeries program to model the problem given by Equation (1.106) for various functions $f(t)$, parameters λ , ω_0^2 , and initial conditions $y(0)$, $y'(0)$. Using the program, calculate tables of the quantities a_n , b_n , c_n , α_n , d_n , and δ_n ; draw the graphs of the individual harmonics or their sum; display bar charts of squared amplitudes c_n^2 and d_n^2 ; and examine the role of energy in the system.

1.37 (The RLC Series Circuit). Consider the circuit consisting of a resistor of resistance R , an inductor with inductance L , a capacitor of capacitance C , and a source of alternating voltage, $V(t)$. The differential equation of motion for the charge, q , in the circuit is

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = V(t). \quad (1.139)$$

A comparison of this equation with equation (1.106) gives

$$\lambda = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad f(t) = \frac{1}{L} V(t).$$

As a specific example, use the values of the parameters $L = 1$ H, $R = 10\Omega$, $C = 2.5 \cdot 10^{-5}$ F, $T = 1/50$ s, and (in volts)

$$V(t) = \begin{cases} 110 & \text{if } t \in [0.005, 0.015], \\ 0 & \text{if } t \in [0, 0.005] \text{ or } t \in [0.015, 0.02]; \end{cases}$$

that is, the voltage is applied in short impulses with a frequency of 50 Hz. For simplicity, set the initial conditions equal to zero: $q(0) = 0$ C and $q'(0) = 0$ C/s. Find time dependence of charge $q(t)$ and current $I(t) = dq/dt$.

Using the TrigSeries program, model this problem for various electric potentials $V(t)$ and parameters of the circuit. The program also allows the user to observe the evolution of the potential energy, $U(t)$, stored in the capacitor and the energy, $K(t)$, stored in the inductor. Remark on any interesting connections between these variables.

1.38. Using the method of Fourier series expansion, find periodic solutions of the following differential equations:

1. $y'' - y = f(x)$, $f(x) = \begin{cases} \operatorname{sgn} x, & -\pi < x < \pi, \\ 0, & x = \pm\pi. \end{cases}$
2. $y'' + y = f(x)$, $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2\pi)$.
3. $y^{(4)} - 6y = f(x)$, $f(x) = \pi^2 - x^2$, $x \in [-\pi, \pi]$.

1.39. Prove the Fourier transform of $f(x) = e^{-a|x|}$ equals $2a/(a^2 + k^2)$, where a is a positive constant.

1.40. Prove the Fourier transform of

$$f(x) = \begin{cases} h, & -a \leq x < a, \\ 0, & x < -a, x \geq a, \end{cases}$$

equals $(2h/k) \sin ak$. The *pulse function* $f(x)$, defined above, can be also written in terms of the Heaviside function:

$$f(x) = h[H(x + a) - H(x - a)].$$

1.41. Show that cosine and sine transforms of the function $f(x) = 1/\sqrt{x}$ are equal to the original function $1/\sqrt{x}$.

1.42. Solve the *integral equation*

$$\int_0^\infty g(z) \sin zx dz = f(x)$$

for the function $g(z)$ when

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leq x \leq \pi, \\ 0 & \text{if } x \geq \pi. \end{cases}$$

Hint. The solution is sine transform of the function

$$\sqrt{\frac{2}{\pi}} f(x).$$

Answer.

$$\frac{\sin \pi x}{1 - x^2}.$$

1.43. Find sine and cosine transforms of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq a, \\ \frac{1}{2} & \text{if } x = a, \\ 0 & \text{if } x > 0. \end{cases}$$

Answer.

$$\hat{f}_c(x) = \sqrt{\frac{2}{\pi}} \frac{\sin az}{z}, \quad \hat{f}_s(x) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos az}{z}.$$

1.44. Find sine and cosine transforms of the function

$$f(x) = \begin{cases} \cos \frac{x}{2} & \text{if } |x| \leq \pi, \\ 0 & \text{if } |x| > \pi. \end{cases}$$

1.45. Find sine and cosine transforms of the function

$$f(x) = \begin{cases} -e^x & \text{if } -1 \leq x < 0, \\ e^{-x} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

1.46. Find sine and cosine transforms of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq -\frac{1}{2}, \\ 0 & \text{if } -\frac{1}{2} \leq x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

1.47. Find the Fourier transform of the function

$$f(x) = \begin{cases} h, & a \leq x < b, \\ 0, & x < a, x \geq b. \end{cases}$$

1.48. Find the Fourier transform of the function

$$f(x) = \begin{cases} \sin x & \text{if } |x| \leq \pi, \\ 0 & \text{if } |x| > \pi. \end{cases}$$

1.49. Find the Fourier transform of the function

$$f(x) = \begin{cases} \cos x & \text{if } 0 < x \leq \pi, \\ 0 & \text{if } x > \pi. \end{cases}$$

1.50. Find the Fourier transform of the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \sin x & \text{if } 0 < x < \pi, \\ 0 & \text{if } x \geq \pi. \end{cases}$$

2

Sturm-Liouville Theory

2.1 The Sturm-Liouville Problem

Many applications in physics and engineering involve the solution of linear, second-order differential equations, which fall into a class of problems known as Sturm-Liouville eigenvalue problems. Such problems consist of a linear, second-order, ordinary differential equation containing a parameter whose value is determined so that the solution to the equation satisfies a given *boundary condition*. Thus, Sturm-Liouville problems are special kinds of *boundary value problems* for certain types of ordinary differential equations. The set of orthogonal functions generated by the solution to such problems may be used as basis functions for the Fourier expansion method of solving partial differential equations as well as other important problems. The special functions we briefly introduce in this chapter arise from one or another Sturm-Liouville problem and will be discussed in more detail later in the book.

To begin, we notice that a homogeneous linear second-order ordinary differential equation, containing the parameter λ multiplied by the function $y(x)$,

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) + \lambda d(x)y(x) = 0, \quad (2.1)$$

can be written in the *Sturm-Liouville form*

$$\frac{d}{dx} [p(x)y'(x)] + [q(x) + \lambda r(x)] y(x) = 0, \quad (2.2)$$

where

$$p(x) = e^{\int \frac{b(x)}{a(x)} dx}, \quad q(x) = \frac{p(x)c(x)}{a(x)}, \quad r(x) = \frac{p(x)d(x)}{a(x)} \quad (2.3)$$

(clearly $a(x) \neq 0$). Here $y(x)$ may represent some physical quantity in which we are interested, such as the amplitude of a wave at a particular location or the temperature at a particular time or location. The other functions in the equation express the physical situation that governs the behavior of the quantity $y(x)$.

Reading Exercise. Show the equivalence of Equations (2.1) and (2.2):

1. Verify that substitution of $p(x)$ from Equations (2.3) into Equation (2.2) gives Equation (2.1);
2. Verify that Equation (2.1) can be written in Sturm-Liouville form by dividing by $a(x)$ and then multiplying by the integrating factor $p(x)$ from Equation (2.3) .

As we will see, many physical problems result in the linear ordinary equation (2.2), where the function $y(x)$ is defined on an interval $[a, b]$ and obeys *homogeneous boundary conditions* of the form

$$\begin{aligned} \alpha_1 y' + \beta_1 y|_{x=a} &= 0, \\ \alpha_2 y' + \beta_2 y|_{x=b} &= 0. \end{aligned} \quad (2.4)$$

Note that the constants α_1 and β_1 cannot both be zero simultaneously, nor can the constants α_2 and β_2 . Note also that the relative signs for α_k and β_k are not arbitrary; we generally must have $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$. This choice of signs (details of which are discussed in the Section 2.2.) is necessary in setting up boundary conditions for various classes of physical problems. The very rare cases for which the signs are different occur in problems where there is explosive behavior, such as an exponential temperature increase. Throughout the book, we consider to be “normal” physical situations in which processes occur smoothly and thus the parameters in boundary conditions are restricted as above.

Equations (2.2) and (2.4) define a Sturm-Liouville problem. Solving this problem involves determining the values of the constant λ for which nontrivial solutions $y(x)$ exist. If $\alpha_k = 0$, the boundary condition simplifies to $y = 0$ (known as the *Dirichlet boundary condition*); if $\beta_k = 0$, the

boundary condition is $y' = 0$ (called the *Neumann boundary condition*); otherwise, the boundary condition is referred to as a *mixed boundary condition*.

Notice that for the function $y(x)$ defined on an infinite or semi-infinite interval, the conditions of Equations (2.4) may not be specified and are often replaced by the condition of regularity or physically reasonable behavior as $x \rightarrow \pm\infty$. For instance, on the interval $[a, \infty)$, the second condition in Equations (2.4) may not be specified but can instead be replaced by the condition of regularity at $x \rightarrow \infty$ (e.g., that $y(\infty)$ be finite).

For the following discussion, $p(x)$, $q(x)$, $r(x)$, and $p'(x)$ are continuous, real functions on an interval $[a, b]$ and $p(x) \geq 0$ and $r(x) \geq 0$ on the interval $[a, b]$. The coefficients α_k and β_k in Equations (2.4) are assumed to be real and independent of λ .

The differential equation (2.2) and boundary conditions (2.4) are *homogeneous*, meaning that all the terms depend in some way on $y(x)$, which is essential for the subsequent development. The trivial solution $y(x) = 0$ is always possible for homogeneous equations, but we seek special values of λ (called *eigenvalues*) for which there are nontrivial solutions $y(x)$ (called *eigenfunctions*) that depend on λ .

If we introduce the differential operator (called the *Sturm-Liouville operator*)

$$Ly(x) = -\frac{d}{dx} [p(x)y'(x)] - q(x)y(x) = -p(x)y''(x) - p'(x)y'(x) - q(x)y(x), \quad (2.5)$$

then Equation (2.2) becomes

$$Ly(x) = \lambda r(x)y(x). \quad (2.6)$$

As seen from Equation (2.5), $Ly(x)$ is a *linear operator*. When $r(x) = 1$, this equation appears as an ordinary *eigenvalue problem* for which we must determine λ and $y(x)$. For $r(x) \neq 1$, we have a similar problem, and the function $r(x)$ is called a *weight function*. As stated previously, the only requirement on $r(x)$ is that it is real and nonnegative.

Equation (2.2) (or 2.6)) and boundary conditions (2.4) constitute the *boundary value problem*. Its eigenvalues, λ_n , are real, and the eigenfunctions form an orthogonal set $\{y_n(x)\}$. To prove it, let us write Equation (2.6) for two eigenfunctions, $y_n(x)$ and $y_m(x)$, and take the complex conjugate of the equation for $y_m(x)$. Notice that in spite of the fact that $p(x)$,

$q(x)$, and $r(x)$ are real, λ and $y(x)$ can be complex. We have

$$Ly_n(x) = \lambda_n r(x) y_n(x)$$

and

$$Ly_m^*(x) = \lambda_m^* r(x) y_m^*(x).$$

Multiplying the first of these equations by $y_m^*(x)$ and the second by $y_n(x)$, we then integrate both from a to b and subtract the two results to obtain

$$\int_a^b y_m^*(x) Ly_n(x) dx - \int_a^b y_n(x) Ly_m^*(x) dx = (\lambda_n - \lambda_m^*) \int_a^b r(x) y_m^*(x) y_n(x) dx. \quad (2.7)$$

Using the definition of L given by Equation (2.5), the left side of Equation (2.7) is

$$\left\{ p(x) \left[\frac{dy_m^*}{dx} \cdot y_n(x) - y_m^*(x) \cdot \frac{dy_n}{dx} \right] \right\}_a^b. \quad (2.8)$$

Reading Exercise. Verify the previous statement.

Then, using the boundary conditions of Equations (2.4), we can easily prove that the expression in Equation (2.8) equals zero.

Reading Exercise. Verify that the expression in Equation (2.8) equals zero.

Thus, we are left with

$$\int_a^b y_m^*(x) Ly_n(x) dx = \int_a^b y_n(x) Ly_m^*(x) dx. \quad (2.9)$$

An operator, L , that satisfies Equation (2.9) may be termed a *Hermitian* or *self-adjoint operator*. Thus we may say that the *Sturm-Liouville linear operator* satisfying homogeneous boundary conditions is *Hermitian*. Many important operators in physics, especially in quantum mechanics, are Hermitian. The main properties of Hermitian operators are that their eigenvalues are real and their eigenfunctions are orthogonal (or can be chosen to be orthogonal). We now supply the proof of these properties.

Using Equation (2.9), Equation (2.7) may be written as

$$(\lambda_n - \lambda_m^*) \int_a^b r(x) y_m^*(x) y_n(x) dx = 0. \quad (2.10)$$

When $m = n$, the integral cannot be zero (recall that $r(x) > 0$); thus, $\lambda_n^* = \lambda_n$ and we have proved that *the eigenvalues of a Sturm-Liouville problem are real*.

Then, for $\lambda_m \neq \lambda_n$, Equation (2.10) is

$$\int_a^b r(x) y_m^*(x) y_n(x) dx = 0, \quad (2.11)$$

and we conclude that *the eigenfunctions corresponding to different eigenvalues of a Sturm-Liouville problem are orthogonal (with the weight function $r(x)$)*.

The squared norm of the eigenfunction $y_n(x)$ is defined to be

$$\|y_n\|^2 = \int_a^b r(x) |y_n(x)|^2 dx. \quad (2.12)$$

Note that the eigenfunctions of Hermitian operators always *can be chosen to be real*. This can be done by creating linear combinations of the functions $y_n(x)$; for example, choose $\sin x$ and $\cos x$ instead of $\exp(\pm ix)$ for the solutions of the equation $y'' + y = 0$. Real eigenfunctions are more convenient to work with because it is easier to match them to boundary conditions that are intrinsically real since they represent physical restrictions.

The above proof fails if $\lambda_m = \lambda_n$ for some $m \neq n$ (in other words there exist different eigenfunctions belonging to the same eigenvalues), in which case we cannot conclude that the corresponding eigenfunctions, $y_m(x)$ and $y_n(x)$, are orthogonal (although in some cases they are). If there are f eigenfunctions that have the same eigenvalues, we have an *f -fold degeneracy* of the eigenvalue. In general, a degeneracy reflects a symmetry of the underlying physical system (examples will be given later in this chapter). For a Hermitian operator it is always possible to construct linear combinations of the eigenfunctions belonging to the same eigenvalue so that these new functions are orthogonal.

If $p(a) \neq 0$ and $p(b) \neq 0$ then $p(x) > 0$ on the closed interval $[a, b]$ (which follows from $p(x) \geq 0$ for $a \leq x \leq b$) and we have the so-called *regular Sturm-Liouville problem*. If $p(a) = 0$, then we do not have the first of the boundary conditions in Equations (2.4); instead, we require $y(x)$ and $y'(x)$ to be finite at $x = a$. Similar situations occur if $p(b) = 0$, or if

both $p(a) = 0$ and $p(b) = 0$. All these cases correspond to the so-called *singular Sturm-Liouville problem*.

If $p(a) = p(b)$ and also instead of Equations (2.4) we have *periodic boundary conditions* $y(a) = y(b)$ and $y'(a) = y'(b)$, we have what is referred to as the *periodic Sturm-Liouville problem*.

The following list summarizes the three types of Sturm-Liouville problems:

1. For $p(x) > 0$ and $r(x) > 0$, we have the *regular* problem;
2. For $p(x) \geq 0$ and $r(x) \geq 0$, we have the *singular* problem;
3. For $p(a) = p(b)$ and $r(x) > 0$, we have the *periodic* problem.

Notice that the interval (a, b) can be infinite, in which case the Sturm-Liouville problem is also classified as singular.

In this book, we discuss orthogonal polynomials, all of which are the orthogonal eigenfunctions of a corresponding Sturm-Liouville problem. Table 2.1 presents the Sturm-Liouville form of the differential equation defining these orthogonal functions and the Bessel functions.

In Table 2.1, the Bessel functions are denoted as $J_n(x)$ and the Neumann functions as $N_n(x)$ (a commonly used alternative notation is $Y_n(x)$). The interval for both can be infinite, $[0, \infty)$ for $J_n(x)$ and $(0, \infty)$ for $N_n(x)$, because the functions $N_n(x)$ diverge as $x \rightarrow 0+$. The Legendre polynomials are denoted by $P_n(x)$. The Chebyshev polynomials of the first kind are denoted as $T_n(x)$ and Chebyshev polynomials of the second kind are denoted as $U_n(x)$. The Jacobi polynomials are denoted as $P_n^{(\alpha, \beta)}(x)$, the Laguerre polynomials as $L_n^{(\alpha)}(x)$ and the Hermite polynomials as $H_n(x)$.

Theorem 2.1 lists several important *properties of the Sturm-Liouville problem*.

Theorem 2.1.

1. *Each regular and each periodic Sturm-Liouville problem has an infinite number of nonnegative, discrete eigenvalues $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. All eigenvalues are real numbers.*
2. *For each eigenvalue of a regular Sturm-Liouville problem, there is only one eigenfunction; for a periodic Sturm-Liouville problem, this property does not hold.*

Function ($n = 0, 1, \dots$)	Equation	Weight function $r(x)$	Range of x
$J_n(x)$ and $N_n(x)$	$\frac{d}{dx} [xy'] + \left(\lambda x - \frac{n^2}{x}\right) y = 0$	x	$[0, b]$ and $(0, b]$
$P_n(x)$	$\frac{d}{dx} \left[(1-x^2) y' \right] + \lambda y = 0$	1	$[-1, 1]$
$T_n(x)$	$\frac{d}{dx} \left[\sqrt{1-x^2} y' \right] + \lambda \frac{y}{\sqrt{1-x^2}} = 0$	$1/\sqrt{1-x^2}$	$[-1, 1]$
$U_n(x)$	$\frac{d}{dx} \left[(1-x^2)^{3/2} y' \right] + \lambda \sqrt{1-x^2} y = 0$	$\sqrt{1-x^2}$	$[-1, 1]$
$P_n^{(\alpha, \beta)}(x)$ ($\alpha, \beta > -1$)	$\frac{d}{dx} \left[(1-x)^{\alpha+1} (1+x)^{\beta+1} y' \right] + \lambda (1-x)^\alpha (1+x)^\beta y = 0$	$(1-x)^\alpha (1+x)^\beta$	$(-1, 1)$
$L_n^{(\alpha)}(x)$ ($\alpha > -1$)	$\frac{d}{dx} \left[x^{\alpha+1} e^{-x} y' \right] + \lambda x^\alpha e^{-x} y = 0$	$x^\alpha e^{-x}$	$[0, \infty)$
$H_n(x)$	$\frac{d}{dx} \left[e^{-x^2} y' \right] + \lambda e^{-x^2} y = 0$	e^{-x^2}	$(-\infty, \infty)$

Table 2.1. Sturm-Liouville equations.

3. For each of the types of Sturm-Liouville problems, the eigenfunctions corresponding to different eigenvalues are linearly independent.
4. For each of the types of Sturm-Liouville problems, the set of eigenfunctions is orthogonal with respect to the weight function $r(x)$ on the interval $[a, b]$.
5. If $q(x) \leq 0$ on $[a, b]$ and $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$, then all $\lambda_n \geq 0$.

Some of these properties have been proven previously, such as property (4) and part of property (1). The remaining part of property (1) should be obvious and will be shown in several examples, as well as property (5). Property (2) can be proved by postulating that there are two eigenfunctions corresponding to the same eigenvalue. We then apply Equations (2.2) and (2.4) to show that these two eigenfunctions coincide or differ at most by some multiplicative constant. We leave this proof to the reader as a reading exercise. Similarly, property (3) is easily proven.

Bessel functions and the orthogonal polynomials arise from singular Sturm-Liouville problems; thus, the first statement in the Theorem 2.1 is not directly applicable to these important cases. In spite of that, singular Sturm-Liouville problems may also have an infinite sequence of discrete eigenvalues, which we will later verify directly for Bessel functions and for the orthogonal polynomials. Notice that there is the possibility for a singular Sturm-Liouville problem to have a continuous range of eigenvalues—in other words, a continuous spectrum—however, we will not encounter such situations in the problems we study in this book.

When Equation (2.11) is satisfied, eigenfunctions $y_n(x)$ form a *complete orthogonal set* on $[a, b]$. This means that any reasonable well-behaved function, $f(x)$, defined on $[a, b]$ can be expressed as a series of eigenfunctions (called a *generalized Fourier series*) of a Sturm-Liouville problem, in which case we may write

$$f(x) = \sum_n^{\infty} a_n y_n(x), \quad (2.13)$$

where it is convenient to start the summation in some cases with $n = 1$ and in other cases with $n = 0$. An expression for the coefficients a_n can be found by multiplying both sides of Equation (2.13) by $r(x)y_n(x)$ and integrating over $[a, b]$ to give

$$a_n = \frac{\int_a^b r(x)f(x)y_n(x)dx}{\|y_n\|^2}. \quad (2.14)$$

As we discussed in Chapter 1 on Fourier series, the sum in Equation (2.13) converges to $f(x)$ in the mean, or

$$\lim_{N \rightarrow \infty} \int_a^b r(x) \left[f(x) - \sum_n^N a_n y_n(x) \right]^2 dx. \quad (2.15)$$

If we now substitute a_n from Equation (2.14) into Equation (2.13), we have

$$f(x) = \sum_n^{\infty} \frac{y_n(x)}{\|y_n\|^2} \int_a^b r(x')f(x')y_n(x')dx'.$$

By interchanging the sum and the integral, we obtain

$$f(x) = \int_a^b f(x')r(x') \sum_n^{\infty} \frac{y_n(x)y_n(x')}{\|y_n\|^2} dx'.$$

From here, we conclude that

$$\sum_n^{\infty} \frac{y_n(x)y_n(x')}{\|y_n\|^2} r(x') = \delta(x - x'), \quad (2.16)$$

where $\delta(x)$ is the Dirac delta function, properties of which are discussed in Chapter 1 (see Equation (1.120)). Equation (2.16) is referred to as the completeness relation for the set of eigenfunctions $y_n(x)$.

The Sturm-Liouville theory also provides a theorem for convergence of the series in Equation (2.13) at every point x of $[a, b]$:

Theorem 2.2. *Let $\{y_n(x)\}$ be the set of eigenfunctions of a regular Sturm-Liouville problem, and let $f(x)$ and $f'(x)$ be piecewise continuous on a closed interval. Then the series Equation (2.13) converges to $f(x)$ at every point where $f(x)$ is continuous and to the value $[f(x_0 + 0) + f(x_0 - 0)]/2$ if x_0 is a point of discontinuity.*

Theorem 2.2 is also valid for the orthogonal polynomials and Bessel functions related to singular Sturm-Liouville problems. This theorem, which is extremely important for applications, is similar to Theorem 1.3 for trigonometric Fourier series, which was discussed in Chapter 1.

Formulas in Equations (2.11) through (2.14) can be written in a more convenient way if we define a *scalar product* of real eigenfunctions φ and ψ as the number given by

$$\varphi \cdot \psi = \int_a^b r(x)\varphi(x)\psi(x)dx. \quad (2.17)$$

This definition of the scalar product has properties identical to those for vectors in linear Euclidian space, a result that can be easily proved:

$$\begin{aligned} \varphi \cdot \psi &= \psi \cdot \varphi, \\ (a\varphi) \cdot \psi &= a(\varphi \cdot \psi), \text{ (where } a \text{ is a number)} \\ \varphi \cdot (a\psi) &= a\varphi \cdot \psi, \\ \varphi \cdot (\psi + \phi) &= \varphi \cdot \psi + \varphi \cdot \phi, \\ \varphi \cdot \varphi &\geq 0. \end{aligned} \quad (2.18)$$

The last property relies on the assumption made for the Sturm-Liouville equation that $r(x) \geq 0$. If φ is continuous on $[a, b]$, then $\varphi \cdot \varphi = 0$ only if φ is zero.

Reading Exercise. Prove the relations given in Equations (2.18).

In terms of the scalar product, the orthogonality of eigenfunctions (defined by Equation (2.11)) means that

$$y_n \cdot y_m = 0 \quad \text{if } n \neq m \quad (2.19)$$

and the formula for the Fourier coefficients in Equation (2.14) becomes

$$a_n = \frac{f \cdot y_n}{y_n \cdot y_n}. \quad (2.20)$$

Functions satisfying the condition

$$\varphi \cdot \varphi = \int_a^b r(x) |\varphi(x)|^2 dx < \infty \quad (2.21)$$

belong to a Hilbert space, L^2 , having infinite dimensionality. The complete orthogonal set of functions $\{y_n(x)\}$ serves as the orthogonal basis in L^2 , where here completeness means that the series in Equation (2.13) converges to $f(x)$ in the mean.

Finally, we briefly discuss Green's functions for the Sturm-Liouville problem. Suppose we are to solve an nonhomogeneous differential equation given by

$$Ly(x) = f(x), \quad (2.22)$$

where L is a Hermitian operator under the boundary conditions defined in Equation (2.4) and $f(x)$ is a specified function. We search for a solution to Equation (2.22) for the function $y(x)$ satisfying the same boundary conditions as in Equation (2.4). Let us assume that we already know the eigenfunctions $y_n(x)$ and eigenvalues λ_n of the related Equation (2.6). Since the eigenfunctions $y_n(x)$ form a complete set, the solution to Equation (2.22) can be written as a superposition of $y_n(x)$; that is,

$$y(x) = \sum_n^{\infty} c_n y_n(x). \quad (2.23)$$

Obviously, the Sturm-Liouville operator L is *linear*, which means that

$$L \sum_n^{\infty} c_n y_n(x) = \sum_n^{\infty} c_n Ly_n(x). \quad (2.24)$$

Then, using Equation (2.6), we have

$$\sum_n^{\infty} c_n \lambda_n r(x) y_n(x) = f(x).$$

Multiplying both sides by $y_m(x)$, integrating over $[a, b]$, and using the orthogonality conditions, we obtain

$$c_n = \frac{1}{\lambda_n} \frac{\int_a^b f(x') y_n(x') dx'}{\|y_n\|^2}, \quad (2.25)$$

where we have defined x' as a new integration variable. Thus, from Equation (2.23), we have

$$y(x) = \int_a^b \sum_n \left[\frac{1}{\lambda_n} y_n(x) y_n(x') \right] f(x') dx', \quad (2.26)$$

where we assume that we may interchange the order of summation and integration and for simplicity we have assumed the functions are normalized so that $\|y_n\|^2 = 1$. Equation (2.26) can thus be written as

$$y(x) = \int_a^b G(x, x') f(x') dx', \quad (2.27)$$

where

$$G(x, x') = \sum_n \frac{1}{\lambda_n} y_n(x) y_n(x') \quad (2.28)$$

is called the *Green's function* of the Sturm-Liouville problem.

The important property of the function $G(x, x')$ is that it satisfies the equation

$$L[G(x, x')] = \delta(x - x'). \quad (2.29)$$

To see this, let us multiply Equation (2.29) by $f(x')$, integrate with respect x' on the interval $[a, b]$, and change the order of the operator L and the integral. This gives

$$L \left[\int_a^b G(x, x') f(x') dx' \right] = \int_a^b \delta(x - x') f(x') dx'. \quad (2.30)$$

On the left side of this equation, the operator L acts on the function $y(x)$ as defined in Equation (2.27) and on the right side, applying the properties of the delta function, we obtain the function $f(x)$. That is, Equation (2.30) is equivalent to Equation (2.22). From Equation (2.30) we see that product $G(x, x')f(x')$ is the response at point x from a point-like source of intensity $f(x')$ at point x' . In other words, the Green's function has the physical sense of a field created by a point source and the solution given by Equation (2.27) corresponds to the superposition principle: the value of the function $y(x)$ at some point x is a sum of the contributions of all point-like sources $G(x, x')f(x')$ located in the interval $a \leq x' \leq b$. Thus, the Green's function in Equation (2.28) (which is the solution of Equation (2.29)) allows us to obtain the solution of nonhomogeneous second-order linear differential equation (2.22) for an arbitrary function $f(x)$ acting as an external source. Examples of the use of the Green's functions are given in Section 2.3.

2.2 Mixed Boundary Conditions

In this section, we discuss the question of the signs of the coefficients in the boundary conditions given in Equations (2.4):

$$\begin{aligned}\alpha_1 y' + \beta_1 y|_{x=a} &= 0, \\ \alpha_2 y' + \beta_2 y|_{x=b} &= 0.\end{aligned}\tag{2.31}$$

As previously stated, the correct signs for the case $a \leq x \leq b$ will be $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$. These restrictions on the signs follow from rather general physical arguments. Consider, for instance, the simple problem of longitudinal oscillations of a spring of length l with one end rigidly fixed at $x = 0$ and the other end attached elastically to a support with an equilibrium location of $x = l$. A given point on the spring, which initially has an equilibrium position x at time $t = 0$, will have the location $\tilde{x} = x + u(x, t)$ at subsequent times, where $u(x, t)$ is the longitudinal displacement from equilibrium at location x and time t . The boundary condition at $x = 0$ for the displacement $u(x, t)$ is

$$u(0, t) = u|_{x=0} = 0\tag{2.32}$$

and has the simple physical meaning that the end at $x = 0$ is not moving and the displacement is equal to zero for all times t .

Now let us discuss the boundary condition at the other of the spring end $x = l$, which is attached to a support in such a way that it moves due to both internal forces and an external elastic force of attachment. The internal elastic force in the spring, $F(x, t)$, will obey Hooke's law. At $x = l$, this force is equal to

$$F(l, t) = k \left. \frac{\partial u}{\partial x} \right|_{x=l} = ku_x|_{x=l}, \quad (2.33)$$

where k is Young's modulus and $k > 0$. Here we denote the partial derivative of the function $u(x, t)$ with respect to x as u_x . When the right end is free there is no external force and the boundary condition at $x = l$ has the simple form

$$u_x|_{x=l} = 0. \quad (2.34)$$

In the case of elastic attachment, the right end experiences small displacements; however, we will assume these displacements are small enough that we may still formulate the boundary condition as a condition on the spring at the location $x = l$. The external elastic force acting on this end is directed against the force of tension in the spring given in Equation (2.33), and again assuming a linear Hooke's law force, will be proportional to the displacement, $u(x, t)$. Thus, we have an external force, $\gamma u|_{x=l}$, where the coefficient γ may be referred to as the rigidity of attachment, and we have $\gamma > 0$. Therefore, the boundary condition at $x = l$ is given by

$$ku_x|_{x=l} = -\gamma u|_{x=l} \quad (2.35)$$

or

$$ku_x + \gamma u|_{x=l} = 0. \quad (2.36)$$

This boundary condition coincides with the second case of Equation (2.31). From Equation (2.36), we have $\gamma/k > 0$; otherwise, we would have a physical inconsistency in the formulation of the problem. This sign corresponds to the positive sign of the ratio in the second case of Equation (2.31); $\beta_2/\alpha_2 > 0$.

Suppose now we let the left end at $x = 0$ also be elastically attached. The elastic restoring force in the spring at this point is $F(x, t)$, as given by

$$F|_{x=0} = ku_x|_{x=0}. \quad (2.37)$$

The elastic attachment force acting on this end is directed against the force of tension in the spring, and its magnitude is $\gamma u|_{x=0}$. Thus, the boundary

condition at $x = 0$ is

$$ku_x - \gamma u|_{x=0} = 0. \quad (2.38)$$

This boundary condition coincides with the first case of Equation (2.31). Because $-\gamma k < 0$, from the physical arguments given above, we have $\beta_1/\alpha_1 < 0$.

Often it is the case that physical problems have nonhomogeneous boundary conditions in which the right sides of Equation (2.31) are not zero. In such cases, they must be converted to homogeneous boundary conditions by introducing some auxiliary function—a task we will have to perform many times in the following chapters, since Sturm-Liouville problems are defined to have only homogeneous boundary conditions.

Let us give one more physical example that has boundary conditions similar to Equations (2.31) and (2.36). Consider the cooling of a uniform rod with insulated lateral surfaces where the ends exchange heat with the environment, which has a temperature $\theta(t)$. Consider a segment of the rod $(x_1, x_1 + \Delta x)$ of length Δx . During a unit of time this segment obtains an amount of heat, $c\rho S\Delta x u_t$, where c is specific heat capacity, ρ is the mass density, S is the cross sectional area of the rod, $u(x, t)$ is the temperature of the rod at location x and time t , and the subscript t denotes a time derivative. This amount of heat is equal to the heat this segment obtains (or loses) during a unit of time through cross sections at x_1 and $x_1 + \Delta x$, given by

$$-\kappa S u_x|_{x=x_1} + \kappa S u_x|_{x=x_1+\Delta x}.$$

The coefficient κ is called the thermal conductivity. Thus, we have

$$c\rho S\Delta x u_t = -\kappa S u_x|_{x=x_1} + \kappa S u_x|_{x=x_1+\Delta x}. \quad (2.39)$$

Let us discuss with more detail the signs in the right side of Equation (2.39). If, at x_1 the gradient of temperature is $u_x > 0$, then in the region $x_1 < x < x_1 + \Delta x$, the temperature is higher than at x_1 and heat is leaving the segment Δx (where the heat flux is directed in the negative direction along the x -axis). In this case the first term on the right of Equation (2.39) should be taken with a minus sign. If $u_x < 0$, then the temperature for $x < x_1$ (to the left of x_1) is higher than in the region $x_1 < x < x_1 + \Delta x$, and the heat flows into Δx directed along the positive x -axis. In this case, the first term on the right of Equation (2.39) still should be taken with a minus sign because $u_x < 0$. Similar consideration can be carried out to justify the sign of the second term in the right side of Equation (2.39).

Using Equation (2.39), we can obtain the boundary conditions at the ends of the rod that are in contact with the environment having temperature $\theta(t)$. Considering the end at $x = l$ first, we write the analogue of Equation (2.39) for a segment $(l - \Delta x, l)$. At the left end, $x = l - \Delta x$ and heat flux coincides with the first term in Equation (2.39): $-\kappa u_x|_{x=l-\Delta x}$. Heat passing per unit time into this segment through the right end at $x = l$ is $SH [u|_{x=l} - \theta(t)]$, where $H > 0$ is the heat exchange coefficient. This expression is positive if the temperature of the rod at $x = l$ is higher than the temperature of the environment (i.e., the heat flux is directed along the positive x -axis and the segment Δx loses heat), and negative in the opposite situation. Therefore, Equation (2.39) in the vicinity of the right end becomes

$$c\rho\Delta xu_t = -\kappa u_x|_{x=l-\Delta x} - H [u|_{x=l} - \theta(t)] . \quad (2.40)$$

Taking the limit as $\Delta x \rightarrow 0$ in Equation (2.40) we obtain the boundary condition for the rod's end at $x = l$:

$$\kappa u_x + H [u - \theta(t)]|_{x=l} = 0. \quad (2.41)$$

This condition looks like the second condition in Equation (2.31) and condition (2.36) if we set $\theta(t) = 0$. Also we see that $H/\kappa > 0$. Similarly, considering a segment $(0, \Delta x)$ we obtain the boundary condition for the left end of the rod at $x = 0$. Heat passing per unit time into this segment through the right end at $x = \Delta x$ is equal to $S\kappa u_x|_{x=\Delta x}$. The amount of heat obtained through the left end at $x = 0$ is $-SH [u|_{x=0} - \theta(t)]$. This heat is positive if the rod temperature is less than the temperature of the environment (i.e., the rod is heating and heat flux is directed along the x -axis). Thus, the equation for the segment $(0, \Delta x)$ of the rod is

$$c\rho\Delta xu_t = -H [u|_{x=0} - \theta(t)] + \kappa u_x|_{x=\Delta x} , \quad (2.42)$$

or, taking the limit $\Delta x \rightarrow 0$,

$$\kappa u_x - \alpha [u - \theta(t)]|_{x=0} = 0. \quad (2.43)$$

In the case $\theta(t) = 0$, the boundary condition in Equation (2.43) coincides with the first of Equations (2.31) and with condition (2.38). Also, we see that $-H/\kappa < 0$. When $\theta(t) \neq 0$, Equations (2.41) and (2.43) are non-homogeneous boundary conditions, and to solve boundary value problems we have to reduce them to being homogeneous.

From these two examples, it is clear that the signs of the ratios of the coefficients in the boundary conditions in Equation (2.31) are very important. Otherwise, we could face physically senseless situations such as heat flowing spontaneously from a cool body to a warm environment.

Next, we briefly discuss similar restrictions on the signs of the coefficients in the boundary conditions for two- and three-dimensional problems. Suppose the domain for the rectangular coordinate x is the interval $[a, b]$ and for coordinate y is the interval $[c, d]$. Many physical problems (e.g., rectangular oscillating membranes, heat conductivity, various electrostatics problems, etc.) reduce to separate Sturm-Liouville problems for the variables x and y . In these cases, we may formulate boundary conditions for all four boundaries independently. For the variable x they are still given by Equation (2.31), and for the variable y they are in a similar form:

$$\begin{aligned}\alpha_3 u' + \beta_3 u|_{y=c} &= 0, \\ \alpha_4 u' + \beta_4 u|_{y=d} &= 0,\end{aligned}\tag{2.44}$$

with identical restrictions on the signs of the coefficients: $\beta_3/\alpha_3 < 0$, $\beta_4/\alpha_4 > 0$.

For three-dimensional cases in Cartesian coordinates, where the variable z varies in the interval $[e, f]$, we have two more boundary conditions, similar to Equations (2.31) and (2.44):

$$\begin{aligned}\alpha_5 u' + \beta_5 u|_{z=e} &= 0, \\ \alpha_6 u' + \beta_6 u|_{z=f} &= 0,\end{aligned}\tag{2.45}$$

with the restrictions $\beta_5/\alpha_5 < 0$ and $\beta_6/\alpha_6 > 0$.

If we consider a circular domain with radius R , $r \leq R$, it is more convenient to use polar coordinates (r, φ) . The boundary condition at the boundary $r = R$ in general is

$$\alpha u' + \beta u|_{r=R} = 0,\tag{2.46}$$

with the sign restrictions on α and β given by $\beta/\alpha > 0$. If the domain is defined by the condition $r \geq R$, then obviously the sign of the ratio of the coefficients in Equation (2.46) should be the opposite, and we have $\beta/\alpha < 0$. Similarly, for an annulus domain, $a \leq r \leq b$, the conditions are

$$\begin{aligned}\alpha_1 u' + \beta_1 u|_{r=a} &= 0, \\ \alpha_2 u' + \beta_2 u|_{r=b} &= 0,\end{aligned}\tag{2.47}$$

with the signs $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$.

For a cylinder of radius R and height l , the homogeneous boundary conditions in cylindrical coordinates (r, φ, z) on the cylinder's surface are

$$\begin{aligned}\alpha_1 u' + \beta_1 u|_{r=R} &= 0, \\ \alpha_2 u' + \beta_2 u|_{z=0} &= 0, \\ \alpha_3 u' + \beta_3 u|_{z=l} &= 0.\end{aligned}\tag{2.48}$$

Coefficient ratios in Equations (2.48) have sign restrictions similar to the situations discussed previously and are given by $\beta_1/\alpha_1 > 0$, $\beta_2/\alpha_2 < 0$, and $\beta_3/\alpha_3 > 0$.

For a sphere of radius R , in spherical polar coordinates (r, φ, θ) , the homogeneous boundary condition on the sphere's surface is

$$\alpha u' + \beta u|_{r=R} = 0,\tag{2.49}$$

where the restriction on the sign of the ratio of coefficients is given by $\beta/\alpha > 0$.

2.3 Examples of Sturm-Liouville Problems

Example 2.1. Solve the equation

$$y''(x) + \lambda y(x) = 0\tag{2.50}$$

on the interval $[0, 1]$ with boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 0.\tag{2.51}$$

Solution. First, comparing Equation (2.50) with Equations (2.5) and (2.6), it is clear that we have a Sturm-Liouville problem with linear operator $L = -d^2/dx^2$ and functions $q(x) = 0$ and $p(x) = r(x) = 1$. (Note that L can be taken with either a negative or positive sign, as is clear from Equations (2.1), (2.5) and (2.6).)

Let us discuss the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$ separately. If $\lambda = 0$, then a general solution to Equation (2.50) is

$$y(x) = C_1 x + C_2$$

and, from the boundary conditions of Equation (2.51), we have $C_1 = C_2 = 0$ (i.e., there exists only the trivial solution $y(x) = 0$). If $\lambda < 0$, then

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

and the boundary conditions of Equation (2.51) again give $C_1 = C_2 = 0$, and therefore the trivial solution $y(x) = 0$. Thus, we have only the possibility $\lambda > 0$, in which case we write $\lambda = \mu^2$ with μ real, and we have a general solution of Equation (2.50) given by

$$y(x) = C_1 \sin \mu x + C_2 \cos \mu x.$$

The boundary condition $y(0) = 0$ requires that $C_2 = 0$, and the boundary condition $y(1) = 0$ gives $C_1 \sin \mu = 0$. From this we must have $\sin \mu = 0$ and $\mu_n = n\pi$ since the choice $C_1 = 0$ again gives the trivial solution. Thus, the eigenvalues are

$$\lambda_n = \mu_n^2 = n^2 \pi^2, \quad n = 1, 2, \dots, \quad (2.52)$$

and the eigenfunctions are $y_n(x) = C_n \sin n\pi x$, where for $n = 0$ we have the trivial solution $y_0(x) = 0$. It is obvious that we can restrict ourselves to only positive values of n since negative values do not give new solutions in the case that the constants C_n are arbitrary. These eigenfunctions are orthogonal over the interval $[0, 1]$ since we can easily show that

$$\int_0^1 \sin n\pi x \cdot \sin m\pi x dx = 0 \text{ for } m \neq n. \quad (2.53)$$

The orthogonality of eigenfunctions follows from the fact that the Sturm-Liouville operator, L , is Hermitian for the boundary conditions given in Equation (2.51).

Reading Exercise. Directly verify that L is Hermitian.

Hint. Evaluate

$$\int_0^1 y_n(x) L y_m(x) dx$$

integrating by parts twice.

The eigenfunctions may be normalized by writing

$$C_n^2 \int_0^1 \sin^2 n\pi x dx = C_n^2 \cdot \frac{1}{2} = 1,$$

which results in the orthonormal eigenfunctions

$$y_n(x) = \sqrt{2} \sin n\pi x, \quad n = 0, 1, 2, \dots \quad (2.54)$$

Thus, we have shown that the boundary value problem consisting of Equations (2.50) and (2.51) has eigenfunctions that are sine functions. In other words, *the expansion in eigenfunctions of the Sturm-Liouville problem for solutions to Equations (2.50) and (2.51) is equivalent to the trigonometric Fourier sine series.*

Reading Exercise. Suggest alternatives to boundary conditions (2.51) that will result in cosine functions as the eigenfunctions for equation (2.50).

Example 2.2. Determine the eigenvalues and corresponding eigenfunctions for the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0, \quad (2.55)$$

$$y'(0) = 0, \quad y(1) = 0. \quad (2.56)$$

Solution. As in Example 2.1, check as a reading exercise that the parameter λ must be positive in order to have nontrivial solutions. Thus, we may write $\lambda = \mu^2$ so that we have oscillating solutions given by

$$y(x) = C_1 \sin \mu x + C_2 \cos \mu x.$$

The boundary condition $y'(0) = 0$ gives $C_1 = 0$, and the boundary condition $y(1) = 0$ gives $C_2 \cos \mu = 0$. If $C_2 = 0$, we have a trivial solution; otherwise, we have $\mu_n = (2n + 1)\pi/2$, for $n = 0, 1, 2, \dots$. Therefore, the eigenvalues are

$$\lambda_n = \mu_n^2 = \frac{(2n + 1)^2 \pi^2}{4}, \quad n = 0, 1, 2, \dots, \quad (2.57)$$

and the eigenfunctions are

$$y_n(x) = C_n \cos \frac{(2n + 1)\pi x}{2}, \quad n = 0, 1, 2, \dots \quad (2.58)$$

We leave it to the reader to prove that the eigenfunctions in Equation (2.58) are orthogonal on the interval $[0, 1]$. The reader may also normalize these eigenfunctions to find the normalization constant C_n , which is equal to $\sqrt{2}$.

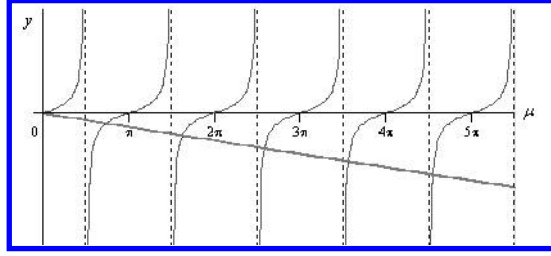


Figure 2.1. The functions $\tan \mu_n$ and $-\mu_n/5$ plotted against μ . The eigenvalues of the Sturm-Liouville problem in Example 2.3 are given by the intersections of these lines.

Example 2.3. Determine the eigenvalues and eigenfunctions for the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0, \quad (2.59)$$

$$y(0) = 0, \quad y'(1) + 5y(1) = 0. \quad (2.60)$$

Solution. As in the previous examples, nontrivial solutions exist only when $\lambda > 0$ (the reader should verify this as a reading exercise). Letting $\lambda = \mu^2$ we obtain a general solution as

$$y(x) = C_1 \sin \mu x + C_2 \cos \mu x.$$

From the boundary condition $y(0) = 0$, we have $C_2 = 0$. The other boundary condition gives $\mu \cos \mu + 5 \sin \mu = 0$. Thus, the eigenvalues are given by the equation

$$\tan \mu_n = -\frac{\mu_n}{5}. \quad (2.61)$$

From Figure 2.1, we see directly that there are an infinite number of discrete eigenvalues. The eigenfunctions

$$y_n(x) = C_n \sin \mu_n x, \quad n = 0, 1, 2, \dots \quad (2.62)$$

are orthogonal so that

$$\int_0^1 \sin \mu_n x \cdot \sin \mu_m x dx = 0 \text{ for } m \neq n. \quad (2.63)$$

The orthogonality condition shown in Equation (2.63) follows from the general theory as a direct consequence of the fact that the operator $L = -d^2/dx^2$ is Hermitian for the boundary conditions (2.60).

The normalized eigenfunctions are

$$y_n(x) = \frac{2\sqrt{\mu_n}}{\sqrt{2\mu_n - \sin 2\mu_n}} \sin \mu_n x. \quad (2.64)$$

Example 2.4. Solve the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < l, \quad (2.65)$$

on the interval $[0, l]$ with periodic boundary conditions

$$y(0) = y(l), \quad y'(0) = y'(l). \quad (2.66)$$

Solution. Again, verify as a reading exercise that nontrivial solutions exist only when $\lambda > 0$ (for which we will have oscillating solutions, as before). Letting $\lambda = \mu^2$, we can write a general solution in the form

$$y(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

The boundary conditions in Equations (2.66) give

$$\begin{cases} C_1(\cos \mu l - 1) + C_2 \sin \mu l = 0, \\ -C_1 \sin \mu l + C_2(\cos \mu l - 1) = 0. \end{cases} \quad (2.67)$$

This system of homogeneous algebraic equations has a nontrivial solution only when its determinant is equal to zero:

$$\begin{vmatrix} \cos \mu l - 1 & \sin \mu l \\ -\sin \mu l & \cos \mu l - 1 \end{vmatrix} = 0 \quad (2.68)$$

which yields

$$\cos \mu l = 1. \quad (2.69)$$

The roots of Equation (2.69) are

$$\lambda_n = \left(\frac{2\pi n}{l} \right)^2, \quad n = 0, 1, 2, \dots \quad (2.70)$$

With these values of λ_n , Equations (2.67) for C_1 and C_2 have two linearly independent nontrivial solutions given by

$$C_1 = \begin{pmatrix} C_1^{(1)} \\ C_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} C_1^{(2)} \\ C_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.71)$$

Substituting each set into the general solution, we obtain the eigenfunctions

$$y_n^{(1)}(x) = \cos \sqrt{\lambda_n} x \quad \text{and} \quad y_n^{(2)}(x) = \sin \sqrt{\lambda_n} x. \quad (2.72)$$

Therefore, for the eigenvalue $\lambda_0 = 0$, we have the eigenfunction $y_0(x) = 1$ and a trivial solution $y(x) \equiv 0$. Each nonzero eigenvalue, λ_n , has two linearly independent eigenfunctions so that for this example we have twofold degeneracy.

Collecting the above results, we have that this boundary value problem with periodic boundary conditions has the following eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{2\pi n}{l} \right)^2, \quad n = 0, 1, 2, \dots \quad (2.73)$$

$$y_0(x) \equiv 1, \quad y_n(x) = \begin{cases} \cos \frac{2\pi n}{l} x, \\ \sin \frac{2\pi n}{l} x, \end{cases} \quad (2.74)$$

(two expressions in Equation (2.74) reflect twofold degeneracy),

$$\|y_0\|^2 = l, \quad \|y_n\|^2 = \frac{l}{2} \quad (\text{for } n = 1, 2, \dots).$$

In particular, when $l = 2\pi$ we have

$$\lambda_n = n^2, \quad y_0(x) \equiv 1, \quad y_n(x) = \begin{cases} \cos nx, \\ \sin nx. \end{cases}$$

From this result, we see that the boundary value problem consisting of Equations (2.65) and (2.66) results in eigenfunctions for this Sturm-Liouville problem which allows *an expansion of the solution equivalent to the complete trigonometric Fourier series expansion*.

Example 2.5. In the following chapters of this book, we will encounter a number of two-dimensional Sturm-Liouville problems. Here we present a simple example for future reference.

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \quad (2.75)$$

where k is a real constant that determines the function $u(x, y)$ with independent variables in domains $0 \leq x \leq l$, $0 \leq y \leq h$. Define the two-dimensional Sturm-Liouville operator in a fashion similar to that for the one-dimensional case.

Solution. We have

$$L = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2}. \quad (2.76)$$

Let the boundary conditions be Dirichlet type so that we have

$$u(0, y) = u(l, y) = u(x, 0) = u(x, h) = 0. \quad (2.77)$$

Reading Exercise. By direct substitution into Equation (2.75) and using boundary conditions (2.77), check that this Sturm-Liouville problem has the eigenvalues

$$k_{nm}^2 = \frac{1}{\pi^2} \left(\frac{n^2}{l^2} + \frac{m^2}{h^2} \right) \quad (2.78)$$

and the corresponding eigenfunctions

$$u_{nm} = \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{h}. \quad (2.79)$$

In the case of a square domain where $l = h$, the eigenfunctions u_{nm} and u_{mn} have the same eigenvalues, $k_{nm} = k_{mn}$, which is a degeneracy reflecting the symmetry of the problem with respect to x and y .

Example 2.6. Find the Green's function of the operator L given by

$$Ly = -y'' - 2y', \quad y(0) = y(1) = 0$$

on the interval $[0, 1]$.

Solution. This Sturm-Liouville problem has the form

$$Ly = -y'' - 2y' = \lambda y, \quad (2.80)$$

$$y(0) = y(1) = 0, \quad (2.81)$$

for which it is easy to check that a nontrivial solution exists only for $\lambda > 1$. In this case a general solution of Equation (2.80) is

$$y(x) = e^{-x} \left[C_1 \cos(\sqrt{\lambda - 1}x) + C_2 \sin(\sqrt{\lambda - 1}x) \right]. \quad (2.82)$$

Substituting Equation (2.82) into the boundary conditions of Equation (2.81) gives, for nontrivial solutions,

$$C_1 = 0 \quad \text{and} \quad \sin(\sqrt{\lambda - 1}) = 0.$$

The second relation gives the eigenvalues, λ_n , of the Sturm-Liouville problem as

$$\lambda_n = \pi^2 n^2 + 1, \quad n = 1, 2, 3, \dots \quad (2.83)$$

When $n = 0$, $\sin(\sqrt{\lambda_n - 1}x) = 0$ for any value of x ; thus, $n = 0$ corresponds to a trivial solution, $y_0(x) = 0$. The eigenfunctions $y_n(x)$, corresponding to the eigenvalues λ_n in Equation (2.83), are

$$y_n(x) = e^{-x} \sin(\sqrt{\lambda_n - 1}x) = e^{-x} \sin n\pi x. \quad (2.84)$$

To find the Green's function $G(x, x')$, substitute Equations (2.83) and (2.84) into Equation (2.29), which gives

$$G(x, x') = \sum_{n=1}^{\infty} \frac{\sin(\pi n x) \sin(\pi n x')}{\pi^2 n^2 + 1} e^{-x} e^{-x'}. \quad (2.85)$$

Example 2.7.

1. Find the Green's function for the equation

$$y'' + y = f(x), \quad 0 \leq x \leq 1, \quad (2.86)$$

if the function $y(x)$ satisfies boundary conditions

$$y(0) = 0, \quad y(1) = 0. \quad (2.87)$$

2. Use this Green's function to find the solution of the boundary problem $y'' + y = f(x)$ if $f(x) = x$.

Solution.

1. Comparing the given equation with Equation (2.2) we see that $p(x) = 1$ and $q(x) = 0$, so $L = -d^2/dx^2 - 1$. A similar boundary value problem with the same boundary conditions (Dirichlet type) was discussed in detail in Example 2.1. We leave it to the reader as a reading exercise to obtain the following eigenfunctions and eigenvalues of the boundary value problem consisting of the equation $Ly = \lambda y$ and boundary conditions in Equation (2.87):

$$\lambda_n = n^2\pi^2 - 1, \quad y_n(x) = \sqrt{2} \sin n\pi x, \quad n = 1, 2, 3, \dots \quad (2.88)$$

Substituting Equations (2.88) into Equation (2.29), we have

$$G(x, x') = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \cdot \sin n\pi x'}{n^2\pi^2 - 1}. \quad (2.89)$$

2. The solution to the equation $y'' + y = f(x)$, which is a particular case of Equation (2.22) with $f(x) = -x$, is given by Equation (2.27); thus,

$$y(x) = \int_a^b G(x, x')(-x')dx' = \sum_{n=1}^{\infty} \frac{2 \sin n\pi x}{n^2\pi^2 - 1} \int_0^1 (-x') \sin n\pi x' dx' = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n^2\pi^2 - 1} \frac{(-1)^n}{n\pi}.$$

We see that this solution satisfies the given boundary conditions.

Example 2.8. Obtain the eigenfunction expansion (generalized Fourier series expansion) of the function $f(x) = x^2(1 - x)$ by using the eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 \leq x \leq \pi/2, \quad (2.90)$$

$$y'(0) = y'(\pi/2) = 0. \quad (2.91)$$

Solution. First, prove as a reading exercise that the eigenvalues and eigenfunctions of this boundary value problem are

$$\lambda = 4n^2, \quad y_n(x) = \cos 2nx, \quad n = 0, 1, 2, \dots \quad (2.92)$$

A Fourier series expansion, given in Equation (2.13), of the function $f(x)$ using the eigenfunctions above is

$$x^2(1-x) = \sum_{n=0}^{\infty} c_n y_n(x) = \sum_{n=0}^{\infty} c_n \cos 2nx. \quad (2.93)$$

Since f and f' are continuous functions, this expansion will converge to $x^2(1-x)$ for $0 < x < \pi/2$, as was shown previously. In Equation (2.90), we see that the function $r(x) = 1$, and thus the coefficients of this expansion obtained from Equation (2.14) are

$$c_0 = \frac{\int_0^{\pi/2} x^2(1-x) dx}{\int_0^{\pi/2} dx} = \frac{\pi^2}{4} \left(\frac{1}{3} - \frac{\pi}{8} \right),$$

$$c_n = \frac{\int_0^{\pi/2} x^2(1-x) \cos 2nx \, dx}{\int_0^{\pi/2} \cos^2 2nx \, dx} = \frac{(-1)^n}{n^4} \left[1 - \frac{3\pi}{4} + \frac{3}{2\pi n^2} \right] - \frac{3}{2\pi n^4}, \quad n = 1, 2, 3, \dots$$

Figure 2.2 shows the partial sum ($n = 10$) of this series, compared with the original function $f(x) = x^2(1-x)$.

Two important special functions, Legendre and Bessel functions, are discussed in detail in Chapters 9 and 8, respectively. In the following two examples, they serve simply as illustrations of Sturm-Liouville problems.

Example 2.9 (Fourier-Legendre Series). The Legendre equation is

$$\frac{d}{dx} [(1-x^2) y'] + \lambda y = 0 \quad (2.94)$$

for x on the closed interval $[-1, 1]$. There are no boundary conditions in a straight form because $p(x) = 1 - x^2$ vanishes at the endpoints. However, we seek a finite solution, a condition that in this case acts as a boundary condition.

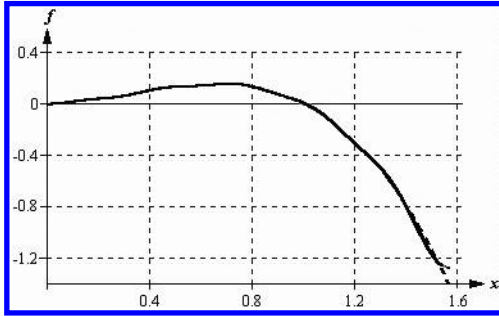


Figure 2.2. Graphs of the function $f(x) = x^2(1 - x)$ (dashed line) and partial sum with $n = 10$ of the Fourier expansion of $f(x)$ (solid line).

Solution. The Legendre polynomials, $P_n(x)$, are the only solutions of Legendre's equation that are bounded on the closed interval $[-1, 1]$. The set of functions $\{P_n(x)\}$, where $n = 0, 1, 2, \dots$, is orthogonal with respect to the weight function $r(x) = 1$ on the interval $[-1, 1]$, in which case the orthogonality relation is

$$\int_{-1}^1 P_n(x)P_m(x)dx = 0 \text{ for } m \neq n. \quad (2.95)$$

The eigenfunctions for this problem are thus $P_n(x)$ with eigenvalues $\lambda = n(n+1)$ for $n = 0, 1, 2, \dots$, as we will see in Chapter 9.

If $f(x)$ is piecewise smooth on $[-1, 1]$, the series

$$\sum_{n=0}^{\infty} c_n P_n(x) \quad (2.96)$$

converges to

$$\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)] \quad (2.97)$$

at any point x_0 on $(-1, 1)$. Because $r(x) = 1$ in Equation (2.14), the coefficients c_n are

$$c_n = \frac{\int_{-1}^1 f(x)P_n(x)dx}{\int_{-1}^1 P_n^2(x)dx}, \quad (2.98)$$

or written in terms of the scalar product,

$$c_n = \frac{f(x) \cdot P_n(x)}{P_n(x) \cdot P_n(x)}. \quad (2.99)$$

Example 2.10 (Fourier-Bessel Series). Solve the Sturm-Liouville problem

$$(xy')' + \left(\lambda x - \frac{\nu^2}{x} \right) y = 0, \quad 0 \leq x \leq 1, \quad (2.100)$$

with boundary conditions such that $y(0)$ is finite and $y(1) = 0$. Here ν is a constant.

Solution. The eigenvalues for this problem are $\lambda = j_n^2$ for $n = 1, 2, \dots$, where $j_1, j_2, j_3, \dots$ are the positive zeros of the functions $J_\nu(x)$, which are Bessel functions of order ν . If $f(x)$ is a piecewise smooth function on the interval $[0, 1]$, then for $0 < x < 1$ it can be resolved in the series

$$\sum_{n=1}^{\infty} c_n J_\nu(j_n x), \quad (2.101)$$

which converges to

$$\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]. \quad (2.102)$$

Since, in this Sturm-Liouville problem, $r(x) = x$, the coefficients c_n are

$$c_n = \frac{\int_0^1 x f(x) J_\nu(j_n x) dx}{\int_0^1 x J_\nu^2(j_n x) dx}, \quad (2.103)$$

or in terms of the scalar product,

$$c_n = \frac{f(x) \cdot J_\nu(j_n x)}{J_\nu(j_n x) \cdot J_\nu(j_n x)}. \quad (2.104)$$

Problems

2.1. Find eigenvalues and eigenfunctions of the Sturm-Liouville problem for the equation $y''(x) + \lambda y(x) = 0$ with the following boundary conditions:

1. $y(0) = 0$ and $y'(l) = 0$;
2. $y'(0) = 0$ and $y(\pi) = 0$;
3. $y'(0) = 0$ and $y'(1) = 0$;
4. $y'(0) = 0$, $y'(1) + y(1) = 0$;
5. $y'(0) + y(0) = 0$, $y(1) = 0$;
6. $y'(0) = 0$, $y'(l) = 0$;
7. $y(-l) = y(l)$, $y'(-l) = y'(l)$ (periodic boundary conditions). As was noted above, if the boundary conditions are periodic then the eigenvalues can be degenerate. Show that in this problem two linearly independent eigenfunctions exist for each eigenvalue.

2.2.

1. Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + y' + \lambda y = 0, \quad y(0) = 0, \quad y(5) = 0.$$

2. Write this differential equation in self-adjoint form and determine an expression for the orthogonality relation.

2.3.

1. Find the eigenvalues and eigenfunctions of the boundary value problem

$$x^2 y'' + x y' + \lambda y = 0, \quad 1 \leq x \leq 10, \quad y(1) = 0, \quad y(10) = 0.$$

2. Write this differential equation in self-adjoint form and determine an expression (2.11) for the orthogonality relation.

2.4. The Hermite differential equation $y'' - 2xy' + 2ny = 0$, $n = 0, 1, 2, \dots$ has polynomial solutions $H_n(x)$. Write this differential equation in self-adjoint form and determine an expression (2.11) for the orthogonality relation.

2.5. Write the Chebyshev equation in the form of a Sturm-Liouville equation and identify $p(x)$, $q(x)$, and $r(x)$. Write the orthogonality condition for Chebyshev polynomials of the first and second kind.

2.6. Classify the following Sturm-Liouville problems as regular, singular, or periodic:

1. $(xy')' + \lambda xy = 0$ on the interval $[0, \infty)$;
2. $[(1 - x^2)y']' + \lambda y(x) = 0$ on the interval $[-1, 1]$;
3. $(1 - x^2)y'' - xy' + \lambda y = 0$ on the interval $[-1, 1]$.

2.7. The Laguerre differential equation

$$xy'' + (1 - x)y' + ny = 0, \quad n = 0, 1, 2, \dots,$$

has polynomial solutions $L_n(x)$. Write this equation in self-adjoint form and give the orthogonality relation.

2.8. Consider the regular Sturm-Liouville problem given by

$$\frac{d}{dx} [(1 + x^2) y'] + \frac{\lambda}{1 + x^2} y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

1. With the substitution $x = \tan \theta$, find the eigenvalues and eigenfunctions of the boundary value problem.
2. Give an orthogonality relation.

2.9. Find the Green's function for the equation $y''(x) + y(x) = f(x)$ on the interval $[0, 1]$ if the function $y(x)$ satisfies boundary conditions $y(0) = 0$ and $y'(1) = 0$.

2.10. Using the Green's function of Problem 2.9, solve the equation $y''(x) + y(x) = x$ satisfying the same boundary conditions $y(0) = 0$ and $y'(1) = 0$.

2.11. Find the Green's function for the equation $y''(x) + y(x) = f(x)$ on the interval $[0, 1]$ if the function $y(x)$ satisfies boundary conditions $y'(0) = 0$ and $y'(1) = 0$.

2.12. Using the Green's function of Problem 2.11, solve the equation $y''(x) + y(x) = x$ satisfying the same boundary conditions $y'(0) = 0$ and $y'(1) = 0$.

2.13. Find the Green's functions for the operators L acting on functions $y(x)$ defined on the interval $[0, 1]$ for the following cases:

1. $L = -d^2/dx^2$, $y(0) = 0$, $y'(1) + y(1) = 0$;
2. $L = -d^2/dx^2$, $y'(0) - y(0) = 0$, $y(1) = 0$;
3. $L = -d^2/dx^2 - 1$, $y'(0) - y(0) = 0$, $y'(1) + y(1) = 0$;

4. $L = -d^2/dx^2 + 1, \quad y(0) = 0, \quad y(1) = 0;$

5. $L = -d^2/dx^2 + 1, \quad y(0) = 0, \quad y'(1) = 0.$

2.14. Find the Green's functions for the operators L acting on functions $y(x)$ defined on the interval $[1, 2]$ for the following cases:

1. $L = -x^2 d^2/dx^2 - 2xd/dx, \quad y'(1) = 0, \quad y(2) = 0;$

2. $L = -xd^2/dx^2 - d/dx, \quad y'(1) = 0, \quad y(2) = 0;$

3. $L = -x^3 d^2/dx^2 - 3x^2 d/dx - 2x^2, \quad y(1) = 0, \quad 2y'(2) + 2y(2) = 0.$

2.15. Find the Green's functions for the operators L on the interval $[0, \pi/4]$ for the following cases:

1. $Ly = -(y' \cos^2 x)', \quad y(0) = 0, \quad y(\pi/4) = 0;$

2. $Ly = -(y' \cos^2 x)', \quad y'(0) - y(0) = 0, \quad y'(\pi/4) + y(\pi/4) = 0.$

2.16. Find the Green's functions for the operators L acting on functions $y(x)$ defined on the interval $[0, 1]$ for the following cases:

1. $L = -d^2/dx^2, \quad y(0) = y(1), \quad y'(0) = y'(1);$

2. $L = -d^2/dx^2 - \pi^2, \quad y(0) = y(1) = 0;$

3. $L = -d^2/dx^2 - \pi^2, \quad y'(0) = y'(1) = 0;$

4. $L = -d^2/dx^2 + 1, \quad y(0) = y(1), \quad y'(0) = y'(1).$

5. $L = -d^2/dx^2 - 4d/dx, \quad y'(0) - y(0) = 0, \quad y(1) = 0;$

6. $L = -d^2/dx^2 - 2d/dx + 2, \quad y(0) = 0, \quad y(1) = 0;$

7. $L = -d^2/dx^2 + 4d/dx, \quad y'(0) - y(0) = 0, \quad y'(1) + y(1) = 0;$

8. $L = -2d^2/dx^2 + 4d/dx, \quad y(0) = 0, \quad y(1) = 0.$

2.17. Find the Green's function and solutions of the following boundary value problem on the interval $[0, \pi]$: $y'' + 4y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$

2.18. Find the Green's function and solutions of the following boundary value problem on the interval $[1, 2]$: $x^4 y'' + x^3 y' + 2x^2 y = f(x), \quad y(1) = 0, \quad y'(2) + y(2) = 0.$

2.19. Using the Green's function method, find the solution of the boundary value problem given by $(1 + \cos x)y'' - y \sin x = f(x), \quad 0 \leq x \leq \pi/2, \quad 2y'(0) - y(0) = 0, \quad y(\pi/2) = 0.$

2.20. Find the Green's functions for the following operators L :

1. $Ly = -y'(x) - y(x)$, $0 \leq x \leq 1$, $y(0) = y(1)$;
2. $Ly = -y''(x) + 2y(x)/x^2$, $0 \leq x < \infty$, $y(0) = 0$, $|y(x \rightarrow \infty)| < \infty$.

2.21. Find the Green's functions of the oscillation equation $y''(x) + \lambda^2 y(x) = f(x)$, $0 \leq x \leq 1$ for the following boundary conditions:

1. $y(0) = y(1) = 0$;
2. $y'(0) = y'(1) = 0$;
3. $y(0) = y(1)$, $y'(0) = y'(1)$.

2.22. Obtain the Fourier series expansion of the functions

1. $f(x) = x(1 - x)$,
2. $f(x) = x$ for $0 \leq x \leq 0.5$ and $f(x) = 1 - x$ for $0.5 \leq x \leq 1$,
3. $f(x) = \cos(2\pi x) - 1$

using the eigenfunctions of the Sturm-Liouville problem 2.13.4 as the complete set of functions. Compare and discuss the speeds of the convergence for these functions, $f(x)$.

2.23. Obtain the Fourier series expansion of

1. $f(x) = x(x - 2)$,
2. $f(x) = \pi x + \sin(\pi x)$,
3. $f(x) = x^3 - 3x$,
4. $f(x) = \cos(\pi x) - 1$

using the eigenfunctions of the Sturm-Liouville problem 2.13.5 as the complete set of functions. Compare and discuss the speeds of the convergence for these functions $f(x)$.

2.24. Obtain the Fourier series expansion of

1. $f(x) = x(x - 3/2)$,
2. $f(x) = \pi x/2 + \sin(\pi x)$,
3. $f(x) = 1 - x - \cos(\pi x)$

using the eigenfunctions of the Sturm-Liouville problem 2.13.1 as the complete set of functions. Compare and discuss the speeds of the convergence for these functions $f(x)$.

2.25. Obtain the Fourier series expansion of

1. $f(x) = 2x^2 - x - 1$,
2. $f(x) = x + \cos(\pi x)$,
3. $f(x) = \pi(1 - x)/2 + \sin(\pi x)$

using the eigenfunctions of the Sturm-Liouville problem 2.13.2 as the complete set of functions. Compare and discuss the speeds of the convergence for these functions $f(x)$.

2.26. Obtain the Fourier series expansion of

1. $f(x) = x^2 - x - 1$,
2. $f(x) = (2x - 1)/3 + \cos(\pi x)$

using the eigenfunctions of the Sturm-Liouville problem 2.13.3 as the complete set of functions. Compare and discuss the speeds of the convergence for these functions $f(x)$.

2.27. Expand the following functions, $f(x)$, in a Fourier series using the eigenfunctions of the boundary value problem indicated below. Notice that, contrary to Problems 2.22 through 2.26, the functions $f(x)$ do not satisfy the boundary conditions of the corresponding Sturm-Liouville problem. Determine how this affects the speeds of the convergence of the respective series.

1. $f(x) = x(x - 1)$. Use the eigenfunctions of Problem 2.13.1;
2. $f(x) = x - 1$. Use the eigenfunctions of Problem 2.13.2;
3. $f(x) = \sin(\pi x)$. Use the eigenfunctions of Problem 2.13.3;
4. $f(x) = \cos(\pi x/2)$. Use the eigenfunctions of Problem 2.13.4;
5. $f(x) = \sin(\pi x)$. Use the eigenfunctions of Problem 2.13.5.

3

One-Dimensional Hyperbolic Equations

Different physical phenomena may have some common features and be described by the same or similar equations. When this is the case, a study of one physical system may lead to a deeper understanding of a similar but less familiar system. In this chapter, we study systems related to the problem of oscillations of a string. Starting with the string system as the paradigm case, we go on to examine the related systems of longitudinal oscillations of a thin rod, torsional oscillations of a cylinder, acoustic waves, waves of a fluid in a shallow channel and electrical oscillations in a circuit.

We start with the classical problem of small transverse oscillations of a thin, stretched string. The equilibrium position of the string is located along the x -axis and the string itself moves perpendicular to the x -axis with no stretching in the x direction. Let $u(x, t)$ represent displacements of the string from the equilibrium so that a graph of the function $u(x, t)$ gives the string's amplitude at the location x and at the instant t (Figure 3.1). For small oscillations, we assume that the displacement $u(x, t)$ and derivative $u_x(x, t)$ are small and their second powers and products can be neglected compared to the magnitude of $u(x, t)$. Here and in the following discussion, a subscript will be used to denote a derivative.

3.1 Derivation of the Basic Equations

To derive the equation describing the oscillations of a string, first consider an interval $(x, x + \Delta x)$ corresponding to a segment, AB , of the string (Figure 3.1).

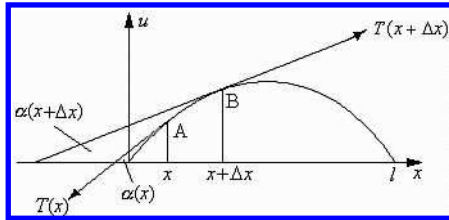


Figure 3.1. Schematic picture of string of length l with arbitrary displacement.

Points on the string are constrained to move perpendicular to the x direction, which means the tension forces at points A and B are directed along tangent lines at those points and the sum of their x components equals zero:

$$T(x) \cos \alpha(x) - T(x + \Delta x) \cos \alpha(x + \Delta x) = 0.$$

For small oscillations, we may use the approximation

$$\cos \alpha(x) = \frac{1}{\sqrt{1 + \tan^2 \alpha(x)}} = \frac{1}{\sqrt{1 + u_x^2(x)}} \approx 1,$$

in which case

$$T(x) \approx T(x + \Delta x);$$

that is, the value of tension T does not depend on x , and for all x and t we have $T \approx T_0$, where T_0 is the tension in the equilibrium state.

As can be seen from Figure 3.1, at point x the vertical component of the force of tension is

$$T_y = -T_0 u_x(x, t). \quad (3.1)$$

The same expression with a positive sign holds at the point $x + \Delta x$. The signs of the tension at x and $x + \Delta x$ depend on the orientation of the segment, AB , of the string and are opposite for the two ends of the segment. This relation will be important in deriving boundary conditions.

The sum of the vertical components of the forces of tension at points A and B is

$$Y = T_0 [\sin \alpha(x + \Delta x) - \sin \alpha(x)].$$

For small oscillations, we may use the expansion

$$\sin \alpha(x) = \frac{\tan \alpha(x)}{\sqrt{1 + \tan^2 \alpha(x)}} = \frac{u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} \approx \frac{\partial u}{\partial x},$$

so that

$$Y = T_0 \left[\left(\frac{\partial u}{\partial x} \right)_{x=x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_{x=x} \right].$$

In the limit as Δx becomes small we may replace the terms inside the brackets by a partial derivative, arriving at

$$Y = T_0 \frac{\partial^2 u}{\partial x^2} dx.$$

For a string with a position-dependent density, $\rho(x)$, the force Y on segment Δx is equal to mass times acceleration or

$$Y = \rho(x) \frac{\partial^2 u}{\partial t^2} dx.$$

If there is also an external force $F(x, t)$ per unit length acting on the string perpendicular to the x -axis, we obtain the equation for *forced oscillations of a string*:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + F(x, t). \quad (3.2)$$

For the case of a constant linear mass density with $\rho = \text{const}$, i.e., for a uniform string, this equation can be written as

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (3.3)$$

where $a = \sqrt{T_0/\rho}$, $f(x, t) = F(x, t)/\rho$. In the case where the external force is gravity, we have $f(x, t) = -mg/l\rho = -g$ in Equation (3.3). If there is no external force ($F(x, t) \equiv 0$) we have the equation for free oscillations of a string

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (3.4)$$

which is referred to as the nonhomogeneous *wave equation*.

It may be immediately verified that

$$u_1(x, t) = f_1(x - at) \quad \text{and} \quad u_2(x, t) = f_2(x + at)$$

are solutions of Equation (3.4), where f_1 and f_2 are arbitrary, twice differentiable functions. Each of these solutions has a simple physical interpretation. In the first case, the displacement $u = u_1$ at point x and time t is the

same as that at point $x + a\Delta t$ at time $t + \Delta t$. Thus, the disturbance moves in the direction of increasing x with velocity a . The quantity a is therefore the “velocity of propagation” of the disturbance, or the *wave speed*. In the second case, the displacement $u = u_2$ at point x at time t is found at the point with coordinate $x - a\Delta t$ at a later time $t + \Delta t$. This disturbance therefore travels in the direction of decreasing x with velocity a . The general solution of Equation (3.4) can be written as the sum of solutions, such as $u_1(x, t)$ and $u_2(x, t)$:

$$u(x, t) = f_1(x - at) + f_2(x + at).$$

This solution is described with more details in Section 3.8, devoted to the method of D’Alembert.

Reading Exercise. Select two arbitrary, twice differentiable functions $f_1(x - at)$ and $f_2(x + at)$ (e.g., sine or exponential functions) and show that they are solutions to the equation (3.4). Also show that their sum is a solution.

Equation (3.4) describes the simplest situation with no external forces and no dissipation. For a string vibrating in an elastic medium with a linear force per unit length of $F = -\alpha u$, where α is the Hooke’s law coefficient, we have the wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - \alpha u. \quad (3.5)$$

When a string oscillates in a medium with friction proportional to the speed of the string, the force per unit length, F , is given by $F = -ku_t$, where k is a coefficient of friction. For this case, the equation contains the time derivative $u_t(x, t)$ and we have the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - 2\kappa \frac{\partial u}{\partial t}, \quad (3.6)$$

where $2\kappa = k/\rho$.

3.2 Boundary and Initial Conditions

Partial (as well as ordinary) differential equations such as those in Section 3.1 generally have an infinite number of solutions. Thus, when a physical

problem is described by such an equation, additional conditions are needed to find the unique solution defining the behavior of the system. These additional conditions are determined by the physical nature of the system and should obey the following demands:

1. They should guarantee the *uniqueness* of the solution (i.e., there should not be two different functions satisfying the equation and additional conditions).
2. They should guarantee the *stability* of the solution (i.e., any small variations of these additional conditions or the coefficients of the differential equation result in only insignificant variations in the solution). In other words, the solution should depend *continuously* on additional conditions and the coefficients of the equation.

These additional conditions may be classified as two distinct types; initial conditions and boundary conditions.

Initial conditions characterize the function satisfying the equation at the initial moment $t = 0$. Equations that are second order in time have two initial conditions. For example, in the problem of transverse oscillations of a string, the initial conditions define the string's shape and speed distribution at zero time:

$$u(x, 0) = \varphi(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad (3.7)$$

where $\varphi(x)$ and $\psi(x)$ are specified functions of x .

Boundary conditions characterize the behavior of the function satisfying the equation at the boundary of the physical region of interest for all moments of time t . In most cases, the boundary conditions for partial differential equations give the function $u(x, t)$ and/or the normal component of its gradient along the boundary.

Let us consider various boundary conditions for transverse oscillations of a string over the finite interval $0 \leq x \leq l$ from a physical point of view.

1. If the left end of the string, located at $x = 0$, is rigidly fixed, the boundary condition at $x = 0$ is

$$u(0, t) = 0.$$

A similar condition exists for the right end of the string, located at $x = l$, if it is fixed. These are called *fixed end* boundary conditions.

2. If the motion of left end of the string is driven with the function $g(t)$, then

$$u(0, t) = g(t),$$

in which case we have *driven end* boundary conditions.

3. If the end at $x = 0$ can move and experiences a force, $f(t)$, that varies with time (e.g., a string attached to a ring that is driven up and down on a vertical rod), then from Equation (3.1), we have

$$-T_0 u_x(0, t) = f(t).$$

If this boundary condition is applied, instead, to the right end of the string located at $x = l$, the left-hand side of this formula will have a positive sign. These are called *forced end* boundary conditions and are different from driven end conditions because the slope rather than the position is specified as the initial condition.

4. If the end at $x = 0$ moves freely, but is still attached (e.g., a string attached to a ring that can slide up and down on a vertical rod with no friction), then the slope at the end will be zero. In this case, the last equation gives

$$u_x(0, t) = 0$$

with similar equations for the right end. The conditions in this case are called *free end* boundary conditions.

5. If the left end is attached to a surface that can stretch, we have an *elastic boundary*, in which case we add a vertical component of elastic force $-ku(0, t)$, to the left-hand side of Equation (3.1). This results in the boundary condition

$$u_x(0, t) - hu(0, t) = 0, \quad h = \frac{k}{T_0}.$$

For the right end we have

$$u_x(l, t) + hu(l, t) = 0.$$

If the point to which the string is elastically attached is also moving and its deviation from the initial position is described by the function $\zeta(t)$, the boundary condition becomes

$$u_x(0, t) - h[u(0, t) - \zeta(t)] = 0. \quad (3.8)$$

It can be seen that for stiff attachment (large k) when even a small shift of the end causes strong tension, the boundary condition of Equation (3.8) becomes $u(0, t) = g(t)(k = \infty)$ with $g(t) = \xi(t)$. For weak attachment (small k , weak tensions), this condition (3.8) becomes the condition for a free end $u_x(0, t) = 0(k = 0)$.

In general, for one-dimensional problems, the boundary conditions at the ends $x = 0$ and $x = l$ can be summarized in the form

$$\alpha_1 u_x + \beta_1 u|_{x=0} = g_1(t), \quad \alpha_2 u_x + \beta_2 u|_{x=l} = g_2(t), \quad (3.9)$$

where $g_1(t)$ and $g_2(t)$ are known functions, and $\alpha_1, \beta_1, \alpha_2, \beta_2$ are (real) constants. As discussed in Chapter 2, due to physical constraints the normal restrictions on these constants are $\beta_1/\alpha_1 < 0$ and $\beta_2/\alpha_2 > 0$.

When functions on the right-hand sides of Equation (3.9) are zero (i.e., $g_{1,2}(t) \equiv 0$), the boundary conditions are said to be *homogeneous*. In this case, if $u_1(x, t), u_2(x, t), \dots, u_n(x, t)$ satisfy these boundary conditions, then any linear combination of these functions

$$C_1 u_1(x, t) + C_2 u_2(x, t) + \dots + C_n u_n(x, t)$$

(where $C_1, \dots, C_n$ are constants) also satisfies these conditions. This property will be used frequently in the following discussion.

We may classify the above physical notions of boundary conditions as formally belonging to one of three main types:

1. *Boundary conditions of the first kind (Dirichlet boundary conditions).* For this case, we are given $u|_{x=a} = g(t)$, where here and below $a = 0$ or l . This describes a given boundary regime; for example, if $g(t) = 0$ we have fixed ends.
2. *Boundary conditions of the second kind (Neumann boundary conditions).* In this case, we are given $u_x|_{x=a} = g(t)$, which describes a given force acting at the ends of the string; for example, if $g(t) = 0$ we have free ends.
3. *Boundary conditions of the third kind (mixed boundary conditions).* Here we have $u_x \pm hu|_{x=a} = g(t)$ (minus sign for $a = 0$, plus sign for $a = l$); for example, an elastic attachment for the case $h = \text{const}$.

Applying these three conditions alternately to the two ends of the string result in nine types of boundary problems. A list classifying all possible combinations of boundary conditions can be found in Appendix A.

As mentioned previously, the initial and boundary conditions completely determine the solution of the wave equation. It can be proved that under certain conditions of smoothness of the functions $\varphi(x)$, $\psi(x)$, $g_1(t)$, and $g_2(t)$ defined in Equations (3.7) and (3.9), a unique solution always exists; therefore, these conditions are, in general, necessary. The following sections investigate many examples of the dependence of the solutions on the boundary conditions.

In some physical situations, either the initial conditions or the boundary conditions may be ignored, leaving only one condition to determine the solution. For instance, suppose the point M_0 is rather distant from the boundary and the boundary conditions are given such that the influence of these conditions at M_0 is exposed after a rather long time interval. In such cases, if we investigate the situation for a relatively short time interval, instead of a complete problem, we can ignore the boundaries and study the *initial value problem* (or the *Cauchy problem*). Formally, these solutions are for an infinite region, but they apply to a finite string for times short enough that the boundary conditions have not had time to have an effect. For instance, in the one-dimensional case for short time periods, we may ignore the boundary conditions and search for the solution of the equation

$$u_{tt} = a^2 u_{xx} + f(x, t) \text{ for } -\infty < x < \infty, t > 0,$$

with the initial conditions

$$\left. \begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x) \end{aligned} \right\} \text{ for } -\infty < x < \infty.$$

Similarly, if we study a process close enough to one boundary (at one end for the one-dimensional case) and rather far from the other boundary, for some characteristic time of that process the boundary condition at the distant end may be insignificant. For the one-dimensional case, we arrive at a boundary value problem for a semi-infinite region, $0 \leq x < \infty$, where in addition to the differential equation we have additional conditions

$$\begin{aligned} u(0, t) &= g(t), \quad t > 0, \\ \left. \begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x) \end{aligned} \right\} \quad 0 < x < \infty. \end{aligned}$$

In contrast, it is clear physically that for a substantially large time, the behavior of the system mostly depends on the boundary conditions. These are problems “*with no initial conditions*” (the *steady-state regime*). For the one-dimensional case, they are formulated in the following way: Find the solution of the equation at hand (e.g., the homogeneous wave equation) for the region $0 \leq x \leq l$ for times $t \gg 0$ with the boundary conditions

$$u(0, t) = g_1(t), \quad u(l, t) = g_2(t).$$

Here as well as in the previous situation, other kinds of boundary conditions as described above can be applied.

3.3 Other Boundary Value Problems: Longitudinal Vibrations of a Thin Rod

In this section, we consider other boundary value problems that are similar to the vibrating string problem. The intent here is to show the similarity in the approach to solving physical problems, which on the surface appear quite different but in fact have a similar mathematical structure.

We start this section by considering a thin elastic rod of cylindrical, rectangular, or other regular cross section. In this case, forces applied along the axis, perpendicular to the (rigid) cross section, will cause changes in the length of the rod. We will assume that the forces act along the rod axis and each cross-sectional area can move only in the x direction. Such assumptions can be justified if the transverse dimensions are substantially smaller compared to the length of the rod and the forces acting on the rod are comparatively weak.

3.3.1 Derivation of the Basic Equations

If a force compresses the rod along its axis and is then released, the rod will begin to vibrate along this axis. Let the ends of the rod be located at the points $x = 0$ and $x = l$ when it is at rest. The location of some cross section at rest will be given by x . Let the function $u(x, t)$ be the longitudinal shift of this cross section from equilibrium at time t . The derivative $u_x(x, t)$ gives the relative length change at location x .

We start with the element of the rod between two cross sections S and S_1 (Figure 3.2), with coordinates at rest of x and $x + dx$. The forces of

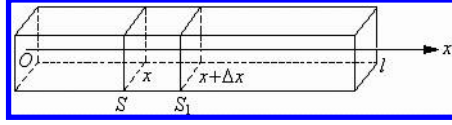


Figure 3.2. Arbitrary segment of a rod of length Δx and rectangular cross section.

tension at these cross sections, T_x and T_{x+dx} , act along the x -axis. For small vibrations, the resultant of these two forces is

$$T_{x+dx} - T_x = ES \left. \frac{\partial u}{\partial x} \right|_{x+dx} - ES \left. \frac{\partial u}{\partial x} \right|_x \approx ES \frac{\partial^2 u}{\partial x^2} dx,$$

where E is the elasticity modulus of the rod and S is its cross-sectional area. The acceleration of this element is $\frac{\partial^2 u}{\partial t^2}$. Together, these two equations give the equation of longitudinal motion of a cross-sectional element as

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial^2 u}{\partial x^2} dx, \quad (3.10)$$

where ρ is the rod density. Using the notation

$$a = \sqrt{E/\rho}, \quad (3.11)$$

we obtain the differential equation for *longitudinal free oscillations of a uniform rod* as

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \quad (3.12)$$

As discussed earlier, solutions of such hyperbolic equations have a wave character with the speed of wave propagation, a , given by (3.11).

If there is also an external force per unit volume, $F(x, t)$, we obtain instead the equation

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial^2 u}{\partial x^2} dx + F(x, t) S dx,$$

or, by collecting terms, we get

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho} F(x, t). \quad (3.13)$$

This is the equation for *forced oscillations of a uniform rod*. Note the similarity of Equation (3.13) to Equation (3.3) for a string under forced oscillations; the two equations are equivalent.

The initial conditions are similar to those for a string

$$u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x),$$

which are the initial deflection and initial speed of points of a rod, respectively.

3.3.2 Boundary Conditions for a Rod

The boundary conditions for a rod are the following:

1. For a rod with *rigid fixed ends*, with the left end located at $x = 0$, the boundary condition is

$$u(0, t) = 0.$$

2. If the left end at $x = 0$ is *driven* by the function $g_1(t)$, then

$$u(0, t) = g_1(t),$$

where $g_1(t)$ is a given function of t .

3. If the left end is *free*, the tension at that location is zero, $T(a, t) = 0$. Then, from $T(x, t) = ES \cdot u_x(x, t)$, the condition follows that

$$u_x(0, t) = 0.$$

4. If the either end of the rod is attached to an elastic material (a wall that “gives” in the horizontal direction), we say the rod has a boundary condition with an *elastic end*. For the right end of the rod, located at $x = l$, the force on the element $(l - \Delta x, l)$ from the rest of the rod (from the left) is

$$-ES \frac{\partial u}{\partial x}(l - \Delta x, t),$$

and from the right there is an elastic force from the wall of

$$-ku(l, t),$$

where k is elasticity coefficient of the rod-wall connection. Newton's law for this element gives

$$\rho_0 S \Delta x \frac{\partial^2 u}{\partial t^2} = -ES \frac{\partial u}{\partial x}(l - \Delta x, t) - ku(l, t).$$

In the limit $\Delta x \rightarrow 0$, we obtain the boundary condition for the right end at $x = l$

$$ESu_x(l, t) + ku(l, t) = 0,$$

or

$$u_x(l, t) + hu(l, t) = 0,$$

where $h = k/ES$.

For the left end of the rod, located at $x = 0$, the sign of h in the boundary condition is the opposite. If we take the element $(0, \Delta x)$ and apply a force from the left of

$$-ku(0, t)$$

with a force from the right of

$$ES \frac{\partial u}{\partial x}(\Delta x, t),$$

the equation for this element is

$$\rho_0 S \Delta x \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial u}{\partial x}(\Delta x, t) - ku(0, t).$$

In the limit $\Delta x \rightarrow 0$, we obtain

$$u_x(0, t) - hu(0, t) = 0.$$

3.4 Torsional Oscillations of an Elastic Cylinder

Next, we treat small torsional oscillations of an elastic uniform cylinder. For this system, we imagine each cross section twisting about the cylinder axis while remaining in the same location on the axis.

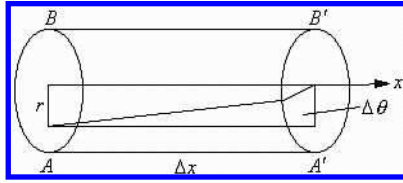


Figure 3.3. Elastic cylinder with one end twisted by an angle $\Delta\theta$ relative to the other end.

In each cross section, normal forces along the axis are assumed to be absent. The tangent forces on a cross-sectional plane create rotational motion around the axis. We assign the origin to the center of the leftmost cross section, which is at rest, and the x -axis is directed along the cylinder axis (Figure 3.3). A cross section AB at distance x from the origin rotates by some angle θ that depends on x and time. A second cross section $A'B'$ at a distance Δx from AB rotates by the angle $\theta + \Delta\theta$. Thus, the shift of $A'B'$ relative to AB is $\Delta\theta$.

3.4.1 Derivation of the Basic Equation

The torque M_x , applied to the cross section AB located at x is proportional to the shear modulus, G , and the moment of inertia per unit mass of the cross section AB relative to its center, J_0 , and to the degree of twist as a function of position, $d\theta/dx$ at the cross section:

$$M_x = GJ_0 \frac{d\theta}{dx}.$$

The torque will also be equal to the product of the moment of inertia of the segment Δx times the angular acceleration, $\partial^2\theta/\partial t^2$, so that the angle θ depends on both x and t , $\theta = \theta(x, t)$. Using the notation K for the momentum of inertia of the cylinder per unit length, we have the moment of inertia of the segment as $K\Delta x$. Adding the two torques acting (in opposite directions) on each end of the segment Δx , we get

$$M_{x+\Delta x} - M_x = GJ_0 \left[\frac{\partial\theta}{\partial x}(x + \Delta x, t) - \frac{\partial\theta}{\partial x}(x, t) \right] = GJ_0 \frac{\partial^2\theta}{\partial x^2} \Delta x.$$

Thus, the dynamic equation for a small segment of the cylinder is

$$K\Delta x \frac{\partial^2\theta}{\partial t^2} = GJ_0 \frac{\partial^2\theta}{\partial x^2} \Delta x,$$

or

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2}, \quad \text{where } a = \sqrt{GJ_0/K}. \quad (3.14)$$

If we compare Equation (3.14) and Equation (3.12), we see that we obtained *the same equations as for the string oscillations*, Equation (3.4). Only the definitions of the terms are different; the equations and the boundary conditions are, mathematically speaking, the same.

For a cylinder with uniform density ρ , we find for the moment of inertia K that

$$K = \rho \int_0^l dx \iint r^2 d\sigma = \rho J_0,$$

which gives $a = \sqrt{G/\rho}$, where a is the speed of a torsional wave on the rod.

3.4.2 Initial Conditions and Examples of Boundary Conditions for a Twisted Rod

Initial conditions are given by specifying an initial twist and twist velocity, respectively, for segments along the cylinder:

$$\theta(x, 0) = \varphi(x), \quad \frac{\partial \theta}{\partial t}(x, 0) = \psi(x).$$

Boundary conditions are similar to those for longitudinal oscillations of a rod:

1. When the ends of the cylinder are *rigidly fixed* the boundary conditions are given by

$$\theta(0, t) = 0, \quad \theta(l, t) = 0.$$

2. When the ends of the cylinder are *free*, the boundary conditions are given by

$$\frac{\partial \theta}{\partial x}(0, t) = 0, \quad \frac{\partial \theta}{\partial x}(l, t) = 0.$$

3. If the ends of the cylinder are *fixed elastically* to a flexible wall, the boundary conditions are

$$\frac{\partial \theta}{\partial x}(0, t) - h\theta(0, t) = 0, \quad \frac{\partial \theta}{\partial x}(l, t) + h\theta(l, t) = 0.$$

3.5 Acoustic Waves

In this section, we discuss a related problem: acoustic waves in an ideal gas. Under appropriate physical constraints, we arrive at a wave equation similar to the previous two examples but in a higher dimensional space.

3.5.1 Derivation of the Basic Equation

We start by considering small fluctuations of an ideal gas. We will assume all changes to be adiabatic; that is, the pressure $p(\vec{r}, t)$ and gas density $\rho(\vec{r}, t)$ are related by the equation of state

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma \quad (3.15)$$

where p_0 and ρ_0 are the equilibrium pressure and density, respectively, and $\gamma = c_p/c_v$ is the ratio of molar specific heats at constant pressure and constant volume for the gas.

For small fluctuations in density ($\rho - \rho_0$ small) we can linearize Equation (3.15) by expanding it in a Taylor series around ρ_0 :

$$p = p_0 \left[1 + \gamma(\rho - \rho_0)/\rho_0 \right]. \quad (3.16)$$

This equation is a particular case of a more general situation when small fluctuation in density and pressure are linearly proportional, in which case we may use

$$\tilde{p} = \alpha^2 \tilde{\rho}, \quad (3.17)$$

where $\tilde{p}(x, t) = p(x, t) - p_0$ is a “pressure perturbation,” and $\tilde{\rho}(x, t) = \rho(x, t) - \rho_0$ is a “density perturbation.” The coefficient of proportionality, the derivative

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho} \right)_S,$$

is determined by the compressibility of the medium. For an ideal medium, the compressions and expansions are adiabatic and the index S indicates the derivative is taken at constant entropy S . This derivative has the dimension of speed and is a characteristic of the medium. The medium could be a gas, fluid, solid, or plasma; thus, the results of this section are applicable to rather different physical situations and substances, which all obey the relation in Equation (3.17).

Newton's law of motion for a volume of gas ΔV is

$$\rho \frac{\partial \vec{v}}{\partial t} \Delta V + \text{grad } p \Delta V = \vec{F} \Delta V$$

where $\vec{F}(\vec{r}, t)$ is an external force (taken per unit volume) and $\vec{v}$ is the velocity of the volume element ΔV . (See Appendix D for a review of vector calculus concepts and formulas needed in this section.) Approximating the density, ρ , by the equilibrium density ρ_0 , we obtain

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \vec{F} - \text{grad } p, \quad (3.18)$$

which is called the Euler equation.

The *continuity equation* for small fluctuations in a medium without sources or sinks (i.e., mass is neither created nor destroyed) is

$$\frac{\partial \rho}{\partial t} + \rho_0 \text{div } \vec{v} = 0, \quad (3.19)$$

where we have replaced ρ by ρ_0 . Equations (3.17) and (3.18) together with Equation (3.19) give a closed system of equations and are the bases for what is known as the “linear acoustic” approximation for acoustic waves.

By differentiating Equation (3.19) by t and changing the order of derivative operators, we get

$$\frac{\partial^2 \rho}{\partial t^2} + \rho_0 \text{div } \frac{\partial \vec{v}}{\partial t} = 0.$$

Together with equation (3.18), the identity $\text{div grad} = \nabla^2$ (see Appendix D), and Equation (3.17), we have

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \nabla^2 \rho - \text{div } \vec{F}. \quad (3.20)$$

By differentiating Equation (3.17) twice by t and using Equation (3.20), we obtain

$$p_{tt} = \alpha^2 (\alpha^2 \nabla^2 \rho - \text{div } \vec{F})$$

which yields

$$\frac{\partial^2 p}{\partial t^2} = \alpha^2 \nabla^2 p - \alpha^2 \text{div } \vec{F}. \quad (3.21)$$

Next, we may replace Equation (3.21) with one involving $\vec{u}(\vec{r}, t)$, the displacement vector of elementary volume ΔV , and $\vec{v}(\vec{r}, t)$, the corresponding velocity vector. First, by differentiating Equation (3.18) by t , changing the order of operators $\partial/\partial t$ and ∇ , and using Equation (3.17), we obtain

$$\rho_0 \frac{\partial^2 \vec{v}}{\partial t^2} = \frac{\partial \vec{F}}{\partial t} - \alpha^2 \nabla \frac{\partial \rho}{\partial t}.$$

Then, with Equation (3.19), we have

$$\frac{\partial^2 \vec{v}}{\partial t^2} = \alpha^2 \text{grad div } \vec{v} + \frac{1}{\rho_0} \frac{\partial \vec{F}}{\partial t}. \quad (3.22)$$

For an ideal medium oscillating with small amplitudes, the motion is conservative so that $\nabla \times \vec{v} = 0$; thus, with the identity known from vector calculus

$$\text{grad div } \vec{v} = \nabla^2 \vec{v} + \nabla \times \nabla \times \vec{v},$$

Equation (3.22) gives

$$\frac{\partial^2 \vec{v}}{\partial t^2} = \alpha^2 \nabla^2 \vec{v} + \frac{1}{\rho_0} \frac{\partial \vec{F}}{\partial t}. \quad (3.23)$$

Since $\vec{u}$ and $\vec{v}$ are related by $\vec{v} = \vec{u}_t$, the equation describing vector $\vec{u}$ is

$$\frac{\partial^2 \vec{u}}{\partial t^2} = \alpha^2 \nabla^2 \vec{u} + \frac{1}{\rho_0} \vec{F}. \quad (3.24)$$

Equations (3.20) through (3.24) are called the *acoustic equations*.

If there is no external force, the density ρ and pressure p satisfy the homogeneous differential equations

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \nabla^2 \rho, \quad \frac{\partial^2 p}{\partial t^2} = \alpha^2 \nabla^2 p. \quad (3.25)$$

As we can see, a nonstationary perturbation in pressure or density generated in some volume of a medium does not stay localized but propagates in the medium in all directions with speed α as a *sound wave*. For a gas, the speed is a function of pressure and density, $\alpha = \sqrt{\gamma p_0 / \rho_0}$, as seen from Equation (3.16).

The energy per unit volume of the oscillating medium is

$$E = \frac{\rho_0 v^2}{2} + \frac{\bar{p}^2}{2\rho_0 \alpha^2}. \quad (3.26)$$

Thus, if we have the solution of the boundary value problem, the expression in Equation (3.26) makes it possible to find the energy excitations of different modes.

In the following discussion, we limit the acoustic equations to the case of *one-dimensional oscillations*. As a physical example, we assume that the substance (a gas, for instance) makes small longitudinal oscillations along a tube in such a way that all elementary volume elements move parallel to the tube axis, x . For this case, pressure and density are functions of x and t , and the external force has only one component, $F(x, t)$. The acoustic equations become

$$\begin{aligned} \frac{\partial^2 \rho}{\partial t^2} &= \alpha^2 \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial F}{\partial x}, & \frac{\partial^2 p}{\partial t^2} &= \alpha^2 \frac{\partial^2 p}{\partial x^2} - \alpha^2 \frac{\partial F}{\partial x}, \\ \frac{\partial^2 v}{\partial t^2} &= \alpha^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{\rho_0} \frac{\partial F}{\partial t}, & \frac{\partial^2 u}{\partial t^2} &= \alpha^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho_0} F. \end{aligned} \quad (3.27)$$

Note that we do not need to solve these equations simultaneously. Depending on the boundary conditions, we can solve one of these equations and then, using the relations above between p and ρ and u and v , obtain the other quantities. Note also the relation between Equations (3.27) and the equations derived above for a string and a vibrating rod. In the case where there is no external force ($F = 0$) in Equation (3.27), we arrive at the wave equation.

3.5.2 Boundary Conditions for a Tube Filled with Gas

Physically, we can see that for one-dimensional problems of gas in a tube, the only relevant boundary conditions are the conditions at the ends of the tube.

1. If the ends of the tube (located at $x = 0$ and $x = l$) are closed by *rigid walls* that cannot be penetrated, the boundary conditions for

u , v , p , and ρ are (using the index x to denote a derivative)

$$\begin{aligned} p_x(0, t) = p_x(l, t) &= 0, \\ \rho_x(0, t) = \rho_x(l, t) &= 0, \\ u(0, t) = u(l, t) &= 0, \\ v(0, t) = v(l, t) &= 0. \end{aligned}$$

The conditions for p and ρ are equivalent due to Equation (3.17) and follow from the Euler equation (with zero external forces at the boundary). The conditions for u and v are simply the physical limitation that the gas not penetrate the walls.

2. If the ends of the tube are *open*, the boundary conditions are

$$\begin{aligned} \tilde{p}(0, t) = \tilde{p}(l, t) &= 0, \\ \tilde{\rho}(0, t) = \tilde{\rho}(l, t) &= 0, \\ u_x(0, t) = u_x(l, t) &= 0, \\ v_x(0, t) = v_x(l, t) &= 0. \end{aligned}$$

The first two expressions mean that for an open tube, pressure and density at the ends do not change. Then, with $\text{div } \vec{v} = 0$, we obtain the conditions for u and v .

3. If the ends are closed by massless pistons attached by springs with elasticity coefficient κ and moving with no friction inside the tube, we have *elastic boundaries*. The springs act on the pistons with an elastic force $-\kappa u(l, t)$ when a piston is displaced by distance u .

At the right end, $x = l$, we have the boundary condition for $u(x, t)$ given by

$$u_x(l, t) + hu(l, t) = 0,$$

with a constant h that depends on the properties of the springs and the area of the piston.

At the left end, $x = 0$, and we have

$$u_x(0, t) - hu(0, t) = 0.$$

For $v(x, t)$ under the same conditions, we have

$$v_x(0, t) - hv(0, t) = 0, \quad v_x(l, t) + hv(l, t) = 0.$$

For $\tilde{p}(x, t)$ and $\tilde{\rho}(x, t)$, the following conditions can be obtained:

$$\tilde{p}_{tt}(0, t) - h^* \tilde{p}_x(0, t) = 0, \quad \tilde{p}_{tt}(l, t) + h^* \tilde{p}_x(l, t) = 0,$$

$$\tilde{\rho}_{tt}(0, t) - h^* \tilde{\rho}_x(0, t) = 0, \quad \tilde{\rho}_{tt}(l, t) + h^* \tilde{\rho}_x(l, t) = 0,$$

where $h^* = h\gamma$. The same conditions are valid for the functions $p(x, t)$ and $\rho(x, t)$.

Reading Exercise. Consider the motion of a medium characterized by the force potential $U(\vec{r})$ and velocity potential φ ; thus, $\vec{F} = -\nabla U$ and $\vec{v} = -\nabla \varphi$. U and $\vec{F}$ are given as potential and force per unit volume. Check that the equation that follows from the Euler equation is

$$\rho_0 \frac{\partial \varphi}{\partial t} = -U - p + \text{const}(\vec{r}).$$

(The constant does not depend on $\vec{r}$, but can depend on t .) Derive an equation for φ . The advantage of such an equation is that it is written for a scalar quantity φ rather than for vector $\vec{v}$. Formulate the boundary conditions for φ for the boundary conditions (1) and (2).

3.6 Waves in a Shallow Channel

Here we consider small perturbations of the surface of a liquid contained in a shallow channel. The channel's length is l and h is the equilibrium depth. Small volume elements located at x are shifted from the equilibrium position by distance $\zeta(x, t)$ in the horizontal direction and $\eta(x, t)$ in the vertical direction (Figure 3.4). Again we will see that a set of equations related to those derived previously will emerge from the physical situation at hand.

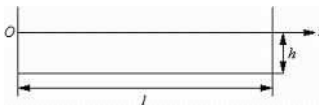


Figure 3.4. Shallow tank of water of depth h and length l .

3.6.1 Derivation of the Equations for Displacements $\zeta(x, t)$ and $\eta(x, t)$

If we neglect the pressure changes due to the wave motion itself, the pressure has only a hydrostatic nature (i.e., it is related only to the depth of the liquid). We consider only small and not very fast waves so that second powers of speed and its derivatives can be neglected. The liquid is also assumed to be incompressible. At a fixed depth y from the upper (free) surface, the pressure at location x is $p = p_0 + \rho g(h + \eta - y)$, where g is the acceleration of gravity. Taking the derivative, we have

$$\frac{\partial p}{\partial x} = g\rho \frac{\partial \eta}{\partial x}. \quad (3.28)$$

For the volume element $\Delta V = \Delta x \Delta y \Delta z$ between cross sections at x and Δx , the equation of motion (Newton's law) is

$$\rho \Delta V \frac{\partial v}{\partial t} = -\Delta V \frac{\partial p}{\partial x} \Big|_{x+\theta \Delta x}, \quad 0 < \theta < 1.$$

By dividing by ΔV and letting Δx approach zero, we obtain

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}.$$

Together with Equation (3.28), this gives

$$\frac{\partial v}{\partial t} = -g \frac{\partial \eta}{\partial x}. \quad (3.29)$$

The continuity equation for mass conservation is

$$\rho l \eta + \frac{\partial}{\partial x}(\rho l h \zeta) = 0,$$

which, for an uncompressible liquid with density $\rho = \text{const}$, gives

$$\eta = -h \frac{\partial \zeta}{\partial x}. \quad (3.30)$$

By definition, $\frac{\partial v}{\partial t} = \frac{\partial^2 \zeta}{\partial t^2}$, where v is the wave's speed in the horizontal direction. From Equation (3.29), we then have

$$\frac{\partial^2 \zeta}{\partial t^2} = -g \frac{\partial \eta}{\partial x}, \quad (3.31)$$

and taking the derivative of Equation (3.30) gives

$$-g \frac{\partial \eta}{\partial x} = gh \frac{\partial^2 \zeta}{\partial x^2}. \quad (3.32)$$

Finally, by equating Equations (3.31) and (3.32) we obtain the wave equation

$$\zeta_{tt} - a^2 \zeta_{xx} = 0, \text{ where } a^2 = gh.$$

Similarly for $\eta(x, t)$, we can obtain

$$\eta_{tt} - a^2 \eta_{xx} = 0.$$

Reading Exercise. Provide the steps for the derivation of the wave equation for $\eta(x, t)$.

3.6.2 Examples of Initial and Boundary Conditions

We may have an initial horizontal displacement of the liquid in the channel and an initial velocity distribution for the horizontal displacement given, respectively, by

$$\zeta(x, 0) = \varphi(x), \quad \zeta_t(x, 0) = \psi(x),$$

in which case

$$\eta(x, 0) = -h\varphi'(x), \quad \eta_t(x, 0) = -h\psi'(x).$$

The simplest boundary conditions are for an undisturbed liquid at the ends of the channel:

$$\zeta(0, t) = \zeta(l, t) = 0 \quad \text{and} \quad \eta_x(0, t) = \eta_x(l, t) = 0.$$

Here again, we see that a particular physical system, in this case shallow water waves, can be described by a set of differential equations and boundary conditions that we have encountered before. Solutions to this and related equations will be considered in the following sections.

3.7 Electrical Oscillations in a Circuit

Now we consider the boundary value problem for a current and a voltage in a circuit. To simplify the task, suppose that the circuit is a long thin wire with resistance, capacitance, and inductance distributed uniformly along its length. Thus we consider the physical quantities R , C , and L as well as the possibility of leakage, G , where these quantities are defined to be per unit length. Initially, at $t = 0$, the current is $\varphi(x)$, and the voltage is $\psi(x)$ at locations x , along the wire.

3.7.1 Derivation of the Basic Equations

Let the functions $i(x, t)$ and $V(x, t)$ represent, respectively, current and voltage at a location x along the wire and at time t . Applying Ohm's law for an element $(x, x + \Delta x)$ with nonzero self-inductance, we obtain

$$V(x, t) - V(x + \Delta x, t) = i(x, t)R\Delta x + i_t(x, t)L\Delta x.$$

By dividing by Δx and then taking the limit as Δx goes to zero, we get

$$-V_x(x, t) = Ri(x, t) + Li_t(x, t). \quad (3.33)$$

Charge conservation provides one more equation relating the functions $i(x, t)$ and $V(x, t)$. To charge a piece of wire of length x to $x + \Delta x$ that has capacitance $C\Delta x$, we multiply capacitance by the voltage change during time t to $t + \Delta t$. The charge lost due to leakage during the time Δt is $G\Delta x\Delta tV(x, t)$. Using charge conservation, we have

$$[i(x, t) - i(x + \Delta x, t)]\Delta t = C\Delta x[V(x, t + \Delta t) - V(x, t)] + G\Delta x\Delta tV(x, t).$$

By dividing by $\Delta x\Delta t$ and taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, we get

$$-i_x(x, t) = CV_t(x, t) + GV(x, t). \quad (3.34)$$

Using Equations (3.33) and (3.34) we can obtain individual equations for $i(x, t)$ and $V(x, t)$. Differentiating both sides of Equation (3.33) by t , and both sides of Equation (3.34) by x , then multiplying the first equation by $-C$ and substituting the first equation into the second gives

$$-i_{xx}(x, t) = -RCi_t(x, t) - LCi_{tt}(x, t) + GV_x(x, t).$$

Substituting $V_x(x, t) = -Ri(x, t) - Li_t(x, t)$ from Equation (3.33) into this equation, we finally obtain the so-called *telegraph equation*:

$$i_{xx}(x, t) = LCi_{tt}(x, t) + (RC + LG)i_t(x, t) + RGi(x, t). \quad (3.35)$$

Similarly, we have that the function $V(x, t)$ obeys the equation

$$V_{xx}(x, t) = LCV_{tt}(x, t) + (RC + LG)V_t(x, t) + RGV(x, t). \quad (3.36)$$

Reading Exercise. Derive Equation (3.36) for voltage.

These equations are similar to the equations describing string oscillations. From this, we see that Equations (3.35) and (3.36) describe electrical oscillations in the *RLC* circuit. When $R = 0$ and $G = 0$, the equations have the simplest form. In the case of the current equation, we have

$$\frac{\partial^2 i}{\partial t^2} = a^2 \frac{\partial^2 i}{\partial x^2},$$

where $a^2 = 1/LC$, and once again we have the equation of free oscillations of a string, Equation (3.4). If $G = 0$, the equation resulting is similar to Equation (3.6), which describes oscillations in a medium with the force of resistance proportional to speed:

$$\frac{\partial^2 i}{\partial t^2} = a^2 \frac{\partial^2 i}{\partial x^2} - 2\kappa \frac{\partial i}{\partial t},$$

where $a^2 = 1/LC$ and $2\kappa = (R/L + G/C)$.

3.7.2 Examples of Initial Conditions for Current, $i(x, t)$

The equation for current contains the second-order time derivative; thus, we need two initial conditions. One initial condition is the initial current in the wire

$$i(x, 0) = \varphi(x).$$

To find the second initial condition for $i_t(x, 0)$, we set $t = 0$ in Equation (3.35) to get

$$i_t(x, 0) = -\frac{1}{L}V_x(x, 0) - \frac{R}{L}i(x, 0).$$

The condition $V(x, 0) = \psi(x)$ gives $V_x(x, 0) = \psi'(x)$. With $i(x, 0) = \varphi(x)$, we finally obtain

$$i_t(x, 0) = -\frac{1}{L}[\psi'(x) - R\varphi(x)].$$

This is the second initial condition for the current, $i(x, t)$.

3.7.3 Examples of Initial Conditions for Voltage, $V(x, t)$

If we know the voltage at various points in the circuit at some instant of time, we have an initial condition for $V(x, t)$, which is

$$V(x, 0) = \psi(x).$$

To find the second initial condition for $V(x, t)$, we set $t = 0$ in Equation (3.36) to get

$$V_t(x, 0) = -\frac{1}{C}i_x(x, 0) - \frac{G}{C}V(x, 0).$$

The condition $i(x, 0) = \varphi(x)$ gives $i_x(x, 0) = \varphi'(x)$. With $V(x, 0) = \psi(x)$, we obtain

$$V_t(x, 0) = -\frac{1}{C}[\varphi'(x) - G\psi(x)].$$

This is the second initial condition for the voltage $V(x, t)$.

3.7.4 Boundary Conditions for $i(x, t)$ and $V(x, t)$

The following examples are boundary conditions for the most common physical cases.

1. One end of the wire is attached to a battery with electromagnetic force (*emf*) E , and the other end is grounded. The boundary conditions are

$$V(0, t) = E, \quad V(l, t) = 0.$$

2. A sinusoidal voltage with frequency ω is applied to the left end of the wire, and the other end is insulated. The boundary conditions are

$$V(0, t) = E \sin \omega t, \quad i(l, t) = 0.$$

3. The ends are each attached to devices with resistances R_0 and R_l , and self-inductances L_0 and L_l , respectively. The boundary conditions are

$$V(0, t) = E - R_0 i_0 - L_0 \frac{di_0}{dt} \quad \text{and} \quad V(l, t) = R_l i_l + L_l \frac{di_l}{dt},$$

where E is *emf* of a battery and i_0 and i_l are currents at the two ends of the conductor.

4. The ends of the wire are each attached to capacitors C_0 and C_l , respectively. The boundary conditions are

$$V(0, t) = E - \frac{1}{C_0} \int i_0 dt \quad \text{and} \quad i(l, t) = C_l \frac{dV_l}{dt},$$

where V_l is the voltage at the end of the wire.

3.8 Traveling Waves: D'Alembert Method

In this section, we study waves propagating along infinite ($-\infty < x < \infty$) and semi-infinite ($x \geq 0$) intervals. For these cases, a physically intuitive method to solve the wave equation, *D'Alembert's method*, avoids using the more mathematically sophisticated Fourier method. Our physical model will be waves on a string; however, as shown in the previous sections, the same wave equation describes many different physical phenomena and the results derived here apply to those cases as well.

3.8.1 The Infinite String

It is clear that oscillations in the central part of a very long string do not depend on the behavior of its ends. If we excite a long tensed cord by raising and then releasing its central part, waves propagating to the right and left will be created. This simple picture changes only when the waves reach the ends and begin to travel in opposite directions after reflection. Thus, in the discussion in this section, we do not take the ends of the string into account and we do not consider reflected waves.

For free oscillations of a string, we have the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (3.37)$$

where $a^2 = T/\rho$; T and ρ are the tension and linear mass density, respectively; and u is the displacement of the string from equilibrium as a function of location and time as in the previous discussions.

Let us introduce two new variables (called the *characteristics* of the wave equation)

$$\xi = x - at \quad \text{and} \quad \eta = x + at. \quad (3.38)$$

The differentiation operators in (3.37) can be expressed in terms of ξ and η as

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},$$

and similarly for $\partial/\partial t$. Because $\partial/\partial \xi$ and $\partial/\partial \eta$ commute, we have

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right)^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \eta^2} = \frac{\partial^2}{\partial \xi^2} + 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2},$$

in which case

$$\frac{\partial}{\partial t} = -a \frac{\partial}{\partial \xi} + a \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{1}{a^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2} - 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}.$$

Substituting these expressions into the wave Equation (3.37), we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \quad (3.39)$$

Writing this as

$$\frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) = 0,$$

we conclude that $\partial/\partial \eta$ must be independent of ξ , so that

$$\frac{\partial u}{\partial \eta} = g(\eta),$$

where $g(\eta)$ is an arbitrary function of η . Integration with respect to η now yields

$$u = \int g(\eta) d\eta + f_1(\xi),$$

where $f_1(\xi)$ is an arbitrary function of ξ . Using the notation

$$f_2(\eta) = \int g(\eta) d\eta,$$

we can write

$$u = f_1(\xi) + f_2(\eta),$$

Returning to the variables (x, t) , we have

$$u = f_1(x - at) + f_2(x + at). \quad (3.40)$$

It is easily verified that the function $u(x, t)$, defined by Equation (3.40), is a solution of Equation (3.37) for any twice differentiable functions f_1 and f_2 . As discussed earlier, the function f_1 represents the wave moving with constant speed, a , to the right (along the x -axis), and the function f_2 represents a wave similarly moving to the left.

The solution in Equation (3.40) of Equation (3.37) is called *D'Alembert's solution*. In order to describe the solution for free oscillations in a particular physical case, we must find the functions f_1 and f_2 for that particular situation. Since we are considering boundaries at infinity, these functions will be determined only by the initial conditions of the string.

3.8.2 The Cauchy Problem

The Cauchy problem for the infinite string is defined by the case for which the solutions of Equation (3.37) satisfy the conditions

$$u|_{t=0} = \varphi(x) \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x). \quad (3.41)$$

Substituting Equation (3.40) into Equation (3.41) gives

$$\begin{aligned} f_1(x) + f_2(x) &= \varphi(x), \\ -af_1'(x) + af_2'(x) &= \psi(x). \end{aligned}$$

From the second relation

$$-af_1(x) + af_2(x) = \int_0^x \psi(x)dx + aC$$

(where C is an arbitrary constant), we have

$$f_1(x) = \frac{1}{2} \left[\varphi(x) - \frac{1}{a} \int_0^x \psi(x)dx - C \right], \quad \text{and} \quad f_2(x) = \frac{1}{2} \left[\varphi(x) + \frac{1}{a} \int_0^x \psi(x)dx + C \right].$$

Thus,

$$u(x, t) = \frac{1}{2} \left[\varphi(x - at) - \frac{1}{a} \int_0^{x-at} \psi(y)dy - C + \varphi(x + at) + \frac{1}{a} \int_0^{x+at} \psi(y)dy + C \right],$$

which finally gives

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy. \quad (3.42)$$

If the function $\psi(x)$ is differentiable and $\varphi(x)$ is twice differentiable, this solution satisfies Equation (3.37) and initial conditions (3.41). (The reader may check this as a reading exercise.) The method of construction of the solution in Equation (3.42) proves its uniqueness. The solutions will be stable solutions if they depend continuously on the variables x and t , which may be defined more formally in the following way. A solution is continuous if, given any $\varepsilon > 0$, we can choose δ such that if we replace $\varphi(x)$ and $\psi(x)$ with $\varphi^{(1)}(x)$ and $\psi^{(1)}(x)$, respectively, satisfying the inequalities

$$|\varphi(x) - \varphi^{(1)}(x)| < \delta \quad \text{and} \quad |\psi(x) - \psi^{(1)}(x)| < \delta$$

for all x , then

$$|u(x, t) - u^{(1)}(x, t)| < \varepsilon$$

for all x and $0 \leq t \leq T$ (where T is an arbitrary time). In other words, the absolute value of the differences between the new and the original solutions is smaller than ε for any assigned time interval. This immediately follows from D'Alembert's formula:

$$\begin{aligned} |u(x, t) - u^{(1)}(x, t)| &= \left| \frac{1}{2} [\varphi(x - at) + \varphi(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy \right. \\ &\quad \left. - \frac{1}{2} [\varphi^{(1)}(x - at) + \varphi^{(1)}(x + at)] - \frac{1}{2a} \int_{x-at}^{x+at} \psi^{(1)}(y) dy \right| \\ &\leq \left| \frac{1}{2} [\varphi(x - at) - \varphi^{(1)}(x - at)] \right| + \left| \frac{1}{2} [\varphi(x + at) - \varphi^{(1)}(x + at)] \right| \\ &\quad + \frac{1}{2a} \int_{x-at}^{x+at} |\psi(y) - \psi^{(1)}(y)| dy \\ &\leq 2 \cdot \frac{1}{2} \delta + \frac{1}{2a} \cdot [(x + at) - (x - at)] \cdot \delta = \delta(1 + t) \leq \delta(1 + T) < \varepsilon \end{aligned}$$

if, for any given $\varepsilon > 0$, we take positive $\delta < \varepsilon/(1 + T)$. Thus, the solution is unique and depends continuously on the initial conditions (i.e., is

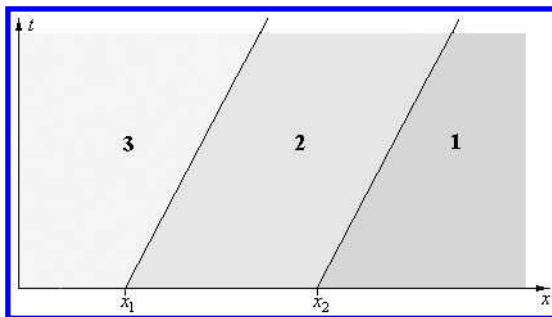


Figure 3.5. Characteristics $x_1 = x - at$ and $x_2 = x - at$.

stable). Problems with such properties are often referred to as well-posed problems.

Notice that very often the functions $\varphi(x)$ and $\psi(x)$ are not differentiable on the entire axis and as a result, Equation (3.42) cannot be a solution to Equation (3.37), strictly speaking. But suppose that we change $\varphi(x)$ and $\psi(x)$ a little to make them smoother and differentiable the necessary number of times. After this change, because of the property of the continuous dependence of the solutions on the initial conditions, Equation (3.42) gives the solution to Equation (3.37), and this new solution with slightly modified initial conditions differs insignificantly from the original solution. In such cases, when a formal solution cannot be obtained because the initial conditions are not differentiable the necessary number of times but the problem depends on the initial conditions in a continuous way, we may still use the formal solution in Equation (3.42). In this case, the solution is called the *generalized solution*. We will use a similar approach, not only for the problems in this section, but throughout the book.

The D'Alembert formula is very helpful in visualizing the propagation of initial disturbances by graphical superposition. The method based on this formula is called the *method of traveling waves*. Let us look at the D'Alembert method with the help of the phase plane, (x, t) . Let the initial deflection be zero except on the interval $[x_1, x_2]$. The characteristics $\xi = x - at$ and $\eta = x + at$ split the phase plane into several zones, six in this case, comprising three to the right and three to the left.

Figure 3.5 shows the three zones for the wave moving to the right. The boundaries of these zones are the characteristic lines, ξ , corresponding to the values of x equal to x_1 and x_2 . The line $x_1 = x - at$ corresponds to

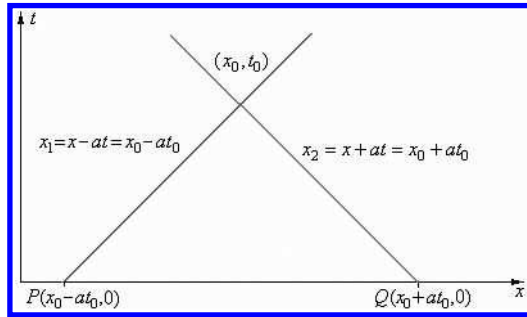


Figure 3.6. The characteristic triangle with vertex at point (x_0, t_0) .

the rear edge of the wave moving to the right, and the line $x_2 = x + at$ corresponds to the front edge of the same wave. Along these lines, the combination $x - at$ has constant values x_1 and x_2 . It is obvious that this wave, $f_1(x - at)$, appears in zone 2, but it is zero in zones 1 and 3 because it has not yet reached zone 1 and has already left zone 3.

Similarly, the wave moving to the left, $f_2(x + at)$, appears in one out of three zones obtained by splitting the phase plane along the characteristic lines $x_1 = x - at$ and $x_2 = x + at$.

Reading Exercise. Plot the phase plane corresponding to the initial deflection given below in the upper part of Figure 3.7 (solid line) and indicate all six zones corresponding to the waves $f_1(x - at)$ and $f_2(x + at)$.

Another way to write the solution of Equation (3.37) is to use the *characteristic triangle* shown in Figure 3.6. Suppose we want to find the solution at some point (x_0, t_0) . The vertex of this triangle is the point (x_0, t_0) , and the two sides are given by the equations $x - at = x_0 - at_0$ and $x + at = x_0 + at_0$. The base of this triangle (i.e., the line between the points $P(x_0 - at_0, 0)$ and $Q(x_0 + at_0, 0)$) determines the wave amplitude at point (x_0, t_0) as follows from Equation (3.42):

$$u(x_0, t_0) = \frac{\varphi(P) + \varphi(Q)}{2} + \frac{1}{2a} \int_P^Q \psi(y) dy. \quad (3.43)$$

We now discuss, in more detail, two physical situations that are typically encountered: waves created by a displacement and waves created by a pulse.

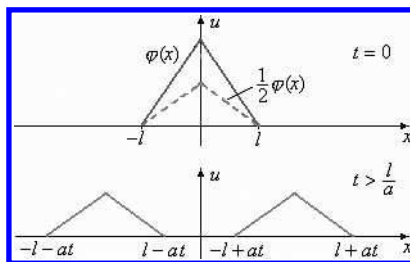


Figure 3.7. Propagation of an initial arbitrary displacement.

For the case of waves created by a displacement, we let the initial speeds of points on the string be zero, but the initial displacements are not zero. The solution given by Equation (3.42) in this case is

$$u(x, t) = \frac{1}{2} [\varphi(x - at) + \varphi(x + at)] . \quad (3.44)$$

The first term, $\frac{1}{2}\varphi(x - at)$, is a constant shape disturbance propagating with speed a in the positive x direction, and the term $\frac{1}{2}\varphi(x + at)$ is the same shaped disturbance moving in the opposite direction. Suppose that the initial disturbance exists only on a limited interval, $-l \leq x \leq l$. This kind of disturbance is shown schematically in Figure 3.7, where the function $\varphi(x)$ is plotted with a solid line in the upper part of the figure. The dashed line shows the function $\frac{1}{2}\varphi(x)$. We can consider $u(x, t = 0)$ given by Equation (3.44) as two independent disturbances, $\frac{1}{2}\varphi(x)$, each propagating in opposite directions with unchanged amplitude. Initially, at $t = 0$, the profiles of both waves coincide, after which they separate and the distance between them increases. The bottom part of Figure 3.7 shows these waves after some time $t > l/a$. As they pass a given section of the string, this part returns to rest. Only two intervals of the string of length $2l$ each are deflected at any instant t .

Reading Exercise. Let the initial displacement be $\varphi(x) = \exp(-4x^2)$. Use equation (3.44) to obtain

$$u(x, t) = \frac{1}{2} [\exp(-4(x - at)^2) + \exp(-4(x + at)^2)] .$$

For the case of waves created by a pulse, we let the initial displacement be zero, $\varphi(x) = 0$, but the initial velocities are given as functions of

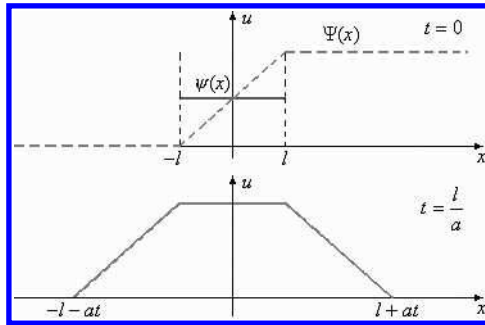


Figure 3.8. Propagation of an initial pulse.

position, $\psi(x)$. In other words, points on the string are given some initial velocity by an external agent. An example of a distribution of velocities is schematically shown in Figure 3.8, where, to simplify the plot, a constant function $\psi(x)$ on $-l \leq x \leq l$ was chosen. The function $\psi(x)$ is plotted with a solid line in the upper part of Figure 3.8.

Consider the definite integral, $\Psi(x)$, of the function $\psi(x)$, which is zero on the interval $-\infty < x \leq -l$. For $x \geq l$, the function $\Psi(x)$ is constant and equal to

$$\int_{-l}^l \psi(x) dx.$$

The graph for $\Psi(x)$ is shown with a dashed line in Figure 3.8. From Equation (3.42), we have

$$u(x, t) = \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy = \Psi(x+at) - \Psi(x-at). \quad (3.45)$$

Equation (3.45) means that we again have two waves moving in opposite directions, but now the waves have opposite signs.

At some location x for large enough time, we have $x+at > l$ and the wave $\Psi(x+at)$ becomes a constant; and, at the same instant of time, $x-at < -l$ and the wave $\Psi(x-at)$ is equal to zero. As a result, the perturbation propagates in both directions but, contrary to the case of a wave created by a displacement, none of the elements of the string return to the initial position existing at $t = 0$. The bottom part of Figure 3.8

illustrates this situation for an instant of time $t = l/a$. When $t > l/a$, we have a similar trapezoid for $u(x, t)$ (with the same height as at $t = l/a$), which expands in both directions uniformly with time.

Example 3.1. Let $\varphi(x) = 0$, $\psi(x) = \psi_0 = \text{const}$ for $x_1 < x < x_2$, and $\psi(x) = 0$ outside of this interval.

Solution. In this case, we have

$$\Psi(x) = \frac{1}{2a} \int_{x_1}^x \psi(y) dy = \begin{cases} 0, & x < x_1, \\ \frac{\psi_0}{2a}(x - x_1), & x_1 < x < x_2, \\ \frac{\psi_0}{2a}(x_2 - x_1) = \text{const}, & x > x_2. \end{cases} \quad (3.46)$$

With the help of the program Waves (included on the CD), different scenarios may be modeled by using the method of traveling waves, which is essentially a straightforward application of the D'Alembert formula. Arbitrary shapes of $\varphi(x)$ and $\psi(x)$, much more complex than Example 3.1, may be examined using the program. A description of how to use this program is found in Appendix E.

Reading Exercise. Solve this problem with the help of the program Waves. Using the parameter values $a = 0.5$, $l = 1$, $\psi_0 = 1$, generate the wave propagation. Explain why the maximum deflection is $u_{max} = 2$, and why, for the point $x = 0$, this maximum deflection value is reached at time $t = l/a = 2$.

Example 3.2. An infinite stretched string is excited by the initial deflection

$$\varphi(x) = \frac{1}{x^2 + 1}$$

with no initial velocities. Find the vibrations of the string. Write an analytical solution representing the motion of the string for $t > 0$, and simulate the vibrations using the program Waves.

Solution. In the given problem, the initial speeds of points on the string are zero ($\psi(x) \equiv 0$), so the solution given by Equation (3.44) in this case is

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} = \frac{1}{2} \left[\frac{1}{(x - at)^2 + 1} + \frac{1}{(x + at)^2 + 1} \right].$$

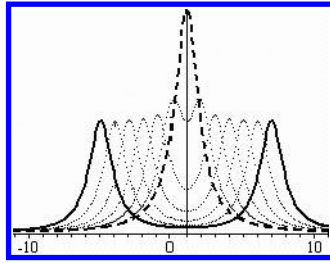


Figure 3.9. Graph of the solution to Example 3.2.

Figure 3.9 shows the solution for the case when $a^2 = 0.25$. The dashed line represents the initial deflection and the solid line is the string profile at time $t = 12$. The dotted lines show the evolution of the string profile within the period of time from $t=0$ to $t=12$. The graph in Figure 3.9 was obtained with the program Waves.

Example 3.3. An infinite stretched string is initially at rest. The initial distribution of velocities is given by

$$\psi(x) = \begin{cases} x, & x \in [-l, l], \\ 0, & x \notin [-l, l]. \end{cases}$$

Find the vibrations of the string. Write an analytical solution representing the motion of the string for $t > 0$, and simulate the vibrations by using the program Waves.

Solution. In the given problem the initial deflection of the string is zero ($\varphi(x) \equiv 0$), so the solution given by Equation (3.45) in this case is

$$u(x, t) = \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy = \Psi(x+at) - \Psi(x-at).$$

For the following sections of the string, we have:

For $-\infty < x < -l$,

$$u(x, t) = \begin{cases} \frac{(x+at)^2 - l^2}{4a}, & -\frac{l+x}{a} \leq t \leq \frac{l-x}{a}, \\ 0, & \text{otherwise.} \end{cases}$$

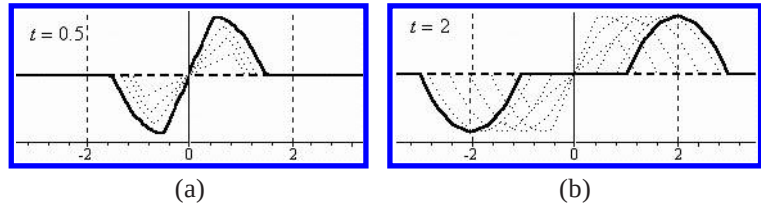


Figure 3.10. Graph of the solution to Example 3.3. (a) $t = 0.5$. (b) $t = 2$.

For $-l < x < 0$,

$$u(x, t) = \begin{cases} t, & 0 \leq t \leq \frac{l+x}{a}, \\ \frac{(x+at)^2 - l^2}{4a}, & \frac{l+x}{a} < t \leq \frac{l-x}{a}, \\ 0, & \frac{l-x}{a} < t < +\infty. \end{cases}$$

For $0 < x < l$,

$$u(x, t) = \begin{cases} t, & 0 \leq t \leq \frac{l-x}{a}, \\ \frac{l^2 - (x-at)^2}{4a}, & \frac{l-x}{a} < t \leq \frac{l+x}{a}, \\ 0, & \frac{l+x}{a} < t < +\infty. \end{cases}$$

For $l < x < +\infty$,

$$u(x, t) = \begin{cases} \frac{l^2 - (x-at)^2}{4a}, & \frac{x-l}{a} \leq t \leq \frac{x+l}{a}, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 3.10 shows the solution for the case when $a^2 = 1$, $l = 1$. The solid line represents in (a) the string profile at time $t = 0.5$, and in (b) the string at time $t = 2$. This solution was obtained with the program Waves.

3.8.3 Nonhomogeneous Equations

We very briefly present the solution of the nonhomogeneous equation for the infinite string. The solution of Equation (3.47),

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = F(x, t), \quad (3.47)$$

consists of two parts; the solution in Equation (3.40) of the homogeneous Equation (3.37) (i.e., two waves propagating in positive and negative x directions) plus a particular solution of the nonhomogeneous Equation (3.47). The particular solution will have the form

$$\frac{1}{2a} \iint_{\Delta} F(x, t) dS, \quad (3.48)$$

which is a double integral over the characteristic triangle with the vertex at point (x, t) . We present the result in Equation (3.48) without proof, mainly because D'Alembert's method, despite being intuitively clear, does not give a general approach for solving the nonhomogeneous case, as does, for example, the Fourier method (i.e., the Fourier series method for finite dimensions and the Fourier integral for infinite dimensions). The Fourier method allows the solution of many other types of problems and is discussed in greater detail in other chapters for various kinds of partial differential equations.

3.8.4 Modifications of the General Form of the Wave Equation

We finish Section 3.8 with a brief consideration of equations that are more general than the wave Equation (3.37) given by $u_{tt} = a^2 u_{xx}$. The question we want to address is what general types of equations can be satisfied by arbitrary traveling waves $f(x \pm at)$, where f is an arbitrary function and a is a constant speed.

The exponential function

$$u(x, t) = A e^{i(\omega t - kx)} \quad (3.49)$$

describes a so-called plane wave moving along the x -axis and is a solution to the wave Equation (3.37) if $a = \omega/k$, as may be verified by the reader as a reading exercise. Changing the sign of x corresponds to the wave propagating along the negative x -axis. The parameter ω is the wave frequency (the wave is monochromatic), k is the wave number, and A is the amplitude. The wave number is related to the wavelength λ by the relation $k = 2\pi/\lambda$. The plane wave can be also described by the real or imaginary parts of Equation (3.49)—in other words, cosine or sine functions.

As we saw in Chapter 1 on Fourier analysis, any arbitrary wave form may be constructed from a sum of appropriate harmonic waves such as

plane waves. If we have a superposition or sum of plane waves, the speed of each component wave is the speed with which its phase, $\delta = \omega t - kx$, moves. Points of the plane wave with the same amplitude have the same phase; thus the speed can be found from the condition $d\delta = \omega dt - k dx = 0$. From here, we obtain a speed

$$v = \frac{dx}{dt} = \frac{\omega}{k}, \quad (3.50)$$

which is called the *phase speed*; $v = \text{const.}$ From Equation (3.50), we have $\omega = kv$, which is called the *dispersion relation*. We note that for the simple wave equation (3.37), the dispersion equation gives the simple linear relationship $a = \omega/k$ between wave speed and frequency.

In addition to the phase speed, we may introduce the *group speed* for the superposition as

$$v_g = \frac{dx}{dt} = \frac{d\omega}{dk}. \quad (3.51)$$

From the dispersion relation, we see that for plane wave solutions to the simplest wave equation (3.37), the phase speed and group speed are the same. This will no longer hold true for the modifications to the wave equation (3.37), which we discuss below.

Consider the modified wave equation with constant coefficients in the general form

$$u_{tt} - a^2 u_{xx} + b_1 u_t + b_2 u_x + cu = 0. \quad (3.52)$$

The physical significance of the individual terms are considered next.

If we introduce a new function, $v(x, t)$, into Equation (3.52) by using the substitution

$$u(x, t) = e^{\lambda x + \mu t} v(x, t) \quad (3.53)$$

with $\lambda = b_1/2$ and $\mu = b_2/2$, Equation (3.52) reduces to a substantially simpler form, known as the Klein-Gordon equation, given by

$$v_{tt} - a^2 v_{xx} + cv = 0. \quad (3.54)$$

Reading Exercise. Prove the statement about the Klein-Gordon equation.

It is immediately seen that Equation (3.54) does not allow a solution in the form $f(x \pm at)$ if $c \neq 0$. Physically, this result is related to the phenomenon of wave dispersion, which we now discuss in more detail.

Suppose that, in order to create a particular wave shape on the string as an initial condition, we form a superposition of simple plane waves given by Equation (3.49) with slightly different frequencies and wave numbers. Such a superposition physically describes a group of waves or a wave packet such as a modulated signal.

Reading Exercise. Obtain a modulated wave by adding two waves, $f_1 = \cos(\omega_1 t - k_1 x)$ and $f_2 = \cos(\omega_2 t - k_2 x)$, with close frequencies and wave numbers, $\Delta\omega = \omega_1 - \omega_2 \ll \omega_{1,2}$, $\Delta k = k_1 - k_2 \ll k_{1,2}$, and show that it gives a (nonharmonic) wave

$$f_1 + f_2 \approx 2a \cos \frac{t\Delta\omega - x\Delta k}{2} \cos(\omega_1 t - k_1 x)$$

with a frequency approximately equal to $\omega_1 \approx \omega_2$ and with an amplitude that slowly varies with the (low) frequency $\Delta\omega$. (When two waves of similar frequency combine to form a wave with slowly varying amplitude, the resulting phenomena are called *beats*.) It is clear that the group phase of a resultant wave is $\delta_g = (t\Delta\omega - x\Delta k)/2$; thus, the group speed can be defined from the condition $d\delta_g = 0$. This gives $v_g = dx/dt \approx \Delta x/\Delta t = \Delta\omega/\Delta k$, in agreement with the definition in Equation (3.51).

Notice another physically important case, when the group and phase speeds are different. In quantum mechanics, a particle with energy E and momentum $p = mv$ is also considered as a wave with frequency ω and wave number k . The relationships are given by the De Broglie formulas: $E = \hbar\omega$ and $p = \hbar k$ (here $\hbar$ is Planck's constant). The corresponding plane wave (3.49) in this case is

$$\psi(x, t) = Ae^{i(\omega t - kx)} = Ae^{i(Et - px)/\hbar}$$

and is called a *wave function*. Using the relationship between energy and momentum $E = p^2/2m$, we have for the phase speed $v = \omega/k = E/p = p/2m = v/2$ —half of an actual particle's speed. Contrary to this, the group speed gives the right result: $v_g = d\omega/dk = dE/dp = p/m = v$. For a relativistic particle, $E = \sqrt{p^2 c^2 + m^2 c^4}$ and the phase speed is greater than the speed of light, c , which does not make sense. Contrary to that, $v_g = d\omega/dk = dE/dp = pc^2/E = v$. For a photon, with $m = 0$, $E = pc$ and the phase and group speeds coincide.

For the wave equation (3.37), the individual components of a superposition all have the same speed, so there is no modification of the shape as

the packet propagates. For Equation (3.54), however, harmonics in the superposition have different frequencies and they move with different speeds, with the result that the shape of such a wave form distorts over time. This phenomenon is called dispersion and occurs in many physical situations (e.g., color dispersion for light moving through glass and the deterioration of digital signals in both wires and fiber optic cable over long distances).

It is easy to demonstrate that solutions to Equation (3.54) exhibit the dispersion phenomena. Substituting the plane wave (3.49) into Equation (3.54), we obtain $-\omega^2 + k^2 a^2 + c = 0$, resulting in the *dispersion relation* $\omega = \sqrt{c + k^2 a^2}$. Here we see that if $c \neq 0$, the group speed $d\omega/dk$, given by Equation (3.51), and the speed of the individual components $v = \omega/k$ are different as was claimed above. The dispersion relation, then, is an equation that indicates how different the wave packet speed (group speed) will be from the speed of the individual components. The last one, $v = \omega/k = aw/\sqrt{\omega^2 + c^2}$, due to dispersion, is not constant but depends on the frequency, $v = v(\omega)$. In Example 3.4, we obtain this result for a particular choice of initial conditions.

Example 3.4. Consider an infinite thin wire with resistance R , capacitance C , inductance L , and leakage G distributed uniformly along its length. Assuming the condition $GL = CR$ and the initial voltage and current in the wire

$$V(x, 0) = f_1(x) = e^{-x^2} \quad \text{and} \quad i(x, 0) = f_2(x) = e^{-x^2},$$

find electrical voltage and current oscillations in the wire. Write an analytical solution and simulate the oscillations by using the program Waves.

Solution. The Cauchy problem is defined by

$$\left. \begin{aligned} V_x + Li_t + Ri &= 0, \\ i_x + CV_t + GV &= 0, \end{aligned} \right\} \quad -\infty < x < \infty, \quad 0 < t < \infty,$$

$$V(x, 0) = f_1(x), \quad i(x, 0) = f_2(x), \quad -\infty < x < \infty.$$

Eliminating current $i(x, t)$ from these equations and assuming the condition $GL = CR$, we obtain the second-order equation for defining voltage $V(x, t)$:

$$V_{tt} - \frac{1}{LC}V_{xx} + 2\frac{G}{C}V_t + \frac{G^2}{C^2}V = 0.$$

The initial conditions for the voltage $V(x, t)$ are

$$V(x, 0) = f_1(x), \quad V_t(x, 0) = -\frac{1}{C} [f_2'(x) + G f_1(x)].$$

To obtain the last initial condition, we put $t = 0$ in the second equation of the system of equations for V and i and take into account $i_x(x, 0) = f_2'(x)$.

When $G = 0$, we have the wave equation in its simplest form. When $G \neq 0$, with the help of the substitution $u(x, t) = e^{Gt/C} V(x, t)$, the problem reduces to the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 u}{\partial x^2}$$

under the conditions

$$u(x, 0) = f_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = -\frac{1}{C} f_2'(x).$$

The solution given by D'Alembert's Equation (3.42) is

$$\begin{aligned} u(x, t) &= \frac{f_1(x - at) + f_1(x + at)}{2} + \frac{\sqrt{LC}}{2a} \int_{x-at}^{x+at} \left(-\frac{1}{C}\right) f_2'(y) dy \\ &= \frac{f_1(x - at) + f_1(x + at)}{2} - \frac{1}{2} \sqrt{L/C} [f_2(x + at) - f_2(x - at)] \\ &= \frac{1}{2} \left[f_1(x - at) + \sqrt{L/C} f_2(x - at) \right] + \frac{1}{2} \left[f_1(x + at) - \sqrt{L/C} f_2(x + at) \right]. \end{aligned}$$

Here $a = 1/\sqrt{LC}$. Finally, substituting the functions $f_1(x) = e^{-x^2}$ and $f_2(x) = e^{-x^2}$, we obtain

$$u(x, t) = \frac{1}{2} \left[e^{-(x-at)^2} \left(1 + \sqrt{L/C}\right) + e^{-(x+at)^2} \left(1 - \sqrt{L/C}\right) \right].$$

Figure 3.11, obtained with the program Waves, shows the solution for function $u(x, t)$ for the case when $C = 1$, $L = 4$, so that $a^2 = 0.25$. The dashed line represents $u(x, 0)$ and the solid line is $u(x, t = 10)$. The dotted lines in between them show the solution within the period of time from $t = 0$ to $t = 10$. Having the solution for $u(x, t)$, it is easy to obtain $V(x, t)$ and $i(x, t)$. The voltage and current are propagating along the wire with constant speed and decay exponentially because of leaking. This solution is essentially the plane wave, $f(x \pm at)$, and exhibits no dispersion.

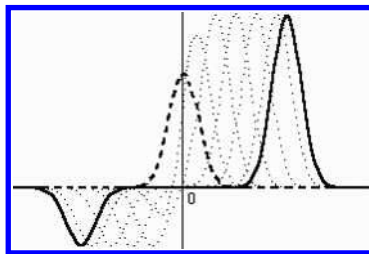


Figure 3.11. Graph of the solution to Example 3.4.

The situation is quite different if the condition $GL = CR$ does not hold. In this case, for both V and i , we obtain equations similar to Equation (3.54), which, as discussed, exhibit dispersion phenomena, so an electrical signal propagates along the cable with distortion.

Reading Exercise. Prove that the substitution $i(x, t) = e^{\mu t}u(x, t)$, with $\mu = (GL + CR)/2CL$, reduces Equation (3.35) to $u_{xx} - CLu_{tt} + cu = 0$, where $c = -(CR - LG)^2/4$. Thus, if $GL \neq CR$, this is the particular case of Equation (3.54). Solve this equation with the program Waves and discuss the dispersion phenomena. Do the same for $V(x, t)$.

At this point, we return to Equation (3.52) and briefly discuss the physical significance of some of the cases when $c = 0$. In the case that $b_1 < 0$ and $b_2 = 0$, we have an equation that describes waves moving on a string with a velocity-dependent external friction term. Physically, this might represent a string immersed in a viscous liquid. In this case, substituting a plane wave into Equation (3.52) yields a dispersion relation given by $\omega^2 = ak^2 - ikb_1$, which contains an imaginary number. This complex frequency may be written generically as $\omega = \omega_r + i\omega_i$, where ω_r and ω_i are real numbers. Substituting this into the plane wave Equation (3.49), we have $u(x, t) = Ae^{-\omega_i t}e^{i(kx - \omega_r t)}$, which indicates that harmonic plane waves will decrease exponentially over time in the presence of friction.

The case $b_2 < 0$ and $b_1 = 0$ with $c = 0$ in Equation (3.51) yields a similar result, known as *dissipation*. In this case, the dispersion relation again yields a complex number, which indicates that waves will decay in amplitude as they propagate on the string. Verification of this result is left as a reading exercise. The source of energy loss in the case of dissipation is friction internal to the string due to bending forces (in contrast to external friction in the previous case).

It is natural at this point to ask whether terms representing physical situations exist that may be included in the wave equation and would cause the amplitude of an initial plane wave to *increase* over time. The answer is affirmative, and a simple example of such a term, referred to as a nonlinearity, is $-\alpha(\partial^2 u^2(x, t)/\partial x^2)$. We thus have a representative nonlinear wave equation given by

$$u_{tt} - a^2 u_{xx} - \alpha u_{xx}^2 = 0. \quad (3.55)$$

Although plane waves are not solutions to this equation, we may again find a dispersion relation by substituting a plane wave into the equation and solving for the frequency, ω . In this case, we get $\omega^2 = k^2(a^2 + 4\alpha A e^{i(kx - \omega t)})$. Using a Taylor series expansion for the exponential gives an approximate dispersion relation of $\omega \approx k \sqrt{a^2 + 4\alpha A}$. From this result, we have the interesting fact that, for a superposition of harmonic waves, the velocity of each component is amplitude dependent. This means that taller wave components travel faster, with the result that the top of a wave packet will advance faster than the base so that the wave becomes steeper over time on its leading edge. It will also be the case that taller components of an initial superposition of plane waves will catch up with slower components, and the packet will grow in amplitude as it propagates. In the case of water waves, this eventually results in a breaking wave, although for the string cases we are considering here we have limited the motion to be perpendicular to the string axis. In fact, Equation (3.55) models shallow water waves, where the parameter α is inversely related to the water depth. It is not surprising, then, that we see waves on the ocean increase in size and steepness (to the point of breaking) as they approach the shore, where there is a larger degree of nonlinearity as the sea depth decreases. The effects of nonlinearity also occur for very large waves approaching the shore, such as tsunamis, leading to devastating results.

It might seem that having both nonlinear and dispersive effects in the same system would result in even less stable behavior of waves on a string but this turns out not to be the case. As a final example, we examine two cases in which the decaying effects of dispersion and the amplitude-increasing effects of nonlinearity exactly balance to yield a stable propagating wave form known as a *solitary wave*.

Consider the wave equation modified in the following way:

$$u_{tt} - a^2 u_{xx} - cu + \alpha u^3 = 0. \quad (3.56)$$

This equation, known as the *phi-4 equation*, is the continuum limit of a model of a discrete system that has been used to describe interatomic forces of a chain of atoms or molecules with two equilibrium positions. The dispersion relation, $\omega \approx \sqrt{a^2 k^2 - c + \alpha A^2}$, tells us that for the right combination of parameters and amplitude, we have $c = \alpha A^2$, so that wave speed and group speed are equal. The conclusion is that waves that propagate with a stable shape can exist in systems that combine dispersion and nonlinearity.

Because Equation (3.56) is nonlinear, the D'Alembert or Fourier methods will not yield a solution. One known solution is the so-called kink (plus sign) or antikink (minus sign) solution:

$$u(x, t) = \pm \sqrt{\frac{c}{\alpha}} \tanh \left[\frac{\sqrt{c/2}}{a} \frac{(x - x_0 - vt)}{\sqrt{1 - v^2/a^2}} \right], \quad (3.57)$$

a result the reader may verify as a reading exercise. Here v is the speed of the kink and x_0 locates the kink's center. Notice that the limit of the solution as x goes to $\pm\infty$ has a value of $\pm\sqrt{c/\alpha}$, that is, the two ends of the string are in different locations. Physically, one can imagine a string that lies with one end in a trough or trench at a distance $\sqrt{c/\alpha}$ from $x = 0$ and the other end in a second trough of the same depth that runs parallel to the first at a distance $-\sqrt{c/\alpha}$ from $x = 0$. At some point, the string crosses the hump between the two parallel troughs; this is the location of the kink (or antikink, if it crosses in the other direction). The kink, then, is a stable wave form because it is in a minimum energy configuration and may propagate with no loss of energy, as do traveling harmonic waves in the linear system.

In other systems, solitary waves exist that have the additional property that they maintain their shape after collision with other solitary waves in the same system. Solitary waves with this property are referred to as *solitons* and have been used to model physical phenomena as diverse as the movement of crystal dislocations, boundaries between magnetic domains, elementary particles, and magnetic flux propagation in Josephson junctions. A famous example of such a system is the sine-Gordon equation given by

$$u_{tt} - a^2 u_{xx} + \alpha \sin(u) = 0 \quad (3.58)$$

with dispersion relation $\omega \approx \sqrt{\alpha + a^2 k^2}$ and solutions

$$u(x, t) = 4 \tan^{-1} \left[\exp \left(\pm \frac{\sqrt{\alpha}(x - x_0 - vt)}{a \sqrt{1 - v^2/a^2}} \right) \right]. \quad (3.59)$$

Reading Exercise. Verify that Equation (3.59) is a solution to the sine-Gordon equation and that the given dispersion relation is correct.

Reading Exercise. The three-dimensional Klein-Gordon equation

$$\nabla^2 u - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = m^2 u$$

has been used in quantum mechanics and is a relativistic analogue of the Schrödinger equation for a free particle with mass m , where a is the speed of light. As has been discussed earlier in this section, this equation (in one dimension) also describes a vibration of a string in an elastic medium.

1. Solve this equation in one dimension for an infinite interval $-\infty < x < \infty$.
2. Generalize this solution for three-dimensional infinite space.

3.9 Semi-infinite String Oscillations and the Use of Symmetry Properties

In this section, we extend the methods of Section 3.8 to include strings that are semi-infinite; that is, we will no longer ignore the ends of the string. The boundary value problem for oscillations of a string originally in equilibrium along a semi-infinite length $x \geq 0$ can be formulated as the equation for the displacement function $u(x, t)$ (here we consider only the simplest wave equation),

$$u_{tt} - a^2 u_{xx} = 0, \quad x > 0, \quad t > 0, \quad (3.60)$$

where there are two initial conditions given by

$$u(x, 0) = \varphi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x), \quad (3.61)$$

and a boundary condition given by

$$\alpha u_x + \beta u|_{x=0} = g(t), \quad (3.62)$$

where $|\alpha| + |\beta| \neq 0$. As mentioned in Section 2.1, when $\alpha = 0$, we have what is known as the *Dirichlet problem*; the conditions $\beta = 0$ are referred to as the *Neumann problem*; and the case $\alpha \neq 0, \beta \neq 0$ is a *mixed problem*.

Because of the linearity of this type of boundary value problem, we may search for the solution $u(x, t)$ as the sum $u = u_1 + u_2$, where $u_1(x, t)$ is a solution of the problem with homogeneous boundary conditions and nonhomogeneous initial conditions, and $u_2(x, t)$ is a solution of the problem with nonhomogeneous boundary conditions and homogeneous initial conditions.

3.9.1 Boundary Value Problems with Homogeneous Boundary Conditions

For homogeneous boundary conditions in the case of a semi-infinite string, it is useful to consider an approach that uses symmetry arguments. The extension of initial conditions on the real semi-infinite string to an imaginary infinite string is sometimes referred to as the *method of images*. We apply this idea in the following cases.

Homogeneous Dirichlet Boundary Condition. Assume that the end of a string at $x = 0$ is *rigidly fixed*. The boundary condition (called the *homogeneous Dirichlet condition*) in this case is

$$u(0, t) = 0. \quad (3.63)$$

From the initial condition $u(x, 0) = \varphi(x)$ and the boundary condition of Equation (3.63), it follows that $\varphi(0) = 0$.

Let us extend the functions $\varphi(x)$ and $\psi(x)$ onto the negative semiaxis $x < 0$ in such a way that the extended functions, $\Phi(x)$ and $\Psi(x)$, defined on the entire (infinite) axis are odd functions, given as

$$\Phi(x) = \begin{cases} \varphi(x), & x > 0, \\ -\varphi(-x), & x < 0, \end{cases} \quad \Psi(x) = \begin{cases} \psi(x), & x > 0, \\ -\psi(-x), & x < 0. \end{cases} \quad (3.64)$$

As shown in Section 3.8.2, the solution to the Cauchy problem with initial conditions $\Phi(x)$ and $\Psi(x)$ for Equation (3.60) on an infinite axis $-\infty < x < \infty$ can be expressed by using the D'Alembert formula:

$$u(x, t) = \frac{1}{2} [\Phi(x + at) + \Phi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(\xi) d\xi. \quad (3.65)$$

It is easy to demonstrate that the function $u(x, t)$ given by Equation (3.65) is the solution to the problem defined by Equations (3.60), (3.61), and (3.63) for $x \geq 0$. First, the function $u(x, t)$ satisfies the homogeneous wave Equation (3.60) for all $-\infty < x < \infty$ (including $x > 0$). Second, the function $u(x, t)$ satisfies initial conditions (3.61) because $\Phi(x) \equiv \varphi(x)$, and $\Psi(x) \equiv \psi(x)$ for $x \geq 0$. And, finally, $u(x, t)$ satisfies the boundary condition (3.63) at $x = 0$:

$$u(0, t) = \frac{1}{2} [\Phi(at) + \Phi(-at)] + \frac{1}{2a} \int_{-at}^{at} \Psi(\xi) d\xi = 0.$$

Here the first term equals zero because the function $\Phi(x)$ is odd, and the second term equals zero because it is the integral of an odd function over symmetric limits.

We now can rewrite Equation (3.65), expressing functions $\Phi(x)$ and $\Psi(x)$ by using the functions $\varphi(x)$ and $\psi(x)$ given in Equations (3.64).

- If $x + at > x - at > 0$, then

$$\Phi(x \pm at) = \varphi(x \pm at), \text{ and } \Psi(x \pm at) = \psi(x \pm at).$$

- If $x - at < 0$, then

$$\Phi(x - at) = -\varphi(at - x), \text{ and } \Psi(x - at) = -\psi(at - x).$$

Therefore, the solution to this problem is

$$u(x, t) = \begin{cases} \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi, & t < x/a, \\ \frac{\varphi(x + at) - \varphi(at - x)}{2} + \frac{1}{2a} \int_{at-x}^{x+at} \psi(\xi) d\xi, & t \geq x/a. \end{cases} \quad (3.66)$$

Points of the string with coordinates $x > at$ are not affected by the boundary condition at $x = 0$ because of the limited speed of wave propagation, in which case Equation (3.66) reduces to its second term, coinciding with the solution for the case of the infinite string.

Reading Exercise. Verify that Equation (3.66) satisfies initial and boundary conditions (3.61) and (3.63), respectively.

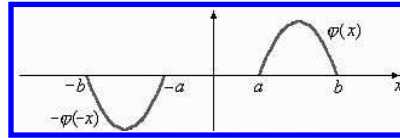


Figure 3.12. Odd continuation of the function $\varphi(x)$ to the (infinite) x -axis.

Let us now give a detailed geometrical illustration for the solution of a homogeneous wave equation in the case of the Dirichlet boundary condition with a rigidly fixed end, $u(0, t) = 0$. Here we consider two cases: (1) $\psi(x) = 0$ (the wave is due to an initial deflection), and (2) $\varphi(x) = 0$ (the wave is due to an initial pulse).

For the case of waves created by an initial deflection, suppose that function $\varphi(x)$ is not zero on some interval (a, b) , where $0 < a < b$. First, let us extend $\varphi(x)$ in an odd sense onto the entire x -axis, as shown in Figure 3.12.

After the string is released, the beginning of the process is completely analogous to the case of an infinite string (see Figure 3.7 in Section 3.8.2). An initial arbitrary deflection is presented as a sum of two component waves (shown as a bold curve in the upper panel of Figure 3.13) propagating in opposite directions.

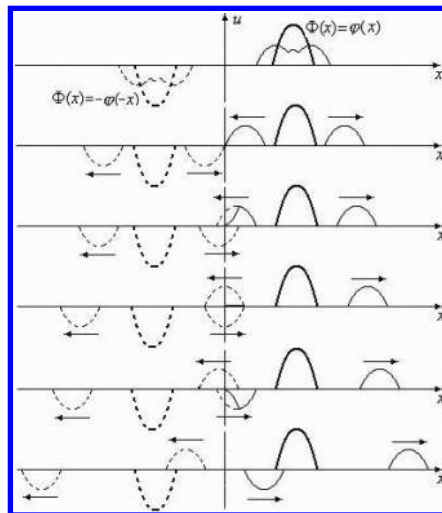


Figure 3.13. Propagation of the initial displacement.

When the portion running to the left along the positive semiaxis ($x > 0$) reaches the point $x = 0$, the portion of the extended function running to the right (with an oppositely directed deflection, or *inverse phase*) along the negative semiaxis $x < 0$ will reach the same point. When these component waves collide at $x = 0$, they pass through each other, adding together to form a shape that is equivalent to a reflection on a semi-infinite string from a fixed end at $x = 0$. This process is schematically shown in Figure 3.13. The profile of the component wave running to the left toward the point $x = 0$ (solid thin curve) collides with the component of the extended wave coming from the right (dashed thin curve) with the opposite sign (or phase). The result for the semi-infinite axis (the real problem) is two waves traveling to the right, the second of which is inverted (the two solid thin curves in the bottom panel).

For the case of waves created by a pulse, we now consider a semi-infinite string, $x \geq 0$, fixed at the point $x = 0$, where the initial deflection is zero but initial speeds exist for points on the string in an interval (a, b) , where $\psi(x) = v_0 = \text{const}$. As in the previous case, we may extend this distribution of speeds onto $x < 0$ to create an *odd* function of speeds on the infinite x -axis. Waves will be created on the two intervals (a, b) and $(-b, -a)$ so that initially the situation is similar to that of an infinite string.

As before, the component of the wave running to the left along the positive semiaxis, $x > 0$ will reach the point $x = 0$ at the same moment as the component running to the right from the interval $(-b, -a)$ with opposite phase. The resulting real, trapezoid shaped wave traveling to the right on the positive semiaxis is shown schematically in the lower panel of Figure 3.14 for the initial rectangular-shaped velocity distribution.

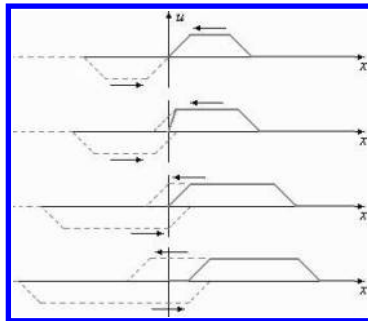


Figure 3.14. Propagation of waves created by an initial velocity pulse.

Homogeneous Neumann Boundary Condition. Next we consider the problem of the propagation of an initial perturbation when the end $x = 0$ is free (the *Neumann boundary condition*):

$$u_{tt} - a^2 u_{xx} = 0, \quad x > 0, \quad t > 0, \quad (3.67)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (3.68)$$

$$u_x(0, t) = 0. \quad (3.69)$$

In this case, we extend the functions $\varphi(x)$ and $\psi(x)$ onto the negative semiaxis $x < 0$ in such a way that the extended functions, $\Phi(x)$ and $\Psi(x)$, defined on the entire infinite axis are even:

$$\Phi(x) = \begin{cases} \varphi(x), & x > 0, \\ \varphi(-x), & x < 0, \end{cases} \quad \Psi(x) = \begin{cases} \psi(x), & x > 0, \\ \psi(-x), & x < 0. \end{cases} \quad (3.70)$$

The solution of the Cauchy problem for Equation (3.67) with initial conditions $\Phi(x)$ and $\Psi(x)$ on an infinite axis, $-\infty < x < \infty$, is again given by D'Alembert's formula in Equation (3.65). It is easily seen that the function $u(x, t)$, given by Equation (3.65) is the solution to the problem defined by Equations (3.67) through (3.69) for $x \geq 0$.

Indeed, $u(x, t)$ satisfies the homogeneous wave Equation (3.67) on $-\infty < x < \infty$ and therefore on $x > 0$. The function $u(x, t)$ satisfies initial conditions (3.68) because $\Phi(x) \equiv \varphi(x)$, and $\Psi(x) \equiv \psi(x)$ on $x \geq 0$. At $x = 0$, we have

$$\frac{\partial u}{\partial x}(0, t) = \frac{1}{2} \left[\frac{\partial \Phi}{\partial x}(at) + \frac{\partial \Phi}{\partial x}(-at) \right] + \frac{1}{2a} [\Psi(at) - \Psi(-at)].$$

Using the fact that the derivative of an even function is an odd function,

$$\Phi'(x) = -\Phi'(-x),$$

we see that the first term is equal to zero because function $\Phi'(x)$ is odd. The second term is also zero because the function $\Psi(x)$ is even and we have $u_x(0, t) = 0$.

As in the previous case, we may rewrite Equation (3.65) expressing functions $\Phi(x)$ and $\Psi(x)$ with the functions $\varphi(x)$ and $\psi(x)$ by using Equation (3.70). Therefore, the solution to problem expressed by Equations (3.67) through (3.69) is given by

$$u(x, t) = \begin{cases} \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi, & t < x/a, \\ \frac{\varphi(x + at) + \varphi(at - x)}{2} + \frac{1}{2a} \left\{ \int_0^{x+at} \psi(\xi) d\xi + \int_0^{at-x} \psi(\xi) d\xi \right\}, & t \geq x/a. \end{cases} \quad (3.71)$$

Reading Exercise. Verify that Equation (3.71) satisfies the initial and the boundary conditions given in Equations (3.68) and (3.69).

Similarly, as was done for a fixed end, we can consider two cases: (1) $\psi(x) = 0$, and (2) $\varphi(x) = 0$. In the case of a free end, the initial deflection or velocities should be extended as even functions, which results in a reflected wave on the positive (real) semiaxis with the same phase, in contrast to an inverted phase for the case of reflection from a fixed end. Several examples are discussed later in this section.

Homogeneous Mixed Boundary Conditions. Now consider the case where at the end, $x = 0$, of a semi-infinite string, we have a *mixed homogeneous boundary condition*, that is, a problem described by equations

$$u_{tt} - a^2 u_{xx} = 0, \quad x > 0, \quad t > 0, \quad (3.72)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (3.73)$$

$$u_x(0, t) - hu(0, t) = 0, \quad h = \text{const.} \quad (3.74)$$

Physically, this may be thought of as a semi-infinite string where the end at $x = 0$ is attached elastically, such as by a spring.

Once again the method of solution is based on extending the functions $\varphi(x)$ and $\psi(x)$ for the initial conditions onto the negative semiaxis $x < 0$ in such a way that functions

$$f_1(x) \equiv \Phi'(x) - h \Phi(x), \quad f_2(x) \equiv \Psi'(x) - h \Psi(x)$$

become *odd* on the entire x -axis. Functions $\Phi(x)$ and $\Psi(x)$ are the continuations of $\varphi(x)$ and $\psi(x)$ on the entire x -axis; the portion located on the negative semiaxis may be referred to as the *image* of the real initial conditions on the positive x -axis. To arrive at the solution, we must first

prove that the function

$$u(x, t) = \frac{1}{2} [\Phi(x + at) + \Phi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(\xi) d\xi \quad (3.75)$$

is the solution to the problem expressed by Equations (3.72) through (3.74) for $x \geq 0$.

The function $u(x, t)$ satisfies the homogeneous wave equation on domain $-\infty < x < \infty$; therefore, it satisfies this equation for $x > 0$. Also the function $u(x, t)$ satisfies initial conditions in Equation (3.73) because $\Phi(x) \equiv \varphi(x)$, $\Psi(x) \equiv \psi(x)$ for $x \geq 0$.

There is a little more work in establishing that the function $u(x, t)$ satisfies boundary conditions (3.74). Consider the linear combination $u_x(x, t) - hu(x, t)$ and a change of variables in Equation (3.75) using the substitution $\xi + x$ for ξ . We have

$$\begin{aligned} u_x - hu &= \frac{1}{2} [\Phi'(x + at) - h\Phi(x + at)] + \frac{1}{2} [\Phi'(x - at) - h\Phi(x - at)] \\ &\quad + \frac{1}{2a} \int_{-at}^{at} \left[\frac{d\Psi(\xi + x)}{dx} - h\Psi(\xi + x) \right] d\xi. \end{aligned}$$

Taking into account that

$$\frac{d\Psi(x + \xi)}{dx} \equiv \frac{d\Psi(x + \xi)}{d\xi},$$

and replacing ξ by $\xi - x$ in the integral we obtain

$$\begin{aligned} u_x - hu &= \frac{1}{2} [\Phi'(x + at) - h\Phi(x + at)] + \frac{1}{2} [\Phi'(x - at) - h\Phi(x - at)] + \int_{x-at}^{x+at} \left[\frac{d\Psi(\xi)}{d\xi} - h\Psi(\xi) \right] d\xi \\ &= \frac{f_1(x + at) + f_1(x - at)}{2} + \int_{x-at}^{x+at} f_2(\xi) d\xi. \end{aligned}$$

At the boundary $x = 0$, we have

$$u_x(x, t) - hu(x, t)|_{x=0} = \frac{f_1(at) + f_1(-at)}{2} + \frac{1}{2a} \int_{-at}^{+at} f_1(\xi) d\xi = 0, \quad (3.76)$$

where the last equality holds because the functions $f_1(x)$ and $f_2(x)$ are odd.

Thus, again we have shown that the function $u(x, t)$ represented by D'Alembert's formula in Equation (3.75) is the solution to the problem of Equations (3.72) through (3.74).

Next, we obtain expressions for the functions $\Phi(x)$ and $\Psi(x)$ for $x < 0$ (where we already have $\Phi(x) \equiv \varphi(x)$, $\Psi(x) \equiv \psi(x)$ for $x > 0$). To define $\Phi(x)$ for $x < 0$, we have the Cauchy problem

$$\Phi'(x) - h\Phi(x) = f(x), \quad x < 0, \quad (3.77)$$

$$\Phi(0) = \varphi(0),$$

where

$$f(x) = -\varphi'(-x) + h\varphi(-x).$$

The solution to Equation (3.77) is

$$\begin{aligned} \Phi(x) &= e^{hx} \left[\varphi(0) + \int_0^x e^{-hz} f(z) dz \right] \\ &= e^{hx} \left[\varphi(0) - \int_0^x e^{-hz} \varphi'(-z) dz + h \int_0^x e^{-hz} \varphi(-z) dz \right], \quad x \leq 0. \end{aligned}$$

Evaluating the first integral by parts, we have

$$-\int_0^x e^{-hz} \varphi'(-z) dz = \int_0^x e^{-hx} d(\varphi'(-z)) = e^{-hx} \varphi(-x) - \varphi(0) + h \int_0^x e^{-hz} \varphi(-z) dz.$$

Then, for $x \leq 0$,

$$\begin{aligned} \Phi(x) &= e^{hx} \left[\varphi(0) + e^{-hx} \varphi(-x) - \varphi(0) + h \int_0^x e^{-hz} \varphi(-z) dz + h \int_0^x e^{-hz} \varphi(-z) dz \right] \\ &= \varphi(-x) + 2h \int_0^x e^{-hz} \varphi(-z) dz. \end{aligned}$$

Thus, we may write the function $\Phi(x)$ as

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0, \\ \varphi(-x) + 2h \int_0^x e^{h(x-z)} \varphi(-z) dz, & x \leq 0. \end{cases}$$

Similarly, we have the Cauchy problem for the function $\Psi(x)$ for $x < 0$:

$$\Psi'(x) - h\Psi(x) = g(x), \quad x < 0, \quad (3.78)$$

$$\Psi(0) = \psi(0),$$

where $g(x) = -\psi'(-x) + h\psi(-x)$.

The above yields the following for the function $\Psi(x)$:

$$\Psi(x) = \begin{cases} \psi(x), & x \geq 0, \\ \psi(-x) + 2h \int_0^x e^{h(x-z)} \psi(-z) dz, & x \leq 0. \end{cases}$$

Reading Exercise. Obtain the last expression for function $\Psi(x)$.

Substituting these expressions for functions $\Phi(x)$ and $\Psi(x)$ into Equation (3.75), we obtain, for $x > 0$ and $t \geq x/a$,

$$\begin{aligned} u(x, t) = & \frac{\varphi(x + at) + \varphi(at - x)}{2} + h e^{h(x-at)} \int_0^{x-at} e^{-hz} \varphi(-z) dz \\ & + \frac{1}{2a} \left\{ \int_0^{x+at} \psi(s) ds + \int_0^{at-x} \psi(s) ds \right\} \\ & + \frac{h}{a} \int_{x-at}^0 e^{-hs} ds \int_0^s e^{-hz} \psi(-z) dz. \end{aligned} \quad (3.79)$$

Evaluating the last integral in Equation (3.79) by parts yields

$$\begin{aligned}
 \frac{h}{a} \int_{x-at}^0 e^{-hs} ds \int_0^s e^{-hz} \psi(-z) dz &= -\frac{h}{a} \int_0^{x-at} e^{hs} ds \int_0^s e^{-hz} \psi(-z) dz \\
 &= -\frac{1}{a} \int_0^{x-at} \left\{ \int_0^s e^{-hz} \psi(-z) dz \right\} d e^{hs} \\
 &= -\frac{1}{a} e^{h(x-at)} \int_0^{x-at} e^{-hz} \psi(-z) dz + \frac{1}{a} \int_0^{x-at} \psi(-z) dz \\
 &= -\frac{1}{a} e^{h(x-at)} \int_0^{x-at} e^{-hz} \psi(-z) dz - \frac{1}{a} \int_0^{at-x} \psi(s) ds.
 \end{aligned} \tag{3.80}$$

Substituting Equation (3.80) into Equation (3.79) we have for $x > 0$, and $t \geq x/a$ the expression

$$\begin{aligned}
 u(x, t) &= \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{at-x}^{x=at} \psi(s) ds \\
 &\quad + h e^{-h(x-at)} \int_0^{x-at} e^{-hz} \varphi(-z) dz - \frac{1}{a} e^{-h(x-at)} \int_0^{x-at} e^{-hz} \psi(-z) dz.
 \end{aligned}$$

Finally, the solution of the problem expressed in Equations (3.72) through (3.74) is

$$u(x, t) = \begin{cases} \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi, & 0 < t < x/a, \\ \frac{\varphi(x+at) + \varphi(at-x)}{2} + \frac{1}{2a} \int_{at-x}^{x+at} \psi(\xi) d\xi \\ \quad + \int_0^{at-x} e^{h[z-(at-x)]} \left[-h\varphi(z) + \frac{1}{a}\psi(z) \right] dz, & t \geq x/a. \end{cases} \tag{3.81}$$

Reading Exercise. Check that Equation (3.81) satisfies the initial and the boundary conditions given in Equations (3.73) and (3.74), respectively.

Example 3.5. A semi-infinite string stretched along $x \geq 0$ is excited by the initial deflection

$$\varphi(x) = \begin{cases} \sin \frac{\pi x}{l}, & x \in [0, l], \\ 0, & x \notin [0, l], \end{cases}$$

with no initial velocities. Find the vibrations of the string for the cases when the end ($x = 0$) of the string (1) is rigidly fixed, (2) is free, and (3) is elastically fixed with coefficient h . Write an analytical solution, representing the motion of the string for $t > 0$, and simulate the vibrations by using the program Waves.

Solution. We are seeking a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } x > 0, t > 0$$

that satisfies the initial conditions

$$u|_{t=0} = \varphi(x) = \begin{cases} \sin \frac{\pi x}{l}, & x \in [0, l], \\ 0, & x \notin [0, l], \end{cases} \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x) = 0.$$

1. Let the end of the string at $x = 0$ be rigidly fixed (the *boundary condition of Dirichlet type*):

$$u(0, t) = 0.$$

For this problem the initial speeds of points on the string are zero ($\psi(x) \equiv 0$), so the solution is given by Equation (3.66) which in this case yields

$$u(x, t) = \begin{cases} \frac{1}{2} \left[\sin \frac{\pi}{l}(x + at) + \sin \frac{\pi}{l}(x - at) \right], & 0 < t < x/a, \\ \frac{1}{2} \left[\sin \frac{\pi}{l}(x + at) - \sin \frac{\pi}{l}(at - x) \right], & t \geq x/a. \end{cases}$$

Therefore:

- (a) The string profile for $0 < t < \frac{l}{2a}$ is

$$u(x, t) = \begin{cases} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l}, & 0 < x \leq l - at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & l - at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

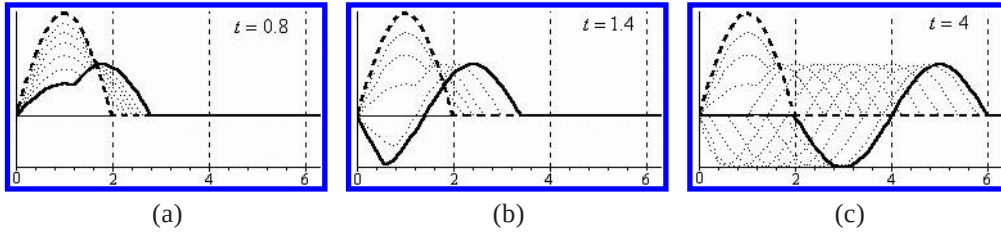


Figure 3.15. Graphs of the solutions to Example 3.5 for the case of the Dirichlet boundary condition. (a) $t = 0.8$. (b) $t = 1.4$. (c) $t = 4$.

(b) The string profile for $\frac{l}{2a} < t < \frac{l}{a}$ is

$$u(x, t) = \begin{cases} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l}, & 0 < x \leq l - at, \\ -\frac{1}{2} \sin \frac{\pi(at-x)}{l}, & l - at < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

(c) The string profile for $t > \frac{l}{a}$ is

$$u(x, t) = \begin{cases} 0, & x \leq at - l, \\ -\frac{1}{2} \sin \frac{\pi(at-x)}{l}, & at - l < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

Figure 3.15 shows the solution for the case when $a^2 = 1$, $l = 2$. The dashed line represents the initial deflection of the string and the solid line is the string profile at times (a) $t = 0.8$, (b) $t = 1.4$, and (c) $t = 4$. This solution was obtained with the program Waves.

- Let the end $x = 0$ of the string be free (Neumann boundary conditions):

$$\frac{\partial u}{\partial x}(0, t) = 0.$$

The initial speeds of points on the string are again zero ($\psi(x) \equiv 0$), so the solution is given by Equation (3.71), which yields

$$u(x, t) = \begin{cases} \frac{1}{2} \left[\sin \frac{\pi}{l}(x + at) + \sin \frac{\pi}{l}(x - at) \right], & 0 < t < x/a, \\ \frac{1}{2} \left[\sin \frac{\pi}{l}(x + at) + \sin \frac{\pi}{l}(at - x) \right], & t \geq x/a. \end{cases}$$

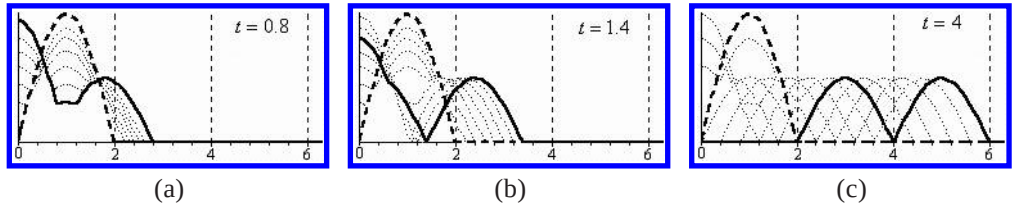


Figure 3.16. Graphs of the solution to Example 3.5 for the case of Neumann boundary conditions. (a) $t = 0.8$. (b) $t = 1.4$. (c) $t = 4$.

Therefore:

(a) The string profile for $0 < t < \frac{l}{2a}$ is

$$u(x, t) = \begin{cases} \cos \frac{\pi x}{l} \sin \frac{\pi at}{l}, & 0 < x \leq at, \\ \sin \frac{\pi x}{l} \cos \frac{\pi at}{l}, & at < x \leq l - at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & l - at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

(b) The string profile for $\frac{l}{2a} < t < \frac{l}{a}$ is

$$u(x, t) = \begin{cases} \cos \frac{\pi x}{l} \sin \frac{\pi at}{l}, & 0 < x \leq l - at, \\ \frac{1}{2} \sin \frac{\pi(at-x)}{l}, & l - at < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

(c) The string profile for $t > \frac{l}{a}$ is

$$u(x, t) = \begin{cases} 0, & x \leq at - l, \\ \frac{1}{2} \sin \frac{\pi(at-x)}{l}, & at - l < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

Figure 3.16 shows the solution for the case when $a^2 = 1$ and $l = 2$. The dashed line represents the initial deflection of the string and the solid line is the string profile at times (a) $t = 0.8$, (b) $t = 1.4$, and (c) $t = 4$. This solution was obtained with the program Waves.

3. Let the end $x = 0$ of the string be elastically fixed with elastic coefficient h (*mixed boundary conditions*):

$$\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0.$$

The initial speeds of points on the string are again zero ($\psi(x) \equiv 0$), so the solution is given by Equation (3.81) which yields

$$u(x, t) = \begin{cases} \frac{\varphi(x + at) + \varphi(x - at)}{2}, & 0 < t < x/a, \\ \frac{\varphi(x + at) + \varphi(at - x)}{2} - h \int_0^{at-x} e^{h[z-(at-x)]} \varphi(z) dz, & t \geq x/a. \end{cases}$$

If $0 < t \leq \frac{x}{a}$, then

$$u(x, t) = \frac{1}{2} \sin \frac{\pi(x + at)}{l} + \frac{1}{2} \sin \frac{\pi(x - at)}{l} = \sin \frac{\pi x}{l} \cos \frac{\pi at}{l}.$$

If $\frac{x}{a} \leq t \leq \frac{x+l}{a}$, then $0 \leq at - x \leq l$ and

$$u(x, t) = \frac{1}{2} \sin \frac{\pi(x + at)}{l} + \frac{1}{2} \sin \frac{\pi(at - x)}{l} + F_1(x, t),$$

where

$$\begin{aligned} F_1(x, t) &= -he^{-h(at-x)} \int_0^{at-x} e^{hz} \sin \frac{\pi z}{l} dz \\ &= \frac{hl^2}{\pi^2 + l^2 h^2} \left[-h \sin \frac{\pi(at-x)}{l} + \frac{\pi}{l} \cos \frac{\pi(at-x)}{l} - \frac{\pi}{l} e^{-h(at-x)} \right]. \end{aligned}$$

Finally, if $t > \frac{x+l}{a}$, the solution is

$$u(x, t) = \frac{1}{2} \sin \frac{\pi(x + at)}{l} + \frac{1}{2} \sin \frac{\pi(at - x)}{l} + F_2(x, t),$$

where

$$F_2(x, t) = -he^{-h(at-x)} \int_0^l e^{hz} \sin \frac{\pi z}{l} dz = \frac{\pi hl}{\pi^2 + l^2 h^2} e^{-h(at-x)} (e^{hl} + 1).$$

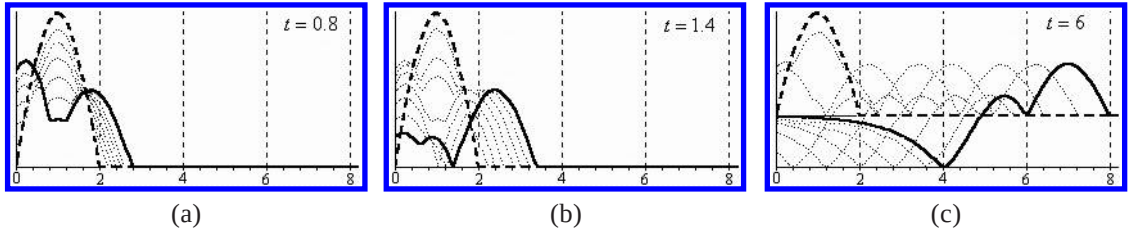


Figure 3.17. Graphical representation of the solution for Example 3.5 for the case of mixed boundary conditions. (a) $t = 0.8$. (b) $t = 1.4$. (c) $t = 6$.

Therefore:

(a) The string profile for $0 < t < \frac{l}{2a}$ is

$$u(x, t) = \begin{cases} \frac{1}{2} \sin \frac{\pi(x+at)}{l} + F_1(x, t), & x \leq at, \\ \sin \frac{\pi x}{l} \cdot \cos \frac{\pi at}{l}, & at < x \leq l - at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & l - at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

(b) The string profile for $\frac{l}{2a} < t < \frac{l}{a}$ is

$$u(x, t) = \begin{cases} \frac{1}{2} \sin \frac{\pi(x+at)}{l} + F_1(x, t), & x \leq l - at, \\ F_1(x, t), & l - at < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

(c) The string profile for $t > \frac{l}{a}$ is

$$u(x, t) = \begin{cases} F_2(x, t), & x \leq at - l, \\ F_1(x, t), & at - l < x \leq at, \\ \frac{1}{2} \sin \frac{\pi(x-at)}{l}, & at < x \leq l + at, \\ 0, & x > l + at. \end{cases}$$

Figure 3.17 shows the solution for the case when $a^2 = 1$, $l = 2$, and $h = 1$. The dashed line represents the initial deflection of the string and the solid line is the string profile at times (a) $t = 0.8$, (b) $t = 1.4$, and (c) $t = 6$. This solution was obtained with the program Waves.

3.9.2 Boundary Value Problems with Nonhomogeneous Boundary Conditions

In the general case of nonhomogeneous boundary conditions for a semi-infinite string, the solution may be formulated as a sum, each term of which satisfies only one condition, either boundary or initial. Physical examples include semi-infinite strings and rods driven on one end by various methods.

Dirichlet Boundary Conditions. Consider a boundary value problem for the homogeneous wave equation on a string with *homogeneous initial conditions* and a *nonhomogeneous boundary condition of the Dirichlet type*:

$$\begin{aligned} u_{tt} &= a^2 u_{xx}, \quad x > 0, \quad t > 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \quad u(0, t) = g(t). \end{aligned} \quad (3.82)$$

Let us solve the problem expressed in Equations (3.82) by using the method of traveling waves. Because the equation is homogeneous and given the zero initial conditions of Equation (3.82), the only source of perturbations is the boundary condition given by the function $g(t)$. This, then, might represent a string being shaken at one end where we are given the motion of the end as $g(t)$.

The solution can be found in the form of a wave propagating to the right, given by

$$u(x, t) = f(x - at),$$

where f is an arbitrary but sufficiently smooth function. To define the function f , we first apply the boundary condition

$$u(0, t) = f(-at) = g(t).$$

Changing to the variable z as the argument of the function f , we obtain

$$f(z) = g\left(-\frac{z}{a}\right),$$

so that

$$u(x, t) = g\left(-\frac{x - at}{a}\right) = g\left(t - \frac{x}{a}\right).$$

This function is defined only for $x - at \leq 0$ because $g(t)$ exists only for $t \geq 0$. The phase plane showing the region of existence of solutions is shown in Figure 3.18.

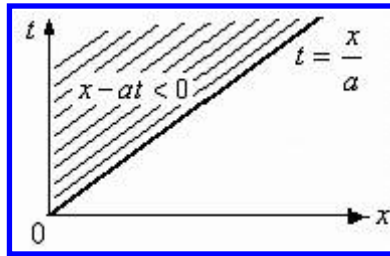


Figure 3.18. Solutions are defined only for values of x and t below the solid line in the lower region of the phase plane.

We may, however, find $u(x, t)$ for all values of its variables by extending the function $g(t)$ to negative values of t , as $g(t) = 0$ for $t < 0$. Then the function f is given by

$$f(z) = \begin{cases} 0, & z \geq 0, \\ g(-z/a), & z < 0. \end{cases}$$

Substituting $z = x - at$, we have

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq x/a, \\ g(t - x/a), & t > x/a. \end{cases} \quad (3.83)$$

This solution is defined for all values of its variables and also satisfies homogeneous initial conditions. From the first initial condition, we have

$$u(x, 0) = f(x) = 0 \text{ for } x \geq 0.$$

The second initial condition is also satisfied:

$$u_t(x, 0) = -a f'(x) = 0 \text{ for } x \geq 0.$$

Note that Equation (3.83) has a simple physical meaning. For a point on the string located at a distance x from the end $x = 0$ and for time $t < x/a$, the perturbation is zero because the displacement has not yet reached this point. For $t \geq x/a$, the perturbation propagating to the right from the end $x = 0$ has passed through this point. The displacement profile everywhere on the string and for all times is defined by the boundary condition function, $g(t)$.

Neumann Boundary Conditions. Now consider a boundary value problem for the homogeneous wave equation on a string with *homogeneous initial conditions* and *nonhomogeneous Neumann boundary conditions* defined by the equations

$$\begin{aligned} u_{tt} &= a^2 u_{xx}, \quad x > 0, \quad t > 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \quad u_x(0, t) = g(t). \end{aligned} \quad (3.84)$$

It should be obvious that the boundary perturbation creates a wave propagating to the right with speed a , and the solution has the form

$$u(x, t) = f(x - at).$$

From this, we have

$$\frac{\partial u}{\partial x}(x, t) = f'(x - at),$$

and we can determine the function f from the boundary condition

$$\frac{\partial u}{\partial x}(0, t) = f'(-at) = g(t).$$

Then, as in the previous section, we have

$$f'(z) = g\left(-\frac{z}{a}\right), \text{ and } f(z) = -a \int_0^{-z/a} g(\tau) d\tau;$$

thus,

$$u(x, t) = f(x - at) = -a \int_0^{t-x/a} g(\tau) d\tau.$$

Because $g(t)$ is defined for $t \geq 0$, the function $u(x, t)$ is defined for $x - at \leq 0$ only.

To find $u(x, t)$ for all values of its arguments, we extend the function $g(t)$ to negative values of t , letting $g(t) = 0$ for $t < 0$. This yields

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq x/a, \\ -a \int_0^{t-x/a} g(\tau) d\tau, & t > x/a, \end{cases} \quad (3.85)$$

which is defined for all values of its arguments and also satisfies the initial conditions.

The physical meaning of Equation (3.85) is analogous to the physical meaning of Equation (3.83) for boundary conditions of the Dirichlet type. For time $t < x/a$, the perturbation has not yet reached the points on the string that are situated at a distance x .

The problem defined in Equation (3.84) may also be treated as a mathematical model of a small longitudinal oscillation of a semi-infinite elastic rod. The boundary condition of the second kind can be thought of physically as a force $f(t) = k(0)g(t)$, applied to the end $x = 0$, where $k(0)$ is the rod's elasticity at point $x = 0$. Suppose now that the force acts at the end $x = 0$ during the interval of time (t_1, t_2) , that is,

$$g(t) = \begin{cases} \tilde{g}(t), & t \in (t_1, t_2), \\ 0, & t \notin (t_1, t_2). \end{cases}$$

Then from Equation (3.63), we find

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq t_1 + x/a, \\ -a \int_{t_1}^{t-x/a} \tilde{g}(\tau) d\tau, & t_1 + x/a < t \leq t_2 + x/a, \\ -a \int_{t_1}^{t_2} \tilde{g}(\tau) d\tau, & t > t_2 + x/a. \end{cases}$$

It follows from this expression that for $t < t_1 + x/a$ the perturbation at the point x is equal to zero; that is, in this time interval a boundary perturbation does not influence the rod at point x because the perturbation does not have enough time to reach this point. At time $t = t_1 + x/a$, the perturbation appears at point x and depends on t until time $t = t_2 + x/a$. After time $t = t_2 + x/a$, the perturbation at point x becomes time-independent and the system returns to equilibrium.

Mixed Boundary Conditions. Now consider a boundary value problem for the homogeneous wave equation on a semi-infinite string with *homogeneous initial conditions* and *nonhomogeneous mixed boundary conditions*

given by

$$\begin{aligned} u_{tt} &= a^2 u_{xx}, \quad x > 0, \quad t > 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \quad u_x(0, t) - hu(0, t) = g(t), \end{aligned} \quad (3.86)$$

where h is some constant.

Again we apply the idea of traveling waves to the problem. Much the same way as in the case of the Dirichlet boundary conditions in Equation (3.82), the homogeneity of the equation and the initial conditions in Equation (3.86) imply that the solution can be presented as a wave running to the right, or

$$u(x, t) = f(x - at),$$

where f is a sufficiently smooth function.

Substituting the function $f(x - at)$ into boundary condition (3.86) yields the ordinary differential equation

$$f'(-at) - hf(-at) = g(t), \quad t > 0,$$

where the prime indicates a derivative with respect to the entire argument. From the first initial condition, we have

$$u(x, 0) = f(x) = 0, \quad \text{for } x \geq 0.$$

Making the substitution $z = -at$ and taking into account that $f(0) = 0$, we obtain the Cauchy problem for the function $f(z)$:

$$\begin{aligned} f'(z) - hf(z) &= g(-z/a), \quad z < 0, \\ f(0) &= 0. \end{aligned}$$

The reader may verify as a reading exercise that the solution of this problem can be written in the form

$$f(z) = e^{hz} \int_0^z e^{-hs} g(-s/a) ds, \quad z < 0.$$

Making the variable transformation $\xi = -s/a$, in the last integral, we get

$$f(z) = -ae^{hz} \int_0^{-z/a} e^{ha\xi} g(\xi) d\xi, \quad z < 0.$$

Putting $z = x - at$, in the last formula, we obtain the final expression for displacements:

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq x/a, \\ -ae^{h(x-at)} \int_0^{t-x/a} e^{ha\xi} g(\xi) d\xi, & t \geq x/a. \end{cases} \quad (3.87)$$

Notice that setting $h = 0$ reduces Equation (3.87) to Equation (3.85), which is the solution of the boundary value problem for a homogeneous wave equation on a semi-infinite string with Neumann, nonhomogeneous boundary conditions.

This concludes the discussion of the three main types of nonhomogeneous boundary conditions when the initial conditions are homogeneous. The other cases, when the initial conditions are nonhomogeneous but the boundary conditions are homogeneous, have been discussed in the previous section. As was stated before, if both initial and boundary conditions are nonhomogeneous, the solution $u(x, t)$ has the form $u = u_1 + u_2$ where $u_1(x, t)$ is a solution for homogeneous boundary conditions and nonhomogeneous initial conditions, and $u_2(x, t)$ is a solution for nonhomogeneous boundary conditions and homogeneous initial conditions. Various examples follow.

Example 3.6. Let a semi-infinite string along the positive x -axis be initially at rest. For $t > 0$, the end of the string at $x = 0$ is driven with the function

$$g(t) = Ae^{-ct} \sin \omega t.$$

Find the displacements $u(x, t)$ of the string for $t > 0$. Write an analytical solution and simulate the vibrations by using the program Waves.

Solution. We are to find a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x > 0, t > 0$$

that satisfies the zero initial conditions

$$u|_{t=0} = 0 \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0$$

and the nonhomogeneous Dirichlet boundary condition

$$u(0, t) = e^{-ct} \sin \omega t.$$

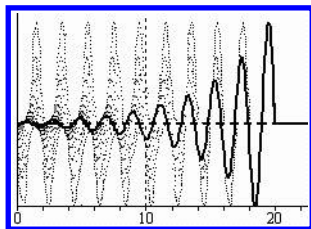


Figure 3.19. Graph of the solution to Example 3.6.

The solution to the problem is given by Equation (3.83), which in this case is

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq x/a, \\ Ae^{-c(t-x/a)} \sin \omega(t - x/a), & t > x/a. \end{cases}$$

Figure 3.19 shows the solution for the case when $a^2 = 1$, $A = 5$, $c = 0.2$, and $\omega = 3$. The dashed line represents the initial deflection and the solid line is the string profile at time $t = 20$. The dotted lines show the evolution of the string profile within the period of time from 0 to 20. As is clear from the discussion, at the instant of time $t = x/a$, vibrations with frequency ω appear at the location x of the string.

Example 3.7. A semi-infinite elastic rod occupies the region $0 \leq x < \infty$. For $t > 0$, a longitudinal force

$$F(t) = Ae^{-ct} \cos \omega t$$

acts on the end $x = 0$. Find the longitudinal displacements $u(x, t)$ of cross sections of the rod. Write formulas for the analytical solution representing the motion of cross sections and simulate the vibrations using the program Waves. Consider a harmonic force $g(t) = A \cos \omega t$ applied to the end of a rod at $x = 0$.

Solution. We have to find a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0$$

that satisfies the zero initial conditions

$$u|_{t=0} = 0 \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0,$$

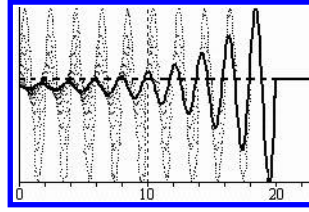


Figure 3.20. Graph of the solution to Example 3.7.

and nonhomogeneous Neumann boundary condition

$$\frac{\partial u}{\partial x}(0, t) = Ae^{-ct} \cos \omega t.$$

Evaluating the integral

$$\int_0^{t-x/a} Ae^{-c\tau} \cos \omega \tau d\tau = -\frac{A}{c^2 + \omega^2} \left\{ e^{-c(t-x/a)} [c \cos \omega(t-x/a) - \omega \sin \omega(t-x/a)] - c \right\},$$

the solution to the problem defined by Equations (3.85) takes the form

$$u(x, t) = \begin{cases} 0, & 0 \leq t \leq x/a, \\ \frac{aA}{c^2 + \omega^2} \left\{ e^{-c(t-x/a)} [c \cos \omega(t-x/a) - \omega \sin \omega(t-x/a)] - c \right\}, & t > x/a. \end{cases}$$

Figure 3.20 shows the solution for the case when $a^2 = 1$, $A = 5$, $c = 0.2$, and $\omega = 3$. The dashed line represents the initial deflection and the solid line is the string profile at time $t = 20$. The dotted lines in between them show the evolution of the string profile within the period of time from 0 to 20. As is clear from the last formula, after the moment of time $t = x/a$, vibrations with frequency ω appear at point x of the string.

3.10 Finite Intervals: The Fourier Method for One-Dimensional Wave Equations

In this and the following sections, we introduce a powerful Fourier method for solving hyperbolic equations for *finite intervals*. First, we apply this method to the one-dimensional homogeneous wave equation with various boundary conditions.

3.10.1 The Fourier Method for Homogeneous Equations

The Fourier method, or the method of separation of variables, is one of the most widely used methods for analytical solution of boundary value problems in mathematical physics. The method gives a solution in terms of a series of eigenfunctions of the corresponding Sturm-Liouville problem (discussed in Chapter 2).

Let us apply the method in the case of the general one-dimensional homogeneous hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} + \gamma u = 0, \quad (3.88)$$

where a , κ , and γ are constants. As we discussed when deriving Equations (3.5) and (3.6), for physical situations the requirement is: $\kappa \geq 0$, $\gamma \geq 0$. Here we will work on a finite interval, $0 \leq x \leq l$, and obviously $t \geq 0$.

To obtain unique solutions of Equation (3.88), additional boundary and initial conditions must be imposed on the function $u(x, t)$. Some of these will be homogeneous, some not. Initially, we will search for a solution of Equation (3.88) satisfying the *homogeneous* boundary conditions

$$P_1[u] \equiv \alpha_1 u_x + \beta_1 u|_{x=0} = 0, \quad P_2[u] \equiv \alpha_2 u_x + \beta_2 u|_{x=l} = 0, \quad (3.89)$$

with constants α_1 , β_1 , α_2 , and β_2 , and initial conditions

$$u(x, t)|_{t=0} = \varphi(x), \quad \left. \frac{\partial u}{\partial t}(x, t) \right|_{t=0} = \psi(x), \quad (3.90)$$

where $\varphi(x)$ and $\psi(x)$ are given functions. As discussed in Chapter 2, normally there are physical restrictions on the signs of the coefficients in Equation (3.89) so that we have $\alpha_1/\beta_1 < 0$ and $\alpha_2/\beta_2 > 0$. In the following discussion, we use the notations $P_1[u]$ and $P_2[u]$ defined as in Equation (3.89) for boundary conditions.

We begin by assuming that a nontrivial (non-zero) solution of Equation (3.88) can be found that is a product of two functions, one depending only on x , and another depending only on t :

$$u(x, t) = X(x)T(t). \quad (3.91)$$

This assumption that the variables x and t can be separated will be justified if it leads to a unique solution of the boundary value problem consisting of

Equation (3.88) and the conditions in Equations (3.89) and (3.90). Substituting Equation (3.91) into Equation (3.88), we obtain

$$X(x)T''(t) + 2\kappa X(x)T'(t) - a^2 X''(x)T(t) + \gamma X(x)T(t) = 0,$$

or, by rearranging terms,

$$\frac{T''(t) + 2\kappa T'(t) + \gamma T(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}, \quad (3.92)$$

where primes indicate derivatives with t or x . The left side of this equation depends only on t , and the right side only on x , which is possible only if each side of Equation (3.92) equals a constant. By using $-\lambda$ for this constant, we obtain

$$\frac{T''(t) + 2\kappa T'(t) + \gamma T(t)}{a^2 T(t)} \equiv \frac{X''(x)}{X(x)} = -\lambda.$$

Thus, Equation (3.92) gives two ordinary linear second-order homogeneous differential equations:

$$T''(t) + 2\kappa T'(t) + (a^2 \lambda + \gamma)T(t) = 0, \quad (3.93)$$

and

$$X''(x) + \lambda X(x) = 0. \quad (3.94)$$

Thus, we see that we have successfully separated the variables, resulting in separate equations for functions $X(x)$ and $T(t)$. These equations share the common parameter λ .

To find λ , we apply the boundary conditions. Homogenous boundary condition of Equation (3.89), imposed on $u(x, y)$, give the homogeneous boundary conditions on the function $X(x)$:

$$\alpha_1 X' + \beta_1 X \Big|_{x=0} = 0, \quad \alpha_2 X' + \beta_2 X \Big|_{x=l} = 0. \quad (3.95)$$

This result therefore leads to the *Sturm-Liouville boundary problem*, which may be stated in the present case as the following:

Find values of the parameter λ (eigenvalues) for which non-trivial (not identically equal to zero) solutions to Equation (3.94), $X(x)$ (eigenfunctions), satisfying boundary conditions (3.95) exist.

Let us briefly remind the reader of the main properties of eigenvalues and eigenfunctions of the Sturm-Liouville problem given in Equations (3.94) and (3.95).

1. There exists an *infinite set* of real nonnegative discrete eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{X_n(x)\}$. The eigenvalues increase as the number n increases:

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots \quad (\lim \lambda_n = +\infty).$$

2. Eigenfunctions corresponding to different eigenvalues are *linearly independent and orthogonal*:

$$\int_0^l X_i(x)X_j(x)dx = 0, \quad i \neq j. \quad (3.96)$$

3. *The completeness property* states that any function $f(x)$ that is twice differentiable on $(0, l)$ and satisfies the homogeneous boundary conditions in Equations (3.95) can be resolved in an absolutely and uniformly converging series with eigenfunctions of the boundary value problem given in Equations (3.94) and (3.95):

$$f(x) = \sum_{n=1}^{\infty} f_n X_n(x), \quad f_n = \frac{1}{\|X_n\|^2} \int_0^l f(x)X_n(x)dx.$$

Eigenvalues and eigenfunctions of a boundary value problem depend on the type of the boundary conditions: Dirichlet, Neumann, or mixed. In principle, the particular values of the constants α_i and β_i in Equations (3.95) determine one of these three possible types. All possible variants are presented in Appendix A. For all of these variants, the solution for the problem given in Equations (3.94) and (3.95) is

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x, \quad (3.97)$$

where the coefficients C_1 and C_2 are determined from the system of Equations (3.95):

$$\begin{cases} C_1\beta_1 + C_2\alpha_1\sqrt{\lambda} = 0, \\ C_1[-\alpha_2\sqrt{\lambda}\sin\sqrt{\lambda}l + \beta_2\cos\sqrt{\lambda}l] + C_2[\alpha_2\sqrt{\lambda}\cos\sqrt{\lambda}l + \beta_2\sin\sqrt{\lambda}l] = 0. \end{cases} \quad (3.98)$$

This system of linear homogeneous algebraic equations for coefficients C_1 and C_2 has a nontrivial solution only when its determinant equals zero:

$$(\alpha_1\alpha_2\lambda + \beta_1\beta_2)\tan\sqrt{\lambda}l - \sqrt{\lambda}(\alpha_1\beta_2 - \beta_1\alpha_2) = 0. \quad (3.99)$$

It is easy to determine (for instance by using graphical methods) that Equation (3.99) has an infinite number of roots $\{\lambda_n\}$, which conforms to general Sturm-Liouville theory. For each root λ_n , we obtain a nonzero solution of Equations (3.98).

It is often convenient to present the solution in a form that allows us to consider not only a mixed boundary condition but two other kinds of boundary conditions as well, when some of the constants α_i or β_i may be equal to zero. We will do this in the following way. Using the first expression in Equations (3.98), we represent C_1 and C_2 as

$$C_1 = C\alpha_1\sqrt{\lambda_n}, \quad C_2 = -C\beta_1,$$

where $C \neq 0$ is an arbitrary constant (because the determinant is equal to zero, the same C_1 and C_2 satisfy the second equation of the system of Equations (3.98)). For these constraints, the choice

$$C = \frac{1}{\sqrt{\lambda_n\alpha_1^2 + \beta_1^2}}$$

(with positive square root and $\alpha_1^2 + \beta_1^2 \neq 0$) allows us to obtain a simple set of coefficients C_1 and C_2 . For Dirichlet boundary conditions, we assign $\alpha_1 = 0$, $\beta_1 = -1$ (we may also use $\beta_1 = 1$ because the overall sign of $X_n(x)$ is not important) so that $C_1 = 0$ and $C_2 = 1$. For Neumann boundary conditions $\beta_1 = 0$, $\alpha_1 = 1$, and we have $C_1 = 1$, $C_2 = 0$.

With these choices of C_1 and C_2 , the functions $X_n(x)$ are bounded by the values ± 1 . The alternative often used for the coefficients C_1 and C_2 corresponds to the normalizations $\|X_n\|^2 = 1$. Here and elsewhere in the book we will use the first choice because in this case the graphs for $X_n(x)$ are easier to plot.

From the above discussion, the eigenfunctions of the Sturm-Liouville problem given by the equations

$$X'' + \lambda X = 0,$$

$$\alpha_1 X' + \beta_1 X \Big|_{x=0} = 0, \quad \alpha_2 X' + \beta_2 X \Big|_{x=l} = 0$$

can be written as

$$X_n(x) = \frac{1}{\sqrt{\alpha_1^2 \lambda_n + \beta_1^2}} \left[\alpha_1 \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x - \beta_1 \sin \sqrt{\lambda_n} x \right]. \quad (3.100)$$

The orthogonality property in Equation (3.96) can be easily verified by the reader as an exercise. The square norms of eigenfunctions are

$$\|X_n\|^2 = \int_0^l X_n^2(x) dx = \frac{1}{2} \left[l + \frac{(\beta_2 \alpha_1 - \beta_1 \alpha_2)(\lambda_n \alpha_1 \alpha_2 - \beta_1 \beta_2)}{(\lambda_n \alpha_1^2 + \beta_1^2)(\lambda_n \alpha_2^2 + \beta_2^2)} \right]. \quad (3.101)$$

The eigenvalues are $\lambda_n = (\mu_n/l)^2$, where μ_n is the n th root of the equation

$$\tan \mu = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) l \mu}{\mu^2 \alpha_1 \alpha_2 + l^2 \beta_1 \beta_2}. \quad (3.102)$$

This equation remains unchanged when the sign of μ changes, which indicates that positive and negative roots are placed symmetrically on the μ axis. Because the eigenvalues λ_n do not depend on the sign of μ , it is enough to find only *positive roots* μ_n of Equation (3.102) since negative roots do not give new values of λ_n .

It is not difficult to demonstrate (for instance, by using a graphical solution of this equation) that $\mu_{n+1} > \mu_n$, again in accordance with Sturm-Liouville theory. As an example, we demonstrate how to find the roots of this equation in the case of mixed boundary conditions. The cases for which both boundary conditions are of Dirichlet type or Neumann type, or one is Dirichlet type and the other is Neumann type, are simpler, and we leave them to the reader as simple reading exercises. The obtained results can be compared with those collected in Appendix A.

Let us represent the mixed boundary conditions in the following form:

$$\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial x}(l, t) + hu(l, t) = 0.$$

The eigenvalues are determined from the equation for $\tan \mu$, Equation (3.102), where $\alpha_1 = 1$, $\beta_1 = -h$, $\alpha_2 = 1$, and $\beta_2 = h$:

$$\tan \mu = \frac{2hl\mu}{\mu^2 - h^2 l^2}. \quad (3.103)$$

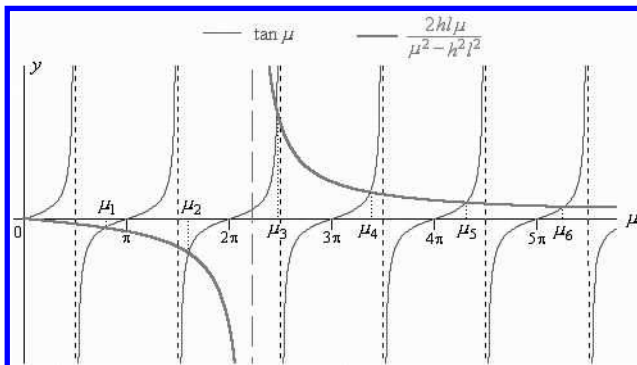


Figure 3.21. Example of a graphical solutions for eigenvalues of a Sturm-Liouville problem.

This equation has an infinite number of roots, μ_n . Figure 3.21 shows curves of the two functions $y = \tan \mu$ and $y = 2hl\mu/(\mu^2 - h^2l^2)$, plotted on the same set of axes. The values of μ at the intersection points of these curves are the roots of the eigenvalue equation.

For the values $l = 100$ and $h = 0.07$, the first six positive roots of this equation are

$$\begin{aligned} \mu_1 = 2.464 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad \mu_2 = 5.036 \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \quad \mu_3 = 7.752 \in \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right), \\ \mu_4 = 10.59 \in \left(\frac{7\pi}{2}, \frac{9\pi}{2}\right), \quad \mu_5 = 13.25 \in \left(\frac{9\pi}{2}, \frac{11\pi}{2}\right), \quad \mu_6 = 16.51 \in \left(\frac{11\pi}{2}, \frac{13\pi}{2}\right). \end{aligned}$$

The line $\mu = hl$ (the dashed line in Figure 3.21) is the asymptote of the graph $y = 2hl\mu/(\mu^2 - h^2l^2)$. In our example $\mu = 7 \in (3\pi/2, 5\pi/2)$, and we notice that this interval includes *two* roots, μ_2 and μ_3 , whereas each of the other intervals, $((2k-1)\pi/2, (2k+1)\pi/2)$, contains *one* root of Equation (3.103). This is a commonly encountered situation.

Each eigenvalue

$$\lambda_n = \left(\frac{\mu_n}{l}\right)^2, \quad n = 1, 2, 3, \dots$$

corresponds to an eigenfunction

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right].$$

The norms of these eigenfunctions are

$$\|X_n\|^2 = \frac{l}{2} + \frac{h}{\lambda_n + h^2}.$$

Now consider the next step of the solution: the determination of the function $T(t)$. Equation (3.93) is an ordinary linear second-order homogeneous differential equation. Its general solution, for $\lambda = \lambda_n$, is given by

$$T_n(t) = a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t), \quad (3.104)$$

where a_n and b_n are arbitrary constants. Two particular solutions, $y_n^{(1)}(t)$ and $y_n^{(2)}(t)$, are

$$y_n^{(1)}(t) = \begin{cases} e^{-\kappa t} \cos \omega_n t & \text{if } \kappa^2 < a^2 \lambda_n + \gamma, \\ e^{-\kappa t} \cosh \omega_n t & \text{if } \kappa^2 > a^2 \lambda_n + \gamma, \\ e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_n + \gamma, \end{cases} \quad (3.105)$$

and

$$y_n^{(2)}(t) = \begin{cases} e^{-\kappa t} \sin \omega_n t & \text{if } \kappa^2 < a^2 \lambda_n + \gamma, \\ e^{-\kappa t} \sinh \omega_n t & \text{if } \kappa^2 > a^2 \lambda_n + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_n + \gamma, \end{cases} \quad (3.106)$$

where $\omega_n = \sqrt{|a^2 \lambda_n + \gamma - \kappa^2|}$. If $\kappa^2 = a^2 \lambda_n + \gamma$, we define $\omega_n = 1$ to unify further formulas.

Reading Exercise. Verify Equations (3.105) and (3.106).

It is clear that each function

$$u_n(x, t) = T_n(t)X_n(x) = \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] X_n(x)$$

is a solution of Equation (3.88) and satisfies boundary conditions in Equation (3.89).

The next step is to compose the series

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] X_n(x), \quad (3.107)$$

which will allow us to satisfy the initial conditions in Equation (3.90). If this series and the series obtained from differentiating this one by x and t

converges uniformly, its sum gives a solution of Equation (3.88) and satisfies the boundary conditions in Equation (3.89). The initial conditions in Equation (3.90) give

$$u|_{t=0} = \varphi(x) = \sum_{n=1}^{\infty} a_n X_n(x), \quad (3.108)$$

and

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x) = \sum_{n=1}^{\infty} [\omega_n b_n - \kappa a_n] X_n(x). \quad (3.109)$$

Equations (3.108) and (3.109) also give the expansions of the functions $\varphi(x)$ and $\psi(x)$ in a series using the eigenfunctions of the boundary value problem given in Equations (3.94) and (3.95).

Assuming uniform convergence of the series (Equations (3.108) and (3.109)), we can find coefficients a_n and b_n . By multiplying both sides of these equations by $X_n(x)$, integrating from 0 to l , and using the orthogonality condition defined in Equation (3.96), we obtain

$$a_n = \frac{1}{\|X_n\|^2} \int_0^l \varphi(x) X_n(x) dx \quad (3.110)$$

and

$$b_n = \frac{1}{\omega_n} \left[\frac{1}{\|X_n\|^2} \int_0^l \psi(x) X_n(x) dx + \kappa a_n \right]. \quad (3.111)$$

The series given in Equation (3.107) with these coefficients gives the solution of the original problem given in Equations (3.88) through (3.90) describing *free oscillations* of a string. This series can be considered as the expansion of the (initially unknown) solution function $u(x, t)$ into a Fourier series using an orthogonal system of functions $\{X_n(x)\}$. The series converges for “reasonably smooth” boundary and initial conditions. Note here that the success of this method is based on the following details: the functions $\{X_n(x)\}$ are orthogonal to each other and form a complete set (i.e., they form a basis set for an expansion); the functions $\{X_n(x)\}$ satisfy the same boundary conditions as the solutions, $u(x, t)$; and solutions to linear equations obey the superposition principle (i.e., sums of solutions are also solutions).

The partial solutions $u_n(x, t) = T_n(t)X_n(x)$ represent the so-called *standing waves* on the string. The points where $X_n(x)$ has maximum or minimum values are the *antinodes* of this standing wave. The points where $X_n(x) = 0$ are not moving and are called *nodes* of the harmonic $u_n(x, t)$. The general solution, $u(x, t)$, is a *superposition of standing waves*; thus, any oscillation can be presented in this way.

We give two examples for homogeneous wave equations with homogeneous boundary conditions. Here we want to demonstrate the formulation of the problem rather than the details of the solution.

Example 3.8. The ends of a uniform string of linear density ρ and length l are fixed, and all external forces are neglected. Displace the string from equilibrium by shifting the point $x = x_0$ by a distance h at time $t = 0$ and then release it with zero initial speed. Find the displacements $u(x, t)$ of the string for times $t > 0$.

Solution. The boundary value problem is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0, \quad a = \sqrt{\frac{T}{\rho}}, \quad 0 < x < l, \quad t > 0, \\ u(x, 0) &= \begin{cases} \frac{h}{x_0}x, & 0 < x \leq x_0, \\ \frac{h(l-x)}{l-x_0}x, & x_0 < x < l, \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0, \\ u(0, t) &= 0, \quad u(l, t) = 0. \end{aligned}$$

The eigenvalues and eigenfunctions are those of the Dirichlet problem on $0 < x < l$:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

Figure 3.22 shows the first four eigenfunctions (for $l = 100$).

Using Equations (3.110) and (3.111), we obtain

$$a_n = \frac{2hl^2}{\pi^2 x_0(l-x_0)n^2} \sin \frac{n\pi x_0}{l}, \quad b_n = 0.$$

Therefore, vibrations on the string are given by the series

$$u(x, t) = \frac{2hl^2}{\pi^2 x_0(l-x_0)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}.$$

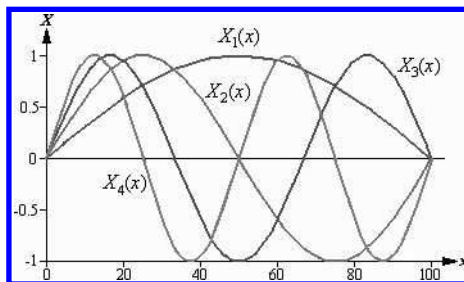


Figure 3.22. Eigenfunctions $X_1(x)$ through $X_4(x)$ for Example 3.8.

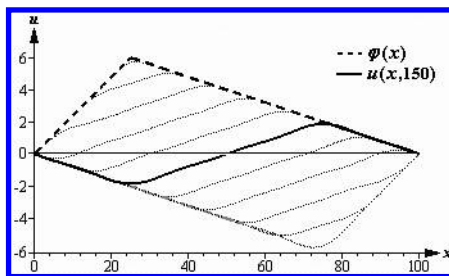


Figure 3.23. Solutions, $u(x, t)$, at various time intervals (dotted curves) for Example 3.8.

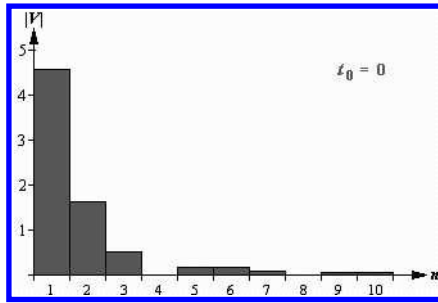
Figure 3.23 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.8. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $h = 6$, and $x_0 = 25$.

Here and in the following examples, the dashed line represents the initial deflection of the string and the solid line is the deflection of the string at time $t = 150$. The dotted lines show the deflection of the string within the period of time from 0 until 150.

In the solution $u(x, t)$ the terms for which $\sin n\pi x_0/l = 0$ vanish from the series; that is, the solution does not contain overtones for which $x = x_0$ is a node. For instance, if x_0 is at the middle of the string, the solution does not contain harmonics with even numbers.

Reading Exercise. Verify the preceding statement.

The formula for $u(x, t)$ can be presented in a more physically intuitive form. Let us denote $\sqrt{\lambda_n}$ as k_n , where $k_n = n\pi/l$, called *wave numbers*. The frequencies $\omega_n = n\pi a/l$ give the *frequency spectrum* of the problem.

Figure 3.24. Bar charts of $|V_n(x)|$ for Example 3.8.

The solution to the problem now can be written as

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos \omega_n t \cdot \sin k_n x,$$

where the amplitude

$$A_n = \frac{D}{n^2} \sin \frac{n\pi x_0}{l}, \quad \text{and} \quad D \equiv \frac{2hl^2}{\pi^2 x_0(l - x_0)}.$$

The first harmonic, $u_1(x, t)$, is zero for all x when $\omega_1 t = \pi/2$ (i.e., when $t = l/2a$). It becomes zero again for $t = 3l/2a$, $t = 5l/2a$, etc. The second harmonic $u_2(x, t)$ is zero for all x for the moments of time $t = l/4a$, $t = 3l/4a$, etc.

The bar chart in Figure 3.24 represents $|V_n(t)| = |A_n| \cos \omega_n t$ (in units of D) for the case $x_0 = l/4 = 25$ and $t = 0$. The terms with numbers $n = 4k$ vanish from the series. Note that the amplitudes, A_n , decrease as $1/n^2$.

Example 3.9. A homogeneous rod of density ρ and length l elastically fixed at the end $x = l$ is stretched by a longitudinal force $F_0 = \text{const}$, applied to the end at $x = 0$. At time $t = 0$, the force F_0 stops acting. Find the longitudinal oscillations of the rod if initial velocities are zero, the resistance of a medium is absent, and there are no external forces.

Solution. Assuming that Hooke's law is applicable, we can find initial displacements of locations along the rod, $u|_{t=0} = \varphi(x)$. Because in each cross section the force of tension T is constant and equals F_0 we have $\varphi'(x) = -F_0/ES$, where E is Young's modulus and S is the cross-sectional area of the rod. Integrating and using $\varphi'(l) + h\varphi(l) = 0$ (since the rod is elastically

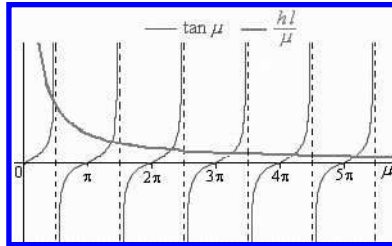


Figure 3.25. Graphical solution of the eigenvalue equation for Example 3.9.

fixed at $x = l$), we obtain

$$\varphi(x) = \frac{F_0}{ES} \left(l + \frac{1}{h} - x \right).$$

Here, h is the elasticity coefficient. If F_0 is a compressing force, the expression for $\varphi'(x)$ will have the opposite sign. Thus, we return to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$

with initial and boundary conditions

$$u(x, 0) = \frac{F_0}{ES} \left(l + \frac{1}{h} - x \right), \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) + hu(l, t) = 0.$$

The boundary conditions are the Neumann type at $x = 0$ (the free end) and the mixed type at $x = l$ (an elastic connection). The eigenvalues are easily obtained (they may be found in Appendix A) and are

$$\lambda_n = \left(\frac{\mu_n}{l} \right)^2, \quad n = 1, 2, 3, \dots,$$

where μ_n is the n th root of the equation $\tan \mu = hl/\mu$.

Figure 3.25 shows curves of the two functions, $\tan \mu$ and hl/μ , plotted on the same set of axes. The eigenvalues are the squares of the values of μ at the intersection points of these curves divided by length l .

Each eigenvalue corresponds to an eigenfunction $X_n(x) = \cos \sqrt{\lambda_n}x$ with the norm

$$\|X_n\|^2 = \frac{1}{2} \left(l + \frac{h}{\lambda_n + h^2} \right).$$

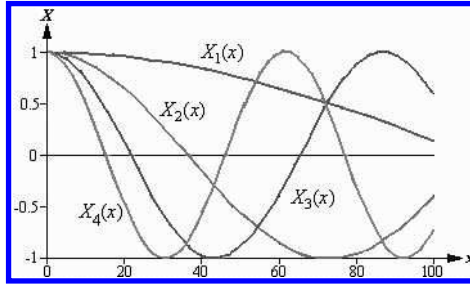
Figure 3.26. Eigenfunctions $X_1(x)$ through $X_4(x)$ for Example 3.9.

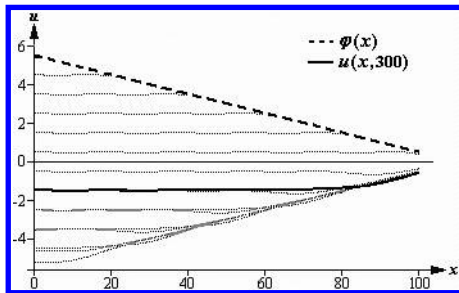
Figure 3.26 shows the first four eigenfunctions (for $l = 100$). Because the initial speeds are zero, all coefficients, b_n , are zero. Coefficients a_n are found by using Equation (3.110):

$$a_n = \frac{1}{\|X_n\|^2} \int_0^l \varphi(x) \cos \sqrt{\lambda_n} x dx = \frac{2F_0}{ES} \frac{1 - \cos \sqrt{\lambda_n} l + (\sqrt{\lambda_n}/h) \sin \sqrt{\lambda_n} l}{\lambda_n [l + h/(\lambda_n + h^2)]}.$$

Finally, we have the series that describes the rod's oscillations

$$u(x, t) = \frac{2F_0}{ES} \sum_{n=1}^{\infty} \frac{1 - \cos \sqrt{\lambda_n} l + (\sqrt{\lambda_n}/h) \sin \sqrt{\lambda_n} l}{\lambda_n [l + h/(\lambda_n + h^2)]} \cos \sqrt{\lambda_n} at \cdot \cos \sqrt{\lambda_n} x.$$

Figure 3.27 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.9. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $E = 1$, $S = 1$, $h = 0.1$, and $F_0 = 0.05$.

Figure 3.27. Graphical solutions, $u(x, t)$, at various time intervals (dotted curves) for Example 3.9.

Notice that the program Waves uses the same formulas as those used to solve a problem analytically. The only numeric calculations the program performs are numeric evaluations of the coefficients of the Fourier series and numeric solutions of transcendental equations for eigenvalues in the cases of mixed boundary conditions. A description of how to use the program is found in Appendix E.

Reading Exercise. Use the Fourier method to find the eigenvalues, λ_n , and eigenfunctions, $X_n(x)$, for the Klein-Gordon equation,

$$\nabla^2 u - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = m^2 u$$

for the one-dimensional case and for $0 \leq x \leq l$. Provide eigenvalues for the boundary conditions

1. $u(0) = u(l) = 0$;
2. $u(0) = 0, \quad u_x(l) = 0$.

3.10.2 The Fourier Method for Nonhomogeneous Equations

Next, we apply the Fourier method to the nonhomogeneous linear equation

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} + \gamma u = f(x, t), \quad (3.112)$$

where $f(x, t)$ is a given function.

First, let the function $u(x, t)$ satisfy the *homogeneous boundary conditions* given in Equation (3.89) and initial conditions given in Equation (3.90). We shall search for the solution as a sum of two functions

$$u(x, y) = u_1(x, y) + u_2(x, y), \quad (3.113)$$

where $u_1(x, t)$ is the solution of the homogeneous equation satisfying both (homogeneous) boundary and initial conditions

$$\frac{\partial^2 u_1}{\partial t^2} + 2\kappa \frac{\partial u_1}{\partial t} - a^2 \frac{\partial^2 u_1}{\partial x^2} + \gamma u_1 = 0, \quad (3.114)$$

$$u_1(x, t)|_{t=0} = \varphi(x), \quad \left. \frac{\partial u_1}{\partial t}(x, t) \right|_{t=0} = \psi(x), \quad (3.115)$$

$$\alpha_1 u_{1_x} + \beta_1 u_1 \Big|_{x=0} = 0, \quad \alpha_2 u_{1_x} + \beta_2 u_1 \Big|_{x=l} = 0, \quad (3.116)$$

and $u_2(x, t)$ is the solution of the nonhomogeneous equation satisfying the same boundary conditions and *zero initial conditions*

$$\frac{\partial^2 u_2}{\partial t^2} + 2\kappa \frac{\partial u_2}{\partial t} - a^2 \frac{\partial^2 u_2}{\partial x^2} + \gamma u_2 = f(x, t), \quad (3.117)$$

$$u_2(x, t)|_{t=0} = 0, \quad \frac{\partial u_2}{\partial t}(x, t) \Big|_{t=0} = 0, \quad (3.118)$$

$$\alpha_1 u_{2_x} + \beta_1 u_2 \Big|_{x=0} = 0, \quad \alpha_2 u_{2_x} + \beta_2 u_2 \Big|_{x=l} = 0. \quad (3.119)$$

Clearly, the function $u_1(x, t)$ represents *free oscillations* (i.e., the oscillations due to initial perturbation only) and the function $u_2(x, t)$ represents *forced oscillations* (i.e., the oscillations due to an external force when initial disturbances are zero).

The methods for finding the solution in the case of free oscillations, $u_1(x, t)$, have been discussed in Section 3.10.1. Therefore, we turn our attention in this section to finding the forced oscillation solutions, $u_2(x, t)$. Similar to the case of free oscillations, let us search the solution for the function $u_2(x, t)$ as the series

$$u_2(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \quad (3.120)$$

where $X_n(x)$ are eigenfunctions of the corresponding homogeneous boundary value problem and $T_n(t)$ are unknown functions of t .

We choose the functions $T_n(t)$ in such a way that the series (3.120) satisfies the nonhomogeneous Equation (3.117) and zero initial conditions given in Equation (3.118). Substituting the series (3.120) into Equation (3.117), we obtain (assuming that this series can be differentiated the necessary number of times)

$$\sum_{n=1}^{\infty} \{ [T_n''(t) + 2\kappa T_n'(t) + \gamma T_n(t)] X_n(x) - a^2 T_n(t) X_n''(x) \} = f(x, t).$$

Because $X_n(x)$ are the eigenfunctions of the corresponding homogeneous boundary value problem, they satisfy

$$X_n''(x) + \lambda_n X_n(x) = 0.$$

Substituting $X_n''(x) = -\lambda_n X_n(x)$ into the previous equation, we obtain

$$\sum_{n=1}^{\infty} [T_n''(t) + 2\kappa T_n'(t) + (a^2 \lambda_n + \gamma) T_n(t)] X_n(x) = f(x, t). \quad (3.121)$$

Because of the completeness of the set of functions $\{X_n(x)\}$, we can expand the function $f(x, t)$ on the interval $(0, l)$ into a Fourier series of the functions $X_n(x)$ so that

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) X_n(x), \quad (3.122)$$

where

$$f_n(t) = \frac{1}{\|X_n\|^2} \int_0^l f(x, t) X_n(x) dx. \quad (3.123)$$

Comparing the series in Equation (3.121) and that in Equation (3.122), we obtain an ordinary differential equation with constant coefficients for the functions $T_n(t)$, given by

$$T_n''(t) + 2\kappa T_n'(t) + (a^2 \lambda_n + \gamma) T_n(t) = f_n(t), \quad n = 1, 2, 3, \dots, \quad (3.124)$$

In order that the solutions represented by the series $u_2(x, t)$ given in Equation (3.120) satisfy initial conditions in Equation (3.118), it is clear that functions $T_n(t)$ should satisfy the conditions

$$T_n(0) = 0, \quad T_n'(0) = 0, \quad n = 1, 2, 3, \dots \quad (3.125)$$

Solutions of the Cauchy problem given in Equations (3.124) and (3.125) for functions $T_n(t)$ can be written in different forms, one of which is an integral representation that is very convenient for our further purposes:

$$T_n(t) = \frac{1}{\omega_n} \int_0^t f_n(\tau) Y_n(t - \tau) d\tau. \quad (3.126)$$

Here

$$Y_n(t) = \begin{cases} e^{-\kappa t} \sin \omega_n t & \text{if } \kappa^2 < a^2 \lambda_n + \gamma, \\ e^{-\kappa t} \sinh \omega_n t & \text{if } \kappa^2 > a^2 \lambda_n + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_n + \gamma, \end{cases} \quad (3.127)$$

where

$$\omega_n = \sqrt{|a^2\lambda_n + \gamma - \kappa^2|}$$

if $\kappa^2 \neq a^2\lambda_n + \gamma$, and $\omega_n = 1$ if $\kappa^2 = a^2\lambda_n + \gamma$. To prove that the representation in Equation (3.126) yields Equation (3.124), we differentiate Equation (3.126) by t twice to get

$$\omega_n T'_n(t) = \int_0^t f_n(\tau) \frac{\partial}{\partial t} Y_n(t - \tau) d\tau + Y_n(0) f_n(t),$$

and

$$\omega_n T''_n(t) = \int_0^t f_n(\tau) \frac{\partial^2}{\partial t^2} Y_n(t - \tau) d\tau + Y'_n(0) f_n(t).$$

By using these formulas, we can prove that Equation (3.124) and the initial conditions in Equation (3.125) are satisfied.

Reading Exercise. Verify this last result for $\kappa = 0$ (i.e., for $Y_n(t) = \sin \omega_n t$).

Using Equation (3.123) for $f_n(\tau)$, we obtain from (3.126)

$$T_n(t) = \frac{1}{\omega_n \|X_n\|^2} \int_0^t d\tau \int_0^l f(x, \tau) X_n(x) Y_n(t - \tau) dx. \quad (3.128)$$

With this expression for $T_n(t)$, the series (3.120) gives—assuming that it, as well as the series obtained by differentiating by x and t up to two times, converges uniformly—the solution of the boundary value problem given in Equations (3.117) through (3.119).

Combining the results for the functions $u_1(x, t)$ and $u_2(x, t)$, the solution of the forced oscillations problem with initial conditions (3.90) and homogeneous boundary conditions (3.89) is

$$u(x, t) = u_1(x, t) + u_2(x, t) = \sum_{n=1}^{\infty} \left\{ \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] + T_n(t) \right\} X_n(x), \quad (3.129)$$

where the $T_n(t)$ are defined by Equation (3.128) and a_n and b_n have been defined in the previous section.

Let us consider two examples of nonhomogeneous problems with homogeneous boundary conditions.

Example 3.10. Consider a homogeneous string $0 \leq x \leq l$ with rigidly fixed ends, no friction, and density ρ . Starting at time $t = 0$, a uniformly distributed harmonic force with linear density

$$F(x, t) = F_0 \sin \omega t$$

acts on the string. Find the resulting oscillations and investigate the resonance behavior.

Solution. The boundary value problem that models this process is

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \frac{F_0}{\rho} \sin \omega t, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = u(l, t) = 0.$$

The boundary conditions here are homogeneous Dirichlet conditions; thus, the eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

Because the initial conditions are homogeneous ($\varphi(x) = \psi(x) = 0$), the solution is described by the series

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}. \quad (3.130)$$

We can find $T_n(t)$ by using Equation (3.126):

$$T_n(t) = \frac{1}{\omega_n} \int_0^t f_n(\tau) Y_n(t - \tau) d\tau,$$

where the quantities $\omega_n = n\pi a/l$ are the natural frequencies and $Y_n(t) = \sin n\pi at/l$.

We then have

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \frac{F_0}{\rho} \sin \omega t \int_0^l \sin \frac{n\pi x}{l} dx,$$

which gives

$$f_{2n-1}(t) = \frac{4F_0}{(2n-1)\pi\rho} \sin \omega t, \quad f_{2n}(t) = 0,$$

and

$$\begin{aligned} T_{2n-1}(t) &= \frac{l}{(2n-1)\pi a} \cdot \frac{4F_0}{(2n-1)\pi\rho} \int_0^t \sin \omega \tau \sin \frac{(2n-1)\pi a}{l}(t-\tau) d\tau \\ &= \frac{4F_0 l}{a\pi^2(2n-1)^2\rho} \frac{\omega_{2n-1} \sin \omega t - \omega \sin \omega_{2n-1} t}{\omega_{2n-1}^2 - \omega^2}, \end{aligned}$$

$$T_{2n}(t) = 0.$$

These expressions apply when, for any n , the frequency of the external force, ω , is not equal to any of the natural frequencies of the string. Substituting $T_n(t)$ into Equation (3.130), we obtain the solution

$$u(x, t) = \frac{4F_0 l}{\rho a \pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \frac{\omega_{2n-1} \sin \omega t - \omega \sin \omega_{2n-1} t}{\omega_{2n-1}^2 - \omega^2} \sin \frac{n\pi x}{l}. \quad (3.131)$$

Because ω_n is proportional to n , terms with $\sin \omega t$, which are purely periodic with frequency ω , decrease as $1/n$, whereas terms with $\sin \omega_{2n-1} t$, which represent oscillations (eigenmodes) with different frequencies, decrease as $1/n^2$.

Figure 3.28 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.10. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $\rho = 1$, $F_0 = 0.025$, and $\omega = 0.3$ (the frequency of the external force).

If for some $n = k$, the frequency $\omega = \omega_{2k-1}$, and we have a case of *resonance*. To proceed, we apply L'Hôpital's rule to this term, taking

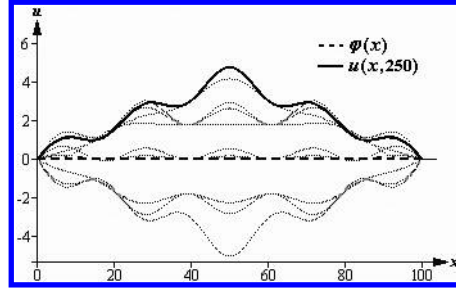


Figure 3.28. Graph of solutions, $u(x, t)$, at various time intervals for Example 3.10.

the derivatives of the numerator and denominator with respect to ω_{2k-1} to yield

$$-\frac{2F_0l}{\rho a \pi^2 (2k-1)^2} \frac{\omega_{2n-1} t \cos \omega_{2k-1} t - \sin \omega_{2n-1} t}{\omega_{2k-1}} = \frac{2F_0l^2}{\rho a^2 \pi^3 (2k-1)^3} [\sin \omega_{2n-1} t - \omega_{2n-1} t \cos \omega_{2k-1} t].$$

In this case, we can rewrite the solution in the form

$$u(x, t) = \frac{4F_0l}{\rho a \pi^2} \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{1}{(2n-1)^2} \frac{\omega_{2n-1} \sin \omega t - \omega \sin \omega_{2n-1} t}{\omega_{2n-1}^2 - \omega^2} \sin \frac{n\pi x}{l} + \frac{2F_0l^2}{\rho a^2 \pi^3 (2k-1)^3} [\sin \omega t - \omega t \cos \omega t] \sin \frac{(2k-1)\pi x}{l}.$$

The last term describes the string's behavior for small values of t ; that is, the amplitude increases until the approximation of small oscillations is not applicable.

Figure 3.29 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.10 in the case of resonance in the fifth mode:

$$\omega = \omega_5 = \frac{5\pi}{100}.$$

Intuitively, it is clear that under the action of a periodic external force, the motion eventually becomes periodic with the frequency coinciding

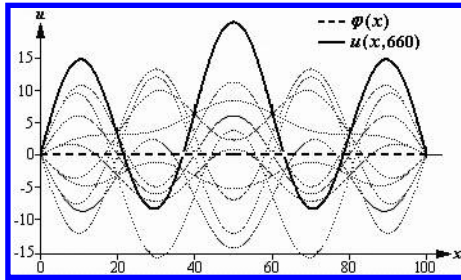


Figure 3.29. Graph of solutions $u(x, t)$, at various time intervals for Example 3.10 in the case of resonance.

with the force's frequency. The case in which the eigenmodes are practically gone and the motion is governed by external influences only is referred to as steady-state motion. This also means that the role of initial conditions becomes negligible after some time interval that depends on the parameters of the problem.

Let us now obtain a periodic solution for Example 3.10. Contrary to the solution obtained above in the form of an infinite series, a steady-state solution is described by a finite formula, which is very convenient and important for the analysis of the properties of the solution. Let us write the equation as

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = A \sin \omega t$$

and seek a steady-state solution in the form

$$u(x, t) = X(x) \sin \omega t.$$

For $X(x)$, we obtain an ordinary linear differential equation

$$X''(x) + \left(\frac{\omega}{a}\right)^2 X(x) = -\frac{A}{a^2},$$

which has a general solution

$$X(x) = c_1 \cos kx + c_2 \sin kx - \frac{A}{\omega^2},$$

where $k = \omega/a$. The boundary conditions $X(0) = X(l) = 0$ give

$$c_1 = \frac{A}{\omega^2} \quad \text{and} \quad c_2 = \frac{A}{\omega^2} \frac{1 - \cos kl}{\sin kl}.$$

Then, using standard trigonometric formulas for double angles we obtain

$$u(x, t) = \frac{2A}{\omega^2} \frac{\sin\left(\frac{kx}{2}\right) \sin\left[\frac{k(l-x)}{2}\right]}{\cos\left(\frac{kl}{2}\right)} \sin \omega t. \quad (3.132)$$

Reading Exercise. Should this solution (Equation (3.132)) be identical with the first term in the solution obtained with the Fourier method (Equation (3.131))? To verify, expand this solution into a Fourier sine series. Using the program Waves, play with the parameters to see when you can neglect the terms in the second sum in the solution obtained with the Fourier method.

Example 3.11. The upper end of an elastic homogeneous heavy rod is rigidly attached to the ceiling of free falling elevator. When the elevator reaches the speed v_0 it stops instantly. Set up the boundary value problem for vibrations of the rod.

Solution. The mathematical model for this problem is:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} &= g, \quad 0 < x < l, \quad t > 0, \\ u(x, 0) &= 0, \quad \frac{\partial u}{\partial t}(x, 0) = v_0, \\ u(0, t) &= 0, \quad \frac{\partial u}{\partial x}(l, t) = 0, \end{aligned}$$

where g is the acceleration of gravity.

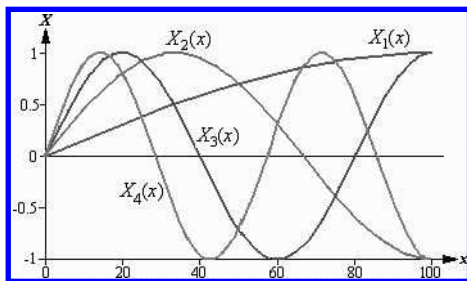
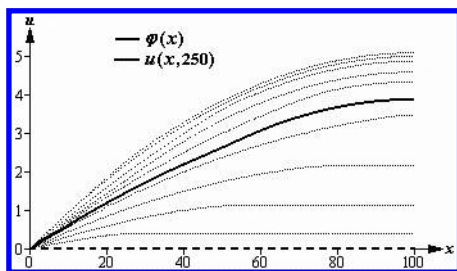
The boundary conditions are Dirichlet type at $x = 0$ (the secured end) and Neumann type at $x = l$ (the free end). From our previous work, we have eigenvalues and eigenfunctions for the problem given by

$$\lambda_n = \left[\frac{(2n-1)\pi}{2l} \right]^2, \quad X_n(x) = \sin \frac{(2n-1)\pi x}{2l}, \quad n = 1, 2, 3, \dots$$

Figure 3.30 shows the first four eigenfunctions (for $l = 100$).

From Equations (3.110), (3.111), (3.123), and (3.126), we have

$$a_n = 0, \quad b_n = \frac{8v_0 l}{a\pi^2(2n-1)^2},$$

Figure 3.30. Eigenfunctions $X_1(x)$ through $X_4(x)$ for Example 3.11.Figure 3.31. Solutions, $u(x, t)$, at various times for Example 3.11.

$$f_n(t) = \frac{4g}{\pi(2n-1)}, \quad T_n(t) = \frac{16gl^2}{a^2\pi^3(2n-1)^3} \left[1 - \cos \frac{(2n-1)a\pi t}{2l} \right].$$

Thus, a general solution of this boundary value problem is

$$u(x, t) = \frac{8l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \times \left\{ \frac{2gl}{a\pi(2n-1)} \left[1 - \cos \frac{(2n-1)a\pi t}{2l} \right] + v_0 \sin \frac{(2n-1)\pi at}{2l} \right\} \sin \frac{(2n-1)\pi x}{2l}.$$

Figure 3.31 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.11. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $g = 0.0005$, and $v_0 = 0.01$.

3.10.3 The Fourier Method for Equations with Nonhomogeneous Boundary Conditions

With the preceding development, we are prepared to study a general boundary value problem for a nonhomogeneous equation with nonhomogeneous

boundary conditions defined by

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} + \gamma u = f(x, t), \quad (3.133)$$

$$u(x, t)|_{t=0} = \varphi(x), \quad u_t(x, t)|_{t=0} = \psi(x), \quad (3.134)$$

$$\alpha_1 u_x + \beta_1 u|_{x=0} = g_1(t), \quad \alpha_2 u_x + \beta_2 u|_{x=l} = g_2(t), \quad (3.135)$$

where functions $\varphi(x)$, $\psi(x)$, $g_1(t)$, and $g_2(t)$ are given.

The Fourier method cannot be applied to this problem directly because the boundary conditions are nonhomogeneous. However, we can reduce this problem to the previously investigated case with homogeneous boundary conditions.

To proceed, let us search for the solution as a sum of two functions

$$u(x, t) = v(x, t) + w(x, t),$$

where $v(x, t)$ is a new, unknown function and the function $w(x, t)$ is chosen in a way that satisfies the given nonhomogeneous conditions

$$\alpha_1 w_x + \beta_1 w|_{x=0} = g_1(t), \quad \alpha_2 w_x + \beta_2 w|_{x=l} = g_2(t). \quad (3.136)$$

The function $w(x, t)$ should also have the necessary number of continuous derivatives in x and t .

For the function $v(x, t)$, we obtain the boundary value problem with *homogeneous boundary conditions*

$$\frac{\partial^2 v}{\partial t^2} + 2\kappa \frac{\partial v}{\partial t} - a^2 \frac{\partial^2 v}{\partial x^2} + \gamma v = f^*(x, t),$$

$$v(x, t)|_{t=0} = \varphi^*(x), \quad v_t(x, t)|_{t=0} = \psi^*(x), \quad (3.137)$$

$$\alpha_1 v_x + \beta_1 v|_{x=0} = 0, \quad \alpha_2 v_x + \beta_2 v|_{x=l} = 0, \quad (3.138)$$

where

$$f^*(x, t) = f(x, t) - \frac{\partial^2 w}{\partial t^2} - 2\kappa \frac{\partial w}{\partial t} + a^2 \frac{\partial^2 w}{\partial x^2}, \quad (3.139)$$

$$\varphi^*(x) = \varphi(x) - w(x, 0), \quad (3.140)$$

$$\psi^*(x) = \psi(x) - w_t(x, 0). \quad (3.141)$$

The solution to this problem has been described above.

Reading Exercise. Verify Equations (3.137)–(3.139).

For the auxiliary function, $w(x, t)$, a number of choices are possible. One criterion is to simplify the form of the equation for the function, $v(x, t)$. To this end, we search for $w(x, t)$ in with the form

$$w(x, t) = P_1(x)g_1(t) + P_2(x)g_2(t), \quad (3.142)$$

where $P_1(x)$ and $P_2(x)$ are polynomials of first or second order. Coefficients of these polynomials should be chosen in such a way that the function $w(x, t)$ satisfies the given boundary conditions in Equation (3.135); therefore, the function $v(x, t)$ satisfies homogeneous boundary conditions.

- **Case 1.** The parameters β_1 and β_2 in Equation (3.135) are not zero simultaneously.

Note that we are looking for the simplest functions, $w(x, t)$, that can be chosen to satisfy the given boundary conditions. Let us take $P_{1,2}(x)$ as polynomials of first order and search for the function $w(x, t)$ in the form

$$w(x, t) = (\gamma_1 + \delta_1 x)g_1(t) + (\gamma_2 + \delta_2 x)g_2(t). \quad (3.143)$$

Substituting Equation (3.143) the into boundary conditions given by Equation (3.136), we obtain two equations, which may be written as

$$\begin{cases} \alpha_1[\delta_1 g_1(t) + \delta_2 g_2(t)] + \beta_1 [\gamma_1 g_1(t) + \gamma_2 g_2(t)] = g_1(t), \\ \alpha_2[\delta_1 g_1(t) + \delta_2 g_2(t)] + \beta_2 [(\gamma_1 + \delta_1 l)g_1(t) + (\gamma_2 + \delta_2 l)g_2(t)] = g_2(t), \end{cases}$$

or as

$$\begin{cases} (\alpha_1 \delta_1 + \beta_1 \gamma_1 - 1)g_1(t) + (\alpha_1 \delta_2 + \beta_1 \gamma_2)g_2(t) = 0, \\ (\alpha_2 \delta_1 + \beta_2 \gamma_1 + \beta_2 \delta_1 l)g_1(t) + (\alpha_2 \delta_2 + \beta_2 \gamma_2 + \beta_2 \delta_2 l - 1)g_2(t) = 0. \end{cases}$$

To be true, the coefficients of $g_1(t)$ and $g_2(t)$ should be zero, in which case the following system of equations is valid for arbitrary t :

$$\begin{cases} \alpha_1 \delta_1 + \beta_1 \gamma_1 - 1 = 0, \\ \alpha_1 \delta_2 + \beta_1 \gamma_2 = 0, \\ \alpha_2 \delta_1 + \beta_2 \gamma_1 + \beta_2 \delta_1 l = 0, \\ \alpha_2 \delta_2 + \beta_2 \gamma_2 + \beta_2 \delta_2 l - 1 = 0. \end{cases} \quad (3.144)$$

From this system of equations, we obtain coefficients γ_1 , δ_1 , γ_2 , and δ_2 as

$$\gamma_1 = \frac{\alpha_2 + \beta_2 l}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \quad \delta_1 = \frac{-\beta_2}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \quad (3.145)$$

$$\gamma_2 = \frac{-\alpha_1}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \quad \delta_2 = \frac{\beta_1}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}. \quad (3.146)$$

Reading Exercise. Obtain the above results for Case 1.

- **Case 2.** The parameters $\beta_1 = \beta_2 = 0$, or $u_x|_{x=0} = g_1(t)$, $u_x|_{x=l} = g_2(t)$. In this case, the system of Equations (3.144) is inconsistent, which is why the polynomials $P_{1,2}(x)$ in the expression for $w(x, t)$ must be of higher order. Let us take the second-order polynomials

$$P_1(x) = \gamma_1 + \delta_1 x + \xi_1 x^2 \quad \text{and} \quad P_2(x) = \gamma_2 + \delta_2 x + \xi_2 x^2.$$

Then, as in Case 1, Equations (3.136) yield a system of simultaneous equations that, to be valid for arbitrary t , leads to an expression for $w(x, t)$ given by

$$w(x, t) = \left[x - \frac{x^2}{2l} \right] g_1(t) + \frac{x^2}{2l} g_2(t).$$

Reading Exercise. Obtain the above results for Case 2.

Such a choice satisfies the given boundary conditions

$$\alpha_1 w_x + \beta_1 w|_{x=0} = g_1(t) \quad \text{and} \quad \alpha_2 w_x + \beta_2 w|_{x=l} = g_2(t).$$

Reading Exercise. Prove the previous statement.

Combining all described types of boundary conditions, we obtain nine different auxiliary functions, which are given in Appendix B.

The following two examples are applications of the wave equation with nonhomogeneous boundary conditions.

Example 3.12. The left end of a string is moving according to

$$g(t) = A \sin \omega t,$$

and the right end, $x = l$, is secured. Initially, the string is at rest. Describe oscillations when there are no external forces and the resistance of the surrounding medium is zero. Find the solution in the case of resonance.

Solution. The boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = A \sin \omega t, \quad u(l, t) = 0.$$

Let us search for the solution as the sum

$$u(x, t) = v(x, t) + w(x, t),$$

where we use an auxiliary function

$$w(x, t) = \left(1 - \frac{x}{l}\right) A \sin \omega t,$$

which satisfies the boundary conditions. This obvious result can also be obtained from Equations (3.143) which give $\gamma_1 = 1$, $\delta_1 = -1/l$, $\gamma_2 = 0$, and $\delta_2 = 1/l$. For $v(x, t)$, we obtain the boundary value problem for the nonhomogeneous oscillation equation with external force

$$f^*(x, t) = A\omega^2 \left(1 - \frac{x}{l}\right) \sin \omega t,$$

initial conditions

$$\varphi^*(x, t) = 0, \quad \psi^*(x, t) = -A\omega \left(1 - \frac{x}{l}\right),$$

and homogeneous boundary conditions.

The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

From Equations (3.110), (3.111), and (3.123), we have

$$a_n = 0, \quad b_n = \frac{1}{\|X_n\|^2} \int_0^l \psi^*(x) \sin \frac{n\pi x}{l} dx = -\frac{2A\omega l}{an^2\pi^2},$$

$$f_n(t) = \frac{2A\omega^2}{n\pi} \sin \omega t.$$

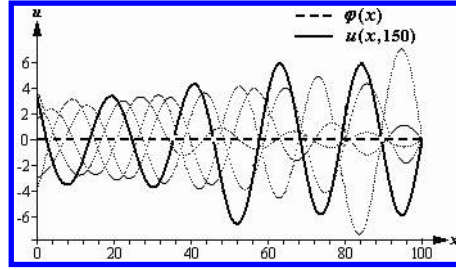


Figure 3.32. Graph of solutions $u(x, t)$ at various time intervals for Example 3.12.

We can find $T_n(t)$ by using Equation (3.128):

$$T_n(t) = \frac{2A\omega^2 l}{a\pi^2 n^2} \frac{\omega \sin \omega_n t - \omega_n \sin \omega t}{\omega^2 - \omega_n^2}.$$

These expressions apply when, for any n , the frequency of the external force $f^*(x, t)$ is not equal to any of the natural frequencies $\omega_n = n\pi a/l$ of the string (i.e., $\omega \neq \omega_n$). In this case, the solution is

$$\begin{aligned} u(x, t) &= w(x, t) + \sum_{n=1}^{\infty} [T_n(t) + b_n \sin \omega_n t] \sin \frac{n\pi x}{l} \\ &= \left(1 - \frac{x}{l}\right) A \sin \omega t + \frac{2A\omega l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\omega_n (\omega_n \sin \omega_n t - \omega \sin \omega t)}{\omega^2 - \omega_n^2} \sin \frac{n\pi x}{l}. \end{aligned}$$

Figure 3.32 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.12. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $A = 4$, and $\omega = 0.3$ (the frequency of the external force).

If, for some $n = k$, the frequency $\omega = \omega_k$, we have a case of *resonance*. For the k th term, we have

$$b_k = -\frac{2A\omega l}{ak^2\pi^2} = -\frac{2A\omega}{k\pi\omega_k} = -\frac{2A}{k\pi},$$

$$\begin{aligned} T_k(t) &= \frac{1}{\omega_k} \int_0^t f_k(\tau) \sin \omega_k(t - \tau) d\tau = \frac{2A\omega^2}{\omega n \pi} \int_0^t \sin \omega t \sin \omega(t - \tau) d\tau \\ &= \frac{A\omega}{n\pi} \int_0^t [\cos \omega(2\tau - t) - \cos \omega t] d\tau = \frac{A}{\pi k} (\sin \omega t - \omega t \cos \omega t), \end{aligned}$$

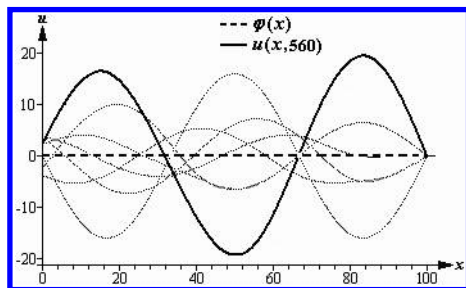


Figure 3.33. Solutions $u(x, t)$ at various time intervals for Example 3.12 in the case of resonance.

so that

$$T_k(t) + b_k \sin \omega_k t = -\frac{A}{k\pi} (\sin \omega t + \omega t \cos \omega t).$$

Thus, we can rewrite the solution in the form

$$\begin{aligned} u(x, t) &= w(x, t) + [T_k(t) + b_k \sin \omega_k t] \sin \frac{k\pi x}{l} + \sum_{\substack{n=1 \\ n \neq k}}^{\infty} [T_n(t) + b_n \sin \omega_n t] \sin \frac{n\pi x}{l} \\ &= \left(1 - \frac{x}{l}\right) A \sin \omega t - \frac{A}{k\pi} (\sin \omega t + \omega t \cos \omega t) \sin \frac{k\pi x}{l} \\ &\quad + \frac{2A\omega l}{a\pi^2} \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{1}{n^2} \frac{\omega_n (\omega_n \sin \omega_n t - \omega \sin \omega t)}{\omega^2 - \omega_n^2} \sin \frac{n\pi x}{l}. \end{aligned}$$

Figure 3.33 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.12 in the case of resonance in the third mode: $\omega = \omega_3 = 3\pi/100$.

In the case of periodic boundary conditions, a steady-state solution clearly is also periodic with the same period.

Reading Exercise. Obtain a steady-state solution for Example 3.12.

Hint. Search for a solution in the form $u(x, t) = v(x, t) + w(x, t)$, and then substitute $v(x, t) = X(x) \sin \omega t$.

Example 3.13. Find the longitudinal vibrations of a rod, $0 \leq x \leq l$, with the left end fixed. To the right end at $x = l$, the force $F(t) = At$ ($A = \text{const}$) is applied starting at time $t = 0$. There is no friction.

Solution. We must solve the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0$$

with the boundary and initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) = \frac{A}{ES}t,$$

where E is Young's modulus, and S is the cross sectional area of the rod.

Again, we assume

$$u(x, t) = v(x, t) + w(x, t).$$

By inspection, we see we may choose the function $w(x, t)$ to have the form

$$w(x, t) = \frac{A}{ES}xt$$

such that $w(x, t)$ satisfies both boundary conditions (this choice also can be formally obtained as previously explained). For $v(x, t)$, we obtain the boundary value problem

$$\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = 0,$$

$$v(x, t)|_{t=0} = 0, \quad \frac{\partial v}{\partial t}(x, t)|_{t=0} = -\frac{A}{ES}x,$$

$$v(0, t) = 0, \quad \frac{\partial v}{\partial x}(l, t) = 0.$$

$$\lambda_n^2 = \left[\frac{(2n-1)\pi}{2l} \right]^2, \quad X_n(x) = \sin \frac{(2n-1)\pi x}{2l}, \quad n = 1, 2, 3, \dots$$

in which case the solution for $v(x, t)$ is

$$v(x, t) = -\frac{16Al^2}{ESa\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} \sin \frac{(2n-1)a\pi t}{2l} \sin \frac{(2n-1)\pi x}{2l}.$$

Thus,

$$u(x, t) = \frac{A}{ES}xt - \frac{16Al^2}{ESa\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} \sin \frac{(2n-1)a\pi t}{2l} \sin \frac{(2n-1)\pi x}{2l}.$$

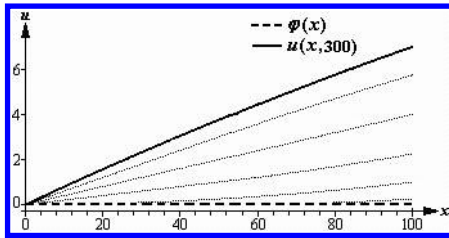


Figure 3.34. Graph of solutions $u(x, t)$ at various time intervals for Example 3.13

Figure 3.34 shows the spatial-time-dependent solution $u(x, t)$ for Example 3.13. This solution was obtained with the program Waves for the case when $a^2 = 1$, $l = 100$, $E = 1$, $S = 1$, $A = 0.0002$.

3.11 Generalized Fourier Solutions

To review the previous sections, we found that the solution of the nonhomogeneous wave equation with initial conditions and nonhomogeneous boundary conditions defined in Equations (3.133) through (3.135) is given by

$$u(x, t) = w(x, t) + \sum_{n=0}^{\infty} \left\{ \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] + T_n(t) \right\} X_n(x). \quad (3.147)$$

If the function $u(x, t)$ in Equation (3.147) is continuous on the closed interval $[0, l]$, the initial condition $u(x, t)|_{t=0} = \varphi(x)$ at $x = 0$ and $x = l$ leads to the relations

$$\alpha_1 \varphi_x + \beta_1 \varphi|_{x=0} = g_1(0) \quad \text{and} \quad \alpha_2 \varphi_x + \beta_2 \varphi|_{x=l} = g_2(0). \quad (3.148)$$

Similarly, if the first derivatives of $u(x, t)$ are continuous on the closed interval $[0, l]$, the second initial condition $u_t(x, t)|_{t=0} = \psi(x)$ gives the relations

$$\alpha_1 \psi_x + \beta_1 \psi|_{x=0} = \frac{\partial g_1}{\partial t}(0) \quad \text{and} \quad \alpha_2 \psi_x + \beta_2 \psi|_{x=l} = \frac{\partial g_2}{\partial t}(0). \quad (3.149)$$

If the consistency relations in Equations (3.148) and (3.149) are satisfied; if the functions $f(x, t)$, $\varphi(x)$, $\psi(x)$, $g_1(x, t)$, and $g_2(x, t)$ are “reasonably” smooth; and if second derivatives of the function $u(x, t)$ given by Equation (3.147) are continuous on $(0, l)$, then Equation (3.147) gives the so-

called *classical solution* of the boundary value problem defined by Equations (3.133) through (3.135).

In many physical applications, some of these conditions are not satisfied. For example, even in the simple problem of free oscillations of a string with fixed ends initially having a triangle shape (where the first derivative of the function $\varphi(x)$ has a discontinuity at the vertex point x_0), the assumption of smoothness does not hold. Another difficulty arises in the simple case of a string excited by a sharp blow from a hammer, for which the initial velocity distribution function $\psi(x)$ has a discontinuity at the strike point.

In certain of these cases we may define *generalized solutions* of the boundary value problem in Equations (3.133) through (3.135) as the functions $\tilde{u}(x, t)$ with the following properties. First, we find the uniformly convergent sequence $u_N(x, t)$ defined by

$$u_N(x, t) = w(x, t) + \sum_{n=0}^N \left\{ \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] + T_n(t) \right\} X_n(x),$$

which are solutions of Equation (3.133). The functions $u_N(x, t)$ also satisfy boundary conditions (3.135) with initial conditions $\varphi_N(x)$ and $\psi_N(x)$, which are the partial sums of Fourier series for the functions $\varphi(x)$ and $\psi(x)$. In the case for $N \rightarrow \infty$, we have

$$\lim_{N \rightarrow \infty} \int_0^l [\varphi(x) - \varphi_N(x)]^2 dx = \lim_{N \rightarrow \infty} \int_0^l [\psi(x) - \psi_N(x)]^2 dx = 0,$$

and $u_N(x, t)$ converges to $\tilde{u}(x, t)$. It can be demonstrated that generalized solutions of the boundary value problem given by Equations (3.133) through (3.135) are unique.

For real, physical problems, we more often work with generalized solutions rather than with classical solutions. Functions $f(x, t)$, $\varphi(x)$, $\psi(x)$ always have some limited precision, so the solution in Equation (3.133) is actually an approximation. The significant point is that the generalized solution differs from the “true” or classical solution by a very small degree in the sense discussed above. A simple example is the problem of string oscillations in a gravitational field. The external force (nonhomogeneous term) is not zero at the ends of the string, and if the ends are fixed, the boundary conditions are not consistent with the equation. But in this

case, the situation is not at all hopeless because the generalized solution converges uniformly at any point in the interval. Similarly, if initial velocities and boundary conditions are not consistent, the generalized solution converges uniformly everywhere, and only the series for the derivative of $u(x, t)$ (the speed) converges poorly.

Reading Exercise. Solve the problem of a string oscillating in a gravity field if the ends are fixed and initially the string was displaced at some point from the equilibrium position and released with zero speed. Solve the problem analytically and then by using the program Waves.

Hint. It is clear that the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g$$

does not yield classical boundary and initial conditions (see Problem 3.2 at the end of the chapter). Discuss the behavior of the analytical solution (which is clearly a generalized solution) at the end of the interval.

3.12 Energy of the String

3.12.1 Kinetic and Potential Energies

Next we consider the energy associated with the motion of the string. The calculation of kinetic and potential energies is essential to the formulation of the problem in terms of classical Lagrangian or Hamiltonian mechanics. The kinetic energy density (energy per unit length) of the string is

$$\frac{1}{2} \rho \left(\frac{\partial u}{\partial t} \right)^2$$

and the total kinetic energy then is

$$E_{\text{kin}}(t) = \frac{1}{2} \int_0^l \rho \left(\frac{\partial u}{\partial t} \right)^2 dx, \quad (3.150)$$

the integral being taken over the whole length of the string.

Next, we determine an expression for the potential energy. If a portion of the string of initial length dx is stretched to a length ds when displaced, the increase in length is

$$ds - dx = \left[\sqrt{1 + \left(\frac{\partial u}{\partial x} \right)^2} - 1 \right] dx.$$

In the approximation of “small vibrations” ($\partial u / \partial x \ll 1$), this becomes

$$\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx.$$

Since this stretching takes place against a force of tension T , the potential energy gain, which is the work done against tension, is

$$\frac{1}{2} T \left(\frac{\partial u}{\partial x} \right)^2 dx.$$

This is the potential energy density; thus, the total potential energy is

$$E_{\text{pot}}(t) = \frac{1}{2} \int_0^l T \left(\frac{\partial u}{\partial x} \right)^2 dx. \quad (3.151)$$

The total energy is therefore

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \frac{\rho}{2} \int_0^l \left[\left(\frac{\partial u}{\partial t} \right)^2 + a^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] dx, \quad (3.152)$$

where we have used the fact that $T = \rho a^2$.

As we saw earlier, the Fourier series solution of the one-dimensional problem of forced oscillations with nonhomogeneous boundary conditions is

$$u(x, t) = w(x, t) + \sum_{n=0}^{\infty} \left\{ \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] + T_n(t) \right\} X_n(x).$$

The kinetic energy may thus be written as

$$E_{\text{kin}}(t) = \frac{\rho}{2} \left\{ \int_0^l \left(\frac{\partial w}{\partial t} \right)^2 dx + 2 \sum_{n=0}^{\infty} \left[\frac{dT_n}{dt} + a_n \frac{dy_n^{(1)}}{dt} + b_n \frac{dy_n^{(2)}}{dt} \right] \int_0^l \frac{\partial w}{\partial t} X_n(x) dx \right. \\ \left. + \sum_{n=0}^{\infty} \left[\frac{dT_n}{dt} + a_n \frac{dy_n^{(1)}}{dt} + b_n \frac{dy_n^{(2)}}{dt} \right]^2 \int_0^l X_n^2(x) dx \right\}$$

and the potential energy is

$$E_{\text{pot}}(t) = \frac{T}{2} \left\{ \int_0^l \left(\frac{\partial w}{\partial x} \right)^2 dx + 2 \sum_{n=0}^{\infty} \left[T_n(t) + a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] \int_0^l \frac{\partial w}{\partial x} \frac{dX_n}{dx} dx \right. \\ \left. + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_n(t) + a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] \left[T_m(t) + a_m y_m^{(1)}(t) + b_m y_m^{(2)}(t) \right] \right. \\ \left. \times \int_0^l \frac{dX_n}{dx} \frac{dX_m}{dx} dx \right\}.$$

3.12.2 Energy in the Harmonics

In the case of homogeneous boundary conditions, we have $w(x, t) \equiv 0$. It is also easy to see that for homogeneous boundary conditions of Dirichlet or Neumann type (e.g., the ends of a string are rigidly fixed or free), the derivatives of functions $X_n(x)$ are orthogonal:

$$\int_0^l \frac{dX_n}{dx} \frac{dX_m}{dx} dx = 0.$$

The physical significance of this equation is that in this case the individual terms in the Fourier series solution are independent, and energy in a harmonic cannot be exchanged with the energies associated with the other terms.

The n th harmonic is

$$u_n(x, t) = \left\{ T_n(t) + \left[a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right] \right\} X_n(x).$$

Its kinetic energy is

$$\begin{aligned} E_{\text{kin}}^{(n)}(t) &= \frac{1}{2} \int_0^l \rho \left(\frac{\partial u_n}{\partial t} \right)^2 dx = \frac{\rho}{2} \left[\frac{dT_n}{dt} + a_n \frac{dy_n^{(1)}}{dt} + b_n \frac{dy_n^{(2)}}{dt} \right]^2 \int_0^l X_n^2(x) dx \\ &= \frac{\rho}{2} \left[\frac{dT_n}{dt} + a_n \frac{dy_n^{(1)}}{dt} + b_n \frac{dy_n^{(2)}}{dt} \right]^2 \|X_n\|^2 \end{aligned}$$

and its potential energy is

$$\begin{aligned} E_{\text{pot}}^{(n)}(t) &= \frac{1}{2} \int_0^l T \left(\frac{\partial u_n}{\partial x} \right)^2 dx = \frac{T}{2} \left[T_n(t) + a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right]^2 \int_0^l \left(\frac{dX_n}{dx} \right)^2 dx \\ &= \frac{T}{2} \left[T_n(t) + a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) \right]^2 \left\| \frac{dX_n}{dx} \right\|^2. \end{aligned}$$

It is evident that there are no terms representing interactions of the harmonics in either the kinetic or potential energy expressions. The total energy of the n th harmonic is the sum of these two energies.

Consider, as an example, free vibrations without damping governed by the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

In this case,

$$T_n(t) = 0, \quad y_n^{(1)}(t) = \cos a \sqrt{\lambda_n} t, \quad y_n^{(2)}(t) = \sin a \sqrt{\lambda_n} t,$$

and the kinetic energy of the n th harmonic is

$$E_{\text{kin}}^{(n)}(t) = \frac{\rho a^2 \lambda_n}{2} \left[-a_n \sin a \sqrt{\lambda_n} t + b_n \cos a \sqrt{\lambda_n} t \right]^2 \|X_n\|^2.$$

The potential energy of the n th harmonic is

$$E_{\text{pot}}^{(n)}(t) = \frac{T}{2} \left[a_n \cos a \sqrt{\lambda_n} t + b_n \sin a \sqrt{\lambda_n} t \right]^2 \left\| \frac{dX_n}{dx} \right\|^2.$$

Taking into account $T = \rho a^2$ and that for boundary conditions of Dirichlet or Neumann type we have

$$\left\| \frac{dX_n}{dx} \right\|^2 = \lambda_n \|X_n\|^2 = \frac{\lambda_n l}{2},$$

the potential energy of the n th harmonic can be written as

$$E_{\text{pot}}^{(n)}(t) = \frac{\rho a^2 \lambda_n l}{4} \left[a_n \cos a \sqrt{\lambda_n} t + b_n \sin a \sqrt{\lambda_n} t \right]^2.$$

The potential and kinetic energies of a harmonic taken together give a constant, the total energy, which is

$$E_{\text{tot}}^{(n)}(t) = E_{\text{kin}}^{(n)}(t) + E_{\text{pot}}^{(n)}(t) = \frac{\rho a^2 \lambda_n l}{4} (a_n^2 + b_n^2).$$

In each harmonic, the energy oscillates between kinetic and potential forms as the string itself oscillates. The periods of the oscillations of the string are

$$\tau_n = \frac{2\pi}{a \sqrt{\lambda_n}}$$

and those of the energy are half that value.

Reading Exercise. We leave it to the reader to work out few more results:

1. If both ends of the string are rigidly fixed, or both ends are free, then $\lambda_n = (\pi n/l)^2$, and the total energy is

$$E_{\text{tot}}^{(n)}(t) = \frac{\rho a^2 \pi^2 n^2}{4l} (a_n^2 + b_n^2).$$

2. If one end of the string is rigidly fixed and the other is free, then $\lambda_n = [\pi(2n+1)/2l]^2$, and the total energy is

$$E_{\text{tot}}^{(n)}(t) = \frac{\rho a^2 \pi^2 (2n+1)^2}{16l} (a_n^2 + b_n^2).$$

3. As an example of another physical system, consider energy oscillations in the RLC circuit. Magnetic and electric energies are respectively $E_L = Li^2/2$ and $E_C = CV^2/2$.

With $i = \partial q / \partial t$, this gives (for a wire from 0 to l)

$$E_L = \frac{1}{2} \int_0^l L \left(\frac{\partial q}{\partial t} \right)^2 dx \quad \text{and} \quad E_C = \frac{1}{2} \int_0^l R^2 C \left(\frac{\partial q}{\partial t} \right)^2 dx.$$

Reading Exercise. Using the material described in Section 3.10, obtain the expressions for these energies for boundary conditions of the Dirichlet and Neumann types.

The following examples look at the energy of a string under various conditions. The reader should keep in mind that identical equations apply to many other physical problems, so the solutions shown here are more general than for vibrations on a string.

Example 3.14. The ends of a uniform string of length l are fixed and all external forces, including the gravitational force, are neglected. Displace the string from equilibrium by shifting the point $x = x_0$ by a distance h at time $t = 0$, and then release it with zero initial speed. Find the kinetic, potential, and total energies of the string as functions of time.

Solution. This example has been solved previously (see Example 3.8). Here we present the graphs of total, kinetic, and potential energies of the string. The wave equation and solutions are given by

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad a = \sqrt{\frac{T_0}{\rho}}, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = \begin{cases} \frac{h}{x_0}x, & 0 \leq x \leq x_0, \\ \frac{h(l-x)}{l-x_0}x, & x_0 < x \leq l, \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = u(l, 0) = 0.$$

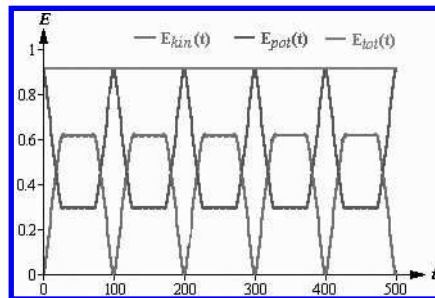


Figure 3.35. Energies of the string for Example 3.14.

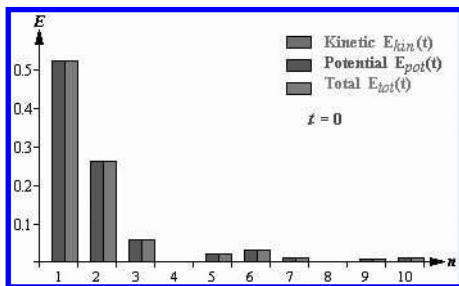


Figure 3.36. Distribution of kinetic, potential and total energies at $t = 0$ for Example 3.14.

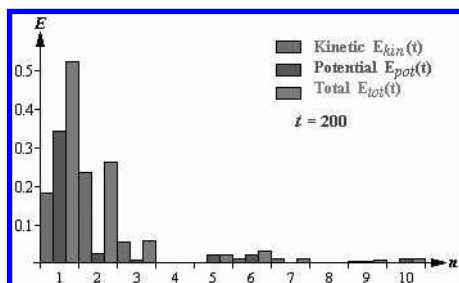


Figure 3.37. Distribution of kinetic, potential and total energies at $t = 200$ for Example 3.14.

Graphs of the energies on the string may be obtained with the program Waves (see Figures 3.35, 3.36, and 3.37). The following values of the parameters were chosen as a specific case: $a^2 = 1$, $\kappa = 0$, $l = 100$, $h = 6$, and $x_0 = 25$.

We may solve the same problem when the force of friction is not zero. Here we again use the program Waves with the same values of the parameters as above but with the friction coefficient, $\kappa = 0.001$. The result is shown in Figure 3.38.

Example 3.15. A homogeneous rod of density ρ and length l is elastically fixed at the end $x = l$. A longitudinal force, $F_0 = \text{const}$, is applied to the end $x = 0$ so that the rod is stretched. At time $t = 0$, the force F_0 stops acting. Find the longitudinal oscillations of the rod if the initial velocities are zero, the resistance of the external medium is absent, and there are no external forces. Find the kinetic, potential, and total energies of the string as functions of time.

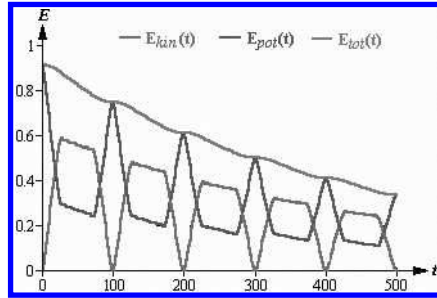


Figure 3.38. Energies of the string for Example 3.14 with friction included.

Solution. This example has also been solved previously (see Example 3.9). Here we present the graphs of total, kinetic, and potential energies of the string. The wave equation and solutions are given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = \frac{F_0}{ES} \left(l + \frac{1}{h} - x \right), \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) + hu(l, t) = 0.$$

The graphs of the energies may be obtained with the program Waves for the following values of the parameters: $a^2 = 1$, $\kappa = 0$, $l = 100$, $E = 1$, $S = 1$, $h = 0.1$ (elasticity coefficient), and $F_0 = 0.05$ (the value of longitudinal force), as shown in Figure 3.39.

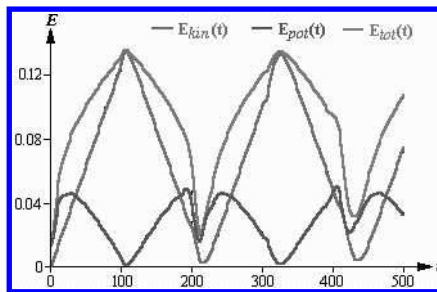


Figure 3.39. Energies of the rod for Example 3.15.

Reading Exercise. Explain qualitatively the energy transforms shown in Figure 3.39.

Example 3.16. Consider a homogeneous string, $0 \leq x \leq l$, with rigidly fixed ends and no friction. Starting at time $t = 0$, a uniformly distributed harmonic force with linear density

$$F(x, t) = F_0 \sin \omega t$$

acts on the string. Find the resulting oscillations and investigate the resonance behavior. Find the kinetic, potential, and total energies of the string as functions of time.

Solution. This example has also been solved previously (see Example 3.10). Here we present the energy graphs of the string. The wave equation and solutions are given by

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \frac{F_0}{\rho} \sin \omega t, \quad 0 < x < l, \quad t > 0,$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = 0, \quad u(l, t) = 0.$$

Graphs of the energies may be obtained with the program Waves for the following values of the parameters: $a^2 = 1$, $\kappa = 0$, $l = 100$, $\rho = 1$, $F_0 = 0.025$, $\omega = 0.3$ (the frequency of the external force), as shown in Figures 3.40 and 3.41.

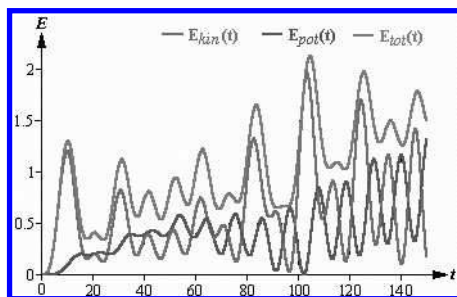


Figure 3.40. Energies of the string for Example 3.16.

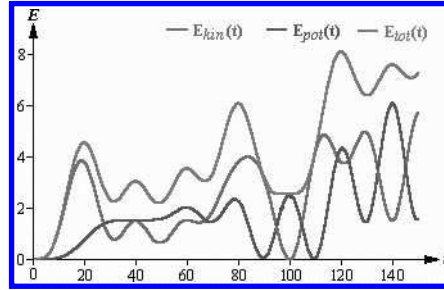


Figure 3.41. Energies of the string for the values of the parameters in Example 3.16 but with a driving frequency equal to the fifth harmonic $\omega = \omega_5 = 5\pi/100$.

Problems

Small Transverse Oscillations of Strings

3.1. A uniform string on the interval $0 \leq x \leq l$ with rigidly fixed ends is at rest under constant force F_0 applied perpendicular to the x axis at point x_0 . At $t = 0$, the force F_0 instantly ceases (Figure 3.42). Set up the boundary value problem to find the resulting oscillations of the string.

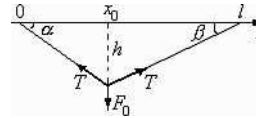


Figure 3.42. Schematic for Problem 3.1.

Hint. To find the initial deviation, h , use the equilibrium condition along the vertical axis

$$T(\sin \alpha + \sin \beta) = F_0.$$

For small deviations, T is constant, $\sin \alpha \approx \tan \alpha$, and $\sin \beta \approx \tan \beta$. With

$$\tan \alpha = \frac{h}{x_0} \quad \text{and} \quad \tan \beta = \frac{h}{l - x_0},$$

we get

$$h = \frac{F_0 x_0 (l - x_0)}{lT}.$$

3.2. A uniform string has fixed ends at $x = 0$ and $x = l$. At $x = x_0$, it is displaced by a small distance h from the equilibrium position and released with zero speed at $t = 0$. Take into account gravity and a resistance force proportional to speed, and set up the boundary value problem.

Answer. The equation is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - 2\kappa \frac{\partial u}{\partial t} + g,$$

and the boundary and initial conditions are

$$u(x, 0) = \begin{cases} \frac{h}{x_0}x, & 0 < x < x_0, \\ \frac{h(l-x)}{l-x_0}, & x_0 < x < l, \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < l,$$

$$u(0, t) = u(l, t) = 0.$$

3.3. A string with length l rigidly fixed at the ends lies at rest in the horizontal direction. At $t = 0$, it is excited by a sharp blow from a hammer, which strikes the string with impulse I at point x_0 . Set up the boundary value problem for small oscillations in a medium with resistance coefficient κ .

Answer. The equation and the boundary conditions are

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \frac{I}{\rho} \delta(x - x_0),$$

$$u(0, t) = u(l, t) = 0.$$

The coefficient of the delta function, $\delta(x - x_0)$, is obtained from the condition that the total impulse at $t = 0$ is I , or

$$\int \frac{\partial u}{\partial t}(x, 0) \rho dx = I.$$

3.4. Repeat Problem 3.3, but with a hammer that has a width of 2δ and provides the string with an initial speed, v_0 , on the interval $(x_0 - \delta, x_0 + \delta)$.

Answer. The equation and boundary conditions are

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \begin{cases} 0, & 0 < x < x_0 - \delta, \\ v_0, & x_0 - \delta < x < x_0 + \delta, \\ 0, & x_0 + \delta < x < l, \end{cases}$$

$$u(0, t) = u(l, t) = 0.$$

3.5. Repeat Problem 3.4, but with a hammer that has a concave shape schematically described by the function

$$\psi(x) = \begin{cases} v_0 \cos \frac{\pi(x-x_0)}{2\delta}, & x_0 - \delta < x < x_0 + \delta, \\ 0, & \text{otherwise.} \end{cases}$$

3.6. Repeat Problem 3.5 but with a hammer that is soft, such that the force on the string is described by

$$f(x, t) = \begin{cases} F_0 \cos \frac{\pi(x-x_0)}{2\delta} \sin \frac{\pi t}{\tau}, & |x-x_0| < \delta, \quad 0 \leq t \leq \tau, \\ 0, & \text{otherwise.} \end{cases}$$

Hint. The hammer is not ideally rigid; thus, the force changes in time:

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t).$$

If the hammer is moving slowly, we can consider its speed to be approximately zero; thus,

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad u(0, t) = u(l, t) = 0.$$

3.7. A uniform string with length l is fixed at the end $x = 0$, and its other end at $x = l$ is attached to a massless ring that moves with no friction along a vertical rod such that the tangent line to the string at this end is always horizontal. Initially, the ring is displaced by a small distance h and is released at $t = 0$. Set up the boundary value problem for small oscillations in a medium with resistance κ .

3.8. Repeat Problem 3.7 but with the tangent line to the end $x = l$ making an angle α with the x -axis. Let the initial displacement be

$$u(x, 0) = \frac{x^2 \tan \alpha}{l}$$

and initial speed be zero. The resistance coefficient is, again, κ .

Longitudinal Oscillations of Uniform Elastic Rods

3.9. The right end $x = l$ of a rod of density ρ and length l is elastically fixed. The end $x = 0$ is stretched by a constant force F_0 directed along the rod axis. At time $t = 0$, the force stops acting. Set up the boundary value problem for oscillations of the rod with no external resistance.

3.10. Consider a rod of length l that hangs vertically. The end $x = 0$ is rigidly fixed and the end $x = l$ has a mass M added to it at $t = 0$. Set up the boundary value problem for oscillations of the rod with no external resistance.

3.11. Set up the boundary value problem to model longitudinal oscillations of a spring with free ends under uniform initial compression.

3.12. A rod of density ρ and length l has a fixed end at $x = 0$ and free end at $x = l$. It is excited by a horizontal blow with impulse I at one end. The rod is oscillating in a medium with the resistance proportional to speed. Set up the boundary value problem for oscillations of the rod.

3.13. Starting with $t = 0$, the left end $x = 0$ of a rod is moving horizontally according to the function $\varphi(t)$. A force $\Phi = \Phi(t)$ is applied to its right end at $x = l$ along the rod axis at $t = 0$, before which the rod was free and at rest. Set up the boundary value problem for rod oscillations with resistance proportional to speed.

Answer. Equation and boundary conditions are

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \\ u(0, t) &= \varphi(t), \quad u_x(l, t) = \frac{\Phi(t)}{ES}.\end{aligned}$$

3.14. Mass M is attached to the end, $x = l$, of a vertically hanging rod. Set up the boundary value problem assuming that besides this mass there is also an external force. Consider two cases:

1. The force is distributed along the rod with force density $F(x, t)$.
2. The force $F_0(t)$ is acting at the point $x = x_0$.

Torsional Oscillations of an Elastic Cylinder

3.15. Set up the boundary value problem to model torsional oscillations of an elastic cylinder that results in nonhomogeneous equations similar to Equations (3.3), (3.5), and (3.6).

3.16. Derive the boundary conditions in Section 3.4.2 for a twisted rod.

Acoustic Waves

- 3.17.** Provide arguments to support the validity of the derivation leading to Equation (3.26) for the energy of oscillations.
- 3.18.** Derive the boundary conditions found in Section 3.5.2 for gas in a tube.

Waves in a Shallow Channel

- 3.19.** Derive the wave equation for the vertical component of the speed of a liquid's surface elements in a shallow channel.
- 3.20.** Discuss the physical implications of the nonhomogeneous equation for waves in liquids.

Electric Oscillations in Circuits

- 3.21.** A charged insulated wire of length l has an electric potential $V_0 = \text{const}$. Resistance, self-inductance, leakage (per unit time) and capacitance are denoted by R , L , G , and C , respectively. At $t = 0$, the left end at $x = 0$ is grounded and the end at $x = l$ remains insulated. Set up the boundary value problem for the voltage $V(x, t)$ in the wire for $t > 0$.
- 3.22.** A conductor of length l is perfectly insulated ($G = 0$), and R , L , and C are known. The current at $t = 0$ is absent, and the voltage is $V(x, t)|_{t=0} = f(x)$. The left end at $x = 0$ is insulated and the right end at $x = l$ is grounded. Set up the boundary value problem for the current in the wire for $t > 0$.
- 3.23.** Initially, the current and voltage in an insulated conductor are zero. The left end at $x = 0$ is insulated and the right end at $x = l$ is grounded. R , L , G , and C are given. At time $t = 0$, a charge Q is supplied to the conductor at point $x = x_0$. Set up the boundary value problem for the voltage $V(x, t)$ in the wire for $t > 0$.
- 3.24.** Initially, the current and voltage in an insulated conductor are zero. The left end of the conductor at $x = 0$ is insulated and the right end at $x = l$ is attached to a source of emf, $E(t) = E_0 \sin \omega t$ at $t = 0$ ($E_0 = \text{const}$). R , L , G , and C are given. Set up the boundary value problem for the voltage $V(x, t)$ in the wire for $t > 0$.
- 3.25.** Derive Equation (3.36).
- 3.26.** For the trapezoid shown in Figure 3.9, plot the shape for subsequent moments of time $t = 1/2a$, $t = 3l/2a$, $t = 2l/a$.

3.27. In Section 3.7.1, we derived equations that describe electrical oscillations in an electric circuit. Consider an infinite electric line, $-\infty < x < \infty$, with parameters R , C , and L . As previously discussed, these equations do not yield a solution in the form of traveling waves. In the particular case, $G/C = R/L$ (the condition of an electrical line without waveform distortion), however, these equations yield the solution in a form of damped wave: $u(x, t) = e^{-\lambda t} f(x - at)$, where f is an arbitrary function, $a = \sqrt{1/CL}$ is a wave propagation speed, and $\lambda = R/L = G/C$ is the damping coefficient. Show that the solution has the form of a damped wave for arbitrary initial conditions for current $i(x, 0) = \varphi(x)$ and voltage $V(x, 0) = \psi(x)$, $-\infty < x < \infty$.

The following selection of problems should be solved by using the software Waves included with this book, as well as analytically. Instructions for the use of the program can be found in Appendix E. To solve problems with the software, numerical values for parameters such as length, density, tension, coefficients in expressions for initial conditions and other constants will need to be supplied. The algorithms used in the software assume that oscillations will remain small. Dimensionless parameters may be used, but for the physicist or engineer it is also useful, if possible, to model a realistic situation with, for example, speed in meters per second or tension in Newtons. The text of the software is written for the case of a vibrating string, but it is easy to model, for instance, electrical oscillations by inputting the corresponding “electrical” parameters.

One of the advantages of using the software is that a range of parameters may be explored very quickly. For example, it is very simple to (1) model oscillations in different frequency intervals in an electric circuit; (2) find a medium amount of resistance so that the main mode is reduced by a factor of two during some number of periods; (3) explore the process of energy exchange between two modes when the frequency of an external force is close to the frequency gap; or (4) study the role of a parameter’s effect in the behavior of different modes. This opens the way to exploring a wide variety of different variants of basically the same problem.

It is always useful to solve a problem analytically, and if this solution is correct it should give the same results that the software provides. The two solutions cannot be compared in detail without expressing the analytical solution in animated graphical form, but from the analytical solution one can often determine the general behavior, for instance the zeros (nodes) of harmonics, which can indicate the correctness of the solution.

For Problems 3.28 through 3.40, assume that the equation of motion has the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

In all problems, choose appropriate values of parameters and interval limits. Simulate the vibrations for a range of values of wave speed, for example, $a^2 = 2, 1, \frac{1}{2}$.

3.28. An infinite stretched string is excited by the initial displacement

$$u(x, 0) = \varphi(x).$$

Assume no initial velocities, and find the subsequent vibrations for each of the following initial shapes, $\varphi(x)$:

1. $\varphi(x) = \begin{cases} h(x+l)/l, & x \in [-l, 0], \\ h(l-x)/l, & x \in [0, l], \\ 0, & x \notin [-l, l]; \end{cases}$
2. $\varphi(x) = \begin{cases} A(l^2 - x^2), & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$
3. $\varphi(x) = \begin{cases} A \sin(\pi x/l), & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$
4. $\varphi(x) = A/(1 + 4x^2);$
5. $\varphi(x) = Ae^{-x^2};$
6. $\varphi(x) = Axe^{3x-x^2};$
7. $\varphi(x) = Ae^{-x^2} \sin x.$

3.29. An infinite stretched string is initially at rest. Assume at time $t = 0$ the initial distribution of velocities is given by

$$u_t(x, 0) = \psi(x).$$

Find the vibrations for each of the following initial velocities $\psi(x)$:

1. $\psi(x) = \begin{cases} v_0, & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$
2. $\psi(x) = \begin{cases} x, & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$
3. $\psi(x) = \begin{cases} A \sin(\pi x/l), & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$
4. $\psi(x) = A/(1 + 4x^2);$
5. $\psi(x) = Ae^{-x^2};$
6. $\psi(x) = Axe^{3x-x^2};$
7. $\psi(x) = Ae^{-x^2} \sin x.$

3.30. Consider an infinite thin wire with resistance R , capacitance C , inductance L , and leakage G per unit length. Find the electrical current oscillations in the wire if $GL = CR$ (the condition required if an electrical line is not to have wave-form distortion). The initial voltage and current in the wire are

$$V(x, 0) = f_1(x), \quad i(x, 0) = f_2(x), \quad -\infty < x < +\infty.$$

Find the electrical current oscillations in the wire for each of the following sets of initial functions:

1. $f_1(x) = e^{-|x|}$ and $f_2(x) = \cos \pi x$;
2. $f_1(x) = \sin \pi x$ and $f_2(x) = e^{-|x|}$;
3. $f_1(x) = \sin \pi x$ and $f_2(x) = \cos \pi x$;
4. $f_1(x) = 1/(x^2 + 1)$ and $f_2(x) = e^{-x^2}$;
5. $f_1(x) = e^{-x^2}$ and $f_2(x) = 1/(x^2 + 1)$.

Hint. To define the voltage, $V(x, t)$, see Example 3.4 from Section 3.8. The problem for defining the current, $i(x, t)$, in the wire can be found in an analogous way.

3.31. Let an infinite string be at rest prior to $t = 0$. At time $t = 0$, it is excited by a sharp blow from a hammer that transmits an impulse I at point $x = x_0$ to the string. Find the vibrations of the string for $t > 0$. Write formulas representing the motion of the string and simulate the vibrations using the program Waves for $x_0 = 1$, $I = 10$.

3.32. The wave $\varphi(x - at)$ moves along an infinite string. Consider this wave as the initial disturbance of the string at time $t = 0$ and find the vibrations of the string for $t > 0$. Find the displacements, $u(x, t)$, of points on the string for each of the following initial wave functions $\varphi(x)$:

1.
$$\varphi(x) = \begin{cases} h(x+l)/l, & x \in [-l, 0], \\ h(l-x)/l, & x \in [0, l], \\ 0, & x \notin [-l, l]; \end{cases}$$
2.
$$\varphi(x) = \begin{cases} A(l^2 - x^2), & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$$
3.
$$\varphi(x) = \begin{cases} A \sin(\pi x/l), & x \in [-l, l], \\ 0, & x \notin [-l, l]; \end{cases}$$
4. $\varphi(x) = A/(1 + 4x^2)$;
5. $\varphi(x) = Ae^{-x^2}$;

6. $\varphi(x) = Axe^{3x-x^2}$;

7. $\varphi(x) = Ae^{-x^2} \sin x$.

Compare the solutions with the respective answers to Problem 3.28.

Hint. Initial conditions in Problem 3.28 are

$$u(x, 0) = \varphi(x) \neq 0, \quad u_t(x, 0) = \psi(x) \equiv 0, .$$

But this problem, with a moving wave at time $t = 0$, is described by nonzero “initial” displacement and speed (it is assumed that for $t < 0$ the wave already exists), in which case

$$u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x, 0) = -a\varphi'(x), \quad -\infty < x < +\infty.$$

3.33. Consider a semi-infinite elastic rod in the region $x \geq 0$. Assume that at the initial time $t = 0$, it is excited by the local initial deflection, $u(x, 0) = \varphi(x)$, defined on the interval $[a, b]$. The initial distribution of velocities is zero; that is, $u_t(x, 0) = 0$. Find the displacements, $u(x, t)$, of points on the string for the following cases:

1. the end at $x = 0$ is rigidly fixed,
2. the end at $x = 0$ is free,
3. the end at $x = 0$ is elastically fixed, and
4. for each of the following initial deflections $\varphi(x)$:

$$(a) \quad \varphi(x) = \begin{cases} 2h(x-a)/(b-a), & x \in [a, (a+b)/2], \\ 2h(b-x)/(b-a), & x \in [(a+b)/2, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(b) \quad \varphi(x) = \begin{cases} -x^2 + (a+b)x - ab, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(c) \quad \varphi(x) = \begin{cases} \sin \pi(x-a)/(b-a), & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(d) \quad \varphi(x) = \begin{cases} 1/\{[x-(a+b)/2]^2 + 1\}, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(e) \quad \varphi(x) = \begin{cases} \exp[-(x-(a+b)/2)^2], & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}$$

3.34. A semi-infinite string $x \geq 0$ is initially at rest. At time $t = 0$, the string is struck by a hammer in the interval $[a, b]$ along the string such that the initial distribution of velocities is given by $u_t(x, 0) = \psi(x)$. Find the displacements, $u(x, t)$, of points on the string for the following cases:

1. the end at $x = 0$ is rigidly fixed,
2. the end at $x = 0$ is free,
3. the end at $x = 0$ is elastically fixed, and
4. for each of the following initial functions $\psi(x)$:

$$(a) \quad \psi(x) = \begin{cases} v_0, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(b) \quad \psi(x) = \begin{cases} x - a, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(c) \quad \psi(x) = \begin{cases} \sin \pi(x - a)/(b - a), & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(d) \quad \psi(x) = \begin{cases} 1/\{[x - (a + b)/2]^2 + 1\}, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(e) \quad \psi(x) = \begin{cases} \exp[-(x - (a + b)/2)^2], & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}$$

3.35. A wave $\varphi(x - at)$ moves along a semi-infinite string $x \geq 0$. Consider this wave as the initial disturbance of the string at time $t = 0$ and find the vibrations of the string for $t > 0$. Find the displacements, $u(x, t)$, of string points for the following cases:

1. the end at $x = 0$ is rigidly fixed,
2. the end at $x = 0$ is free,
3. the end at $x = 0$ is elastically fixed, and
4. for each of the following initial functions $\varphi(x)$:

$$(a) \quad \varphi(x) = \begin{cases} 2h(x - a)/(b - a), & x \in [a, (a + b)/2], \\ 2h(b - x)/(b - a), & x \in [(a + b)/2, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$(b) \quad \varphi(x) = \begin{cases} -x^2 + (a + b)x - ab, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases}$$

$$\begin{aligned}
\text{(c)} \quad \varphi(x) &= \begin{cases} \sin \pi(x-a)/(b-a), & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases} \\
\text{(d)} \quad \varphi(x) &= \begin{cases} 1/\{[x-(a+b)/2]^2 + 1\}, & x \in [a, b], \\ 0, & x \notin [a, b]; \end{cases} \\
\text{(e)} \quad \varphi(x) &= \begin{cases} \exp[-(x-(a+b)/2)^2], & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}
\end{aligned}$$

Hint. See the hint for Problem 3.32.

3.36. Consider a semi-infinite string in the region $x \geq 0$. For $t > 0$, the end $x = 0$ of the string is driven with the function $u(0, t) = g(t)$. Find the displacements, $u(x, t)$, of points on the string for each of the following functions $g(t)$:

1. $g(t) = A \sin \omega t$;
2. $g(t) = A \cos \omega t$;
3. $g(t) = A(\sin \omega t + \cos \omega t)$;
4. $g(t) = A(1 - \cos \omega t)$;
5. $g(t) = A(1 - e^{-ct} \sin \omega t)$.

3.37. Let a semi-infinite elastic rod occupy the region $x \geq 0$. For $t > 0$ the end at $x = 0$ is subjected to the longitudinal force $F(t)$. Find the longitudinal displacements $u(x, t)$ of cross sections of the rod for each of the following functions, $g(t)$:

1. $g(t) = A \sin \omega t$;
2. $g(t) = A \cos \omega t$;
3. $g(t) = A(\sin \omega t + \cos \omega t)$;
4. $g(t) = A(1 - \cos \omega t)$;
5. $g(t) = A(1 - e^{-ct} \sin \omega t)$.

3.38. Consider a semi-infinite thin wire with resistance R , capacitance C , inductance L , and leakage G per unit length. Suppose that $GL = CR$. Find the electrical current oscillations in the wire for $t > 0$ if, prior to $t = 0$, the voltage and current in the wire are

$$V(x, t) = e^{-(R/L)t} f(x + at) \quad i(x, t) = -e^{-(R/L)t} \sqrt{C/L} f(x + at).$$

Solve the problem for the two cases in which at time $t = 0$ the end $x = 0$ is

1. grounded,
2. insulated, and
3. for each of the following initial functions $f(x)$:

(a) $f(x) = A \sin \omega x$;

(b) $f(x) = A \cos \omega x$;

(c) $f(x) = A(\sin \omega x + \cos \omega x)$;

(d) $f(x) = A(1 - \cos \omega x)$;

(e) $f(x) = A(1 - e^{-x} \sin \omega x)$.

3.39. Consider a semi-infinite electrical line without waveform distortion, that is, $GL = CR$. At time $t = 0$, the end at $x = 0$ of the wire is attached to a battery with emf

$$E(t) = E_0 \sin \omega t, \quad t > 0.$$

The initial voltage and current in the wire are zero. Find the voltage and current in the wire as a function of time.

3.40. A semi-infinite pipe with constant cross section occupies the region $x \geq 0$ and is filled with a fluid at rest. Starting at time $t = 0$ the fluid flux $w(x, t)$ at the end $x = 0$ quickly changes to a quantity A and then remains steady. Find the flux and pressure in the pipe as functions of time.

Hint. Neglecting resistance to flow, the equations that model the problem are

$$\begin{cases} -p_x = w_t, \\ -p_t = a^2 w_x. \end{cases}$$

Eliminating $p(x, t)$ from these equations, we obtain a boundary value problem for w given by

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2},$$

$$w(x, 0) = 0, \quad \frac{\partial w}{\partial t}(x, 0) = 0, \quad x > 0,$$

$$w(0, t) = A, \quad t > 0.$$

Small Transverse Oscillations in Strings. *Note:* Problems 3.41 through 3.51 refer to uniform finite strings with the ends at $x = 0$ and $x = l$ (all problems can be easily reformulated for various oscillating one-dimensional systems).

3.41. The initial shape of a string with fixed ends is $u(x, 0) = \varphi(x)$, and the initial speed is given by $u_t(x, 0) = \psi(x)$. No external forces or dissipation is present. Describe string oscillations for the following cases:

1. $\varphi(x) = \frac{x(l-x)}{8l}$, $\psi(x) = 0$;
2. $\varphi(x) = A \sin \frac{\pi x}{l}$, $\psi(x) = 0$;
3. $\varphi(x) = 0$, $\psi(x) = v_0 = \text{const}$;
4. $\varphi(x) = 0$, $\psi(x) = v_0 \frac{x(l-x)}{100}$;
5. $\varphi(x) = \frac{x(l-x)}{10l}$, $\psi(x) = v_0 x(l-x)$.

(If φ , ψ , and v_0 and other parameters are not in dimensionless form, constants such as A here and below may be introduced to keep correct dimensions. The coefficients, such as $1/10$ and $1/100$, may also be changed by any suitable number.)

Find the tension T so that the period of the main harmonic is $l/2a$, $l/4a$, $l/8a$. Search for harmonics (low or high) that are the most sensitive to small variations in the input parameters.

3.42. A string with fixed ends is displaced at point $x = x_0$ by a small distance h from equilibrium and released at $t = 0$ without initial speed. No external forces or dissipation act on the string. Describe the string oscillations. Find the location of x_0 so that the following overtones are absent:

1. third,
2. fifth,
3. seventh.

Hint. Start with the analytical solutions given in the text (e.g., Example 3.8).

3.43. The ends of a string are rigidly fixed. The string is excited by the sharp blow of a hammer, supplying an impulse I at point x_0 . No external forces act but the surrounding medium supplies a resistance with coefficient κ . Model the string oscillations. Find x_0 such that the energies of seventh and eighth overtones are reduced. Find the value of the coefficient, κ , such that oscillations decay (with precision of about 5%) during (1) two periods, (2) three periods, and (3) four periods of the main mode (this is most easily seen by looking at the energy of a harmonic).

Comment. The loudness of a sound is characterized by the energy or amplitude of the oscillations; tone by the period of oscillations; timbre by the ratio of energies of the main mode and overtones. The presence of high overtones destroys the harmony of a sound, producing dissonance. Low overtones, in contrast, give a sense of completeness to a sound.

3.44. The end at $x = 0$ of a string is fixed, and the end at $x = l$ is attached to a massless ring that can slide along a frictionless rod perpendicular to the x -axis such that the tangent line to the string is always horizontal. The initial shape and speed of the string are $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$. No external forces act except that the surrounding medium has a resistance with coefficient κ . Find string oscillations for the following initial conditions:

1. $\varphi(x) = hx/l$, $\psi(x) = 0$;
2. $\varphi(x) = (1/20) \sin(\pi x/2l)$, $\psi(x) = 0$;
3. $\varphi(x) = 0$, $\psi(x) = v_0 = \text{const}$;
4. $\varphi(x) = 0$, $\psi(x) = v_0 x(l-x)/l^2$;
5. $\varphi(x) = x(l-x)/10l$, $\psi(x) = v_0 \cos(\pi x/l)$.

Find the coefficient, κ , such that oscillations decay (with precision of about 10%) during (1) two periods, (2) three periods, and (3) four periods of the main mode.

3.45. The ends of a string are rigidly fixed. An external force $F(t)$ begins to act at point x_0 at time $t = 0$. Model string oscillations for zero initial conditions where the string is in a resisting medium with resistance coefficient κ .

Explore the time evolution of the string for the following cases:

1. $F(t) = F_0$;
2. $F(t) = F_0 \sin \omega t$;
3. $F(t) = F_0 \cos \omega t$;
4. $F(t) = F_0 e^{-At} \sin \omega t$;
5. $F(t) = F_0 e^{-At} \cos \omega t$.

3.46. The ends of a string are fixed. Starting at $t = 0$, a uniformly distributed force $F(x, t) = F_0 \sin \omega t$ is applied. Model the string oscillations for zero initial conditions and resistance coefficient, κ . Study the phenomena of resonance for this situation.

3.47. The left end of a string is rigidly fixed but the right end is free. The initial conditions are zero with no resistance. Find the string oscillations if an external uniform force with constant density acts on the string.

3.48. The ends of a string are rigidly fixed. The initial conditions are zero with no resistance. Find the string oscillations if an external uniform force $f(x, t)$ acts on the string for the following cases:

1. $f(x) = f_0 x(x - l)$;
2. $f(x, t) = f_0 x e^{-t}$;
3. $f(x, t) = f_0 x(x - l) \sin \omega t$;
4. $f(x, t) = f_0 x(x - l) \cos \omega t$;
5. $f(x, t) = f_0 x(x - l) e^{-At} \sin \omega t$.

3.49. A string with fixed ends is displaced at point $x = x_0$ by a small distance h from equilibrium position and released at $t = 0$ without initial speed. The resistance force of the medium is proportional to the speed. Describe the subsequent string oscillations, assuming there is a gravitational field. Evaluate the time when oscillations terminate (with some accuracy) and find the coordinate of the bottom point of the string profile when the oscillations stopped (i.e., locate these coordinates on the screen after oscillations stopped).

3.50. The left end of a string is rigidly fixed, and the right end can freely move along the vertical direction in such a way that the tangent line at this end makes an angle $-\alpha$ to the x -axis ($u_x(l, t) = -\tan \alpha$). The initial displacement is

$$u(x, 0) = -\frac{x^2 \tan \alpha}{l}$$

and the initial speed is zero. Find the string oscillations if the resistance coefficient of the medium is κ .

3.51. The left end of a string is driven according to $u(0, t) = g_1(t)$, the right end is fixed. Model the string oscillations for zero initial conditions without external forces for the following functions:

1. $g_1(t) = A \sin \omega t$;
2. $g_1(t) = A(1 - \cos \omega t)$;
3. $g_1(t) = A e^{-0.01t} \sin \omega t$.

Longitudinal Oscillations in Rods. Problems 3.52, 3.53, and 3.54 are formulated in terms of waves in a thin elastic straight rod of density ρ , length l (the ends at $x = 0$ and $x = l$), and elasticity modulus E .

3.52. A rod with free ends is excited at $t = 0$ by a hammer strike with impulse I along its axis. The resistance of the surrounding medium is proportional to speed. Describe the oscillations $u(x, t)$. Find the coefficient κ such that oscillations decay (with precision of about 5%) during (1) two periods, (2) three periods, and (3) four periods of the main mode.

3.53. A rod is stretched by a force f applied to its right end at $x = l$ along the rod's axis. The ends are elastically attached with elasticity coefficients h_1 and h_2 . At $t = 0$, the force F instantly stops acting. Describe the oscillations for small h_1 and h_2 ("soft" attachment) and large h_1 and h_2 ("stiff" attachment). Compare with a similar case with free ends.

3.54. Starting at $t = 0$, the end of a string at $x = 0$ is moving according to $u(0, t) = g(t)$, and an external force $F = F(t)$ is applied to the end at $x = l$ along the axis. Assume zero initial conditions and an embedding medium that has a resistance proportional to speed. Describe the oscillations $u(x, t)$ for the following cases:

1. $F(t) = F_0(1 - \cos \omega t)$, $g(t) = g_0$;
2. $F(t) = F_0$, $g(t) = g_0 \sin \omega t$;
3. $F(t) = F_0 e^{-0.01t} \sin \omega t$, $g(t) = g_0 \sin \omega t$.

Hint. The equation and the boundary conditions are:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \\ u(0, t) &= g(t), \quad u_x(l, t) = \frac{F(t)}{ES}. \end{aligned}$$

Electrical Oscillations in a Circuit

3.55. A charged insulated wire of length l experiences an electric potential $V_0 = \text{const}$. Resistance, self-inductance, leakage (per unit time) and capacitance are designated by R , L , G , and C , respectively. At $t = 0$, the end $x = 0$ is grounded, and the end at $x = l$ remains insulated. Find the voltage $V(x, t)$ in the wire for $t > 0$. Use physically reasonable values of L , R , C , G , and V_0 . By varying these parameters, find which combinations of them are physically not interesting—that is, they lead to very small (or very big) periods such that oscillations are practically impossible to observe. Try to model the oscillations in centimeter or meter units.

3.56. A conductor of length l is perfectly insulated ($G = 0$), and R , L , and C are known. Current at $t = 0$ is absent, and the voltage is $V(x, t)|_{t=0} = V(x)$. The end at $x = 0$ is insulated and the end at $x = l$ is grounded. Find the current in the wire for $t > 0$ for the following voltages:

1. $V(x) = V_0 \sin(\pi x/l)$;
2. $V(x) = V_0 (1 - \cos(\pi x/l))$;
3. $V(x) = V_0 x(l - x)$;
4. $V(x) = V_0 / [(x - l/2)^2 + 1]$;
5. $V(x) = V_0 \exp[-A(x - l/2)^2]$.

3.57. Initially, the current and voltage in an insulated conductor are zero. The left end at $x = 0$ is insulated and the right end at $x = l$ is attached to a source of emf $E(t)$ at $t = 0$. The parameters R , L , G , and C are known. Find the voltage, $V(x, t)$, in the wire for $t > 0$ for the following applied emfs:

1. $E(t) = E_0 \sin \omega t$;
2. $E(t) = E_0 (1 - \cos \omega t)$;
3. $E(t) = E_0 e^{-0.01t} \sin \omega t$.

Acoustic Waves

3.58. At the end $x = l$ of a filled tube of length $0 \leq x \leq l$, the fluid flux $\Phi(x, t)$ changes sharply at $t = 0$ by an amount $A = \text{const}$. The end at $x = 0$ is connected to a large reservoir where the pressure does not change. Assuming uniform (or zero) flow and pressure before $t = 0$, find w and p in the tube for $t > 0$.

Answer. The following is a brief solution. The flux is $\Phi = \rho_0 v S$. The equations below model the problem:

$$\begin{aligned} -\frac{\partial p}{\partial x} &= \frac{\partial \Phi}{\partial t} + 2\kappa \Phi, \\ -\frac{\partial p}{\partial t} &= a^2 \frac{\partial \Phi}{\partial x}. \end{aligned} \tag{3.153}$$

The constant κ characterizes the resistance to the flow per unit length. Two conditions for pressure perturbation, $\tilde{p}(x, 0) = 0$ and $\tilde{p}(0, t) = 0$, are obvious, but we will not use them. Excluding $p(x, t)$ from Equation (3.153), we obtain the boundary value problem for Φ :

$$\frac{\partial^2 \Phi}{\partial t^2} + 2\kappa \frac{\partial \Phi}{\partial t} - a^2 \frac{\partial^2 \Phi}{\partial x^2} = 0; \tag{3.154}$$

$$\tilde{\Phi}(x, 0) = 0, \quad \frac{\partial \tilde{\Phi}}{\partial t}(x, 0) = 0,$$

$$\frac{\partial \tilde{\Phi}}{\partial x}(0, t) = 0, \quad \tilde{\Phi}(l, t) = A$$

($\tilde{\Phi}$ is a flux perturbation; it is clear that for derivatives, we can use Φ instead of $\tilde{\Phi}$). The solution is

$$\Phi(x, t) = A - \frac{4A}{\pi} e^{-\kappa t} \sum_{n=0}^{+\infty} \frac{1}{2n+1} \left(\cos \tilde{\omega}_n t + \frac{\kappa}{\tilde{\omega}_n} \sin \tilde{\omega}_n t \right) \sin \frac{(2n+1)\pi(t-x)}{2l},$$

where

$$\tilde{\omega}_n = \sqrt{\left[\frac{(2n+1)\pi a}{2l} \right]^2 - \kappa^2}.$$

Then the pressure can be found with Equation (3.153):

$$\begin{aligned} p(l, t) &= p(0, t) - \int_0^l \left(\frac{\partial \Phi}{\partial t} + 2\kappa \Phi \right) dx \\ &= - \left\{ 2\kappa x A + \frac{4aA}{\pi} e^{-\kappa t} \sum_{n=0}^{+\infty} \frac{\sin(\tilde{\omega}_n t - 2\varphi_n)}{(2n+1) \cos \varphi_n} \right\}, \end{aligned}$$

where $\tan \varphi_n = \kappa / \tilde{\omega}_n$.

3.59. Find the steady oscillations of pressure in a tube if the end at $x = l$ is closed by a permeable elastic cap and the flux of injected fluid through this end changes harmonically with time. The pressure at the end $x = 0$ remains constant (this end is open or connected to a large reservoir).

Hint. Equations (3.153) and (3.154) in Problem 3.58 describe this problem. The boundary conditions for w are

$$\frac{\partial w}{\partial x}(0, t) = 0, \quad \frac{\partial w}{\partial x}(l, t) + h w(l, t) = A \sin \omega t.$$

The initial conditions for a steady regime are not important.

3.60. A tube filled with an ideal gas and opened from the end at $x = 0$ is moving along its axis with a constant speed v_0 . The other end at $x = l$ is closed. At time $t = 0$, the tube is instantly stopped. Find $u(x, t)$.

Hint. The problem reduces to the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

where $a^2 = \gamma p_0 / \rho_0$ is the speed of sound, and

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = v_0, \quad u(0, t) = 0, \quad u_x(l, t) = 0.$$

3.61. A tube is filled with fluid and the end at $x = 0$ is closed with a wall. The end at $x = l$ is closed with a piston oscillating along the tube axis with a constant frequency and small amplitude. Set up the problem and find the displacement $u(x, t)$ and velocity $v(x, t)$ as functions of time and location.

Hint. Because the amplitude of oscillations at the end at $x = l$ is small, this end can be considered as having the approximately constant coordinate $x = l$. The boundary conditions for velocity are

$$v(0, t) = 0 \quad \text{and} \quad v(l, t) = A\omega \sin \omega t.$$

When we search for a steady solution (at large t), the role of the initial conditions is negligible. Otherwise we need them, and one of the possibilities is

$$v(x, 0) = 0 \quad \text{and} \quad v_t(x, 0) = 0.$$

Are these initial and boundary conditions consistent with each other? If not, how could they be made consistent (by changing the boundary conditions)? Solve the problem for both cases and compare the convergence of their Fourier series. (Read Section 3.11 on generalized solutions again.)

Torsional Oscillations of an Elastic Cylinder

3.62. A uniform elastic cylinder ($0 \leq x \leq l$) is deformed in such a way that its cross sections have small angles of rotation. The cylinder is then released at $t = 0$. The ends of the cylinder are rigidly fixed. Find the rotation along the length of the cylinder for $t > 0$ for the following sets of initial conditions:

1. $\varphi(x) = Ax(l - x)/l, \quad \psi(x) = 0;$
2. $\varphi(x) = A \sin(\pi x/l), \quad \psi(x) = 0;$
3. $\varphi(x) = 0, \quad \psi(x) = Ax;$
4. $\varphi(x) = 0, \quad \psi(x) = Ax(x - l);$
5. $\varphi(x) = A(x^2 - x), \quad \psi(x) = Ax(x - l).$

Auxiliary Functions $w(x, t)$ for Different Types of Boundary Conditions

3.63 through 3.72. Obtain the expressions for $w(x, t)$ for all possible combinations of boundary conditions in Appendix B (there are nine variants).

4

Two-Dimensional Hyperbolic Equations

In this chapter, we consider physical problems related to two-dimensional flexible surfaces called membranes. A membrane may be defined as a thin film that bends, but, in the present development, does not stretch. The boundary of the membrane may be fixed or free or have forces applied to it. We also consider cases in which the membrane interacts with the material in which it is embedded and is thus subject to external forces such as driving forces or friction. Examples of membranes include drum heads, flags, trampolines, soap films, and biological barriers such as cellular membranes. Surfaces of liquids may be treated as membranes under the appropriate circumstances.

Our development of the behavior of a membrane will parallel our previous discussion of a vibrating string, but in this case the motion is of a two-dimensional object oscillating in a third direction. First, let us consider a membrane in equilibrium in the x - y plane limited by a smooth, closed boundary L under tension T , which acts tangent to the surface of the membrane. In this chapter, we treat external forces acting on the membrane in a direction perpendicular to the x - y plane only, except at the boundary of the membrane. Under the action of such a force, or in the case of an initial perturbation from equilibrium, points on the membrane move to a new position, which we describe by its distance from equilibrium, $u = u(x, y)$ at location (x, y) . The distance of the membrane surface from equilibrium may also vary in time, so that the displacement $u(x, y, t)$ is a function of time as well as location.

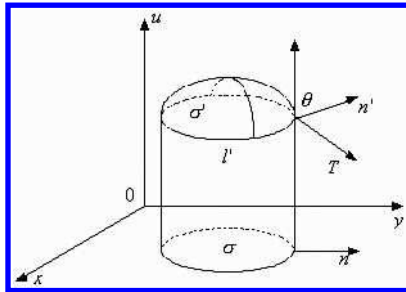


Figure 4.1. Small surface element of a membrane, σ , displaced from equilibrium into stretched element, σ' . The angle θ is between the force of tension T , which is tangent to the curved surface element and the direction of the displacement u . The vectors n and n' are the normal vectors to the surfaces σ and σ' , respectively.

We consider only cases where the curvature of the membrane is small, in which case we can neglect powers of u^2 (and higher orders) and its derivatives: $u^2 \approx 0$, $u_x^2 \approx 0$, etc.

Figure 4.1 shows a small section, σ , of the membrane whose equilibrium position is limited by the closed curve l . When the membrane is displaced from the equilibrium position, this section is deformed to the area σ' , limited by the closed curve l' , as shown in Figure 4.1. The new area σ' at some instant of time is given by

$$\sigma' = \iint_{\sigma} \sqrt{1 + u_x^2 + u_y^2} dx dy \approx \iint_{\sigma} dx dy = \sigma.$$

From this result, we see that, for small oscillations with low curvature, we may neglect changes of area of the membrane. As was the case for small vibrations of a string, this allows us to assume that the tension T in the membrane does not vary with x or y .

4.1 Derivation of the Equations of Motion

4.1.1 Equations of Motion

To derive an equation of motion for the membrane, let us consider its fragment, the deformed area σ' limited by the curve l' . The tension T acting on this area is evenly distributed on the contour l' and is perpendicular to the contour and tangent to the surface of the deformed area. For a segment ds'

of the curve l' , the tension will be Tds' acting on the segment. Since motions of the membrane are constrained to be perpendicular to the x - y plane, we consider the component of the tension in the direction u (perpendicular to the x - y plane), which is $Tds' \cos \theta$, where θ is the angle between T and the direction of the displacement u . For small oscillations of the membrane, $\cos \theta$ is approximately equal to $\partial u / \partial n$, where n is the normal to the curve l , the boundary of the original equilibrium area σ . From this approximation, we have that the component of tension acting on element ds' of contour l' in the direction of displacement u is

$$T \frac{\partial u}{\partial n} ds'.$$

We now integrate over the contour l' to find the component of tension acting on area element σ' and perpendicular to the equilibrium surface as

$$T \int_{l'} \frac{\partial u}{\partial n} ds'.$$

For small oscillations of the membrane, $ds \approx ds'$ (i.e., the boundary l does not deform much as the element σ is stretched). Using Green's formula (see Appendix D), we have, in rectangular coordinates,

$$T \int_l \frac{\partial u}{\partial n} ds = \iint_{\sigma} T \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy. \quad (4.1)$$

The above discussion includes only forces due to the original tension on the membrane. If an additional external force per unit area $F(x, y, t)$ (which may vary in time) acts parallel to the direction $u(x, y, t)$, then the component in the u direction of this force acting on area σ' of the membrane is given by

$$\iint_{\sigma} F(x, y, t) dx dy. \quad (4.2)$$

The two forces in Equations (4.1) and (4.2) cause an acceleration of the area element σ' . If the mass per area of the membrane is given by the surface density, $\rho(x, y)$, the right-hand side of Newton's second law for the motion of this area element becomes

$$\iint_{\sigma} \rho(x, y) \frac{\partial^2 u}{\partial t^2} dx dy.$$

Setting the forces acting on this element equal to the mass times acceleration of the area element, we have

$$\iint_{\sigma} \left[\rho(x, y) \frac{\partial^2 u}{\partial t^2} - T \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + F(x, y, t) \right] dx dy = 0.$$

We began with an arbitrary surface element, σ , from which it follows that

$$\rho(x, y) \frac{\partial^2 u}{\partial t^2} - T \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = F(x, y, t). \quad (4.3)$$

Equation (4.3) is the linear partial differential equation that describes *small, transverse, forced oscillations of a membrane*.

In the case of a membrane of uniform mass density ($\rho = \text{const}$), we may write Equation (4.3) as

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t), \quad (4.4)$$

where $a = \sqrt{T/\rho}$ and $f(x, y, t) = F(x, y, t)/\rho$. In cases for which the external force is absent (i.e., $F(x, y, t) = 0$), then from Equation (4.4), we obtain the *homogeneous* equation for *free oscillations of a uniform membrane*, given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (4.5)$$

If, in addition to the internal tension, the membrane is subject to an external linear restoring force proportional to displacement, we may add a force $F = -\alpha u$ per unit of area of the membrane, where α is the spring coefficient of the ambient material. For such a membrane embedded in a springy or spongy environment, Equation (4.4) becomes

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \gamma u = f(x, y, t), \quad (4.6)$$

where $\gamma = \alpha/\rho$.

If the membrane is embedded in a material that produces a drag on the motion of the membrane, such as is the case for biological membranes, which are normally immersed in a liquid environment, a friction term must be added to Equation (4.4). Friction forces are generally proportional to

velocity, and we then have $F = -ku_t$ as the force per unit area of membrane where k is the coefficient of friction. The equation of oscillation in this case includes the time derivative of displacement, $u_t(x, y, t)$, and we have

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t), \quad (4.7)$$

where $2\kappa = k/\rho$.

Equations (4.4) through (4.7) are all linear partial differential equations of hyperbolic type. In the following discussion, we solve these equations for various cases and give examples. First we consider the physical limitations presented by requirements at the boundaries of the membrane.

4.1.2 Boundary and Initial Conditions

The equations of motion, Equations (4.4), (4.5), (4.6) and (4.7), are not by themselves sufficient to entirely specify the motion of a membrane. Additional conditions need to be specified—namely, initial conditions and boundary conditions.

If the position and velocity of points on the membrane are known at some initial time, $t = 0$, and are given by the functions $\varphi(x, y)$ and $\psi(x, y)$, respectively, we have the *initial conditions*

$$u|_{t=0} = \varphi(x, y) \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x, y). \quad (4.8)$$

As in the case of the vibrating string, we may be given, along with initial conditions, information about the behavior of the membrane at its edges at all times t , in which case we have *boundary conditions*. The following outline presents several variants of conditions on the boundary, L , of the membrane.

1. If the edge of the membrane is rigidly fixed, then the motion of the membrane on the border, L , does not occur, and we have as the boundary condition

$$u|_L = 0,$$

which is referred to as a *fixed edge* boundary condition.

2. If the behavior over time of the displacement, $u(x, y, t)$, of the boundary is given by some function $g(t)$, then we have

$$u|_L = g(t),$$

which is called a *driven edge* boundary condition.

3. In the case of a boundary that is free (e.g., the edges of a flag under small oscillations) so that the displacement is only in a direction perpendicular to the x - y plane, we have *free edge* boundary conditions, given by

$$\left. \frac{\partial u}{\partial n} \right|_L = 0.$$

4. The edge may also be subject to a force with linear density f_1 in the x - y plane, which affects the tension at the boundary. In this case, we have the *stretched edge* boundary condition,

$$\left(-T \frac{\partial u}{\partial n} + f_1 \right) \Big|_L = 0. \quad (4.9)$$

5. If the force density, f_1 , in the stretched edge condition given by Equation (4.10) is a linear spring-like force (e.g., the boundary of a trampoline fixed to its support with springs), we may write $-ku$ for f_1 and we have

$$\left(\frac{\partial u}{\partial n} + hu \right) \Big|_L = 0, \quad (4.10)$$

where $h = k/T$.

6. If the edges to which a membrane is elastically attached are moving in some prescribed way, the right sides of Equations (4.9) and (4.10) will contain some function of time, $g(t)$, describing the motion of the edges. In this case, we have nonhomogeneous boundary conditions.

We may combine all of these conditions (1)–(6) in a generic form given by

$$\alpha \frac{\partial u}{\partial n} + \beta u \Big|_L = g(t). \quad (4.11)$$

The Dirichlet boundary condition corresponds to $\alpha = 0$, $\beta = 1$, in which case we have the driven edge situation. The Neumann boundary condition corresponds to $\alpha = 1$, $\beta = 0$, and we have the stretched edge condition, or the free edge if $g(t) = 0$. Mixed boundary conditions, when both $\alpha \neq 0$ and $\beta \neq 0$, correspond to two cases: if $g(t) = 0$, they are homogeneous, and if $g(t) \neq 0$, they are nonhomogeneous. Clearly, the types of boundary conditions can vary along the boundary, and we will consider such a situation in Section 4.2.

4.2 Oscillations of a Rectangular Membrane

In this section, we solve variations of Equations (4.4) through (4.7) for several specific examples of boundary or initial conditions by using the method of Fourier series. Let us start with a membrane that has an initial equilibrium shape of a rectangle limited by the straight lines $x = 0$, $x = l_x$, $y = 0$, and $y = l_y$ (Figure 4.2).

We begin with the most general case of a membrane subject to friction forces, a linear restoring force, and external forcing, $f(x, y, t)$. From the previous discussion, we see that the equation of motion for such a problem is given by

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \gamma u = f(x, y, t) \quad (4.12)$$

with generic boundary conditions given on the boundary of the rectangle as

$$\begin{aligned} P_1[u] \equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} &= g_1(y, t), & P_2[u] \equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} &= g_2(y, t), \\ P_3[u] \equiv \alpha_3 \frac{\partial u}{\partial y} + \beta_3 u \Big|_{y=0} &= g_3(x, t), & P_4[u] \equiv \alpha_4 \frac{\partial u}{\partial y} + \beta_4 u \Big|_{y=l_y} &= g_4(x, t), \end{aligned} \quad (4.13)$$

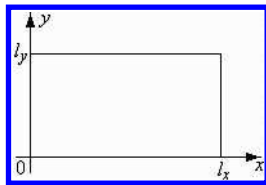


Figure 4.2. Rectangular membrane in its equilibrium position.

where $g_1(y, t), \dots, g_4(x, t)$ are the given functions of time and respective variable, and $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4$, and β_4 are constants subject to the same restrictions from physical arguments that we saw in Chapter 2. We also consider the initial conditions

$$u|_{t=0} = \varphi(x, y) \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x, y), \quad (4.14)$$

where $\varphi(x, y)$ and $\psi(x, y)$ are given functions.

As in the case for the movement of a string, we use the Fourier method and separation of variables to solve Equation (4.12). In a manner exactly parallel to that for the solution of a vibrating string, but instead for an object described initially by two spatial dimensions, we obtain solutions in the form of a series of eigenfunctions of the corresponding Sturm-Liouville problem.

4.2.1 The Fourier Method for Homogeneous Equations with Homogeneous Boundary Conditions

We start with the *homogeneous equation* (i.e., no external forcing)

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \gamma u = 0 \quad (4.15)$$

with *homogeneous boundary conditions*

$$\begin{aligned} P_1[u] &\equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = 0, & P_2[u] &\equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} = 0, \\ P_3[u] &\equiv \alpha_3 \frac{\partial u}{\partial y} + \beta_3 u \Big|_{y=0} = 0, & P_4[u] &\equiv \alpha_4 \frac{\partial u}{\partial y} + \beta_4 u \Big|_{y=l_y} = 0, \end{aligned} \quad (4.16)$$

and initial conditions in Equation (4.14) given by

$$u|_{t=0} = \varphi(x, y), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x, y).$$

To begin, we first assume that nontrivial (nonzero) solutions can be written as the product of two functions, one a function of time the second a function of x and y :

$$u(x, y, t) = V(x, y)T(t). \quad (4.17)$$

Substituting Equation (4.17) into Equation (4.15), we get

$$V(x, y)T''(t) + 2\kappa V(x, y)T'(t) - a^2[V_{xx}(x, y) + V_{yy}(x, y)]T(t) + \gamma V(x, y)T(t) = 0,$$

or, upon rearranging terms,

$$\frac{T''(t) + 2\kappa T'(t) + \gamma T(t)}{a^2 T(t)} = \frac{V_{xx}(x, y) + V_{yy}(x, y)}{V(x, y)}, \quad (4.18)$$

where we have used the shorthand notation for the derivatives in x and y

$$V_{xx}(x, y) \equiv \frac{\partial^2 V}{\partial x^2} \quad \text{and} \quad V_{yy}(x, y) \equiv \frac{\partial^2 V}{\partial y^2}.$$

The left side of Equation (4.18) is a function of t only, and the right side is a function of x and y only, which is possible only if both sides are equal to some constant value. Denoting this constant as $-\lambda$, we have

$$\frac{T''(t) + 2\kappa T'(t) + \gamma T(t)}{a^2 T(t)} \equiv \frac{V_{xx}(x, y) + V_{yy}(x, y)}{V(x, y)} = -\lambda. \quad (4.19)$$

Using the left side of Equation (4.19) for the function $T(t)$, we get the homogeneous linear differential equation of second order

$$T''(t) + 2\kappa T'(t) + (a^2 \lambda + \gamma)T(t) = 0, \quad (4.20)$$

where primes denote derivatives with respect to time. For the function $V(x, y)$, we have the equation

$$V_{xx}(x, y) + V_{yy}(x, y) + \lambda V(x, y) = 0 \quad (4.21)$$

with boundary conditions

$$\begin{aligned} P_1[V] &\equiv \alpha_1 V_x + \beta_1 V|_{x=0} = 0, & P_2[V] &\equiv \alpha_2 V_x + \beta_2 V|_{x=l_x} = 0, \\ P_3[V] &\equiv \alpha_3 V_y + \beta_3 V|_{y=0} = 0, & P_4[V] &\equiv \alpha_4 V_y + \beta_4 V|_{y=l_y} = 0. \end{aligned}$$

To solve Equation (4.21) for $V(x, y)$, we again make the assumption that the variables are independent, and we attempt to separate them by using the substitution

$$V(x, y) = X(x)Y(y). \quad (4.22)$$

Reading Exercise. Show that substituting Equation (4.22) into Equation (4.21) and applying the boundary conditions yields

$$X''(x) + \lambda_x X(x) = 0, \quad (4.23)$$

with boundary conditions

$$\alpha_1 X'(0) + \beta_1 X(0) = 0, \quad \alpha_2 X'(l_x) + \beta_2 X(l_x) = 0;$$

and

$$Y''(y) + \lambda_y Y(y) = 0, \quad (4.24)$$

with boundary conditions

$$\alpha_3 Y'(0) + \beta_3 Y(0) = 0, \quad \alpha_4 Y'(l_y) + \beta_4 Y(l_y) = 0,$$

where λ_x and λ_y are constants from the division of variables linked by the correlation

$$\lambda_x + \lambda_y = \lambda.$$

Hint. The boundary conditions for $X(x)$ and $Y(y)$ follow from the corresponding conditions for the function $V(x, y)$. For example, from the condition

$$\begin{aligned} \alpha_1 V_x(0, y) + \beta_1 V(0, y) &= \alpha_1 X'(0)Y(y) + \beta_1 X(0)Y(y) \\ &= [\alpha_1 X'(0) + \beta_1 X(0)] Y(y) = 0 \end{aligned}$$

it follows that

$$\alpha_1 X'(0) + \beta_1 X(0) = 0,$$

since $Y(y) \neq 0$ (i.e., we are searching for nontrivial solutions).

Solutions to Equations (4.23) and (4.24) (given in Appendix A) depend on the boundary conditions and have the generic form

$$X(x) = C_1 \cos \sqrt{\lambda_x} x + C_2 \sin \sqrt{\lambda_x} x \quad (4.25)$$

and

$$Y(y) = D_1 \cos \sqrt{\lambda_y} y + D_2 \sin \sqrt{\lambda_y} y.$$

Applying the boundary conditions for Equation (4.23) allows us to determine the coefficients C_1 and C_2 .

Reading Exercise. Show that an application of the boundary conditions for Equation (4.23) by using the generic solutions in Equation (4.25) lead to the following conditions for C_1 and C_2 :

$$\begin{cases} C_1\beta_1 + C_2\alpha_1\sqrt{\lambda_x} = 0, \\ C_1 \left[-\alpha_2\sqrt{\lambda_x} \sin \sqrt{\lambda_x}l_x + \beta_2 \cos \sqrt{\lambda_x}l_x \right] \\ + C_2 \left[\alpha_2\sqrt{\lambda_x} \cos \sqrt{\lambda_x}l_x + \beta_2 \sin \sqrt{\lambda_x}l_x \right] = 0. \end{cases} \quad (4.26)$$

The system of linear homogeneous Equations (4.26) has nonzero solutions only in the case that its determinant is zero. It is left for the reader as a reading exercise to show that setting the determinant equal to zero yields

$$(\alpha_1\alpha_2\lambda_x + \beta_1\beta_2) \tan \sqrt{\lambda_x}l_x - \sqrt{\lambda_x}(\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \quad (4.27)$$

As was the case previously with the string, Equation (4.27) has an infinite number of roots $\{\lambda_{xn}\}$. For each root λ_{xn} , we obtain a nonzero solution of Equation (4.26), which yields

$$C_1 = \frac{\alpha_1\sqrt{\lambda_{xn}}}{\sqrt{\lambda_{xn}\alpha_1^2 + \beta_1^2}} \quad \text{and} \quad C_2 = -\frac{\beta_1}{\sqrt{\lambda_{xn}\alpha_1^2 + \beta_1^2}}.$$

Similarly, boundary conditions applied to Equation (4.24) for coefficients D_1 and D_2 yield the system of linear homogeneous equations

$$\begin{cases} D_1\beta_3 + D_2\alpha_3\sqrt{\lambda_y} = 0, \\ D_1 \left[-\alpha_4\sqrt{\lambda_y} \sin \sqrt{\lambda_y}l_y + \beta_4 \cos \sqrt{\lambda_y}l_y \right] \\ + D_2 \left[\alpha_4\sqrt{\lambda_y} \cos \sqrt{\lambda_y}l_y + \beta_4 \sin \sqrt{\lambda_y}l_y \right] = 0. \end{cases} \quad (4.28)$$

Again, we have nonzero solutions to Equations (4.28) only in the case that the determinant is zero, which gives

$$(\alpha_3\alpha_4\lambda_y + \beta_3\beta_4) \tan \sqrt{\lambda_y}l_y - \sqrt{\lambda_y}(\alpha_3\beta_4 - \beta_3\alpha_4) = 0$$

with an infinite number of roots $\{\lambda_{yn}\}$. For each root λ_{yn} , we find nonzero solutions of Equation (4.28) to be

$$D_1 = \frac{\alpha_3\sqrt{\lambda_{ym}}}{\sqrt{\lambda_{ym}\alpha_3^2 + \beta_3^2}} \quad \text{and} \quad D_2 = -\frac{\beta_3}{\sqrt{\lambda_{ym}\alpha_3^2 + \beta_3^2}}.$$

Collecting the above results we have eigenfunctions for the Sturm-Liouville problem defined by Equations (4.23) and (4.24) given by

$$X_n(x) = \frac{1}{\sqrt{\alpha_1^2 \lambda_{xn} + \beta_1^2}} \left[\alpha_1 \sqrt{\lambda_{xn}} \cos \sqrt{\lambda_{xn}} l_x - \beta_1 \sin \sqrt{\lambda_{xn}} l_x \right] \quad (4.29)$$

and

$$Y_m(y) = \frac{1}{\sqrt{\alpha_3^2 \lambda_{ym} + \beta_3^2}} \left[\alpha_3 \sqrt{\lambda_{ym}} \cos \sqrt{\lambda_{ym}} l_y - \beta_3 \sin \sqrt{\lambda_{ym}} l_y \right] \quad (4.30)$$

(square roots should be taken with positive signs).

The eigenvalues of the problem are

$$\lambda_{xn} = \left[\frac{\mu_{xn}}{l_x} \right]^2 \quad \text{and} \quad \lambda_{ym} = \left[\frac{\mu_{ym}}{l_y} \right]^2, \quad (4.31)$$

where μ_{xn} is the n th root of the equation

$$\tan \mu_x = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) l_x \mu_x}{\mu_x^2 \alpha_1 \alpha_2 + l_x^2 \beta_1 \beta_2}, \quad (4.32)$$

and μ_{ym} is the m th root of the equation

$$\tan \mu_y = \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3) l_y \mu_y}{\mu_y^2 \alpha_3 \alpha_4 + l_y^2 \beta_3 \beta_4}. \quad (4.33)$$

The norms of the eigenfunctions are given by

$$\|X_n\|^2 = \int_0^{l_x} X_n^2(x) dx = \frac{1}{2} \left[l_x + \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)(\lambda_{xn} \alpha_1 \alpha_2 - \beta_1 \beta_2)}{(\lambda_{xn} \alpha_1^2 + \beta_1^2)(\lambda_{xn} \alpha_2^2 + \beta_2^2)} \right], \quad (4.34)$$

and

$$\|Y_m\|^2 = \int_0^{l_y} Y_m^2(y) dy = \frac{1}{2} \left[l_y + \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3)(\lambda_{ym} \alpha_3 \alpha_4 - \beta_3 \beta_4)}{(\lambda_{ym} \alpha_3^2 + \beta_3^2)(\lambda_{ym} \alpha_4^2 + \beta_4^2)} \right], \quad (4.35)$$

in which case functions $X_n(x)$ and $Y_m(y)$ are bounded by the values ± 1 .

Returning to the two-dimensional problem defined by Equation (4.21) and subsequent boundary conditions, we make the following observations based on the linearity of the original differential Equation (4.15). If λ_{xn} and $X_n(x)$ are eigenvalues and eigenfunctions of Equation (4.23), and λ_{ym} and $Y_m(y)$ are eigenvalues and eigenfunctions of Equation (4.24), then

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} \quad (4.36)$$

and

$$V_{nm}(x, y) = X_n(x)Y_m(y) \quad (4.37)$$

are eigenvalues and eigenvectors, respectively, of the problem in Equation (4.21). Note that the success of the method of separation of variables is based on the fact that the sets of orthogonal functions $X_n(x)$ and $Y_m(y)$ satisfy the same boundary conditions as $u(x, y, t)$. The functions $V_{nm}(x, y)$ are orthogonal, and the norms are given by

$$\|V_{nm}\|^2 = \|X_n\|^2 \|Y_m\|^2. \quad (4.38)$$

The system of eigenfunctions, V_{nm} , given in Equation (4.37) form a complete set of basis functions for a two-dimensional rectangular membrane. By this, we mean that any smooth (i.e., twice-differentiable) shape of the deformed rectangular membrane with the generic boundary conditions given above can be expanded in a converging series of the functions V_{nm} .

We now return to the equation describing the time evolution of the membrane, Equation (4.20). This is an ordinary linear differential equation of second order that we have seen previously for one-dimensional oscillations. It should be clear in this case, however, that $T(t)$ now depends on two indexes corresponding to the eigenfunctions $X(x)$ and $Y(y)$. Specifically, we may write $\lambda = \lambda_{nm}$ and denote $T(t)$ as $T_{nm}(t)$, which is defined by

$$T_{nm}(t) = a_{nm}y_{nm}^{(1)}(t) + b_{nm}y_{nm}^{(2)}(t), \quad (4.39)$$

where a_{nm} and b_{nm} are arbitrary constants. Similar to the case for the one-dimensional problem (Equations (3.105) and (3.106)), we have

$$y_{nm}^{(1)}(t) = \begin{cases} e^{-\kappa t} \cos \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \cosh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases} \quad (4.40)$$

and

$$y_{nm}^{(2)}(t) = \begin{cases} e^{-\kappa t} \sin \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \sinh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases} \quad (4.41)$$

where $\omega_{nm} = \sqrt{|a^2 \lambda_{nm} + \gamma - \kappa^2|}$ if $\kappa^2 \neq a^2 \lambda_{nm} + \gamma$, and $\omega_{nm} = 1$ otherwise.

Reading Exercise. Following the arguments used in Chapter 3 for the one-dimensional case, verify Equations (4.40) and (4.41).

Thus, eigenfunctions for the free oscillations of a rectangular membrane may be written as

$$u_{nm}(x, y, t) = T_{nm}(t)V_{nm}(x, y) = \left[a_{nm}y_{nm}^{(1)}(t) + b_{nm}y_{nm}^{(2)}(t) \right] V_{nm}(x, y), \quad (4.42)$$

which will form a complete set of solutions to equation (4.15) satisfying the boundary conditions in Equation (4.16).

To satisfy the initial conditions of Equation (4.14), we first expand the displacement $u(x, y, t)$ in a Fourier series as

$$u(x, y, t) = \sum_n \sum_m \left[a_{nm}y_{nm}^{(1)}(t) + b_{nm}y_{nm}^{(2)}(t) \right] V_{nm}(x, y). \quad (4.43)$$

If this series, as well as the series obtained by twice differentiation term by term with respect the variables x , y , and t , converges uniformly, then its sum will be the solution of Equation (4.15) satisfying boundary conditions (4.16). Under the same assumptions, we may also expand the initial conditions given by Equation (4.14) by using the same set of basis functions:

$$u|_{t=0} = \varphi(x, y) = \sum_n \sum_m a_{nm} V_{nm}(x, y), \quad (4.44)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x, y) = \sum_n \sum_m [\omega_{nm} b_{nm} - \kappa a_{nm}] V_{nm}(x, y). \quad (4.45)$$

Equations (4.44) and (4.45) show that functions $\varphi(x, y)$ and $\psi(x, y)$ can be expanded in a complete set of functions, $V_{nm}(x, y)$, which form the solution of the Sturm-Liouville boundary value problem of Equation (4.21). Again, supposing the series in Equations (4.44) and (4.45) converge uni-

formly, we may determine the coefficients a_{nm} and b_{nm} by making use of the orthogonality of the eigenfunctions $V_{nm}(x, y)$. First, we multiply Equations (4.44) and (4.45) by $V_{nm}(x, y)$ and integrate over x from 0 to l_x and over y from 0 to l_y . This yields the Fourier coefficients

$$a_{nm} = \frac{1}{\|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} \varphi(x, y) V_{nm}(xy) dx dy \quad (4.46)$$

and

$$b_{nm} = \frac{1}{\omega_{nm}} \left[\frac{1}{\|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} \psi(x, y) V_{nm}(xy) dx dy + \kappa a_{nm} \right]. \quad (4.47)$$

The coefficients a_{nm} and b_{nm} substituted into the series (4.43) yield a complete solution to Equation (4.15) with boundary conditions in Equation (4.16), and initial conditions (4.14), under the assumption that the series in Equation (4.43) converges uniformly and can be differentiated term by term in x , y , and t . Therefore, we may say that Equation (4.43) completely describes the *free oscillations* of a membrane. This solution thus has the form of a Fourier series on the orthogonal system of functions $\{V_{nm}(x, y)\}$, each function of which represents a mode characterized by two numbers, n and m . This series converges under “reasonable” assumptions about initial and boundary conditions.

Particular solutions $u_{nm}(x, y, t) = T_{nm}(t)V_{nm}(x, y)$, where the time and space components are separate, are called *standing wave* solutions and are analogous to standing waves on a one-dimensional string. The profile of the standing wave is defined by the function $V_{nm}(x, y)$ which varies as a function of time $T_{nm}(t)$. Lines along which $V_{nm}(x, y) = 0$ do not change with time and are called *node lines* of the standing wave. Loose sand placed on a vibrating membrane will collect along node lines because there is no motion at those locations. Locations where $V_{nm}(x, y)$ has a relative maximum or minimum at some instant of time are called *antinodes* of the standing wave. The general solution, $u(x, y, t)$, is an infinite sum of these standing waves as was the case for the vibrating string. This property of being able to construct arbitrary shapes from a sum of component waves (or modes) is referred to as the *superposition of standing waves* and is a general property of linear systems of all dimensions.

Consider the simple case of a rectangular membrane with its sides clamped at the boundary. The vibrations are caused only by the initial conditions; thus, we want to solve the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (4.48)$$

satisfying the boundary conditions

$$\begin{aligned} u(0, y, t) &= u(l_x, y, t) = 0, \\ u(x, 0, t) &= u(x, l_y, t) = 0, \end{aligned}$$

and initial conditions in Equation (4.14).

We leave it to the reader to check as a reading exercise that, using the results from the generic case investigated above, eigenvalues and eigenfunctions for this problem are

$$\lambda_{xn} = \left(\frac{n\pi}{l_x} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l_x}, \quad \|X_n\|^2 = \frac{l_x}{2}, \quad n = 1, 2, 3, \dots,$$

and

$$\lambda_{ym} = \left(\frac{m\pi}{l_y} \right)^2, \quad Y_m(y) = \sin \frac{m\pi y}{l_y}, \quad \|Y_m\|^2 = \frac{l_y}{2}, \quad m = 1, 2, 3, \dots,$$

with

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right),$$

and we have that

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} \quad \text{with} \quad \|V_{nm}\|^2 = \frac{l_x l_y}{4}. \quad (4.49)$$

It is obvious that these functions form a complete set of orthogonal functions for oscillations of the rectangular membrane. The time evolution function is

$$T_{nm}(t) = a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t,$$

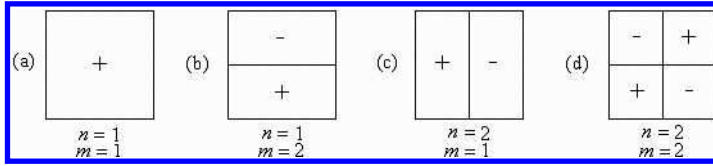


Figure 4.3. Modes of vibration $V_{nm}(x, y) = X_n(x)Y_m(y)$. Plus signs indicate motion out of the page; minus signs indicate simultaneous motion into the page.

where the frequencies are

$$\omega_{nm} = a \lambda_{nm} = a\pi \sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}.$$

Each pair of integers (n, m) corresponds to a particular characteristic mode (called a normal mode) of vibration of the membrane. An arbitrary membrane deflection may then be represented as a superposition of normal modes:

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{nm} V_{nm} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} \cos(\omega_{nm} t + \delta_{nm}),$$

where we have introduced coefficients c_{nm} and phase shifts δ_{nm} via the relations $a_{nm} = c_{nm} \cos \delta_{nm}$ and $b_{nm} = -c_{nm} \sin \delta_{nm}$.

If the membrane vibrates in one of its normal modes, then all points on the membrane participate in harmonic motion with frequency ω_{nm} . As an example, consider the (2,1) mode (i.e., $n = 2, m = 1$). The eigenfunction is

$$V_{21}(x, y) = X_2(x)Y_1(y) = \sin \frac{2\pi x}{l_x} \sin \frac{\pi y}{l_y}.$$

The only nodal line is the straight line $x = l_x/2$. Similarly, the (1,2) mode ($n = 1, m = 2$) has the nodal line $y = l_y/2$ (see Figure 4.3). Nodal lines split the membrane into zones, and all points of each zone move with the same phase, all up or all down (labeled with + and -) at some instant (although not necessarily with the same amplitude).

Generally speaking, each node vibrates with its own frequency, ω_{nm} . However, if l_y/l_x is a rational number, two or more modes could possess the same frequency. As an example, consider a square membrane on which $l_x = l_y$, in which case, $\omega_{12} = \omega_{21}$. This frequency is said to be

twofold degenerate, by which we mean there are two linearly independent eigenfunctions corresponding the same eigenvalue.

Next, we consider two examples of physical problems for free oscillations of a membrane with homogeneous boundary conditions.

Example 4.1. Find the transverse oscillations of a uniform rectangular membrane ($0 \leq x \leq l_x, 0 \leq y \leq l_y$) having fixed edges and with an initial displacement of

$$u(x, y, 0) = Axy(l_x - x)(l_y - y),$$

assuming interactions with the surrounding medium can be neglected and the initial velocities of points on the membrane are zero.

Solution. This is an example of the case discussed above with the specified initial conditions; thus, the problem reduces to solutions of Equation (4.48), rewritten as

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0,$$

with initial and boundary conditions

$$u(x, y, 0) = Axy(l_x - x)(l_y - y), \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

$$u(0, y, t) = u(l_x, y, t) = u(x, 0, t) = u(x, l_y, t) = 0.$$

As we obtained in Equation (4.49), the eigenfunctions are

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y}, \quad \text{with } \|V_{nm}\|^2 = \frac{l_x l_y}{4}.$$

The three-dimensional view shown in Figure 4.4 depicts two eigenfunctions, $V_{11}(x, y)$ and $V_{22}(x, y)$, chosen as examples. The values of λ for these modes are $\lambda_{11} = 0.891$ and $\lambda_{22} = 3.564$, respectively.

We leave it to the reader as a reading exercise to check, by using Equations (4.46) and (4.47), that the expressions for the coefficients of the series are

$$a_{nm} = \begin{cases} \frac{64A l_x^2 l_y^2}{\pi^2 n^2 m^2} & \text{if } n \text{ and } m \text{ are odd,} \\ 0 & \text{if } n \text{ or } m \text{ are even,} \end{cases}$$

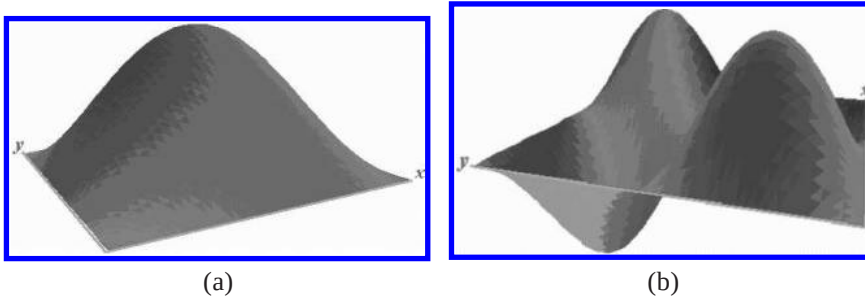


Figure 4.4. Eigenfunctions (a) $V_{11}(x, y)$ and (b) $V_{22}(x, y)$ for a membrane with fixed edges.

and $b_{nm} = 0$, and the time evolution is given by

$$T_{nm} = a_{nm} \cos \omega_{nm} t = \begin{cases} \frac{64Al_x^2 l_y^2}{\pi^2 n^2 m^2} \cos \omega_{nm} t & \text{if } n \text{ and } m \text{ are odd,} \\ 0 & \text{if } n \text{ or } m \text{ are even,} \end{cases}$$

where $\omega_{nm} = a \lambda_{nm} = a\pi \sqrt{(n/l_x)^2 + (m/l_y)^2}$. Consequently, the displacements of the membrane as a function of time for this problem can be expressed by the series

$$u(x, y, t) = \frac{64Al_x^2 l_y^2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos \omega_{(2n-1)(2m-1)} t}{(2n-1)^2 (2m-1)^2} \sin \frac{(2n-1)\pi x}{l_x} \sin \frac{(2m-1)\pi y}{l_y}.$$

Figure 4.5 shows two snapshots of the solution at the times $t = 7$ and $t = 10$. This solution was obtained with the program Waves for the case

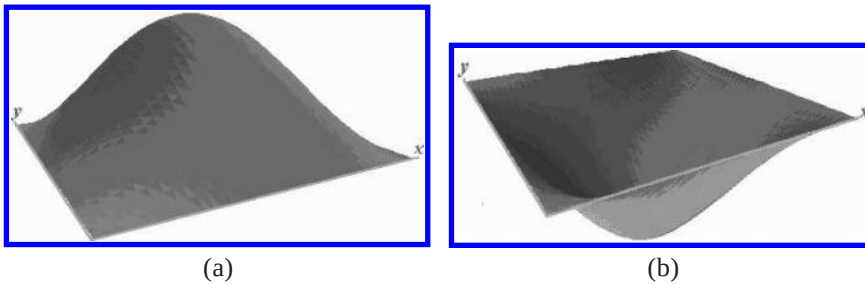


Figure 4.5. Graph of the membrane in Example 4.1 at (a) $t = 7$, (b) $t = 10$.

$a^2 = 1$, $l_x = 4$, $l_y = 6$, and $A = 0.01$. A description of the program Waves and directions for its use are given in Appendix E.

Example 4.2. A uniform rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) has edges at $x = l_x$ and $y = l_y$ that are free, and edges at $x = 0$ and $y = 0$ that are firmly fixed. Find the transverse oscillations of the membrane caused by an initial displacement $u(x, y, 0) = Axy$, assuming interactions with the surrounding medium can be neglected and the initial velocities of points on the membrane are zero.

Solution. This problem reduces to finding the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0,$$

with initial and boundary conditions

$$u(x, y, 0) = Axy, \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

$$u(0, y, t) = \frac{\partial u}{\partial x}(l_x, y, t) = 0, \quad u(0, y, t) = \frac{\partial u}{\partial y}(x, l_y, t) = 0.$$

Eigenvalues and eigenfunctions of the problem are

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{(2n-1)^2}{4l_x^2} + \frac{(2m-1)^2}{4l_y^2} \right], \quad n, m = 1, 2, \dots$$

and

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{(2n-1)\pi x}{2l_x} \sin \frac{(2m-1)\pi y}{2l_y}, \quad \|V_{nm}\|^2 = \frac{l_x l_y}{4}.$$

The three-dimensional picture shown in Figure 4.6 depicts two eigenfunctions, $V_{11}(x, y)$ and $V_{22}(x, y)$, chosen as examples for this problem. The values of λ are $\lambda_{11} = 0.223$ and $\lambda_{22} = 2.005$.

By using Equations (4.46) and (4.47), we obtain the coefficients

$$a_{nm} = (-1)^{n+m} \frac{64Al_x l_y}{\pi^4 (2n-1)^2 (2m-1)^2} \text{ and } b_{nm} = 0.$$

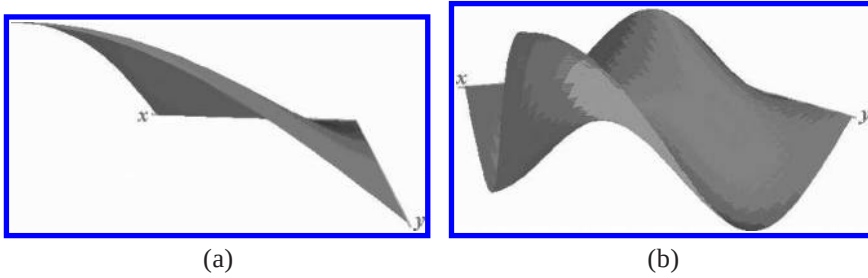


Figure 4.6. Sample eigenfunctions (a) $V_{11}(x, y)$ and (b) $V_{22}(x, y)$ for Example 4.2.

In this case, displacements of the membrane as a function of time are expressed by the series

$$u(x, y, t) = \frac{64Al_x l_y}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m} \cos \omega_{nm} t}{(2n-1)^2 (2m-1)^2} \sin \frac{(2n-1)\pi x}{2l_x} \sin \frac{(2m-1)\pi y}{2l_y},$$

where

$$\omega_{nm} = a\lambda_{nm} = a\pi \sqrt{\frac{(2n-1)^2}{4l_x^2} + \frac{(2m-1)^2}{4l_y^2}}.$$

Figure 4.7 shows two snapshots of the solution at the times $t = 7$ and $t = 10$. This solution was obtained with the program Waves for the case $A = 0.01$, $a^2 = 1$, $l_x = 4$, and $l_y = 6$. A description of the program Waves and directions for its use are given in Appendix E.

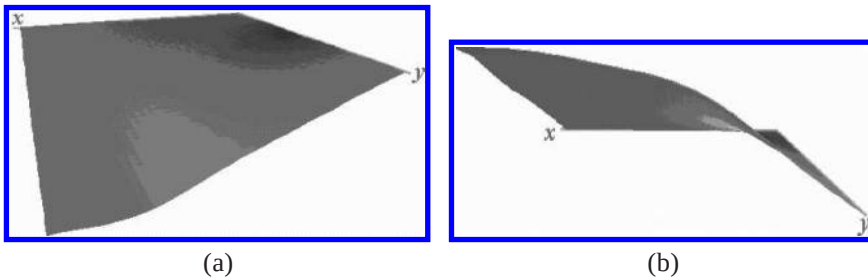


Figure 4.7. Graph of the membrane in Example 4.2 at (a) $t = 7$, (b) $t = 10$.

4.2.2 The Fourier Method for Nonhomogeneous Equations

Building on the previous sections, we now consider the problem of solutions of the *nonhomogeneous* linear Equation (4.12) for a two-dimensional membrane:

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \gamma u = f(x, y, t),$$

where $f(x, y, t)$ is a given function. First we search for solutions that satisfy the *homogeneous boundary conditions* in Equation (4.16) given by

$$\begin{aligned} P_1[u] &\equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = 0, & P_2[u] &\equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} = 0, \\ P_3[u] &\equiv \alpha_3 \frac{\partial u}{\partial y} + \beta_3 u \Big|_{y=0} = 0, & P_4[u] &\equiv \alpha_4 \frac{\partial u}{\partial y} + \beta_4 u \Big|_{y=l_y} = 0, \end{aligned}$$

and *nonhomogeneous initial conditions* given in Equation (4.14) as

$$u|_{t=0} = \varphi(x, y) \quad \text{and} \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(x, y).$$

Because the equation of membrane oscillations is linear, the displacement, $u(x, y, t)$, may be written as the sum

$$u(x, y, t) = u_1(x, y, t) + u_2(x, y, t),$$

where $u_1(x, y, t)$ is the solution of the *homogeneous* equation with *homogeneous* boundaries and *nonhomogeneous* initial conditions:

$$\frac{\partial^2 u_1}{\partial t^2} + 2\kappa \frac{\partial u_1}{\partial t} - a^2 \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) + \gamma u_1 = 0, \quad (4.50)$$

$$\begin{aligned} P_1[u_1] &\equiv \alpha_1 \frac{\partial u_1}{\partial x} + \beta_1 u_1 \Big|_{x=0} = 0, & P_2[u_1] &\equiv \alpha_2 \frac{\partial u_1}{\partial x} + \beta_2 u_1 \Big|_{x=l_x} = 0, \\ P_3[u_1] &\equiv \alpha_3 \frac{\partial u_1}{\partial y} + \beta_3 u_1 \Big|_{y=0} = 0, & P_4[u_1] &\equiv \alpha_4 \frac{\partial u_1}{\partial y} + \beta_4 u_1 \Big|_{y=l_y} = 0, \end{aligned} \quad (4.51)$$

$$u_1|_{t=0} = \varphi(x, y), \quad \left. \frac{\partial u_1}{\partial t} \right|_{t=0} = \psi(x, y). \quad (4.52)$$

The function $u_2(x, y, t)$ is the solution of the *nonhomogeneous* equation with *homogeneous* boundary conditions and initial conditions:

$$\frac{\partial^2 u_2}{\partial t^2} + 2\kappa \frac{\partial u_2}{\partial t} - a^2 \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) + \gamma u_2 = f(x, y, t), \quad (4.53)$$

$$\begin{aligned} P_1[u_2] &\equiv \alpha_1 \frac{\partial u_2}{\partial x} + \beta_1 u_2 \Big|_{x=0} = 0, & P_2[u_2] &\equiv \alpha_2 \frac{\partial u_2}{\partial x} + \beta_2 u_2 \Big|_{x=l_x} = 0, \\ P_3[u_2] &\equiv \alpha_3 \frac{\partial u_2}{\partial y} + \beta_3 u_2 \Big|_{y=0} = 0, & P_4[u_2] &\equiv \alpha_4 \frac{\partial u_2}{\partial y} + \beta_4 u_2 \Big|_{y=l_y} = 0, \end{aligned} \quad (4.54)$$

$$u_2|_{t=0} = 0, \quad \left. \frac{\partial u_2}{\partial t} \right|_{t=0} = 0. \quad (4.55)$$

In other words, the solution $u_1(x, y, t)$ is for the case of *free oscillations* (i.e., such oscillations that occur only as a consequence of an initial perturbation), and the solution $u_2(x, y, t)$ is for the case of *forced oscillations* (i.e., such oscillations that occur under the action of an external force $f(x, y, t)$ when initial perturbations are absent).

The problem of free oscillations was considered in the previous section, for which case the solution $u_1(x, y, t)$ is known. To proceed, we need only to find the solution $u_2(x, y, t)$ for forced oscillations. As in the case for free oscillations, we may expand $u_2(x, y, t)$ in the series

$$u_2(x, y, t) = \sum_n \sum_m T_{nm}(t) V_{nm}(x, y), \quad (4.56)$$

where $V_{nm}(x, y)$ are eigenfunctions of the corresponding homogeneous boundary problem and $T_{nm}(t)$ are, at this stage, unknown functions of t . Any choice of functions $T_{nm}(t)$ satisfies the homogeneous boundary conditions in Equation (4.54) for the function $u_2(x, y, t)$ because the functions $V_{nm}(x, y)$ satisfy these conditions.

To find the functions $T_{nm}(t)$ we proceed as follows. Substituting the series in Equation (4.56) into Equation (4.53), we have

$$\sum_n \sum_m [T''_{nm}(t) + 2\kappa T'_{nm}(t) + (a^2 \lambda_{nm} + \gamma)T_{nm}(t)] V_{nm}(x, y) = f(x, y, t). \quad (4.57)$$

We may also expand the function $f(x, y, t)$ in a Fourier series by using the basis functions $V_{nm}(x, y)$ on the rectangle $[0, l_x; 0, l_y]$:

$$f(x, y, t) = \sum_n \sum_m f_{nm}(t) V_{nm}(x, y), \quad (4.58)$$

where the coefficients of expansion are given by

$$f_{nm}(t) = \frac{1}{\|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} f(x, y, t) V_{nm}(x, y) dx dy. \quad (4.59)$$

Comparing the expansions in Equations (4.57) and (4.58) for the same function $f(x, y, t)$, we obtain a differential equation for functions $T_{nm}(t)$:

$$T''_{nm}(t) + 2\kappa T'_{nm}(t) + (a^2 \lambda_{nm} + \gamma)T_{nm}(t) = f_{nm}(t). \quad (4.60)$$

The solution $u_2(x, y, t)$, defined by the series in Equation (4.56) and satisfying the initial conditions in Equation (4.55) require that the functions $T_{nm}(t)$ in turn satisfy the conditions

$$T_{nm}(0) = 0, \quad T'_{nm}(0) = 0. \quad (4.61)$$

The solution of the Cauchy problem defined by Equations (4.60) and (4.61) may be written as

$$T_{nm}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}(\tau) Y_{nm}(t - \tau) d\tau, \quad (4.62)$$

where

$$Y_{nm}(t) = \begin{cases} e^{-\kappa t} \sin \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \sinh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases} \quad (4.63)$$

with $\omega_{nm} = \sqrt{|a^2 \lambda_{nm} + \gamma - \kappa^2|}$ if $\kappa^2 \neq a^2 \lambda_{nm} + \gamma$, and $\omega_{nm} = 1$ otherwise.

Reading Exercise. Verify the formulas given in Equations (4.62) and (4.63).

We may substitute the expression for $f_{nm}(p)$ given in Equation (4.59) to yield, finally,

$$T_{nm}(t) = \frac{1}{\omega_{nm} \|V_{nm}\|^2} \int_0^t d\tau \int_0^{l_x} \int_0^{l_y} f(x, y, \tau) V_{nm}(x, y) Y_{nm}(t - \tau) dx dy. \quad (4.64)$$

Substituting the above formulas for $T_{nm}(t)$ into the series (4.56) yields the solution of the boundary value problem defined by Equations (4.53) through (4.55) under the condition that the series in Equation (4.56) and the series obtained from Equation (4.56) by term-by-term differentiation (up to second order with respect to x , y , and t) converge uniformly. Thus, the solution of the original problem of forced oscillations with zero initial conditions is given by

$$\begin{aligned} u(x, y, t) &= u_1(x, y, t) + u_2(x, y, t) \\ &= \sum_n \sum_m \left\{ T_{nm}(t) + \left[a_{nm} y_{nm}^{(1)}(t) + b_{nm} y_{nm}^{(2)}(t) \right] \right\} V_{nm}(x, y), \end{aligned}$$

where the coefficients $T_{nm}(t)$ are defined by Equation (4.62) and a_{nm} and b_{nm} are defined in Section 4.2.1 for free oscillations.

We now consider examples of solutions of physical problems involving nonhomogeneous equations of oscillations with homogeneous boundary conditions.

Example 4.3. Find the transverse oscillations of a rectangular membrane $[0, l_x; 0, l_y]$ with fixed edges, subjected to a transverse driving force

$$F(t) = A \sin \omega t,$$

attached at the point (x_0, y_0) , $0 < x_0 < l_x$, $0 < y_0 < l_y$. Assume the reaction of the surrounding medium can be ignored.

Solution. The boundary problem may be defined as

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{A}{\rho} \delta(x - x_0) \delta(y - y_0) \sin \omega t$$

with initial conditions

$$u(x, y, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

and Dirichlet homogeneous boundary conditions

$$u(0, y, t) = u(l_x, y, t) = 0 \quad \text{and} \quad u(x, 0, t) = u(x, l_y, t) = 0.$$

From the previous development, eigenvalues and eigenfunctions for these boundary conditions are

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right),$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y}, \quad \|V_{nm}\|^2 = \frac{l_x l_y}{4}.$$

Using the initial conditions, $\varphi(x, y) = \psi(x, y) = 0$, the solution $u(x, y, t)$ is defined by the series

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{nm}(t) \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y},$$

where

$$T_{nm}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}(\tau) \sin \omega_{nm}(t - \tau) d\tau,$$

$$f_{nm}(t) = \frac{4A}{\rho l_x l_y} \sin \omega t \sin \frac{n\pi x_0}{l_x} \sin \frac{m\pi y_0}{l_y},$$

and

$$\omega_{nm} = a \lambda_{nm} = a\pi \sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}.$$

Figure 4.8 shows the solution profile $u(x, 2.5, t)$ for Example 4.3. This solution was obtained with the program Waves for the case $a^2 = 1$, $l_x = 4$, $l_y = 6$, $\rho = 1$, $A = 0.5$, $x_0 = 1.5$, $y_0 = 2.5$, and $\omega = 1.5$ (the frequency of the external force).

If the frequency of the driving force is not equal to any of the natural frequencies of the membrane (i.e., $\omega \neq \omega_{nm}$, $n, m = 1, 2, 3, \dots$), then

$$T_{nm}(t) = \frac{4A}{\rho l_x l_y (\omega_{nm}^2 - \omega^2)} \sin \frac{n\pi x_0}{l_x} \sin \frac{m\pi y_0}{l_y} \left[\sin \omega t - \frac{\omega}{\omega_{nm}} \sin \omega_{nm} t \right]$$

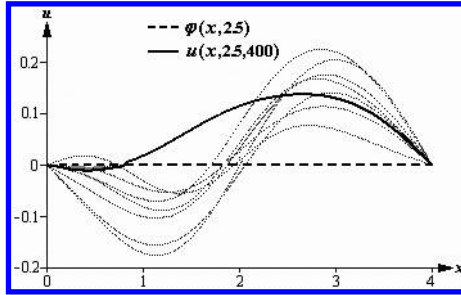


Figure 4.8. Solution profile $u(x, 2.5, t)$ for Example 4.3 at a driving frequency other than resonance.

and

$$u(x, y, t) = \frac{4A}{\rho l_x l_y} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(\omega_{nm}^2 - \omega^2)} \left[\sin \omega t - \frac{\omega}{\omega_{nm}} \sin \omega_{nm} t \right] \\ \times \sin \frac{n\pi x_0}{l_x} \sin \frac{m\pi y_0}{l_y} \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y}.$$

In the case of *resonance*, where the frequency of the driving force *does* coincide with one of the normal mode frequencies of the membrane (n_0, m_0) (i.e., $\omega = \omega_{n_0 m_0}$), we have

$$T_{n_0 m_0}(t) = \frac{2A}{\rho l_x l_y \omega} \sin \frac{n_0 \pi x_0}{l_x} \sin \frac{m_0 \pi y_0}{l_y} \left[\sin \omega t - \omega t \frac{\omega}{\omega_{n_0 m_0}} \cos \omega t \right]$$

and

$$u(x, y, t) = \frac{4A}{\rho l_x l_y} \sum_{n \neq n_0}^{\infty} \sum_{m \neq m_0}^{\infty} \frac{1}{(\omega_{nm}^2 - \omega^2)} \left[\sin \omega t - \frac{\omega}{\omega_{nm}} \sin \omega_{nm} t \right] \\ \times \sin \frac{n\pi x_0}{l_x} \sin \frac{m\pi y_0}{l_y} \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} \\ + \frac{2A}{\rho l_x l_y \omega} \left[\sin \omega t - \omega t \frac{\omega}{\omega_{n_0 m_0}} \cos \omega t \right] \sin \frac{n_0 \pi x_0}{l_x} \sin \frac{m_0 \pi y_0}{l_y} \sin \frac{n_0 \pi x}{l_x} \sin \frac{m_0 \pi y}{l_y}.$$

Figure 4.9 shows the solution profile $u(x, 2.5, t)$ for the case of resonance where the frequency of the external force $\omega = \omega_{23} = \pi/\sqrt{2}$. Other parameters are the same as in Figure 4.8.

If the frequency, $\omega_{n_0 m_0}$, is a multiple of a natural frequency, then instead of one resonance term appearing in the solution there will be several.

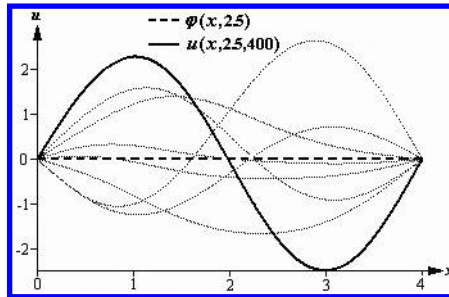


Figure 4.9. Solution profile $u(x, 2.5, t)$ for Example 4.3 at resonance.

Example 4.4. Find the oscillations of the surface of a body of water in a reservoir with a rectangular surface $[0, l_x; 0, l_y]$ under the action of a variable external pressure on the surface given by

$$p_0(x, y, t) = -A\omega \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y} \sin \omega t,$$

if the depth of water in the equilibrium state is h .

Solution. The vertical displacement is described by Equation (4.4):

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -\frac{A\omega}{\rho} \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y} \sin \omega t, \quad a^2 = gh$$

under initial conditions

$$u(x, y, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(l_x, y, t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 0, t) = \frac{\partial u}{\partial y}(x, l_y, t) = 0.$$

Eigenvalues and eigenfunctions for these boundary conditions are

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right),$$

$$V_{nm}(x, y) = \cos \frac{n\pi x}{l_x} \cos \frac{m\pi y}{l_y}, \quad \|V_{nm}\|^2 = \begin{cases} l_x l_y & \text{if } n = 0, m = 0, \\ l_x l_y / 2 & \text{if } n = 0, m > 0, \text{ or } n > 0, m = 0, \\ l_x l_y / 4 & \text{if } n > 0, m > 0. \end{cases}$$

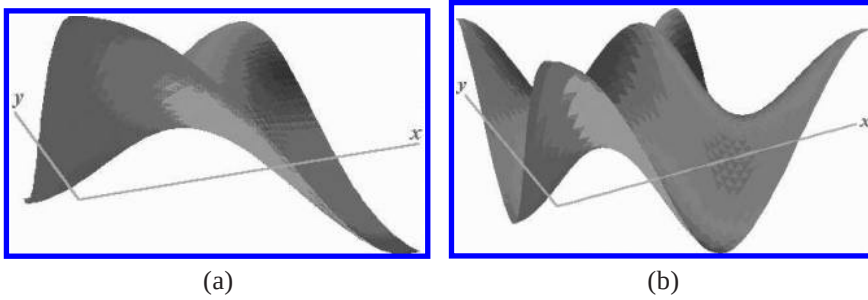


Figure 4.10. Eigenfunctions (a) $V_{11}(x, y)$ and (b) $V_{22}(x, y)$ for the free surface described in Example 4.4.

The three-dimensional picture shown in Figure 4.10 depicts two eigenfunctions, $V_{11}(x, y)$ and $V_{22}(x, y)$, chosen as examples. The values of λ for these modes are $\lambda_{11} = 0.891$ and $\lambda_{22} = 3.564$, respectively.

As a result of the initial conditions being equal to zero, the solution $u(x, y, t)$ is defined by the series

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T_{nm}(t) \cos \frac{n\pi x}{l_x} \cos \frac{m\pi y}{l_y},$$

where

$$T_{nm}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}(\tau) \sin \omega_{nm}(t - \tau) d\tau,$$

$$f_{nm}(t) = -\frac{A\omega \sin \omega t}{\rho} \cdot \frac{1}{||V_{nm}||^2} \int_0^{l_x} \int_0^{l_y} \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y} \cos \frac{n\pi x}{l_x} \cos \frac{m\pi y}{l_y} dx dy,$$

and

$$\omega_{nm} = a\lambda_{nm} = a\pi \sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}.$$

It is obvious that $f_{nm}(t) \neq 0$ only when $n = m = 1$, and that $f_{11}(t) = -\frac{A\omega}{\rho} \sin \omega t$. Thus, we have

$$T_{11}(t) = -\frac{A\omega}{\rho (\omega_{11}^2 - \omega^2)} \left[\sin \omega t - \frac{\omega}{\omega_{11}} \sin \omega_{11} t \right]$$

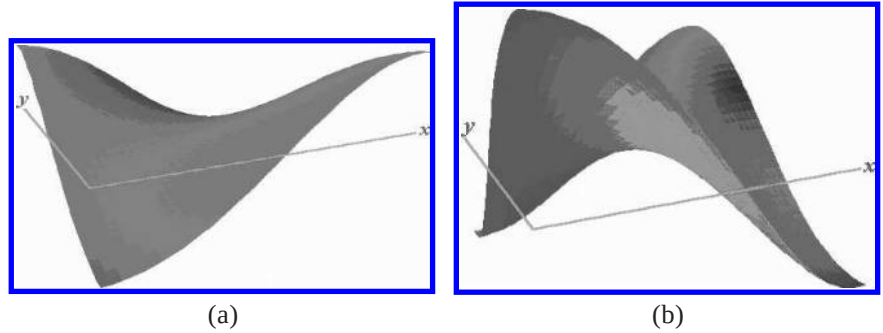


Figure 4.11. Graph of the membrane in Example 4.4 at (a) $t = 0.8$, (b) $t = 1.6$. The values of the parameters are $A = 0.1$, $\omega = 5$, $h = 1$, $g = 9.8$, $l_x = 4$, $l_y = 6$.

and

$$u(x, y, t) = -\frac{A\omega}{\rho(\omega_{11}^2 - \omega^2)} \left[\sin \omega t - \frac{\omega}{\omega_{11}} \sin \omega_{11} t \right] \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y}.$$

4.2.3 The Fourier Method for Equations with Nonhomogeneous Boundary Conditions

Consider now the boundary problem of forced oscillations of a membrane given by Equation (4.12) with *nonhomogeneous boundary and initial conditions* given by Equations (4.13) and (4.14):

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \gamma u = f(x, y, z),$$

$$P_1[u] \equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = g_1(y, t), \quad P_2[u] \equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} = g_2(y, t),$$

$$P_3[u] \equiv \alpha_3 \frac{\partial u}{\partial y} + \beta_3 u \Big|_{y=0} = g_3(x, t), \quad P_4[u] \equiv \alpha_4 \frac{\partial u}{\partial y} + \beta_4 u \Big|_{y=l_y} = g_4(x, t),$$

$$u|_{t=0} = \varphi(x, y), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(x, y).$$

We consider situations when the boundary conditions along the edges of a membrane are consistent at the corners of a membrane (which would

be required in a physical occurrence). The following conforming conditions are valid:

$$P_3[g_1] \equiv \alpha_3 \frac{\partial g_1}{\partial y} + \beta_3 g_1 \Big|_{y=0} = P_1[g_3] \equiv \alpha_1 \frac{\partial g_3}{\partial y} + \beta_1 g_3 \Big|_{x=0},$$

$$P_4[g_1] \equiv \alpha_4 \frac{\partial g_1}{\partial y} + \beta_4 g_1 \Big|_{y=l_y} = P_1[g_4] \equiv \alpha_1 \frac{\partial g_4}{\partial y} + \beta_1 g_4 \Big|_{x=0},$$

$$P_3[g_2] \equiv \alpha_3 \frac{\partial g_2}{\partial y} + \beta_3 g_2 \Big|_{y=0} = P_2[g_3] \equiv \alpha_2 \frac{\partial g_3}{\partial y} + \beta_2 g_3 \Big|_{x=l_x},$$

$$P_4[g_2] \equiv \alpha_4 \frac{\partial g_2}{\partial y} + \beta_4 g_2 \Big|_{y=l_y} = P_2[g_4] \equiv \alpha_2 \frac{\partial g_4}{\partial y} + \beta_2 g_4 \Big|_{x=l_x}.$$

If in some particular problem such consistency conditions do not hold, the way to resolve such a difficulty may require a smearing of boundary conditions at the corners, using simple smooth functions to make them consistent. The solutions of an actual problem and the modified one will differ at the vicinity of the corners, but hopefully this difference will be not that substantial at points far enough away from these corners. We will not consider such problems for hyperbolic equations; however, as a reading exercise, the reader can consider Dirichlet boundary conditions that are inconsistent in one of the corners and suggest an adequate smearing procedure. Other ways to treat inconsistent boundary conditions are considered in Chapters 6 and 7 for parabolic and elliptic equations.

Let us return to consistent boundary conditions. Notice that it is not possible to use the Fourier method immediately since the boundary conditions are nonhomogeneous. However, this problem is easily reduced to the problem with zero boundary conditions. To proceed, let us search for solutions of the problem in the form

$$u(x, y, t) = v(x, y, t) + w(x, y, t), \quad (4.65)$$

where $v(x, y, t)$ is an unknown function and the function $w(x, y, t)$ satisfies the given nonhomogeneous boundary conditions

$$\begin{aligned} P_1[w] &\equiv \alpha_1 w_x + \beta_1 w|_{x=0} = g_1(y, t), & P_2[w] &\equiv \alpha_2 w_x + \beta_2 w|_{x=l_x} = g_2(y, t), \\ P_3[w] &\equiv \alpha_3 w_y + \beta_3 w|_{y=0} = g_3(x, t), & P_4[w] &\equiv \alpha_4 w_y + \beta_4 w|_{y=l_y} = g_4(x, t), \end{aligned} \quad (4.66)$$

and possesses the necessary number of continuous derivatives with respect to x , y , and t .

For the function $v(x, y, t)$, we have the following boundary value problem (check this result as a reading exercise):

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} + 2\kappa \frac{\partial v}{\partial t} - a^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \gamma v &= f^*(x, y, t), \\ P_1[v] \equiv \alpha_1 v_x + \beta_1 v|_{x=0} &= 0, \quad P_2[v] \equiv \alpha_2 v_x + \beta_2 v|_{x=l_x} = 0, \\ P_3[v] \equiv \alpha_3 v_y + \beta_3 v|_{y=0} &= 0, \quad P_4[v] \equiv \alpha_4 v_y + \beta_4 v|_{y=l_y} = 0, \\ v|_{t=0} &= \varphi^*(x, y), \quad \frac{\partial v}{\partial t} \Big|_{t=0} = \psi^*(x, y), \end{aligned}$$

where

$$f^*(x, y, t) = f(x, y, t) - \frac{\partial^2 w}{\partial t^2} - 2\kappa \frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \gamma w, \quad (4.67)$$

$$\varphi^*(x, y) = \varphi(x, y) - w(x, y, 0), \quad (4.68)$$

and

$$\psi^*(x, y) = \psi(x, y) - w_t(x, y, 0). \quad (4.69)$$

Solutions of this problem were considered in Section 4.2.2.

Let us search for the auxiliary function $w(x, y, t)$ in the form

$$\begin{aligned} w(x, y, t) &= g_1(y, t)\overline{X} + g_2(y, t)\overline{\overline{X}} + g_3(x, t)\overline{Y} + g_4(x, t)\overline{\overline{Y}} \\ &+ A(t)\overline{X}\overline{Y} + B(t)\overline{\overline{X}}\overline{\overline{Y}} + C(t)\overline{X}\overline{\overline{Y}} + D(t)\overline{\overline{X}}\overline{\overline{Y}}. \end{aligned} \quad (4.70)$$

We choose the functions $\left\{ \overline{X}, \overline{\overline{X}} \right\}$ in such a way that the function $\overline{X}(x)$ satisfies homogeneous boundary condition at $x = l_x$, and the function $\overline{\overline{X}}(x)$ satisfies homogeneous boundary condition at $x = 0$,

$$P_2 \left[\overline{X}(l_x) \right] = 0 \quad \text{and} \quad P_1 \left[\overline{\overline{X}}(0) \right] = 0.$$

Also, it is convenient to normalize the functions $\overline{X}(x)$ and $\overline{\overline{X}}(x)$ so that

$$P_1 \left[\overline{X}(0) \right] = 1 \quad \text{and} \quad P_2 \left[\overline{\overline{X}}(l_x) \right] = 1.$$

The final choice of functions $\{\bar{X}, \bar{\bar{X}}\}$ depends on the type of boundary conditions for the function $u(x, y, t)$.

Suppose β_1 and β_2 are not both zero. In this case, we can search for $\bar{X}(x)$ and $\bar{\bar{X}}(x)$ as polynomials of first order:

$$\bar{X}(x) = \gamma_1 + \delta_1 x \quad \text{and} \quad \bar{\bar{X}}(x) = \gamma_2 + \delta_2 x \quad (4.71)$$

This choice yields the system of equations

$$\begin{cases} P_1 [\bar{X}] \equiv \alpha_1 \frac{\partial \bar{X}}{\partial x} + \beta_1 \bar{X} \Big|_{x=0} = \beta_1 \gamma_1 + \alpha_1 \delta_1 = 1, \\ P_2 [\bar{X}] \equiv \alpha_2 \frac{\partial \bar{X}}{\partial x} + \beta_2 \bar{X} \Big|_{x=l_x} = \beta_2 \gamma_1 + (\alpha_2 + \beta_2 l_x) \delta_1 = 0, \end{cases}$$

and

$$\begin{cases} P_1 [\bar{\bar{X}}] \equiv \alpha_1 \frac{\partial \bar{\bar{X}}}{\partial x} + \beta_1 \bar{\bar{X}} \Big|_{x=0} = \beta_1 \gamma_2 + \alpha_1 \delta_2 = 0, \\ P_2 [\bar{\bar{X}}] \equiv \alpha_2 \frac{\partial \bar{\bar{X}}}{\partial x} + \beta_2 \bar{\bar{X}} \Big|_{x=l_x} = \beta_2 \gamma_2 + (\alpha_2 + \beta_2 l_x) \delta_2 = 1. \end{cases}$$

From these conditions, we may find a unique solution for coefficients γ_1 , δ_1 , γ_2 , and δ_2 :

$$\begin{aligned} \gamma_1 &= \frac{\alpha_2 + \beta_2 l_x}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_1 &= \frac{-\beta_2}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \\ \gamma_2 &= \frac{-\alpha_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_2 &= \frac{\beta_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}. \end{aligned} \quad (4.72)$$

Reading Exercise. Obtain Equations (4.72) for γ_1 , δ_1 , γ_2 , and δ_2 .

If $\beta_1 = \beta_2 = 0$, we search for $\bar{X}(x)$ and $\bar{\bar{X}}(x)$ as polynomials of second order, from which it is easy to see that

$$\bar{X}(x) = x - \frac{x^2}{2l_x} \quad \text{and} \quad \bar{\bar{X}}(x) = \frac{x^2}{2l_x} \quad (4.73)$$

will serve our needs.

Similarly, we choose the functions $\{\bar{Y}, \bar{\bar{Y}}\}$ in such a way that the function $\bar{Y}(y)$ satisfies the homogeneous condition at $y = l_y$, and the function $\bar{\bar{Y}}(y)$ satisfies the homogeneous condition at $y = 0$:

$$P_3 [\bar{Y}(l_y)] = 0 \quad \text{and} \quad P_4 [\bar{\bar{Y}}(0)] = 0.$$

As in the case of the variable x , we may choose the normalization condition for $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ as

$$P_3 [\bar{Y}(0)] = 1 \quad \text{and} \quad P_4 [\bar{\bar{Y}}(l_y)] = 1.$$

Suppose β_3 and β_4 are not both zero. In this case, we search for the functions $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ as polynomials of first order:

$$\bar{Y}(y) = \gamma_3 + \delta_3 y \quad \text{and} \quad \bar{\bar{Y}}(y) = \gamma_4 + \delta_4 y. \quad (4.74)$$

For coefficients $\gamma_3, \delta_3, \gamma_4$, and δ_4 , we have

$$\begin{aligned} \gamma_3 &= \frac{\alpha_4 + \beta_4 l_y}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}, & \delta_3 &= \frac{-\beta_4}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3} \\ \gamma_4 &= \frac{-\alpha_3}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}, & \delta_4 &= \frac{\beta_3}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}. \end{aligned} \quad (4.75)$$

If $\beta_3 = \beta_4 = 0$, then $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ can be taken as polynomials of second order:

$$\bar{Y}(y) = y - \frac{y^2}{2l_y} \quad \text{and} \quad \bar{\bar{Y}}(y) = \frac{y^2}{2l_y}.$$

By using the above results, we can find the coefficients $A(t)$, $B(t)$, $C(t)$, and $D(t)$ in the auxiliary function $w(x, y, t)$.

1. At the boundary $x = 0$, we have

$$P_1[w]_{x=0} = g_1(y, t) + (P_1[g_3(0, t)] + A) \bar{Y} + (P_1[g_4(0, t)] + B) \bar{\bar{Y}}.$$

2. At the boundary $x = l_x$, we have

$$P_2[w]_{x=l_x} = g_2(y, t) + (P_2[g_3(l_x, t)] + C) \bar{Y} + (P_2[g_4(l_x, t)] + D) \bar{\bar{Y}}.$$

3. At the boundary $y = 0$, we have

$$P_3[w]_{y=0} = g_3(x, t) + (P_3[g_1(0, t)] + A) \bar{X} + (P_3[g_2(0, t)] + C) \bar{\bar{X}}.$$

4. At the boundary $y = l_y$, we have

$$P_4[w]_{y=l_y} = g_4(x, t) + (P_4[g_1(l_y, t)] + B) \bar{X} + (P_4[g_2(l_y, t)] + D) \bar{\bar{X}}.$$

To simplify the above, we may choose

$$A(t) = -P_3[g_1(y, t)]_{y=0}, \quad B(t) = -P_4[g_1(y, t)]_{y=l_y},$$

$$C(t) = -P_3[g_2(y, t)]_{y=0}, \quad D(t) = -P_4[g_2(y, t)]_{y=l_y}.$$

For the above choices, the boundary conditions conform at the edges:

$$P_3[g_1(y, t)]_{y=0} = P_1[g_3(x, t)]_{x=0}, \quad P_3[g_2(y, t)]_{y=0} = P_2[g_3(x, t)]_{x=l_x},$$

$$P_4[g_1(y, t)]_{y=l_y} = P_1[g_4(x, t)]_{x=0}, \quad P_4[g_2(y, t)]_{y=l_y} = P_2[g_4(x, t)]_{x=l_x}.$$

It is easy to check that the above auxiliary functions, $w(x, y, t)$, satisfy the given boundary conditions. We leave it to the reader to verify as a reading exercise that:

$$P_1[w]_{x=0} = g_1(y, t), \quad P_2[w]_{x=l_x} = g_2(y, t),$$

$$P_3[w]_{y=0} = g_3(x, t), \quad P_4[w]_{y=l_y} = g_4(x, t).$$

We now consider an example of the solution of a specific problem for oscillations with nonhomogeneous boundary conditions.

Example 4.5. Find oscillations of a homogeneous rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) if the boundary conditions are given by

$$u(0, y, t) = u(l_x, y, t) = 0 \quad \text{and} \quad u(x, 0, t) = u(x, l_y, t) = h \sin \frac{\pi x}{l_x}.$$

Initially the membrane has shape and velocity given, respectively, by

$$\varphi(x, y, 0) = h \sin \frac{\pi x}{l_x} \quad \text{and} \quad \psi(x, y, 0) = v_0 \sin \frac{\pi x}{l_x}.$$

Solution. To proceed, we must solve the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

under the conditions

$$\begin{aligned} u(x, y, 0) &= h \sin \frac{\pi x}{l_x}, \quad \frac{\partial u}{\partial t}(x, y, 0) = v_0 \sin \frac{\pi x}{l_x}, \\ u(0, y, t) &= u(l_x, y, t) = 0, \quad u(x, 0, t) = u(x, l_y, t) = h \sin \frac{\pi x}{l_x}. \end{aligned}$$

We search for a solution of this problem as the sum

$$u(x, y, t) = v(x, y, t) + w(x, y, t),$$

where $w(x, y, t)$ is chosen to be

$$w(x, y, t) = h \sin \frac{\pi x}{l_x},$$

which satisfies the boundary conditions of the problem and therefore obviously provides homogeneous boundary conditions for the function $v(x, y, t)$ (this result can be obtained from the general formulas presented in this section).

Figures 4.12 and 4.13 present the graphs of the auxiliary function and profile of the membrane for the following values of the parameters: $a^2 = 1$, $l_x = 4$, $l_y = 6$, $v_0 = 0.5$, $h = 0.1$.

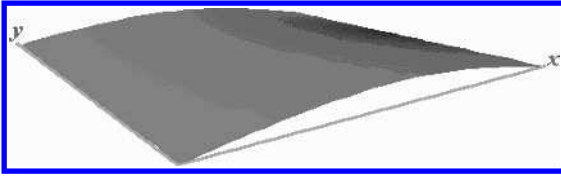
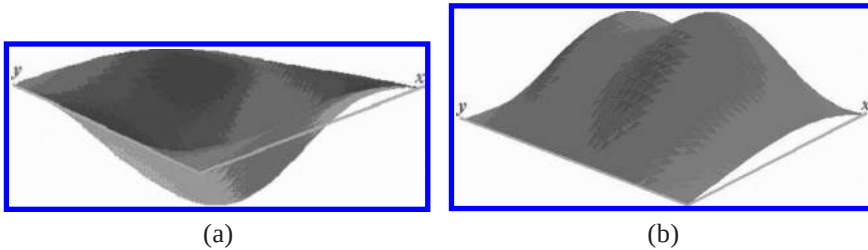
For the function $v(x, y, t)$, we have the boundary problem for the non-homogeneous equation of oscillation where

$$f^*(x, y, t) = -a^2 \frac{h\pi^2}{l_x^2} \sin \frac{\pi x}{l_x},$$

$$\varphi^*(x, y) = 0, \quad \psi^*(x, y) = v_0 \sin \frac{\pi x}{l_x},$$

and homogeneous boundary conditions. The solution to the problem is thus

$$\begin{aligned} u(x, y, t) &= h \sin \frac{\pi x}{l_x} + \sum_{m=1}^{\infty} \left\{ \frac{4v_0}{(2m-1)\pi\omega_{1(2m-1)}} \sin \omega_{1(2m-1)}t \right. \\ &\quad \left. - \frac{4ha^2\pi}{(2m-1)l_x^2\omega_{1(2m-1)}^2} [1 - \cos \omega_{1(2m-1)}t] \right\} \sin \frac{\pi x}{l_x} \sin \frac{(2m-1)\pi y}{l_y}. \end{aligned}$$

Figure 4.12. Graph of the auxiliary function $w(x, y, t)$ for Example 4.5.Figure 4.13. Graph of the membrane in Example 4.5 at (a) $t = 5$, (b) $t = 8.5$.

4.3 The Fourier Method Applied to Small Transverse Oscillations of a Circular Membrane

Suppose a membrane in its equilibrium position has the form of a circle with radius l , is located in the x - y plane, and has its center at the origin of coordinates. As before, for the case of rectangular membranes, we consider transverse oscillations of the membrane only, for which all points on the membrane move perpendicular to the x - y plane. In polar (or circular) coordinates with variables r and φ , the displacement, u , of points on the membrane will be a function of r , φ , and time t : $u = u(r, \varphi, t)$.

The following discussion for a circular or polar coordinate system parallels the development for a rectangular membrane found in Section 4.2. The Laplace operator in polar coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2},$$

with the result that the equation of oscillations of a membrane in polar coordinates has the form

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) + \gamma u = f(r, \varphi, t). \quad (4.76)$$

The domains of the independent variables are $0 \leq r \leq l$, $0 \leq \varphi < 2\pi$, and $0 \leq t < \infty$. A discussion of the Laplace operator in polar coordinates can be found in Appendix D.

Boundary conditions in polar coordinates are particularly simple, and in general form can be written as

$$\alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = g(\varphi, t), \quad (4.77)$$

where α and β are constants that are not zero simultaneously (i.e., $|\alpha| + |\beta| \neq 0$). If $\alpha = 0$, we have Dirichlet boundary conditions, and if $\beta = 0$ we have Neumann boundary conditions. If $\alpha \neq 0$ and $\beta \neq 0$, we have mixed boundary conditions. As was discussed in Chapter 2, from physical arguments it will normally be the case that $\beta/\alpha > 0$.

It should be clear from our previous discussion that in order to describe the membrane oscillation for $t > 0$, we will need to know the initial displacement and initial velocity of the membrane to solve various physical problems. In a manner analogous to the rectangular case, the initial conditions may be stated as

$$u|_{t=0} = \phi(r, \varphi) \quad \text{and} \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(r, \varphi). \quad (4.78)$$

Thus the deviation of points of a membrane with coordinates (r, φ) at some arbitrary initial moment of time is $\phi(r, \varphi)$, and the initial velocities of these points are given by the function $\psi(r, \varphi)$. It should be clear from physical arguments that the solution $u(r, \varphi, t)$ is to be single-valued; periodic in φ with period 2π ; and remains finite at all points of the membrane, including the center of the membrane, where $r = 0$.

4.3.1 The Fourier Method for Homogeneous Equations with Homogeneous Boundary Conditions

We begin by assuming the membrane is set in motion by some combination of initial displacements and/or initial velocities. To solve the equation of

motion for the displacement from equilibrium, $u(r, \varphi, t)$, of the membrane at all points and times, we start by solving the *homogeneous equation* of oscillations

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) + \gamma u = 0 \quad (4.79)$$

with *homogeneous* boundary condition

$$\alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = 0 \quad (4.80)$$

and *nonhomogeneous* initial conditions from Equations (4.78)

$$u|_{t=0} = \phi(r, \varphi) \quad \text{and} \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(r, \varphi).$$

Let us represent the function $u(r, \varphi, t)$ as a product of two functions. The first depends only on r and φ , and we denote it as $V(r, \varphi)$; the second depends only on t and we denote it as $T(t)$:

$$u(r, \varphi, t) = V(r, \varphi)T(t). \quad (4.81)$$

By substituting Equation (4.81) in Equation (4.79) and separating variables, we obtain

$$\frac{T''(t) + 2\kappa T'(t) + \gamma T(t)}{a^2 T(t)} \equiv \frac{1}{V(r, \varphi)} \left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} \right] = -\lambda,$$

where λ is a separation of variables constant (we already know that a choice of minus sign before λ is convenient). Thus, the function $T(t)$ satisfies the ordinary linear homogeneous differential equation of second order,

$$T''(t) + 2\kappa T'(t) + (a^2 \lambda + \gamma)T(t) = 0, \quad (4.82)$$

and the function $V(r, \varphi)$ satisfies the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \lambda V = 0, \quad (4.83)$$

with

$$\alpha \frac{\partial V(l, \varphi)}{\partial r} + \beta V(l, \varphi) = 0 \quad (4.84)$$

as a boundary condition. Using physical arguments, we also require that the solutions remain finite so that

$$|V(0, \varphi)| < \infty, \quad (4.85)$$

and that the solutions be periodic, which may be defined as

$$V(r, \varphi) = V(r, \varphi + 2\pi). \quad (4.86)$$

For the boundary value problem defined by Equations (4.83) through (4.86), we again may separate the variables—in this case r and φ —by using the substitution

$$V(r, \varphi) = R(r)\Phi(\varphi). \quad (4.87)$$

By substituting Equation (4.87) into Equation (4.83) and dividing by $R(r)\Phi(\varphi)$, we obtain

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} + \lambda r^2 \equiv -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \nu,$$

where ν is another separation of variables constant. The equations result in two eigenvalue problems, each of which is one-dimensional. The first is given by

$$\Phi''(\varphi) + \nu \Phi(\varphi) = 0, \quad (4.88)$$

with the boundary condition

$$\Phi(\varphi) = \Phi(\varphi + 2\pi), \quad (4.89)$$

and the second is given by

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda - \frac{\nu}{r^2} \right) R = 0, \quad (4.90)$$

with the boundary condition

$$\alpha \frac{\partial R}{\partial r} + \beta R \Big|_{r=l} = 0, \quad |R(0)| < \infty. \quad (4.91)$$

Note that with a change of φ by the value 2π (see Equation (4.89)), we return to the same point on the membrane; consequently, the displacement of the membrane $u(r, \varphi, t)$ does not change under the substitution φ by $\varphi + 2\pi$.

Boundary condition (4.89) indicates that ν cannot be negative; otherwise, solutions of Equation (4.88) would include exponentially increasing or decreasing functions, which are not periodic.

If $\nu = 0$, then Equation (4.88) reduces to $\Phi''(\varphi) = 0$, having the solution $\Phi(\varphi) = C_1\varphi + C_2$, which will satisfy condition (4.89) only in the case that $C_1 = 0$. In other words, $\Phi(\varphi) = C_2$, where C_2 is a constant and, in particular, we may take $\Phi(\varphi) \equiv 1$. For the above choices, $\nu_0 = 0$ is the eigenvalue, and the function $\Phi_0(\varphi) \equiv 1$ is the corresponding eigenfunction.

If $\nu > 0$, linear-independent solutions of Equation (4.88) are

$$\Phi^{(1)}(\varphi) = \cos \sqrt{\nu}\varphi \quad \text{and} \quad \Phi^{(2)}(\varphi) = \sin \sqrt{\nu}\varphi.$$

The functions $\Phi^{(1)}(\varphi)$ and $\Phi^{(2)}(\varphi)$ each have period $2\pi/\sqrt{\nu}$. To maintain the periodic boundary conditions, this period must equal 2π or an integer number times 2π , which will be the case only if $\sqrt{\nu}$ is an integer number. We require, therefore, that $\nu = n^2$ where $n = 1, 2, 3, \dots$. We may use these integer values of ν to label the eigenvalues for the equation as ν_n and corresponding eigenfunctions by $\Phi_n^{(1)}(\varphi)$ and $\Phi_n^{(2)}(\varphi)$. Thus, the solutions to Equation (4.88) are given by

$$\nu = n^2, \quad \Phi_n^{(1)}(\varphi) = \cos n\varphi, \quad \Phi_n^{(2)}(\varphi) = \sin n\varphi, \quad n = 0, 1, 2, \dots \quad (4.92)$$

Note that the boundary value problem given by Equations (4.88) and (4.89) is the Sturm-Liouville problem with periodic boundary conditions on the interval $[0, 2\pi]$, which is why the two linearly independent eigenfunctions $\cos n\varphi$ and $\sin n\varphi$ belong to the same eigenvalue, $\nu_n = n^2$.

To find the solutions $R(r)$, we use Equation (4.90) with homogeneous boundary conditions (4.91), where the eigenvalues $\nu = n^2$ found previously for $\Phi(\varphi)$ are now included in Equation (4.90). Equation (4.90) is known as the Bessel equation, solutions to which are discussed in detail in Chapter 8. The second of the conditions given in Equation (4.91) imposed on the function $R(r)$ is that it should be bounded at $r = 0$, which is a singular point of the equation. For singular points, this kind of restriction may be considered as a boundary condition.

For the present we do not need to know much about the Bessel equation or its solutions; the relevant issue is that the general solution of Equation (4.90) with $\nu = n^2$ is

$$R_n(r) = d_{1n} J_n(\sqrt{\lambda} r) + d_{2n} N_n(\sqrt{\lambda} r), \quad (4.93)$$

where $J_n(\sqrt{\lambda} r)$ is the Bessel function of order n and $N_n(\sqrt{\lambda} r)$ is the Neumann function of order n , $n = 0, 1, 2, \dots$. (Note that the functions N_n are sometimes designated as Y_n in the literature.) Clearly, only positive λ are allowed. The functions $J_n(\sqrt{\lambda} r)$ are bounded as $r \rightarrow 0$, whereas the functions $N_n(\sqrt{\lambda} r)$ diverge as $r \rightarrow 0$. The physical requirement that the function $R(r)$ be bounded leads to the result that the coefficients of $N_n(\sqrt{\lambda} r)$ must equal zero, in which case we have $d_{2n} = 0$.

From the homogeneous boundary condition given in Equation (4.91), we have

$$\alpha \lambda J'_n(\sqrt{\lambda} l) + \beta J_n(\sqrt{\lambda} l) = 0.$$

Setting $\sqrt{\lambda} l \equiv \mu$, we obtain a transcendental equation defining μ , given by

$$\alpha \mu J'_n(\mu) + \beta J_n(\mu) = 0, \quad (4.94)$$

which has an infinite number of *positive* roots (see Chapter 8) that we label as

$$\mu_0^{(n)}, \quad \mu_1^{(n)}, \quad \mu_2^{(n)}, \quad \dots$$

The corresponding values of λ are thus

$$\lambda_{nm} = \left(\frac{\mu_m^{(n)}}{l} \right)^2, \quad n, m = 0, 1, 2, \dots \quad (4.95)$$

We see therefore that we need only positive roots, $\mu_m^{(n)}$, because negative roots do not give new values of λ_{nm} .

The eigenfunctions are

$$R_{nm}(r) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right). \quad (4.96)$$

The index $m = 0$ corresponds to the first root of Equation (4.94). (Note that very often the roots are labeled with the starting value $m = 1$ in the literature.) The eigenvalues λ_{nm} and eigenfunctions $R_{nm}(r)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem given in Equations (4.90) and (4.91) with $\nu = n^2$.

For our current purposes, we note here several properties of Bessel functions, J_n , and eigenfunctions given in Equation (4.96), a detailed discussion of which is presented in Chapter 8.

1. Eigenfunctions $R_{nm}(r)$ belonging to different eigenvalues λ_{nm} for some fixed value of n are orthogonal with weight r :

$$\int_0^l r R_{nm_1}(r) R_{nm_2}(r) dr = 0 \quad (4.97)$$

or

$$\int_0^l r J_n \left(\mu_{m_1}^{(n)} r / l \right) J_n \left(\mu_{m_2}^{(n)} r / l \right) dr = 0 \text{ for } m_1 \neq m_2. \quad (4.98)$$

2. The norm of the eigenfunction $R_{nm}(r)$, $\|R_{nm}\|^2 = \left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2$, is

$$\|R_{nm}\|^2 = \int_0^l r J_n^2 \left(\frac{\mu_m^{(n)} r}{l} \right) dr = \frac{l^2}{2} \left\{ \left[J_n' \left(\mu_m^{(n)} \right) \right]^2 + \left[1 - \frac{n^2}{\left(\mu_m^{(n)} \right)^2} \right] J_n^2 \left(\mu_m^{(n)} \right) \right\}. \quad (4.99)$$

To arrive at Equation (4.99), the above expression for $\|R_{nm}\|^2$ we evaluate integral

$$I = \int x J_n^2(x) dx.$$

By using integration by parts, we obtain

$$I = \int x J_n^2(x) dx = \int J_n^2(x) d \left(\frac{x^2}{2} \right) = \frac{x^2}{2} J_n^2(x) - \int x^2 J_n(x) J_n'(x) dx.$$

From the Bessel equation

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0,$$

we have

$$x^2 J_n(x) = -x^2 J_n''(x) - x J_n'(x) + n^2 J_n(x);$$

thus,

$$\begin{aligned} I &= \frac{x^2}{2} J_n^2(x) + \int x J_n'(x) \frac{d}{dx} (x J_n'(x)) dx - n^2 \int J_n(x) J_n'(x) dx \\ &= \frac{x^2}{2} J_n^2(x) + \frac{x^2}{2} [J_n'(x)]^2 - \frac{n^2}{2} J_n^2(x) = \frac{x^2}{2} [J_n'(x)]^2 + \frac{x^2 - n^2}{2} J_n^2(x). \end{aligned}$$

From here, we obtain the result shown in Equation (4.99):

$$\begin{aligned} \|R_{nm}\|^2 &= \int_0^l r J_n^2\left(\frac{\mu_m^{(n)}}{l} r\right) dr = \frac{l^2}{\left(\mu_m^{(n)}\right)^2} \int_0^{\mu_m^{(n)}} x J_n^2(x) dx \\ &= \frac{l^2}{2} \left\{ \left[J_n'(\mu_m^{(n)}) \right]^2 + \left[1 - \frac{n^2}{\left(\mu_m^{(n)}\right)^2} \right] J_n^2(\mu_m^{(n)}) \right\}. \end{aligned}$$

Now, with Equation (4.99), we may use the boundary condition

$$\alpha \mu J_n'(\mu) + \beta l J_n(\mu) = 0$$

to find the explicit expressions for the norm of the eigenfunctions for different values of coefficients α and β .

1. For the *Dirichlet* boundary condition, $\alpha = 0$ and $\beta = 1$, in which case eigenvalues are obtained from the equation

$$J_n(\mu) = 0,$$

and we have

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[J_n'(\mu_m^{(n)}) \right]^2. \quad (4.100)$$

2. For the *Neumann* boundary condition, $\alpha = 1$ and $\beta = 0$, in which case eigenvalues are obtained from the equation

$$J_n'(\mu) = 0,$$

and we have

$$\|R_{nm}\|^2 = \frac{l^2}{2 \left(\mu_m^{(n)}\right)^2} \left[\left(\mu_m^{(n)}\right)^2 - n^2 \right] J_n^2(\mu_m^{(n)}). \quad (4.101)$$

3. For the *mixed* boundary condition, $\alpha = 1$ and $\beta = h$, in which case eigenvalues are obtained from the equation

$$\mu J'_n(\mu) + hl J_n(\mu) = 0.$$

From this, we may write

$$J'_n \left(\mu_m^{(n)} \right) = -\frac{hl}{\mu_m^{(n)}} J_n \left(\mu_m^{(n)} \right).$$

Substituting into Equation (4.99), we obtain

$$\begin{aligned} \|R_{nm}\|^2 &= \frac{l^2}{2} \left\{ \left[J'_n \left(\mu_m^{(n)} \right) \right]^2 + \left[1 - \frac{n^2}{\left(\mu_m^{(n)} \right)^2} \right] J_n^2 \left(\mu_m^{(n)} \right) \right\} \\ &= \frac{l^2}{2} \left[1 + \frac{l^2 h^2 - n^2}{\left(\mu_m^{(n)} \right)^2} \right] J_n^2 \left(\mu_m^{(n)} \right). \end{aligned} \quad (4.102)$$

Similarly, substituting

$$J_n \left(\mu_m^{(n)} \right) = -\frac{\mu_m^{(n)}}{hl} J'_n \left(\mu_m^{(n)} \right)$$

in Equation (4.99), we obtain

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[1 + \frac{\left(\mu_m^{(n)} \right)^2 - n^2}{l^2 h^2} \right] \left[J'_n \left(\mu_m^{(n)} \right) \right]^2. \quad (4.103)$$

Equation (4.102) is suitable for calculations involving small h ($h \rightarrow 0$), whereas Equation (4.103) is appropriate for large h ($h \rightarrow \infty$). It is easily verified as a reading exercise that as $h \rightarrow 0$, Equation (4.102) transforms into Equation (4.101), and as $h \rightarrow \infty$, Equation (4.103) transforms into Equation (4.100).

Returning to the boundary value problem for the circular membrane, we obtain for each eigenvalue

$$\lambda_{nm} = \left(\frac{\mu_m^{(n)}}{l} \right)^2,$$

two linearly independent eigenfunctions

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi \text{ and } V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi. \quad (4.104)$$

Since

$$\int_0^{2\pi} d\varphi = 2\pi, \quad \|\cos n\varphi\|^2 = \int_0^{2\pi} \cos^2 n\varphi d\varphi = \pi$$

and

$$\|\sin n\varphi\|^2 = \int_0^{2\pi} \sin^2 n\varphi d\varphi = \pi (n > 0),$$

the norms of eigenfunctions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are

$$\|V_{0m}^{(1)}\|^2 = 2\pi \|R_{nm}(r)\|^2 \quad (4.105)$$

and

$$\|V_{nm}^{(1)}\|^2 = \|V_{nm}^{(2)}\|^2 = \pi \|R_{nm}(r)\|^2 \text{ for } n > 0. \quad (4.106)$$

This completely defines λ_{nm} , $V_{nm}^{(1)}$, and $V_{nm}^{(2)}$, the eigenvalues and eigenfunctions of Equations (4.83) through (4.86) for the problem of free oscillations of a circular membrane for the case of homogeneous boundary conditions.

To determine the time evolution of the oscillating membrane, we return to Equation (4.82). This is an ordinary linear homogeneous differential equation of second order. Substituting $\lambda = \lambda_{nm}$ in Equation (4.82) and denoting the corresponding solution of this equation as $T_{nm}(t)$, we have

$$T_{nm}''(t) + 2\kappa T_{nm}'(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}(t) = 0. \quad (4.107)$$

This linear second-order equation with constant coefficients has two linearly independent solutions:

$$y_{nm}^{(1)}(t) = \begin{cases} e^{-\kappa t} \cos \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \cosh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases} \quad (4.108)$$

and

$$y_{nm}^{(2)}(t) = \begin{cases} e^{-\kappa t} \sin \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \sinh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases} \quad (4.109)$$

with $\omega_{nm} = \sqrt{a^2 \lambda_{nm} + \gamma - \kappa^2}$ if $\kappa^2 \neq a^2 \lambda_{nm} + \gamma$, and $\omega_{nm} = 1$ otherwise.

A general solution of Equation (4.107) is a linear combination of $y_{nm}^{(1)}(t)$ and $y_{nm}^{(2)}(t)$. Collecting the functions $\Phi(\varphi)$, $R(r)$, and $T(t)$ and substituting them into identity (4.81) gives solutions to Equation (4.79) in the form of a product of functions of one variable satisfying the given boundary conditions:

$$\begin{aligned} u_{nm}^{(1)}(r, \varphi, t) &= T_{nm}^{(1)} V_{nm}^{(1)}(r, \varphi) = \left[a_{nm} y_{nm}^{(1)} + b_{nm} y_{nm}^{(2)} \right] V_{nm}^{(1)}(r, \varphi), \\ u_{nm}^{(2)}(r, \varphi, t) &= T_{nm}^{(2)} V_{nm}^{(2)}(r, \varphi) = \left[c_{nm} y_{nm}^{(1)} + d_{nm} y_{nm}^{(2)} \right] V_{nm}^{(2)}(r, \varphi). \end{aligned} \quad (4.110)$$

To find solutions to the equation of motion for a membrane satisfying not only the boundary conditions in Equation (4.80) but also various initial conditions, let us sum these functions as a series, superposing all $u_{nm}^{(1)}(r, \varphi, t)$ and $u_{nm}^{(2)}(r, \varphi, t)$:

$$\begin{aligned} u(r, \varphi, t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[a_{nm} y_{nm}^{(1)}(t) + b_{nm} y_{nm}^{(2)}(t) \right] V_{nm}^{(1)}(r, \varphi) \right. \\ &\quad \left. + \left[c_{nm} y_{nm}^{(1)}(t) + d_{nm} y_{nm}^{(2)}(t) \right] V_{nm}^{(2)}(r, \varphi) \right\}. \end{aligned} \quad (4.111)$$

If this series and the series obtained from it by twice differentiating term by term with respect to the variables r , φ , and t converges uniformly then its sum will be a solution to Equation (4.79), satisfying boundary condition (4.80).

To satisfy the initial conditions given in Equation (4.78), we require that

$$u|_{t=0} = \phi(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[a_{nm} V_{nm}^{(1)}(r, \varphi) + c_{nm} V_{nm}^{(2)}(r, \varphi) \right] \quad (4.112)$$

and

$$\begin{aligned} \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ [\omega_{mn} b_{mn} - \kappa a_{mn}] V_{nm}^{(1)}(r, \varphi) \right. \\ \left. + [\omega_{mn} d_{mn} - \kappa c_{mn}] V_{nm}^{(2)}(r, \varphi) \right\}. \end{aligned} \quad (4.113)$$

We next substitute the expressions (4.104) for $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ into the series (4.112) and (4.113) to get

$$\begin{aligned} \phi(r, \varphi) = \sum_{m=0}^{\infty} a_{0m} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) + \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} a_{nm} J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \right] \cos(n\varphi) \\ + \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} a_{nm} J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \right] \sin(n\varphi) \end{aligned} \quad (4.114)$$

and

$$\begin{aligned} \psi(r, \varphi) = \sum_{m=0}^{\infty} [\omega_{0m} a_{0m} - \kappa a_{0m}] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) \\ + \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} [\omega_{nm} b_{nm} - \kappa a_{nm}] J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \right] \cos(n\varphi) \\ + \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} [\omega_{nm} d_{nm} - \kappa c_{nm}] J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \right] \sin(n\varphi). \end{aligned} \quad (4.115)$$

It is clear that the above series give the expansion of the periodic functions $\phi(r, \varphi)$ and $\psi(r, \varphi)$ in a Fourier series on the interval $(0, 2\pi)$ and that the coefficients of $\cos n\varphi$ and $\sin n\varphi$ in Equations (4.114) and (4.115) are

Fourier coefficients. Using this fact, we obtain the following relations:

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \phi(r, \varphi) d\varphi &= \sum_{m=0}^{\infty} a_{0m} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right), \\ \frac{1}{\pi} \int_0^{2\pi} \phi(r, \varphi) \cos n\varphi d\varphi &= \sum_{m=0}^{\infty} a_{nm} J_n \left(\frac{\mu_m^{(n)}}{l} r \right), \\ \frac{1}{\pi} \int_0^{2\pi} \phi(r, \varphi) \sin n\varphi d\varphi &= \sum_{m=0}^{\infty} c_{nm} J_n \left(\frac{\mu_m^{(n)}}{l} r \right),\end{aligned}\quad (4.116)$$

and

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \psi(r, \varphi) d\varphi &= \sum_{m=0}^{\infty} [\omega_{0m} b_{0m} - \kappa a_{0m}] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right), \\ \frac{1}{\pi} \int_0^{2\pi} \psi(r, \varphi) \cos n\varphi d\varphi &= \sum_{m=0}^{\infty} [\omega_{nm} b_{nm} - \kappa a_{nm}] J_n \left(\frac{\mu_m^{(n)}}{l} r \right), \\ \frac{1}{\pi} \int_0^{2\pi} \psi(r, \varphi) \sin n\varphi d\varphi &= \sum_{m=0}^{\infty} [\omega_{nm} d_{nm} - \kappa c_{nm}] J_n \left(\frac{\mu_m^{(n)}}{l} r \right).\end{aligned}\quad (4.117)$$

To determine the coefficients in Equation (4.116), we first consider the expansion of an arbitrary function, $F(r)$, in terms of Bessel functions, that is, a Fourier-Bessel series given by

$$F(r) = \sum_{m=0}^{\infty} C_m J_n \left(\frac{\mu_m^{(n)}}{l} r \right), \quad (4.118)$$

where $\mu_m^{(n)}$ are positive roots of

$$\alpha \mu J'_n(\mu) + \beta l J_n(\mu) = 0. \quad (4.119)$$

Because of the orthogonality of Bessel functions, the coefficients C_m of the Bessel series are given by the formula

$$C_m = \frac{1}{\left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2} \int_0^l F(r) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) r dr. \quad (4.120)$$

With Equation (4.120), it is easy to obtain the following expressions for Fourier coefficients in Equations (4.116) and (4.117) in terms of Bessel functions:

$$\begin{aligned} a_{0m} &= \frac{1}{2\pi \left\| J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr d\varphi, \\ a_{nm} &= \frac{1}{\pi \left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \cos n\varphi r dr d\varphi, \\ c_{nm} &= \frac{1}{\pi \left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin n\varphi r dr d\varphi, \end{aligned}$$

and

$$\begin{aligned} \omega_{0m} b_{0m} - \kappa a_{0m} &= \frac{1}{\pi \left\| J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \psi(r, \varphi) J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr d\varphi, \\ \omega_{nm} b_{nm} - \kappa a_{nm} &= \frac{1}{\pi \left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \psi(r, \varphi) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \cos n\varphi r dr d\varphi, \\ \omega_{nm} d_{nm} - \kappa c_{nm} &= \frac{1}{\pi \left\| J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \right\|^2} \int_0^l \int_0^{2\pi} \psi(r, \varphi) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin n\varphi r dr d\varphi, \end{aligned}$$

$$n = 1, 2, \dots, \quad m = 0, 1, 2, \dots$$

Using the expressions (4.104) for the functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$, after simple algebra, we have for the coefficients:

$$\begin{aligned} a_{nm} &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \\ b_{nm} &= \frac{1}{\omega_{nm}} \left[\frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \psi(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi + \kappa a_{nm} \right], \\ c_{nm} &= \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi, \\ d_{nm} &= \frac{1}{\omega_{nm}} \left[\frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^l \int_0^{2\pi} \psi(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi + \kappa c_{nm} \right]. \end{aligned} \quad (4.121)$$

Reading Exercise. Obtain Equations (4.121).

By substituting these values for the coefficients a_{nm} , b_{nm} , c_{nm} , and d_{nm} into the series (4.11), we obtain solutions of the problem defined in Equations (4.79), (4.80), and (4.78) under the assumption that the series (4.11), and the series obtained from it by twice differentiation term by term with respect to the variables r , φ , and t , converge uniformly.

Equation (4.11) gives the evolution of *free oscillations* of a circular membrane when the boundary condition is homogeneous. It can be considered as the expansion of the (unknown) function $u(r, \varphi, t)$ in a Fourier series using the orthogonal system of functions $\{V_{nm}(r, \varphi)\}$. This series converges under sufficiently reasonable assumptions about initial and boundary conditions.

The particular solutions

$$u_{nm}(r, \varphi, t) = T_{nm}^{(1)}(t) V_{nm}^{(1)}(r, \varphi) + T_{nm}^{(2)}(t) V_{nm}^{(2)}(r, \varphi)$$

are called *standing wave* solutions. We see, then, that the profile of a standing wave depends on the functions $V_{nm}(r, \varphi)$; the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ change only the amplitude of the standing wave over time, as was the case for standing waves on a string and on the rectangular membrane. Lines on the membrane defined by $V_{nm}(r, \varphi) = 0$ remain at rest for

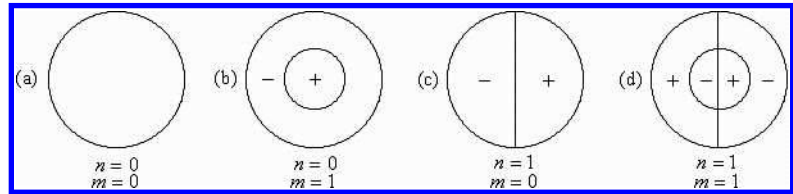


Figure 4.14. Sketch of the first few modes of vibration for the mode $V_{nm}^{(1)}(r, \varphi)$.

all times and are called *nodal lines* of the standing wave $V_{nm}(r, \varphi)$. Points where $V_{nm}(r, \varphi)$ reaches a relative maximum or minimum for all times are called *antinodes* of this standing wave. From the above discussion of the Fourier expansion, we see that an arbitrary motion of the membrane may be thought of as an infinite sum of these standing waves.

Each harmonic, $u_{nm}(r, \varphi, t)$, in the double series in Equation (4.11) represents a natural mode of vibration called a normal mode. Each frequency ω_{nm} has two modes, one with $\cos n\varphi$ and the other with $\sin n\varphi$, exhibiting a twofold degeneracy. The exceptions are the cases of radial symmetry, where $n = 0$ and they are nondegenerate.

Each normal mode possesses a characteristic pattern of nodal lines. The first few of these normal vibration modes for

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi$$

are sketched in Figure 4.14 with similar sketches for the modes $V_{nm}^{(2)}(r, \varphi)$. In the fundamental mode of vibration corresponding to $\mu_0^{(0)}$, the membrane vibrates as a whole; while in the mode of vibration corresponding to $\mu_1^{(0)}$, the membrane vibrates in two parts, as shown with the part labeled with a plus sign initially above the equilibrium level and the part labeled with a minus sign initially below the equilibrium level. The nodal line in this case is a circle, which remains at rest as the two sections reverse location. The mode characterized by $\mu_0^{(1)}$ is equal to zero when $\varphi = \pm\pi/2$ and is positive and negative as shown.

4.3.2 Radial Oscillations of a Membrane

Oscillations of a circular membrane are said to be radial if they do not depend on the polar angle φ (i.e., the deviation of an arbitrary point M from its position of equilibrium at time t depends only on t and the distance

between point M and the center of the membrane). Solutions for radial oscillations will have a simpler form than that for more general types of oscillations. Physically, we see that radial oscillations will occur when initial displacements and initial velocities do not depend on φ , but rather are functions only of r :

$$u(r, \varphi, t)|_{t=0} = \phi(r), \quad \left. \frac{\partial u}{\partial t}(r, \varphi, t) \right|_{t=0} = \psi(r). \quad (4.122)$$

In this case, all coefficients— a_{nm} , b_{nm} , c_{nm} , and d_{nm} , with $n \geq 1$ —equal zero. We may easily verify this, for example, for a_{nm} :

$$\begin{aligned} a_{nm} &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \phi(r) V_{nm}^{(1)}(r, \varphi) r dr d\varphi \\ &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \phi(r) J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \cos n\varphi r dr d\varphi. \end{aligned}$$

Because $\int_0^{2\pi} \cos n\varphi d\varphi = 0$ for any integer $n \geq 1$, we have $a_{nm} = 0$.

Similarly, $b_{nm} = 0$ for $n \geq 1$, and $c_{nm} = 0$, $d_{nm} = 0$ for all n . Thus, the solution does not contain the functions $V_{nm}^{(2)}(r, \varphi)$. If $n = 0$, the coefficients a_{0m} and b_{0m} are nonzero and the formulas used to calculate them can be simplified. In this case, we have

$$a_{0m} = \frac{1}{\|V_{0m}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \phi(r) J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr d\varphi.$$

By putting factors that are independent of φ outside the integral and using $\int_0^{2\pi} d\varphi = 2\pi$, we have, after simplification,

$$a_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l \phi(r) J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr. \quad (4.123)$$

Similarly, we find

$$b_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l \psi(r) J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr. \quad (4.124)$$

Substituting these coefficients into the series in Equation (4.111), we notice that the series reduces from a double series to a single one since all terms in the second sum of this series disappear. Only those terms in the first sum remain, for which $n = 0$, making it necessary to sum only on m but not on n . The final result is

$$\begin{aligned} u(r, \varphi, t) &= \sum_{m=0}^{\infty} \left[a_{0m} y_{0m}^{(1)}(t) + b_{0m} y_{0m}^{(2)}(t) \right] V_{0m}^{(1)}(r, \varphi) \\ &= \sum_{m=0}^{\infty} \left[a_{0m} y_{0m}^{(1)}(t) + b_{0m} y_{0m}^{(2)}(t) \right] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right). \end{aligned} \quad (4.125)$$

Thus, for radial oscillations, the solution contains only Bessel functions of zeroth order.

Next, we consider several examples of solutions of problems for the homogeneous equation of oscillation with homogeneous boundary conditions.

Example 4.6. Find the transverse oscillations of a circular membrane with radius l with a fixed edge. Assume the initial displacement has the form of a paraboloid of rotation, initial velocities are zero, and the reaction of the environment is small enough to be neglected.

Solution. Drawing from the above discussion, we have the following boundary value problem of a circular membrane with a fixed edge:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] &= 0, \\ 0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0, \\ u(r, 0) &= A \left(1 - \frac{r^2}{l^2} \right), \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = 0. \end{aligned}$$

The oscillations of the membrane are radial since the initial displacement and the initial velocities do not depend on the polar angle φ . Thus, only terms with $n = 0$ are not zero.

The boundary conditions of the problem are of the *Dirichlet type*, in which case the eigenvalues $\mu_m^{(0)}$ are the solutions of the equation

$$J_0(\mu) = 0,$$

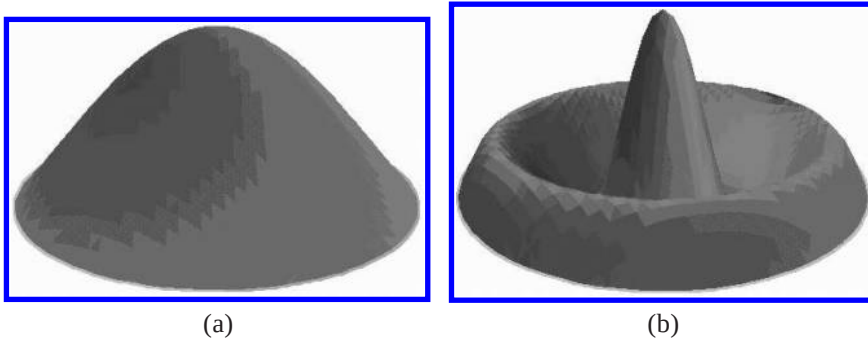


Figure 4.15. Two eigenfunctions, (a) $V_{00}^{(1)}(r, \varphi)$ and (b) $V_{02}^{(1)}(r, \varphi)$, for the Dirichlet boundary conditions in Example 4.6.

and the eigenfunctions are

$$V_{0m}^{(1)}(r, \varphi) = J_0\left(\frac{\mu_m^{(0)}}{l}r\right).$$

The three-dimensional picture shown in Figure 4.15 depicts the two eigenfunctions for the given problem,

$$V_{00}^{(1)}(r, \varphi) = J_0\left(\frac{\mu_0^{(0)}}{l}r\right) \quad \text{and} \quad V_{02}^{(1)}(r, \varphi) = J_0\left(\frac{\mu_2^{(0)}}{l}r\right)$$

(these two eigenfunctions are chosen as examples); the corresponding values of λ are $\lambda_{00} = 1.4458$ and $\lambda_{02} = 18.7218$.

The solution, $u(r, \varphi, t)$, is given by the series

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} a_{0m} \cos \frac{a\mu_m^{(0)}t}{l} J_0\left(\frac{\mu_m^{(0)}}{l}r\right).$$

The coefficients a_{0m} are given by Equation (4.123):

$$a_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l A\left(1 - \frac{r^2}{l^2}\right) J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr.$$

We may calculate the norm of $V_{0m}^{(1)}$ by using Equation (4.100) and the fact that the derivative of $J_0(x)$ gives $-J_1(x)$, a property of Bessel functions

that is valid only for $J_0(x)$. We thus have

$$\left\| V_{0m}^{(1)} \right\|^2 = \pi l^2 \left[J_0' \left(\mu_m^{(0)} \right) \right]^2 = \pi l^2 \left[J_1 \left(\mu_m^{(0)} \right) \right]^2.$$

To calculate the coefficients a_{0m} of the expansion, we use the following properties of Bessel functions of the first kind:

$$\int x^n J_{n-1}(x) dx = x^n J_n(x), \quad (4.126)$$

and

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x). \quad (4.127)$$

Calculating the integrals for the coefficients in a_{0m} , we have, taking into account that $J_0(\mu_m^{(0)}) = 0$,

$$\begin{aligned} \int_0^l J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr &= \frac{l^2}{\left(\mu_m^{(0)} \right)^2} x J_1(x) \Big|_0^{\mu_m^{(0)}} = \frac{l^2}{\mu_m^{(0)}} J_1 \left(\mu_m^{(0)} \right), \\ \frac{1}{l^2} \int_0^l r^2 J_0 \left(\frac{\mu_m^{(0)} r}{l} \right) r dr &= \frac{1}{l^2} \frac{l^4}{\left(\mu_m^{(0)} \right)^4} \int_0^{\mu_m^{(0)}} x^2 [2J_1(x) - xJ_2(x)] dx \\ &= \frac{l^2}{\left(\mu_m^{(0)} \right)^4} [2x^2 J_2(x) - x^3 J_3(x)]_0^{\mu_m^{(0)}} \\ &= \frac{l^2}{\left(\mu_m^{(0)} \right)^4} [(x^3 - 4x)J_1(x) - 6x^2 J_0(x)]_0^{\mu_m^{(0)}} \\ &= \left[\frac{l^2}{\mu_m^{(0)}} - \frac{4l^2}{\left(\mu_m^{(0)} \right)^3} \right] J_1 \left(\mu_m^{(0)} \right). \end{aligned}$$

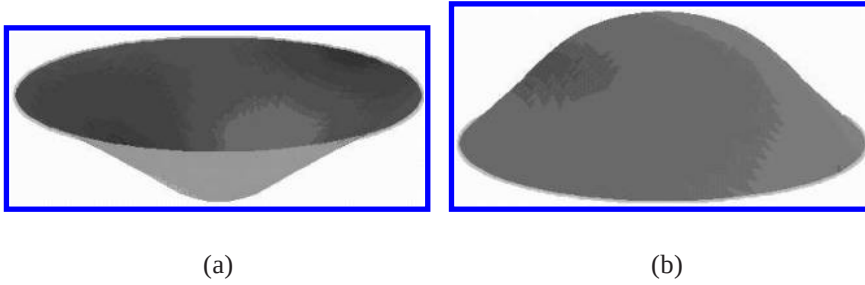


Figure 4.16. Graph of the membrane in Example 4.6 at (a) $t = 2.5$ and (b) $t = 5$.

The final result is

$$\begin{aligned}
 a_{0m} &= \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l A \left(1 - \frac{r^2}{l^2}\right) J_0\left(\frac{\mu_m^{(0)} r}{l}\right) r dr \\
 &= \frac{2\pi A}{\pi l^2 \left[J_1\left(\mu_m^{(0)}\right)\right]^2} \left[\frac{l^2}{\mu_m^{(0)}} - \frac{l^2}{\mu_m^{(0)}} + \frac{4l^2}{\left(\mu_m^{(0)}\right)^3} \right] J_1\left(\mu_m^{(0)}\right) = \frac{8A}{\left(\mu_m^{(0)}\right)^3 J_1\left(\mu_m^{(0)}\right)}.
 \end{aligned}$$

Using this result, we may describe the oscillations of the membrane by the series in Bessel functions of zeroth order:

$$u(r, \varphi, t) = 8A \sum_{m=0}^{\infty} \frac{1}{\left(\mu_m^{(0)}\right)^3 J_1\left(\mu_m^{(0)}\right)} \cos \frac{a\mu_m^{(0)} t}{l} J_0\left(\frac{\mu_m^{(0)} r}{l}\right).$$

Figure 4.16 shows two snapshots of the solution at the times $t = 2.5$ and $t = 5$. This solution was obtained with the program Waves for the case $a^2 = 1$, $l = 2$ and $A = 0.1$.

Example 4.7. Find the transverse oscillations of a homogeneous circular membrane of radius l with a rigidly fixed edge on which the oscillations are initiated by a localized impact, normal to the surface of the membrane. This impact is applied at the point (r_0, φ_0) , where $r_0 < l$, and supplies an impulse I to the membrane. Any initial displacement is absent and the reaction of the environment is negligible.

Solution. The boundary value problem describing the oscillations of the membrane reduces to the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right],$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0$$

under the conditions

$$u(r, \varphi, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = \frac{I}{\rho} \delta(r - r_0) \delta(\varphi - \varphi_0), \quad u(l, \varphi, t) = 0.$$

The product $\delta(r - r_0) \delta(\varphi - \varphi_0)$ is a δ -function in two (polar) dimensions; by multiplying by the area element in polar coordinates given by $r dr d\varphi$ and integrating over this area, we obtain 1 or 0, depending on whether the point (r_0, φ_0) belongs to this area or not, respectively.

The boundary condition of the problem is of the *Dirichlet type*, the eigenvalues are given by equation $J_n(\mu) = 0$, and the eigenfunctions are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \cos n\varphi, \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin n\varphi.$$

The eigenfunctions' squared norms can be calculated by using Equations (4.100) and (4.105):

$$\|V_{nm}^{(1)}\|^2 = \|V_{nm}^{(2)}\|^2 = \varepsilon_n \pi \frac{l^2}{2} \left[J_n' \left(\mu_m^{(n)} \right) \right]^2, \quad \varepsilon_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n > 0. \end{cases}$$

The initial displacement of the membrane is zero, in which case the solution $u(r, \varphi, t)$ is given by the series

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [b_{nm} \cos n\varphi + d_{nm} \sin n\varphi] \cdot J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin \frac{a \mu_m^{(n)} t}{l}. \quad (4.128)$$

Next, we calculate the coefficients b_{nm} and d_{nm} :

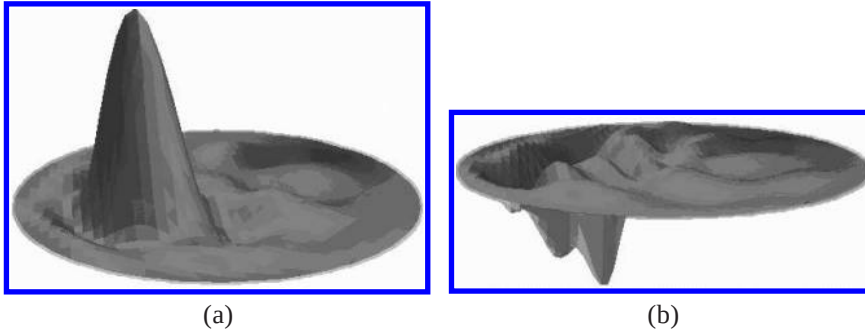


Figure 4.17. Graph of the membrane in Example 4.7 at (a) $t = 0.3$ and (b) $t = 4.3$.

$$\begin{aligned}
 b_{nm} &= \frac{I}{\rho \omega_{nm} \|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \delta(r - r_0) \delta(\varphi - \varphi_0) \cos n\varphi J_n \left(\frac{\mu_m^{(n)} r}{l} \right) r dr d\varphi \\
 &= \frac{2I \cos n\varphi_0}{\varepsilon_n a \pi \rho l \mu_m^{(n)} [J'_n(\mu_m^{(n)})]^2} J_n \left(\mu_m^{(n)} r_0 / l \right),
 \end{aligned} \tag{4.129}$$

and

$$\begin{aligned}
 d_{nm} &= \frac{I}{\rho \omega_{nm} \|V_{nm}^{(2)}\|^2} \int_0^l \int_0^{2\pi} \delta(r - r_0) \delta(\varphi - \varphi_0) \sin n\varphi J_n \left(\frac{\mu_m^{(n)} r}{l} \right) r dr d\varphi \\
 &= \frac{2I \sin n\varphi_0}{\varepsilon_n a \pi \rho l \mu_m^{(n)} [J'_n(\mu_m^{(n)})]^2} J_n \left(\mu_m^{(n)} r_0 / l \right).
 \end{aligned}$$

Therefore, the evolution of the displacements of points on the membrane is described by the series

$$u(r, \varphi, t) = \frac{2I}{a \pi \rho l} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\cos n(\varphi - \varphi_0) J_n \left(\mu_m^{(n)} r_0 / l \right)}{\varepsilon_n \mu_m^{(n)} [J'_n(\mu_m^{(n)})]^2} J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin \frac{a \mu_m^{(n)} t}{l}.$$

Figure 4.17 shows two snapshots of the solution at the times $t = 0.3$ and $t = 4.3$. This solution was obtained with the program Waves for the case $a^2 = 1$, $l = 2$, $r_0 = 1$, $\varphi_0 = \pi$, $I = 10$, and $\rho = 1$.

Example 4.8. A container in the form of a vertical circular cylinder of radius l with a horizontal bottom is partially filled with water and moves for a long time with velocity $v_0 = \text{const}$ in a direction perpendicular to the axis of the container.

Find the oscillations of the surface of the water in the container for $t > 0$ if, at time $t = 0$, the container stops instantly. Assume that for times $t < 0$, the water is not moving relative to the container. The pressure on the free surface of the water may be considered to be constant and other interactions with the environment may be neglected.

Solution. The equation describing the surface before $t = 0$ when the container is moving at constant speed is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right],$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0.$$

The initial and boundary conditions are

$$u(r, \varphi, 0) = v_0 r \cos \varphi, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = 0, \quad \text{and} \quad \frac{\partial u}{\partial r}(l, \varphi, t) = 0.$$

The boundary condition of the problem is of the *Neumann type*, in which case eigenvalues are given by the equation

$$J'_n(\mu) = 0.$$

Obviously, in this case $\lambda_{00} = 0$ and $V_{00}^{(1)} = 0$. The other eigenfunctions are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \cos n\varphi \quad \text{and} \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)} r}{l} \right) \sin n\varphi.$$

The three-dimensional picture shown in Figure 4.18 shows the two eigenfunctions

$$V_{11}^{(1)}(r, \varphi) = J_1 \left(\mu_1^{(1)} r/l \right) \cos \varphi \quad \text{and} \quad V_{13}^{(1)}(r, \varphi) = J_1 \left(\mu_3^{(1)} r/l \right) \cos \varphi$$

(chosen as examples) for Example 4.8; the corresponding values of λ are $\lambda_{11} = 7.1061$ and $\lambda_{13} = 34.2574$.

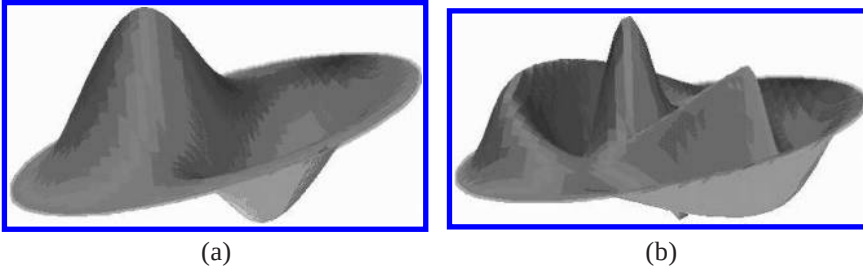


Figure 4.18. Eigenfunctions (a) $V_{11}^{(1)}(r, \varphi)$ and (b) $V_{13}^{(1)}(r, \varphi)$ for the Neumann boundary condition of Example 4.8.

The eigenfunctions' squared norms can be calculated by using Equations (4.100) and (4.105):

$$\|V_{nm}^{(1)}\|^2 = \|V_{nm}^{(2)}\|^2 = \varepsilon_n \pi \frac{l^2}{2(\mu_m^{(n)})^2} \left[(\mu_m^{(n)})^2 - n^2 \right] J_n^2(\mu_m^{(n)}),$$

where

$$\varepsilon_n = \begin{cases} 2, & \text{if } n = 0, \\ 1, & \text{if } n > 0. \end{cases}$$

Let us find the coefficients of the series. Because the second initial condition is homogeneous and $\kappa = 0$, we have $b_{nm} = d_{nm} = 0$ for all n and m . For the remaining coefficients, we have

$$\begin{aligned} a_{nm} &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} v_0 r \cos \varphi V_{nm}^{(1)}(r, \varphi) r dr d\varphi \\ &= \frac{v_0}{\|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \cos \varphi \cos n\varphi d\varphi \int_0^l r^2 J_n \left(\frac{\mu_m^{(n)} r}{l} \right) dr, \\ c_{nm} &= \frac{v_0}{\|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \cos \varphi \sin n\varphi d\varphi \int_0^l r^2 J_n \left(\frac{\mu_m^{(n)} r}{l} \right) dr. \end{aligned}$$

Obviously, $a_{nm} = 0$ for $n \neq 1$ and $c_{nm} = 0$ for all n . Thus, the expansion contains only the Bessel function J_1 :

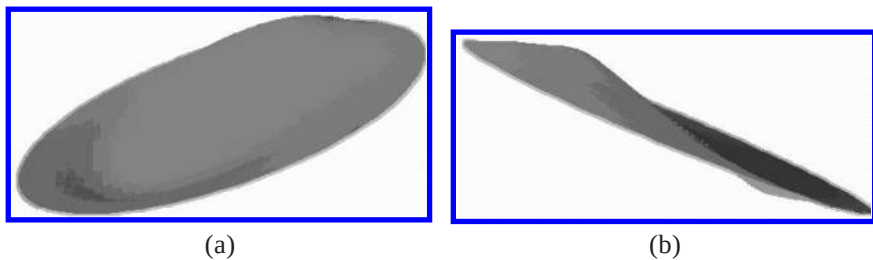


Figure 4.19. Graph of the membrane in Example 4.8 at (a) $t = 1.4$ and (b) $t = 5.4$.

$$\begin{aligned}
 a_{1m} &= \frac{2v_0 \left(\mu_m^{(1)}\right)^2}{l^2 \left[\left(\mu_m^{(1)}\right)^2 - 1 \right] J_1^2 \left(\mu_m^{(1)}\right)} \int_0^l r^2 J_1 \left(\frac{\mu_m^{(1)}}{l} r \right) dr \\
 &= \frac{2v_0 l \mu_m^{(1)} J_2 \left(\mu_m^{(1)}\right)}{\left[\left(\mu_m^{(1)}\right)^2 - 1 \right] J_1^2 \left(\mu_m^{(1)}\right)}.
 \end{aligned}$$

From here, we obtain

$$u(r, \varphi, t) = v_0 \cos \varphi \sum_{m=0}^{\infty} a_{1m} J_1 \left(\frac{\mu_m^{(1)}}{l} r \right) \cos \frac{a \mu_m^{(1)} t}{l}.$$

Figure 4.19 shows two snapshots of the solution at the times $t = 1.4$ and $t = 5.4$. This solution was obtained with the program Waves for the case $a^2 = 0.25$, $l = 2$, and $v_0 = 5$.

Example 4.9. The periphery of a flexible circular membrane of radius l is fixed elastically with coefficient h . The initial displacement is zero. Find the transversal vibrations of the membrane if the initial velocities of the membrane are described by the function

$$\psi(r) = A(l - r).$$

Solution. The boundary value problem consists of the solution of the equation

$$\begin{aligned}
 \frac{\partial^2 u}{\partial t^2} - a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] &= 0, \\
 0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0,
 \end{aligned}$$

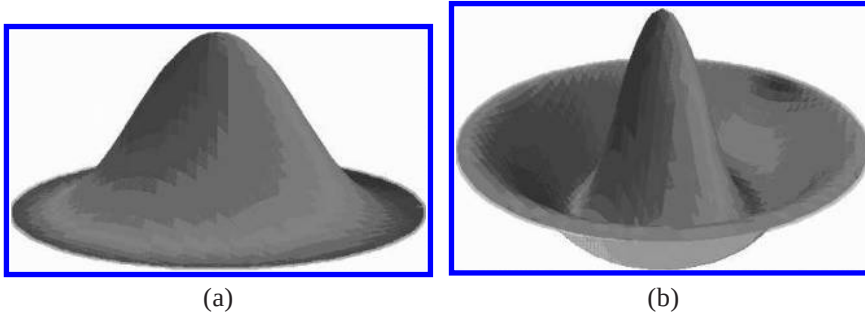


Figure 4.20. Eigenfunctions (a) $V_{01}^{(1)}(r, \varphi)$ and (b) $V_{02}^{(1)}(r, \varphi)$ for the mixed boundary condition of Example 4.9.

with the conditions

$$u(r, \varphi, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = A(l - r), \quad \left. \frac{\partial u}{\partial r} + hu \right|_{r=l} = 0.$$

The oscillations of the membrane are radial since the initial functions do not depend on the polar angle φ , thus only terms with $n = 0$ are not zero.

The boundary condition of the problem is of *mixed type*, in which case eigenvalues $\mu_m^{(0)}$ are given by the roots of the eigenvalue equation

$$\mu J_0'(\mu) + hJ_0(\mu) = 0.$$

The eigenfunctions (see [Figure 4.20](#)) are

$$V_{0m}^{(1)}(r, \varphi) = J_0\left(\frac{\mu_m^{(0)} r}{l}\right).$$

The oscillations of the membrane are radial since the initial functions do not depend on the polar angle φ ; thus, only terms with $n = 0$ are not zero.

The initial displacement is zero, so the coefficients $a_{0m} = 0$. The coefficients b_{0m} are given by Equation (4.124), which results in

$$b_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l A(l - r) J_0\left(\frac{\mu_m^{(0)} r}{l}\right) r dr.$$

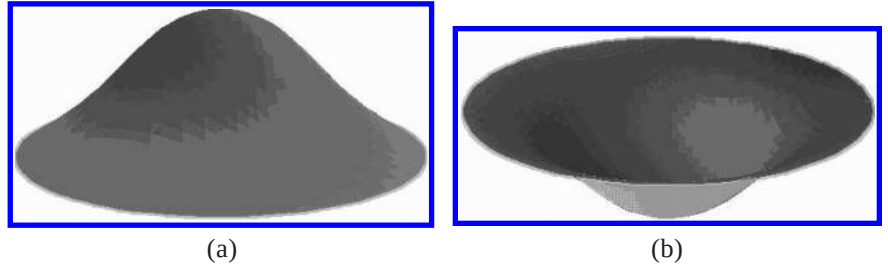


Figure 4.21. Graph of the membrane in Example 4.9 at (a) $t = 2$, (b) $t = 10$.

We may calculate the norm of $V_{0m}^{(1)}$ by using Equation (4.103) and taking into account that $J'_0(x) = -J_1(x)$. We then have

$$\|V_{0m}^{(1)}\|^2 = 2\pi \frac{(\mu_m^{(0)})^2 + l^2 h^2}{2h^2} \left[J'_0(\mu_m^{(0)}) \right]^2 = 2\pi \frac{(\mu_m^{(0)})^2 + l^2 h^2}{2h^2} J_1^2(\mu_m^{(0)}).$$

Combining the above equations yields

$$\begin{aligned} b_{0m} &= \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l A(l-r) J_0\left(\frac{\mu_m^{(0)} r}{l}\right) r dr \\ &= \frac{2h^2 A}{\left[(\mu_m^{(0)})^2 + l^2 h^2\right] J_1^2(\mu_m^{(0)})} \left[\frac{l^3}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) - \int_0^l r^2 J_0\left(\frac{\mu_m^{(0)} r}{l}\right) r dr \right]. \end{aligned}$$

Finally, we may describe the oscillations of the membrane by the series in Bessel functions of zeroth order:

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} b_{0m} \sin \frac{a \mu_m^{(0)} t}{l} J_0\left(\frac{\mu_m^{(0)} r}{l}\right).$$

Figure 4.21 shows two snapshots of the solution at the times $t = 2$ and $t = 10$. This solution was obtained with the program Waves for the case $a^2 = 0.25$, $l = 2$, $h = 1$, and $A = 5$.

4.3.3 The Fourier Method for Nonhomogeneous Equations

In this section, we demonstrate the application of Fourier's method to *nonhomogeneous* equations in polar coordinates for the case of a circular membrane Equation (4.76):

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) + \gamma u = f(r, \varphi, t),$$

where $f(r, \varphi, t)$ is a given function.

First, find solutions satisfying the *homogeneous* boundary condition (Equation (4.80))

$$\alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = 0$$

and *nonhomogeneous* initial conditions (Equation (4.78))

$$u|_{t=0} = \phi(r, \varphi) \text{ and } \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(r, \varphi).$$

We begin, as before, by searching for a solution in the form of the sum

$$u(r, \varphi, t) = u_1(r, \varphi, t) + u_2(r, \varphi, t), \quad (4.130)$$

where $u_1(r, \varphi, t)$ is the solution to the *homogeneous* equation with *homogeneous* boundary and *nonhomogeneous* initial conditions, given by

$$\frac{\partial^2 u_1}{\partial t^2} + 2\kappa \frac{\partial u_1}{\partial t} - a^2 \left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r} \frac{\partial u_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_1}{\partial \varphi^2} \right) + \gamma u_1 = 0, \quad (4.131)$$

$$\alpha \frac{\partial u_1}{\partial r} + \beta u_1 \Big|_{r=l} = 0, \quad (4.132)$$

$$u_1|_{t=0} = \phi(r, \varphi), \quad \frac{\partial u_1}{\partial t} \Big|_{t=0} = \psi(r, \varphi), \quad (4.133)$$

and $u_2(r, \varphi, t)$ is the solution to the *nonhomogeneous* equation with *zero boundary and initial conditions*, given by

$$\frac{\partial^2 u_2}{\partial t^2} + 2\kappa \frac{\partial u_2}{\partial t} - a^2 \left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_2}{\partial \varphi^2} \right) + \gamma u_2 = f(r, \varphi, t), \quad (4.134)$$

$$\alpha \frac{\partial u_2}{\partial r} + \beta u_2 \Big|_{r=l} = 0, \quad (4.135)$$

$$u_2|_{t=0} = 0, \quad \frac{\partial u_2}{\partial t} \Big|_{t=0} = 0. \quad (4.136)$$

Physically, the solution $u_1(r, \varphi, t)$ represents *free oscillations*; that is, oscillations that occur only due to an initial perturbation. The solution $u_2(r, \varphi, t)$ represents *forced oscillations*, or oscillations that result from the action of external forces when initial perturbations are absent.

Methods for finding the solution $u_1(r, \varphi, t)$ for free oscillations were considered in the Section 4.3.2; our task here need only be to find the solutions $u_2(r, \varphi, t)$ for forced oscillations. As in the case of free oscillations, we search for the solution $u_2(r, \varphi, t)$ in the form of the series

$$u_2(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_{nm}^{(1)}(t) V_{nm}^{(1)}(r, \varphi) + T_{nm}^{(2)}(t) V_{nm}^{(2)}(r, \varphi) \right], \quad (4.137)$$

where $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are eigenfunctions of the corresponding homogeneous boundary value problem, and $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ are, for the moment, unknown functions of time, t .

The zero boundary condition given in Equation (4.135) for the function $u_2(r, \varphi, t)$ is satisfied for any choice of $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ under the restriction of uniform convergence of the series because it is known to be satisfied by the functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$. However, the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ must also be selected so that the series (4.137) satisfies the nonhomogeneous Equation (4.134) and the homogeneous initial conditions (4.136).

By substituting the series in Equation (4.137) into Equation (4.134), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{d^2}{dt^2} T_{nm}^{(1)}(t) + 2\kappa \frac{d}{dt} T_{nm}^{(1)}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(1)}(t) \right] V_{nm}^{(1)}(r, \varphi) \\ & + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{d^2}{dt^2} T_{nm}^{(2)}(t) + 2\kappa \frac{d}{dt} T_{nm}^{(2)}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(2)}(t) \right] V_{nm}^{(2)}(r, \varphi) = f(r, \varphi, t). \end{aligned} \quad (4.138)$$

Next, we expand the function $f(r, \varphi, t)$ in a Fourier series with functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ as the basis functions:

$$f(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[f_{nm}^{(1)}(t) V_{nm}^{(1)}(r, \varphi) + f_{nm}^{(2)}(t) V_{nm}^{(2)}(r, \varphi) \right], \quad (4.139)$$

where

$$f_{nm}^{(1)}(t) = \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \int_0^l f(r, \varphi, t) V_{nm}^{(1)} r dr d\varphi, \quad (4.140)$$

and

$$f_{nm}^{(2)}(t) = \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \int_0^l f(r, \varphi, t) V_{nm}^{(2)} r dr d\varphi. \quad (4.141)$$

Comparing the series (4.138) with Equation (4.139) for the same function $f(r, \varphi, t)$, we obtain the following differential equations, which will determine the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$:

$$\frac{d^2}{dt^2} T_{nm}^{(1)}(t) + 2\kappa \frac{d}{dt} T_{nm}^{(1)}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(1)}(t) = f_{nm}^{(1)}(t), \quad (4.142)$$

$$\frac{d^2}{dt^2} T_{nm}^{(2)}(t) + 2\kappa \frac{d}{dt} T_{nm}^{(2)}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(2)}(t) = f_{nm}^{(2)}(t). \quad (4.143)$$

The solution $u_2(r, \varphi, t)$ defined by the series (4.137) satisfies the initial conditions (4.136), which impose on the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ the conditions

$$\begin{cases} T_{nm}^{(1)}(0) = 0, \\ \frac{d}{dt} T_{nm}^{(1)}(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} T_{nm}^{(2)}(0) = 0, \\ \frac{d}{dt} T_{nm}^{(2)}(0) = 0 \end{cases} \quad n, m = 0, 1, 2, \dots \quad (4.144)$$

As for the one-dimensional case, solutions of the Cauchy problems defined in Equations (4.142), (4.143), and (4.144) for functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ can be written in the form

$$T_{nm}^{(1)}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}^{(1)}(\tau) Y_{nm}(t - \tau) d\tau, \quad (4.145)$$

$$T_{nm}^{(2)}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}^{(2)}(\tau) Y_{nm}(t - \tau) d\tau, \quad (4.146)$$

where

$$Y_{nm}(t) = \begin{cases} e^{-\kappa t} \sin \omega_{nm} t & \text{if } \kappa^2 < a^2 \lambda_{nm} + \gamma, \\ e^{-\kappa t} \sinh \omega_{nm} t & \text{if } \kappa^2 > a^2 \lambda_{nm} + \gamma, \\ t e^{-\kappa t} & \text{if } \kappa^2 = a^2 \lambda_{nm} + \gamma, \end{cases}$$

with $\omega_{nm} = \sqrt{|a^2 \lambda_{nm} + \gamma - \kappa^2|}$ if $\kappa^2 \neq a^2 \lambda_{nm} + \gamma$, and $\omega_{nm} = 1$ otherwise. By substituting $f_{nm}^{(1)}(\tau)$ and $f_{nm}^{(2)}(\tau)$ from Equations (4.140) and (4.141) into Equations (4.145) and (4.146), we obtain

$$T_{nm}^{(1)}(t) = \frac{1}{\omega_{nm} \|V_{nm}^{(1)}\|^2} \int_0^t d\tau \int_0^{2\pi} \int_0^l f(r, \varphi, \tau) V_{nm}^{(1)}(r, \varphi) Y_{nm}(t - \tau) r dr d\varphi, \quad (4.147)$$

$$T_{nm}^{(2)}(t) = \frac{1}{\omega_{nm} \|V_{nm}^{(2)}\|^2} \int_0^t d\tau \int_0^{2\pi} \int_0^l f(r, \varphi, \tau) V_{nm}^{(2)}(r, \varphi) Y_{nm}(t - \tau) r dr d\varphi. \quad (4.148)$$

Substituting the expressions for $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ from Equations (4.147) and (4.148) in the series in Equation (4.137), we obtain solutions to the boundary value problem defined in Equations (4.134) through (4.136), assuming that Equation (4.137) and the series obtained from it by twice differentiating term by term with respect to the variables r , φ , and t converges uniformly. Thus, the solution of the problem of forced oscillations with zero boundary conditions is

$$\begin{aligned}
u(r, \varphi, t) &= u_1(r, \varphi, t) + u_2(r, \varphi, t) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[T_{nm}^{(1)}(t) + \left(a_{nm} y_{nm}^{(1)}(t) + b_{nm} y_{nm}^{(2)}(t) \right) \right] \cdot V_{nm}^{(1)}(r, \varphi) \right. \\
&\quad \left. + \left[T_{nm}^{(2)}(t) + \left(c_{nm} y_{nm}^{(1)}(t) + d_{nm} y_{nm}^{(2)}(t) \right) \right] \cdot V_{nm}^{(2)}(r, \varphi) \right\}, \tag{4.149}
\end{aligned}$$

where coefficients $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ are defined by Equations (4.147) and (4.148), and a_{nm} , b_{nm} , c_{nm} , and d_{nm} are as obtained previously in the discussion in Section 4.3.1.

4.3.4 Radial Oscillations of a Circular Membrane

In the case of radial oscillations of a circular membrane, the solution of the nonhomogeneous equation becomes simpler. For radial oscillations, the initial displacement and initial velocity do not depend on φ and are thus functions of r and t only; the same holds for the function f :

$$u(r, \varphi, t)|_{t=0} = \phi(r), \quad \left. \frac{\partial u}{\partial t}(r, \varphi, t) \right|_{t=0} = \psi(r), \text{ and } f(r, \varphi, t) \equiv f(r, t).$$

For radial oscillations, the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ are zero for $n \geq 1$. We may easily verify that this is true for $T_{nm}^{(1)}(t)$, for example. For this case, we have

$$\begin{aligned}
f_{nm}^{(1)}(t) &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} f(r, t) V_{nm}^{(1)} r dr d\varphi \\
&= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} f(r, t) J_n \left(\mu_m^{(n)} r / l \right) \cos n\varphi \cdot r dr d\varphi,
\end{aligned}$$

and because $\int_0^{2\pi} \cos n\varphi d\varphi = 0$ for any nonzero integer n , we have that $f_{nm}^{(1)} = 0$. Consequently,

$$T_{nm}^{(1)}(t) = \frac{1}{\omega_{nm}} \int_0^t f_{nm}^{(1)}(p) Y_{nm}(t-p) dp = 0.$$

Similarly, $f_{nm}^{(2)} = 0$, and therefore $T_{nm}^{(2)} = 0$ for all nonzero n .

If $n = 0$, then coefficients $f_{0m}^{(1)}$ are nonzero, but the integral may be simplified in this case. We remove factors independent of φ from the integral and, by using $\int_0^{2\pi} d\varphi = 2\pi$, we obtain

$$\begin{aligned} f_{0m}^{(1)}(t) &= \frac{1}{\|V_{0m}^{(1)}\|^2} \int_0^l \int_0^{2\pi} f(r, t) J_0\left(\mu_m^{(0)} r/l\right) r dr d\varphi \\ &= \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l f(r, t) J_0\left(\mu_m^{(0)} r/l\right) r dr. \end{aligned}$$

Substituting these functions into the series (4.137), we find that it reduces from a double series to a single series. Finally, for the case of radial oscillations, Equation (4.149) becomes (remembering that for radial oscillations, $V_{nm}^{(2)} = 0$)

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} \left\{ T_{0m}^{(1)}(t) + \left[a_{0m} y_{0m}^{(1)}(t) + b_{0m} y_{0m}^{(2)}(t) \right] \right\} \cdot J_0\left(\frac{\mu_m^{(0)}}{l} r\right).$$

Example 4.10 is a specific oscillating membrane problem in which the solution involves a nonhomogeneous equation with a homogeneous boundary condition.

Example 4.10. Find the transverse oscillations of a homogeneous circular membrane of radius l with a rigidly fixed edge when a variable pressure

$$p = p_0 \cos \omega t$$

acts on one side of the membrane. Assume that initial deviations and initial velocities are absent and that the reaction of the environment is negligibly small.

Solution. The boundary problem modeling the evolution of such oscillations leads to the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] = \frac{p_0}{\rho} \cos \omega t,$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0$$

under zero initial and boundary conditions given by

$$u(r, \varphi, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = 0.$$

From the previous discussion in Section 4.3.3, we have that if the eigenvalues $\mu_m^{(0)}$ are roots of the equation $J_0(\mu) = 0$, the eigenfunctions are

$$V_{0m}^{(1)}(r, \varphi) = J_0\left(\frac{\mu_m^{(0)}}{l}r\right).$$

For this case, the oscillations are radial because the initial conditions are homogeneous and the external pressure is a function of t only. Consequently, the solution $u(r, \varphi, t)$ is defined by the series

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} T_{0m}^{(1)}(t) \cdot J_0\left(\frac{\mu_m^{(0)}}{l}r\right),$$

where

$$T_{0m}^{(1)}(t) = \frac{1}{\omega_{0m}} \int_0^t f_{0m}^{(1)}(p) \sin \omega_{0m}(t - p) dp. \quad (4.150)$$

Using the formulas

$$\left\| V_{0m}^{(1)} \right\|^2 = \pi l^2 \left[J_0'(\mu_m^{(0)}) \right]^2 = \pi l^2 \left[J_1(\mu_m^{(0)}) \right]^2$$

and

$$\int_0^l J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr = \frac{l^2}{\left[\mu_m^{(0)}\right]^2} \int_0^{\mu_m^{(0)}} x J_0(x) dx = \frac{l^2}{\mu_m^{(0)}} J_1(\mu_m^{(0)}),$$

we find

$$f_{0m}^{(1)}(t) = \frac{p_0 \cos \omega t}{\rho \left\| V_{0m}^{(1)} \right\|^2} 2\pi \int_0^l J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr = \frac{2p_0}{\rho \mu_m^{(0)} J_1(\mu_m^{(0)})} \cos \omega t. \quad (4.151)$$

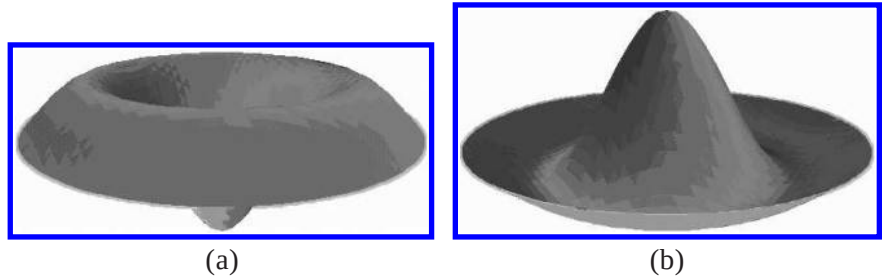


Figure 4.22. Graph of the membrane in Example 4.10 at (a) $t = 7$, (b) $t = 10.5$.

Using Equation (4.151) in Equation (4.150) gives

$$\begin{aligned}
 T_{0m}^{(1)}(t) &= \frac{2p_0}{\rho \omega_{0m} \mu_m^{(0)} J_1\left(\mu_m^{(0)}\right)} \int_0^t \cos \omega p \sin \omega_{0m}(t-p) dp \\
 &= -\frac{2p_0}{\rho \mu_m^{(0)} J_1\left(\mu_m^{(0)}\right) (\omega^2 - \omega_{0m}^2)} [\cos \omega t - \cos \omega_{0m} t].
 \end{aligned}$$

Finally, then, the deflection of the membrane as a function of time is described by the series

$$u(r, \varphi, t) = \frac{2p_0}{\rho} \sum_{m=0}^{\infty} \frac{\cos \omega_{0m} t - \cos \omega t}{(\omega^2 - \omega_{0m}^2) \mu_m^{(0)} J_1\left(\mu_m^{(0)}\right)} J_0\left(\frac{\mu_m^{(0)}}{l} r\right).$$

Figure 4.22 shows two snapshots of the solution at the times $t = 7$ and $t = 10.5$. This solution was obtained with the program Waves for $a^2 = 1$, $l = 2$, $p_0 = 0.25$, and $\omega = 3$.

4.3.5 The Fourier Method for Equations with Nonhomogeneous Boundary Conditions

Consider now the general boundary value problem for equations describing the forced oscillations of a circular membrane with nonhomogeneous boundary and initial conditions given by Equations (4.76) through (4.78). We have

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) + \gamma u = f(r, \varphi, t),$$

$$\alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = g(\varphi, t),$$

$$u|_{t=0} = \phi(r, \varphi), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(r, \varphi).$$

It is not possible to apply the Fourier method directly to this problem because the boundary conditions are nonhomogeneous. However, we may reduce the problem to one with zero boundary conditions.

We search for the solution of the problem in the form of the sum

$$u(r, \varphi, t) = v(r, \varphi, t) + w(r, \varphi, t),$$

where $v(r, \varphi, t)$ is a new, unknown function, and the function $w(r, \varphi, t)$ is chosen so that it satisfies the given nonhomogeneous boundary condition

$$\alpha \frac{\partial w}{\partial r} + \beta w \Big|_{r=l} = g(\varphi, t)$$

and has the necessary number of continuous derivatives in r , φ , and t .

For the function $v(r, \varphi, t)$, we obtain following boundary value problem:

$$\frac{\partial^2 v}{\partial t^2} + 2\kappa \frac{\partial v}{\partial t} - a^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} \right) + \gamma v = f^*(r, \varphi, t),$$

$$\alpha \frac{\partial v}{\partial r}(l, \varphi, t) + \beta v(l, \varphi, t) = 0,$$

$$v(r, \varphi, t)|_{t=0} = \phi^*(r, \varphi), \quad \frac{\partial v}{\partial t}(r, \varphi, t) \Big|_{t=0} = \psi^*(r, \varphi),$$

where

$$f^*(r, \varphi, t) = f(r, \varphi, t) - \frac{\partial^2 w}{\partial t^2} - 2\kappa \frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} \right) - \gamma w, \quad (4.152)$$

$$\phi^*(r, \varphi) = \phi(r, \varphi) - w(r, \varphi, 0), \quad (4.153)$$

$$\psi^*(r, \varphi) = \psi(r, \varphi) - \frac{\partial w}{\partial t}(r, \varphi, 0). \quad (4.154)$$

Solutions to this problem were considered in Section 4.3.3.

Reading Exercise. Verify Equations (4.152) through (4.154).

The function $w(r, \varphi, t)$ can be chosen in different forms, the simplest of which is

$$w(r, \varphi, t) = (c_0 + c_1 r^2)g(\varphi, t),$$

where c_0 and c_1 are constants. These constants must be chosen so that the function $w(r, \varphi, t)$ satisfies the given boundary conditions.

1. For the case of Dirichlet boundary conditions $\alpha = 0$ and $\beta = 1$:

(a) The boundary condition is $u(l, \varphi, t) = g(\varphi, t)$. The auxiliary function (with $c_0 = 0$, $c_1 = 1/l^2$) is given by

$$w(r, \varphi, t) = \frac{r^2}{l^2} g(\varphi, t). \quad (4.155)$$

(b) The boundary condition is $u(l, \varphi, t) = g(t)$ or $u(l, \varphi, t) = g_0 = \text{const.}$

The auxiliary function (with $c_0 = 1$, $c_1 = 0$) is given by

$$w(r, \varphi, t) = g(t) \text{ or } w(r, \varphi, t) = g_0. \quad (4.156)$$

2. For the case of Neumann boundary conditions $\alpha = 1$ and $\beta = 0$:

$$\frac{\partial u}{\partial r}(l, \varphi, t) = g(\varphi, t),$$

and the auxiliary function is (with $c_0 = 0$, $c_1 = 1/2l$)

$$w(r, \varphi, t) = \frac{r^2}{2l} \cdot g(\varphi, t) + C, \quad (4.157)$$

where C is an arbitrary constant.

3. For the case of mixed boundary conditions $\alpha = 1$ and $\beta = h$:

$$\frac{\partial}{\partial r} u(l, \varphi, t) + hu(l, \varphi, t) = g(\varphi, t),$$

and the auxiliary function is (with $c_0 = 0$, $c_1 = 1/(2l + hl^2)$)

$$w(r, \varphi, t) = \frac{r^2}{l(2 + hl)} g(\varphi, t). \quad (4.158)$$

Reading Exercise. We leave it to the reader to check that the chosen function $w(r, \varphi, t)$ in Equations (4.155) through (4.158) satisfies the boundary condition

$$\alpha \frac{\partial w}{\partial r}(l, \varphi, t) + \beta w(l, \varphi, t) = g(\varphi, t). \quad (4.159)$$

We next consider an example of an oscillating membrane problem where we need to find solutions with nonhomogeneous boundary conditions.

Example 4.11. Find the oscillations of a circular membrane ($0 \leq r \leq l$) in an environment without resistance and with zero initial conditions, where the motion is caused by movement at its edge described by

$$u(l, \varphi, t) = A \sin \omega t, \quad t \geq 0.$$

Solution. The boundary problem modeling the evolution of such oscillations is given by the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] = 0,$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0$$

with zero initial conditions

$$u(r, \varphi, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = 0$$

and boundary condition

$$u(l, \varphi, t) = A \sin \omega t.$$

Here, $\alpha = 0$ and $\beta = 1$, so, by using Equation (4.156), we may immediately write

$$w(r, \varphi, t) = A \sin \omega t.$$

Note that for this initial boundary value problem we are searching for the solution in the form of the sum

$$u(r, \varphi, t) = w(r, \varphi, t) + v(r, \varphi, t).$$

For the function $v(r, \varphi, t)$, we obtain the following from the boundary conditions and the above definitions given in Equations (4.152) through (4.154):

$$f^*(r, \varphi, t) = A\omega^2 \sin \omega t,$$

$$\phi^*(r, \varphi) = 0, \quad \psi^*(r, \varphi) = -A\omega.$$

The function $f^*(r, \varphi, t)$ does not depend on the polar angle φ . We therefore see that the function $v(r, \varphi, t)$ defines radial oscillations of the membrane only. The solution $v(r, \varphi, t)$ is thus given by the series

$$v(r, \varphi, t) = \sum_{m=0}^{\infty} \left[T_{0m}^{(1)}(t) + b_{0m} \sin \omega_{0m} t \right] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right),$$

where $\omega_{0m} = a\sqrt{\lambda_{0m}} = a\mu_m^{(0)}/l$ and $\mu_m^{(0)}$ are the roots of equation $J_0(\mu) = 0$.

To determine the coefficients b_{0m} , we have

$$\begin{aligned} b_{0m} &= \frac{1}{\omega_{0m} \|V_{0m}^{(1)}\|^2} \int_0^l \int_0^{2\pi} (-A\omega) J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) r dr d\varphi \\ &= -\frac{A\omega}{\omega_{0m} \pi l^2 \left[J_1 \left(\mu_m^{(0)} \right) \right]^2} \cdot 2\pi \cdot \frac{l^2 J_1 \left(\mu_m^{(0)} \right)}{\left(\mu_m^{(0)} \right)^2} = -\frac{2A\omega}{\omega_{0m} \left(\mu_m^{(0)} \right)^2 J_1 \left(\mu_m^{(0)} \right)}. \end{aligned}$$

Next, we determine the function $f_{0m}^{(1)}(t)$:

$$\begin{aligned} f_{0m}^{(1)}(t) &= \frac{1}{\|V_{0m}^{(1)}\|^2} \int_0^l \int_0^{2\pi} A\omega^2 \sin \omega t J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) r dr d\varphi \\ &= \frac{A\omega^2 \sin \omega t}{\pi l^2 \left[J_1 \left(\mu_m^{(0)} \right) \right]^2} \cdot 2\pi \cdot \frac{l^2 J_1 \left(\mu_m^{(0)} \right)}{\left(\mu_m^{(0)} \right)^2} = \frac{2A\omega^2 \sin \omega t}{\left(\mu_m^{(0)} \right)^2 J_1 \left(\mu_m^{(0)} \right)}. \end{aligned}$$

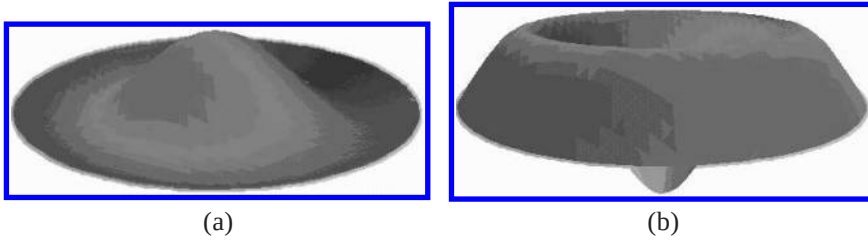


Figure 4.23. Graph of membrane in Example 4.11 at (a) $t = 2$ and (b) $t = 20$.

From this, we have

$$\begin{aligned}
 T_{0m}^{(1)}(t) &= \frac{1}{\omega_{0m}} \int_0^t f_{0m}^{(1)}(\tau) \sin \omega_{0m}(t - \tau) d\tau = \frac{2A\omega^2}{\omega_{0m} \left(\mu_m^{(0)}\right)^2 J_1 \left(\mu_m^{(0)}\right)} \\
 &\times \int_0^t \sin \omega \tau \sin \omega_{0m}(t - \tau) d\tau = \frac{2A\omega^2 [\omega \sin \omega_{0m}t - \omega_{0m} \sin \omega t]}{\omega_{0m} \left(\mu_m^{(0)}\right)^2 (\omega^2 - \omega_{0m}^2) J_1 \left(\mu_m^{(0)}\right)}.
 \end{aligned}$$

Finally, we may express the solution to the wave equation by the series

$$\begin{aligned}
 u(r, \varphi, t) &= w(r, \varphi, t) + v(r, \varphi, t) \\
 &= A \sin \omega t + \sum_{m=0}^{\infty} \left[T_{0m}^{(1)}(t) + b_{0m} \sin \omega_{0m}t \right] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) \\
 &= A \sin \omega t + 2A\omega \sum_{m=0}^{\infty} \frac{[\omega_{0m} \sin \omega_{0m}t - \omega \sin \omega t]}{\left(\mu_m^{(0)}\right)^2 J_1 \left(\mu_m^{(0)}\right) (\omega^2 - \omega_{0m}^2)} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right).
 \end{aligned}$$

Figure 4.23 shows two snapshots of the solution at the times $t = 2$ and $t = 20$. This solution was obtained with the program Waves for $a^2 = 1$, $l = 2$, $A = 0.2$, and $\omega = 3$.

Problems

In Problems 4.1 through 4.26, we consider transverse oscillations of a rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), located in the x - y plane. Solve these problems analytically and then study them in detail with the program Waves. Directions on how to use the program are found in Appendix E. The reader should

assign values for the parameters such as the values of surface mass density ρ , tension T , and coefficient of resistance of the environment, κ . It might be interesting to consider realistic values for the parameters in the following problems. Note that the oscillations should remain small.

The program Waves allows the user to see the oscillation of individual harmonics. The problems discussed refer to membranes, but note that many other, similar physical problems are described by the same two-dimensional hyperbolic equation; for example, oscillations of the surface of a liquid (analogous to a membrane with free boundaries), axial oscillations of a gas in a cylindrical tube, among others.

In all of these problems, the membrane has the initial form $u(x, y, 0) = \varphi(x, y)$ at $t = 0$ and is released with initial velocity $u_t(x, y, 0) = \psi(x, y)$. In Problems 4.1 through 4.5 external forces and resistance of the embedding medium are absent.

Using the program Waves, begin with a prototype of free transverse oscillations of a membrane and investigate the behavior of the membrane for each of the given problems. For each problem change the tension, T , so that the initial (fundamental) period of oscillation increases to: (1) twice the original frequency and (2) four times the original frequency. Also, investigate changing the size of the membrane determined by l_x and l_y under fixed tension T .

4.1. The membrane is fixed along its edges. The initial conditions are

$$\varphi(x, y) = A \sin \frac{\pi x}{l_x} \sin \frac{\pi y}{l_y}, \quad \psi(x, y) = 0.$$

4.2. The edge at $x = 0$ of the membrane is free and other edges are fixed. The initial conditions are

$$\varphi(x, y) = A \cos \frac{\pi x}{2l_x} \sin \frac{\pi y}{l_y}, \quad \psi(x, y) = 0.$$

4.3. The edge at $y = 0$ of the membrane is free and other edges are fixed. The initial conditions are

$$\varphi(x, y) = A \sin \frac{\pi x}{l_x} \cos \frac{\pi y}{2l_y}, \quad \psi(x, y) = 0.$$

4.4. The edges $x = 0$ and $y = 0$ of membrane are fixed, and edges $x = l_x$ and $y = l_y$ are free. The initial conditions are

$$\varphi(x, y) = 0, \quad \psi(x, y) = Axy \left(l_x - \frac{x}{2} \right) \left(l_y - \frac{y}{2} \right).$$

4.5. The edges $x = 0$, $x = l_x$, and $y = l_y$ of membrane are fixed, and the edge $y = 0$ is elastically constrained with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = A x(l_x - x) [y^2(l_y + 1) - l_y^2(y + 1)], \quad \psi(x, y) = 0.$$

In Problems 4.6 through 4.10, external forces are absent and the resistance of the embedding environment is proportional to velocity with proportionality constant κ .

Using the program Waves, begin with a prototype of free transverse oscillations of a membrane using a coefficient of resistance, κ , so that oscillations decay to zero (with a precision of 5%) during: (1) three periods of oscillation, (2) five periods of oscillation, and (3) ten periods of oscillations. Apply the following boundary conditions to the prototype.

4.6. The edges $x = 0$, $y = 0$, and $y = l_y$ of the membrane are fixed, and the edge $x = l_x$ is attached elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = 0, \quad \psi(x, y) = Axy(l_y - y)[(x - l_x)(l_x + 1) - l_x].$$

4.7. The edges $x = 0$ and $y = 0$ of the membrane are fixed, the edge $y = l_y$ is free, and the edge $x = l_x$ is attached elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = 0, \quad \psi(x, y) = Axy \left(l_x - \frac{2x}{3} \right) \left(l_y - \frac{y}{2} \right).$$

4.8. The edges $x = l_x$ and $y = l_y$ are fixed, the edge $x = 0$ is free, and the edge $y = 0$ is attached elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = A(l_x^2 - x^2)(y - l_y)^2, \quad \psi(x, y) = 0.$$

4.9. The edges $y = 0$ and $y = l_y$ are free, the edge $x = 0$ is fixed, and the edge $x = l_x$ is attached elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = Axy^2(3l_y - 2y)[(l_x + 1)(l_x - x) + l_x], \quad \psi(x, y) = 0.$$

4.10. The edges $x = 0$ and $y = l_y$ are free, the edge $x = l_x$ is attached elastically with coefficient of elasticity $h = 1$ and the edge $y = 0$ is attached elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\varphi(x, y) = A[l_x^2 + 2l_x - x^2][(l_y - y)^2 + 2l_y - l_y^2], \quad \psi(x, y) = 0.$$

In Problems 4.11 through 4.15 at the initial instant of time, $t = 0$, the membrane is set in motion by a blow, which applies an impulse I at the point (x_0, y_0) ($0 < x_0 < l_x$, $0 < y_0 < l_y$).

Using the program Waves, begin with a prototype of free transverse oscillations of a membrane with no external force, zero initial deviations from equilibrium, and resistance proportional to velocity, determined by a coefficient of resistance, $\kappa > 0$.

4.11. The membrane is fixed along its edges.

4.12. The edge $x = l_x$ of membrane is free and the other edges are fixed.

4.13. The edges $x = 0$ and $y = l_y$ of the membrane are free, and the edges $x = l_x$ and $y = 0$ are fixed.

4.14. The edges $x = l_x$, $y = 0$, and $y = l_y$ of the membrane are fixed, and the edge $x = 0$ is attached elastically with coefficient of elasticity $h = 1$.

4.15. The edges $y = 0$ and $y = l_y$ are free, the edge $x = l_x$ is fixed, and the edge $x = 0$ is attached elastically with coefficient of elasticity $h = 1$.

In problems 4.16 through 4.20, assume that a force density $f(x, y, t)$ acts, the initial displacement from equilibrium as well as the initial velocity are zero, and the resistance of the embedding medium is absent ($\kappa = 0$).

Solve the equations of motion with the given boundary conditions for each case given below. Using the program Waves, begin with a prototype of free transverse oscillations of a membrane with an external force.

4.16. The membrane is fixed along its boundaries. The external force is

$$f(x, y, t) = Ax \sin \frac{2\pi y}{l_y} (1 - \sin \omega t).$$

4.17. The edge of the membrane at $x = 0$ is free, and other edges are fixed. The external force is

$$f(x, y, t) = A(l_x - x) \sin \frac{\pi y}{l_y} \sin \omega t.$$

4.18. The edges $x = l_x$ and $y = l_y$ of the membrane are free, and edges $x = 0$ and $y = 0$ are fixed. The external force is

$$f(x, y, t) = Ax \sin \frac{\pi y}{l_y} \cos \omega t.$$

4.19. The edges $x = 0$, $y = 0$, and $y = l_y$ of the membrane are fixed, and the edge $x = l_x$ is attached elastically with elasticity coefficient $h = 1$. The external force is

$$f(x, y, t) = Axy \cos \frac{\pi y}{2l_y} (1 - \sin \omega t).$$

4.20. The edges $x = l_x$ and $y = 0$ are free, the edge $x = 0$ is attached elastically with elasticity coefficient $h = 1$, and the edge $y = l_y$ is attached elastically with elasticity coefficient $h = 1$. The external force is

$$f(x, y, t) = A \sin \frac{\pi x}{l_x} \cos \frac{\pi y}{2l_y} \sin \omega t.$$

In Problems 4.21 through 4.25, the boundary of the membrane is driven. Assume the initial velocities are zero and the resistance of the embedding material is absent ($\kappa = 0$). The initial shape of the membrane, $u(x, y, 0) = \varphi(x, y)$, and nonhomogeneous terms in the mixed boundary condition are given. Solve the equations of motion with the given boundary conditions for each case. Using the program Waves, begin with a prototype of free transverse oscillations of a membrane with forced boundaries.

4.21. The motion of the edge $x = 0$ of the membrane is given by

$$g_1(y, t) = A \sin \frac{\pi y}{l_y} \cos \omega t,$$

and the other edges are fixed. Initially, the membrane has the shape

$$\varphi(x, y) = A \left(1 - \frac{x}{l_x} \right) \sin \frac{\pi y}{l_y}.$$

4.22. The motion of the edge $y = l_y$ of the membrane is given by

$$g_4(x, t) = A \sin \frac{\pi x}{l_x} \sin \omega t,$$

and the other edges are fixed. Initially, the membrane has the shape

$$\varphi(x, y) = Axy \cos \frac{\pi x}{2l_x}.$$

4.23. Edges $x = 0$ and $x = l_x$ are fixed, the edge $y = 0$ is free, and the edge $y = l_y$ moves as

$$g_4(x, t) = A \cos \frac{\pi x}{2l_y} \cos \omega t.$$

Initially, the membrane has the shape

$$\varphi(x, y) = Ax(l_x - x) \sin \frac{\pi y}{l_y}.$$

4.24. The edges $x = 0$ and $y = l_y$ are fixed, and the edge $y = 0$ is subject to the action of a harmonic force causing displacements

$$g_3(x, t) = A \sin \frac{\pi x}{l_x} \cos \omega t.$$

Initially, the membrane has the shape

$$\varphi(x, y) = Ax(y - l_y) \cos \frac{\pi x}{2l_x}.$$

4.25. The edges $y = 0$ and $y = l_y$ are fixed, the edge $x = 0$ is free, and the edge $x = l_x$ undergoes harmonic motion given by

$$g_2(y, t) = Ay \sin^2 \frac{\pi y}{l_y} (1 - \sin \omega t).$$

Initially, the membrane has the shape

$$\varphi(x, y) = Ay(l_y - y) \cos \frac{\pi y}{l_y}.$$

4.26. Find the natural frequencies of vibration of a rectangular membrane described by the Klein-Gordon equation,

$$\nabla^2 u - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = m^2 u.$$

Find the eigenfunctions for this membrane with

1. fixed edges,
2. free edges.

In the following problems, consider transverse oscillations of a homogeneous *circular* membrane of radius l , located in the x - y plane. Find analytical solutions and also use the program Waves to find solutions. It is more convenient to solve problems using dimensionless variables, but it is also useful to suggest realistic physical values for surface mass density ρ of the membrane, tension T , and the coefficient of resistance of the environment κ . Note that the physical coefficients should be chosen so that oscillations are small.

For all problems, the displacement of the membrane, $u(r, \varphi, 0) = \phi(r, \varphi)$, is given at some initial moment of time, $t = 0$, and the membrane is released with initial velocity $u_t(r, \varphi, 0) = \psi(r, \varphi)$.

In Problems 4.27 through 4.32, external forces and resistance of the environment are absent. Find solutions and simulate the transverse oscillations of the membrane using the program Waves for the following cases. Change the tension T so that the period of oscillation (considered as the fundamental) increases to: (1) two times the fundamental, and (2) four times the fundamental. Obtain the same result by changing the radius of the membrane l for a constant tension T .

4.27. The membrane is fixed along its contour. The initial conditions are

$$\phi(r, \varphi) = Ar(l^2 - r^2) \sin \varphi, \quad \psi(r, \varphi) = 0.$$

4.28. The membrane is fixed along its contour. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar(l^2 - r^2) \cos 4\varphi.$$

4.29. The edge of the membrane is free. The initial conditions are

$$\phi(r, \varphi) = Ar \left(l - \frac{r}{2} \right) \sin 3\varphi, \quad \psi(r, \varphi) = 0.$$

4.30. The edge of the membrane is free. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar \left(l - \frac{r}{2} \right) \cos 2\varphi.$$

4.31. The edge of the membrane is fixed elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\phi(r, \varphi) = Ar [(l + 2)(1 - r) + r^2] \sin(\pi - \varphi), \quad \psi(r, \varphi) = 0.$$

4.32. The edge of the membrane is fixed elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar [(l + 2)(1 - r) + r^2] \cos(\pi + \varphi).$$

In Problems 4.33 through 4.38, external forces are absent, but the coefficient of resistance of the environment (a force of resistance proportional to velocity) is κ .

For the following cases, simulate free transverse oscillations of the membrane. Choose the coefficient of resistance κ so that oscillations decay to zero (to within 5% accuracy) during: (1) four periods of oscillation, (2) six periods of oscillation, and (3) ten periods of oscillation.

4.33. The membrane is fixed along its contour. The initial conditions are

$$\phi(r, \varphi) = Ar^3 (l^2 - r^2) \sin 3\varphi, \quad \psi(r, \varphi) = 0.$$

4.34. The membrane is fixed along its contour. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar^2 (l^2 - r^2) \sin 2\varphi.$$

4.35. The edge of the membrane is free. The initial conditions are

$$\phi(r, \varphi) = Ar \left(l - \frac{r}{2} \right) \sin 5\varphi, \quad \psi(r, \varphi) = 0.$$

4.36. The edge of the membrane is free. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar \left(l - \frac{r}{2} \right) \cos 4\varphi.$$

4.37. The edge of the membrane is fixed elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\phi(r, \varphi) = Ar \left[r - \frac{l(2 + hl)}{1 + hl} \right] \sin 3\varphi, \quad \psi(r, \varphi) = 0.$$

4.38. The edge of the membrane is fixed elastically with coefficient of elasticity $h = 1$. The initial conditions are

$$\phi(r, \varphi) = 0, \quad \psi(r, \varphi) = Ar \left[r - \frac{l(2 + hl)}{1 + hl} \right] \cos 3\varphi.$$

In Problems 4.39 through 4.41, the membrane is excited at time $t = 0$ by a sharp impact from a hammer, transferring to the membrane an impulse I at a point (r_0, φ_0) , where $0 < r_0 < l$ and $0 \leq \varphi_0 < 2\pi$.

For the following situations, simulate free transverse oscillations of the membrane, assuming the initial displacement is zero, external forcing is absent, and the environment causes a resistance proportional to velocity ($\kappa > 0$).

4.39. The membrane is fixed along its contour.

4.40. The edge of the membrane is free.

4.41. The edge of the membrane is attached elastically with elastic coefficient $h = 1$.

In Problems 4.42 through 4.46 for the given external force, simulate the transverse oscillations of the membrane caused by a transverse force with force density $f(r, \varphi, t)$ continuously applied to the membrane. Assume the initial displacement is zero and resistance of environment is absent ($\kappa = 0$).

4.42. The membrane is fixed along its contour. The external force is

$$f(r, \varphi, t) = A (\sin \omega t + \cos \omega t).$$

4.43. The membrane is fixed along its contour. The external force is

$$f(r, \varphi, t) = A(l - r) \sin \omega t.$$

4.44. The edge of the membrane is free. The external force is

$$f(r, \varphi, t) = Ar \cos \omega t.$$

4.45. The edge of the membrane is free. The external force is

$$f(r, \varphi, t) = A(l^2 - r^2) \sin \omega t.$$

4.46. The edge of the membrane is fixed elastically with coefficient of elasticity $h = 1$. The external force is

$$f(r, \varphi, t) = A(l - r) \cos \omega t.$$

In Problems 4.47 through 4.51, simulate the transverse oscillations of a membrane caused by its borders being displaced according to the function $g(\varphi, t)$ (the nonhomogeneous term in the boundary condition). Assume external forces, initial velocities, and resistance of the environment are absent ($\kappa = 0$). The initial displacement is given by $u(r, \varphi, 0) = \phi(r, \varphi)$.

4.47. $g(\varphi, t) = A \sin^2 \omega t$, $\phi(r, \varphi) = Ar(l - r) \sin \varphi$.

4.48. $g(\varphi, t) = A(\sin \omega t + \cos \omega t)$, $\phi(r, \varphi) = A(l^2 - r^2)$.

4.49. $g(\varphi, t) = A(1 - \cos \omega t)$, $\phi(r, \varphi) = A(l^2 - r^2) \sin \varphi$.

4.50. $g(\varphi, t) = A \sin^2 \omega t$, $\phi(r, \varphi) = Ar(l - \frac{r}{2}) \cos 2\varphi$.

4.51. $g(\varphi, t) = A(1 - \cos \omega t)$, $\phi(r, \varphi) = Ar(l - r) \sin 3\varphi$.

4.52. Find the natural frequencies of a circular membrane described by the Klein-Gordon equation

$$\nabla^2 u - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = m^2 u.$$

Find the eigenfunctions of the membrane for the case of

1. fixed edges,
2. free edges.

5

One-Dimensional Parabolic Equations

In this chapter, we consider a general class of equations known as parabolic equations and their solutions. We start in Section 5.1 with a comparison of two specific physical examples: heat conduction and diffusion. More general properties of parabolic equations and their solutions are discussed in subsequent sections.

5.1 Physical Problems Described by Parabolic Equations: Boundary Value Problems

We begin in Section 5.1 with the specific example of heat conduction in its most general form and continue in Section 5.1.2 with a second example involving diffusion. We show that the mathematical formulation of these two different physical problems leads to identical parabolic equations. For both cases, we look at a variety of possible physical situations.

5.1.1 Heat Conduction

Heat may be defined as the flow of energy through a body (we consider *solid bodies* first) due to a difference in temperature. This is a kinetic process at the molecular level and involves energy transfer due to molecular collisions. In this chapter we introduce the equations that model heat transfer in solids where there is no macroscopic mass transfer. Heat flow

through a solid due to a temperature change is assumed to obey the linear heat flow equation established by Fourier:

$$\vec{q} = -\kappa \nabla T \quad (5.1)$$

Here $\vec{q}$ is the heat flux (or current density), which is the heat (or energy) that flows through a cross-sectional surface area of the solid during one time unit. The quantity ∇T is the temperature gradient (the difference in temperature along a line parallel to the flux), and the coefficient κ is called the thermal conductivity. In the International System of Units (SI), temperature is measured in K; flux is measured in $J/(m^2s)$; and thermal conductivity is measured in $J/(msK)$, where K is kelvin, J is joules, m is meters, and s is seconds. Since $\vec{q}$ and ∇T are oppositely directed (heat flows down a temperature gradient), it must be the case that $\kappa > 0$. Generally, κ is a function of temperature, T , and pressure, p . In the solid mediums we consider here, p remains constant and κ does not depend on p . In the problems in this section, κ is considered as a constant independent of T also. The equation we derive to describe the temperature evolution is the consequence of Equation (5.1) and the law of conservation of energy:

The amount of heat absorbed by a volume must equal the total heat flow into the volume through its surface.

For an arbitrary volume V enclosed by a surface S , the heat flux into the solid per unit time is given by

$$-\oint_S \vec{q} \cdot d\vec{S},$$

where the minus sign corresponds to the conventional definition of a surface element $d\vec{S}$ directed outward, normal to the surface (see [Figure 5.1](#)). In a time interval Δt , the total heat flux is Δt times the heat flux per unit time. This heat flow raises the temperature within the volume V by ΔT so that

$$-\oint_S \vec{q} \cdot d\vec{S} \Delta t = \iiint_V (c\rho \Delta T) dV, \quad (5.2)$$

where ρ is the mass density of the medium and c is its heat capacity.

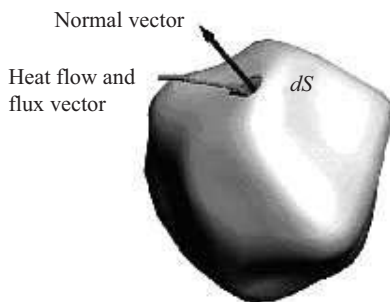


Figure 5.1. Heat flow schematic.

We may modify the above expression by using Gauss's divergence theorem (shown in Appendix D), given by

$$-\oint_S \vec{q} \cdot d\vec{S} = \iiint_V \text{div} \vec{q} dV, \quad (5.3)$$

which relates the divergence of a current density that (in this case) describes heat flow through the surface enclosing this volume with flow out of the volume. Here we define flow out of the volume as positive.

From Equations (5.2) and (5.3), we obtain the *continuity equation*,

$$\rho c \frac{\partial T}{\partial t} + \text{div} \vec{q} = 0, \quad (5.4)$$

where, in the limit of very small time intervals, Δt , we have replaced $\Delta T / \Delta t$ by $\partial T / \partial t$. In words, Equation (5.4) may be stated as:

The amount of heat (or energy) obtained by a unit volume of a body during some unit time interval equals the negative of the divergence of the current density.

Using the relation

$$\text{div} \vec{q} = \text{div} [-\kappa \nabla T] = -\kappa \nabla^2 T,$$

Equations (5.1) and (5.4) lead to the *heat conduction equation* given by

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T. \quad (5.5)$$

The coefficient $\chi = \kappa/c\rho$ is called the thermal diffusivity. For water (ice) at 0° Celsius, $\chi_{water} \approx 1.4 \cdot 10^{-7} m^2/s$; for wood, it is two to three times this value; for typical mineral materials, it is about $10^{-6} m^2/s$; and for metals, it is about two orders of magnitude greater (for instance, for aluminum, $\chi_{Al} \approx 0.86 \cdot 10^{-4} m^2/s$). Equation (5.5) is applicable to solids when the heat transfer is entirely due to heat conduction, and for other mediums if macroscopic mass motion is negligible.

In cases when the heat distribution within a nonuniformly heated body is maintained in a steady state—for instance, with the help of some external heat source—the temperature in the solid becomes time independent and the heat equation reduces to Laplace’s equation, given by

$$\nabla^2 T = 0. \quad (5.6)$$

For cases when the thermal coefficient, κ , is not constant, instead of Equation (5.6) we have, for the steady-state situation,

$$\text{div}(\kappa \nabla T) = 0.$$

We may also write an equation for the case where a medium contains a heat source (e.g., heating by an electric current) or a heat absorber. If Q is the rate at which heat is added (or removed) per unit time and unit volume, we have an additional term

$$\iiint_V Q dV \cdot \Delta t$$

on the left-hand side of Equation (5.2) so that the heat conduction equation, Equation (5.5), becomes

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T + \frac{Q}{\rho c}. \quad (5.7)$$

5.1.2 The Diffusion Equation

Diffusion is a mixing process that occurs when one substance is introduced into a second substance. The introduced quantity, by means of molecular or atomic transport, spreads from locations where its concentration is higher to locations where the concentration is lower. Examples include the diffusion of perfume molecules into a room when the bottle is opened

and the diffusion of neutrons in a nuclear reactor. Given sufficient time, diffusion will lead to an equalizing of the concentration of the introduced substance. We may imagine situations, however, where equilibrium is not reached—for example, by continually adding more of the introduced substance at one location and/or removing it at another location.

Experiments show that, for low-concentration gradients, diffusion obeys Fick's law, in which the current density of each component of a mixture is proportional to the concentration gradient of this component:

$$\vec{I} = -D\nabla q. \quad (5.8)$$

Here ∇q is the concentration gradient, where q is measured in kg or moles per m^3 (note that the symbol q in this section represents a different physical quantity than previously). Current density, I , is measured in kg or moles or molecules per m^2s , and D , the coefficient of diffusion, is measured in m^2/s . The case $I = \text{constant}$ is the case of steady-state diffusion, in which a substance is introduced and removed at the same rate.

Despite very high molecular speeds at room temperatures, diffusion is a slow process because the vast number of intermolecular collisions leads to very long zigzag paths for each molecule. For example, for the mixing of two gases, H_2 and O_2 , $D \approx 6.8 \times 10^{-5} m^2/s$. The coefficient D can also be expressed by using molecular motion characteristics such as the mean free path length; λ (the average distance between collisions); and the average velocity, $\langle v \rangle$, due to the ambient temperature: $D \approx \lambda \langle v \rangle$.

Let us next derive an equation that describes changes of concentration of the introduced substance. The continuity equation

$$\frac{\partial q}{\partial t} + \text{div } \vec{I} = 0, \quad (5.9)$$

is a statement of conservation of mass:

Any increase in the amount of molecules in some volume must equal the amount of molecules entering through the surface enclosing this volume.

By substituting $\vec{I}$ from Equation (5.8) into Equation (5.9), we obtain the diffusion equation

$$\frac{\partial q}{\partial t} = D\nabla^2 q. \quad (5.10)$$

If the introduced substance is being created (or destroyed) (e.g., for example in chemical reactions), we may describe this action by some function f on the right-hand side of the continuity Equation (5.10). The diffusion equation then takes the form

$$\frac{\partial q}{\partial t} = D \nabla^2 q + f. \quad (5.11)$$

If we now compare Equations (5.10) and (5.11) to Equations (5.5) and (5.7), we see they are identical equations mathematically, known as parabolic equations. Only the connection of the mathematical quantities to physical quantities is different. As is often the case in physics and engineering, one set of equations may be applied to several different physical situations. We need only solve this equation once to have solutions to several different types of physical problems simply by changing the definitions of various terms. Note that most of the equations we consider in this book have multiple applications.

5.1.3 The One-Dimensional Parabolic Equation

Now we consider in more detail the one-dimensional parabolic equation. To simplify the discussion, we consider the heat conduction equation, but the same equation with different definitions for the components applies to the case of diffusion. For the one-dimensional case, the temperature, which we will denote as $u(x, t)$, depends on one space coordinate, x , along the length of the rod and time, t .

We start by considering heat conduction leading to a temperature variation within a rigid rod (or bar) of length l (see [Figure 5.2](#)). Assume that the temperature does not vary with the cross section, in which case we can

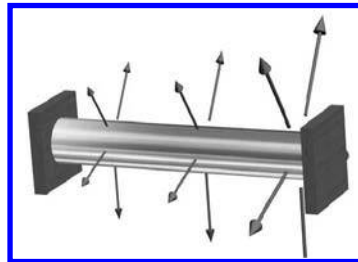


Figure 5.2. Heated rod.

use the one-dimensional equation for heat flow. If the bar does not have a uniform temperature initially, then heat will be conducted from places of higher temperature to places of lower temperature. In the simplest case, when the lateral surface and the ends of the bar are insulated, conduction leads to a completely uniform final heat distribution at all points in the bar. If heat exchange with the environment occurs, or heat is generated in some region of the bar, the temperature distribution becomes more complicated, as is shown next.

We first consider a bar with the lateral surface (i.e., the sides, as opposed to the ends) insulated, and examine the effects of environmental interaction with the ends. Let the function $u(x, t)$ represent the temperature of a cross section of the rod with coordinate x at time t . We derive the equation for the function $u(x, t)$ employing only the conservation of energy and Fourier's law.

Select a region of the bar bounded on the ends by cross sections at locations x and $x + \Delta x$. According to Fourier's law the amount of heat that flows through cross section S at the coordinate x during time Δt is $dQ = -\kappa S(\partial u / \partial x) \Delta t$. If we neglect infinitesimally small high-order quantities, then value of the partial derivative $\partial u / \partial x$ at point $x + \Delta x$ is

$$\frac{\partial u}{\partial x} + d_x \left(\frac{\partial u}{\partial x} \right) = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \Delta x.$$

Therefore, the heat flow through cross section S at $x + \Delta x$ is

$$-\kappa S \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \Delta x \right) \Delta t.$$

Taking the difference between incoming and outgoing heat currents, we obtain the change of the quantity of energy (heat), ΔQ , in the chosen region of the bar during time Δt :

$$\Delta Q = -\kappa S \frac{\partial u}{\partial x} \Delta t + \kappa S \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \Delta x \right) \Delta t = \kappa S \frac{\partial^2 u}{\partial x^2} \Delta x \Delta t. \quad (5.12)$$

We may also suppose, during the same time interval, that the temperature has changed by the quantity $(\partial u / \partial t) \Delta t$. This change of temperature implies a heat flow, which can be expressed as follows:

$$\Delta Q = c \rho S \Delta x \frac{\partial u}{\partial t} \Delta t. \quad (5.13)$$

Setting Equations (5.12) and (5.13) for the heat flow, ΔQ , equal, we obtain

$$c\rho \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

Finally, with the definition $\kappa/c\rho = a^2$ we obtain the heat conduction equation for a homogeneous bar without heat sources:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \quad (5.14)$$

As can be seen from Equation (5.5), $a^2 = \kappa/c\rho$ is the thermal diffusivity, χ .

Now suppose that energy in the form of heat can be generated or absorbed in some regions of the bar. Let the function $F(x, t)$ describe the density of heat sources (or sinks if $F(x, t) < 0$) such that the amount of heat generated within the region of the bar $(x, x + \Delta x)$ during time interval $(t, t + \Delta t)$ equals

$$F(x, t)\Delta x\Delta t.$$

For example, if a steady electric current within the bar generates heat, we have a heating effect per unit length given by $F(x, t) = \text{const} = I^2 R$, where I is electric current and R is resistance per unit length.

Having composed an equation of heat balance, we obtain, from the above definitions,

$$c\rho S \frac{\partial U}{\partial t} \Delta x \Delta t = kS \frac{\partial^2 u}{\partial x^2} \Delta x \Delta t + F(x, t)\Delta x \Delta t.$$

Using the notation $f(x, t) = F(x, t)/c\rho S$, we arrive at the nonhomogeneous heat equation describing heat conduction with the presence of sources or absorbers of heat within a rod:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t). \quad (5.15)$$

Now consider the possibility of heat exchange with the environment in the case that the lateral surfaces of the rod are not insulated. The heat exchange in this case obeys Newton's law of cooling, which says that the flux across the surface is proportional to the temperature difference between the surface and the surrounding medium. For the one-dimensional case, this

means that the amount of heat lost by the bar per unit length and unit time, is

$$q = H(u - \theta),$$

where $\theta(x, t)$ is the temperature of the environment (when $\theta = \text{const}$, it can be considered zero since the temperature scale in this case is arbitrary). The quantity H is called the coefficient of heat exchange, or coefficient of surface heat transfer. The units of q are $[q] = J/(m \cdot s)$ and the dimensions of H and κ are the same (notice from the units that this is a one-dimensional example). Now we can write the net effective density of heat sources at x and time t as

$$\tilde{F} = F(x, t) - H(u - \theta),$$

where $F(x, t)$ is the density of heat sources inside the bar. Thus the heat conduction equation with lateral heat exchange has the following form:

$$u_t = a^2 u_{xx} - \gamma u + f(x, t), \quad (5.16)$$

where $\gamma = H/c\rho S$ and $f(x, t) = \gamma\theta(x, t) + F(x, t)/c\rho S$.

Let us now, as an example of the various applications of this equation, switch to the physical case of diffusion, where $u(x, t)$ represents the concentration of the introduced substance at a given location and time instead of temperature. We may also allow the particles of the diffusing substance to be unstable in the sense that they may disappear (such as an unstable gas or a gas being absorbed) or multiply (as with neutron diffusion). If the rates of these processes at each point in space are proportional to the concentration, the process is described by

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \beta u, \quad (5.17)$$

where D is the coefficient of diffusion, and β is the coefficient of disintegration ($\beta < 0$) or multiplication ($\beta > 0$).

We may also consider the possibility of a diffusing, stable substance participating in the motion of the material in which it is diffusing (which can be a *fluid* or a *gas*). Suppose the fluid and diffusing substance flow along the x -axis with velocity v , or in the simplest case, a sample of fluid containing the introduced substance, moves along the x -axis. Selecting the element $(x, x + \Delta x)$ and considering the amount of substance that flows

through cross sections at x and $x + \Delta x$ due to diffusion as well as fluid motion, we derive an equation that includes the first derivative of u with respect to x in addition to the second derivative (which was already present due to diffusion alone):

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x}. \quad (5.18)$$

Later we show how Equation (5.18), with the help of proper substitution, can be reduced to the standard heat conduction Equation (5.14).

5.1.4 Initial and Boundary Conditions

Knowledge of the heat conduction equation is not enough to determine the temperature inside a body at some moment in time. We also must know the initial temperature distribution within the body (the initial condition; see Figure 5.3), and heat regime at the surface of the body (boundary conditions). We encounter this situation several more times in this book: *General solutions to the applicable mathematical equations must be limited by using arguments from physical constraints to be useful in modeling a physical situation.*

Let us discuss the one-dimensional case first. Knowledge of the initial temperature distribution means that, at time $t = 0$, we are given some function $\varphi(x)$ such that

$$u(x, 0) = \varphi(x). \quad (5.19)$$

Boundary conditions on the cross sections of the ends (i.e., at points $x = 0$ and $x = l$) can be specified in several ways. We discuss three common, generic kinds of boundary conditions here, which correspond to the three different heat regimes at the ends of the bar.

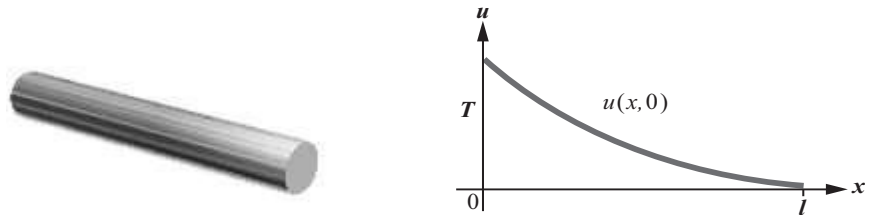


Figure 5.3. Example of initial conditions.

1. *Dirichlet condition.* The temperature at the end, $x = a$ (here $a = 0$ or l), of the bar changes by a specified law given by

$$u(a, t) = g(t), \quad (5.20)$$

where $g(t)$ is a known function of time t . In particular, g is constant if the end of the bar is maintained at a steady temperature.

2. *Neumann condition.* The heat current is given at the end, $x = a$, of the bar, in which case

$$q = -\kappa \frac{\partial u}{\partial x}.$$

This may be written as

$$\left. \frac{\partial u}{\partial x} \right|_{x=a} = g(t), \quad (5.21)$$

where $g(t)$ is a known function. If one of the ends of the bar is insulated, then the coefficient of heat exchange equals zero, and the boundary condition at this end takes the form

$$\left. \frac{\partial u}{\partial x} \right|_{x=a} = 0. \quad (5.22)$$

3. *Mixed condition.* In this case, the ends of the bar are exchanging heat with the environment, which has a temperature, θ . Actual heat exchange in real physical situations is very complicated, but we may simplify the problem by assuming that it obeys Newton's law, $q = H(u - \theta)$, where H is a positive constant. This law assumes that heat flows from hot objects to cooler objects with a heat current proportional to the temperature difference. The amount of heat transferred from the cross section at the end during time Δt is $H(u - \theta)S\Delta t$.

According to the law of conservation of energy, the amount of heat loss must equal the heat flow through the end's cross section as the result of heat conduction in the bar. The heat current flowing through a cross section of the bar along the x -axis is $-\kappa S(\partial u / \partial x)\Delta t$. At the right end of the bar, the direction of flux flowing to the environment coincidences with the positive x -axis; therefore, the current equals $-\kappa S\Delta t[\partial u / \partial x]_{x=l}$.

On the left end, these directions are opposite, and therefore the current is $-\kappa S \Delta t [\partial u / \partial x]_{x=0}$. It may also be the case that the external environments at the ends of the bar are different. In this case, boundary conditions at the ends become

$$\begin{aligned} \kappa S \Delta t \frac{\partial u}{\partial x} \Big|_{x=0} &= H_1 [u|_{x=0} - \theta_1], \\ -\kappa S \Delta t \frac{\partial u}{\partial x} \Big|_{x=l} &= H_2 [u|_{x=l} - \theta_2], \end{aligned} \quad (5.23)$$

where the temperatures of the environment at the left and right ends, θ_1 and θ_2 , are considered to be known functions of time. In the simplest case, they are constants. Thus, in the case of free heat exchange, the boundary conditions at the $x = a$ end of the rod has the form

$$\frac{\partial u}{\partial x} \pm hu \Big|_{x=a} = g(t), \quad (5.24)$$

with the given function $g(t)$ and $h = \text{const}$.

An algorithm for posing the problem of heat conduction within a uniform (isotropic) rod can be set up in the following way: *Find the solution of the heat conduction equation satisfying the initial condition*

$$u|_{t=0} = \varphi(x), \quad (5.25)$$

and one of the boundary conditions (Equations (5.20), (5.21) or (5.24)).

The generalization of the initial and boundary conditions to the three-dimensional case should be obvious. The initial condition is given by the function $\varphi(x, y, z)$ (or $\varphi(\vec{r})$). If this function is continuous, we must find a solution of the heat equation such that $\lim_{t \rightarrow 0} u(x, t) = \varphi(x)$ at all points of the body. If the initial distribution is discontinuous at points or surfaces, these discontinuities disappear after a short time and the solution must converge to the values determined by the initial temperature at all points where the distribution is continuous. We discuss such situations in more detail in the following section.

The three main types of boundary conditions for the three-dimensional case are the following:

- **Case 1.** The *prescribed surface temperature* condition, which means that the function

$$u(x, y, z, t)|_s = g(x, y, z, t)|_s \quad (5.26)$$

is known on some surface S . In the simplest case, this equals some constant (the surface S is *isothermal*).

- **Case 2.** The *prescribed flux across the surface* condition is given by

$$\left. \frac{\partial u(x, y, z, t)}{\partial n} \right|_S = g(x, y, z, t)|_S, \quad (5.27)$$

which means that the spatial rate of change of $u(x, y, z, t)$ in the direction of the outward normal to the surface is a given function of time at all points of the surface. When there is no flux across the surface, we have

$$\left. \frac{\partial u(x, y, z, t)}{\partial n} \right|_S = 0. \quad (5.28)$$

- **Case 3.** The *mixed* condition means that instead of Equation (5.23) (in the one-dimensional case), we have

$$\left[\kappa \frac{\partial u}{\partial n} + H(u - \theta) \right]_S = 0, \quad (5.29)$$

or

$$\left[\frac{\partial u}{\partial n} + h(u - \theta) \right]_S = 0 \quad (5.30)$$

with a constant coefficient of surface heat transfer, $h = H/\kappa$. (Notice that the dimension of the constant H in Equation (5.29) is different than that in Equation (5.23) since we now have a problem in three-dimensional space). If there is also an additional (prescribed) flux, F , the surface Equation (5.30) is replaced by

$$\left[\frac{\partial u}{\partial n} + h \left(u - \theta - \frac{F}{H} \right) \right]_S = 0, \quad (5.31)$$

which has the same form as Equation (5.30) with a trivial replacement of θ by $\theta + F/H$. As $h \rightarrow 0$, Equation (5.30) tends to the boundary condition of the second kind and as $h \rightarrow \infty$ it tends to the condition of the first kind. As discussed earlier, $h > 0$ for most physical situations. Also notice that the solutions for $h \neq 0$ often are not valid in the limiting case $h = 0$.

In many cases, the flux of heat from the surface is not a linear function of the temperature difference between the surface and its surroundings. An example of such a case is black-body radiation. A body at absolute temperature T loses heat at the rate $\sigma E(T^4 - T_0^4)$, where T_0 is the surrounding temperature, σ is the Stefan-Boltzmann constant, and E is the emissivity of the surface. If $T - T_0$ is not large, $(T^4 - T_0^4)$ can be approximately replaced by $(T - T_0)T_0^3$, and we can use one of the formulas above.

We have focused here on the heat conduction equation because the associated terminology is more concrete and intuitively fruitful than that for diffusion. But because the diffusion and heat conduction equations have identical forms, the solutions to diffusion problems can be obtained by a trivial replacement of D and q by χ and T , respectively. The boundary condition in Equation (5.20) corresponds to the concentration maintained at the ends, condition (5.21) corresponds to an impenetrable end, and condition (5.24) corresponds to a semipermeable end (when diffusion through this end is similar to that described by Newton's law for heat exchange). An analogue from chemistry is the case of a reaction on the boundary of a body when the speed of the reaction (i.e., the speed of creation or absorption of one of the chemical components) is proportional to the concentration of this component. For the three-dimensional case, the condition for Equation (5.20) means that if the component of the diffusion current normal to the surface $I = -D\nabla q$ is zero, we have $I = -D\nabla q$ on the surface. Or, stated another way, if a surface absorbs the original material, making it the diffusing material we have $q = 0$ on the surface.

5.2 The Principle of the Maximum, Correctness, and the Generalized Solution

It is very important to determine whether a problem is correctly formulated, that is, whether it has a unique solution that continuously depends on initial and boundary conditions. These demands are clear because in general we expect, for physical processes, that a small change of parameters should cause only a small change in the solution. In other words, we want to know whether the boundary and initial conditions we use are appropriate so that the mathematical solution corresponds to the physical

problem at hand. Clearly, in real situations these conditions are known only within some limited accuracy. If, mathematically, a small variation of the initial and boundary conditions leads to a large change in the solution, the usefulness of the obtained solution is rather doubtful.

5.2.1 The Principle of the Maximum

Let us consider the heat conduction equation in a bounded domain V with the boundary surface S when there are internal heat sources with density $f(x, y, z, t)$. The temperature $u(x, y, z, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u + f. \quad (5.32)$$

Suppose the following boundary and initial conditions are given:

$$u|_S = g(Q, t), \quad u|_{t=0} = \varphi(P), \quad (5.33)$$

where the functions g and φ are continuous and the values of g for $t = 0$ on the surface S coincide with the values of φ (here Q is a point on the surface S and P is a point in the domain V). We state without proof the following theorems:

Theorem 5.1. *Assuming the heat influx is nonnegative so that $f(x, y, z, t) \geq 0$, the function $u(x, y, z, t)$ attains its lowest value either at $t = 0$ or on the boundary of the domain V .*

Similarly, if the function f is nonpositive everywhere (i.e., $f(x, y, z, t) \leq 0$) then the function $u(x, y, z, t)$ attains its maximum value either at $t = 0$ or on the boundary of the domain V .

We thus have the *principle of the maximum*:

A function, $u(x, y, z, t)$, satisfying the homogeneous heat conduction equation (i.e., with no sources or absorbers of heat) takes its maximum value and its minimum value either at the initial moment $t = 0$ or on the boundary of the domain V .

The physical sense of this statement should be clear. If the temperature on the boundary at the initial moment does not exceed a value U , then, in the interior of the body, no temperature higher than U can be attained. This follows from thermodynamics: heat does not flow from a region of lower

temperature to a region of higher temperature for an isolated system; the temperature of a given region cannot spontaneously increase.

Several consequences follow from the principle of the maximum. The most important are the *uniqueness of the solution* of the heat conduction equation under the conditions in Equation (5.33) and the continuous dependence of this solution on the right-hand terms of the boundary and initial conditions. This may be referred to as the *physical determination* of the boundary value problem, and such problems are said to be well-posed problems.

Proofs of the above statements are very simple. If there were two solutions satisfying the homogeneous equation, then their difference would vanish for $t = 0$ and on the surface S . According to Theorem 5.1, both the maximum and the minimum of this difference would be zero, meaning that the difference itself would be zero. Therefore, the problem cannot have two different solutions.

In a similar way, let us prove that the problem is well posed. If the absolute value of the difference of two functions that include the boundary conditions does not exceed a certain positive number ε , then the absolute value of the difference of the corresponding solutions of the homogeneous equation with small boundary values also will not exceed ε . A similar argument may be constructed for initial conditions.

Theorem 5.2. *The solution of Equation (5.32) depends continuously not only on the conditions of Equation (5.33) but also on the function f . More precisely: If*

$$|\varphi| < \frac{\varepsilon_0}{2}, \quad |g| < \frac{\varepsilon_0}{2} \text{ and } |f| < \frac{\varepsilon_0}{2t_0},$$

then in any (finite) interval $0 < t < t_0$ for all $P \subset V$, we have

$$|u| < \varepsilon_0.$$

So far, we have discussed the question of the uniqueness and the physical determination of the first boundary value problem for a bounded interval. Similar results can be obtained for other types of boundary value problems, or for a problem without initial conditions, as shown next.

5.2.2 Generalized Solutions

As stated previously, the heat conduction equation (as well as other problems of mathematical physics) has a unique and physically reasonable solution only if there are considerable physical restrictions placed on the

generic solutions to the equation. The boundary value problems of the first, second, and third kinds provide such unique solutions (they are called *classical, or true, solutions*) if functions f , φ , and g are smooth enough and the boundary and initial conditions (if they exist) are consistent with each other. For instance, for Equation (5.33), the values of g for $t = 0$ on the surface S should coincide with the values of φ . If these conditions do not hold, we may still have a so-called *generalized solution* which we now discuss. Suppose we have three sequences of continuous functions

$$f_n, \quad \varphi_n, \quad \text{and} \quad g_n,$$

which uniformly approach the continuous functions f , φ , and g , respectively. By *uniform convergence* here we mean that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly to its sum $f(x)$ on the interval $[a, b]$ if, for any arbitrarily small $\varepsilon > 0$, we can find a number N such that for all $n \geq N$ the remainder of the series

$$\left| \sum_{n=N}^{\infty} f_n(x) \right| \leq \varepsilon$$

for all $x \in [a, b]$. This indicates that the series approaches its sum uniformly with respect to x .

Let the equation

$$\frac{\partial u_n}{\partial t} = a^2 \nabla^2 u_n + f_n$$

with the conditions

$$u_n|_{t=0} = \varphi_n, \quad u_n|_S = f_n$$

have a solution u_n (as we have shown, this solution is unique). By Theorem 5.2, the difference $u_m - u_n$ is arbitrarily small for large m and n ; thus, the sequence u_n converges uniformly to a function u that satisfies the boundary and initial conditions. This function u is called the *generalized solution*. Such a solution is unique because there cannot be two sets of sequences $u_n^{(1)}$ and $u_n^{(2)}$ for which functions $f_n^{(1)}$ and $f_n^{(2)}$, $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$, and $g_n^{(1)}$ and $g_n^{(2)}$ would converge to the same limit while the sequences themselves would converge to different limits.

In practice, it is usually sufficient to find a generalized solution to a given physical problem. In any case, in physical situations we know parameter values only with some limited precision. If the functions describing the initial and boundary conditions are not continuous or disagree with each other at some points or surfaces, these discontinuities disappear from the generalized solution after a short time, and the solution converges to the values determined by the initial and boundary conditions at all points where the distribution is continuous. Therefore, the generalized solution, even though it is not the true solution, will differ insignificantly from the true solution. As we demonstrate in Section 5.3, in cases when the initial and boundary conditions are not continuous or disagree with each other, we may still find a generalized solution sufficient for the purpose of solving the physical problem at hand.

5.3 The Fourier Method of Separation of Variables for the Heat Conduction Equation

In the following discussion, we consider several applications of the Fourier method to parabolic equations. We treat the heat conduction case, note that the same arguments may be applied to other parabolic equations.

5.3.1 A Simplification of the General Equation

In this section, we solve the one-dimensional heat conduction (or diffusion) equation in general form by using the method of Fourier, taking into account various possible boundary and initial conditions. We start by considering the boundary value problem for the one-dimensional heat conduction equation in its most general form on the bounded interval $[0, l]$ given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + \xi \frac{\partial u}{\partial x} - \gamma u + f(x, t) \quad (5.34)$$

with initial condition

$$u(x, 0) = \varphi(x), \quad 0 < x < l \quad (5.35)$$

and boundary conditions

$$\begin{aligned} P_1[u] &\equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = g_1(t), \\ P_2[u] &\equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l} = g_2(t), \end{aligned} \quad t > 0. \quad (5.36)$$

At the end $x = 0$, when $\alpha_1 = 0$, we have the Dirichlet condition; when $\beta_1 = 0$ we have the Neumann condition; and when both constants α_1 and β_1 are not zero, we have the mixed condition (clearly the constants α_1 and β_1 cannot be zero simultaneously). The same holds at the end $x = l$. As discussed in Chapter 2, physical limitations most often lead to the restrictions $\alpha_1/\beta_1 < 0$ and $\alpha_2/\beta_2 > 0$ for the signs of the coefficients in the boundary conditions of Equation (5.36).

Physically, the function $f(x, t)$ describes heat sources or sinks, and the term γu describes heat exchange with the environment. The term ξu_x corresponds to an advection, an additional heat transfer due to the motion of the medium with speed $(-\xi)$. Examples include convective motion in fluids or gases and the additional motion of diffusing particles due to a gravitational field.

Substituting

$$u(x, t) = e^{\mu x} v(x, t),$$

where $\mu = -\frac{\xi}{2a^2}$, into Equation (5.34) yields the equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} - \tilde{\gamma} v + \tilde{f}(x, t),$$

where

$$\tilde{\gamma} = \gamma + \frac{\xi^2}{4a^2} \text{ and } \tilde{f}(x, t) = e^{-\mu x} f(x, t).$$

The initial condition for the function $v(x, t)$ has the form

$$v(x, 0) = \tilde{\varphi}(x),$$

with $\tilde{\varphi}(x) = e^{-\mu x} \varphi(x)$. Boundary conditions for the function $v(x, t)$ are

$$P_1[v] \equiv \alpha_1 \frac{\partial v}{\partial x} + \tilde{\beta}_1 v \Big|_{x=0} = g_1(t) \text{ and } P_2[v] \equiv \alpha_2 \frac{\partial v}{\partial x} + \tilde{\beta}_2 v \Big|_{x=l} = \tilde{g}_2(t),$$

where

$$\tilde{\beta}_1 = \beta_1 + \mu \cdot \alpha_1, \quad \tilde{\beta}_2 = \beta_2 + \mu \cdot \alpha_2, \text{ and } \tilde{g}_2(t) = e^{-\mu l} \cdot g_2(t).$$

Reading Exercise. Derive the equation for $v(x, t)$ and boundary and initial conditions that accompany this equation.

It is clear from the above discussion that, for most commonly encountered physical problems, we may restrict our study here to the heat conduction equation in the simplified form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u + f(x, t) \quad (5.37)$$

with initial condition given by Equation (5.35) and boundary conditions given by Equation (5.36).

5.3.2 The Fourier Method for Homogeneous Equations

Let us first find the solution of the *homogeneous equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u, \quad 0 < x < l, \quad t > 0 \quad (5.38)$$

that satisfies the initial condition

$$u(x, t)|_{t=0} = \varphi(x) \quad (5.39)$$

(where $\varphi(x)$ is a given function), and has *homogeneous boundary conditions*

$$P_1[u] = \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = 0 \text{ and } P_2[u] = \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l} = 0. \quad (5.40)$$

The Fourier method of separation of variables supposes that a solution of Equation (5.38) can be found as a product of two functions, one depending only on x and the second depending only on t :

$$u(x, t) = X(x)T(t). \quad (5.41)$$

Substituting Equation (5.41) into Equation (5.38), we obtain

$$X(x) [T'(t) + \gamma T(t)] - a^2 X''(x)T(t) = 0.$$

The variables can be separated, and denoting a *separation constant* as $-\lambda$, we obtain

$$\frac{T'(t) + \gamma T(t)}{a^2 T(t)} \equiv \frac{X''(x)}{X(x)} = -\lambda. \quad (5.42)$$

Thus Equation (5.42) gives two ordinary linear homogeneous differential equations, a first-order equation for function $T(t)$:

$$T'(t) + (a^2\lambda + \gamma) T(t) = 0; \quad (5.43)$$

and a second-order equation for function $X(x)$:

$$X''(x) + \lambda X(x) = 0. \quad (5.44)$$

To find the allowed values of λ , we apply the boundary conditions. The homogenous boundary condition of Equation (5.40) imposed on $u(x, t)$ yields homogeneous boundary conditions on the function $X(x)$ given by

$$\alpha_1 X' + \beta_1 X \Big|_{x=0} = 0, \quad \alpha_2 X' + \beta_2 X \Big|_{x=l} = 0. \quad (5.45)$$

Thus, we obtain the Sturm-Liouville boundary problem for eigenvalues λ and the corresponding eigenfunctions $X(x)$. As we know from Chapter 2 and previous discussion in this chapter, there exist *infinite* sets of real non-negative discrete spectra of eigenvalues $\{\lambda_n\}$ and corresponding sets of eigenfunctions $\{X_n(x)\}$ (clearly $\lambda = 0$ is also possible if $\beta_1 = \beta_2 = 0$).

As we obtained in Chapter 3, the eigenvalues of the Sturm-Liouville problem stated in Equations (5.44) and (5.45) are

$$\lambda_n = \left(\frac{\mu_n}{l} \right)^2, \quad (5.46)$$

where μ_n is n th nonnegative root of the equation

$$\tan \mu = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) l \mu}{\mu^2 \alpha_1 \alpha_2 + l^2 \beta_1 \beta_2}. \quad (5.47)$$

The corresponding eigenfunctions can be written as

$$X_n(x) = \frac{1}{\sqrt{\alpha_1^2 \lambda_n + \beta_1^2}} \left[\alpha_1 \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x - \beta_1 \sin \sqrt{\lambda_n} x \right], \quad (5.48)$$

which are bounded by the values ± 1 , and their norms are

$$\|X_n\|^2 = \int_0^l X_n^2(x) dx = \frac{1}{2} \left[l + \frac{(\beta_2 \alpha_1 - \beta_1 \alpha_2)(\lambda_n \alpha_1 \alpha_2 - \beta_1 \beta_2)}{(\lambda_n \alpha_1^2 + \beta_1^2)(\lambda_n \alpha_2^2 + \beta_2^2)} \right]. \quad (5.49)$$

For example, if zero temperature is maintained steadily on the both ends of the bar (Dirichlet boundary conditions), then $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$, and Equation (5.47) takes the form $\tan \mu = 0$. Hence, for this case we obtain

$$\mu_n = n\pi \quad (n = 1, 2, 3, \dots),$$

and eigenvalues and eigenfunctions of this problem are determined by the formulas

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}.$$

It is clear that because λ_n contains the factor n^2 , it is not necessary to keep negative values of n . Eigenvalues and eigenfunctions for other kinds of boundary conditions are listed in Appendix A.

Now we consider Equation (5.43). It is a linear first-order differential equation, and the general solution with $\lambda = \lambda_n$ is

$$T_n(t) = C_n e^{-(a^2 \lambda_n + \gamma)t}, \quad (5.50)$$

where C_n is an arbitrary constant. Nonnegative values of λ_n are required so that the solution cannot grow to infinity with time. Now we have that each function

$$u_n(x, t) = T_n(t)X_n(x) = C_n e^{-(a^2 \lambda_n + \gamma)t} X_n(x)$$

is a solution of Equation (5.38) satisfying boundary conditions (5.40). To satisfy the initial conditions (5.39), we compose the series

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-(a^2 \lambda_n + \gamma)t} X_n(x). \quad (5.51)$$

If this series, as well as the series obtained by differentiating twice by x and once by t , converges uniformly, the sum gives a solution to Equation (5.38) and satisfies the boundary conditions of Equation (5.40). The initial condition in Equation (5.39) gives

$$u|_{t=0} = \varphi(x) = \sum_{n=1}^{\infty} C_n X_n(x), \quad (5.52)$$

where we have expanded the function $\varphi(x)$ in a series of the eigenfunctions of the boundary value problem given by Equations (5.44) and (5.45).

Assuming uniform convergence of the series in Equation (5.51), we can find the coefficients C_n . By multiplying both sides of Equation (5.52) by $X_n(x)$, integrating from 0 to l , and imposing the orthogonality condition of the functions $X_n(x)$, we obtain

$$C_n = \frac{1}{\|X_n\|^2} \int_0^l \varphi(x) X_n(x) dx. \quad (5.53)$$

If the series in Equation (5.51) and the series obtained from it by differentiating by t and twice differentiating by x are uniformly convergent, then by substituting the values of the coefficients, C_n , into the series in Equation (5.51), we obtain the solution of the problem stated in Equations (5.38) through (5.40).

Equation (5.51) gives a solution for *free heat exchange* (heat exchange without sources of heat within the body). It can be considered as the decomposition of an unknown function, $u(x, t)$, into a Fourier series over an orthogonal set of functions $\{X_n(x)\}$. This series converges well for “reasonable” assumptions about initial and boundary conditions.

In the following examples, we consider involving homogeneous equations with homogeneous boundary conditions. These and subsequent examples (whose analytical solutions are rather lengthy) can easily be simulated with the program Heat. The reader is encouraged to experiment with the parameters and choose interesting physical conditions for a deeper understanding of the physical situations being modeled. Instructions for the use of this program are found in Appendix E. The program uses the same formulas obtained and used while solving the problem analytically. The only numeric calculations the program performs are a numeric evaluation of the coefficients of the Fourier series, such as in Equation (5.53), and a numeric solution of transcendental equations for eigenvalues in the cases of mixed boundary conditions (for instance see Example 5.3 in this section).

Example 5.1. Let zero temperature be maintained on both the ends, $x = 0$ and $x = l$, of a uniform isotropic bar of length l with a heat-insulated lateral surface. Initially, the temperature distribution inside the bar is given by

$$u(x, 0) = \varphi(x) = \begin{cases} \frac{x}{l} u_0 & \text{for } 0 < x \leq \frac{l}{2}, \\ \frac{l-x}{l} u_0 & \text{for } \frac{l}{2} < x < l, \end{cases}$$

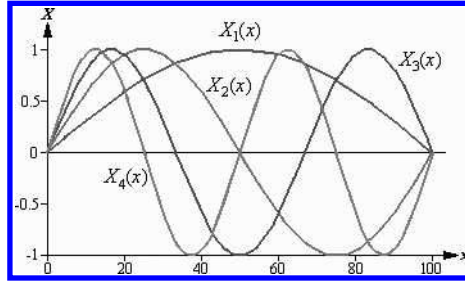


Figure 5.4. Eigenfunctions $X_1(x)$ through $X_4(x)$ for Example 5.1.

where $u_0 = \text{const}$. There are no sources of heat inside the bar. Find the temperature distribution for the interior of the bar for time $t > 0$.

Solution. The problem is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

with consistent initial and boundary conditions

$$u(x, 0) = \varphi(x), \quad u(0, t) = u(l, t) = 0.$$

The boundary conditions of the problem are Dirichlet homogeneous boundary conditions; therefore, eigenvalues and eigenfunctions of the problem are

$$\lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

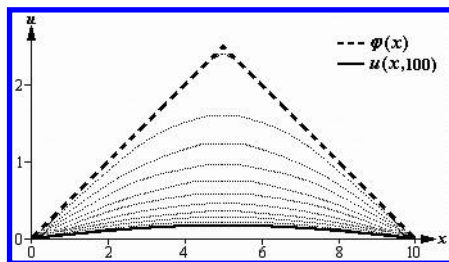
Figure 5.4 shows the first four eigenfunctions of the given boundary value problem.

Applying Equation (5.53), we obtain (for $n = 1, 2, 3, \dots$)

$$C_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx = \frac{4u_0}{n^2\pi^2} \sin \frac{n\pi}{2} = \begin{cases} 0, & n = 2k, \\ \frac{4u_0}{(2k-1)^2\pi^2} (-1)^k, & n = 2k-1. \end{cases}$$

Hence, the distribution of temperature inside the bar for some moment is described by the series

$$u(x, t) = \frac{4u_0}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^2} e^{-\frac{a^2(2k-1)^2\pi^2}{l^2}t} \sin \frac{(2k-1)\pi x}{l}.$$

Figure 5.5. Solution $u(x, t)$ for Example 5.1.

The series (as well as the series in several of the following examples) converges rather rapidly because it has coefficients that decrease as $1/k^2$.

Figure 5.5 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.1. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 5$, and $a^2 = 0.25$. All parameters are dimensionless.

The dashed line represents the initial temperature and the solid line is temperature at time $t = 100$. The dotted lines in between show the temperature evolution within the period of time from 0 until 100.

Reading Exercise. Choose values of parameters in real physical units and estimate how much time it takes to reach equilibrium for different materials and for samples of different size. Make sure that the answers are physically reasonable.

Hint. Clearly, the approach to equilibrium (the time for temperature fluctuations to decay) is governed by the factor $\exp[-(a^2 \lambda_n^2 + \gamma)t]$.

Example 5.2. Consider the case when the ends, $x = 0$ and $x = l$, of a bar are thermally insulated from the environment. The lateral surface is also insulated. In this case, the derivatives of temperature with respect to x on the ends of the bar equal zero. Initially, the temperature is distributed as in Example 5.1:

$$\varphi(x) = \begin{cases} \frac{x}{l} u_0 & \text{for } 0 < x \leq \frac{l}{2}, \\ \frac{l-x}{l} u_0 & \text{for } \frac{l}{2} < x < l, \end{cases}$$

where $u_0 = \text{const}$. Sources of heat are absent. Find the temperature distribution inside the bar for $t > 0$.

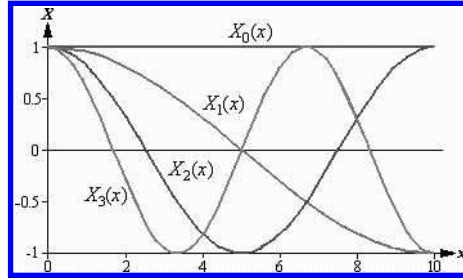


Figure 5.6. Eigenfunctions $X_0(x)$ through $X_3(x)$ for Example 5.2.

Solution. We are to solve the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

with initial and boundary conditions, respectively, of

$$u(x, 0) = \varphi(x) \text{ and } \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(l, t) = 0.$$

Notice that the initial and boundary conditions in this case are not consistent (they are contradictory); therefore, we can obtain only a generalized solution. The boundary conditions of the problem are Neumann homogeneous boundary conditions. We leave it to the reader as a reading exercise to obtain the formulas that define the problem (for all possible situations, see Appendix A). The eigenvalues and eigenfunctions obtained from these formulas are:

$$\lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l},$$

$$\|X_n\|^2 = \begin{cases} l, & n = 0, \\ l/2, & n > 0, \end{cases} \quad n = 0, 1, 2, \dots$$

Figure 5.6 shows the first four eigenfunctions of the given boundary value problem.

By applying Equation (5.53), we obtain

$$C_0 = \frac{u_0}{l^2} \left[\int_0^{l/2} x dx + \int_{l/2}^l (l-x) dx \right] = \frac{u_0}{4},$$

$$C_n = \frac{2u_0}{l^2} \int_0^{l/2} x \cos \frac{n\pi x}{l} dx + \frac{2u_0}{l^2} \int_{l/2}^l (l-x) \cos \frac{n\pi x}{l} dx$$

$$= \begin{cases} 0 & \text{for } n = 2k-1, \\ \frac{u_0}{k^2\pi^2} [(-1)^k - 1] & \text{for } n = 2k, \end{cases} \quad k = 1, 2, 3, \dots$$

Thus, the only nonzero C_n are those for which $n = 2k$ with $k = 2m+1$, $m = 0, 1, 2, \dots$ (i.e., $n = 4m+2$); so we have

$$C_{4m+2} = -\frac{2u_0}{(2m+1)^2\pi^2}.$$

Finally, the temperature distribution inside the bar for some moment is expressed by the series

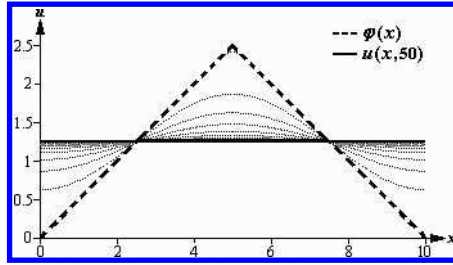
$$u(x, t) = \frac{u_0}{4} - \frac{2u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\frac{(4m+2)^2 a^2 \pi^2}{l^2} t} \cos \frac{(4m+2)\pi x}{l}.$$

At $x = l/4$ and $x = 3l/4$, all cosine terms equal zero; hence, at these points $u = u_0/4$ for any $t \geq 0$. It is also clear that

$$\int_0^l u(x, t) dx = \frac{1}{4} u_0 l.$$

Notice that this area is proportional to the amount of energy (heat) in the bar. The insulated ends of the bar correspond to a graph of $u(x, t)$ that has horizontal tangents at $x = 0$ and $x = l$. As $t \rightarrow \infty$, the first term of the series dominates. From this and from physical considerations, we conclude that $u \rightarrow u_0/4$ as $t \rightarrow \infty$.

Figure 5.7 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.2. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 5$, and $a^2 = 0.25$.

Figure 5.7. Solution $u(x, t)$ for Example 5.2.

Example 5.3. Consider the situation in which the heat flux is governed by Newton's law of cooling, and a constant temperature environment occurs at each end of a uniform isotropic bar of length l ($0 \leq x \leq l$) with an insulated lateral surface. The initial temperature of the bar is equal to $u_0 = \text{const}$. Internal sources of heat in the bar are absent. Find the temperature distribution inside the bar for $t > 0$.

Solution. We need to solve the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

for the initial condition

$$u(x, 0) = u_0$$

and boundary conditions

$$\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) + hu(l, t) = 0,$$

where $h > 0$ is the coefficient of heat exchange with the environment. Obviously, as in Example 5.2, we can obtain only a generalized solution.

The boundary conditions of this problem are mixed homogeneous boundary conditions, so eigenvalues are

$$\lambda_n = \left(\frac{\mu_n}{l} \right)^2, \quad n = 1, 2, 3, \dots,$$

where μ_n is the n th root of the equation

$$\tan \mu = \frac{2hl\mu}{\mu^2 - h^2l^2}.$$

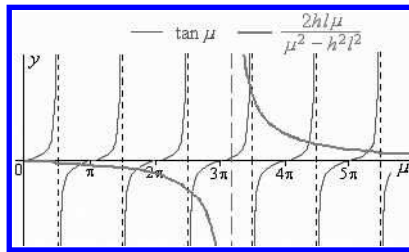


Figure 5.8. Graphical solution of the eigenvalue equation for Example 5.3 ($h = 1$).

Figure 5.8 shows curves of the two functions,

$$y = \tan \mu \text{ and } y = \frac{2hl\mu}{\mu^2 - h^2l^2},$$

plotted on the same set of axes.

Each eigenvalue corresponds to an eigenfunction

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right]$$

with the norm

$$\|X_n\|^2 = \frac{l}{2} + \frac{h}{\lambda_n + h^2}.$$

Figure 5.9 shows the first four eigenfunctions of the given boundary value problem.

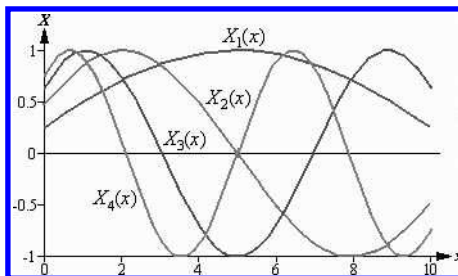
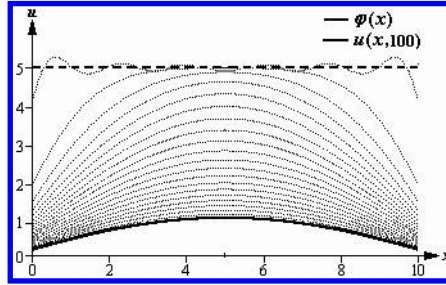


Figure 5.9. Eigenfunctions $X_1(x)$ through $X_4(x)$ for Example 5.3 ($h = 1$).

Figure 5.10. Solution $u(x, t)$ for Example 5.3.

By applying Equation (5.53), we obtain

$$\begin{aligned} C_n &= \frac{1}{\|X_n\|^2} \int_0^l \frac{u_0}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right] dx \\ &= \frac{2u_0 \sqrt{\lambda_n + h^2}}{l(\lambda_n + h^2) + 2h} \left[\sin \sqrt{\lambda_n} l - \frac{h}{\sqrt{\lambda_n}} (\cos \sqrt{\lambda_n} l - 1) \right]. \end{aligned}$$

Hence, the temperature distribution inside the bar for some moment of time is expressed by the series

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 \lambda_n t} \frac{1}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right].$$

Figure 5.10 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.3. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 5$, $h = 1$, and $a^2 = 0.25$.

In this example, the boundary and initial conditions do not match each other; as a result, at $t = 0$, the temperature $u(x, 0)$ given by the solution in the form of an eigenfunction expansion does not converge uniformly. The convergence is poor at points close to the ends of the rod, and the solution appears to have unphysical oscillations of temperature initially. Increasing the number of terms in the series smoothes these oscillations at all points except the ends. These oscillations do not occur physically and, in fact, they disappear from the solution for any finite time $t > 0$, in which case $u(x, t)$ converges rapidly to a physically reasonable result.

5.3.3 The Fourier Method for Nonhomogeneous Equations

In this section, we consider the *nonhomogeneous* linear equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u + f(x, t), \quad (5.54)$$

where $f(x, t)$ is a known function with initial condition

$$u(x, t)|_{t=0} = \varphi(x) \quad (5.55)$$

and *homogeneous boundary conditions*

$$P_1[u] \equiv \alpha_1 u_x + \beta_1 u|_{x=0} = 0, \quad P_2[u] \equiv \alpha_2 u_x + \beta_2 u|_{x=l} = 0. \quad (5.56)$$

To start, let us express the function $u(x, y)$ as the sum of two functions:

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where $u_1(x, t)$ satisfies the *homogeneous equation* with the given (homogeneous) boundary conditions and the specified (nonhomogeneous) initial condition:

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} - \gamma u_1,$$

$$u_1(x, t)|_{t=0} = \varphi(x),$$

$$P_1[u_1] \equiv \alpha_1 u_{1x} + \beta_1 u_1|_{x=0} = 0, \quad P_2[u_1] \equiv \alpha_2 u_{1x} + \beta_2 u_1|_{x=l} = 0.$$

The function $u_2(x, t)$ satisfies the nonhomogeneous equation with *zero boundary and initial conditions*:

$$\frac{\partial u_2}{\partial t} = a^2 \frac{\partial^2 u_2}{\partial x^2} - \gamma u_2 + f(x, y), \quad (5.57)$$

$$u_2(x, t)|_{t=0} = 0, \quad (5.58)$$

$$P_1[u_2] \equiv \alpha_1 u_{2x} + \beta_1 u_2|_{x=0} = 0, \quad P_2[u_2] \equiv \alpha_2 u_{2x} + \beta_2 u_2|_{x=l} = 0. \quad (5.59)$$

The function $u_1(x, t)$ represents *free heat exchange*, that is, the heat propagation due to a nonhomogeneous initial temperature distribution inside the

bar. The function $u_2(x, t)$ represents *heat propagation* due to emission or absorption of heat when the initial temperature of the bar is zero.

The methods for finding $u_1(x, t)$ have been discussed in Section 5.3.2; therefore, we concentrate our attention here on finding the solutions $u_2(x, t)$. As for the case of free heat exchange inside the bar, let us expand the function $u_2(x, t)$ as a series

$$u_2(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \quad (5.60)$$

where $X_n(x)$ are the eigenfunctions of the corresponding homogeneous boundary value problem and $T_n(t)$ are unknown functions of t .

Boundary conditions in Equation (5.59) for $u_2(x, t)$ are valid for any choice of functions $T_n(t)$ (when the series converges uniformly) because they are valid for the functions $X_n(x)$. By substituting the series in Equation (5.60) into Equation (5.57), we obtain

$$\sum_{n=1}^{\infty} [T'_n(t) + (a^2 \lambda_n + \gamma) T_n(t)] X_n(x) = f(x, t). \quad (5.61)$$

Using the completeness property, we can expand the function $f(x, t)$, as a function of x , into a Fourier series of the functions $X_n(x)$ on the interval $(0, l)$ such that

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) X_n(x). \quad (5.62)$$

From the orthogonality property of the functions $X_n(x)$, we find that

$$f_n(t) = \frac{1}{\|X_n\|^2} \int_0^l f(x, t) X_n(x) dx. \quad (5.63)$$

By comparing the two expansions in Equations (5.61) and (5.62) for the same function $f(x, t)$, we obtain differential equations for the functions $T_n(t)$:

$$T'_n(t) + (a^2 \lambda_n + \gamma) T_n(t) = f_n(t). \quad (5.64)$$

In order for $u_2(x, t)$ given by Equation (5.60) to satisfy the initial conditions (5.58), it is necessary that the functions $T_n(t)$ obey the condition

$$T_n(0) = 0. \quad (5.65)$$

The solution to the ordinary differential equation of first order, Equation (5.64), with the initial condition given by Equation (5.65), can be represented in the integral form

$$T_n(t) = \int_0^t f_n(\tau) e^{-(a^2 \lambda_n + \gamma)(t-\tau)} d\tau, \quad (5.66)$$

or

$$T_n(t) = \int_0^t f_n(\tau) Y_n(t - \tau) d\tau, \text{ where } Y_n(t - \tau) = e^{-(a^2 \lambda_n + \gamma)(t-\tau)}.$$

To check that this solution is correct, let us differentiate it with respect to t :

$$\begin{aligned} T'_n(t) &= \int_0^t f_n(\tau) \frac{\partial}{\partial t} Y_n(t - \tau) d\tau + Y_n(0) f_n(t) \\ &= -(a^2 \lambda_n + \gamma) \int_0^t f_n(\tau) e^{-(a^2 \lambda_n + \gamma)(t-\tau)} d\tau + f_n(t) = -(a^2 \lambda_n + \gamma) T_n(t) + f_n(t), \end{aligned}$$

which coincides with Equation (5.64).

In the following discussion, we show another way to obtain the solution of Equation (5.64) that is instructive in that it demonstrates the method of variation of a constant. The solution of the linear nonhomogeneous equation

$$T'(t) + pT(t) = f(t)$$

consists of a general solution of the homogeneous equation

$$T'(t) + pT(t) = 0,$$

which is

$$T(t) = ce^{-pt},$$

and a particular solution of the nonhomogeneous equation. Let us denote this solution as $\tilde{T}(t)$ and search for it in the same form as the solution of

the homogeneous equation, but replacing the constant c by an unknown function $c(t)$:

$$\tilde{T}(t) = c(t)e^{-pt}.$$

Then

$$\tilde{T}'(t) = c'(t)e^{-pt} - c(t)pe^{-pt},$$

and by substituting $\tilde{T}(t)$ and $\tilde{T}'(t)$ into the nonhomogeneous equation

$$c'(t)e^{-pt} - c(t)pe^{-pt} + c(t)pe^{-pt} = f(t),$$

we obtain

$$c'(t) = e^{pt}f(t),$$

and thus

$$c(t) = \int_0^t f(\tau)e^{p\tau}d\tau.$$

From this result, we may write

$$\tilde{T}(t) = \int_0^t f(\tau)e^{-p(t-\tau)}d\tau,$$

and a general solution of the nonhomogeneous equation is

$$T(t) = \int_0^t f(\tau)e^{-p(t-\tau)}d\tau + ce^{-pt}.$$

To find the coefficient c , we use the initial condition $T(0) = 0$, which gives

$$T(t) = c = 0.$$

Thus, the solution of the Cauchy problem is

$$T(t) = \int_0^t f(\tau)e^{-p(t-\tau)}d\tau.$$

With $p = a^2\lambda_n + \gamma$, this equation yields Equation (5.66), the same solution as before.

If the series in Equation (5.60) and the series obtained from it by term-by-term differentiation with respect to t and twice differentiation by x are uniformly convergent, then by substituting the expressions found for $T_n(t)$ in the series in Equation (5.60), we obtain solutions of the boundary value problem given in Equations (5.57) through (5.59). Thus, the solution of the nonhomogeneous heat conduction problem for a bar with boundary conditions equal to zero has the form

$$u(x, t) = u_1(x, t) + u_2(x, t) = \sum_{n=1}^{\infty} \left[T_n(t) + C_n e^{-(a^2 \lambda_n + \gamma)t} \right] X_n(x), \quad (5.67)$$

where the functions $T_n(t)$ are defined by Equation (5.66) and coefficients C_n have been found earlier when we considered the homogeneous heat equation.

In the following examples, we consider solutions for the nonhomogeneous heat conduction equation with homogeneous boundary conditions.

Example 5.4. A point-like heat source with power $Q = \text{const}$ is located at x_0 ($0 < x_0 < l$) in a uniform isotropic bar with insulated lateral surfaces. The initial temperature of the bar is zero. Temperatures at the ends of the bar are maintained at zero. Find the temperature inside the bar for $t > 0$.

Solution. The boundary value problem modeling heat propagation for this example is

$$\begin{aligned} \frac{\partial u}{\partial t} &= a^2 \frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho} \delta(x - x_0), \\ u(x, 0) &= 0, \quad u(0, t) = u(l, t) = 0, \end{aligned}$$

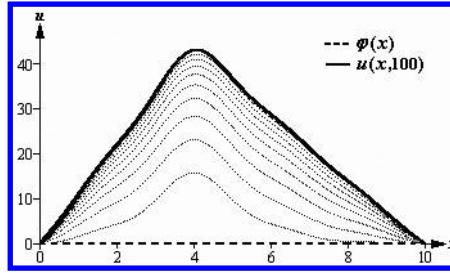
where $\delta(x - x_0)$ is the delta function.

The boundary conditions are Dirichlet homogeneous boundary conditions, so the eigenvalues, eigenfunctions, and norm of the problem are

$$\lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad (n = 1, 2, 3, \dots).$$

In the case of homogeneous initial conditions, $\varphi(x) = 0$, so we have $C_n = 0$ and the solution $u(x, t)$ is defined by the series

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}, \quad (5.68)$$

Figure 5.11. Solution $u(x, t)$ for Example 5.4.

where f_n and T_n are defined by Equations (5.63) and (5.66):

$$f_n(t) = \frac{2}{l} \int_0^l \frac{Q}{c\rho} \delta(x - x_0) \sin \frac{n\pi x}{l} dx = \frac{2Q}{lc\rho} \sin \frac{n\pi x_0}{l} dx,$$

$$T_n(t) = \frac{2Ql}{c\rho a^2 n^2 \pi^2} \sin \frac{n\pi x_0}{l} \left(1 - e^{-\frac{n^2 a^2 \pi^2}{l^2} t} \right).$$

Substituting the expression for $T_n(t)$ into the general formula of Equation (5.68), we obtain the solution of the problem:

$$u(x, t) = \frac{2Ql}{c\rho a^2 \pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x_0}{l} \left(1 - e^{-\frac{n^2 a^2 \pi^2}{l^2} t} \right) \sin \frac{n\pi x}{l}.$$

Figure 5.11 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.4. This solution was obtained with the program Heat for the case when $l = 10$, $Q/c\rho = 5$, $x_0 = 4$, and $a^2 = 0.25$.

Reading Exercise. Find the sum of the series for $u(x, t)$ as $t \rightarrow \infty$ and obtain a piecewise linear function with the vertex at x_0 , which approximates the solid line in Figure 5.11.

The curves shown in Figure 5.12 depict time traces of the temperature at various points on the rod. Note that after time $t \approx 250$, the solution does not depend on time. This is the so-called steady-state temperature regime. The lowest value of temperature is attained at the rod ends in accordance with the principle of the maximum. The principle can also be observed to apply in several of the following examples.

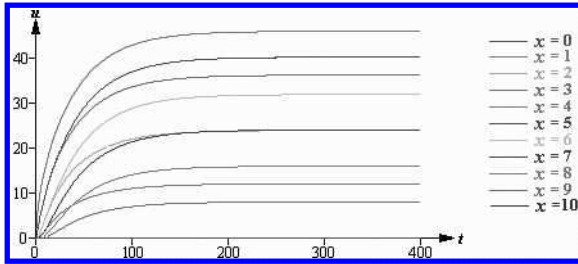


Figure 5.12. Time traces of temperature at points on the rod for Example 5.4.

Example 5.5. A heat source moves along a bar with constant velocity v_0 . The heat flux from the source to the bar is described by

$$q(t) = Ae^{-\gamma t},$$

where γ is the coefficient of heat exchange in the heat conduction equation. Convective heat transfer occurs on the lateral surface of the bar, and the temperature of the environment equals zero. Find the temperature of the bar for $t > 0$ if the initial temperature is zero and the temperature of the ends is maintained at zero.

Solution. The solution to this problem is described by the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u + f(x, t), \quad (5.69)$$

where

$$f(x, t) = \begin{cases} \frac{A}{c\rho S} e^{-\gamma t} \delta(x - v_0 t) & \text{for } 0 < t < l/v_0, \\ 0 & \text{for } t > l/v_0. \end{cases}$$

The initial and boundary conditions are

$$u(x, 0) = 0, \quad u(0, t) = u(l, t) = 0.$$

The boundary conditions are Dirichlet homogeneous boundary conditions, and the eigenvalues and eigenfunctions are thus given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

Substituting

$$u(x, t) = e^{-\gamma t} v(x, t)$$

into Equation (5.69) yields a boundary problem of the form

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + \tilde{f}(x, t),$$

where

$$\tilde{f}(x, t) = e^{\gamma t} f(x, t) = \begin{cases} \frac{A}{c\rho S} \delta(x - v_0 t) & \text{for } 0 < t < l/v_0, \\ 0 & \text{for } t > l/v_0, \end{cases}$$

for conditions

$$v(x, 0) = 0, \quad v(0, t) = v(l, t) = 0.$$

Because $\varphi(x) = 0$ for the zero initial condition, we have $C_n = 0$. Functions $f_n(t)$ and $T_n(t)$ are given by Equations (5.63) and (5.66); hence,

$$f_n(t) = \frac{2}{l} \int_0^l \frac{A}{c\rho S} \delta(x - v_0 t) \sin \frac{n\pi x}{l} dx = \frac{2A}{lc\rho S} \sin \frac{n\pi v_0 t}{l}.$$

- **Case 1.** If $0 < t \leq l/v_0$, we have

$$\begin{aligned} T_n(t) &= \int_0^t f_n(\tau) e^{-\frac{n^2 a^2 \pi^2}{l^2} (t-\tau)} d\tau = \frac{2A}{c\rho S} \frac{a^2}{l \left(v_0^2 + \frac{a^4 n^2 \pi^2}{l^2} \right)} \\ &\times \left[\sin \frac{n\pi v_0 t}{l} - \frac{lv_0}{a^2 n \pi} \cos \frac{n\pi v_0 t}{l} + \frac{lv_0}{a^2 n \pi} e^{-\frac{n^2 a^2 \pi^2}{l^2} t} \right]. \end{aligned}$$

- **Case 2.** If $t > l/v_0$, we have

$$\begin{aligned} T_n(t) &= \int_0^{l/v_0} f_n(\tau) e^{-\frac{n^2 a^2 \pi^2}{l^2} (t-\tau)} d\tau = \frac{2A}{c\rho S} \frac{v_0}{n\pi \left(v_0^2 + \frac{a^4 n^2 \pi^2}{l^2} \right)} \\ &\times \left[e^{-\frac{n^2 a^2 \pi^2}{l^2} t} - (-1)^n e^{-\frac{n^2 a^2 \pi^2}{l^2} \left(t - \frac{l}{v_0} \right)} \right]. \end{aligned}$$

With these $T_n(t)$, we obtain the solution of the problem, which is

$$u(x, t) = e^{-\gamma t} \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}.$$

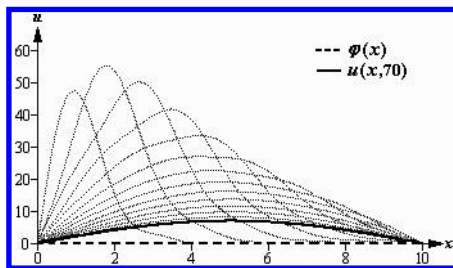
Figure 5.13. Solution $u(x, t)$ for Example 5.5.

Figure 5.13 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.5. This solution was obtained with the program Heat for the case when $l = 10$, $A/c\rho S = 50$, $v_0 = 0.2$, $\gamma = 0.01$, and $a^2 = 0.25$.

Reading Exercise. Show that when the heat source moves slowly (i.e., $lv_0/a^2 \ll 1$), the solution in Case 1 gives the same result as the steady-state solution in Example 5.4. This relation between the two problems can be understood if we realize that when the heater moves slowly, there is enough time for the bar to reach a steady-state temperature distribution at any time.

Hint. Keep only the first term in the brackets in $T_n(t)$; $v_0 t$ in this problem corresponds to x_0 in Example 5.4.

5.3.4 The Fourier Method for Nonhomogeneous Equations with Nonhomogeneous Boundary Conditions

Now we consider the general boundary problem for heat conduction, Equation (5.37), given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u + f(x, t)$$

with nonhomogeneous initial (Equation 5.35)) and boundary (Equation (5.36)) conditions

$$u(x, t)|_{t=0} = \varphi(x),$$

$$P_1[u] \equiv \alpha_1 u_x + \beta_1 u|_{x=0} = g_1(t), \quad P_2[u] \equiv \alpha_2 u_x + \beta_2 u|_{x=l} = g_2(t).$$

We cannot apply the Fourier method directly to obtain a solution of the problem because the boundary conditions are nonhomogeneous. However, the problem can easily be reduced to a problem with boundary conditions equal to zero in the following way.

Let us search for the solution of the problem in the form

$$u(x, t) = v(x, t) + w(x, t), \quad (5.70)$$

where $v(x, t)$ is a new unknown function, and the function $w(x, t)$ is chosen so that it satisfies the given nonhomogeneous boundary conditions

$$P_1[w] \equiv \alpha_1 w_x + \beta_1 w|_{x=0} = g_1(t),$$

$$P_2[w] \equiv \alpha_2 w_x + \beta_2 w|_{x=l} = g_2(t).$$

Also, $w(x, t)$ must have the required number of continuous derivatives for x and t .

For the function $v(x, t)$, we obtain the following boundary value problem:

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} - \gamma v + f^*(x, t),$$

$$v(x, t)|_{t=0} = \varphi^*(x),$$

$$P_1[v] \equiv \alpha_1 v_x + \beta_1 v|_{x=0} = 0, \quad P_2[v] \equiv \alpha_2 v_x + \beta_2 v|_{x=l} = 0,$$

where

$$f^*(x, t) = f(x, t) - \frac{\partial w}{\partial t} + a^2 \frac{\partial^2 w}{\partial x^2} - \gamma w,$$

$$\varphi^*(x) = \varphi(x) - w(x, 0).$$

The solution of such a problem with homogeneous boundary conditions has been considered in Section 5.3.3.

The auxiliary function $w(x, t)$ is ambiguously defined. The simplest way to proceed is to use polynomials and construct it in the form

$$w(x, t) = P_1(x)g_1(t) + P_2(x)g_2(t),$$

where $P_1(x)$ and $P_2(x)$ are polynomials of first or second order. Coefficients of these polynomials will be chosen so that the function $w(x, t)$ satisfies the given boundary conditions. We have the following possibilities.

- **Case 1.** If β_1 and β_2 in Equation (5.36) are not zero simultaneously, we may seek the function $w(x, t)$ in the form

$$w(x, t) = (\gamma_1 + \delta_1 x)g_1(t) + (\gamma_2 + \delta_2 x)g_2(t).$$

Substituting $w(x, t)$ for $u(x, t)$ in the boundary conditions (5.36), and taking into account that the derived system of equations must be valid for arbitrary t , we obtain the coefficients γ_1 , δ_1 , γ_2 , and δ_2 :

$$\begin{aligned} \gamma_1 &= \frac{\alpha_2 + \beta_2 l_x}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_1 &= \frac{-\beta_2}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \\ \gamma_2 &= \frac{-\alpha_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_2 &= \frac{\beta_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}. \end{aligned} \quad (5.71)$$

Reading Exercise. We leave it to the reader to obtain the results in Equation (5.71)

- **Case 2.** If $\beta_1 = \beta_2 = 0$, that is, if

$$\begin{cases} u_x(0, t) = g_1(t) \\ u_x(l, t) = g_2(t), \end{cases}$$

the auxiliary function has the form

$$w(x, t) = \left[x - \frac{x^2}{2l} \right] \cdot g_1(t) + \frac{x^2}{2l} \cdot g_2(t). \quad (5.72)$$

Reading Exercise. Verify Case 2.

Reading Exercise. Prove that, defined as in Equation (5.72), the auxiliary functions $w(x, t)$ satisfy the boundary conditions in Equation (5.36).

Combining the different kinds of boundary conditions listed above, we obtain nine different auxiliary functions, which are listed in Appendix B.

The following examples consider problems that involve the nonhomogeneous heat conduction equation with nonhomogeneous boundary conditions.

Example 5.6. Find the temperature change in a homogeneous isotropic bar of length l ($0 \leq x \leq l$) with a heat-insulated lateral surface during free heat exchange if the initial temperature is given by

$$u(x, 0) = \varphi(x) = u_0 \frac{x^2}{l^2}.$$

The left end of the bar at $x = 0$ is insulated, and at the right end ($x = l$), the temperature is held constant:

$$u(l, t) = u_0, \text{ where } u_0 = \text{const} > 0.$$

Solution. The problem is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with the conditions

$$u(x, 0) = \varphi(x) = u_0 \frac{x^2}{l^2},$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(l, t) = u_0.$$

The solution of the problem will be of the form

$$u(x, t) = v(x, t) + w(x, t).$$

The auxiliary function can be easily obtained from the general formulas above (do this as a reading exercise; the answers are found in Appendix B). We find

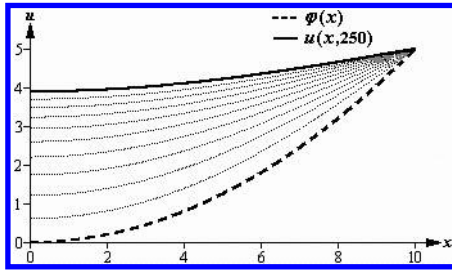
$$w(x, t) = u_0,$$

which corresponds to the steady-state regime as $t \rightarrow \infty$. The eigenvalues and eigenfunctions are easy to obtain (they are also found in Appendix A), and we leave it to the reader as a reading exercise to check that they are:

$$\lambda_n = \left[\frac{(2n-1)\pi}{2l} \right]^2, \quad X_n(x) = \cos \frac{(2n-1)\pi x}{2l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad (n = 1, 2, 3, \dots).$$

Following the same logic as in previous problems, for the function $v(x, t)$, we obtain the conditions

$$f^*(x, t) = 0,$$

Figure 5.14. Solution $u(x, t)$ for Example 5.6.

$$\varphi^*(x) = u_0 \frac{x^2}{l^2} - u_0 = u_0 \left(\frac{x^2}{l^2} - 1 \right).$$

We apply Equation (5.53) to obtain

$$\begin{aligned} C_n &= \frac{2}{l} \int_0^l \varphi^*(x) \cos \frac{(2n-1)\pi x}{2l} dx \\ &= \frac{2}{l} \int_0^l u_0 \left(\frac{x^2}{l^2} - 1 \right) \cos \frac{(2n-1)\pi x}{2l} dx = -\frac{32u_0}{(2n-1)^3 \pi^3} (-1)^n. \end{aligned}$$

With these coefficients, we obtain the solution of the problem:

$$u(x, t) = u_0 - \frac{32u_0}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} e^{-\frac{(2n-1)^2 a^2 \pi^2}{l^2} t} \cos \frac{(2n-1)\pi x}{2l}.$$

Figure 5.14 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.6. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 5$, and $a^2 = 0.25$.

Example 5.7. The initial temperature of a homogeneous isotropic bar of length l ($0 \leq x \leq l$) is

$$u(x, 0) = u_0 = \text{const.}$$

There exists a steady heat flux from the environment into the ends of the bar, which is given by

$$\frac{\partial u}{\partial x}(0, t) = Q_1 = -\frac{q_1}{\kappa S}, \quad \frac{\partial u}{\partial x}(l, t) = Q_2 = \frac{q_2}{\kappa S}.$$

Convective heat transfer occurs with the environment through the lateral surface. Find the temperature $u(x, t)$ of the bar for $t > 0$.

Solution. The problem is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u, \quad a^2 = \frac{\kappa}{c\rho}, \quad \gamma = \frac{h}{c\rho}$$

with initial and boundary conditions

$$u(x, 0) = \varphi(x) = u_0,$$

$$\frac{\partial u}{\partial x}(0, t) = Q_1, \quad \frac{\partial u}{\partial x}(l, t) = Q_2.$$

The boundary conditions of the problem are Neumann boundary conditions, and the eigenvalues and eigenfunctions are, respectively,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l}, \quad \|X_n\|^2 = \begin{cases} l, & n = 0, \\ l/2, & n > 0. \end{cases}$$

The auxiliary function is (see Appendix B):

$$w(x, t) = \left(x - \frac{x^2}{2l}\right) Q_1 + \frac{x^2}{2l} Q_2 = xQ_1 + \frac{x^2}{2l} (Q_2 - Q_1).$$

We leave it to the reader as a reading exercise to obtain the auxiliary function and the eigenvalues and eigenfunctions.

Similar to the previous examples, we have the conditions

$$f^*(x, t) = \frac{a^2}{l} (Q_2 - Q_1) - \gamma \left[\left(x - \frac{x^2}{2l}\right) Q_1 + \frac{x^2}{2l} Q_2 \right],$$

$$\varphi^*(x) = u_0 - \left(x - \frac{x^2}{2l}\right) Q_1 - \frac{x^2}{2l} Q_2.$$

Applying Equations (5.53), (5.63), and (5.66), we obtain

$$C_0 = \frac{1}{l} \int_0^l \left[u_0 - \left(x - \frac{x^2}{2l} \right) Q_1 - \frac{x^2}{2l} Q_2 \right] dx = u_0 - \frac{l}{6} (2Q_1 + Q_2),$$

$$C_n = \frac{2}{l} \int_0^l \left[u_0 - Q_1 x + (Q_1 - Q_2) \frac{x^2}{2l} \right] \cos \frac{n\pi x}{l} dx = \frac{2l}{n^2 \pi^2} [Q_1 - (-1)^n Q_2],$$

$$f_0(t) = \frac{1}{l} \int_0^l f^*(x, t) dx = \frac{a^2(Q_2 - Q_1)}{l} - \frac{\gamma l}{6} (2Q_1 + Q_2),$$

$$T_0(t) = \int_0^t f_0(\tau) e^{-\gamma(t-\tau)} d\tau = \frac{1}{\gamma} \left[\frac{a^2(Q_2 - Q_1)}{l} - \frac{\gamma l}{6} (2Q_1 + Q_2) \right] (1 - e^{-\gamma t}).$$

For $n > 0$,

$$f_n(t) = \frac{2}{l} \int_0^l f^*(x, t) \cos \frac{n\pi x}{l} dx = \frac{2l\gamma}{n^2 \pi^2} [Q_1 - (-1)^n Q_2],$$

$$T_n(t) = \int_0^t f_n(\tau) e^{-(a^2 \lambda_n + \gamma)(t-\tau)} d\tau = \frac{2l\gamma}{n^2 \pi^2 (a^2 \lambda_n + \gamma)} [Q_1 - (-1)^n Q_2] (1 - e^{-(a^2 \lambda_n + \gamma)t}).$$

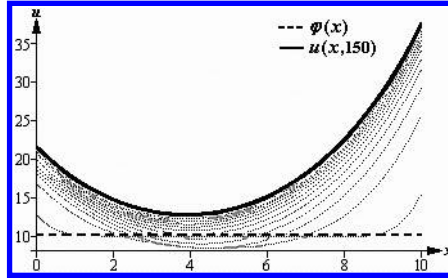
If $\gamma = 0$, then

$$f_0(t) = \frac{a^2}{l} (Q_2 - Q_1), \quad T_0(t) = \frac{a^2}{l} (Q_2 - Q_1)t, \quad f_n(t) = 0, \quad \text{and} \quad T_n(t) = 0, \quad \text{for } n > 0.$$

By substituting the expressions for C_n and $T_n(t)$ into Equation (5.67) and adding auxiliary function $w(x, t)$, we obtain the solution

$$u(x, t) = v(x, t) + w(x, t) = xQ_1 + \frac{x^2}{2l} (Q_2 - Q_1) + \sum_{n=0}^{\infty} \left[T_n(t) + C_n e^{-(a^2 \lambda_n + \gamma)t} \right] \cos \frac{n\pi x}{l}.$$

Figure 5.15 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.7. This solution was obtained with the program Heat for the case when $l = 10$, $\gamma = 0.02$, $u_0 = 10$, $Q_1 = -5$, $Q_2 = 10$, and $a^2 = 0.25$.

Figure 5.15. Solution $u(x, t)$ for Example 5.7.

Example 5.8. Find the temperature distribution inside a thin homogeneous isotropic bar of length l ($0 \leq x \leq l$) with an insulated lateral surface if the initial temperature is zero. The temperature is maintained at zero on the right end of the bar ($x = l$), and on the left it changes as governed by

$$u(0, t) = u_0 \cos \omega t,$$

where u_0 and ω are known constants. There are no sources or absorbers of heat inside the bar.

Solution. The temperature is given by a solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with conditions

$$u(x, 0) = 0,$$

$$u(0, t) = u_0 \cos \omega t, \quad u(l, t) = 0.$$

For these Dirichlet boundary conditions, the eigenvalues, eigenfunctions, and norm of the problem are

$$\lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

$$u(x, t) = v(x, t) + w(x, t).$$

The auxiliary function follows from the general case (also see Appendix B):

$$w(x, t) = \left(1 - \frac{x}{l} \right) u_0 \cos \omega t.$$

Then

$$f^*(x, t) = u_0 \omega \left(1 - \frac{x}{l}\right) \sin \omega t \text{ and } \varphi^*(x) = -u_0 \left(1 - \frac{x}{l}\right).$$

Applying Equations (5.53), (5.63) and (5.66), we obtain

$$C_n = \frac{2}{l} \int_0^l \varphi^*(x) \sin \frac{n\pi x}{l} dx = -\frac{2u_0}{n\pi},$$

$$f_n(t) = \frac{2}{l} \int_0^l f^*(x) \sin \frac{n\pi x}{l} dx = \frac{2u_0\omega}{n\pi} \sin \omega t,$$

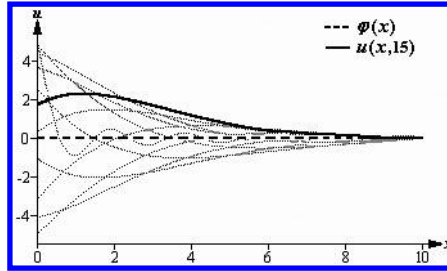
$$\begin{aligned} T_n(t) &= \int_0^t f_n(\tau) e^{-a^2 \lambda_n(t-\tau)} d\tau = \frac{2u_0\omega}{n\pi} \int_0^t \sin \omega \tau e^{-a^2 \lambda_n(t-\tau)} d\tau \\ &= \frac{2u_0\omega}{n\pi(a^4 \lambda_n^2 + \omega^2)} \left[a^2 \lambda_n \sin \omega t - \omega \cos \omega t + \omega e^{-a^2 \lambda_n t} \right]. \end{aligned}$$

Substituting the expressions for C_n and $T_n(t)$ into the general formulas, we obtain the solution:

$$\begin{aligned} u(x, t) &= \left(1 - \frac{x}{l}\right) u_0 \cos \omega t + \sum_{n=1}^{\infty} \left[T_n(t) + C_n e^{-a^2 \lambda_n t} \right] \sin \frac{n\pi x}{l} = \left(1 - \frac{x}{l}\right) u_0 \cos \omega t \\ &+ \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(a^4 \lambda_n^2 + \omega^2)} \left[a^2 \lambda_n \omega \sin \omega t - \omega^2 \cos \omega t + a^2 \lambda_n e^{-a^2 \lambda_n t} \right] \sin \frac{n\pi x}{l}. \end{aligned}$$

Figure 5.16 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.8. This solution was obtained with the program Heat for the case when $l = 10$, $\omega = 0.5$, $u_0 = 5$, and $a^2 = 4$.

Reading Exercise. Investigate the behavior of the solution to Example 5.8, which has many important physical applications. This can be done with help of the program Heat. Try large values for l and ω . Observe that for these cases, heat does not propagate far from the left end of the bar. Determine the behavior of the solution for $t \rightarrow \infty$. In Chapter 6 we demonstrate a way to find an analytical solution for the steady-state regime for large t by separating the variables in the form $\operatorname{Re} X(x) e^{i\omega t}$.

Figure 5.16. Solution $u(x, t)$ for Example 5.8.

Reading Exercise. Check that for $t \rightarrow \infty$ the solution obtained in the previous reading exercise is a sum of purely periodic harmonics and can be presented in the form

$$u(x, t) = \left(1 - \frac{x}{l}\right) u_0 \cos \omega t + \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \delta_n \cdot \sin(\omega t - \delta_n) \cdot \sin \frac{n\pi x}{l},$$

where the phase shifts are given by $\delta_n = \tan^{-1}(\omega/a^2 \lambda_n)$.

5.3.5 Boundary Problems without Initial Conditions

Next we consider frequently encountered physical situations for which knowledge of initial conditions is not important. For example, the influence of the initial conditions clearly decreases with time for cases in which heat propagates through a body. If the moment of interest is long enough after the initial time, the temperature of a bar, for example, is for all purposes defined by the boundary conditions because the effects of the initial conditions have had time to decay. In this case, we may suppose that after a long enough time the initial condition vanishes.

This brings us to boundary problems *without initial conditions*. This situation also frequently applies when *boundary conditions change periodically*, for instance, as in Example 5.8. In these cases, we may assume that after a large interval of time the temperature of a body varies periodically with the same frequency as the boundary condition. Generally, this time can be estimated as $t \gg 1/\omega$. After this time, the initial temperature can always be assumed to be equal to zero (even if it is not). In problems such as Example 5.8, we can specify this time; as can be seen from the solution, temperature becomes steady when $t \gg (l/a)^2$.

Another situation in which initial conditions are not important occurs when the equation contains periodically changing terms, so the solution varies with the same frequency after a long enough time. Example 5.9 below presents such a situation.

Example 5.9. Consider the motion of fluid between two parallel plates, located at $x = 0$ and $x = H$, under a periodically changing pressure gradient parallel to the y -axis. Clearly, this is a one-dimensional problem; the function for which we are searching is the y -component of the fluid speed, $u(x, t)$. Since we are searching for a steady-state regime, we assume the solution does not depend on the initial condition, and formally we set $u(x, 0) = 0$.

Solution. This problem is described by the equation that follows from the Navier-Stokes equation for fluid motion given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + b \cos \omega t, \quad (5.73)$$

where $a^2 \equiv \nu$ is the coefficient of kinematic viscosity. The boundary conditions

$$u(0, t) = u(H, t) = 0$$

correspond to zero velocity at the plates, and the initial condition is

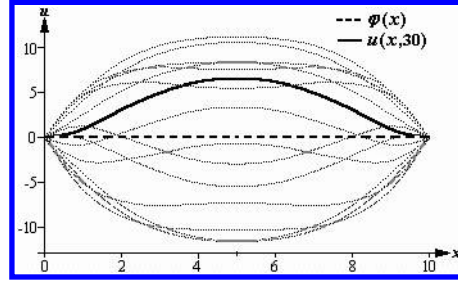
$$u(x, 0) = 0.$$

We have seen the eigenvalues, eigenfunctions, and norm of the problem a number of times before:

$$\lambda_n = \left(\frac{n\pi}{H} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{H}, \quad \|X_n\|^2 = \frac{H}{2}, \quad n = 1, 2, 3, \dots$$

The coefficient $C_n = 0$, since $\varphi(x) = 0$. Applying Equations (5.63) and (5.66), we obtain

$$\begin{aligned} f_n(t) &= \frac{2}{H} \int_0^H b \cos \omega t \cdot \sin \frac{n\pi x}{H} dx = \frac{2b \cos \omega t}{n\pi} [1 - (-1)^n], \\ T_n(t) &= \frac{2b [1 - (-1)^n]}{n\pi} \int_0^t \cos \omega \tau e^{-\frac{a^2 n^2 \pi^2}{H^2} (t-\tau)} d\tau \\ &= \frac{2b H^2 [1 - (-1)^n]}{n\pi [a^4 n^4 \pi^4 + \omega^2 H^4]} \times \left[a^2 n^2 \pi^2 \cos \omega t + \omega H^2 \sin \omega t - a^2 n^2 \pi^2 e^{-\frac{a^2 n^2 \pi^2}{H^2} t} \right]. \end{aligned}$$

Figure 5.17. Solution $u(x, t)$ for Example 5.9.

Obviously, when $n = 2k$, $T_{2k}(t) = 0$. Also,

$$a^2 n^2 \pi^2 \cos \omega t + \omega H^2 \sin \omega t = \sqrt{a^4 n^4 \pi^4 + \omega^2 H^4} \sin(\omega t + \theta_n),$$

where $\theta_n = \arctan \frac{a^2 n^2 \pi^2}{\omega H^2}$. Hence, the solution of the problem has the form

$$u(x, t) = \frac{4bH^2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \left\{ \frac{\sin(\omega t + \theta_{2k-1})}{\sqrt{a^4(2k-1)^4 \pi^4 + \omega^2 H^4}} - \frac{-a^2(2k-1)^2 \pi^2}{[a^4(2k-1)^4 \pi^4 + \omega^2 H^4]} e^{-\frac{a^2(2k-1)^2 \pi^2}{H^2} t} \right\} \sin \frac{(2k-1)\pi x}{H}.$$

Figure 5.17 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.9. This solution was obtained with the program Heat for the case when $H = 10$, $\omega = 0.5$, $b = 5$, and $a^2 = \nu = 1$.

Clearly, as $t \rightarrow \infty$, we have

$$u(x, t) \rightarrow \frac{4bH^2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(\omega t + \theta_{2k-1}) \sin \frac{(2k-1)\pi x}{H}}{(2k-1) \sqrt{a^4(2k-1)^4 \pi^4 + \omega^2 H^4}}.$$

We also can see that $u(0, t) = u(H, t) = 0$, as it should be. In addition, $u(x, t)$ takes its maximum value at $x = H/2$. Notice that the initial conditions and the equation formally disagree, but in this periodic problem the role of the initial condition becomes negligible for times $t \gg H^2/a^2$.

The curves shown in Figure 5.18 depict time traces of the fluid speed. A careful examination of the graphs shows that after time $t \approx 35$ the solution no longer depends on the initial condition.

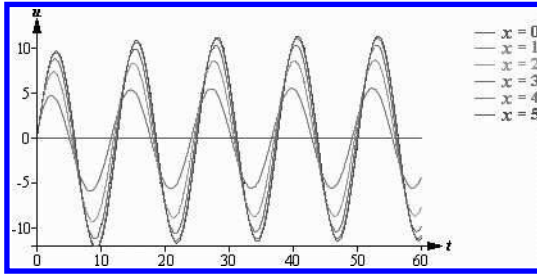


Figure 5.18. Time traces of fluid speed for Example 5.9.

In this and similar problems it is easy to obtain a steady-state (terminal) solution. From the physical point of view, it follows that such a solution is periodic with frequency ω because in systems with dissipation (described by parabolic equations), the internal oscillations decay exponentially with time (which is exactly the opposite to the case of hyperbolic equations). This allows us to search for a solution in the form

$$u(x, t) = \text{Re}\{X(x) \exp(i\omega t)\}. \quad (5.74)$$

Notice that we cannot look for a time dependence for the solution in the form of an external force, $\cos \omega t$, only, as we did for hyperbolic equations. The reason is that in cases where the equation has a first derivative in time, the sine and cosine functions mix, and we have to take both into account (or resort to using the real part of an exponential by using the Re operator). While performing intermediate operations in the following discussion, we may omit the symbol Re and wait to take the real part close to the final step. The function $X(x)$ is complex and the system response will be, as follows from Equation (5.74), shifted in phase relative to the external influence.

Substituting Equation (5.74) into the equation of motion, we obtain the ordinary differential equation

$$X'' - \frac{i\omega}{a^2} X = -\frac{b}{a^2}. \quad (5.75)$$

First, we solve the homogeneous equation,

$$X'' - \frac{i\omega}{a^2} X = 0. \quad (5.76)$$

The characteristic equation for this linear equation has two roots:

$$\frac{\sqrt{i\omega}}{a} = \pm \frac{\sqrt{\omega/2}}{a} (1 + i). \quad (5.77)$$

Thus, a general solution to the homogeneous Equation (5.76) is

$$X(x) = C_1 \exp[q(1+i)x] + C_2 \exp[-q(1+i)x], \quad q \equiv \frac{1}{a} \sqrt{\frac{\omega}{2}}. \quad (5.78)$$

A particular solution of the nonhomogeneous Equation (5.75) is $(-ib/\omega)$, in which case we may write a solution of this equation that satisfies zero boundary conditions as

$$X(x) = -\frac{ib}{\omega} \left(1 - \frac{\cos \alpha \left(x - \frac{H}{2} \right)}{\cos \alpha \frac{H}{2}} \right), \quad \alpha = q(1-i). \quad (5.79)$$

We thus have

$$u(x, t) = \operatorname{Re} X \cos \omega t - \operatorname{Im} X \sin \omega t. \quad (5.80)$$

To obtain a final form, we use the identity

$$\cos z = \cos(x + iy) = \cos x \cos iy + \sin x \sin iy = \cos x \cosh y + i \sin x \sinh y \quad (5.81)$$

to yield the results for $\operatorname{Re} X$ and $\operatorname{Im} X$ that, with Equation (5.80), give the final answer for the steady-state solution:

$$\begin{aligned} \operatorname{Re} X &= -\frac{b}{\omega} \frac{\sin qx \sinh q(x-H) - \sin q(x-H) \sinh qx}{\cos qH + \cosh qH}, \\ \operatorname{Im} X &= -\frac{b}{\omega} \left(1 - \frac{\cos qx \cosh q(x-H) + \cos q(x-H) \cosh qx}{\cos qH + \cosh qH} \right). \end{aligned} \quad (5.82)$$

Next, we consider two examples of solutions of *diffusion problems*.

Example 5.10. Let the pressure and temperature of air in a cylinder of length l ($0 \leq x \leq l$) be equal to the atmospheric pressure. The end of the cylinder at $x = 0$ is opened at the instant $t = 0$, and the other, at $x = l$, remains closed. The concentration of some gas in the external environment is constant ($u_0 = \text{const}$). Find the concentration of gas in the cylinder for $t > 0$ if at the instant $t = 0$ the gas begins to diffuse into the cylinder through the opened end.

Solution. This problem can be represented by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a^2 = D,$$

under conditions

$$u(x, 0) = 0, \quad u(0, t) = u_0, \quad DS \frac{\partial u}{\partial x}(l, t) = 0,$$

where D is the diffusion coefficient.

Clearly, the eigenvalues, eigenfunctions, and norm of the problem are (see Appendix A):

$$\lambda_n = \left[\frac{(2n-1)\pi}{2l} \right]^2, \quad X_n(x) = \sin \frac{(2n-1)\pi x}{2l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots,$$

and the solution will be of the form

$$u(x, t) = v(x, t) + w(x, t).$$

In general, for a specific problem an auxiliary function is easily obtained from the general formulas for $w(x, t)$ found in Appendix B. We may often guess what the function must look like based on the physical observation that we are searching for a terminal or steady-state solution. In the present case,

$$w(x, t) = u_0.$$

We also have

$$f^*(x, t) = 0, \quad \varphi^*(x) = -u_0.$$

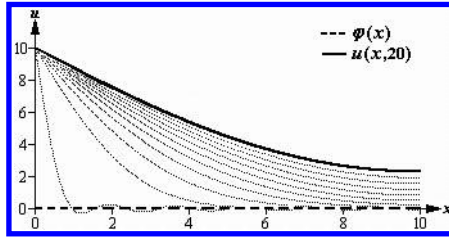
Applying Equation (5.53), we obtain

$$C_n = -\frac{2u_0}{l} \int_0^l \sin \frac{(2n-1)\pi x}{2l} dx = -\frac{4u_0}{(2n-1)\pi}.$$

Substituting the expression for C_n into the general formula, we obtain the final solution:

$$\begin{aligned} u(x, t) &= w(x, t) + \sum_{n=1}^{\infty} C_n e^{-\frac{a^2(2n-1)^2\pi^2}{4l^2}t} \sin \frac{(2n-1)\pi x}{2l} \\ &= u_0 - \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-\frac{a^2(2n-1)^2\pi^2}{4l^2}t} \sin \frac{(2n-1)\pi x}{2l}. \end{aligned}$$

Figure 5.19 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.10. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 10$, and $a^2 = D = 1$.

Figure 5.19. Solution $u(x, t)$ for Example 5.10.

Example 5.11. As in Example 5.10, let the pressure and temperature of air in a cylinder of length l ($0 \leq x \leq l$) be equal to that of the atmosphere, and the concentration of some gas in the external environment be constant, $u_0 = \text{const}$. Both ends are closed by semipermeable partitions. Gas diffuses through these ends into the cylinder from time $t = 0$. Find the concentration of diffusing gas in the cylinder for $t > 0$, assuming that the amount of diffused gas decreases through some chemical reaction and that the speed of dissociation at each point is proportional to the concentration of the gas at that point.

Solution. The problem is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u, \quad a^2 = D,$$

where γ is the coefficient of decay. The initial condition

$$u(x, 0) = 0,$$

and the boundary conditions corresponding to semipermeable partitions (which allow some freedom for gas to diffuse through them) are

$$\frac{\partial u}{\partial x}(0, t) - h[u(0, t) - u_0] = 0, \quad \frac{\partial u}{\partial x}(l, t) + h[u(l, t) - u_0] = 0.$$

The boundary conditions can be written in the form

$$\left. \frac{\partial u}{\partial x} - hu \right|_{x=0} = -hu_0, \quad \left. \frac{\partial u}{\partial x} + hu \right|_{x=l} = hu_0.$$

Substituting $u(x, t) = e^{-\gamma t} v(x, t)$ into the above equations leads to the boundary value problem

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad v(x, 0) = 0,$$

$$\left. \frac{\partial v}{\partial x} - hv \right|_{x=0} = -hu_0 e^{\gamma t}, \quad \left. \frac{\partial v}{\partial x} + hv \right|_{x=l} = hu_0 e^{\gamma t}.$$

The solution will be of the form

$$v(x, t) = y(x, t) + w(x, t),$$

where an auxiliary function $w(x, t)$ is

$$w(x, t) = u_0 e^{\gamma t}.$$

(Derive this result as a reading exercise and verify your answer by consulting Appendix B). In this case, we have

$$f^*(x, t) = -\gamma u_0 e^{\gamma t}, \quad \varphi^*(x) = -u_0.$$

The eigenvalues and eigenfunctions of a problem with mixed boundary conditions are defined by the formulas (see Appendix A):

$$\lambda_n = \left(\frac{\mu_n}{l} \right)^2, \quad n = 1, 2, 3, \dots,$$

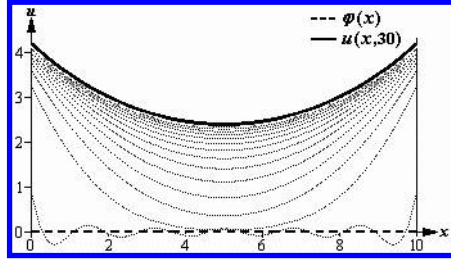
where μ_n is the n th root of the equation $\tan \mu = 2hl\mu/(\mu^2 - h^2l^2)$,

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right], \quad \|X_n\|^2 = \frac{l}{2} + \frac{h}{\lambda_n + h^2}.$$

Applying Equations (5.75), (5.63) and (5.66), we obtain the coefficients as

$$\begin{aligned} C_n &= \frac{1}{\|X_n\|^2} \int_0^l (-u_0) \frac{1}{\sqrt{\lambda_n + h^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right] dx \\ &= -\frac{2u_0 \sqrt{\lambda_n + h^2}}{l(\lambda_n + h^2) + 2h} \left[\sin \sqrt{\lambda_n} l - \frac{h}{\sqrt{\lambda_n}} (\cos \sqrt{\lambda_n} l - 1) \right], \end{aligned}$$

$$\begin{aligned} f_n(t) &= \frac{1}{\|X_n\|^2} \int_0^l (-\gamma u_0 e^{\gamma t}) X_n(x) dx \\ &= -e^{\gamma t} \frac{2\gamma u_0 \sqrt{\lambda_n + h^2}}{l(\lambda_n + h^2) + 2h} \left[\sin \sqrt{\lambda_n} l - \frac{h}{\sqrt{\lambda_n}} (\cos \sqrt{\lambda_n} l - 1) \right], \end{aligned}$$

Figure 5.20. Solution $u(x, t)$ for Example 5.11.

$$\begin{aligned}
 T_n(t) &= \int_0^t f_n(p) e^{-a^2 \lambda_n (t-p)} dx = -\frac{2\gamma u_0 \sqrt{\lambda_n + h^2}}{l(\lambda_n + h^2) + 2h} \\
 &\times \left[\sin \sqrt{\lambda_n} l - \frac{h}{\sqrt{\lambda_n}} (\cos \sqrt{\lambda_n} l - 1) \right] \int_0^t e^{-a^2 \lambda_n (t-p) + \gamma t} dx \\
 &= -\frac{2\gamma u_0 \sqrt{\lambda_n + h^2}}{[l(\lambda_n + h^2) + 2h] (a^2 \lambda_n + \gamma)} \\
 &\times \left[\sin \sqrt{\lambda_n} l - \frac{h}{\sqrt{\lambda_n}} (\cos \sqrt{\lambda_n} l - 1) \right] (e^{\gamma t} - e^{-a^2 \lambda_n t}).
 \end{aligned}$$

Substituting these expressions for C_n and $T_n(t)$ into Equation (5.67), adding $w(x, t)$ and multiplying by $e^{-\gamma t}$, we obtain the solution

$$\begin{aligned}
 u(x, t) &= e^{-\gamma t} w(x, t) + e^{-\gamma t} \sum_{n=1}^{\infty} \left[T_n(t) + C_n e^{-a^2 \lambda_n t} \right] \frac{\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x}{\sqrt{\lambda_n + h^2}} \\
 &= u_0 - 2u_0 \sum_{n=1}^{\infty} \frac{\left[\sin \sqrt{\lambda_n} l - \left(h / \sqrt{\lambda_n} \right) (\cos \sqrt{\lambda_n} l - 1) \right]}{[l(\lambda_n + h^2) + 2h] (a^2 \lambda_n + \gamma)} \\
 &\times \left[\gamma + a^2 \lambda_n e^{-(a^2 \lambda_n t + \gamma)t} \right] \left(\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h \sin \sqrt{\lambda_n} x \right).
 \end{aligned}$$

Figure 5.20 shows the spatial-time-dependent solution $u(x, t)$ for Example 5.11. This solution was obtained with the program Heat for the case when $l = 10$, $u_0 = 5$, $h = 1$, $\gamma = 0.05$, and $a^2 = D = 1$.

Reading Exercise. Find the asymptotes of the solution to Example 5.11 for small and large γ and h . Simulate this problem with the program Heat for different values of these parameters.

This problem is also useful in showing how it is possible to find the analytical solution at $t \rightarrow \infty$ in a closed form, rather than a series. Let us use the notation $U(x)$ for the part of the solution that is independent of time. The equation for this function is

$$U'' = k^2 U, \quad k^2 = \frac{\gamma}{a^2},$$

and the boundary conditions are

$$U'(0) - h[U(0) - u_0] = 0 \text{ and } U'(l) + h[U(l) - u_0] = 0.$$

The solution to this boundary value problem for an ordinary differential equation that is symmetric about the central line is

$$U = A \cosh k \left(\frac{l}{2} - x \right).$$

The constant A is determined from either initial condition:

$$A \left[k \sinh \frac{kl}{2} + h \cosh k \frac{l}{2} \right] = hu_0.$$

Finally, the asymptotic solution as $t \rightarrow \infty$ is

$$U(x) = \frac{hu_0 \cosh k \left(\frac{l}{2} - x \right)}{k \sinh \frac{kl}{2} + h \cosh k \frac{l}{2}}.$$

5.4 Heat Conduction in an Infinite Bar

In some physical situations, we may relax the demands for boundary conditions that were treated in the previous sections. Consider, for example, the process of heat conduction in a very long rod. If the time for a given heat transfer process is not very long, the influence of boundary conditions on the region of the bar considerably distant from its ends is virtually nonexistent. Temperatures in regions of the rod far from the ends will be

determined largely by the initial temperature distribution. In this case, the exact length of bar does not matter since variations in length do not influence the temperature of our region of interest, and we may suppose that the bar has infinite length. Thus, we can formulate the problem by using only initial conditions (which is known as the Cauchy problem) about temperature distribution along an infinite straight line.

Our task in this section, then, is to find a solution of the heat conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) \quad (5.83)$$

in the region $-\infty < x < \infty$ for $t > 0$ that satisfies the condition

$$u(x, 0) = \varphi(x), \quad (5.84)$$

where $\varphi(x)$ is a prescribed function. Because Equations (5.83) and (5.84) represent a linear problem, the solution $u(x, t)$ can be written as the sum of two functions; one is the solution of a homogeneous equation with nonhomogeneous initial conditions, and the other is the solution of a nonhomogeneous equation with homogeneous (zero) initial conditions.

To begin, consider the problem for the *homogeneous* heat conduction equation ($f(x, t) \equiv 0$) with *nonhomogeneous* initial conditions:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad (5.85)$$

$$u(x, 0) = \varphi(x). \quad (5.86)$$

To solve Equation (5.85) for a finite interval, we used the method of separation of variables in Section 5.3. The standard separation of variables procedure given by $u(x, t) = X(x)T(t)$, which can be applied for bounded (or finite) rods, leads to a discrete spectrum of eigenvalues, λ_n . As there are no boundary conditions for an infinite rod, the separation of variables constant is an arbitrary, continuous parameter, and the integral of $X(x)T(t)$ over this parameter also satisfies Equation (5.85). Therefore, in place of the Fourier series approach used in the finite rod case, we will use the Fourier transform approach to solve the problem defined by Equations (5.85) and (5.86) for the infinite rod.

We denote the Fourier transforms of functions $u(x, t)$ and $\varphi(x)$ as $U(k, t)$ and $\Phi(k)$, respectively, and define them by

$$U(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx, \quad (5.87)$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-ikx} dx. \quad (5.88)$$

It is clear that the Fourier transform is defined only for functions satisfying certain restrictions. In practice, the functions should have, at worst, finite discontinuities or “mild” infinite discontinuities. The improper integrals in Equations (5.87) and (5.88) should also converge at infinity, which requires that $u(x, t)$ and $\varphi(x)$ have zero limits as $x \rightarrow \pm\infty$. A very common sufficient (but not necessary) condition is the requirement that the function is absolutely integrable, meaning that the integral of its absolute value converges on an infinite interval $(-\infty, +\infty)$.

To proceed, we will assume that the conditions for the existence of the Fourier transforms given in Equations (5.87) and (5.88) are satisfied and the function $u(x, t)$ and its partial derivatives approach zero quickly enough as $x \rightarrow \pm\infty$. Also, assuming the same is valid for partial derivatives of $u(x, t)$, the integral

$$\frac{\partial U(k, t)}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u(x, t)}{\partial t} e^{-ikx} dx$$

converges.

First, we multiply the heat Equation (5.85) and initial condition (5.86) by $(1/\sqrt{2\pi})e^{-ikx}$ and integrate over x from $-\infty$ to $+\infty$. Since the Fourier transform of $\partial^2 u(x, t)/\partial x^2$ is equal to $-k^2 U(k, t)$, we determine that the function $U(k, t)$ satisfies the ordinary differential equation

$$\frac{\partial U}{\partial t} + a^2 k^2 U = 0.$$

From Equation (5.86), we obtain the initial condition for function $U(k, t)$ given by

$$U(k, 0) = \Phi(k).$$

As a result, the solution of the Cauchy problem for $U(k, t)$ may be written in the form

$$U(k, t) = \Phi(k)e^{-a^2k^2t}. \quad (5.89)$$

We next substitute Equation (5.88) for $\Phi(k)$ into this and return to the original function by applying the inverse Fourier transform. Changing the order of integration, we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k, t)e^{ikx} dk = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a^2k^2t+ik(x-\xi)} dk \right\} \varphi(\xi) d\xi.$$

Denote

$$G(x, \xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a^2k^2t+ik(x-\xi)} dk$$

and evaluate the integral using the variable substitution $z = ak\sqrt{t} - i\frac{x-\xi}{2a\sqrt{t}}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \exp[-a^2k^2t + ik(x-\xi)] dk &= \int_{-\infty}^{\infty} \exp \left[- \left(ak\sqrt{t} - i\frac{x-\xi}{2a\sqrt{t}} \right)^2 - \frac{(x-\xi)^2}{4a^2t} \right] dk \\ &= \frac{1}{a\sqrt{t}} e^{-\frac{(x-\xi)^2}{4a^2t}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{a\sqrt{t}} e^{-\frac{(x-\xi)^2}{4a^2t}}. \end{aligned}$$

Thus, we may write

$$G(x, \xi, t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2t}}. \quad (5.90)$$

The function $G(x, \xi, t)$, as given by Equation (5.90), is called the *fundamental solution* of the heat conduction equation on an infinite interval. It is also known as the *Green's function*. Note that only the difference of the arguments x and ξ appears in G ; thus the Green's function is often written as $G(x - \xi, t)$.

Finally, the solution of problems given in Equations (5.85) and (5.86) may be expressed by the formula

$$u(x, t) = \int_{-\infty}^{\infty} G(x, \xi, t) \varphi(\xi) d\xi \quad (5.91)$$

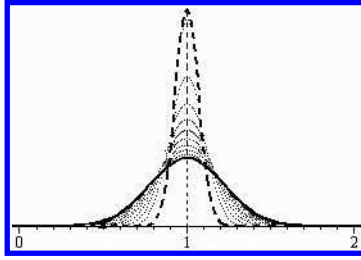


Figure 5.21. The Green's function $G(x, \xi, t_k)$. Parameters are $a^2 = 0.25$, $\xi = 1$, and $0.01 = t_0 < t_1 < \dots < t_n = 0.1$.

or

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(\xi) d\xi. \quad (5.92)$$

The integral expression given in Equation (5.92) is known as the Poisson integral.

Figure 5.21 shows the plot of the Green's function $G(x, \xi, t)$ as a function of x at the successive moments t_k . The area under each of the curves $G(x, \xi, t_k)$ is

$$\int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = 1.$$

Physically, this means that the amount of heat in the rod, $Q = c\rho$, does not change.

It is important to notice that the Green's function, $G(x, x_0, t)$, gives the solution $u(x, t) = G(x, x_0, t)$ for Equation (5.85) for the case when the initial temperature distribution is given by $\varphi(x) = \delta(x - x_0)$, as is immediately seen from Equation (5.92).

Next, let us consider a *nonhomogeneous* heat equation with a *homogeneous* initial condition $\varphi(x) \equiv 0$ given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (5.93)$$

$$u(x, 0) = 0. \quad (5.94)$$

Let $F(k, t)$ be the Fourier transform of $f(x, t)$:

$$F(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, t) e^{-ikx} dx. \quad (5.95)$$

Following the same procedure as before, we obtain the Cauchy problem for Fourier transforms given by

$$\frac{\partial U}{\partial t} + a^2 k^2 U = F, \quad (5.96)$$

$$U(k, 0) = 0.$$

The solution will be of the form

$$U(k, t) = \int_0^t e^{-a^2 k^2(t-\tau)} F(k, \tau) d\tau. \quad (5.97)$$

Reading Exercise. Verify that Equation (5.97) is a solution to Equation (5.96).

If we substitute the expression in Equation (5.95) for $F(k, t)$ into the original equation, apply the inverse Fourier transform, and change the order of integration, we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k, t) e^{ikx} dk \\ &= \int_0^t \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a^2 k^2(t-\tau) + ik(x-\xi)} dk \right\} f(\xi, \tau) d\xi d\tau. \end{aligned}$$

Solving the integral,

$$\int_{-\infty}^{\infty} e^{-a^2 k^2(t-\tau) + ik(x-\xi)} dk = \frac{\sqrt{\pi}}{a\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}},$$

we obtain the solution of the problem defined by Equations (5.93) and (5.94):

$$\begin{aligned} u(x, t) &= \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{f(\xi, \tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} G(x, \xi, t-\tau) f(\xi, \tau) d\xi d\tau, \end{aligned} \quad (5.98)$$

where the Green's function, $G(x, \xi, t-\tau)$, is defined by Equation (5.90). In the case of $f(x, t) = \delta(x-x_0)\delta(t-t_0)$, the integral in Equation (5.98) gives

$$u(x, t) = G(x, x_0, t-t_0).$$

That is, the Green's function gives the solution of heat Equation (5.93) with a zero initial condition and a delta function distribution for the initial temperature distribution $f(x, t)$.

Therefore, we have that the solution of the problem defined by Equations (5.83) and (5.84) for the nonhomogeneous heat equation with nonhomogeneous initial conditions can be represented as the sum of the solutions of two problems: The problem defined in Equations (5.85) and (5.86) for the *homogeneous* equation ($f(x, t) \equiv 0$) with *nonhomogeneous* initial conditions, and the problem defined in Equations (5.93) and (5.94) for the *nonhomogeneous* equation with a *homogeneous* initial condition ($\varphi(x) \equiv 0$). Formally, we may write this as

$$u(x, t) = \int_{-\infty}^{\infty} G(x, \xi, t) \varphi(\xi) d\xi + \int_0^t \int_{-\infty}^{\infty} G(x, \xi, t-\tau) f(\xi, \tau) d\xi d\tau. \quad (5.99)$$

Now we solve several example problems by using the heat equation for an infinite interval and the formal solutions derived in this section.

Example 5.12. Let the initial temperature distribution in an infinite uniform rod be described by the function

$$\varphi(x) = \begin{cases} u_0(1+x) & \text{for } x \in [-1, 0], \\ u_0(1-x) & \text{for } x \in [0, 1], \\ 0 & \text{for } x \notin [-1, 1], \end{cases}$$

where $u_0 = \text{const.}$ The lateral surface is insulated. Find the temperature distribution for $t > 0$.

Solution. The cooling of the rod is described by the following equation and conditions:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0$$

$$u(x, 0) = \varphi(x).$$

Applying Equation (5.92), we obtain

$$u(x, t) = \frac{u_0}{2a\sqrt{\pi t}} \left[\int_{-1}^0 (1 + \xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \int_0^1 (1 - \xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \right].$$

If we make the substitution $z = (x - \xi)/2a\sqrt{t}$, the solution becomes

$$u(x, t) = \frac{u_0}{\sqrt{\pi}} \left\{ (1 + x) \int_{\frac{x}{2a\sqrt{t}}}^{\frac{x+1}{2a\sqrt{t}}} e^{-z^2} dz + (1 - x) \int_{\frac{x-1}{2a\sqrt{t}}}^{\frac{x}{2a\sqrt{t}}} e^{-z^2} dz \right. \\ \left. - 2a\sqrt{t} \int_{\frac{x}{2a\sqrt{t}}}^{\frac{x+1}{2a\sqrt{t}}} z e^{-z^2} dz + 2a\sqrt{t} \int_{\frac{x-1}{2a\sqrt{t}}}^{\frac{x}{2a\sqrt{t}}} z e^{-z^2} dz \right\}.$$

Thus,

$$u(x, t) = \frac{u_0}{2} \left\{ (1 + x) \left[\operatorname{erf} \left(\frac{x+1}{2a\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] + (1 - x) \left[\operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-1}{2a\sqrt{t}} \right) \right] \right\} + u_0 a \sqrt{\frac{t}{\pi}} \left\{ e^{-\frac{(x+1)^2}{4a^2 t}} - e^{-\frac{x^2}{4a^2 t}} - e^{-\frac{x^2}{4a^2 t}} + e^{-\frac{(x-1)^2}{4a^2 t}} \right\},$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2} dz$ is the error function ($\operatorname{erf}(\infty) = 1$), shown in Figure 5.22.

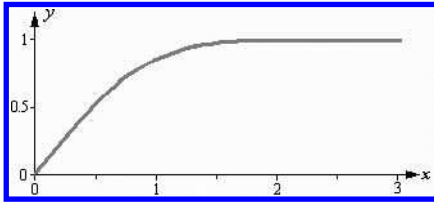


Figure 5.22. The error function $y = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$.

Finally, we obtain the temperature distribution for $t > 0$:

$$u(x, t) = \frac{u_0}{2} \left\{ (1+x) \operatorname{erf} \left(\frac{x+1}{2a\sqrt{t}} \right) - 2 \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) - (1-x) \operatorname{erf} \left(\frac{x-1}{2a\sqrt{t}} \right) \right\} \\ + u_0 a \sqrt{\frac{t}{\pi}} \left\{ e^{-\frac{(x+1)^2}{4a^2t}} - 2e^{-\frac{x^2}{4a^2t}} + e^{-\frac{(x-1)^2}{4a^2t}} \right\}.$$

Clearly, as t approaches infinity, the temperature approaches zero.

Figure 5.23 shows the solution for the case when $a^2 = 0.25$, $u_0 = 2$. The dashed line is the initial temperature distribution and the solid line is the temperature distribution at time $t = 6$. The dotted lines between them show the evolution of temperature within the period of time from 0 until 6 time units. This solution, as well as the figures for the next two experiments, was obtained with the program Heat.

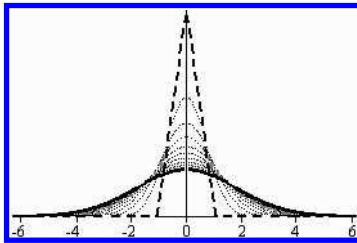


Figure 5.23. Solution for Example 5.12, when $a^2 = 0.25$ and $u_0 = 2$.

Example 5.13. Let the initial temperature distribution inside a rod be described by the function

$$\varphi(x) = u_0 e^{-\beta x^2},$$

where $u_0 = \text{const}$. The lateral surface is insulated. Find the temperature distribution in the rod for $t > 0$.

Solution. The temperature change within the rod is described by the following equations:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0,$$

$$u(x, 0) = u_0 e^{-\beta x^2}.$$

By using Equation (5.92), we obtain:

$$u(x, t) = \frac{u_0}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t} - \beta \xi^2} d\xi.$$

Since

$$\frac{(x-\xi)^2}{4a^2 t} + \beta \xi^2 = \frac{1+4a^2 \beta t}{4a^2 t} \left[\xi - \frac{x}{1+4a^2 \beta t} \right]^2 + \frac{x^2}{1+4a^2 \beta t},$$

we may make the substitution

$$z = \frac{\sqrt{1+4a^2 \beta t}}{2a\sqrt{t}} \left[\xi - \frac{x}{1+4a^2 \beta t} \right], \quad d\xi = \frac{2a\sqrt{t}}{\sqrt{1+4a^2 \beta t}} dz.$$

By using the above substitutions, we obtain

$$u(x, t) = \frac{u_0}{2a\sqrt{\pi t}} \frac{2a\sqrt{t}}{\sqrt{1+4a^2 \beta t}} e^{-\frac{x^2}{1+4a^2 \beta t}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{u_0}{\sqrt{1+4a^2 \beta t}} e^{-\frac{x^2}{1+4a^2 \beta t}},$$

since $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$.

Figure 5.24 shows the solution for the case when $a^2 = 0.25$, $\beta = 0.5$, and $u_0 = 2$. The dashed line is the initial temperature distribution and the solid line is the temperature distribution at time $t = 10$.

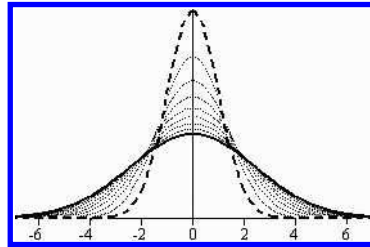


Figure 5.24. Solution for Example 5.13, when $a^2 = 0.25$, $\beta = 0.5$, and $u_0 = 2$.

Example 5.14. Now we consider a situation with heat exchange with the surroundings. Assume that the temperature of the surrounding medium is zero and the heat exchange through the lateral surface of the rod obeys Newton's law of cooling. Let the initial temperature distribution inside the infinite rod be described by the function

$$u(x, 0) = \varphi(x) = \begin{cases} u_0 & \text{for } x \in [x_1, x_2], \\ 0 & \text{for } x \notin [x_1, x_2], \end{cases}$$

where $u_0 = \text{const.}$ Find the temperature distribution inside the rod for $t > 0$.

Solution. The problem is described by the equations

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < +\infty, \quad t > 0,$$

$$u(x, 0) = \varphi(x).$$

Substituting $u(x, t) = e^{-\gamma t} v(x, t)$ yields

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2},$$

$$v(x, 0) = \varphi(x).$$

Applying Equation (5.92), we obtain

$$v(x, t) = \frac{u_0}{2a\sqrt{\pi t}} \int_{x_1}^{x_2} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi.$$

We then make the substitution $z = (x - \xi)/2a\sqrt{t}$, or $d\xi = -2a\sqrt{t}dz$, in which case the solution will be of the form

$$v(x, t) = \frac{u_0}{\sqrt{\pi}} \left[\int_0^{\frac{x-x_1}{2a\sqrt{t}}} e^{-z^2} dz - \int_0^{\frac{x-x_2}{2a\sqrt{t}}} e^{-z^2} dz \right] = \frac{u_0}{2} \left[\operatorname{erf} \left(\frac{x-x_1}{2a\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-x_2}{2a\sqrt{t}} \right) \right].$$

From these formulas, we finally obtain

$$u(x, t) = e^{-\gamma t} \frac{u_0}{2} \left[\operatorname{erf} \left(\frac{x-x_1}{2a\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-x_2}{2a\sqrt{t}} \right) \right].$$

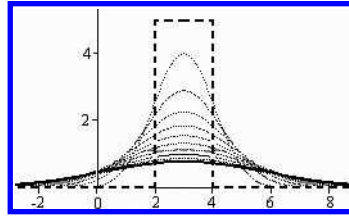


Figure 5.25. Solution for Example 5.14, when $a^2 = 0.25$, $u_0 = 5$, $x_1 = 2$, $x_2 = 4$, and $\gamma = 0.05$.

Figure 5.25 shows the solution for the case when $a^2 = 0.25$, $u_0 = 5$, $x_1 = 2$, $x_2 = 4$, and $\gamma = 0.05$. The dashed line represents the initial deflection and the solid line is the string profile at time $t = 15$.

5.5 Heat Equation for a Semi-infinite Bar

In this section, we extend the method developed in Section 5.4 for heat flow in an infinite rod to the case of a rod that is bounded at one end. Also, in the region of a rod that is close to one end and far away from the other end, the temperature in the rod is, for practical purposes, determined by the temperature condition of the near end and the initial condition. In problems of this kind, we may suppose that the rod has a semi-infinite coordinate range, $0 \leq x < \infty$.

Consider a rod that extends along the x -axis for $x \geq 0$. The boundary value problem for the homogeneous heat conduction equation with boundary conditions of the first, second, and third kind can be written as the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (5.100)$$

with the conditions

$$u|_{t=0} = \varphi(x), \quad (5.101)$$

$$\alpha u_x + \beta u|_{x=0} = g(t), \quad (5.102)$$

where $\varphi(x)$ and $g(t)$ are given functions. When $\alpha = 0$, we have the Dirichlet problem, when $\beta = 0$, we have the Neumann problem; and when both $\alpha \neq 0$ and $\beta \neq 0$, we have the third (mixed) boundary problem.

Due to linearity of the boundary value problem given in Equations (5.100) through (5.102) the solution $u(x, t)$ can be represented as a sum of two functions:

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

where $u_1(x, t)$ is the solution of the problem for the homogeneous equation with *nonhomogeneous* initial and *homogeneous* boundary conditions, and $u_2(x, t)$ is the solution of the problem of the homogeneous equation with *homogeneous* initial and *nonhomogeneous* boundary conditions.

5.5.1 Boundary Value Problems on a Semi-infinite Interval with Homogeneous Boundary Conditions

In order to study the semi-infinite rod, $x > 0$, we use the results obtained in Section 5.4. To do this, we extend the initial condition $\varphi(x)$ on the entire axis $-\infty < x < \infty$ as an even or odd function. Solutions to the actual problems can then be selected from solutions to the associated, infinite rod problem.

- **Case 1.** We start with the case of a steady temperature maintained on the end of the rod at $x = 0$. Clearly, we can choose this temperature as zero. This is the *Dirichlet boundary condition*

$$u(0, t) = 0. \quad (5.103)$$

To apply the methods previously developed for the infinite rod, we first imagine that the semi-infinite rod is extended to the left from $x = 0$ and an initial temperature distribution is defined for $x < 0$ by the same function, φ , taken with a minus sign. We have thus constructed an artificial problem for an infinite rod on the interval, $-\infty < x < \infty$, where the initial temperature condition, $u(x, 0)$, is described by the *odd* function

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \geq 0, \\ -\varphi(-x), & x \leq 0. \end{cases} \quad (5.104)$$

From Equation (5.104), it follows that $\tilde{\varphi}(0) = 0$.

Next, we use our previous results for infinite rods to solve Equation (5.100) for an infinite interval (without boundary conditions) with

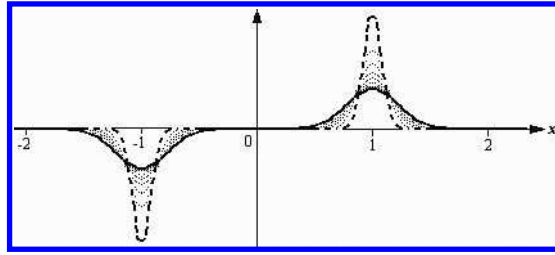


Figure 5.26. The Green's function $G_D(x, \xi, t)$ (Dirichlet boundary condition).

an initial temperature distribution that satisfies Equation (5.104). The solution of the problem was obtained in Section 5.4 and has the form

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} \tilde{\varphi}(\xi) d\xi,$$

for the infinite rod. By changing the limits of integration, we have an expression that applies to the semi-infinite rod:

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \varphi(\xi) d\xi. \quad (5.105)$$

The function

$$G_D(x, \xi, t) = \frac{1}{2a\sqrt{\pi t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \quad (5.106)$$

is known as the *Green's function of the Dirichlet problem* for the heat equation on a semi-infinite straight line.

Figure 5.26 shows the Green's function $G_D(x, \xi, t)$ (the parameters are $\xi = 1$ and $a^2 = 0.25$) as a function of x at the successive moments $0.01 = t_0 < t_1 < \dots < t_n = 0.1$.

From Equation (5.105), we can see the physical significance of the Green's function, $G_D(x, \xi, t)$. If an amount of heat, $Q = c\rho$, appears instantaneously at the initial moment $t = 0$ at point $x = \xi > 0$ while the end at $x = 0$ is maintained at zero temperature, then the function $G_D(x, \xi, t)$ gives the value of the temperature of a semi-infinite rod at time t and location x .

Example 5.15. Consider the cooling process of a semi-infinite rod when the temperature at $x = 0$ is maintained at zero and the temperature distribution at $t = 0$ is constant across the rod (i.e., $\varphi(x) = u_0 = \text{const}$).

Solution. The temperature change in the rod obeys the following boundary value problem:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$

$$u(x, 0) = u_0, \quad u(0, t) = 0.$$

By using Equation (5.105) and substituting $\varphi(x) = u_0 = \text{const}$, we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} u_0 d\xi \\ &= \frac{u_0}{\sqrt{\pi t}} \left\{ \frac{1}{2a\sqrt{t}} \int_0^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi - \frac{1}{2a\sqrt{t}} \int_0^{+\infty} e^{-\frac{(x+\xi)^2}{4a^2 t}} d\xi \right\}. \end{aligned}$$

After making the substitution $z = (\xi \pm x)/2a\sqrt{t}$, we obtain

$$u(x, t) = \frac{u_0}{\sqrt{\pi}} \left\{ \frac{\sqrt{\pi}}{2} \left[1 - \operatorname{erf} \left(-\frac{x}{2a\sqrt{t}} \right) \right] - \frac{\sqrt{\pi}}{2} \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] \right\}.$$

Finally, since $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, we have

$$u(x, t) = u_0 \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right). \quad (5.107)$$

In accordance with Equation (5.107), the temperature distribution smoothes out in space, and each value of temperature propagates to the right at a rate proportional to $\sqrt{t}$.

Figure 5.27 shows the solution $u(x, t)$ for the case when $a^2 = 0.25$ and $u_0 = 5$. The dashed line represents the initial temperature of the rod and the solid line shows the temperature distribution at time $t = 10$.

- **Case 2.** We consider the case of a rod with a heat-insulated end at $x = 0$ such that there is no flux through this end. This is a boundary

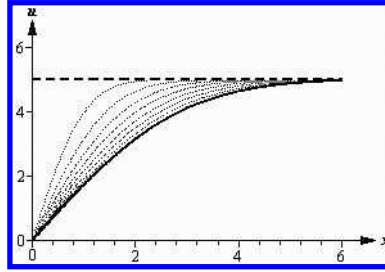


Figure 5.27. Solution for Example 5.15 when $a^2 = 0.25$ and $u_0 = 5$.

problem for the heat equation on a semi-infinite straight line for a homogeneous *Neumann boundary condition* given by

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0. \quad (5.108)$$

For this case, we extend the initial temperature distribution $u(x, 0) = \varphi(x)$ to the left beyond the point $x = 0$ as an *even* function given by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \geq 0, \\ \varphi(-x), & x \leq 0. \end{cases} \quad (5.109)$$

Here we have $\frac{\partial \tilde{\varphi}}{\partial x}(x) = -\frac{\partial \tilde{\varphi}}{\partial x}(-x)$; thus, the derivative $\frac{\partial \tilde{\varphi}}{\partial x}(0) = 0$ at $x = 0$.

Repeating the calculation for the Dirichlet boundary conditions, but using the initial distribution in Equation (5.109) instead of Equation (5.104), we obtain that the general solution is given by the formula

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} + e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \varphi(\xi) d\xi, \quad (5.110)$$

which is distinguished from Equation (5.105) by replacing the difference between the two terms in brackets by their sum. The function

$$G_N(x, \xi, t) = \frac{1}{2a\sqrt{\pi t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} + e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \quad (5.111)$$

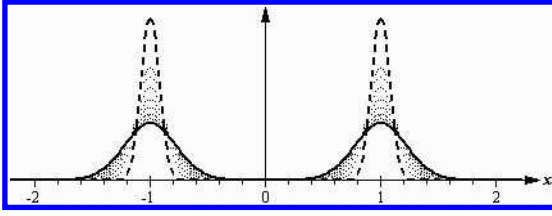


Figure 5.28. The Green's function $G_N(x, \xi, t)$ (Neumann boundary condition).

is the Green's function for the Neumann problem for the heat equation on a semi-infinite axis.

Figure 5.28 shows the Green's function $G_N(x, \xi, t)$ (with parameters $\xi = 1$ and $a^2 = 0.25$) as a function of x at the successive moments $0.01 = t_0 < t_1 < \dots < t_n = 0.1$.

Example 5.16. Consider the cooling of a semi-infinite rod if the end at $x = 0$ is insulated and temperature at the initial time, $t = 0$, is determined by the function

$$\varphi(x) = \begin{cases} u_0, & 0 < x < M, \\ 0, & x \geq M. \end{cases}$$

Solution. The boundary value problem modeling this process has the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$

$$u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial x}(0, t) = 0.$$

By using Equation (5.110), we have

$$\begin{aligned} u(x, t) &= \frac{1}{2a\sqrt{\pi t}} \int_0^{\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} + e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} u_0 d\xi \\ &= \frac{u_0}{\sqrt{\pi t}} \left\{ \frac{1}{2a\sqrt{t}} \int_0^M e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \frac{1}{2a\sqrt{t}} \int_0^M e^{-\frac{(x+\xi)^2}{4a^2 t}} d\xi \right\}. \end{aligned}$$

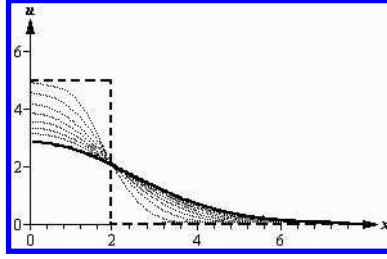


Figure 5.29. Solution for Example 5.16 when $a^2 = 0.25$, $u_0 = 5$, and $M = 2$.

As in Example 5.15, by making the substitution $z = (\xi \pm x)/2a\sqrt{t}$, we have

$$u(x, t) = \frac{u_0}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \left\{ \left[\operatorname{erf} \left(\frac{M-x}{2a\sqrt{t}} \right) + \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] + \left[\operatorname{erf} \left(\frac{M+x}{2a\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] \right\}.$$

We thus obtain

$$u(x, t) = \frac{u_0}{2} \left\{ \operatorname{erf} \left(\frac{M-x}{2a\sqrt{t}} \right) + \operatorname{erf} \left(\frac{M+x}{2a\sqrt{t}} \right) \right\}.$$

In particular, we see that if $M = \infty$ and we have constant temperature everywhere at $t = 0$, then, as it should be,

$$u(x, t) = u_0.$$

Figure 5.29 shows the solution $u(x, t)$ for the case when $a^2 = 0.25$, $u_0 = 5$, and $M = 2$. The dashed line represents the initial temperature of the rod and the solid line shows the temperature distribution at time $t = 10$.

- **Case 3.** Now we consider the possibility of heat exchange with the environment at the left end, $x = 0$, of a semi-infinite rod. Suppose that the temperature of the environment is constant so that we can set it equal to zero, and the lateral surface of the rod is insulated. Here we deal with the *boundary condition of the third kind* (the *mixed boundary condition*), given by

$$u(x, 0) = \varphi(x), \quad \left. \frac{\partial u}{\partial x} - hu \right|_{x=0} = 0 \quad (h = \text{const}). \quad (5.112)$$

As in the case presented in the solution to Example 5.15, we extend the function $\varphi(x)$ to the left beyond the point $x = 0$ and denote a function defined on $-\infty < x < \infty$ as $\tilde{\varphi}(x)$. We perform the extension in such a way that the function $\tilde{\varphi}'(x) - h\tilde{\varphi}(x)$ is odd ($\tilde{\varphi}'(x)$ is the derivative of function $\tilde{\varphi}(x)$ with respect to x).

Obviously, $\tilde{\varphi}(x) \equiv \varphi(x)$ at $x \geq 0$. To define the function $\tilde{\varphi}(x)$ for $x \leq 0$, we form the Cauchy problem

$$\begin{aligned}\tilde{\varphi}'(x) - h\tilde{\varphi}(x) &= -\varphi'(-x) + h\varphi(-x), \quad x < 0, \\ \tilde{\varphi}(0) &= \varphi(0),\end{aligned}$$

which has the solution

$$\tilde{\varphi}(x) = \varphi(-x) + 2h \int_0^x e^{h(x-z)} \varphi(-z) dz, \quad x \leq 0.$$

Thus, the function $\tilde{\varphi}(x)$ has the form

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \geq 0, \\ \varphi(-x) + 2h \int_0^x e^{h(x-z)} \varphi(-z) dz, & x \leq 0. \end{cases} \quad (5.113)$$

The solution of the problem posed in Equation (5.100) for an infinite rod with initial temperature distribution given in Equation (5.113) can be written in the form of Poisson's integral:

$$\begin{aligned}u(x, t) &= \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \tilde{\varphi}(\xi) d\xi = \int_0^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(\xi) d\xi \\ &\quad + \int_{-\infty}^0 \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(-\xi) d\xi + 2h \int_{-\infty}^0 \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \int_0^t e^{h(\xi-z)} \varphi(-z) dz.\end{aligned}$$

After a simple transformation, we obtain the following expression as a solution to the original problem

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} + e^{-\frac{(x+\xi)^2}{4a^2 t}} - 2h \int_0^\infty e^{-\frac{(x+\xi+\eta)^2}{4a^2 t} - h\eta} d\eta \right\} \varphi(\xi) d\xi. \quad (5.114)$$

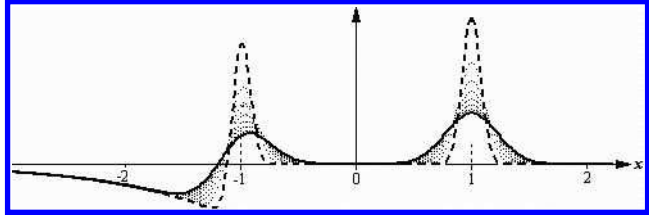


Figure 5.30. The Green's function $G_C(x, \xi, t)$ (mixed boundary condition, $h = 1$).

$$G_C(x, \xi, t) = \frac{1}{2a\sqrt{\pi t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}} - 2h \int_0^\infty e^{-\frac{(x+\xi+\eta)^2}{4a^2t}} - h\eta d\eta \right\} \quad (5.115)$$

is the Green's function of the mixed boundary value problem for the heat equation on a semi-infinite straight line.

Figure 5.30 shows the Green's function $G_C(x, \xi, t)$ (with parameters $\xi = 1$ and $a^2 = 0.25$) as a function of x at the successive moments $0.01 = t_0 < t_1 < \dots < t_n = 0.1$.

Example 5.17. Consider the previous situation for the case of a constant initial temperature, u_0 , for the rod.

Solution. The boundary value problem modeling the cooling of the rod has the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \\ u(x, 0) &= u_0, \quad \frac{\partial u}{\partial x} - hu \Big|_{x=0} = 0. \end{aligned}$$

By using formula (5.114) and setting $\varphi(x) = u_0$, we obtain:

$$\begin{aligned} u(x, t) &= \frac{u_0}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}} - 2h \int_0^\infty e^{-\frac{(x+\xi+\eta)^2}{4a^2t}} - h\eta d\eta \right\} d\xi \\ &= \frac{u_0}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}} \right\} d\xi - \frac{2hu_0}{2a\sqrt{\pi t}} \int_0^{+\infty} \int_0^\infty e^{-\frac{(x+\xi+\eta)^2}{4a^2t}} - h\eta d\eta d\xi. \end{aligned} \quad (5.116)$$

As we have shown in previous examples, the first integral is equal to

$$I_1 = \frac{u_0}{2a\sqrt{\pi t}} \int_0^{+\infty} \left\{ e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}} \right\} d\xi = \frac{u_0}{2} [\operatorname{erf}(\infty) + \operatorname{erf}(\infty)] = u_0. \quad (5.117)$$

Let us calculate the second integral, given by

$$I_2 = -\frac{2hu_0}{2a\sqrt{\pi t}} \int_0^{+\infty} \int_0^{\infty} e^{-\frac{(x+\xi+\eta)^2}{4a^2t}} - h\eta d\eta d\xi.$$

Since

$$\frac{(x + \xi + \eta)^2}{4a^2t} + h\eta = \frac{(x + \xi + \eta + 2ha^2t)^2}{4a^2t} - \frac{4(x + \xi)ha^2t + 4h^2a^4t^2}{4a^2t},$$

we make the substitution

$$z = \frac{x + \xi + \eta + 2ha^2t}{2a\sqrt{t}},$$

which yields:

$$I_2 = -\frac{2hu_0}{\sqrt{\pi}} e^{xh+h^2a^2t} \int_0^{+\infty} e^{h\xi} \int_{\frac{x+\xi}{2a\sqrt{t}}+ha\sqrt{t}}^{\infty} e^{-z^2} dz d\xi. \quad (5.118)$$

We then integrate Equation (5.118) by parts:

$$I_2 = -\frac{2u_0}{\sqrt{\pi}} e^{xh+h^2a^2t} \left\{ - \int_{\frac{x}{2a\sqrt{t}}+ha\sqrt{t}}^{\infty} e^{-z^2} dz + \frac{1}{2a\sqrt{t}} \int_0^{\infty} e^{h\xi - \left(\frac{x+\xi}{2a\sqrt{t}} + ha\sqrt{t}\right)^2} d\xi \right\}. \quad (5.119)$$

Calculation of the first integral in Equation (5.119) yields

$$\int_{\frac{x}{2a\sqrt{t}}+ha\sqrt{t}}^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2} \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} + ha\sqrt{t} \right) \right].$$

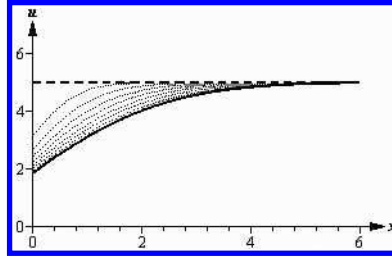


Figure 5.31. Solution for Example 5.17 when $a^2 = 0.25$, $u_0 = 5$, and $h = 1$.

The exponent in the second integral in Equation (5.119) is

$$h\xi - \left(\frac{x + \xi}{2a\sqrt{t}} + ha\sqrt{t} \right)^2 = -\frac{(x + \xi)^2}{4a^2t} - hx - h^2a^2t$$

I_2 is then equal to:

$$\begin{aligned} I_2 &= -\frac{2u_0}{\sqrt{\pi}} e^{xh+h^2a^2t} \left\{ -\frac{\sqrt{\pi}}{2} \left[1 - \operatorname{erf} \left(\frac{x + \xi}{2a\sqrt{t}} + ha\sqrt{t} \right) \right] \right. \\ &\quad \left. + \frac{\sqrt{\pi}}{2} e^{-(xh+h^2a^2t)} \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] \right\} \\ &= u_0 e^{xh+h^2a^2t} \left[1 - \operatorname{erf} \left(\frac{x + \xi}{2a\sqrt{t}} + ha\sqrt{t} \right) \right] - u_0 \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right]. \end{aligned} \quad (5.120)$$

Finally, we obtain the solution of the original problem by combining Equations (5.116), (5.117), and (5.120) to get

$$u(x, t) = u_0 \left\{ \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) + e^{a^2h^2t+xh} \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} + ah\sqrt{t} \right) \right] \right\}.$$

Figure 5.31 shows the solution $u(x, t)$ for the case when $a^2 = 0.25$, $u_0 = 5$, and $h = 1$. The dashed line represents the initial temperature of the rod and the solid line shows the temperature distribution at time $t = 10$.

5.5.2 Boundary Value Problems on a Semi-infinite Line with Nonhomogeneous Boundary Conditions

In the general case of nonhomogeneous boundary conditions, the solution of the problem of a rod of semi-infinite length can be represented as a sum

of two solutions, one of which satisfies only the initial condition, and the other one satisfies only the boundary condition. In the following discussion, we write, without proof, formulas representing the solution of the boundary value problem of the heat equation with zero initial conditions and different kinds of boundary conditions.

- **Case 1.** Let the temperature at $x = 0$ in a rod be given by the function $g(t)$ (the *Dirichlet boundary condition*). The initial and boundary conditions are

$$u(x, 0) = 0, \quad u(0, t) = g(t). \quad (5.121)$$

The solution of the problem has the form

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}} \frac{g(\tau)}{(t-\tau)^{3/2}} d\tau. \quad (5.122)$$

If, in place of τ , we introduce a new variable of integration, ξ , defined by

$$\xi = \frac{x}{2a\sqrt{t-\tau}}, \quad d\xi = \frac{x d\tau}{4a(t-\tau)^{3/2}},$$

$$u(x, t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\xi^2} g\left(t - \frac{x^2}{4a^2\xi^2}\right) d\xi. \quad (5.123)$$

Reading Exercise. Check that Equation (5.123) gives the solution to the boundary value problem.

If $g(t) = g_0 = \text{const}$, then

$$u(x, t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\xi^2} g_0 d\xi = \frac{2}{\sqrt{\pi}} g_0 \frac{\sqrt{\pi}}{2} \left[1 - \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) \right] = g_0 \left[1 - \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) \right].$$

Figure 5.32 shows the solution $u(x, t)$ for the case when $g_0 = 5$ and $a^2 = 0.25$. The dashed line represents the initial temperature of the rod, and the solid line shows the temperature distribution at time $t = 10$.

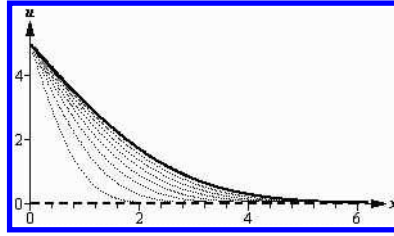


Figure 5.32. Evolution of the temperature distribution for Case 1.

- **Case 2.** If heat flows into the rod through its end $x = 0$ (*Neumann boundary condition*), we have boundary and initial conditions given by

$$u(x, 0) = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=0} = -\frac{1}{\kappa} q(t) = g(t), \quad (5.124)$$

where $q(t)$ is a given function and κ is the coefficient of heat conduction.

The solution has the form

$$u(x, t) = -\frac{a}{\sqrt{\pi}} \int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}} \frac{g(\tau)}{\sqrt{t-\tau}} d\tau. \quad (5.125)$$

Reading Exercise. Check that Equation (5.125) gives the solution to the boundary value problem.

If, in place of τ , we introduce a new variable of integration, ξ , defined by

$$\xi = \sqrt{t - \tau}, \quad d\xi = -\frac{2d\tau}{\sqrt{t - \tau}},$$

$$u(x, t) = \frac{a}{2\sqrt{\pi}} \int_{\sqrt{t}}^0 e^{-\frac{x^2}{4a^2\xi^2}} g(t - \xi^2) d\xi,$$

or

$$u(x, t) = -\frac{a}{2\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-\frac{x^2}{4a^2(\sqrt{t}-\xi)^2}} g[\xi(2\sqrt{t} - \xi)] d\xi. \quad (5.126)$$

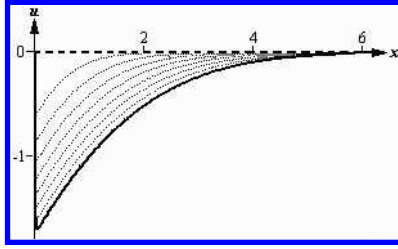


Figure 5.33. Evolution of the temperature distribution for Case 2.

For $g(t) = g_0 = \text{const}$, we have

$$\begin{aligned} u(x, t) &= -g_0 \frac{a}{\sqrt{\pi}} \int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}} \frac{1}{\sqrt{t-\tau}} d\tau \\ &= g_0 \left\{ -2a \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4a^2 t}} + x \cdot \left[1 - \text{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] \right\}. \end{aligned}$$

Figure 5.33 shows the solution $u(x, t)$ of the above problem for the case when $g_0 = 1$, $a^2 = 0.25$. The dashed line represents the initial temperature of the rod, and the solid line shows the temperature distribution at time $t = 10$.

- **Case 3.** Now consider the case when convective heat transfer occurs with the environment through the end of the rod at $x = 0$ (boundary condition of the third kind). The boundary and initial conditions are

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial x} - hu \Big|_{x=0} = g(t). \quad (5.127)$$

The solution is of the form

$$u(x, t) = -\frac{a}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} \left\{ e^{-\frac{x^2}{4a^2(t-\tau)}} - h \int_0^\infty e^{-h\xi - \frac{(x+\xi)^2}{4a^2(t-\tau)}} d\xi \right\} d\tau,$$

or

$$u(x, t) = -\frac{a}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} \left\{ e^{-\frac{x^2}{4a^2(t-\tau)}} - ah\sqrt{\pi(t-\tau)} \cdot e^{xh+a^2h^2(t-\tau)} \right. \\ \left. \times \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t-\tau}} + ah\sqrt{t-\tau} \right) \right] \right\} d\tau. \quad (5.128)$$

If, in place of τ , we introduce the new variable of integration, ξ , defined by

$$\xi = \sqrt{t} - \sqrt{t-\tau}, \quad d\xi = \frac{d\tau}{2\sqrt{t-\tau}},$$

$$u(x, t) = -\frac{a}{2\sqrt{\pi}} \int_0^{\sqrt{t}} g[\xi(2\sqrt{t}-\xi)] \left\{ e^{-\frac{x^2}{4a^2(\sqrt{t}-\xi)^2}} - ah(\sqrt{t}-\xi)\sqrt{\pi} \right. \\ \left. \times e^{xh+a^2h^2(\sqrt{t}-\xi)^2} \left[1 - \operatorname{erf} \left(\frac{x}{2a(\sqrt{t}-\xi)} + ah(\sqrt{t}-\xi) \right) \right] \right\} d\xi. \quad (5.129)$$

Note that if we take $h = 0$ in Equation (5.129), we obtain Equation (5.126) for the solution of the boundary problem of the homogeneous heat conduction equation on a semi-infinite axis with nonhomogeneous Neumann boundary conditions.

If $g(t) = g_0 = \text{const}$, then

$$u(x, t) = I_1 + I_2,$$

where

$$I_1 = g_0 \cdot \left\{ -2a\sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4a^2t}} + x \cdot \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right] \right\},$$

$$I_2 = g_0 a^2 h \int_0^t e^{xh+a^2h^2(t-\tau)} \cdot \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t-\tau}} + ah\sqrt{t-\tau} \right) \right] d\tau.$$

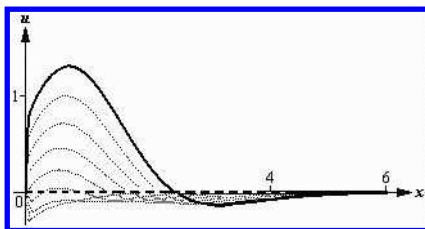


Figure 5.34. Evolution of the temperature distribution for Case 3.

Figure 5.34 shows the solution $u(x, t)$ for the case when $g_0 = 1$ and $a^2 = 0.25$. The dashed line represents the initial temperature of the rod, and the solid line shows the temperature distribution at time $t = 10$.

Reading Exercise. Determine at what time the influence of the boundary condition at $x = 0$ can be neglected. In other words, until what time can the rod still be treated as infinite rather than semi-infinite? This is not a trivial question. On the one hand, this time can be estimated as $t \ll x^2/a^2$ from the expression for the Green's functions for points of the rod with coordinate x . On the other hand, we can consider the speed of propagation of temperature change, which is infinite. Using the program Heat, model several situations: Dirichlet, Neumann, and mixed boundary conditions, homogeneous and nonhomogeneous conditions, and compare the temperatures at different points on the rod.

Problems

Problems 5.1 through 5.30 involve the temperature distribution inside a homogeneous isotropic rod (or bar) of length l ($0 \leq x \leq l$). All of these problems can be solved analytically as well as with the program Heat, in which case values must be assigned for the length l of the rod; mass density ρ ; specific heat capacity c ; coefficient of internal heat-conduction κ ; coefficient of external (with the environment) heat conduction γ , coefficients h_1, h_2 ; and the parameters of functions in each problem. It is simpler to solve these problems assuming dimensionless values for the parameters, but using physical dimensions and realistic values of parameters can lead to insightful solutions. Also, by choosing alternative parameters, the same problem can be made to represent diffusion and other related physical applications.

In Problems 5.1 through 5.9, we consider rods that are thermally insulated over their lateral surfaces. At the initial time, $t = 0$, the temperature distribution is given by $u(x, 0) = \varphi(x)$, $0 < x < l$. There are no heat sources or absorbers inside the rod.

Vary the coefficient of thermal diffusivity, $a^2 = \kappa/(c\rho)$, to obtain the time to reach some given value of temperature (1) two times faster and (2) four times faster. This can be accomplished with the program Heat by using the option “Time Traces of Temperature at Rod Points.” Directions for using the program are found in Appendix E. Note that a significant number of these problems do not have initial and boundary conditions that match each other, in which case you need to search for a generalized solution. Many of the following problems have multiple variants; for example, Problem 5.1 should be solved three separate times—once for each initial temperature function.

5.1. The ends of the rod are kept at zero temperature. The initial temperature of the rod is given as

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x(l - x).$$

5.2. The left end of the rod is kept at zero temperature, and the right end is thermally insulated from the environment. The initial temperature of the rod is

$$\varphi_1(x) = x^2, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x \left(l - \frac{x}{2} \right).$$

5.3. The left end of the rod is thermally insulated, and the right end is kept at zero temperature. The initial temperature of the rod is

$$\varphi_1(x) = x, \quad \varphi_2(x) = 1, \quad \varphi_3(x) = l^2 - x^2.$$

5.4. Both ends of the rod are thermally insulated. The initial temperature of the rod is

$$\varphi_1(x) = x, \quad \varphi_2(x) = l^2 - x^2, \quad \varphi_3(x) = x^2 \left(1 - \frac{2x}{3} \right).$$

5.5. The left end of the rod is kept at the zero temperature and the right end is subject to convective heat transfer with the environment. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x \left(l - \frac{x}{2} \right).$$

5.6. The left end of the rod is subject to convective heat transfer with the environment, which has zero temperature, and the right end is kept at zero temperature. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = l - x, \quad \varphi_3(x) = \frac{x}{3} (l - 2x) + \frac{1}{3}.$$

5.7. The left end of the rod is thermally insulated and the right end is subject to convective heat transfer with the environment, which has a temperature of zero. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = \left(1 - \frac{x^3}{3}\right).$$

5.8. The left end of the rod is subject to convective heat transfer with the environment (whose temperature is zero), and the right end is thermally insulated. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = 1 - \frac{(x-1)^3}{3}.$$

5.9. Both ends of the rod are subject to convective heat transfer with the environment, which has a temperature of zero. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = \left(1 - \frac{x^3}{3}\right).$$

Problems 5.10 through 5.18 consider rods whose lateral surfaces are subject to heat transfer according to Newton's law of cooling. The environment has zero temperature. The initial temperature of the rod is given as $u(x, 0) = \varphi(x)$. Again, there are no heat sources or absorbers inside the rod.

Select the coefficient of external heat conduction, γ , to obtain the time to reach some given value of temperature (1) two times faster and (2) four times faster. This can be accomplished with the program Heat by using the option "Time Traces of Temperature at Rod Points."

5.10. The ends of the rod are kept at a constant temperature, the left end has temperature $u(0, t) = u_1$, and the right end has temperature $u(l, t) = u_2$. The initial temperature of the rod is

$$\varphi_1(x) = x^2, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x \left(l - \frac{x}{2}\right).$$

5.11. The left end of the rod is kept at a constant temperature $u(0, t) = u_1$, and a constant heat flow is supplied to the right end of the rod. The initial temperature of the rod is

$$\varphi_1(x) = x^2, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x \left(l - \frac{x}{2}\right).$$

5.12. A constant heat flow is supplied to the left end of the rod from outside and the right end of the rod is kept at a constant temperature $u(l, t) = u_2$. The initial temperature of the rod is

$$\varphi_1(x) = x, \quad \varphi_2(x) = 1, \quad \varphi_3(x) = l^2 - x^2.$$

5.13. Constant heat flows are supplied to both ends of the rod. The initial temperature of the rod is

$$\varphi_1(x) = x, \quad \varphi_2(x) = l^2 - x^2, \quad \varphi_3(x) = x^2 \left(1 - \frac{2x}{3}\right).$$

5.14. The left end of the rod is kept at a constant temperature $u(0, t) = u_1$, and the right end is subject to convective heat transfer with the environment, which has a constant temperature of θ . The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = 1, \quad \varphi_3(x) = x \left(l - \frac{x}{2}\right).$$

5.15. The left end of the rod is subject to convective heat transfer with an environment of constant temperature $u(0, t) = u_1$, the right end is kept at the constant temperature $u(l, t) = u_2$. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = l - x, \quad \varphi_3(x) = \frac{x}{3} (l - 2x) + \frac{1}{3}.$$

5.16. A constant heat flow is supplied to the left end of the rod from outside, and the right end of the rod is subject to convective heat transfer with an environment of constant temperature θ . The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = \left(1 - \frac{x^3}{3}\right).$$

5.17. The left end of the rod is subject to convective heat transfer with an environment of constant temperature θ , and a constant heat flow is supplied to the right end of the rod. The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = 1 - \frac{(x-1)^3}{3}.$$

5.18. Both ends of the rod are subject to convective heat transfer with an environment of constant temperature θ . The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = \left(1 - \frac{x^3}{3}\right).$$

Problems 5.19 through 5.21 consider rods whose lateral surfaces are subject to heat transfer according to Newton's law. The environment has a constant temperature of θ . One internal source of heat acts at the point x_0 ($0 < x_0 < l$) inside the rod and the power of this source is Q .

5.19. The ends of the rod are kept at a constant temperature, the left end has temperature $u(0, t) = u_1$, and the right end has temperature $u(l, t) = u_2$. The initial temperature of the rod is

$$\varphi_1(x) = x^2, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x \left(l - \frac{x}{2} \right).$$

5.20. Constant heat flows, q_1 and q_2 , are supplied to both ends of the rod from outside. The initial temperature of the rod is

$$\varphi_1(x) = x, \quad \varphi_2(x) = l^2 - x^2, \quad \varphi_3(x) = x^2 \left(1 - \frac{2x}{3} \right).$$

5.21. At both ends of the rod, a convective heat transfer occurs with the environment, which has a constant temperature of θ . The initial temperature of the rod is

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = \left(1 - \frac{x^3}{3} \right).$$

Problems 5.22 through 5.24 consider rods whose lateral surfaces are subject to heat transfer according to Newton's law. The environment has a constant temperature of θ . Internal heat sources and absorbers are active in the rod, and their intensity (per unit mass of the rod) is given by $f(x, t)$. The initial temperature of the rod is zero, $u(x, 0) = 0$.

5.22. The ends of the rod are kept at a constant temperature, the left end has temperature $u(0, t) = u_1$, and the right end has temperature $u(l, t) = u_2$. The intensities of the heat sources and absorbers are

$$\begin{aligned} f_1(x, t) &= A \sin \omega t, & f_2(x, t) &= A e^{-\alpha t} \sin \omega t, \\ f_3(x, t) &= A \cos \omega t, & f_4(x, t) &= A e^{-\alpha t} \cos \omega t, \\ f_5(x, t) &= A \sin \omega t \cos \omega t, & f_6(x, t) &= A e^{-\alpha t} (\sin \omega t + \cos \omega t). \end{aligned}$$

5.23. The constant heat flows, q_1 and q_2 , are supplied to the both ends of the rod from outside. The intensities of the heat sources and absorbers are

$$\begin{aligned} f_1(x, t) &= A \sin \omega t, & f_2(x, t) &= A e^{-\alpha t} \sin \omega t, \\ f_3(x, t) &= A \cos \omega t, & f_4(x, t) &= A e^{-\alpha t} \cos \omega t, \\ f_5(x, t) &= A \sin \omega t \cos \omega t, & f_6(x, t) &= A e^{-\alpha t} (\sin \omega t + \cos \omega t). \end{aligned}$$

5.24. Both ends of the rod are subject to convective heat transfer with the environment at constant temperature θ . The intensities of the heat sources and absorbers are

$$\begin{aligned} f_1(x, t) &= A \sin \omega t, & f_2(x, t) &= A e^{-\alpha t} \sin \omega t, \\ f_3(x, t) &= A \cos \omega t, & f_4(x, t) &= A e^{-\alpha t} \cos \omega t, \\ f_5(x, t) &= A \sin \omega t \cos \omega t, & f_6(x, t) &= A e^{-\alpha t} (\sin \omega t + \cos \omega t). \end{aligned}$$

Problems 5.25 through 5.30 consider rods with thermally insulated lateral surfaces. The initial temperature of the rod is zero, $u(x, 0) = 0$. Generation (or absorption) of heat by internal sources is absent. Find the temperature change inside rod for the following cases:

5.25. The left end of the rod is kept at a constant temperature, $u(0, t) = u_1$, and the temperature of the right end changes according to $g_2(t) = A \cos \omega t$.

5.26. The temperature of the left end changes as $g_1(t) = A \cos \omega t$, and the right end of the rod is kept at a constant temperature, $u(l, t) = u_2$.

5.27. The left end of the rod is kept at a constant temperature, $u(0, t) = u_1$, and the heat flow, $g_2(t) = A \sin \omega t$, is supplied to the right end of the rod from outside.

5.28. The heat flow, $g_1(t) = A \cos \omega t$, is supplied to the left end of the rod from outside, while the right end of the rod is kept at a constant temperature, $u(l, t) = u_2$.

5.29. The left end of the rod is kept at a constant temperature, $u(0, t) = u_1$, and the right end is subject to heat transfer with the environment, which has a temperature that changes as $g_2(t) = A \sin \omega t$.

5.30. The left end of the rod is subject to heat transfer with the environment, which has a temperature that varies as $g_1(t) = A \cos \omega t$, and the right end is kept at a constant temperature, $u(l, t) = u_2$.

Propagation of Heat in an Infinite or Semi-infinite Rod

The remaining problems involve heat flow in infinite or semi-infinite rods. Solve the problems analytically and model the time evolution of the temperature with the program Heat. In all problems, the reader should experiment with values of the rod's mass density ρ , specific heat capacity c , coefficient of thermal conductivity κ , coefficient of heat exchange, and other function parameters. It might be interesting, for example, to examine physical dimensions and realistic values for various parameters in some problems. Hints and instructions for using the program are found in Appendix E.

5.31. Solve the problem of the cooling of a homogeneous infinite rod with an insulated lateral surface, warmed to an initial temperature distribution $u(x, 0) = \varphi(x)$. Find temperature as a function of time and location, $u(x, t)$, of the rod for each of the following initial temperature functions:

$$\begin{aligned} \varphi_1(x) &= \begin{cases} 10, & x < 0, \\ 20, & x \geq 0; \end{cases} & \varphi_2(x) &= \frac{1}{x^2 + 1}; \\ \varphi_3(x) &= A \sin 5x; & \varphi_4(x) &= Ae^{x(2-x)}; \\ \varphi_5(x) &= Axe^{-x^2}; & \varphi_6(x) &= Ae^{-x^2} \sin x. \end{aligned}$$

Write the formulas that represent the temperature distribution of the rod as a function of time, and model the cooling process of the rod for $a^2 = 2, 1, 0.5$.

Comment. In the case where $\varphi_3(x) = A \sin 5x$ and similar cases of periodic functions, in spite of the fact that this function does not approach zero at infinity or a constant, the solution exists and can be easily obtained analytically by separating the variables and taking $u(x, t) = \operatorname{Re} e^{ikx} T(t)$.

5.32. Solve the problem of the cooling of a homogeneous infinite rod, warmed to the initial temperature distribution $u(x, 0) = \varphi(x)$, if heat exchange with the surroundings at zero temperature occurs through the lateral surface of the rod. Assume the heat exchange obeys Newton's law with an effective coefficient of heat exchange $\gamma \geq 0$. Find temperature as a function of time and location, $u(x, t)$, of the bar for each of the following initial temperature functions:

$$\begin{aligned} \varphi_1(x) &= \begin{cases} 10, & x < 0, \\ 20, & x \geq 0; \end{cases} & \varphi_2(x) &= \frac{1}{x^2 + 1}; \\ \varphi_3(x) &= A \sin 5x; & \varphi_4(x) &= A e^{x(2-x)}; \\ \varphi_5(x) &= A x e^{-x^2}; & \varphi_6(x) &= A e^{-x^2} \sin x. \end{aligned}$$

Model the cooling process of the bar for $\gamma = 0.1, 1, 5$.

5.33. The initial temperature distribution of a homogeneous semi-infinite bar is defined by the function $u(x, 0) = \varphi(x)$. The temperature on the end of the bar at $x = 0$ is maintained at zero. Find the temperature as a function of location and time, $u(x, t)$, of the bar for each of the following initial temperature functions:

$$\begin{aligned} \varphi_1(x) &= \begin{cases} -x^2 + 8x - 15, & x \in [3, 5], \\ 0, & x \notin [3, 5]; \end{cases} & \varphi_2(x) &= A x e^{3x-x^2}; \\ \varphi_3(x) &= \begin{cases} \frac{1}{(x-2)^2+1}, & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_4(x) &= (x+1)e^{-x}; \\ \varphi_5(x) &= \begin{cases} \sin \pi(x-1), & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_6(x) &= A e^{-x^2} \sin x. \end{aligned}$$

Model the cooling process of the bar for $a^2 = 2, 1, 0.5$.

5.34. The initial temperature distribution of a homogeneous semi-infinite bar is defined by the function $u(x, 0) = \varphi(x)$. The end of the bar at $x = 0$ is insulated. Find the temperature as a function of location and time, $u(x, t)$, of the bar for each

of the following initial temperature functions:

$$\begin{aligned}\varphi_1(x) &= \begin{cases} -x^2 + 8x - 15, & x \in [3, 5], \\ 0, & x \notin [3, 5]; \end{cases} & \varphi_2(x) &= A x e^{3x-x^2}; \\ \varphi_3(x) &= \begin{cases} \frac{1}{(x-2)^2+1}, & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_4(x) &= (x+1)e^{-x}; \\ \varphi_5(x) &= \begin{cases} \sin \pi(x-1), & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_6(x) &= A e^{-x^2} \sin x.\end{aligned}$$

5.35. The initial temperature distribution of a homogeneous semi-infinite bar is defined by the function $u(x, 0) = \varphi(x)$. Convective heat exchange through the end of the bar at $x = 0$ occurs with the environment, which is at zero temperature. Find the temperature of the bar for each of the following initial temperature functions:

$$\begin{aligned}\varphi_1(x) &= \begin{cases} -x^2 + 8x - 15, & x \in [3, 5], \\ 0, & x \notin [3, 5]; \end{cases} & \varphi_2(x) &= A x e^{3x-x^2}; \\ \varphi_3(x) &= \begin{cases} \frac{1}{(x-2)^2+1}, & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_4(x) &= (x+1)e^{-x}; \\ \varphi_5(x) &= \begin{cases} \sin \pi(x-1), & x \in [1, 2], \\ 0, & x \notin [1, 2]; \end{cases} & \varphi_6(x) &= A e^{-x^2} \sin x.\end{aligned}$$

5.36. The temperature of the end of a semi-infinite bar $0 < x < +\infty$, beginning at $t = 0$, changes according to the function $u(0, t) = g(t)$. The initial temperature of the bar is zero and the lateral surface of the bar is insulated. Find the temperature as a function of location and time, $u(x, t)$, of the bar for each of the following functions:

$$\begin{aligned}g_1(t) &= A \sin \omega t; & g_2(t) &= A \cos \omega t; \\ g_3(t) &= A (\sin \omega t + \cos \omega t); & g_4(t) &= A (1 - e^{-t}); \\ g_5(t) &= A e^{-t} \sin \omega t; & g_6(t) &= A (1 - e^{-t}) \cos \omega t.\end{aligned}$$

5.37. Heat flows into the end of semi-infinite bar at $x = 0$ according to the law $u_x(0, t) = g(t)$. The initial temperature of the bar is zero and the lateral surface of the bar is insulated. Find the temperature as a function of location and time, $u(x, t)$, of the bar for each of the following heat flow functions:

$$\begin{aligned}g_1(t) &= A \sin \omega t; & g_2(t) &= A \cos \omega t; \\ g_3(t) &= A (\sin \omega t + \cos \omega t); & g_4(t) &= A (1 - e^{-t}); \\ g_5(t) &= A e^{-t} \sin \omega t; & g_6(t) &= A (1 - e^{-t}) \cos \omega t.\end{aligned}$$

5.38. Heat exchange with the environment occurs at the end, $x = 0$, of a semi-infinite bar. The corresponding boundary condition is defined by the function $g(t)$. The initial temperature of the bar is equal to zero and the lateral surface of the bar is insulated. Find the temperature as a function of location and time, $u(x, t)$, for each of the following functions:

$$g_1(t) = A \sin \omega t;$$

$$g_2(t) = A \cos \omega t;$$

$$g_3(t) = A (\sin \omega t + \cos \omega t);$$

$$g_4(t) = A (1 - e^{-t});$$

$$g_5(t) = A e^{-t} \sin \omega t;$$

$$g_6(t) = A (1 - e^{-t}) \cos \omega t.$$

6

Parabolic Equations for Higher-Dimensional Problems

6.1 Heat Conduction in More than One Dimension

In this chapter, we extend the discussion of heat flow in the Chapter 5 to cases involving more than one dimension. This section starts with a discussion of an infinite three-dimensional medium, and the chapter continues with two-dimensional examples in subsequent sections.

6.1.1 Heat Conduction in an Infinite Medium

We start with the case of a three-dimensional medium that is infinite in extent and follow the procedure outlined in Chapter 5 for the one-dimensional case. For the present situation, we wish to solve the homogeneous heat-conduction equation in three dimensions given by

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u, \quad (6.1)$$

where $a^2 = \kappa / c\rho$, κ is the thermal conductivity, ρ is the mass density of the medium, and c is its heat capacity. A review of vector calculus and the notations used in this chapter is provided in Appendix D.

For a finite medium, the separation of variables procedure, $u(\vec{r}, t) = X(\vec{r})T(t)$, gives the discrete spectrum of eigenvalues λ_n due to boundary conditions. These boundary conditions are not applicable for infinite medium problems, however. In this situation, it is helpful to express the

function $u(\vec{r}, t)$ as a Fourier integral with respect to the coordinates:

$$u(\vec{r}, t) = \frac{1}{(2\pi)^3} \int u_{\vec{k}}(t) e^{i\vec{k}\vec{r}} d^3\vec{k}, \quad d^3\vec{k} = d\vec{k}_x d\vec{k}_y d\vec{k}_z, \quad (6.2)$$

where the Fourier coefficients are

$$u_{\vec{k}}(t) = \int u(\vec{r}, t) e^{-i\vec{k}\vec{r}} d\vec{r}, \quad d\vec{r} = dx dy dz. \quad (6.3)$$

By substituting Equation (6.2) into Equation (6.1), we obtain

$$\frac{1}{(2\pi)^3} \int \left(\frac{du_{\vec{k}}}{dt} + \vec{k}^2 a^2 u_{\vec{k}} \right) e^{i\vec{k}\vec{r}} d^3\vec{k} = 0.$$

Therefore, for each Fourier component, $u_{\vec{k}}(t)$, Equation (6.1) gives

$$\frac{\partial u_{\vec{k}}}{\partial t} + \vec{k}^2 a^2 u_{\vec{k}} = 0,$$

from which we obtain

$$u_{\vec{k}}(t) = u_{0\vec{k}} e^{-\vec{k}^2 a^2 t}. \quad (6.4)$$

It is clear that the coefficients $u_{0\vec{k}}$ are determined by the initial temperature distribution (initial condition) given by

$$u(\vec{r}, 0) = u_0(\vec{r}) \equiv \varphi(\vec{r}). \quad (6.5)$$

From Equations (6.4) and (6.5), we have

$$u_{0\vec{k}}(t) = \int \varphi(\vec{r}') e^{-i\vec{k}\vec{r}'} d\vec{r}'. \quad (6.6)$$

Thus the temperature distribution as function of coordinates and time is

$$u(\vec{r}, t) = \frac{1}{(2\pi)^3} \int \varphi(\vec{r}') e^{-\vec{k}^2 a^2 t} e^{i\vec{k}(\vec{r}-\vec{r}')} d\vec{r}' d^3\vec{k}. \quad (6.7)$$

The integral over $d^3\vec{k}$ is the product of three integrals, the first of which has the generic form

$$\int_{-\infty}^{\infty} e^{-\alpha \vec{k}_x^2} \cos \beta \vec{k}_x d\vec{k}_x = \left(\frac{\pi}{\alpha} \right)^{1/2} e^{-\beta^2/4\alpha}. \quad (6.8)$$

The other integrals, with $\vec{k}_y$ and $\vec{k}_z$ instead of $\vec{k}_x$, have the same value (the related integrals with sines in place of cosines are zero since the sine function is odd). Finally, we obtain Equation (6.9), which gives the complete solution of the problem since it determines the temperature within the medium for any moment in time if the initial temperature distribution is known:

$$u(\vec{r}, t) = \frac{1}{8(\pi a^2 t)^{3/2}} \int \varphi(\vec{r}') \exp \left[-\frac{(\vec{r} - \vec{r}')^2}{4a^2 t} \right] d\vec{r}'. \quad (6.9)$$

If the initial temperature distribution is a function of only one coordinate, x (e.g., for the case of a thin infinite bar), then performing an integration over y and z in Equation (6.9) yields

$$u(x, t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \varphi(x') \exp \left[-\frac{(x - x')^2}{4a^2 t} \right] dx'. \quad (6.10)$$

Reading Exercise. Check the step between Equations (6.9) and (6.10) and show that the result corresponds to the one-dimensional case discussed in Chapter 5.

Let us consider a useful illustration of the result in Equation (6.9): the case when the temperature at $t = 0$ is zero everywhere except the origin of the coordinate system, where it is infinite. We assume also that the total amount of energy (or heat) is finite and proportional to $\int \varphi(\vec{r}) d\vec{r}$. For an infinite point source at the origin, we may write

$$\varphi(\vec{r}) = A \delta(\vec{r}), \quad A = \text{const},$$

where $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$ is a three-dimensional delta function. From Equation (6.9), we immediately obtain

$$u(\vec{r}, t) = A \frac{1}{8(\pi a^2 t)^{3/2}} \exp \left[-\frac{\vec{r}^2}{4a^2 t} \right]. \quad (6.11)$$

At the origin, the temperature decreases as $t^{-3/2}$, and there is a corresponding temperature rise in the surrounding space. The size of the space where the temperature substantially differs from zero is determined by the exponential in Equation (6.11). That is, it is given by $l \approx a\sqrt{t}$, and l increases

as the square root of the time. If at $t = 0$ the heat was concentrated in the plane at $z = 0$ and does not depend on the y -coordinate, then

$$u(x, t) = A \frac{1}{2\sqrt{\pi a^2 t}} \exp \left[-\frac{x^2}{4a^2 t} \right], \quad (6.12)$$

which follows from Equation (6.10).

The solution in Equation (6.11) with the delta function as the initial condition is the Green's function $G(\vec{r} - \vec{r}', t)$ for homogeneous heat conduction. In terms of the Green's function, the solution in Equation (6.9) can be written as

$$u(\vec{r}, t) = \int \varphi(\vec{r}') G(\vec{r} - \vec{r}', t) d\vec{r}',$$

where

$$G(\vec{r} - \vec{r}', t) = \frac{1}{8(\pi a^2 t)^{3/2}} \exp \left[-\frac{(\vec{r} - \vec{r}')^2}{4a^2 t} \right]. \quad (6.13)$$

We refer the reader to sections in Chapter 5 that discuss heat propagation in one-dimensional infinite objects and present the Green's functions for different situations in detail. The properties of the Green's functions described there also hold for the three-dimensional case. Using the three-dimensional Green's function presented here, it should be clear how to generalize the material presented in this section to, for example, the non-homogeneous heat conduction equation given by

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u + \frac{Q}{c\rho}$$

where Q denotes the density of the heat source.

6.1.2 Heat Conduction in a Semi-infinite Medium

In this section, we apply the method for semi-infinite rods presented in Chapter 5 to the case of a semi-infinite three-dimensional medium. At the end of this section, we provide an interesting application of these ideas to an entirely different physical problem that has the same mathematical expression—alternating current flow in a conductor.

- **Case 1.** Let us consider a three-dimensional medium located at $x > 0$ and begin with the case where constant temperature is maintained on the boundary plane located at $x = 0$. We take this constant

temperature as zero; i.e., the boundary and initial conditions are

$$u(x, y, z, t = 0) = \varphi(x, y, z), \quad u|_{x=0} = 0. \quad (6.14)$$

To apply the methods previously developed to an infinite medium, we first imagine that the medium is extended to the left from $x = 0$ and an initial temperature distribution is defined for $x < 0$ by the same function, φ , taken with a minus sign. Thus, the initial distribution for infinite space is an odd function of x :

$$\varphi(-x, y, z) = -\varphi(x, y, z). \quad (6.15)$$

From Equation (6.15), it follows that $\varphi(0, y, z) = 0$. By symmetry, it is clear that this boundary condition will be valid for $t > 0$.

Now we can solve Equation (6.1) for an infinite medium with an initial temperature distribution that satisfies Equation (6.15). This solution is given by the general Equation (6.9), in which we divide the range of integration over x into two parts, from $-\infty$ to 0 and from 0 to $+\infty$. Using Equation (6.15), we have

$$\begin{aligned} u(\vec{r}, t) = & \frac{1}{8(\pi a^2 t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \varphi(\vec{r}') \exp \left[-\frac{(y - y')^2 + (z - z')^2}{4a^2 t} \right] \\ & \times \left\{ \exp \left[-\frac{(x - x')^2}{4a^2 t} \right] - \exp \left[-\frac{(x + x')^2}{4a^2 t} \right] \right\} d\vec{r}'. \end{aligned} \quad (6.16)$$

If the initial temperature distribution is a function of only x , Equation (6.16) gives

$$u(x, t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_0^{\infty} \varphi(x') \left\{ \exp \left[-\frac{(x - x')^2}{4a^2 t} \right] - \exp \left[-\frac{(x + x')^2}{4a^2 t} \right] \right\} dx'. \quad (6.17)$$

Example 6.1. The temperature is maintained equal to zero on the (boundary) plane at $x = 0$. The initial temperature is constant everywhere for $x > 0$, or $\varphi(x) = u_0$. Find $u(x, t)$.

Solution. By substituting $\xi = (x' \mp x)/2a\sqrt{t}$ in the two integrals in Equation (6.17), with a minus sign in the first and a plus sign in the second, we obtain

$$u(x, t) = \frac{u_0}{2} \left[\operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) - \operatorname{erf} \left(-\frac{x}{2a\sqrt{t}} \right) \right],$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

is the *error function* ($\operatorname{erf}(\infty) = 1$). Since $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, we obtain

$$u(x, t) = u_0 \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right). \quad (6.18)$$

Solution (6.18) could have been written immediately, since it follows from the properties of the error function. It is easy to check that by differentiating Equation (6.18) twice with respect to x and once with respect to t , we obtain the equation $u_t = a^2 u_{xx}$; thus solution (6.18) satisfies both the initial and boundary conditions.

Reading Exercise. The reader is encouraged to check the preceding statement.

Thus, we see that the function

$$u(x, t) = A \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right),$$

where A is an arbitrary constant, is a solution of equation $u_t = a^2 u_{xx}$.

Reading Exercise.

1. Generalize the result in the previous reading exercise to the three-dimensional case.
2. Prove that if a semi-infinite bar is initially at zero temperature, and the end at $x = 0$ is kept at temperature u_0 , the temperature at time t is

$$u(x, t) = u_0 \left[1 - \operatorname{erf} \left(\frac{x}{2a\sqrt{t}} \right) \right].$$

Hint. According to Equation (6.18), the temperature propagates into space at a rate proportional to $\sqrt{t}$. The result (6.18) depends on the single dimensionless parameter $x/2a\sqrt{t}$.

- **Case 2.** Let us consider the case of a thermally insulated boundary plane at $x = 0$, that is, a boundary with no heat flux through it. The boundary and initial conditions are

$$u(x, y, z, t = 0) = \varphi(x), \quad \left. \frac{\partial u}{\partial x} \right|_{x=0} = 0. \quad (6.19)$$

As in Case 1, imagine the medium to extend on both sides of the plane at $x = 0$, but in this case extend the initial temperature distribution, $\varphi(x, y, z)$, as an even function of x :

$$\varphi(-x, y, z) = \varphi(x, y, z), \quad (6.20)$$

for which

$$\frac{\partial \varphi}{\partial x}(x, y, z) = -\frac{\partial \varphi}{\partial x}(-x, y, z) \text{ and } \frac{\partial \varphi}{\partial x}(0, y, z) = 0 \text{ for } x = 0.$$

By symmetry, it is clear that this condition will be satisfied for all $t > 0$. Repeating the calculations from Case 1, but using Equation (6.20) instead of Equation (6.15), we obtain the general solution, which differs from Equations (6.16) and (6.17) by replacing the subtraction of two terms by the addition of two terms:

$$\begin{aligned} u(\vec{r}, t) = & \frac{1}{8(\pi a^2 t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \varphi(\vec{r}') \exp \left[-\frac{(y - y')^2 + (z - z')^2}{4a^2 t} \right] \\ & \times \left\{ \exp \left[-\frac{(x - x')^2}{4a^2 t} \right] + \exp \left[-\frac{(x + x')^2}{4a^2 t} \right] \right\} dx' dy' dz', \end{aligned} \quad (6.21)$$

where

$$u(x, t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_0^{\infty} \varphi(x') \left\{ \exp \left[-\frac{(x - x')^2}{4a^2 t} \right] + \exp \left[-\frac{(x + x')^2}{4a^2 t} \right] \right\} dx'. \quad (6.22)$$

- **Case 3.** Assume that a heat flux enters the medium through its boundary plane at $x = 0$ for a medium located at $x > 0$. That is, the boundary condition is

$$-\kappa \frac{\partial u}{\partial x} \bigg|_{x=0} = q(t), \quad (6.23)$$

with $q(t)$ as a given flux function. Because q does not depend on coordinates y and z , the problem reduces to the one-dimensional case. The initial condition is therefore

$$u(x, y, z, t = 0) = 0. \quad (6.24)$$

To begin, we first solve an auxiliary problem with $q(t) = \delta(t)$. This problem is equivalent to the problem that led to Equation (6.12), that is, to the problem of heat propagation in an infinite medium from a point source that produces a given amount of heat. Indeed, Equation (6.23) means that a unit of energy enters through each unit area of the plane at $x = 0$ at the instant $t = 0$. In this problem, where the initial condition is $u = (2/\rho c)\delta(x)$ for $t = 0$, an amount of heat $\int \rho c u dx = 2$ is concentrated in the same area at time $t = 0$. By symmetry, we may argue that half of this energy flows in the $x > 0$ direction and the other half in the $x < 0$ direction. Since the solutions of both problems are identical, from Equation (6.12), we obtain

$$u(x, t) = \frac{1}{\kappa} \frac{a}{\sqrt{\pi t}} \exp \left[-\frac{x^2}{4a^2 t} \right]. \quad (6.25)$$

The heat conduction equation is linear, so that for arbitrary $q(t)$, instead of $\delta(t)$ the general solution of Equation (6.1) with the conditions in Equations (6.23) and (6.24) is

$$u(x, t) = \frac{1}{\kappa} \int_{-\infty}^t \frac{a}{\sqrt{\pi(t-t')}} q(t') \exp \left[-\frac{x^2}{4a^2(t-t')} \right] dt'. \quad (6.26)$$

Reading Exercise. Check in detail the derivation of Equations (6.25) and (6.26).

In particular, the temperature on the plane at $x = 0$ varies according to

$$u(0, t) = \frac{1}{\kappa} \int_{-\infty}^t \frac{a}{\sqrt{\pi(t-t')}} q(t') dt'. \quad (6.27)$$

By using Equation (6.26), we can solve the problem in which the temperature on the plane $x = 0$ is the given function of time,

$$u|_{x=0} = g(t), \quad (6.28)$$

and the initial temperature is constant (which can be taken as zero):

$$u(-x, y, z, t = 0) = 0. \quad (6.29)$$

Notice that if $u(x, t)$ satisfies Equation (6.1), then so does its derivative, $\partial u / \partial x$. Differentiating Equation (6.26) with respect to x , we obtain

$$-\kappa \frac{\partial u}{\partial x} = \int_{-\infty}^t \frac{x q(t')}{2a \sqrt{\pi(t-t')^3}} \exp \left[-\frac{x^2}{4a^2(t-t')} \right] dt'.$$

According to Equation (6.23), $q(t)$ has the same value at $x = 0$. By writing $u(x, t)$ instead of $-\kappa(\partial u / \partial x)$ and using $g(t)$ instead of $q(t)$, we obtain the solution of the problem as

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_{-\infty}^t \frac{g(t')}{\sqrt{(t-t')^3}} \exp \left[-\frac{x^2}{4a^2(t-t')} \right] dt'. \quad (6.30)$$

Reading Exercise. A radioactive gas is diffusing into the atmosphere from contaminated soil (the boundary of which we can locate at $x = 0$).

1. Provide the arguments to show that the density of the gas in the air, $\rho(x, t)$, is described by the boundary value problem

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} - \lambda \rho, \quad \frac{\partial \rho}{\partial x} \bigg|_{x=0} = -\kappa, \quad \rho(x, 0) = 0, \quad (D, \lambda, \kappa = \text{const}).$$

2. Show that the solution to this problem is

$$\rho(x, t) = \kappa \sqrt{\frac{D}{\pi}} \int_0^t \frac{1}{\sqrt{t'}} \exp \left[-\lambda t' - \frac{x^2}{4Dt'} \right] dt'.$$

- **Case 4.** Let us consider a particular important case when the temperature varies periodically in time on the boundary plane at $x = 0$:

$$u(x = 0, y, z, t) = u_0 \cos \omega t. \quad (6.31)$$

This problem is equivalent to the classic problem of Stokes about waves within an incompressible fluid generated by an infinite, rigid, flat surface harmonically oscillating in its own plane (y - z). If we investigate the process at a time that is sufficiently long from the initial moment, the influence of the initial condition is practically negligible. Thus, this is a problem without an initial condition, and we seek a stationary solution. Formally, we can choose the zero initial condition

$$u(x, y, z, t = 0) = \varphi(x, y, z) = 0.$$

Assume the fluid surface is at $x > 0$ and the plane oscillates along the y -axis (i.e., velocities in the fluid have only a y -component). The fluid velocity satisfies the Navier–Stokes equation, which, for this geometry, reduces to a one-dimensional heat conduction equation

$$u_t = a^2 u_{xx},$$

where $u(x, t)$ is the y -component of fluid velocity and $a^2 = \nu$ is the dynamic viscosity of the fluid.

It is convenient to write the boundary condition as the real part of the complex expression $u = \operatorname{Re} \{ u_0 e^{-i\omega t} \}$. In the following discussion, while performing intermediate (linear) operations, we omit the symbol Re and take the real part of the final result. Thus, we write the boundary condition as

$$u(0, t) = u_0 e^{-i\omega t}. \quad (6.32)$$

It is natural to seek a solution periodic in x and t given by

$$u(x, t) = u_0 e^{i(kx - \omega t)}, \quad (6.33)$$

which satisfies condition (6.32). Substitution of Equation (6.33) into the equation $u_t = a^2 u_{xx}$ gives

$$i\omega = a^2 k^2, \quad k = (1 + i)/\delta, \quad \delta = \sqrt{2a^2/\omega}. \quad (6.34)$$

Thus,

$$u(x, t) = u_0 e^{-x/\delta} \exp [i(x/\delta - \omega t)]. \quad (6.35)$$

(The choice of the sign before the root $\sqrt{i} = +(1 + i)/2$ in the last of Equations (6.34) is determined by the physical requirement that the velocity should be bounded as x increases.)

From this discussion, we see that transverse waves can exist in fluids with nonzero viscosity where the velocity of the fluid is perpendicular to the wave propagation direction. The oscillations damp quickly as the distance from the surface increases. The constant δ is called the penetration depth. At a distance δ from the surface, the wave's amplitude decreases e times; in other words it decreases $e^{2\pi} \approx 540$ times during one wavelength. The penetration depth δ decreases with increasing frequency and increases with increasing viscosity. In a more general case, when a plane wave is moving according to some function $u(x = 0, t) = u_0(t)$ instead of the simple harmonic motion of Equation (6.32), the solution is given by Equation (6.30).

Similarly, the temperature propagation inside a body when the temperature changes periodically according to Equation (6.31) on the boundary at $x = 0$; is described by the same Equation (6.34). Periodically changing surface temperature propagates from the boundary into a body in the form of a temperature wave with the amplitude decreasing exponentially with the depth (Fourier's first law).

An analogous phenomenon exists when alternating current flows in a metal conductor. Alternating current does not flow through a conductor with a uniform cross-sectional profile but concentrates close to the conductor surface (the so-called skin effect). Inside a conductor, the displacement current is insignificant in comparison to conduction current and the charge density is zero, in which case the equations for the electrical and magnetic fields inside a homogeneous conductor become, respectively,

$$\frac{\partial \vec{E}}{\partial t} = a^2 \nabla^2 \vec{E}, \quad \frac{\partial \vec{H}}{\partial t} = a^2 \nabla^2 \vec{H}, \quad (6.36)$$

where $a^2 = c^2/(4\pi\mu\lambda)$. Here c is the speed of light, μ is the magnetic susceptibility, λ is the electric conductivity, and $\rho = 1/\lambda$ is the resistivity of the medium.

Let us consider the same geometry as in the previous problem: a conductor is placed at $x > 0$ where the x -axis is directed inside the conductor and an external electric field is directed along the y -axis, which is parallel to the conductor's surface. By symmetry, it is clear that for a large surface (formally, an infinite plane) the field depends on x (and on time), but does not depend on y and z . If the electric field changes along the plane according to Equation (6.31) or (6.32), then inside the conductor it is given by Equation (6.35). Taking its real part, we obtain the equation for the electric field inside the conductor as

$$E(x, t) = E_0 e^{-x/\delta} \cos [i(x/\delta - \omega t)] , \quad (6.37)$$

where E_0 is the amplitude of the electric field on the surface of the conductor. The field (and current density $j = \lambda E$) are concentrated close to the surface in a layer of thickness δ . For example, a copper conductor ($\mu \approx 1$) with an applied field E with a wavelength of 3000 m (radio frequency) has a penetration depth of $\delta \approx 0.2$ mm. In the case of a direct current, $\omega = 0$, and thus $\delta \rightarrow \infty$ (i.e., a direct current is evenly distributed across the cross section of a conductor). The magnetic field is described by an equation identical to the one for electric field. From these arguments, we see that high-frequency electromagnetic fields do not penetrate deeply into a conductor but concentrate near its surface.

6.2 Heat Conduction within a Finite Rectangular Domain

Next, we consider problems involving parabolic equations for finite, two-dimensional domains such as rectangular and circular plates. Let a heat-conducting, uniform rectangular plate be placed in the horizontal x - y plane with boundaries given by edges along $x = 0$, $x = l_x$, $y = 0$, and $y = l_y$. The plate is assumed to be thin enough that the temperature is the same at all points, with the same x - y coordinates. In order to have a specific example at hand, we discuss the heat problem as before, but it should be remembered that any other physical problem described by a two-dimension parabolic equation can be solved by using the methods discussed next.

Let $u(x, y)$ be the temperature of the plate at the point (x, y) at time t . The heat conduction within such a thin uniform rectangular plate is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + \xi_x \frac{\partial u}{\partial x} + \xi_y \frac{\partial u}{\partial y} - \gamma u + f(x, y, t),$$

$$0 < x < l_x, \quad 0 < y < l_y, \quad t > 0. \quad (6.38)$$

Here a^2 , ξ_x , ξ_y , and γ are real constants. In terms of heat exchange, $a^2 = \kappa/c\rho$ is the thermal diffusivity of the material; $\gamma = h/c\rho$, where h is the heat exchange coefficient (for lateral heat exchange with an external medium); and $f(x, y, t) = Q(x, y, t)/c\rho$, where Q is the density of heat source ($Q < 0$ for locations where the heat is absorbed). The terms with coefficients ξ_x and ξ_y describe heat gain (or loss if they are negative) due to bulk motion of the surrounding medium. Clearly, these coefficients will equal zero for solids but are nonzero for liquids or gases in which bulk movement (advection) of the medium occurs. For example, $(-\xi_x)$ and $(-\xi_y)$ represent the x and y components, respectively, of the speed of the medium for a liquid or gas adjacent to the surface.

The *initial condition* defines the temperature distribution within the membrane at time zero:

$$u(x, y, 0) = \varphi(x, y). \quad (6.39)$$

The *boundary conditions* describe the thermal conditions at the boundary at any time, t . The boundary conditions can be written in a general form as

$$\begin{aligned} P_1[u] \equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} &= g_1(y, t), & P_2[u] \equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} &= g_2(y, t), \\ P_3[u] \equiv \alpha_3 \frac{\partial u}{\partial x} + \beta_3 u \Big|_{y=0} &= g_3(x, t), & P_4[u] \equiv \alpha_4 \frac{\partial u}{\partial x} + \beta_4 u \Big|_{y=l_y} &= g_4(x, t), \end{aligned} \quad (6.40)$$

where $g_1(y, t)$, $g_2(y, t)$, $g_3(x, t)$, and $g_4(x, t)$ are known functions of time and respective variable, and $\alpha_1, \beta_1, \dots, \alpha_4, \beta_4$ are real constants. As discussed in Section 2.2, physical arguments lead to the sign restrictions $\alpha_1/\beta_1 < 0$ and $\alpha_3/\beta_3 < 0$.

As before, there are three main types of boundary conditions (here and in the following discussion, we denote $a = 0$ or l_x and $b = 0$ or l_y):

- **Case 1.** Boundary condition of the first type (*Dirichlet condition*), where we are given the temperature along the y - or x -edge:

$$u(a, y, t) = g(y, t) \quad \text{or} \quad u(x, b, t) = g(x, t).$$

We may also have zero temperature at the edges, in which case $g(y, t) \equiv 0$ or $g(x, t) \equiv 0$.

- **Case 2.** Boundary condition of the second type (*Neumann condition*), where we are given the heat flow along the y - or x -edge:

$$u_x(a, y, t) = g(y, t) \quad \text{or} \quad u_y(x, b, t) = g(x, t).$$

We may also have a thermally insulated edge, in which case $g(y, t) \equiv 0$ or $g(x, t) \equiv 0$.

- **Case 3.** Boundary condition of the third type (*mixed condition*), where there is heat exchange with a medium along the y - or x -edge given by

$$u_x(a, y, t) \pm hu(a, y, t) = g(y, t) \quad \text{or} \quad u_y(x, b, t) \pm hu(x, b, t) = g(x, t). \quad (6.41)$$

We assume that h is a positive constant, in which case the positive sign should be chosen in Equation (6.41) when $a = l_x$, $b = l_y$, and the negative sign chosen when $a = 0$, $b = 0$.

The above boundary value problem can be reduced to the boundary value problem

$$\frac{\partial v}{\partial t} = a^2 \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] - \tilde{\gamma}v + \tilde{f}(x, y, t) \quad (6.42)$$

$$v(x, y, 0) = \tilde{\varphi}(x, y), \quad (6.43)$$

and

$$\begin{aligned} P_1[v] \equiv \alpha_1 \frac{\partial v}{\partial x} + \tilde{\beta}_1 v \Big|_{x=0} &= \tilde{g}_1(y, t), & P_2[v] \equiv \alpha_2 \frac{\partial v}{\partial x} + \tilde{\beta}_2 v \Big|_{x=l_x} &= \tilde{g}_2(y, t), \\ P_3[v] \equiv \alpha_3 \frac{\partial v}{\partial x} + \tilde{\beta}_3 v \Big|_{y=0} &= \tilde{g}_3(x, t), & P_4[v] \equiv \alpha_4 \frac{\partial v}{\partial x} + \tilde{\beta}_4 v \Big|_{y=l_y} &= \tilde{g}_4(x, t), \end{aligned} \quad (6.44)$$

with the help of the substitution

$$u(x, y, t) = e^{\mu x + \nu y} v(x, y, t), \quad (6.45)$$

where

$$\mu = -\frac{\xi_x}{2a^2} \text{ and } \nu = -\frac{\xi_y}{2a^2}.$$

Here,

$$\tilde{\gamma} = \gamma + \frac{\xi_x^2 + \xi_y^2}{4a^2},$$

$$\tilde{f}(x, y, t) = e^{-(\mu x + \nu y)} f(x, y, t),$$

$$\tilde{\varphi}(x, y) = e^{-(\mu x + \nu y)} \varphi(x, y),$$

$$\tilde{\beta}_1 = \beta_1 + \mu \alpha_1, \quad \tilde{g}_1(y, t) = e^{-\nu y} g_1(y, t),$$

$$\tilde{\beta}_2 = \beta_2 + \mu \alpha_2, \quad \tilde{g}_2(y, t) = e^{-(\mu l_x + \nu y)} g_2(y, t),$$

$$\tilde{\beta}_3 = \beta_3 + \nu \alpha_3, \quad \tilde{g}_3(x, t) = e^{-\mu x} g_3(x, t),$$

$$\tilde{\beta}_4 = \beta_4 + \nu \alpha_4, \quad \tilde{g}_4(x, t) = e^{-(\mu x + \nu l_y)} g_4(x, t).$$

Reading Exercise. Make the substitution given in Equation (6.45) and verify the results in Equations (6.42)–(6.44).

From the above results, we see that we need only consider the simpler problem given by Equations (6.38) through (6.40) with the condition $\xi_x = \xi_y = 0$. This general boundary value problem for the rectangular membrane cannot be solved directly by the method of Fourier expansions, since the boundary conditions are nonhomogeneous. However, this problem can be easily reduced to a boundary value problem with zero boundary conditions.

First, we express the solution to the problem defined by Equations (6.38) through (6.40) as the sum of two functions:

$$u(x, y, t) = v(x, y, t) + w(x, y, t), \quad (6.46)$$

where $v(x, y, t)$ is a new, unknown function, and $w(x, y, t)$ is an auxiliary function satisfying the boundary conditions. We seek an auxiliary function, $w(x, y, t)$, in the form

$$\begin{aligned} w(x, y, t) = & g_1(y, t) \overline{X} + g_2(y, t) \overline{\overline{X}} + g_3(x, t) \overline{Y} + g_4(x, t) \cdot \overline{\overline{Y}} \\ & + A(t) \overline{XY} + B(t) \overline{\overline{XY}} + C(t) \overline{\overline{XY}} + D(t) \overline{\overline{\overline{XY}}} \end{aligned} \quad (6.47)$$

where $\overline{X}(x)$, $\overline{\overline{X}}(x)$, $\overline{Y}(y)$ and $\overline{\overline{Y}}(y)$ are polynomials of first or second order, as designated by one or two bars above the symbols. The coefficients of these polynomials will be adjusted in such a way that function $w(x, y, t)$ satisfies the boundary conditions given in Equation (6.40).

The function $v(x, y, t)$ represents heat conduction when heat sources are present within the plate and *boundary conditions are zero*:

$$\frac{\partial v}{\partial t} = a^2 \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] - \gamma v + f^*(x, t), \quad (6.48)$$

$$v(x, y, 0) = \varphi^*(x, y),$$

$$\begin{aligned} P_1[v] &\equiv \alpha_1 \frac{\partial v}{\partial x} + \beta_1 v \Big|_{x=0} = 0, & P_2[v] &\equiv \alpha_2 \frac{\partial v}{\partial x} + \beta_2 v \Big|_{x=l_x} = 0, \\ P_3[v] &\equiv \alpha_3 \frac{\partial v}{\partial x} + \beta_3 v \Big|_{y=0} = 0, & P_4[v] &\equiv \alpha_4 \frac{\partial v}{\partial x} + \beta_4 v \Big|_{y=l_y} = 0, \end{aligned}$$

where

$$f^*(x, y, t) = f(x, y, t) - \frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \gamma w,$$

$$\varphi^*(x, y) = \varphi(x, y) - w(x, y, 0).$$

This problem can be solved by using the Fourier expansion method. The solution, $v(x, y, t)$, can be expressed as the sum of two functions

$$v(x, y, t) = u_1(x, y, t) + u_2(x, y, t), \quad (6.49)$$

where $u_1(x, y, t)$ is the solution of the *homogeneous* equation with the *specified initial conditions*, and $u_2(x, y, t)$ is the solution of a *nonhomogeneous* equation with *zero initial conditions*. For both functions, u_1 and u_2 , the *boundary conditions are zero* (i.e., homogeneous).

The solution $u_1(x, y, t)$ represents the case of *free heat exchange*, that is, heat neither generated within nor lost from the plate, but only transferred by conduction. The solution $u_2(x, y, t)$ represents the *nonfree heat exchange*, that is, the diffusion of heat due to generation (or absorption) of heat by internal sources when the initial distribution of temperature is zero. We find expressions for these two solutions in the following subsections.

6.2.1 The Fourier Method for the Homogeneous Heat Conduction Equation (Free Heat Exchange within a Rectangular Plate)

Let us first find the solution of the homogeneous equation

$$\frac{\partial u_1}{\partial t} = a^2 \left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right] - \gamma u_1, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad t > 0, \quad (6.50)$$

with the initial condition

$$u_1(x, y, 0) = \varphi(x, y), \quad (6.51)$$

and zero boundary conditions given by

$$\begin{aligned} P_1[u_1] &\equiv \alpha_1 \frac{\partial u_1}{\partial x} + \beta_1 u_1 \Big|_{x=0} = 0, & P_2[u_1] &\equiv \alpha_2 \frac{\partial u_1}{\partial x} + \beta_2 u_1 \Big|_{x=l_x} = 0, \\ P_3[u_1] &\equiv \alpha_3 \frac{\partial u_1}{\partial y} + \beta_3 u_1 \Big|_{y=0} = 0, & P_4[u_1] &\equiv \alpha_4 \frac{\partial u_1}{\partial y} + \beta_4 u_1 \Big|_{y=l_y} = 0. \end{aligned} \quad (6.52)$$

This, then, describes the case of *free heat exchange* within the plate.

Let us separate time and spatial variables:

$$u_1(x, y, t) = T(t)V(x, y). \quad (6.53)$$

We leave it to the reader as a reading exercise to obtain: (1) the following equation for the function $T(t)$:

$$T'(t) + (a^2\lambda + \gamma)T(t) = 0, \quad (6.54)$$

and (2) the boundary value problem with homogeneous boundary conditions for the function $V(x, y)$, defined by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \lambda V(x, y) = 0, \quad (6.55)$$

$$\begin{aligned} \alpha_1 V_x(0, y) + \beta_1 V(0, y) &= 0, & \alpha_2 V_x(l_x, y) + \beta_2 V(l_x, y) &= 0, \\ \alpha_3 V_y(x, 0) + \beta_3 V(x, 0) &= 0, & \alpha_4 V_y(0, l_y) + \beta_4 V(0, l_y) &= 0, \end{aligned} \quad (6.56)$$

where λ is the constant of separation of variables.

Next, we can again separate variables,

$$V(x, y) = X(x)Y(y),$$

obtaining the following one-dimensional boundary value problems:

$$\begin{aligned} X''(x) + \lambda_x X(x) &= 0, \\ \alpha_1 X'(0) + \beta_1 X(0) &= 0, \\ \alpha_2 X'(l_x) + \beta_2 X(l_x) &= 0, \end{aligned} \tag{6.57}$$

and

$$\begin{aligned} Y''(y) + \lambda_y Y(y) &= 0, \\ \alpha_3 Y'(0) + \beta_3 Y(0) &= 0, \\ \alpha_4 Y'(l_y) + \beta_4 Y(l_y) &= 0, \end{aligned} \tag{6.58}$$

where the separation of variables constants, λ_x and λ_y , are connected by the relation

$$\lambda_x + \lambda_y = \lambda.$$

Reading Exercise. Obtain the one-dimensional boundary value problems given by Equations (6.57) and (6.58).

For example, the boundary condition

$$\begin{aligned} \alpha_1 V_x(0, y) + \beta_1 V(0, y) &= \alpha_1 X'(0)Y(y) + \beta_1 X(0)Y(y) \\ &= [\alpha_1 X'(0) + \beta_1 X(0)] Y(y) = 0 \end{aligned}$$

leads to

$$\alpha_1 X'(0) + \beta_1 X(0) = 0,$$

assuming $Y(y) \neq 0$ (i.e., we seek nontrivial solutions only).

If λ_{xn} , λ_{ym} , $X_n(x)$, and $Y_m(y)$ are eigenvalues and eigenfunctions, respectively of the boundary value problems for $X(x)$ and $Y(y)$, then

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} \tag{6.59}$$

and

$$V_{nm}(x, y) = X_n(x)Y_m(y) \tag{6.60}$$

are eigenvalues and eigenfunctions of the boundary value problem for $V(x, y)$. The functions $V_{nm}(x, y)$ are orthogonal, and their norms are the products

$$\|V_{nm}\|^2 = \|X_n\|^2 \cdot \|Y_m\|^2.$$

Eigenvalues and eigenfunctions of the boundary value problem depend on the types of boundary conditions. By combining different types of boundary conditions, we can obtain nine different types of boundary value problems for the solution $X(x)$ and nine different types for the solution $Y(y)$ (see Appendix A).

In general, the eigenvalues can be written as follows:

$$\lambda_{xn} = \left(\frac{\mu_{xn}}{l_x} \right)^2, \quad \lambda_{ym} = \left(\frac{\mu_{ym}}{l_y} \right)^2, \quad (6.61)$$

where μ_{xn} is the n th root of the equation

$$\tan \mu_x = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) l_x \mu_x}{\mu_x^2 \alpha_1 \alpha_2 + l_x^2 \beta_1 \beta_2}, \quad (6.62)$$

and μ_{ym} is the m th root of the equation

$$\tan \mu_y = \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3) l_y \mu_y}{\mu_y^2 \alpha_3 \alpha_4 + l_y^2 \beta_3 \beta_4}. \quad (6.63)$$

Similarly, as was obtained in the previous chapters (e.g., Section 4.2.1), the eigenfunctions are

$$\begin{aligned} X_n(x) &= \frac{1}{\sqrt{\alpha_1^2 \lambda_{xn} + \beta_1^2}} \left[\alpha_1 \sqrt{\lambda_{xn}} \cos \sqrt{\lambda_{xn}} x - \beta_1 \sin \sqrt{\lambda_{xn}} x \right], \\ Y_m(y) &= \frac{1}{\sqrt{\alpha_3^2 \lambda_{ym} + \beta_3^2}} \left[\alpha_3 \sqrt{\lambda_{ym}} \cos \sqrt{\lambda_{ym}} y - \beta_3 \sin \sqrt{\lambda_{ym}} y \right]. \end{aligned} \quad (6.64)$$

Now, having the eigenvalues, λ_{nm} , and eigenfunctions, $V_{nm}(x, y)$, of the boundary value problem, we may obtain the solution $u_1(x, y, t)$. The solution of the differential equation

$$T'_{nm}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}(t) = 0 \quad (6.65)$$

is

$$T_{nm}(t) = C_{nm} e^{-(a^2 \lambda_{nm} + \gamma)t}, \quad (6.66)$$

from which we see that the function $u_1(x, y, t)$ can be composed as the infinite series

$$u_1(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{nm}(t) V_{nm}(x, y). \quad (6.67)$$

Using the fact that this function must satisfy the initial condition (6.51) and using the orthogonality condition for functions $V_{nm}(x, y)$, we find coefficients C_{nm} :

$$C_{nm} = \frac{1}{\|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} \varphi(x, y) V_{nm}(x, y) dx dy. \quad (6.68)$$

Reading Exercise. Verify Equation (6.68) by using the initial condition (Equation (6.51)) and the orthogonality condition.

Example 6.2. The initial temperature distribution within a thin uniform rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) with thermally insulated lateral faces is

$$u(x, y, 0) = Axy(l_x - x)(l_y - y), \quad A = \text{const.}$$

Find the distribution of temperature within the plate at any later time if its boundary is kept at constant zero temperature. Generation (or absorption) of heat by internal sources is absent.

Solution. The problem may be modeled by the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right],$$

$$0 < x < l_x, \quad 0 < y < l_y, \quad t > 0$$

under the conditions

$$u(x, y, 0) = \varphi(x, y) = Axy(l_x - x)(l_y - y),$$

$$u(0, y, t) = u(l_x, y, t) = u(x, 0, t) = u(x, l_y, t) = 0.$$

The boundary conditions of the problem are Dirichlet homogeneous boundary conditions; therefore, eigenvalues, eigenfunctions, and norm of the problem are

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right), \quad n, m = 1, 2, 3, \dots$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y},$$

$$\|V_{nm}\|^2 = \|X_n\|^2 \cdot \|Y_m\|^2 = \frac{l_x l_y}{4}.$$

Applying Equation (6.68), we obtain

$$\begin{aligned} C_{nm} &= \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} Axy(l_x - x)(l_y - y) \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y} dx dy \\ &= \begin{cases} \frac{64Al_x^2 l_y^2}{\pi^2 n^2 m^2} & \text{if } n \text{ and } m \text{ are odd,} \\ 0 & \text{if } n \text{ or } m \text{ are even.} \end{cases} \end{aligned}$$

Hence, the distribution of temperatures inside the rectangular plate at some instant of time is described by the series

$$u(x, y, t) = \frac{64Al_x^2 l_y^2}{\pi^6} \sum_{n,m=1}^{\infty} \frac{e^{-\lambda_{nm} a^2 t}}{(2n+1)^2 (2m+1)^2} \sin \frac{(2n+1)\pi x}{l_x} \sin \frac{(2m+1)\pi y}{l_y}.$$

Figure 6.1 shows two snapshots of the solution at the times $t = 0$ and $t = 5$. This solution was obtained with the program Heat for the case $a^2 = 0.25$, $l_x = 4$, $l_y = 6$, and $A = 0.01$. Directions for using the program Heat are found in Appendix E.

6.2.2 The Fourier Method for the Nonhomogeneous Heat Conduction Equation (Rectangular Plate with Internal Sources)

As mentioned earlier in Section 6.2, the solution $u_2(x, y, t)$ represents the *nonfree heat exchange* within the plate—that is, the diffusion of heat due to generation (or absorption) of heat by internal sources (or sinks) for the

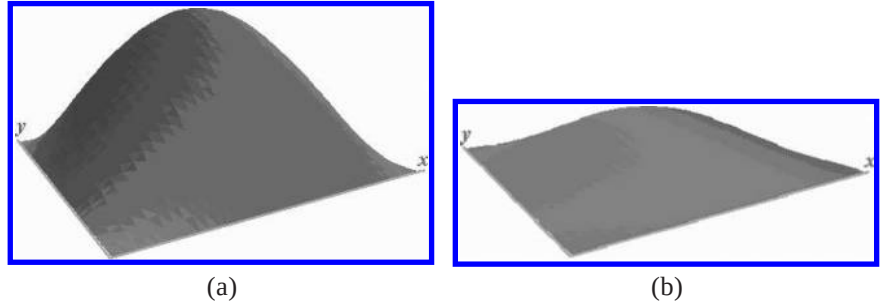


Figure 6.1. Surface graph of plate temperature at (a) $t = 0$ and (b) $t = 5$ for Example 6.2.

case of an initial distribution of temperatures equal to zero. The solution to the general problem of heat conduction in a plate consists of the sum of the *free heat exchange* solutions, $u_1(x, y, t)$, discussed in the previous section, and the solutions, $u_2(x, y, t)$, which are discussed in this section.

The function $u_2(x, y, t)$ is a solution of the *nonhomogeneous* equation

$$\frac{\partial u_2}{\partial t} = a^2 \left[\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right] - \gamma u_2 + f(x, y, t) \quad (6.69)$$

with *zero initial and boundary conditions*. As before, we can separate time and spatial variables to obtain a general solution to Equation (6.69) in the form

$$u_2(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{nm}(t) V_{nm}(x, y), \quad (6.70)$$

where $V_{nm}(x, y)$ are eigenfunctions of the corresponding homogeneous boundary value problem given in Equations (6.50) through (6.52), with solutions given by Equation (6.67) with the definitions of Equation (6.64). Here, $T_{nm}(t)$ are as yet unknown functions of t . Zero boundary conditions for $u_2(x, y, t)$, given by

$$\begin{aligned} P_1[u_2] &\equiv \alpha_1 \frac{\partial u_2}{\partial x} + \beta_1 u_2 \Big|_{x=0} = 0, & P_2[u_2] &\equiv \alpha_2 \frac{\partial u_2}{\partial x} + \beta_2 u_2 \Big|_{x=l_x} = 0, \\ P_3[u_2] &\equiv \alpha_3 \frac{\partial u_2}{\partial y} + \beta_3 u_2 \Big|_{y=0} = 0, & P_4[u_2] &\equiv \alpha_4 \frac{\partial u_2}{\partial y} + \beta_4 u_2 \Big|_{y=l_y} = 0 \end{aligned}$$

are valid for any choice of functions $T_{nm}(t)$ (assuming the series converge uniformly) because they are valid for the functions $V_{nm}(x, y)$. We leave it to the reader to obtain these results as a reading exercise.

We now determine the functions $T_{nm}(t)$ in such a way that the series (6.70) satisfies the nonhomogeneous Equation (6.69) and the homogeneous initial condition. By substituting the series (6.70) into Equation (6.69), we obtain

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [T'_{nm}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}(t)] V_{nm}(x, y) = f(x, y, t). \quad (6.71)$$

We can expand the function $f(x, y, t)$ in a Fourier series of the functions $V_{nm}(x, y)$ in the rectangular region $[0, l_x; 0, l_y]$ such that

$$f(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm}(t) V_{nm}(x, y), \quad (6.72)$$

where

$$f_{nm}(t) = \frac{1}{\|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} f(x, y, t) V_{nm}(x, y) dx dy. \quad (6.73)$$

Comparing the two expansions in Equations (6.71) and (6.72) for the same function $f(x, y, t)$, we obtain differential equations for the functions $T_{nm}(t)$:

$$T'_{nm}(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}(t) = f_{nm}(t). \quad (6.74)$$

In order that the solution represented by the series $u_2(x, y, t)$ given in Equation (6.70) satisfies the zero temperature initial condition, it is necessary that the functions $T_{nm}(t)$ obey the condition

$$T_{nm}(0) = 0. \quad (6.75)$$

As we proved before, the solution of the ordinary differential Equation (6.74) with initial condition (6.75) may be written in the integral form

$$T_{nm}(t) = \int_0^t f_{nm}(\tau) e^{-(a^2 \lambda_{nm} + \gamma)(t-\tau)} d\tau. \quad (6.76)$$

Thus, the solution of the nonhomogeneous heat conduction problem for a thin uniform rectangular plate with boundary conditions equal to zero has the form

$$u(x, y, t) = u_1(x, y, t) + u_2(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[T_{nm}(t) + C_{nm} e^{-(a^2 \lambda_{nm} + \gamma)t} \right] V_{nm}(x, y),$$

where the functions $T_{nm}(t)$ are defined by Equation (6.76) and the coefficients C_{nm} have been found earlier in Equation (6.68).

Example 6.3. Find the temperature, $u(x, y, t)$, of a thin rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) if its boundary is kept at constant zero temperature, the initial temperature distribution within the plate is zero, and one internal source of heat acts at the point (x_0, y_0) in the plate. The value of this source is

$$Q(t) = A \sin \omega t.$$

Assume the plate is thermally insulated over its lateral faces.

Solution. The problem is expressed as

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + \frac{A}{c\rho} \sin \omega t \cdot \delta(x - x_0) \delta(y - y_0),$$

under the conditions

$$u(x, y, 0) = 0,$$

$$u(0, y, t) = u(l_x, y, t) = u(x, 0, t) = u(x, l_y, t) = 0.$$

The boundary conditions of the problem are Dirichlet homogeneous boundary conditions; therefore, eigenvalues, eigenfunctions, and norm of the problem are:

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right), \quad n, m = 1, 2, 3, \dots$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y},$$

$$\|V_{nm}\|^2 = \|X_n\|^2 \cdot \|Y_m\|^2 = \frac{l_x l_y}{4}.$$

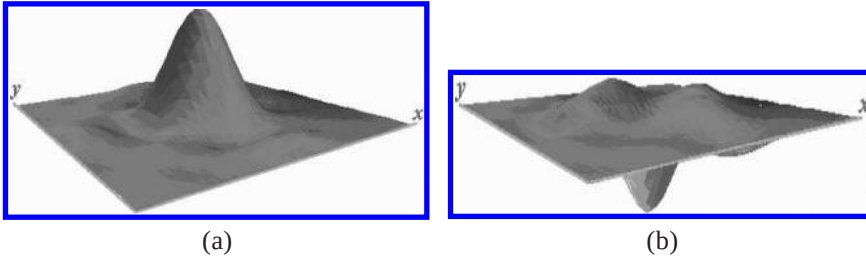


Figure 6.2. Surface graph of plate temperature at (a) $t = 0.6$ and (b) $t = 6.2$ for Example 6.3.

The initial condition is equal to zero, in which case $C_{nm} = 0$. Applying Equation (6.73), we obtain

$$\begin{aligned}
 f_{nm}(t) &= \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} \frac{A}{c\rho} \sin \omega t \cdot \delta(x - x_0) \delta(y - y_0) \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y} dx dy \\
 &= \frac{4A}{c\rho l_x l_y} \sin \omega t \sin \frac{\pi n x_0}{l_x} \sin \frac{\pi m y_0}{l_y}.
 \end{aligned} \tag{6.77}$$

We also have from Equations (6.76) and (6.77) that

$$\begin{aligned}
 T_{nm}(t) &= \int_0^t f_{nm}(\tau) e^{-\lambda_{nm} a^2 (t-\tau)} d\tau = \frac{4A}{c\rho l_x l_y} \sin \frac{\pi n x_0}{l_x} \sin \frac{\pi m y_0}{l_y} \\
 &\quad \times \frac{1}{\omega^2 + (a^2 \lambda_{nm})^2} \left[a^2 \lambda_{nm} \sin \omega t - \omega \cos \omega t + \omega e^{-\lambda_{nm} a^2 t} \right],
 \end{aligned}$$

so that finally we obtain

$$\begin{aligned}
 u(x, y, t) &= \frac{4A}{c\rho l_x l_y} \sum_{n,m=1}^{\infty} \frac{1}{\omega^2 + (a^2 \lambda_{nm})^2} \left[a^2 \lambda_{nm} \sin \omega t - \omega \cos \omega t + \omega e^{-\lambda_{nm} a^2 t} \right] \\
 &\quad \times \sin \frac{\pi n x_0}{l_x} \sin \frac{\pi m y_0}{l_y} \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y}.
 \end{aligned}$$

Figure 6.2 shows snapshots of the solution at times $t = 0.6$ and $t = 6.2$. This solution was obtained with the program Heat for the case $a^2 = 0.25$, $l_x = 4$, $l_y = 6$, $A/c\rho = 120$, $\omega = 5$, $x_0 = 2$, and $y_0 = 3$.

Reading Exercise. Find a stationary solution (periodic in time) of the problem in Example 6.3 and compare it with the solution $u(x, y, t)$ obtained in Example 6.3 when $t \rightarrow \infty$.

Hint. Search for a stationary solution in the form $\operatorname{Re}[F(x, y) \exp(i\omega t)]$; then, for a complex function, $F(x, y)$, you will obtain the Helmholtz equation with zero boundary and initial conditions. A similar problem was discussed in Chapter 5 for the one-dimensional heat equation.

Example 6.4. A heat-conducting, thin, uniform rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. One side of the plate, at $x = 0$, is thermally insulated, and the rest of the boundary is kept at constant zero temperature. The initial temperature distribution within the plate is zero.

Let heat be generated throughout the plate with the intensity of internal sources (per unit mass of the membrane) given by

$$Q(x, y, t) = A(l_x - x) \sin \frac{\pi y}{l_y} \sin t.$$

Find the distribution of temperature within the plate when $t > 0$.

Solution. The problem involves finding the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + \frac{A}{C\rho} (l_x - x) \sin \frac{\pi y}{l_y} \sin t$$

under the conditions

$$u(x, y, 0) = 0,$$

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \quad u(l_x, y, t) = 0, \quad u(x, 0, t) = u(x, l_y, t) = 0.$$

Eigenvalues, eigenfunctions, and norm of the problem are given by

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{(2n-1)^2}{4l_x^2} + \frac{m^2}{l_y^2} \right], \quad n, m = 1, 2, 3, \dots$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \cos \frac{(2n-1)\pi x}{2l_x} \sin \frac{m\pi y}{l_y},$$

$$\|V_{nm}\|^2 = \|X_n\|^2 \cdot \|Y_m\|^2 = \frac{l_x l_y}{4}.$$

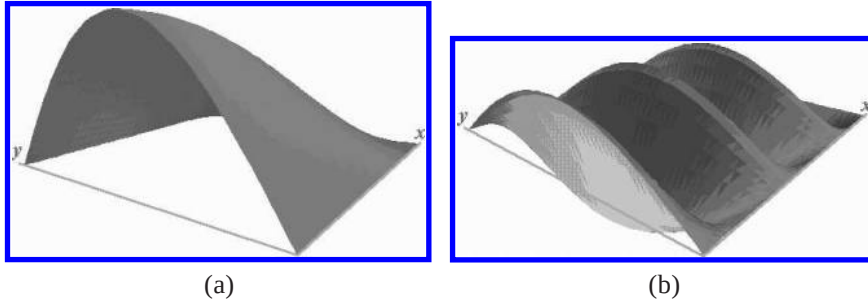


Figure 6.3. Eigenfunctions (a) $V_{11}(x, y)$ and (b) $V_{61}(x, y)$ for Example 6.4.

The three-dimensional picture shown in Figure 6.3 depicts two eigenfunctions (chosen as examples), $V_{11}(x, y)$ and $V_{61}(x, y)$, for the given problem. The values of λ are $\lambda_{11} = 0.428$ and $\lambda_{61} = 18.934$.

The initial condition is equal to zero, in which case $C_{nm} = 0$. Applying Equation (6.73), we obtain

$$f_{n1}(t) = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} f(x, y, t) \cos \frac{(2n-1)\pi x}{2l_x} \sin \frac{\pi y}{l_y} dx dy = \frac{8Al_x}{\pi^2(2n-1)^2} \sin(t),$$

and $f_{nm} = 0$, if $m \neq 1$. Thus, we have

$$\begin{aligned} T_{n1}(t) &= \frac{1}{\|v_{n1}\|^2} \int_0^t f_{n1}(\tau) e^{-\lambda_{n1}a^2(t-\tau)} d\tau \\ &= \frac{8Al_x}{\pi^2(2n-1)^2 \left[1 + (a^2\lambda_{n1})^2\right]} \left\{ a^2\lambda_{n1} \sin(t) - \cos(t) + e^{-\lambda_{n1}a^2t} \right\}, \end{aligned}$$

$T_{nm} = 0$, if $m \neq 1$, and, finally,

$$u(x, y, t) = \frac{8Al_x}{\pi^2} \sin \frac{\pi y}{l_y} \sum_{n=1}^{\infty} \frac{a^2\lambda_{n1} \sin t - \cos t + e^{-\lambda_{n1}a^2t}}{(2n-1)^2 \left[\omega^2 + (a^2\lambda_{n1})^2\right]} \cos \frac{(2n-1)\pi x}{2l_x}.$$

Figure 6.4 shows two snapshots of the solution at the times $t = 3$ and $t = 6$. This solution was obtained with the program Heat for the case $a^2 = 0.25$, $l_x = 4$, $l_y = 6$, and $A = 0.1$.

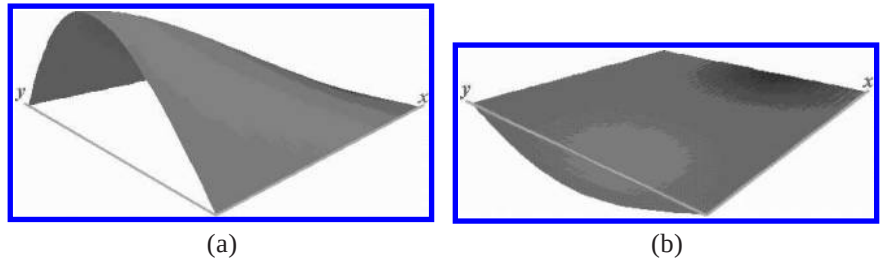


Figure 6.4. Surface graph of plate temperature at (a) $t = 3$ and (b) $t = 6$ for Example 6.4.

6.2.3 The Fourier Method for the Nonhomogeneous Heat Conduction Equation with Nonhomogeneous Boundary Conditions

Now consider the general boundary problem for heat conduction given by

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \gamma u + f(x, y, t), \quad (6.78)$$

with nonhomogeneous initial and boundary conditions

$$u(x, y, t)|_{t=0} = \varphi(x, y),$$

$$\begin{aligned} P_1[u] &\equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = g_1(y, t), & P_2[u] &\equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l_x} = g_2(y, t), \\ P_3[u] &\equiv \alpha_3 \frac{\partial u}{\partial y} + \beta_3 u \Big|_{y=0} = g_3(x, t), & P_4[u] &\equiv \alpha_4 \frac{\partial u}{\partial y} + \beta_4 u \Big|_{y=l_y} = g_4(x, t). \end{aligned}$$

As was mentioned at the beginning of Section 6.2, to deal with homogeneous boundary conditions, we introduce an auxiliary function, $w(x, y, t)$, and search for a solution of the problem in the form

$$u(x, y, t) = v(x, y, t) + w(x, y, t),$$

where $v(x, y, t)$ is a new unknown function, and the function $w(x, y, t)$ is chosen so that it satisfies the given nonhomogeneous boundary conditions.

- **Case 1.** Let the boundary value functions *satisfy consistency conditions* (i.e., the boundary conditions take the same values at the corners of the domain), in which case we have

$$\begin{aligned} P_1[g_3(x, t)]_{x=0} &= P_3[g_1(y, t)]_{y=0}, & P_1[g_4(x, t)]_{x=0} &= P_4[g_1(y, t)]_{y=l_y}, \\ P_2[g_3(x, t)]_{x=l_x} &= P_3[g_2(y, t)]_{y=0}, & P_2[g_4(x, t)]_{x=l_x} &= P_4[g_2(y, t)]_{y=l_y}. \end{aligned} \quad (6.79)$$

In this case, we seek an auxiliary function $w(x, y, t)$ in the form of Equation (6.47) with the coefficients of first- or second-order polynomials $\bar{X}(x)$, $\bar{\bar{X}}(x)$, $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$, adjusted in such a way that the function $w(x, y, t)$ satisfies the boundary conditions.

Functions $\bar{X}(x)$ and $\bar{\bar{X}}(x)$ can be chosen in such a way that

$$\begin{aligned} P_1[\bar{X}(0)] &= 1, & P_2[\bar{X}(l_x)] &= 0, \\ P_1[\bar{\bar{X}}(0)] &= 0, & P_2[\bar{\bar{X}}(l_x)] &= 1. \end{aligned} \quad (6.80)$$

If $\beta_1 \neq 0$ or $\beta_2 \neq 0$, then the functions $\bar{X}(x)$ and $\bar{\bar{X}}(x)$ are polynomials of first order

$$\bar{X}(x) = \gamma_1 + \delta_1 x, \quad \bar{\bar{X}}(x) = \gamma_2 + \delta_2 x.$$

The coefficients of these polynomials γ_1 , δ_1 , γ_2 , and δ_2 are defined uniquely and depend on the types of boundary conditions:

$$\begin{aligned} \gamma_1 &= \frac{\alpha_2 + \beta_2 l_x}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_1 &= \frac{-\beta_2}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \\ \gamma_2 &= \frac{-\alpha_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}, & \delta_2 &= \frac{\beta_1}{\beta_1 \beta_2 l_x + \beta_1 \alpha_2 - \beta_2 \alpha_1}. \end{aligned} \quad (6.81)$$

Reading Exercise. Derive the expressions shown in Equation (6.81).

If $\beta_1 = \beta_2 = 0$, then the functions $\bar{X}(x)$ and $\bar{\bar{X}}(x)$ are polynomials of second order, and we have

$$\bar{X}(x) = x - \frac{x^2}{2l_x}, \quad \bar{\bar{X}}(x) = \frac{x^2}{2l_x}. \quad (6.82)$$

The problem for $Y(y)$ (and its solution) is the same as the problem for $X(x)$. Functions $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ can be chosen in such a way that

$$\begin{aligned} P_3 [\bar{Y}(0)] &= 1, & P_3 [\bar{Y}(l_y)] &= 0, \\ P_4 [\bar{\bar{Y}}(0)] &= 0, & P_4 [\bar{\bar{Y}}(l_y)] &= 1. \end{aligned} \quad (6.83)$$

If $\beta_3 \neq 0$ or $\beta_4 \neq 0$, then the functions $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ are polynomials of first order, and we may write

$$\bar{Y}(y) = \gamma_3 + \delta_3 y, \quad \bar{\bar{Y}}(y) = \gamma_4 + \delta_4 y.$$

The coefficients of these polynomials γ_3 , δ_3 , γ_4 , and δ_4 are defined uniquely and depend on the types of boundary conditions:

$$\begin{aligned} \gamma_3 &= \frac{\alpha_4 + \beta_4 l_y}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}, & \delta_3 &= \frac{-\beta_4}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}, \\ \gamma_4 &= \frac{-\alpha_3}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}, & \delta_4 &= \frac{\beta_3}{\beta_3 \beta_4 l_y + \beta_3 \alpha_4 - \beta_4 \alpha_3}. \end{aligned} \quad (6.84)$$

If $\beta_3 = \beta_4 = 0$, then the functions $\bar{Y}(y)$ and $\bar{\bar{Y}}(y)$ are polynomials of second order, and we have

$$\bar{Y}(y) = y - \frac{y^2}{2l_y}, \quad \bar{\bar{Y}}(y) = \frac{y^2}{2l_y}. \quad (6.85)$$

Coefficients $A(t)$, $B(t)$, $C(t)$, and $D(t)$ of the auxiliary function $w(x, y, t)$ are defined from the boundary conditions:

1. At the edge at $x = 0$, we have

$$P_1[w]_{x=0} = g_1(y, t) + \{P_1[g_3(x, t)]_{x=0} + A(t)\} \bar{Y} + \{P_1[g_4(x, t)]_{x=0} + B(t)\} \bar{\bar{Y}}.$$

2. At the edge at $x = l_x$, we have

$$P_2[w]_{x=l_x} = g_2(y, t) + \{P_2[g_3(x)]_{x=l_x} + C(t)\} \bar{Y} + \{P_2[g_4(x, t)]_{x=l_x} + D(t)\} \bar{\bar{Y}}.$$

3. At the edge at $y = 0$, we have

$$P_3[w]_{y=0} = g_3(x, t) + \{P_3[g_1(y, t)]_{y=0} + A(t)\} \bar{X} + \{P_3[g_2(y, t)]_{y=0} + C(t)\} \bar{\bar{X}}.$$

4. At the edge at $y = l_y$, we have

$$P_4[w]_{y=l_y} = g_4(x, t) + \{P_4[g_1(y, t)]_{y=l_y} + B(t)\} \bar{X} + \{P_4[g_2(y, t)]_{y=l_y} + D(t)\} \bar{\bar{X}}.$$

The conformity conditions for the boundary value functions are valid, so the coefficients are

$$\begin{aligned} A(t) &= -P_1[g_3(x, t)]_{x=0} = -P_3[g_1(y, t)]_{y=0}, \\ B(t) &= -P_1[g_4(x, t)]_{x=0} = -P_4[g_1(y, t)]_{y=l_y}, \\ C(t) &= -P_2[g_3(x, t)]_{x=l_x} = -P_3[g_2(y, t)]_{y=0}, \\ D(t) &= -P_2[g_4(x, t)]_{x=l_x} = -P_4[g_2(y, t)]_{y=l_y}. \end{aligned} \quad (6.86)$$

Reading Exercise. Verify the expressions for the coefficients in Equation (6.86).

Therefore, the auxiliary function, $w(x, y, t)$, satisfies the following given boundary conditions:

$$\begin{aligned} P_1[w]_{x=0} &= g_1(y, t), & P_2[w]_{x=l_x} &= g_2(y, t), \\ P_3[w]_{y=0} &= g_3(x, t), & P_4[w]_{y=l_y} &= g_4(x, t). \end{aligned} \quad (6.87)$$

Reading Exercise. Verify that $w(x, y, t)$ satisfies the boundary conditions in Equation (6.87).

- **Case 2.** Suppose the boundary value functions do not satisfy consistency conditions. Then search for an auxiliary function as the sum of two functions:

$$w(x, y, t) = w_1(x, y, t) + w_2(x, y, t). \quad (6.88)$$

Function $w_1(x, y, t)$ is an auxiliary function satisfying the consistent boundary conditions

$$\begin{aligned} P_1[w_1]_{x=0} &= g_1(y, t), \\ P_2[w_1]_{x=l_x} &= g_2(y, t), \\ P_3[w_1]_{y=0} &= -A(t)\bar{X}(x) - C(t)\bar{\bar{X}}(x), \\ P_4[w_1]_{y=l_y} &= -B(t)\bar{X}(x) - D(t)\bar{\bar{X}}(x), \end{aligned} \quad (6.89)$$

where

$$A(t) = -P_3[g_1(y, t)]_{y=0}, \quad B(t) = -P_4[g_1(y, t)]_{y=l_y},$$

$$C(t) = -P_3[g_2(y, t)]_{y=0}, \quad D(t) = -P_4[g_2(y, t)]_{y=l_y}.$$

Such a function was constructed for Case 1. In Case 2, it has the form

$$w_1(x, y, t) = g_1(y, t)\bar{X} + g_2(y, t)\bar{\bar{X}}. \quad (6.90)$$

Reading Exercise. Verify Equation (6.90).

The function $w_2(x, y, t)$ is a particular solution of the equation

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} = 0 \quad (6.91)$$

(this is the Laplace equation) with the following boundary conditions:

$$P_1[w_2]_{x=0} = 0, \quad P_2[w_2]_{x=l_x} = 0,$$

$$P_3[w_2]_{y=0} = g_3(x, t) + A(t)\bar{X}(x) + C(t)\bar{\bar{X}}(x), \quad (6.92)$$

$$P_4[w_2]_{y=l_y} = g_4(x, t) + B(t)\bar{X}(x) + D(t)\bar{\bar{X}}(x).$$

The solution to this problem has the form

$$w_2(x, y, t) = \sum_{n=1}^{\infty} \{A_n(t)Y_{1n}(y) + B_n(t)Y_{2n}(y)\}X_n(x) \quad (6.93)$$

where λ_{xn} and $X_n(x)$ are eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad 0 < x < l_x$$

$$P_1[X]_{x=0} = P_2[X]_{x=l_x} = 0. \quad (6.94)$$

The coefficients $A_n(t)$ and $B_n(t)$ are defined by the formulas

$$\begin{aligned} A_n(t) &= \frac{1}{\|X_n\|^2} \int_0^{l_x} \left[g_4(x, t) + B(t)\overline{X}(x) + D(t)\overline{\overline{X}}(x) \right] X_n(x) dx, \\ B_n(t) &= \frac{1}{\|X_n\|^2} \int_0^{l_x} \left[g_3(x, t) + A(t)\overline{X}(x) + C(t)\overline{\overline{X}}(x) \right] X_n(x) dx. \end{aligned} \quad (6.95)$$

Thus, we see that eigenvalues and eigenfunctions of this boundary value problem depend on the types of boundary conditions (see Appendix C for a detailed account).

Having the eigenvalues, λ_{xn} , we obtain a similar equation for $Y(y)$, given by

$$Y'' - \lambda_{xn}Y = 0, \quad 0 < y < l_y. \quad (6.96)$$

We choose a fundamental system $\{Y_1, Y_2\}$ of solutions in such a way that

$$\begin{aligned} P_3[Y_1(0)] &= 0, & P_3[Y_1(l_y)] &= 1, \\ P_4[Y_2(0)] &= 1, & P_4[Y_2(l_y)] &= 0. \end{aligned} \quad (6.97)$$

Two particular solutions of Equation (6.96) are $\exp(\pm\sqrt{\lambda_n}y)$ but for the future analysis it is more convenient to choose two linearly independent functions, $Y_1(y)$ and $Y_2(y)$, in the form

$$\begin{aligned} Y_1(y) &= a \sinh \sqrt{\lambda_n}y + b \cosh \sqrt{\lambda_n}y, \\ Y_2(y) &= c \sinh \sqrt{\lambda_n}(l_y - y) + d \cosh \sqrt{\lambda_n}(l_y - y). \end{aligned} \quad (6.98)$$

Reading Exercise. Prove that the two functions in Equation (6.98) are solutions to Equation (6.96).

The values of coefficients a , b , c , and d depend on the types of boundary conditions, $P_3[u]_{y=0}$ and $P_4[u]_{y=l_y}$ (see Appendix C for details). It can be verified that the auxiliary function

$$\begin{aligned}
w(x, y, t) &= w_1(x, y, t) + w_2(x, y, t) \\
&= w_1(x, y, t) + \sum_{n=1}^{\infty} \{A_n(t)Y_{1n}(y) + B_n(t)Y_{2n}(y)\}X_n(x)
\end{aligned}
\tag{6.99}$$

satisfies the given boundary conditions.

Reading Exercise. Prove that the function shown in Equation (6.99) satisfies the appropriate boundary conditions.

It is important to notice that the separation of an unknown function, $u(x, y, t)$, into two functions, $u(x, y, t) = v(x, y, t) + w(x, y, t)$, has a transparent physical meaning. The function $w(x, y, t)$ is a solution to the Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{6.100}$$

with nonhomogeneous boundary conditions, whereas the function $v(x, y, t)$ describes the relaxation of the temperature in the plate to the equilibrium. In particular, for the simple case where the boundary conditions are stationary (i.e., the edges experience no change in temperature), the function w is the final, equilibrium temperature of the body as $t \rightarrow \infty$.

Example 6.5. A heat-conducting, thin, uniform rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. The edge at $y = 0$ of the plate is kept at a constant temperature $u = u_1$, and the edge at $y = l_y$ is at a constant temperature $u = u_2$. The remaining boundary is thermally insulated. The initial temperature distribution within the plate is

$$u(x, y, 0) = u_0 = \text{const.}$$

Find the temperature $u(x, y, t)$ of the plate at any later time, if generation (or absorption) of heat by internal sources is absent.

Solution. The problem may be resolved by solving the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right],$$

under the conditions

$$u(x, y, 0) = \varphi(x, y) = u_0,$$

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(l_x, y, t) = 0, \quad u(x, 0, t) = u_1, \quad u(x, l_y, t) = u_2.$$

The solution to this problem can be expressed as the sum of two functions

$$u(x, y, t) = w(x, y, t) + v(x, y, t),$$

where $v(x, y, t)$ is a new, unknown function and $w(x, y, t)$ is an auxiliary function satisfying the boundary conditions.

The boundary value functions satisfy the conforming conditions; that is,

$$\begin{aligned} g_1|_{y=0} = \frac{\partial g_3}{\partial x} \Big|_{x=0} &= 0, & g_1|_{y=l_y} = \frac{\partial g_4}{\partial x} \Big|_{x=0} &= 0, \\ g_2|_{y=0} = \frac{\partial g_3}{\partial x} \Big|_{x=l_x} &= 0, & g_2|_{y=l_y} = \frac{\partial g_4}{\partial x} \Big|_{x=l_x} &= 0. \end{aligned}$$

An auxiliary function satisfying the given boundary condition has the form

$$\begin{aligned} w(x, y, t) = & g_1(y, t)\overline{X} + g_2(y, t)\overline{\overline{X}} + g_3(x, t)\overline{Y} + g_4(x, t)\overline{\overline{Y}} \\ & + A(t)\overline{XY} + B(t)\overline{\overline{XY}} + C(t)\overline{\overline{XY}} + D(t)\overline{\overline{\overline{XY}}}, \end{aligned}$$

where

$$\begin{aligned} \overline{X}(x) &= x - \frac{x^2}{2l_x}, & \overline{\overline{X}}(x) &= \frac{x^2}{2l_x}, \\ \overline{Y}(y) &= 1 - \frac{y}{l_y}, & \overline{\overline{Y}}(y) &= \frac{y}{l_y}, \\ A(t) &= B(t) = C(t) = D(t) = 0, \end{aligned}$$

in which case we have

$$w(x, y, t) = u_1 + (u_2 - u_1)\frac{y}{l_y}. \quad (6.101)$$

Reading Exercise. Derive the above result, Equation (6.101).

Given Equation (6.101) for $w(x, y, t)$, we see that the separation of the function $u(x, y, t)$ into functions $w(x, y, t)$ and $v(x, y, t)$ is a separation into a stationary solution corresponding to the boundary conditions and the solution describing the relaxation of the temperature to the stationary state.

The relaxation process to a steady state described by the function $v(x, y, t)$ is the solution to the boundary value problem with zero boundary conditions, where the stationary solution is described by

$$f^*(x, y, t) = -\frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0,$$

$$\varphi^*(x, y) = u_0 - u_1 - (u_2 - u_1) \frac{y}{l_y}.$$

Eigenvalues, eigenfunctions, and the norm of the problem can be easily obtained:

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right], \quad n = 0, 1, 2, \dots \quad m = 1, 2, 3, \dots$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \cos \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y},$$

$$\|V_{nm}\|^2 = \|X_n\|^2 \|Y_m\|^2 = \begin{cases} l_x l_y / 2, & n = 0 \\ l_x l_y / 4, & n > 0. \end{cases}$$

The three-dimensional picture shown in Figure 6.5 depicts two eigenfunctions, $V_{01}(x, y)$ and $V_{05}(x, y)$, chosen as examples for the given problem. The values of λ are $\lambda_{01} = 0.274$ and $\lambda_{05} = 6.854$.

Applying Equation (6.68), we obtain

$$C_{nm} = \frac{1}{\|v_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} \left[u_0 - u_1 - (u_2 - u_1) \frac{y}{l_y} \right] \cos \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} dx dy. \quad (6.102)$$

From Equation (6.102) we have

$$\begin{aligned} C_{0m} &= \frac{2}{l_y} \int_0^{l_y} \left[u_0 - u_1 - (u_2 - u_1) \cdot \frac{y}{l_y} \right] \sin \frac{m\pi y}{l_y} dx dy \\ &= \frac{2}{m\pi} \{ (u_0 - u_1) [1 - (-1)^m] + (u_2 - u_1) (-1)^m \}, \end{aligned}$$

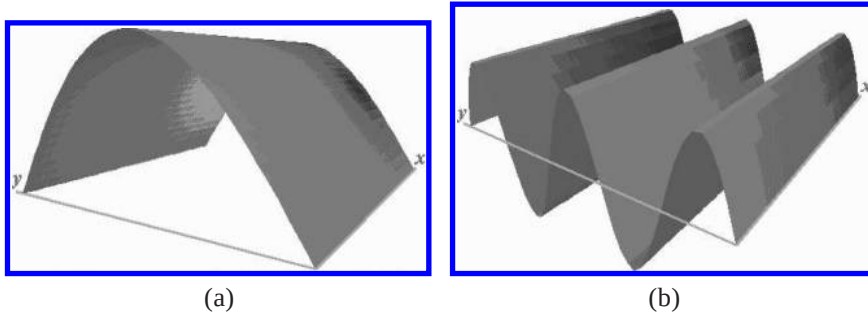


Figure 6.5. Eigenfunctions (a) $V_{01}(x, y)$ and (b) $V_{05}(x, y)$ for the free surface in Example 6.5.

for $n > 0$, $C_{nm} = 0$. In Figure 6.5, we see that the eigenfunctions with index $n = 0$ do not depend on x , and, as we just obtained, the temperature distribution does not depend on x at all. This result could be anticipated from the very beginning, since the initial and boundary conditions do not depend on x . In other words, the solution is actually one-dimensional for this problem. Hence, the distribution of temperature inside the rectangular plate for some instant of time is described by the series

$$u(x, y, t) = u_1 + (u_2 - u_1) \frac{y}{l_y} + \sum_{m=1}^{\infty} C_{0m} e^{-\lambda_{0m} a^2 t} \sin \frac{m \pi y}{l_y}.$$

Figure 6.6 shows two snapshots of the solution at the times $t = 0$ and $t = 10$. This solution was obtained with the program Heat for the case when $a^2 = 0.25$, $l_x = 4$, $l_y = 6$, $u_0 = 10$, $u_1 = 20$, and $u_2 = 50$.

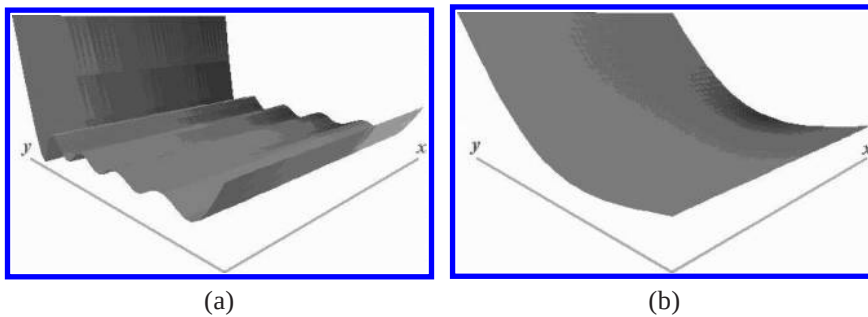


Figure 6.6. Surface graph of temperature at (a) $t = 0$ and (b) $t = 10$ for Example 6.5.

It is interesting to compare the first and second pictures in Figure 6.6. The first one is very rough, the second is smooth. The explanation of the roughness is that at $t = 0$ the initial and boundary conditions do not match (the solution is non-physical). But at any nonzero time the temperature distribution very quickly becomes smooth, as it should be for real, physical situations. Here we see that this method of solution works quite well in approximating a real, physical situation.

Example 6.6. A heat-conducting, thin, uniform rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. The edge at $y = 0$ is thermally insulated; the edge at $y = l_y$ is kept at constant zero temperature; the edge at $x = 0$ is kept at constant temperature $u = u_1$ and the edge at $x = l_x$ is kept at the temperature

$$u(l_x, y, t) = \cos \frac{3\pi y}{2l_y} e^{-t}.$$

The initial temperature distribution within the plate is $u(x, y, 0) = u_0 = \text{const}$. Find the temperature $u(x, y, t)$ of the plate at any later time, if generation (or absorption) of heat by internal sources is absent.

Solution. The problem depends on the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

under the conditions

$$\begin{aligned} u(x, y, 0) &= \varphi(x, y) = u_0, & u(0, y, t) &= u_1, \\ u(l_x, y, t) &= \cos \frac{3\pi y}{2l_y} \cdot e^{-t}, & \frac{\partial u}{\partial y}(x, 0, t) &= 0, & u(x, l_y, t) &= 0. \end{aligned}$$

The solution to this problem can be expressed as the sum of two functions

$$u(x, y, t) = w(x, y, t) + v(x, y, t),$$

where $v(x, y, t)$ is a new, unknown function and $w(x, y, t)$ is an auxiliary function satisfying the boundary conditions.

The boundary value functions do not satisfy the conforming conditions at the point $(0, l_y)$; that is,

$$g_1|_{y=l_y} = u_1 \neq g_4|_{x=0} = 0.$$

So, in this case we shall search for an auxiliary function as the sum of two functions

$$w(x, y, t) = w_1(x, y, t) + w_2(x, y, t),$$

where $w_1(x, y, t)$ is an auxiliary function satisfying the conforming boundary value functions

$$P_1[w_1]_{x=0} = g_1(y, t) = u_1,$$

$$P_2[w_1]_{x=l_x} = g_2(y, t) = \cos \frac{3\pi y}{2l_y} e^{-t},$$

$$P_3[w_1]_{y=0} = \frac{\partial w_1}{\partial y}(x, 0) = g_3(x, t) = 0,$$

$$P_4[w_1]_{y=l_y} = w_1(x, l_y) = u_1 \left(1 - \frac{x}{l_x} \right),$$

and $w_2(x, y, t)$ is a particular solution of the Laplace equation

$$\nabla^2 w_2(x, y, t) = \left[\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} \right] = 0$$

with the following boundary conditions

$$P_1[w_2]_{x=0} = w_2(0, y, t) = 0, \quad P_2[w_2]_{x=l_x} = w_2(l_x, y, t) = 0,$$

$$P_3[w_2]_{y=0} = \frac{\partial w_2}{\partial y}(x, 0, t) = 0,$$

$$P_4[w_2]_{y=l_y} = w_2(x, l_y, t) = u_1 \left(\frac{x}{l_x} - 1 \right).$$

The auxiliary function $w_1(x, y, t)$ is

$$w_1(x, y, t) = u_1 \left(1 - \frac{x}{l_x} \right) + \cos \frac{3\pi y}{2l_y} e^{-t} \frac{x}{l_x} = u_1 + \frac{x}{l_x} \left(\cos \frac{3\pi y}{2l_y} e^{-t} - u_1 \right),$$

as graphed in Figure 6.7.

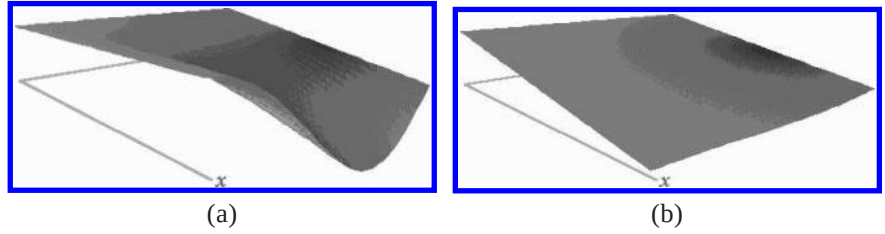


Figure 6.7. Surface graph of the auxiliary function $w_1(x, y, t)$ at (a) $t = 0$ and (b) $t = 3$ for Example 6.6.

The particular solution, $w_2(x, y, t)$, of the problem has the form

$$w_2(x, y, t) = \sum_{n=1}^{\infty} \{A_n(t)Y_{1n}(y) + B_n(t)Y_{2n}(y)\}X_n(x),$$

where λ_{xn} and $X_n(x)$ are eigenvalues and eigenfunctions of the respective Sturm-Liouville problem:

$$\lambda_{xn} = \left[\frac{n\pi}{l_x}\right]^2, \quad X_n(x) = \sin \frac{n\pi x}{l_x}, \quad \|X_n\|^2 = \frac{l_x}{2}, \quad n = 1, 2, 3, \dots$$

The functions $Y_{1n}(y)$ and $Y_{2n}(y)$ for the given boundary conditions are (see Appendix C)

$$Y_{1n}(y) = \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y}, \quad Y_{2n}(y) = -\frac{\sinh \sqrt{\lambda_n} (l_y - y)}{\sqrt{\lambda_n} \cosh \sqrt{\lambda_n} l_y}.$$

Coefficients $A_n(t)$ and $B_n(t)$ are given by

$$A_n = \frac{2}{l_x} \int_0^{l_x} u_1 \left(\frac{x}{l_x} - 1 \right) \sin \frac{n\pi x}{l_x} dx = -\frac{2u_1}{n\pi},$$

$$B_n = 0.$$

Thus,

$$w_2(x, y, t) = -\frac{2u_1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y} \sin \frac{n\pi x}{l_x},$$

as graphed in Figure 6.8.

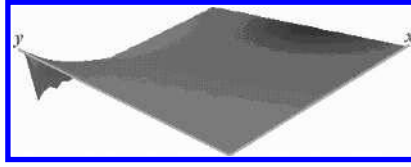


Figure 6.8. Surface graph of the particular solution $w_2(x, y, t)$ to Example 6.6.

The function $v(x, y, t)$ is the solution to the boundary value problem with zero boundary conditions, where

$$f^*(x, y, t) = \frac{x}{l_x} \cos \frac{3\pi y}{2l_y} e^{-t} \left[1 - a^2 \left(\frac{3\pi}{2l_y} \right)^2 \right],$$

$$\varphi^*(x, y) = u_0 - u_1 - \frac{x}{l_x} \left(\cos \frac{3\pi y}{2l_y} e^{-t} - u_1 \right) + \frac{2u_1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y} \sin \frac{n\pi x}{l_x}.$$

Therefore, the eigenvalues, eigenfunctions, and norm of Example 6.6 are

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{n^2}{l_x^2} + \frac{(2m-1)^2}{4l_y^2} \right], \quad n, m = 1, 2, 3, \dots$$

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{n\pi x}{l_x} \cos \frac{(2m-1)\pi y}{2l_y},$$

$$\|V_{nm}\|^2 = \|X_n\|^2 \cdot \|Y_m\|^2 = \frac{l_x l_y}{4}.$$

(As before, these eigenvalues and eigenfunctions can be found in Appendix A or can be easily derived by the reader.)

Applying Equations (6.68) and (6.73), we obtain

$$C_{nm} = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} \varphi^*(x, y) \sin \frac{n\pi x}{l_x} \cos \frac{(2m-1)\pi y}{2l_y} dx dy,$$

$$f_{nm}(t) = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} f^*(x, y, t) \sin \frac{n\pi x}{l_x} \cos \frac{(2m-1)\pi y}{2l_y} dx dy.$$

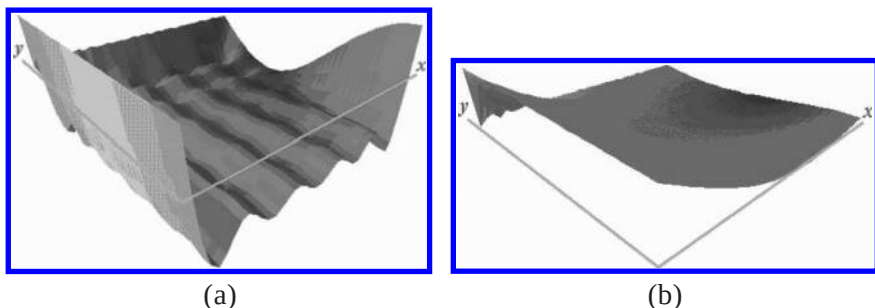


Figure 6.9. Surface graph of the plate temperature at (a) $t = 0$ and (b) $t = 2$ for Example 6.6.

Thus, we have

$$T_{nm}(t) = \int_0^t f_{nm}(\tau) e^{-a^2 \lambda_{nm}(t-\tau)} d\tau$$

and

$$\begin{aligned} u(x, y, t) &= [w_1(x, y, t) + w_2(x, y, t)] + v(x, y, t) \\ &= u_1 + \frac{x}{l_x} \left(\cos \frac{3\pi y}{2l_y} e^{-t} - u_1 \right) - \frac{2u_1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y} \sin \frac{n\pi x}{l_x} \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[T_{nm}(t) + C_{nm} e^{-\lambda_{nm} a^2 t} \right] \sin \frac{n\pi x}{l_x} \cos \frac{(2m-1)\pi y}{2l_y}. \end{aligned}$$

Figure 6.9 shows two snapshots of the solution at the times $t = 0$ and $t = 2$. This solution was obtained with the program Heat for the case when $a^2 = 1$, $l_x = 4$, $l_y = 6$, $u_0 = -1$, and $u_1 = 1$.

6.3 Heat Conduction within a Circular Domain

In this section, we extend the previous discussion of the application of parabolic equations to cases in which there is circular symmetry. As always, note that the mathematics presented is not limited to discussions of heat conduction, but has broader applications such as diffusion.

Let a uniform circular plate be placed in the horizontal x - y plane and bounded at the circular periphery by a radius of length l . The plate is

assumed to be thin enough so that the temperature is the same at all points with the same x - y coordinates. For a circular domain, it is convenient to write the heat conduction equation in polar coordinates. In this case the heat conduction within a thin uniform circular domain is described by the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) - \gamma u + f(r, \varphi, t) \quad (6.103)$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad t > 0.$$

Here, $u(r, \varphi, t)$ is the temperature of the membrane at point (r, φ) at time t .

The *initial condition* defines the temperature distribution within the membrane at time zero and may be expressed in generic form as

$$u(r, \varphi, 0) = \phi(r, \varphi). \quad (6.104)$$

The *boundary condition* describes the thermal condition around the boundary at any time t . In general, the boundary condition can be written as

$$P[u]_{r=l} \equiv \alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = g(\varphi, t). \quad (6.105)$$

It is obvious that boundary value function must be a singled-valued periodic function in φ of period 2π ; that is,

$$g(\varphi + 2\pi, t) = g(\varphi, t).$$

Again, we consider three main types of boundary conditions:

1. Boundary condition of the first type (*Dirichlet condition*),

$$u(l, \varphi, t) = g(\varphi, t), \quad (6.106)$$

where the temperature at the boundary is given or is zero, in which case $g(\varphi, t) \equiv 0$.

2. Boundary condition of the second type (*Neumann condition*),

$$u_r(l, \varphi, t) = g(\varphi, t), \quad (6.107)$$

in which case the heat flow at the boundary is given or the boundary is thermally insulated and $g(\varphi, t) \equiv 0$.

3. Boundary condition of the third type (*mixed condition*),

$$u_r(l, \varphi, t) + hu(l, \varphi, t) = g(\varphi, t), \quad (6.108)$$

in which case the conditions of heat exchange with a medium are specified (here $h = \text{const}$).

If the function $g(\varphi, t)$ on the right-hand side of Equations (6.106), (6.107), and (6.108) is identically zero, then the boundary condition is said to be *homogeneous*.

In the case of a nonhomogeneous boundary condition, we introduce an auxiliary function $w(r, \varphi, t)$ satisfying the given boundary condition and express the solution to the problem as the sum of two functions:

$$u(r, \varphi, t) = v(r, \varphi, t) + w(r, \varphi, t),$$

where $v(r, \varphi, t)$ is a new, unknown function with a zero boundary condition. The construction of function $w(r, \varphi, t)$ is discussed in Section 6.3.3. We seek an auxiliary function $w(r, \varphi, t)$ in the form

$$w(r, \varphi, t) = (c_0 + c_2 r^2)g(\varphi, t),$$

where c_0 and c_2 are real constants. These constants can be adjusted to satisfy the boundary conditions.

The function $v(r, \varphi, t)$ represents heat conduction when heat sources are present within a membrane and boundary conditions are zero and so satisfies the following equations:

$$\frac{\partial v}{\partial t} = a^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} \right) - \gamma v + f^*(r, \varphi, t) \quad (6.109)$$

$$v(r, \varphi, 0) = \phi^*(r, \varphi),$$

$$P[v]_{r=l} \equiv \alpha \frac{\partial v}{\partial r} + \beta v \Big|_{r=l} = 0,$$

where

$$f^*(r, \varphi, t) = f(r, \varphi, t) - \frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} \right) - \gamma w,$$

$$\phi^*(r, \varphi) = \phi(r, \varphi) - w(r, \varphi, 0).$$

We may express the solution $v(r, \varphi, t)$ as the sum of two functions

$$v(r, \varphi, t) = u_1(r, \varphi, t) + u_2(r, \varphi, t), \quad (6.110)$$

where $u_1(r, \varphi, t)$, as in the case of heat flow in the rectangular plate, is the solution of the homogeneous equation (free heat exchange) with the given initial conditions and zero boundary conditions, and $u_2(r, \varphi, t)$ is the solution of the nonhomogeneous equation (heat exchange involving internal sources) with zero initial and boundary conditions.

6.3.1 The Fourier Method for the Homogeneous Heat Conduction Equation (Free Heat Exchange within a Circular Plate)

Let us find the solution of the homogeneous equation

$$\frac{\partial u_1}{\partial t} = a^2 \left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r} \frac{\partial u_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_1}{\partial \varphi^2} \right) - \gamma u_1, \quad (6.111)$$

with the initial condition

$$u_1(r, \varphi, 0) = \phi(r, \varphi) \quad (6.112)$$

and zero boundary conditions

$$P[u_1]_{r=l} \equiv \alpha \frac{\partial u_1}{\partial r} + \beta u_1 \Big|_{r=l} = 0. \quad (6.113)$$

We will see that, as was done previously, we can separate the variables so that

$$u_1(r, \varphi, t) = T(t)V(r, \varphi),$$

where $T(t)$ is the solution of the equation

$$T'(t) + (a^2 \lambda + \gamma)T(t) = 0, \quad (6.114)$$

and $V(r, \varphi)$ is the solution to the following boundary value problem:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \lambda V = 0, \quad (6.115)$$

$$\alpha \frac{\partial V}{\partial r}(l, \varphi) + \beta V(l, \varphi) = 0. \quad (6.116)$$

Two restrictions on $V(r, \varphi)$ are that it be bounded:

$$|V(0, \varphi)| < \infty \quad (6.117)$$

and that it be periodic in φ :

$$V(r, \varphi) = V(r, \varphi + 2\pi).$$

We thus see that the boundary value problem leads to solving a homogeneous equation with homogeneous boundary conditions.

Again, we can separate the variables by writing

$$V(r, \varphi) = R(r)\Phi(\varphi), \quad (6.118)$$

which yields the following two boundary value problems, each of which is one-dimensional:

$$\begin{aligned} \Phi''(\varphi) + \nu\Phi(\varphi) &= 0, \\ \Phi(\varphi) &= \Phi(\varphi + 2\pi), \\ \Phi'(\varphi) &= \Phi'(\varphi + 2\pi) \end{aligned} \quad (6.119)$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda - \frac{\nu}{r^2} \right) R = 0, \quad (6.120)$$

$$\alpha \frac{\partial R}{\partial r} + \beta R \Big|_{r=l} = 0,$$

$$|R(0)| < \infty.$$

Nontrivial periodic solutions for $\Phi(\varphi)$ exist only when $\nu = n^2$, where n is an integer number, and can be presented in the form

$$\Phi_n(\varphi) = D_{1n} \cos n\varphi + D_{2n} \sin n\varphi. \quad (6.121)$$

Each eigenvalue, n , has two linearly independent eigenfunctions, $\cos n\varphi$ and $\sin n\varphi$. This degeneracy exists in this case because the Sturm-Liouville problem for the function $\Phi(\varphi)$ is periodic.

As we already know from Chapter 4, eigenvalues and eigenfunctions of the boundary value problem for $R(r)$ (see Equations (6.120)) have the form

$$\lambda_{nm} = \left(\frac{\mu_m^{(n)}}{l} \right)^2, \quad R_{nm}(r) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right), \quad n, m = 0, 1, 2, \dots, \quad (6.122)$$

where $\mu_m^{(n)}$ is the m th positive root of the equation

$$\alpha \mu J_n'(\mu) + \beta l J_n(\mu) = 0 \quad (6.123)$$

and $J_n(\mu)$ is the Bessel function of the first kind.

By collecting the above results, we may write the eigenfunctions of the given boundary value problem:

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cdot \cos n\varphi \text{ and } V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cdot \sin n\varphi. \quad (6.124)$$

The functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are orthogonal, and their norms are

$$\|V_{nm}\|^2 = \int_0^{2\pi} \int_0^l V_{nm}^2(r, \varphi) r dr d\varphi = \begin{cases} 2\pi \|R_n(r)\|^2 & \text{if } n = 0, \\ \pi \|R_n(r)\|^2 & \text{if } n \neq 0. \end{cases} \quad (6.125)$$

Expressions for the squared norms of the eigenfunctions depend on the coefficients of the boundary condition.

1. If $\alpha = 0$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[J_n' \left(\mu_m^{(n)} \right) \right]^2. \quad (6.126)$$

2. If $\beta = 0$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2 \left(\mu_m^{(n)} \right)^2} \left[\left(\mu_m^{(n)} \right)^2 - n^2 \right] J_n^2 \left(\mu_m^{(n)} \right). \quad (6.127)$$

3. If $\alpha < \beta$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[1 + \frac{\left(\mu_m^{(n)}\right)^2 - n^2}{l^2} \left(\frac{\alpha}{\beta}\right)^2 \right] \left[J_n' \left(\mu_m^{(n)} \right) \right]^2. \quad (6.128)$$

4. If $\alpha > \beta$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2 \left(\mu_m^{(n)}\right)^2} \left[l^2 \left(\frac{\beta}{\alpha}\right)^2 + \left(\mu_m^{(n)}\right)^2 - n^2 \right] J_n^2 \left(\mu_m^{(n)} \right). \quad (6.129)$$

By using the above eigenvalues, λ_{nm} , we can write the solution of the differential equation

$$T_{nm}'(t) + (a^2 \lambda_{nm} + \gamma) T_{nm}(t) = 0 \quad (6.130)$$

as

$$T_{nm}(t) = C_{nm} e^{-(a^2 \lambda_{nm} + \gamma)t}. \quad (6.131)$$

We see that the general solution to the original problem can be expressed as

$$u_1(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[a_{nm} V_{nm}^{(1)}(r, \varphi) + b_{nm} V_{nm}^{(2)}(r, \varphi) \right] e^{-(a^2 \lambda_{nm} + \gamma)t}. \quad (6.132)$$

The coefficients a_{nm} and b_{nm} are defined by using the function that expresses the initial condition and the orthogonality property of functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$:

$$a_{nm} = \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \quad (6.133)$$

$$b_{nm} = \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^l \int_0^{2\pi} \phi(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi. \quad (6.134)$$

Example 6.7. The initial temperature distribution within a very long (infinite) cylinder of radius l is

$$\phi(r, \varphi) = u_0 \left(1 - \frac{r^2}{l^2} \right), \quad u_0 = \text{const.}$$

Find the distribution of temperature within the cylinder if its surface is kept at constant zero temperature. Generation (or absorption) of heat by internal sources is absent.

Solution. The boundary value problem modeling the process of the cooling of an infinite cylinder is

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right],$$

$$u(r, \varphi, 0) = u_0 \left(1 - \frac{r^2}{l^2} \right), \quad u(l, \varphi, t) = 0.$$

The boundary condition of the problem is of the Dirichlet type, so eigenvalues are given by the equation

$$J_n(\mu) = 0,$$

Therefore, if $\mu_m^{(n)}$ are the roots of this equation, the eigenfunctions are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi \quad \text{and} \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi.$$

The initial temperature does not depend on the polar angle φ ; thus, only terms with $n = 0$ are not zero. The solution $u(r, \varphi, t)$ is therefore given by the series

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} a_{0m} e^{-a^2 \lambda_{0m} t} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right).$$

The coefficients a_{0m} are given by Equation (6.133):

$$a_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l u_0 \left(1 - \frac{r^2}{l^2} \right) J_0 \left(\mu_m^{(0)} r / l \right) r dr.$$

We may calculate the norm of $V_{0m}^{(1)}$ by using Equation (6.126) and taking into account that the derivative of $J_0(x)$ gives $-J_1(x)$ (see Chapter 8 for a discussion of the properties of the Bessel function). We thus have

$$\|V_{0m}^{(1)}\|^2 = \pi l^2 \left[J_0'(\mu_m^{(0)}) \right]^2 = \pi l^2 \left[J_1(\mu_m^{(0)}) \right]^2.$$

Calculating the integrals (taking into account that $J_0(\mu_m^{(0)}) = 0$) for the coefficients in a_{0m} , we have

$$\int_0^l J_0(\mu_m^{(0)} r/l) r dr = \frac{l^2}{(\mu_m^{(0)})^2} x J_1(x) \Big|_0^{\mu_m^{(0)}} = \frac{l^2}{\mu_m^{(0)}} J_1(\mu_m^{(0)}),$$

$$\frac{1}{l^2} \int_0^l r^2 J_0(\mu_m^{(0)} r/l) r dr = \left[\frac{l^2}{\mu_m^{(0)}} - \frac{4l^2}{(\mu_m^{(0)})^3} \right] J_1(\mu_m^{(0)}).$$

$$a_{0m} = \frac{2\pi}{\|V_{0m}^{(1)}\|^2} \int_0^l u_0 \left(1 - \frac{r^2}{l^2} \right) J_0(\mu_m^{(0)} r/l) r dr = \frac{8u_0}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})}.$$

By collecting the above results, we may describe the distribution of temperature within the cylinder by the series in Bessel functions of zeroth order given by

$$u(r, \varphi, t) = 8u_0 \sum_{m=0}^{\infty} \frac{e^{-a^2 \lambda_{0m} t}}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} J_0\left(\frac{\mu_m^{(0)}}{l} r\right).$$

On examination of the solution, we see that, due to the exponential nature of the coefficients, the final temperature of the cylinder after a long time will be zero. This is due to dissipation of energy to the surrounding space and could have been anticipated from the physical configuration of the problem.

Figure 6.10 shows two snapshots of the solution at the times $t = 0$ and $t = 3$. This solution was obtained with the program Heat for the case when $a^2 = 0.25$, $l = 2$, and $u_0 = 100$.

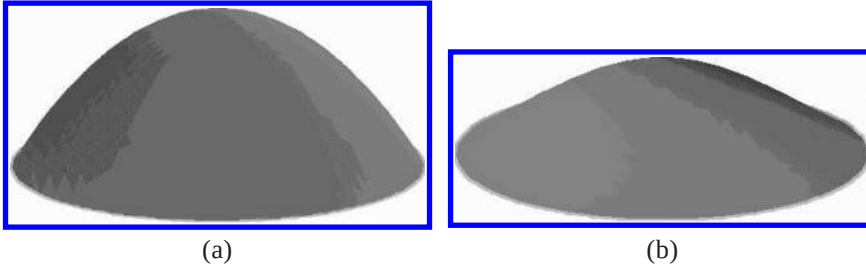


Figure 6.10. Plate temperature at (a) $t = 0$ and (b) $t = 3$ for Example 6.7.

Example 6.8. The initial temperature distribution within a very long (infinite) cylinder of radius l is

$$u(r, \varphi, 0) = u_0 = \text{const.}$$

Find the distribution of temperature within the cylinder if it is subjected to convective heat transfer at its surface according to Newton's law and the temperature of the medium is zero.

Solution. The boundary value problem modeling the process of the cooling of an infinite cylinder is

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right],$$

$$u(r, \varphi, 0) = u_0,$$

$$\frac{\partial u}{\partial r}(l, \varphi, t) + hu(l, \varphi, t) = 0.$$

The boundary condition of the problem is of the mixed type, so eigenvalues of the problem $\lambda_{nm} = (\mu_m^{(n)}/l)^2$ are given by the roots of the equation

$$\mu J_n'(\mu) + hl J_n(\mu) = 0.$$

The eigenfunctions are

$$V_{nm}^{(1)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos(n\varphi), \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin(n\varphi).$$

The coefficients a_{nm} and b_{nm} are given by Equations (6.133) and (6.134):

$$a_{nm} = \frac{u_0}{\|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \cos(n\varphi) d\varphi \int_0^l r J_n \left(\frac{\mu_m^{(n)}}{l} r \right) dr,$$

$$b_{nm} = \frac{u_0}{\|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \sin(n\varphi) d\varphi \int_0^l r J_n \left(\frac{\mu_m^{(n)}}{l} r \right) dr.$$

Obviously, $a_{nm} = 0$ if $n > 0$ and $b_{nm} = 0$ for all n . Let us calculate a_{0m} . First, find

$$\|V_{0m}^{(1)}\|^2 = \pi l^2 \frac{l^2 h^2 + \left(\mu_m^{(0)}\right)^2}{l^2 h^2} \left[J_0' \left(\mu_m^{(0)} \right) \right]^2,$$

or, by taking into account the relation $\mu J_0'(\mu) + h l J_0(\mu) = 0$, we may write

$$\|V_{0m}^{(1)}\|^2 = \pi \left[l^2 h^2 + \left(\mu_m^{(0)}\right)^2 \right] \frac{l^2}{\left(\mu_m^{(0)}\right)^2} J_0^2 \left(\mu_m^{(0)} \right).$$

We thus have

$$a_{0m} = \frac{2u_0 \mu_m^{(0)}}{\left[\left(\mu_m^{(0)}\right)^2 + h^2 l^2 \right] J_0^2 \left(\mu_m^{(0)} \right)} J_1 \left(\mu_m^{(0)} \right).$$

Using the above relations, we may write the distribution of temperature within the cylinder as the series in Bessel functions of zeroth order:

$$u(r, \varphi, t) = 2u_0 \sum_{m=0}^{\infty} \frac{\mu_m^{(0)} J_1 \left(\mu_m^{(0)} \right) e^{-a^2 \lambda_{0m} t}}{\left[\left(\mu_m^{(0)}\right)^2 + h^2 l^2 \right] J_0^2 \left(\mu_m^{(0)} \right)} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right).$$

Figure 6.11 shows snapshots of the solution at the times $t = 0$ and $t = 5$. This solution was obtained with program Heat for the case when $a^2 = 0.25$, $l = 2$, $u_0 = 100$, and $h = 2$. As in Example 6.7, dissipation of energy to the environment brings the final temperature of the plate to zero after very long time periods.

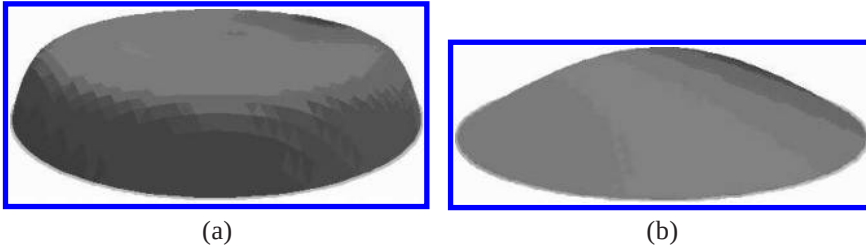


Figure 6.11. Plate temperature at (a) $t = 0$ and (b) $t = 5$ for Example 6.8.

6.3.2 The Fourier Method for the Nonhomogeneous Heat Conduction Equation (Circular Plates with Internal Sources)

The solution $u_2(r, \varphi, t)$ represents the *non-free heat exchange* within the plate; that is, the diffusion of heat due to generation (or absorption) of heat by internal sources when the initial distribution of temperature is zero. The function $u_2(r, \varphi, t)$ is thus the solution of the *nonhomogeneous* equation

$$\frac{\partial u_2}{\partial t} = a^2 \left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_2}{\partial \varphi^2} \right) - \gamma u_2 + f(r, \varphi, t) \quad (6.135)$$

with *initial and boundary conditions equal to zero*.

After the separation of variables, the general solution to this equation clearly is

$$u_2(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_{nm}^{(1)}(t) V_{nm}^{(1)}(r, \varphi) + T_{nm}^{(2)}(t) V_{nm}^{(2)}(r, \varphi) \right], \quad (6.136)$$

where $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are eigenfunctions of the corresponding homogeneous boundary value problem, and $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ are unknown functions of t .

Reading Exercise. Provide the steps leading to Equation (6.136).

Zero boundary conditions for $u_2(r, \varphi, t)$ may be expressed as

$$P[u_2]_{r=l} \equiv \alpha \frac{\partial u_2}{\partial r} + \beta u_2 \Big|_{r=l} = 0$$

and are valid for any choice of functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ (when the series converge uniformly) because they are valid for the functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$.

By substituting the series in Equation (6.136) into the given heat conduction Equation (6.135), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{dT_{nm}^{(1)}}{dt} + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(1)}(t) \right] V_{nm}^{(1)}(r, \varphi) \\ & + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{dT_{nm}^{(2)}}{dt} + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(2)}(t) \right] V_{nm}^{(2)}(r, \varphi) = f(r, \varphi, t). \end{aligned}$$

Here we see that eigenfunctions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$, corresponding to the discrete sequence of eigenvalues, form the complete set of orthogonal systems for expanding the right-hand side of the equation as the Fourier series. This also allows us to expand the function $f(r, \varphi, t)$ within a circle of radius l as

$$f(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[f_{nm}^{(1)}(t) V_{nm}^{(1)}(r, \varphi) + f_{nm}^{(2)}(t) V_{nm}^{(2)}(r, \varphi) \right],$$

where, using the orthogonality of this set of functions, the coefficients are

$$f_{nm}^{(1)}(t) = \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^l \int_0^{2\pi} f(r, \varphi, t) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \quad (6.137)$$

$$f_{nm}^{(2)}(t) = \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^l \int_0^{2\pi} f(r, \varphi, t) V_{nm}^{(2)}(r, \varphi) r dr d\varphi. \quad (6.138)$$

Comparing the expansions, we obtain differential equations for determining the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$:

$$\begin{aligned} \frac{dT_{nm}^{(1)}}{dt} + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(1)}(t) &= f_{nm}^{(1)}(t), \\ \frac{dT_{nm}^{(2)}}{dt} + (a^2 \lambda_{nm} + \gamma) T_{nm}^{(2)}(t) &= f_{nm}^{(2)}(t). \end{aligned} \quad (6.139)$$

In addition, these functions are necessarily subject to the initial conditions:

$$T_{nm}^{(1)}(0) = 0 \quad \text{and} \quad T_{nm}^{(2)}(0) = 0. \quad (6.140)$$

Solving the ordinary differential Equations (6.139) with initial condition (6.140), we can present the functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ (as we did several times previously in Chapter 5) in the form of integral relations given by

$$T_{nm}^{(1)}(t) = \int_0^t f_{nm}^{(1)}(\tau) e^{-(a^2 \lambda_{nm} + \gamma)(t-\tau)} d\tau, \quad (6.141)$$

$$T_{nm}^{(2)}(t) = \int_0^t f_{nm}^{(2)}(\tau) e^{-(a^2 \lambda_{nm} + \gamma)(t-\tau)} d\tau. \quad (6.142)$$

Thus, we have that the solution of the nonhomogeneous equation with zero initial and boundary conditions (that is, the solution to the simplest case of non-free heat exchange) can be written as

$$\begin{aligned} u(r, \varphi, t) &= u_1(r, \varphi, t) + u_2(r, \varphi, t) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[T_{nm}^{(1)}(t) + a_{nm} e^{-(a^2 \lambda_{nm} + \gamma)t} \right] V_{nm}^{(1)}(r, \varphi) \right. \\ &\quad \left. + \left[T_{nm}^{(2)}(t) + b_{nm} e^{-(a^2 \lambda_{nm} + \gamma)t} \right] V_{nm}^{(2)}(r, \varphi) \right\}, \end{aligned} \quad (6.143)$$

where functions $T_{nm}^{(1)}(t)$ and $T_{nm}^{(2)}(t)$ are defined by Equations (6.141) and (6.142), respectively, and coefficients a_{nm} and b_{nm} were found earlier.

Example 6.9. Find the temperature within a thin circular plate of radius l if its boundary is kept at constant zero temperature, the initial temperature distribution within the plate is zero, and one internal source of heat acts at the point (r_0, φ_0) of the plate, where $0 \leq r_0 < l$ and $0 \leq \varphi_0 < 2\pi$. Assume that the value of this source is

$$Q(t) = A \sin \omega t.$$

The plate is thermally insulated over its lateral faces.

Solution. The problem is expressed as

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] + \frac{A}{c\rho} \delta(r - r_0) \delta(\varphi - \varphi_0) \sin \omega t$$

under the conditions

$$u(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = 0.$$

The boundary condition of the problem is of the Dirichlet type, so eigenvalues are given by the equation $J_n(\mu) = 0$, and the eigenfunctions are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\mu_m^{(n)} r / l \right) \cos n\varphi \text{ and } V_{nm}^{(2)}(r, \varphi) = J_n \left(\mu_m^{(n)} r / l \right) \sin n\varphi.$$

The eigenfunctions squared norms can be calculated by using Equation (6.126):

$$\|V_{nm}^{(1)}\|^2 = \|V_{nm}^{(2)}\|^2 = \varepsilon_n \pi \frac{l^2}{2} \left[J_n' \left(\mu_m^{(n)} \right) \right]^2, \quad \varepsilon_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n > 0. \end{cases}$$

The initial temperature of the plate is zero, so we have $a_{nm} = 0$ and $b_{nm} = 0$ for all n and m .

Let us next find $f_{nm}^{(1)}(t)$ and $f_{nm}^{(2)}(t)$:

$$f_{nm}^{(1)}(t) = \frac{A}{c\rho} \frac{2}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \sin \omega t \cos n\varphi_0 J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right),$$

$$f_{nm}^{(2)}(t) = \frac{A}{c\rho} \frac{2}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \sin \omega t \sin n\varphi_0 J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right).$$

We thus have

$$T_{nm}^{(1)}(t) = \frac{A}{c\rho} \frac{2}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \cos n\varphi_0 J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right) I(t)$$

and

$$T_{nm}^{(2)}(t) = \frac{A}{c\rho} \frac{2}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \sin n\varphi_0 J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right) I(t),$$

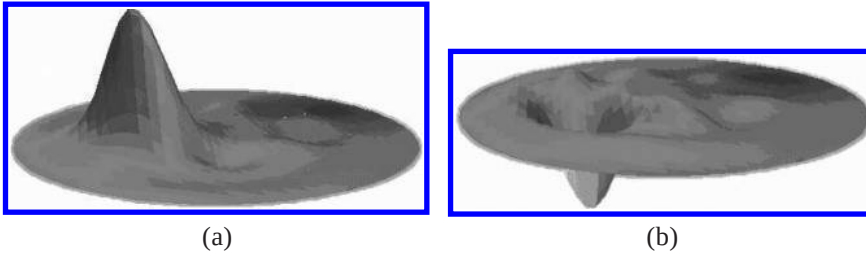


Figure 6.12. Temperature of the plate at (a) $t = 0.5$ and (b) $t = 1$ for Example 6.9.

where we have introduced

$$I(t) = \frac{1}{\left[\omega^2 + (a^2 \lambda_{nm})^2\right]} \left\{ a^2 \lambda_{nm} \sin \omega t - \omega \cos \omega t + \omega e^{-\lambda_{nm} a^2 t} \right\}.$$

Therefore, the evolution of temperature within the plate is described by the series

$$\begin{aligned} u(r, \varphi, t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_{nm}^{(1)} \cos(n\varphi) + T_{nm}^{(2)} \sin(n\varphi) \right] J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \\ &= \frac{2A}{c \rho \pi l^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{I(t)}{\sigma_n \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right) J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n(\varphi - \varphi_0). \end{aligned} \quad (6.144)$$

Figure 6.12 shows snapshots of the solution at the times $t = 0.5$ and $t = 1$. This solution was obtained with the program Heat for the case when $a^2 = 0.25$, $l = 2$, $r_0 = 1$, $\varphi_0 = 1$, $A/c\rho = 100$, and $\omega = 5$.

Reading Exercise. Discuss the result given by Equation (6.144) for the equilibrium (final) state when the source of heat is placed in the center of the plate at $r_0 = 0$. Show that in this case the problem can be reduced to a one-dimensional ordinary differential equation.

Example 6.10. A heat-conducting, thin, uniform, circular plate of radius l is thermally insulated over its lateral faces. The boundary of the plate is kept at constant zero temperature, and the initial temperature distribution

within the plate is zero. Let heat be generated throughout the plate with the intensity of internal sources (per unit mass of the membrane) given by

$$Q(t) = A \cos \omega t.$$

Find the distribution of temperature within the plate when $t > 0$.

Solution. The problem may be expressed by the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] + A \cos \omega t,$$

under the conditions

$$u(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = 0.$$

The boundary condition of the problem is of the Dirichlet type, so eigenvalues are given by

$$\lambda_{nm} = \left(\frac{\mu_m^{(n)}}{l} \right)^2,$$

where $\mu_m^{(n)}$ are the roots of the eigenvalue equation $J_n(\mu) = 0$. The eigenfunctions are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi \quad \text{and} \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi.$$

The initial temperature of the plate is zero and the intensity of internal sources depends only on time, t , so the solution $u(r, \varphi, t)$ is given by the series

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} T_{0m}^{(1)}(t) J_0 \left(\frac{\mu_m^{(n)}}{l} r \right),$$

where

$$T_{0m}^{(1)}(t) = \int_0^t f_{0m}^{(1)}(\tau) e^{-a^2 \lambda_{0m}(t-\tau)} d\tau.$$

Taking into account that

$$\left\| V_{0m}^{(1)} \right\|^2 = \pi l^2 \left[J_0' \left(\mu_m^{(0)} \right) \right]^2 = \pi l^2 \left[J_1 \left(\mu_m^{(0)} \right) \right]^2,$$

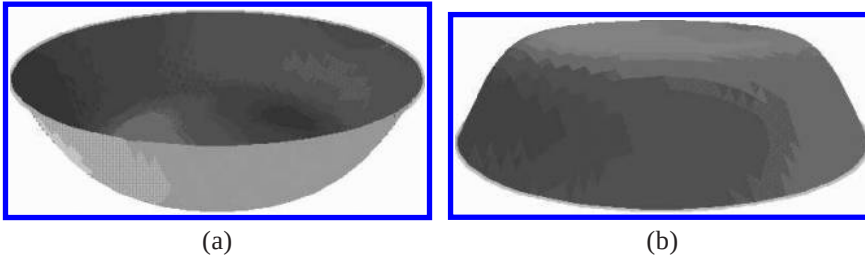


Figure 6.13. Temperature of the plate at (a) $t = 1$ and (b) $t = 3$ for Example 6.10.

$$\int_0^l J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr = \frac{l^2}{\left[\mu_m^{(0)}\right]^2} \int_0^{\mu_m^{(0)}} x J_0(x) dx = \frac{l^2}{\mu_m^{(0)}} J_1\left(\mu_m^{(0)}\right),$$

we find $f_{0m}^{(1)}(t)$ in the form

$$f_{0m}^{(1)}(t) = \frac{A \cos(\omega t)}{\left\|V_{0m}^{(1)}\right\|^2} 2\pi \int_0^l J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr = \frac{2A}{\mu_m^{(0)} J_1\left(\mu_m^{(0)}\right)} \cos(\omega t).$$

Then

$$\begin{aligned} T_{0m}^{(1)}(t) &= \frac{2A}{\mu_m^{(0)} J_1\left(\mu_m^{(0)}\right)} \int_0^t \cos(\omega \tau) e^{-a^2 \lambda_{0m}(t-\tau)} d\tau \\ &= \frac{2A}{\mu_m^{(0)} J_1\left(\mu_m^{(0)}\right) \left[\omega^2 + (a^2 \lambda_{0m})^2\right]} \left[a^2 \lambda_{0m} \cos(\omega t) + \omega \sin(\omega t) - a^2 \lambda_{0m} e^{-a^2 \lambda_{0m} t} \right]. \end{aligned}$$

Therefore, the evolution of temperature within the plate is described by the series

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} T_{0m}^{(1)}(t) J_0\left(\frac{\mu_m^{(n)}}{l}r\right). \quad (6.145)$$

Figure 6.13 shows snapshots of the solution at the times $t = 1$ and $t = 3$. This solution was obtained with the program Heat for the case when $a^2 = 0.25$, $l = 2$, $A = 100$, and $\omega = 5$.

Reading Exercise. Discuss the result given by Equation (6.145) for the equilibrium (final) state. Show that in this case the problem can be reduced to a one-dimensional ordinary differential equation.

6.3.3 The Fourier Method for the Nonhomogeneous Heat Conduction Equation with Nonhomogeneous Boundary Conditions

Finally, we consider the general boundary problem for heat conduction, Equation (6.103), given by

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) - \gamma u + f(r, \varphi, t),$$

with nonhomogeneous initial conditions (6.104) and boundary conditions (6.105) given by

$$u(r, \varphi, 0) = \phi(r, \varphi),$$

$$P[u]_{r=l} \equiv \alpha \frac{\partial u}{\partial r} + \beta u \Big|_{r=l} = g(\varphi, t).$$

To reduce a problem with nonhomogeneous boundary conditions to a problem with homogeneous boundary conditions, we introduce an auxiliary function $w(r, \varphi, t)$ that satisfies the given nonhomogeneous boundary conditions. As always, we search for the solution of the problem as

$$u(r, \varphi, t) = v(r, \varphi, t) + w(r, \varphi, t),$$

where $v(r, \varphi, t)$ is a function satisfying the homogeneous boundary conditions. We seek the auxiliary function $w(r, \varphi, t)$ in the form

$$w(r, \varphi, t) = (c_0 + c_1 r + c_2 r^2) g(\varphi, t),$$

where c_0 , c_1 , and c_2 are real constants. These constants will be adjusted to satisfy the boundary conditions. Because

$$\frac{1}{r} \frac{\partial w}{\partial r}(r, \varphi, t) = \left(\frac{c_1}{r} + 2c_2 \right) g(\varphi, t),$$

and $r = 0$ is a regular point, the coefficient $c_1 \equiv 0$, and the auxiliary function reduces to

$$w(r, \varphi, t) = (c_0 + c_2 r^2) g(\varphi, t).$$

Below we present auxiliary functions for different types of boundary conditions.

1. For $\alpha = 0$ and $\beta = 1$,
 - (a) for the boundary condition $u(l, \varphi, t) = g(\varphi, t)$, we have the auxiliary function $w(r, \varphi, t) = (r^2/l^2)g(\varphi, t)$;
 - (b) for the boundary condition $u(l, \varphi, t) = g(t)$ or $u(l, \varphi, t) = g_0 = \text{const}$, we have the auxiliary function $w(r, \varphi, t) = g(t)$ or $w(r, \varphi, t) = g_0$.
2. For the boundary condition $u_r(l, \varphi, t) = g(\varphi, t)$ ($\alpha = 1$ and $\beta = 0$), we have the auxiliary function $w(r, \varphi, t) = (r^2/2l)g(\varphi, t) + C$, where C is an arbitrary constant.
3. For the boundary condition $u_r(l, \varphi, t) + hu(l, \varphi, t) = g(\varphi, t)$ ($\alpha = 1$ and $\beta = h$), we have the auxiliary function

$$w(r, \varphi, t) = \frac{r^2}{l(2 + lh)}g(\varphi, t).$$

It is easy to verify (we leave it to the reader as a reading exercise) that such auxiliary functions $w(r, \varphi, t)$ satisfy the given boundary condition

$$P[w] \equiv \alpha \frac{\partial w}{\partial r}(r, \varphi, t) + \beta w(r, \varphi, t) \Big|_{r=l} = g(\varphi, t).$$

Example 6.11. The initial temperature distribution within a very long (infinite) cylinder of radius l is

$$u(r, \varphi, 0) = u_0 = \text{const}.$$

Find the distribution of temperature within the cylinder if a constant heat flow,

$$\frac{\partial u}{\partial r}(l, \varphi, t) = Q = \frac{q}{\kappa},$$

is supplied to the surface of the cylinder from the outside starting at time $t = 0$. Generation (or absorption) of heat by internal sources is absent.

Solution. The problem depends on the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right]$$

under the conditions

$$u(r, \varphi, 0) = u_0, \quad \frac{\partial u}{\partial r}(l, \varphi, t) = Q = \frac{q}{\kappa}.$$

An auxiliary function satisfying the given boundary condition is

$$w(r, \varphi, t) = \frac{Q}{2l} r^2.$$

The solution to the problem should be sought in the form

$$u(r, \varphi, t) = w(r, \varphi, t) + v(r, \varphi, t),$$

where the function $v(r, \varphi, t)$ is the solution to the boundary value problem for

$$f^*(r, \varphi, t) = \frac{2a^2}{l} Q,$$

$$\phi^*(r, \varphi) = u_0 - \frac{Q}{2l} r^2.$$

The solution $v(r, \varphi, t)$ may thus be defined by the series

$$v(r, \varphi, t) = \sum_{m=0}^{\infty} \left[T_{0m}^{(1)}(t) + a_{0m} e^{-a^2 \lambda_{0m} t} \right] J_0 \left(\frac{\mu_m^{(0)}}{l} r \right).$$

The initial temperature, $\phi^*(r, \varphi)$, and the right-hand side of the heat equation, $f^*(r, \varphi, t)$, do not depend on the polar angle φ , which is why the solution contains only Bessel functions of zeroth order.

The boundary condition of the problem is of the Neumann type, so eigenvalues are given by the equation

$$J'_n(\mu) = 0,$$

which has roots $\mu_m^{(n)}$. The eigenfunctions and their norms are

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi,$$

$$\left\| V_{nm}^{(1)} \right\|^2 = \left\| V_{nm}^{(2)} \right\|^2 = \frac{\pi l^2}{\left(\mu_m^{(n)} \right)^2} \left[\left(\mu_m^{(n)} \right)^2 - n^2 \right] J_n^2 \left(\mu_m^{(n)} \right),$$

in which case we have

$$\left\| V_{0m}^{(1)} \right\|^2 = \pi l^2 J_0^2 \left(\mu_m^{(0)} \right), \quad \left\| V_{00}^{(1)} \right\|^2 = \pi l^2,$$

$$a_{0m} = \frac{1}{\|V_{0m}^{(1)}\|^2} \int_0^{2\pi} \int_0^l \left(u_0 - \frac{Q}{2l}r^2\right) J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr d\varphi = -\frac{2Ql}{\left(\mu_m^{(0)}\right)^2 J_0\left(\mu_m^{(0)}\right)},$$

$$a_{00} = \frac{1}{\|V_{00}^{(1)}\|^2} \int_0^{2\pi} \int_0^l \left(u_0 - \frac{Q}{2l}r^2\right) r dr d\varphi = u_0 - \frac{Ql}{4}.$$

By applying Equations (6.137) and (6.138), we obtain

$$f_{0m}^{(1)}(t) = \frac{2a^2Q}{l\|V_{0m}^{(1)}\|^2} \int_0^{2\pi} d\varphi \int_0^l J_0\left(\frac{\mu_m^{(0)}}{l}r\right) r dr = \frac{4a^2Q}{l\mu_m^{(0)}J_0^2\left(\mu_m^{(0)}\right)} J_1\left(\mu_m^{(0)}\right) = 0,$$

$$f_{00}^{(1)}(t) = \frac{2a^2Q}{l\|V_{00}^{(1)}\|^2} \int_0^{2\pi} d\varphi \int_0^l r dr = \frac{2a^2Q}{l},$$

and

$$T_{0m}^{(1)}(t) = \int_0^t f_{0m}^{(1)}(\tau) e^{-a^2\lambda_{0m}(t-\tau)} d\tau = 0,$$

$$T_{00}^{(1)}(t) = \int_0^t f_{00}^{(1)}(\tau) d\tau = \frac{2a^2Q}{l}t.$$

Hence, the distribution of temperature within the cylinder at some instant of time is described by the series

$$u(r, \varphi, t) = \frac{Q}{2l}r^2 + u_0 - \frac{Ql}{4} + \frac{2a^2Q}{l}t - 2Ql \sum_{m=1}^{\infty} \frac{e^{-a^2\lambda_{0m}t}}{\left(\mu_m^{(0)}\right)^2 J_0\left(\mu_m^{(0)}\right)} J_0\left(\frac{\mu_m^{(0)}}{l}r\right). \quad (6.146)$$

Figure 6.14 shows snapshots of the solution at the times $t = 0$ and $t = 3$. This solution was obtained with the program Heat for the case when $a^2 = 0.25$, $l = 2$, $u_0 = 10$, and $Q = 2$.

Reading Exercise. Discuss the role and the origin of each term in the solution to Example 6.11 (Equation 6.146).

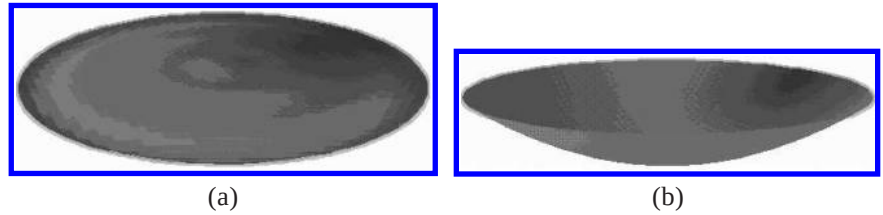


Figure 6.14. Plate temperature at (a) $t = 0$ and (b) $t = 3$ for Example 6.11.

Problems

The problem of propagation of temperature waves on the earth is one of the first examples of the application of heat conduction theory developed by Fourier. The temperature of the earth's surface changes very distinctly over daily (day–night) and yearly (summer–winter) periods. Assume the earth is a homogeneous, three-dimensional, semi-infinite medium with a temperature at its surface that changes periodically. After several temperature variations on the earth's surface, one can ignore the influence of the initial temperature.

6.1. Do temperature fluctuations within the earth's interior have the same period as on the surface? Defend your answer.

6.2. The temperature distribution in the earth takes place with a phase displacement. The time Δt between the occurrence of the temperature maximum (minimum) at depth x is described by formula $(\sqrt{1/2a^2\omega})x$ (Fourier's second law). Derive this formula.

6.3. For two temperature distributions with periods T_1 and T_2 , the corresponding depths x_1 and x_2 in which the relative temperature changes coincide are connected by the equation $x_2 = (\sqrt{T_2/T_1})x_1$ (Fourier's third law). Obtain this formula. Apply it for the daily and yearly variations to compare the depths of penetration.

6.4. The estimation for thermal diffusivity of the earth is $a^2 \approx 0.4 \cdot 10^{-6} \text{ m}^2/\text{s}$. How much time does it take for (the maximum) temperature to reach a 4-m depth?

6.5. Simulate several scenarios for temperature waves and skin effect by using the Heat program. In the library problems that come with the program, try (with different parameters) Problem E.8 for the case of an infinite medium and Problem E.11 for finite objects. By using the program for nonhomogeneous equations, you can simulate situations with sources of heat that are, for instance, generated by radioactive materials distributed in the earth's interior.¹

¹For theoretical details, see H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, second edition, Clarendon Press, Oxford, 1959.

6.6. Solve analytically, as well as with the program Heat, the boundary value problem for the heat (diffusion) equation in a rectangular surface $0 \leq x \leq l_x$, $0 \leq y \leq l_y$ with boundary conditions $u(l_x, y) = \sin^2(\pi y/l_y)$ and $u = 0$ on the other edges. The initial temperature of the rectangle is zero, and there is no heat exchange with the environment. The bottom and top surfaces are thermally insulated (the same is true in all similar problems below).

For Problems 6.7 through 6.9, a thin homogeneous plate with sides of length π lies in the x - y plane, and the center of the plate is located at point O . The edges of the plate are kept at the temperature described by the function of $u(x, y)|_\Gamma$, given in each problem. Find the temperature in the plate if initially the temperature has a constant value A . Notice that the boundary and initial conditions do not match each other in these problems (as well as in many of the following problems); thus, we are searching for generalized solutions.

6.7. $u|_{y=\pi/2} = u|_{y=-\pi/2} = x^2 - x$, $u|_{x=0} = u|_{x=\pi} = 0$.

6.8. $u|_{y=-\pi/2} = x^2$, $u|_{y=\pi/2} = 0.7x$, $u|_{x=0} = u|_{x=\pi} = 0$.

6.9. $u|_{y=\pi/2} = \sin \omega t$, $u|_{x=0} = u|_{x=\pi} = u|_{y=-\pi/2} = 0$.

In Problems 6.10 through 6.12, an infinitely long rectangular cylinder has its central axis along the z -axis, and its cross section is a rectangle with sides of length π . The sides of the cylinder are kept at the temperature described by functions $u(x, y)|_\Gamma$, given in each problem. Find the temperature within the cylinder if initially the temperature is $u(x, y, 0) = Axy$.

6.10. $u|_{y=\pi/2} = u|_{y=-\pi/2} = u|_{y=-\pi/2} = 3x^2$, $u|_{x=0} = u|_{x=\pi} = 0$.

6.11. $u|_{y=\pi/2} = \cos 2x - 2x$, $u|_{y=-\pi/2} = u|_{x=0} = u|_{x=\pi} = 0$.

6.12. $u|_{x=0} = \sin \omega t$, $u|_{x=\pi} = u|_{y=\pi/2} = u|_{y=-\pi/2} = 0$.

For Problems 6.13 and 6.14, solve the boundary value problem for the heat equation in a rectangular surface, $0 \leq x \leq l_x$, $0 \leq y \leq l_y$, with boundary conditions $u_x(0, y, t) = 0$ and $u = 0$ on the other sides.

6.13. The initial temperature of the rectangle is a constant value A .

6.14. The initial temperature of the rectangle is given by $u(x, y, 0) = Axy(l_x - x) \times (l_y - y)$.

6.15. Find the temperature of a long circular cylinder of radius l if its initial temperature is $u|_{t=0} = u_0(1 - r^2/l^2)$ and the surface temperature is kept equal to zero.

6.16. Find the temperature of a long circular cylinder of radius l if its initial temperature is $u|_{t=0} = u_0(1 - r/l)$ and heat flow at the surface is governed by $u_r(l, \varphi, t) = \cos \omega t$.

In Problems 6.17 through 6.19, a thin homogeneous circular plate of radius l lies in the x - y plane with its center at point O . The edge boundary of the plate is kept at the temperature described by a function of the polar angle $\tilde{u}(\varphi)$, given below. Find the temperature distribution in the plate if initially its temperature has a constant value, A .

6.17. $\tilde{u}(\varphi) = \cos \frac{\varphi}{2}$.

6.18. $\tilde{u}(\varphi) = \cos \frac{\varphi}{2} + \frac{\varphi}{\pi}$.

6.19. $\tilde{u}(\varphi) = \cos \frac{\varphi}{2} + \sin \frac{\varphi}{2}$.

For Problems 6.20 and 6.21, a thin homogeneous circular plate of radius l lies in the x - y plane with its center at point O . The edge boundary of the plate is kept at a temperature described by the function of the polar angle, $\tilde{u}(\varphi) = \sin(\varphi/2)$. (Note: The integrals in the coefficients in Problems 6.20 and 6.21 are not easy to evaluate analytically; however, you can quickly find these coefficients with the program Heat.) Find the temperature in the plate if initially the temperature is:

6.20. $u|_{t=0} = u_0 r^2 \cos(Ar/l)$.

6.21. $u|_{t=0} = u_0 \cos \varphi$.

For Problems 6.22 and 6.23, find the temperature in a long circular cylinder of radius l if its surface is insulated (i.e., $u_r(l, \varphi, t) = 0$) and the initial temperature is:

6.22. $u|_{t=0} = u_0(1 - r^2/l^2)$.

6.23. $u|_{t=0} = u_0 r^2 \sin(Ar/l)$.

6.24. In a long circular cylinder of radius l , the generation of heat of density Q (uniformly distributed over a cross-section) occurs starting at time $t = 0$. Find the distribution of temperature over a section, assuming the initial temperature and the temperature of the surface equals zero.

Answer.

$$u(r, t) = \frac{Ql^2}{4k} \left\{ 1 - \left(\frac{r}{l} \right)^2 - 8 \sum_{m=0}^{\infty} \frac{1}{\left(\mu_m^{(0)} \right)^3} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) e^{-\lambda_{0m} t} \right\},$$

where $\mu_m^{(0)}$ are positive roots of the equation $J_0(\mu) = 0$, and $\lambda_{0m} = (\mu_m^{(0)}/l)^2$.

6.25. A long circular cylinder of radius l was heated for a long time by a current generating heat with density Q . Study the cooling of the conductor after the current is disconnected and the temperature exchange with the environment with zero temperature is governed according to Newton's law.

Answer.

$$u(r, t) = \frac{2Qhl^3}{k} \sum_{m=0}^{\infty} \frac{1}{\mu_m^{(0)} \left[\left(\mu_m^{(0)} \right)^3 + h^3 l^3 \right] J_0 \left(\mu_m^{(0)} \right)} J_0 \left(\frac{\mu_m^{(0)}}{l} r \right) e^{-\lambda_{0m} t},$$

where $\mu_m^{(0)}$ are positive roots of the equation $\mu J_0'(\mu) + h l J_0(\mu) = 0$, h is the coefficient of heat exchange, and $\lambda_{0m} = (\mu_m^{(0)} / l)^2$.

For Problems 6.26–6.28, a long circular cylinder of radius l has a periodic heat source placed along the axis of the cylinder (i.e., $Q = A \sin \omega t \cdot \delta(r)$). Find the temperature distribution over the cylinder if the initial temperature is constant and the boundary conditions are:

6.26. $u(l, \varphi, t) = 0$.

6.27. $u_r(l, \varphi, t) = 0$.

6.28. $u_r(l, \varphi, t) + hu(l, \varphi, t) = 0$.

For Problems 6.29 and 6.30, a circular plate of radius l has a periodic heat source, $Q = A \sin \omega t \cdot \delta(r)$, placed at the center of the plate. This source is acting during the limited time, $t_0 < t < t_1$. Find the temperature distribution over the cylinder if the initial temperature is constant and the boundary conditions are:

6.29. $u(l, \varphi, t) = 0$.

6.30. $u_r(l, \varphi, t) = 0$.

6.31. In a thin rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), the edges $x = 0$ and $y = 0$ are kept at constant temperature $u = u_1$, and the edges $x = l_x$ and $y = l_y$ are kept at constant temperature $u = u_2$. Find the heat distribution in the rectangle if the initial temperature is $u(x, y, 0) = u_0 = \text{const}$.

Answer.

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} a^2 t} \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} + u_1 + (u_2 - u_1) \frac{x}{l_x} + 2(u_2 - u_1) \sum_{n=1}^{\infty} \frac{1}{n\pi \sinh \frac{n\pi l_y}{l_x}} \left\{ \sinh \frac{n\pi y}{l_y} + (-1)^n \sinh \frac{n\pi(l_y - y)}{l_y} \right\} \sin \frac{n\pi x}{l_x},$$

where

$$C_{nm} = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} \varphi^*(x, y) \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y} dx dy.$$

$$\begin{aligned} \varphi^*(x, y) &= u_0 - u_1 - (u_2 - u_1) \frac{x}{l_x} \\ &\quad - 2(u_2 - u_1) \sum_{n=1}^{\infty} \frac{1}{n\pi \sinh \frac{n\pi}{l_x} l_y} \left\{ \sinh \frac{n\pi y}{l_x} + (-1)^n \sinh \frac{n\pi(l_y - y)}{l_x} \right\} \sin \frac{n\pi x}{l_x}. \end{aligned}$$

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right], \quad n, m = 1, 2, 3, \dots$$

6.32. Find the heat distribution in a thin rectangular plate, $0 \leq x \leq l_x$, $0 \leq y \leq l_y$, if it is subjected to heat transfer according to Newton's law at its edges. The temperature of the medium is $u_{md} = \text{const}$, the initial temperature of the plate is zero, and there is a constant source of heat, Q , uniformly distributed over the plate.

Hint. The problem is formulated as follows:

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + Q, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad t > 0,$$

$$u(x, y, 0) = 0,$$

$$\left. \frac{\partial u}{\partial x} - h(u - u_{md}) \right|_{x=0} = 0, \quad \left. \frac{\partial u}{\partial x} + h(u - u_{md}) \right|_{x=l_x} = 0,$$

$$\left. \frac{\partial u}{\partial y} - h(u - u_{md}) \right|_{y=0} = 0, \quad \left. \frac{\partial u}{\partial y} + h(u - u_{md}) \right|_{y=l_y} = 0.$$

Answer.

$$u(x, y, t) = w(x, y, t) + v(x, y, t) = u_{md} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_{nm}(t) + C_{nm} e^{-\lambda_{nm} a^2 t} \right] X_n(x) Y_m(y),$$

where

$$C_{nm} = -\frac{u_{md}}{\|V_{nm}\|^2} \frac{\sqrt{\lambda_{xn}} \sin \sqrt{\lambda_{xn}} l_x - h(\cos \sqrt{\lambda_{xn}} l_x - 1)}{\sqrt{\lambda_{xn}(\lambda_{xn} + h^2)}} \times \frac{\sqrt{\lambda_{ym}} \sin \sqrt{\lambda_{ym}} l_y - h(\cos \sqrt{\lambda_{ym}} l_y - 1)}{\sqrt{\lambda_{ym}(\lambda_{ym} + h^2)}},$$

$$T_{nm}(t) = \frac{Q}{a^2 \lambda_{nm} \|V_{nm}\|^2} \left[1 - e^{-a^2 \lambda_{nm} t} \right] \frac{\sqrt{\lambda_{xn}} \sin \sqrt{\lambda_{xn}} l - h(\cos \sqrt{\lambda_{xn}} x - 1)}{\sqrt{\lambda_{xn}(\lambda_{xn} + h^2)}} \\ \times \frac{\sqrt{\lambda_{ym}} \sin \sqrt{\lambda_{ym}} l - h(\cos \sqrt{\lambda_{ym}} x - 1)}{\sqrt{\lambda_{ym}(\lambda_{ym} + h^2)}},$$

$$\|V_{nm}\|^2 = \frac{1}{4} \left[l_x + \frac{2h(\lambda_{xn} + h^2)}{(\lambda_{xn} + h^2)^2} \right] \left[l_y + \frac{2h(\lambda_{ym} + h^2)}{(\lambda_{ym} + h^2)^2} \right],$$

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \frac{\mu_{xn}^2}{l_x^2} + \frac{\mu_{ym}^2}{l_y^2},$$

$$\lambda_{xn} = \left[\frac{\mu_{xn}}{l_x} \right]^2, \quad \lambda_{ym} = \left[\frac{\mu_{ym}}{l_y} \right]^2, \quad n, m = 0, 1, 2, \dots,$$

where μ_{xn} are positive roots of the equation $\tan \mu_x = 2hl_x \mu_x / (\mu_x^2 - h^2 l_x^2)$, and μ_{ym} are positive roots of the equation $\tan \mu_y = 2hl_y \mu_y / (\mu_y^2 - h^2 l_y^2)$.

6.33. An infinitely long rectangular cylinder, $0 \leq x \leq l_x$, $0 \leq y \leq l_y$, with the central axis along the z -axis is placed in a coil. At $t = 0$, a current in the coil turns on and the coil starts to generate an oscillation magnetic field outside the cylinder directed along the z -axis:

$$u(x, y, t) = H_0 \sin \omega t, \quad H_0 = \text{const}, \quad 0 < t < \infty.$$

Find the magnetic field inside the cylinder.

Hint. The problem is formulated as follows:

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right],$$

$$0 < x < l_x, \quad 0 < y < l_y, \quad t > 0$$

$$u(x, y, 0) = 0,$$

$$u|_{x=0} = u|_{x=l_x} = u|_{y=0} = u|_{y=l_y} = H_0 \sin \omega t.$$

Answer.

$$u(x, y, t) = H_0 \sin(\omega t) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_{nm}(t) \sin \frac{n\pi x}{l_x} \sin \frac{m\pi y}{l_y}.$$

where

$$T_{nm}(t) = -\frac{4H_0\omega [1 - (-1)^n][1 - (-1)^m]}{nm\pi^2 [\omega^2 + (a^2\lambda_{nm})^2]} \times \left\{ a^2\lambda_{nm} \cos(\omega t) + \omega \sin(\omega t) - a^2\lambda_{nm} e^{-\lambda_{nm}a^2t} \right\}.$$

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} = \pi^2 \left[\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right], \quad n, m = 1, 2, 3, \dots$$

7

Elliptic Equations

7.1 Elliptic Partial Differential Equations and Related Physical Problems

When studying different stationary (time-independent) processes, we very often encounter elliptic partial differential equations. One of the most common is Laplace's equation:

$$\nabla^2 u = 0. \quad (7.1)$$

The operator ∇^2 is a scalar product of two *gradient operators*, $\nabla^2 = \nabla \cdot \nabla$ (a review of vector calculus is given in Appendix D). In orthogonal Cartesian coordinates,

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z},$$

and *Laplace's equation* is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

The expression on the left side is called the *Laplacian of the function u* , and the expression

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the *Laplacian* (or Laplace's operator).

A function is said to be *harmonic* in a certain region if it and its first and second derivatives are continuous and it satisfies Laplace's equation in

that region. For example, the functions $2xy$, $x^2 - y^2$, and $e^{-x} \cos y$ satisfy the two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (7.2)$$

which is the partial differential equation for a function of two variables, $u(x, y)$. These functions are thus said to be harmonic on the entire x - y plane. Because of the linearity of Laplace's equation, if functions u_1 and u_2 are harmonic, their linear combinations $c_1 u_1 + c_2 u_2$, where $c_{1,2}$ are arbitrary real constants, are also harmonic.

The nonhomogeneous equation

$$\nabla^2 u = -f,$$

with a given function f , of the coordinates is called *Poisson's equation*. In the following discussion, we demonstrate that the choice of a negative sign on the right side corresponds to many physical situations.

Laplace's operator has different forms in different coordinate systems; we present two important versions here (further discussion may be found in Appendix D). The formulas that relate cylindrical (r, φ, z) and Cartesian coordinates are

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z,$$

and the Laplacian in cylindrical coordinates has the form

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \quad (7.3)$$

For the particular case of cylindrical coordinates—a polar coordinate system (r, φ) with no dependence on z —we have

$$x = r \cos \varphi, \quad y = r \sin \varphi,$$

and the Laplacian is

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$

The formulas that relate spherical (r, θ, φ) and Cartesian coordinates are

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

The Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (7.4)$$

Many physical processes, such as stationary heat conduction, diffusion, electrostatics, and hydrodynamics, lead to Laplace's and Poisson's equations. Next, we briefly consider examples of these processes and their associated equations.

7.1.1 Stationary Heat Conduction and Diffusion

For many situations, the heat flow in a body is governed by the heat conduction equation, which may be written as

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T, \quad (7.5)$$

where T is temperature and χ is the thermal diffusivity. If the heat distribution in a body is maintained in equilibrium (i.e., is time-independent), for instance with the help of a fixed, imposed temperature distribution on the body's surface or a steady heat flow through the surface, and the material within the body is not moving, Equation (7.5) reduces to Laplace's equation,

$$\nabla^2 T = 0. \quad (7.6)$$

When a medium contains a heat source or heat absorber, the heat conduction equation is

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T + \frac{Q}{\rho c}, \quad (7.7)$$

where ρ is the mass density, c is the specific heat capacity, and $Q(\vec{r}, t)$ gives the rate at which the heat source radiates or absorbs energy per unit time and unit volume. When the temperature (and Q) is time independent, $\partial T / \partial t = 0$, this equation becomes

$$\nabla^2 T = -\frac{Q}{\rho c \chi},$$

which is a particular example of Poisson's equation:

$$\nabla^2 u = -f.$$

We note here for future discussion that the diffusion equation,

$$\frac{\partial q}{\partial t} = D \nabla^2 q + f, \quad (7.8)$$

where q is the concentration of a material diffusing through a medium, D is the diffusion coefficient, and $f(\vec{r}, t)$ is a source or sink of the diffusing material, has a mathematical form identical to Equation (7.7). For stationary (steady) diffusion, this also becomes Poisson's equation (or Laplace's equation when $f = 0$). We expect, then, that although we may discuss a particular physical application when solving these equations, the solutions we discuss here will apply to both diffusion and heat flow problems.

7.1.2 Potential Fields and the Electrostatic Potential

In this section, we investigate a different physical situation, which also leads to an application of Poisson's or Laplace's equation. The work done by the electric force of an electrostatic field (i.e., a field created by stationary, fixed charges) in moving a sample charge depends only on the location of the initial and final points of the movement, not on the path connecting these points. This property allows us to introduce the very important idea of electric (or electrostatic) potential of an electrostatic field.

We may define the *potential difference* between two points in an electrostatic field, $\vec{E}$, to be equal to the work done by the force of the field in moving a unit charge from an initial point P_0 to a final location P :

$$d\phi = -dW = -\vec{E} \cdot d\vec{s}$$

or

$$\phi(P) - \phi(P_0) = -W = - \int_L^P \vec{E} \cdot d\vec{s} = - \int_{P_0}^P \vec{E} \cdot d\vec{s}, \quad (7.9)$$

where L is an arbitrary line connecting points P_0 and P . Here the minus sign, which is arbitrary, has been included as part of the definition and indicates work done *by* the field *on* the charge (equivalently, one might drop the minus sign and talk about work done by the charge on the field). Such fields and the forces they create are called *conservative* forces or fields. As the definition of potential differences indicates, conservative fields may always be expressed mathematically by using *potentials*, which, as we will see, obey Poisson's and/or Laplace's equation.

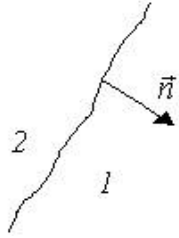


Figure 7.1. Two mediums with a common boundary.

When the initial and final points coincide, we obtain from Equation (7.9):

$$\oint_L \vec{E} \cdot d\vec{s} = 0. \quad (7.10)$$

That is, the integral of a conservative field along any closed loop in the field is equal to zero. Stokes's theorem (see Appendix D) allows us to represent the integral in Equation (7.10) along a closed contour, L , by using the *curl* of vector $\vec{E}$:

$$\oint_L \vec{E} \cdot d\vec{s} = \iint_S (\nabla \times \vec{E}) d\vec{S}.$$

Note that the vector $d\vec{S}$ is perpendicular to the surface element, dS , in the direction defined by the right-hand rule for contour L , and has the length $|d\vec{S}|$. Thus a statement equivalent to Equation (7.10) is that static fields are curl-free:

$$\nabla \times \vec{E} = 0,$$

a result that follows immediately from Maxwell's equations for a static electromagnetic field.

It follows from Equation (7.10) that at any surface the tangent component of the electric field is continuous, or $E_{1t} = E_{2t}$, where the indices 1 and 2 correspond to two mediums separated by a surface. In particular, because the electric field inside a conductor is absent for the static case, in which there are no currents within the conductor, we have $\vec{E} = 0$ resulting in $E_t = 0$ at the conductor's surface. This indicates that the field, $\vec{E}$, is orthogonal to the surface. The electric field outside a charged conductor close to the conductor's surface is equal to $\vec{E} = 4\pi\sigma\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the surface, directed outward, and σ is the

surface charge density. This result follows immediately from Gauss's theorem for an electrostatic field created by a stationary charge distribution with a given charge density. We thus see that $\vec{E}$ has a discontinuity in the normal component at the surface equal to $|\Delta E_n| = 4\pi\sigma$. This equation is valid not only for conductors; it also holds true for the interface between two dielectrics. Returning to the definition of the potential, we see that the potential at the surface must obey the equation

$$\left(\frac{\partial\phi}{\partial n}\right)\Big|_2 - \left(\frac{\partial\phi}{\partial n}\right)\Big|_1 = 4\pi\sigma,$$

where the unit normal (to the interface) $\vec{n}$ is directed into the first medium (see [Figure 7.1](#)).

It is easy to show that the formulas

$$d\phi = -\vec{E} \cdot d\vec{s} = -E_s ds \quad (7.11)$$

(where E_s is the component of vector $\vec{E}$ along an infinitesimal path $d\vec{s}$) and

$$\vec{E} = -\nabla\phi \quad (7.12)$$

are equivalent. From Equation (7.11), we have $\partial\phi/\partial s = -E_s$, where $\partial\phi/\partial s$ denotes a derivative of ϕ in the direction of vector $\vec{s}$; thus, $E_s = -\text{grad}_s\phi$. Because the direction of vector $\vec{s}$ is arbitrary we may drop the letter s , denoting direction, in which case we arrive at Equation (7.12). Including the minus sign, we see that these equations indicate that the electric field $\vec{E}$ is proportional to the decrease of the potential.

The basic feature of an electrostatic situation is that the electric field resulting from a point-like positive charge q is equal to

$$\vec{E} = \frac{q}{r^3}\vec{r} = \frac{q}{r^2}\hat{r},$$

where r is the distance from the charge to the point where the electric field is measured, and $\hat{r} = \vec{r}/r$ is a unit vector directed from q to this point. From the above discussion, we see that at this point the electrostatic potential due to a point charge is

$$\phi = \frac{q}{r}.$$

This result immediately follows from

$$\vec{E} = -\nabla\phi = q \frac{\hat{r}}{r^2}.$$

Because of the above-mentioned linearity of the Laplace's equation, for several charges the net potential is

$$\phi = -\sum_i \frac{q_i}{r_i}.$$

We may generalize these formulas to the case of a continuous charge distribution. If a charged surface has charge density σ per area, then the charge on an elementary area dS is $dq = \sigma dS$, and the potential created at some point r by this surface is the surface integral

$$\phi = \iint \frac{\sigma dS}{r},$$

where r is the distance from an elementary area dS to the point of where the potential is measured.

For charge distributed within a volume with charge density ρ per unit volume, the charge in a volume dV is $dq = \rho dV$ and the potential is given by the volume integral

$$\phi = \iiint \frac{\rho dV}{r}, \quad (7.13)$$

where r is the distance from an elementary volume dV to the point of observation. If charges are distributed within a volume and on its surface (see [Figure 7.2](#)), the potential at point r is

$$\phi = \iiint \frac{\rho dV}{r} + \iint \frac{\sigma dS}{r}. \quad (7.14)$$

Inside a closed conductive surface, the electric field is zero; thus, the electrostatic potential has a constant value everywhere inside a conductor. This constant can be chosen to be zero in the case of a single conductor, but if we consider several conductors the potentials within each of them are still constants but generally they are different for each conductor. Clearly, in the case of a single closed conductive surface with no charges inside, $\rho = 0$ and the surface integral in Equation (7.14) is zero. A useful and interesting property of conductors is that if we place a charge or charges inside that

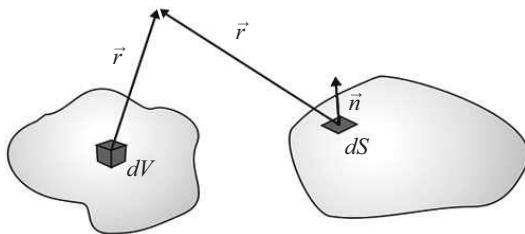


Figure 7.2. Potential due to surface and volume distributions.

create their own potential, the charges will relocate on the inner surface in such a way that the total potential inside remains zero, thus the surface integral in Equation (7.14) now is not equal to zero.

Notice that the integrals in Equations (7.13) and (7.14) converge even at $r = 0$ in spite of the fact that r stays in the denominator. For instance, consider the formula for volume charges in a spherical coordinate system. The elementary volume is

$$dV = r^2 \sin \theta dr d\theta d\varphi$$

and

$$\phi = \iiint \rho r \sin \theta dr d\theta d\varphi;$$

thus, the potential is finite at $r = 0$ also.

Gauss's theorem together with the relation $\vec{E} = -\nabla\phi$ allows us to obtain an equation that relates a potential to the corresponding charge density. Recall that Gauss's theorem gives the electrostatic field created by a stationary charge distribution with charge density ρ :

$$\operatorname{div} \vec{E} = 4\pi\rho.$$

In Cartesian coordinates,

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z},$$

or more generally, independent of a coordinate system,

$$\operatorname{div} \vec{E} = \nabla \cdot \vec{E}.$$

By applying the divergence operator to the equation $\vec{E} = -\nabla\phi$ and using Gauss's equation, we obtain

$$\text{divgrad}\phi = -\text{div}\vec{E} = -4\pi\rho.$$

The left side is

$$\text{divgrad}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$$

or, independent of a coordinate system,

$$\text{divgrad}\phi = \nabla \cdot (\nabla\phi) = \nabla^2\phi.$$

Thus, we have

$$\nabla^2\phi = -4\pi\rho. \quad (7.15)$$

Equation (7.15) is Poisson's equation, in this case for an electrostatic potential. In regions that *do not contain electric charges*—for instance, at points outside of a charged body— $\rho = 0$ and the potential that the body creates at the point $\vec{r}$ obeys Laplace's equation, given by

$$\nabla^2\phi(\vec{r}) = 0.$$

These equations were originally studied by Laplace and Poisson for the case of gravitational fields. These same equations in spherical coordinates are applicable to any field with forces decaying as $1/r^2$ with distance.

We may use Poisson's equation,

$$\nabla^2\phi(\vec{r}) = -4\pi\rho(\vec{r})$$

to find the potential of a collection of charges if their volume distribution is known. The solution of this partial differential equation under appropriate boundary conditions should give the same results as Equation (7.13):

$$\phi(\vec{r}) = \iiint_{V'} \frac{\rho(\vec{r}')}{|\vec{r}' - \vec{r}|} d\vec{r}'. \quad (7.16)$$

It is important to notice that in many cases it is easier to solve Laplace's or Poisson's equations rather than to use the integral representation for ϕ given by the Equation (7.16).

Reading Exercise. Prove that if r is the distance from a fixed point—for simplicity from the origin—to some point in a space, the scalar function $\phi = r^{-1}$ ($r \neq 0$) satisfies Laplace's equation.

Hint.

1. Try a “physical” proof, such as that ϕ is the potential from a point charge and thus it should be a solution of the equation describing an electric potential for this case.
2. Find $\nabla^2 (r^{-1})$ in Cartesian coordinates.

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \frac{\partial}{\partial x} \frac{1}{r} = \frac{x}{r^3}, \quad \frac{\partial^2}{\partial x^2} \frac{1}{r} = -3\frac{x^2}{r^4} + \frac{1}{r^3}, \text{ etc.}$$

Thus,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{1}{r} = 0.$$

3. Repeat for cylindrical coordinates. Note that you cannot just substitute $1/r$ in Equation (7.3) because in this equation, r is the distance to the axis, not to the origin.
4. Repeat for spherical coordinates.

Results 3 and 4 can be easily obtained by using the expressions for the operator ∇^2 in these two coordinate systems (see Appendix D) and noticing that function ϕ depends only on r . Notice that the *potential and its first derivatives (gradient) are finite everywhere in space*. Physically, this must be the case because an infinite discontinuity jump in a potential would result in an infinite electric field, a result that is physically nonsense.

7.1.3 Inviscid Flow of an Incompressible Fluid

As a further example of Poisson's and Laplace's equations, we next consider a flow of an incompressible, inviscid (having no viscosity and therefore no resistance to shear stress) fluid when the velocity of the fluid, $\vec{v}$, depends on the coordinates. Moreover, we do not assume that the flow is stationary; that is, the velocity can also depend on time. Assume that the flow is irrotational, or curl-free:

$$\text{curl } \vec{v} = 0.$$

As discussed in Section 7.1.2, for such cases a velocity potential can be introduced by means of the relation

$$\vec{v} = \text{grad}\phi. \quad (7.17)$$

In other words, this is a potential flow. Note that in fluid mechanics, the minus sign in front of the gradient in this formula is usually omitted.

To show that ϕ satisfies Laplace's equation, consider the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \vec{v} = 0 \quad (7.18)$$

for an incompressible fluid (density $\rho = \text{const}$), which gives $\text{div} \vec{v} = 0$; that is, the flow is source-free. By substituting Equation (7.18) into Equation (7.17) and using

$$\text{div} \vec{v} = \text{div}(\vec{\nabla} \phi) = \nabla^2 \phi,$$

we see that the velocity potential satisfies Laplace's equation:

$$\nabla^2 \phi = 0. \quad (7.19)$$

Flow must stop at the solid surface of a container, in which case we have the boundary condition

$$\frac{d\phi}{dn} = 0, \quad (7.20)$$

where $\vec{n}$ is a vector orthogonal to the surface. If a solid body is moving in a fluid, then on the body's surface we have

$$\frac{d\phi}{dn} = f(\vec{r}, t), \quad (7.21)$$

where $f(\vec{r}, t)$ is the normal to the solid-surface component of the velocity of the body, which is a given function of coordinates and time. Equations (7.21) and (7.22) are particular cases of boundary conditions (both are the Neumann type).

More complex boundary conditions occur at a fluid's free surface or at the interface between two fluids. Let us briefly consider the case of a free surface (the reader should provide the same development for a situation with two fluids in contact as a reading exercise). Compared to the problems studied earlier, this is a new situation because the shape of the free surface is unknown and should be found simultaneously with the potential. In

other words, there exists an additional unknown function, and one needs an additional boundary condition. If the shape of the fluid's surface is given by the equation

$$F(\vec{r}, t) = 0, \quad (7.22)$$

then at any moment of time (and at any point on the free surface)

$$\frac{dF}{dt} = 0. \quad (7.23)$$

This equation can be rewritten as the so-called *kinematic equation*,

$$\frac{\partial F}{\partial t} + \nabla \phi \cdot \nabla F = 0, \quad (7.24)$$

where $d/dt = \partial/\partial t + \vec{v} \cdot \nabla$ is the Lagrangian derivative (derivative along the path of a fluid element), which takes into account the fluid motion.

The physical meaning of Equation (7.23) is straightforward: the normal to the boundary component of the fluid's velocity equals the velocity of the boundary in the same direction. Indeed, it is well known that the normal vector to the free surface is

$$\vec{n} = \frac{\nabla F}{|\nabla F|}.$$

That is, F changes only due to the normal component of the velocity.

As mentioned previously, we still need an equation relating the potential ϕ to the function F . Let us formulate this equation from the so-called *condition of balance of normal stresses*. If the surface tension is negligible, the pressure field is constant along the free surface:

$$p = \text{const.} \quad (7.25)$$

The constant on the right-hand side can be taken as zero since the absolute value of the pressure does not affect the behavior of an incompressible fluid; only the gradient of pressure is important. To find the pressure field within the fluid through the velocity potential, one needs to apply the Bernoulli equation,

$$p = -\rho \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 \right] + \Phi(r, t), \quad (7.26)$$

where $\Phi(\vec{r}, t)$ is the potential of some external forces (e.g., $\Phi = -\rho g z$ in a gravity field). Thus, we arrive at the Laplace equation

$$\nabla^2 \phi = 0, \quad (7.27)$$

with the following boundary conditions at the free surface:

$$F(\vec{r}, t) = 0 \text{ or } \frac{\partial F}{\partial t} + \nabla \phi \cdot \nabla F = 0, \quad -\rho \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 \right] + \Phi(r, t) = 0. \quad (7.28)$$

Note that generally speaking, such a problem is nonlinear. Indeed, the nonlinear term $(\nabla \phi)^2$ enters the boundary condition (7.28). Moreover, Equations (7.28) are prescribed at the moving boundary, which, in turn, is determined from the solution. Such a problem is quite complicated and can be solved analytically only for a few particular cases.

7.1.4 Boundary Conditions: General Considerations

First, we note that problems described by elliptic equations do not contain time; therefore, they do not need initial conditions. Physically, it is also clear that Laplace's and Poisson's equations by themselves are not sufficient to determine, for example, the temperature in all points of a body, or the electric potential outside the conductor. We have to know the heat regime or the charge distribution on the surface of the conductor to solve these problems, and these constitute boundary conditions. From physical reasoning, we see, for instance, that if the temperature distribution on the surface of a body is known, the solution of such a *boundary value problem that consists of Laplace's or Poisson's equation together with a boundary condition* should exist and be unique.

Boundary conditions can be set in several ways, and in fact in our previous discussion of various physical problems, we have already encountered several kinds of boundary conditions. Here we categorize three primary kinds of boundary conditions, which correspond to the three different heat regimes on the surface. (We use temperature terminology here, but the arguments apply equally to other physical systems).

Consider some volume, V , bounded by a surface S . The boundary value problem for a stationary distribution of temperature $u(x, y, z)$ inside the body is stated in the following way:

Find the function $u(x, y, z)$ inside the volume V satisfying the equation

$$\nabla^2 u = -f(x, y, z)$$

and satisfying one of the following kinds of boundary conditions:

1. $u = f_1$ on S (boundary value problem of the first kind),
2. $\partial u / \partial n = f_2$ on S (boundary value problem of the second kind),
3. $\partial u / \partial n + h(u - f_3) = 0$ on S ($h > 0$) (boundary value problem of the third kind), where f_1 , f_2 , f_3 and h are known functions, and $\partial u / \partial n$ is the derivative in the direction of the outward normal to the surface, S .

We may use the above formulation in two distinct ways. If we want to find, for example, the temperature *inside* the volume V for the bounded region, we have what is referred as an *interior boundary value problem*. If instead we want to find, for example, the temperature outside of a heater, or the electrostatic potential for an unbounded region outside the charged volume V , we have an *exterior boundary value problem*.

The physical sense of each of these boundary conditions is clear. The first boundary value problem, when the *surface temperature is prescribed*, is called the *Dirichlet problem*. The second boundary value problem, when the *flux across the surface is prescribed*, is called the *Neumann problem*. The third boundary value problem is called the *mixed problem*. This boundary condition corresponds to the well-known Newton's law of cooling, which governs the heat flux from the surface into the ambient medium.

Obviously, a stationary temperature distribution is possible only if the net heat flow across the boundary is equal to zero. It thus follows that for the *interior Neumann problem*, the function f_2 should obey the additional requirement

$$\iint_S f_2 dS = 0.$$

In a similar way, a boundary value problem can be formulated for the two-dimensional case when an area is bounded by a closed contour L . In

this case the requirement that the net heat flow through the boundary for the interior problem is equal to zero becomes

$$\oint_L f_2 dl = 0.$$

Let us demonstrate the solution of a boundary value problem for a one-dimensional case. When the function $u(x)$ depends on only one variable, Laplace's equation becomes an ordinary differential equation and the solution is trivial.

Example 7.1. Solve the one-dimensional Laplace's equation in Cartesian coordinates, $d^2u/dx^2 = 0$, and apply the Dirichlet boundary conditions to find a solution.

Solution. Integrating the equation gives $u = ax + b$. The Dirichlet problem with boundary conditions $u(x = 0) = u_1$ and $u(x = l) = u_2$ gives the solution as $u(x) = (u_2 - u_1)x/l + u_1$.

Reading Exercise. In Cartesian coordinates, obtain a solution of the one-dimensional Neumann problem for Laplace's equation.

Reading Exercise. In Cartesian coordinates, obtain a solution of the one-dimensional mixed problem for Laplace's equation.

As another example, let us briefly return to the discussion of the type of boundary condition that should be prescribed at the interface separating two mediums. For the sake of simplicity, we restrict ourselves to a flat interface located at $x = 0$, assuming that the first medium is situated at $x < 0$, and the second one (the ambient medium) is at $x > 0$. To proceed, we need to solve Laplace's equation

$$\nabla^2 u = 0$$

in both mediums and account for the continuity conditions for the temperature and the heat flux at the interface given by:

$$T_1 = T_2, \quad \kappa_1 \frac{\partial T_1}{\partial x} = \kappa_2 \frac{\partial T_2}{\partial x} \quad (7.29)$$

at $x = 0$. Here $\kappa_{1,2}$ are the thermal conductivities of the two mediums.

If $\kappa_1 \ll \kappa_2$, the temperature of the ambient medium should be close to a constant; that is, T_{2x} is small in comparison with T_{1x} (where the subscript x denotes a derivative). This means that the second condition gives only a small correction (of the order of κ_1/κ_2) to the temperature distribution at the interface, and a given temperature T_2 has to be prescribed at the interface as a boundary condition for T_1 . From this, we see that we have the Dirichlet problem for the unknown function T_1 .

In the opposite limiting case, $\kappa_1 \gg \kappa_2$, we need to take the second condition in Equations (7.29) into account; that is, the heat flux is fixed at the interface for T_1 , which is the Neumann problem. Of course, the temperature is also continuous at the interface, which changes the temperature of the ambient medium. This leads to an additional correction to T_1 of the order of κ_2/κ_1 when we solve the boundary value problem for T_1 .

In the intermediate situation, $\kappa_1 \approx \kappa_2$, the temperature distribution in the ambient medium also should be obtained along with the solution of the boundary value problem for T_1 . But we can avoid that by assuming a simple linear relation between $T'_2(0)$ and the difference between the temperature for the interface, $T_2(\infty)$, and $T_2(0)$:

$$T'_2(0) = -\alpha [T_2(0) - T_2(\infty)]. \quad (7.30)$$

Often $T_2(\infty)$ is set to be zero. That is, the heat flux from the surface to the second medium is proportional to the local value of the temperature. Taking into account the relation in Equation (7.30), we may eliminate $T_2(0)$ and $T'_2(0)$ from the two boundary conditions (7.29) to find Newton's law (known also as the Biot condition) and to find the boundary condition for $T_1(0)$:

$$T'_1(0) + hT_1(0) = f, \quad h = \alpha \frac{\kappa_2}{\kappa_1}, \quad f = hT'_2(\infty). \quad (7.31)$$

We thus obtain the *mixed problem* for the function T_1 . Equation (7.31) can be easily obtained for one-dimensional cases (see the next reading exercise). A simple generalization of Equation (7.31) to the three-dimensional case leads to Newton's law of cooling:

$$\frac{\partial T_1}{\partial n} + hT_1 = f.$$

In fact, such a condition allows us to take into account the thermal properties of the ambient medium without solving the heat conduction problem

for this medium. This works well for the case of ambient gas in contact with a liquid or solid. However, there are some cases when the direct application of Newton's law leads to improper solutions and the heat transfer in the ambient gas should be taken into account. Notice also that the coefficient h (the heat transfer rate) characterizes not only the properties of contacting materials, but also the geometry of the ambient medium.

Reading Exercise. Consider the similar problem for the two-dimensional case with both $T_1(x, y)$ and $T_2(x, y)$ periodic with respect to y :

$$T_{1,2}(x, y) \sim e^{iky}.$$

Find the distribution of the temperature, which decays at infinity, at $x > 0$, and express constants h and f in the boundary condition (7.31) in terms of k and $\kappa_{1,2}$ by using boundary conditions (7.29).

Reading Exercise. A liquid layer is lying on a solid substrate ($x = 0$) of temperature T_0 and is exposed to the ambient gas above the liquid's free surface at $x = a$. Find the temperature distribution inside the layer by applying the Biot condition (7.31) with $f = 0$.

Reading Exercise. Solve the Dirichlet problem in the case of axial symmetry with boundary conditions $u(r = a) = u_1$, $u(r = b) = u_2$. The result will give a solution to the problem of a stationary distribution of heat between two cylinders with a common axis when the cylinders' surfaces are kept at constant temperatures. The same solution also gives the electric potential between two equipotential cylindrical surfaces. (The solution is a harmonic function between the surfaces with the axis, $r = 0$, excluded.)

Hint. Use Laplace's operator in cylindrical coordinates when there is no dependence on φ and z .

Reading Exercise. Solve the Dirichlet problem in the case of spherical symmetry with boundary conditions $u(r = a) = u_1$, $u(r = b) = u_2$. The result will give a solution to the problem of a static distribution of heat between two spheres with a common center when the surfaces are kept at constant temperatures. It also gives the electric potential between two equipotential spherical surfaces. As in the previous reading exercise, the solution is a harmonic function between the surfaces; the center at $r = 0$ is excluded.

Hint. Use Laplace's operator in spherical coordinates for the case of no dependence on φ and θ .

7.1.5 Principle of the Maximum and the Question of Well-Posed Boundary Value Problems

First let us formulate the *principle of the maximum*:

Theorem 7.1. *If the function u is harmonic in some domain bounded by the surface S , it reaches its maximal (minimal) value at the boundary.*

This principle is an obvious extension of the similar principle of the maximum valid for parabolic equations (see Section 5.2.1). Indeed, by choosing the solution of the boundary value problem for the Laplace's equation as an initial condition for the corresponding parabolic equation, we can conclude that this field does not depend on time. Therefore, the maximal value can be reached only at the boundaries, as stated here.

We will not prove this theorem rigorously, but here we present the main idea of the proof. Let us assume that u has a maximum value (similar consideration can be done for a minimal value) in some interior point M of the domain. In this case, the following necessary and sufficient conditions should be true at M :

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \quad (7.32)$$

$$\frac{\partial^2 u}{\partial x^2} \leq 0, \quad \frac{\partial^2 u}{\partial y^2} \leq 0, \quad \frac{\partial^2 u}{\partial z^2} \leq 0. \quad (7.33)$$

From the conditions in Equations (7.33), we see that $\nabla^2 u \leq 0$, and the equality is possible only if all three second derivatives are zero at M . It can be shown that such an equality cannot be reached at point M (which is an interior point); thus, the maximum value of u is possible only on the boundary.

An important consequence of this theorem is the following:

Theorem 7.1 (cont.). *The solution of the Laplace's equation is equal to zero when the boundary conditions are zero (for the Neumann problem any constant is also a solution).*

The proof of this result is obvious. The harmonic function, u , reaches its maximal value at the boundary; that is, the maximal value of u over the

entire domain is zero. Moreover, its minimal value is also realized at the boundary; that is, the minimal value of u is also zero. Therefore $u = 0$ in whole the domain.

We proceed next to a discussion of the well posed boundary value problem for Laplace's equation. A boundary value problem is said to be *well posed* if its solution *exists*, is *unique* and *continuously depends on the parameters* of the problem. The latter criterion means in our case that a small change in the boundary (and other, if any) conditions implies a small change in the solution; that is, *the solution is stable*. These demands clearly follow from the physical meaning of a boundary value problem.

Let us consider in more detail the interior Dirichlet problem. The function u has prescribed values on the boundary S enclosing volume V and we need to find the solution of Laplace's equation inside the volume V . Assuming that the function $u(x, y, z)$ is continuous in the closed region V bounded by S we can prove the uniqueness of the solution:

Theorem 7.2. *The first interior boundary value problem for Laplace's equation has a unique solution.*

To prove this theorem, assume that there are two different functions, u_1 and u_2 , both satisfying Laplace's equation in the volume V bounded by surface S , and that both are continuous and have the same values on S . From the previous discussion, the difference $u = u_1 - u_2$ is a continuous and harmonic function in the interior of V that is equal to zero on the boundary. As demonstrated earlier, we must have $u = 0$, from which it follows that $u_1 = u_2$. The continuous dependence of the solution on the boundary conditions can be proved by using the properties of harmonic functions.

Often one has to deal with discontinuous boundary conditions, in which case the useful Theorem 7.3 can be proved.

Theorem 7.3. *The first interior boundary value problem for Laplace's equation with piecewise continuous boundary conditions has a unique solution.*

For the exterior problem, an extra demand based on physical considerations is that a solution should be regular (finite) at infinity, as stated in Theorem 7.4.

Theorem 7.4. *The exterior first boundary value problem for Laplace's equation has a unique solution that is regular at infinity.*

Simple arguments based on clear physical examples show that for the three-dimensional case the solution $u(P) \rightarrow 0$ as $P \rightarrow \infty$, and for the two-dimensional case, $u(\infty) = \text{const.}$ An example of the three-dimensional case is the electrostatic potential of a point charge in a space, $\varphi = r^{-1}$. Another example is a sphere of radius R with a temperature that is held constant at u_0 . In this case, the function $u(r) = u_0 R/r$ is the solution of an exterior problem for Laplace's equation. For the two-dimensional problem, with the boundary of a circle of radius R assigned a constant boundary condition, $u|_L = f_0$, we have $u \equiv f_0$ as the unique solution of an exterior problem, and there is no solution that tends to zero at infinity.

Theorems similar to those for the first boundary condition can be proved for the second and third boundary value problem for elliptic equations. Also, notice that a solution of an elliptic boundary value problem is always at least as smooth (in the sense that it has a number of continuous derivatives) as the boundary conditions and other data of a problem. Usually a solution has an infinite number of derivatives in all interior points. This property of elliptic equations is related to the static character of the corresponding physical processes being modeled.

7.2 The Dirichlet Boundary Value Problem for Laplace's Equation in a Rectangular Domain

In the following sections, we extend the previous work on elliptic equations to include examples in several spatial domains. The solutions of elliptic equations for simple domains such as circles, spheres, cylinders, and rectangles can be obtained by using the Fourier method of separation of variables. All of these problems have important physical applications.

Boundary value problems for Laplace's equation in a rectangular domain can be solved with the method of separation of variables. We begin with the Dirichlet problem defined by

$$\nabla^2 u = 0, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad (7.34)$$

$$u(x, y)|_{x=0} = g_1(y), \quad u(x, y)|_{x=l_x} = g_2(y), \quad (7.35)$$

$$u(x, y)|_{y=0} = g_3(x), \quad u(x, y)|_{y=l_y} = g_4(x). \quad (7.36)$$

The boundaries $0, l_x, l_y$ in Equation (7.34) are not included in the domain to avoid a possible difference among the values of function $u(x, y)$ in the corners of the rectangular, given by Equations (7.35) and (7.36). Let us split the problem in Equations (7.34) through (7.36) into two parts, each of which has homogeneous boundary conditions in one variable. To proceed, we introduce

$$u(x, y) = u_1(x, y) + u_2(x, y), \quad (7.37)$$

where $u_1(x, y)$ and $u_2(x, y)$ are the solutions to the following problems on a rectangular boundary:

$$\nabla^2 u_1 = 0, \quad (7.38)$$

$$u_1(x, y)|_{x=0} = u_1(x, y)|_{x=l_x} = 0, \quad (7.39)$$

$$u_1(x, y)|_{y=0} = g_3(x), \quad u_1(x, y)|_{y=l_y} = g_4(x), \quad (7.40)$$

and

$$\nabla^2 u_2 = 0, \quad (7.41)$$

$$u_2(x, y)|_{y=0} = u_2(x, y)|_{y=l_y} = 0, \quad (7.42)$$

$$u_2(x, y)|_{x=0} = g_1(y), \quad u_2(x, y)|_{x=l_x} = g_2(y). \quad (7.43)$$

First, we consider the problem for the function $u_1(x, y)$ and search for a solution in the form

$$u_1(x, y) = X(x)Y(y). \quad (7.44)$$

Substituting Equation (7.44) into Laplace's equation and separating the variables yields

$$\frac{X''(x)}{X(x)} \equiv -\frac{Y''(y)}{Y(y)} = -\lambda. \quad (7.45)$$

We then obtain equations for $X(x)$ and $Y(y)$. With the homogeneous boundary conditions in Equation (7.39), we obtain the one-dimensional Sturm-Liouville problem for $X(x)$:

$$X'' + \lambda X = 0, \quad 0 < x < l_x,$$

$$X(0) = X(l_x) = 0.$$

If $X(x)$ is not identically equal to zero, the solution to this problem is

$$X_n = \sin \sqrt{\lambda_{xn}} x, \quad \lambda_{xn} = \left(\frac{\pi n}{l_x} \right)^2, \quad n = 1, 2, \dots \quad (7.46)$$

With these eigenvalues, λ_{xn} , we obtain an equation for $Y(y)$ from equation (7.45):

$$Y'' - \lambda_{xn}Y = 0, \quad 0 < y < l_y. \quad (7.47)$$

A general solution to this equation can be written in an obvious way as

$$Y_n = \tilde{C}_n^{(1)} \exp\left(\sqrt{\lambda_{xn}}y\right) + \tilde{C}_n^{(2)} \exp\left(-\sqrt{\lambda_{xn}}y\right), \quad (7.48)$$

but there is another way to represent the solution that makes the following analysis easier. Consider two functions:

$$Y_n^{(1)} = \frac{\sinh \sqrt{\lambda_{xn}}y}{\sinh \sqrt{\lambda_{xn}}l_y} \quad \text{and} \quad Y_n^{(2)} = \frac{\sinh \sqrt{\lambda_{xn}}(l_y - y)}{\sinh \sqrt{\lambda_{xn}}l_y}. \quad (7.49)$$

It is easily verified that they both satisfy Equation (7.47) and are linearly independent; thus, they can serve as a fundamental set of particular solutions for this equation. Also, they satisfy the following boundary conditions:

$$Y_n^{(1)}(0) = Y_n^{(2)}(l_y) = 0 \quad \text{and} \quad Y_n^{(1)}(l_y) = Y_n^{(2)}(0) = 1. \quad (7.50)$$

Thus, in addition to the solution given in Equation (7.48), a general solution of Equation (7.47) can be written as

$$Y_n = A_n Y_n^{(1)}(y) + B_n Y_n^{(2)}(y). \quad (7.51)$$

Reading Exercise. Transform the solution in Equation (7.48) to Equation (7.51), obtaining the constants $\tilde{C}_n^{(1,2)}$ in terms of A_n and B_n .

Hint. Use the formula $\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta$.

Now we can write a general solution of Laplace's equation satisfying the homogeneous boundary conditions at the boundaries $x = 0$ and $x = l_x$ in Equation (7.39), as a series in the functions $Y_n^{(1)}(y)$ and $Y_n^{(2)}(y)$:

$$u_1 = \sum_{n=1}^{\infty} \left[A_n Y_n^{(1)}(y) + B_n Y_n^{(2)}(y) \right] \sin \sqrt{\lambda_{xn}}x. \quad (7.52)$$

The coefficients of this series are determined from the boundary conditions given in Equation (7.41). By using the conditions from Equation (7.51), we obtain

$$\begin{aligned} u_1(x, y)|_{y=0} &= \sum_{n=1}^{\infty} B_n \sin \sqrt{\lambda_{xn}} x = g_3(x), \\ u_1(x, y)|_{y=l_y} &= \sum_{n=1}^{\infty} A_n \sin \sqrt{\lambda_{xn}} x = g_4(x). \end{aligned} \quad (7.53)$$

Expanding the functions $g_{3,4}(x)$ in a Fourier series with basis functions $\sin \sqrt{\lambda_{xn}} x$, we obtain

$$g_{3,4}(x) = \sum_{n=1}^{\infty} G_n^{(3,4)} \sin \sqrt{\lambda_{xn}} x, \quad (7.54)$$

where

$$G_n^{(3,4)} = \frac{2}{l_x} \int_0^{l_x} g_{3,4}(\xi) \sin \sqrt{\lambda_{xn}} \xi d\xi. \quad (7.55)$$

By substituting the expansion in Equation (7.54) into condition (7.53) and equating the coefficients of the same basis functions, we obtain

$$A_n = G_n^{(4)}, \quad B_n = G_n^{(3)}, \quad (7.56)$$

which completes the solution of the problem given in Equations (7.38) through (7.40).

Obviously, the solution of the similar problem given in Equations (7.41) through (7.43) can be obtained from Equations (7.52), (7.55), and (7.56) by replacing y for x and l_y for l_x . Carrying out this procedure yields

$$u_2 = \sum_{n=1}^{\infty} \left[C_n X_n^{(1)}(x) + D_n X_n^{(2)}(x) \right] \sin \sqrt{\lambda_{yn}} y, \quad (7.57)$$

where

$$X_n^{(1)}(x) = \frac{\sinh \sqrt{\lambda_{yn}} x}{\sinh \sqrt{\lambda_{yn}} l_x}, \quad X_n^{(2)}(x) = \frac{\sinh \sqrt{\lambda_{yn}} (l_x - x)}{\sinh \sqrt{\lambda_{yn}} l_x}, \quad \lambda_{yn} = \left(\frac{\pi n}{l_y} \right)^2,$$

and

$$C_n = \frac{2}{l_y} \int_0^{l_y} g_2(\xi) \sin \sqrt{\lambda_{yn}} \xi d\xi, \quad D_n = \frac{2}{l_y} \int_0^{l_y} g_1(\xi) \sin \sqrt{\lambda_{yn}} \xi d\xi. \quad (7.58)$$

Example 7.2. Find a steady-state temperature distribution inside a rectangular material that has boundaries maintained under the following conditions:

$$T(x, y)|_{x=0} = T_0 + (T_3 - T_0) \frac{y}{l_y}, \quad T(x, y)|_{x=l_x} = T_1 + (T_2 - T_1) \frac{y}{l_y}, \quad (7.59)$$

and

$$T(x, y)|_{y=0} = T_0 + (T_1 - T_0) \frac{x}{l_x}, \quad T(x, y)|_{y=l_y} = T_3 + (T_2 - T_3) \frac{x}{l_x}; \quad (7.60)$$

that is, at the corners of the rectangle, the temperatures are T_0, T_1, T_2, T_3 , and on the boundaries, the temperature is given by linear functions.

Solution. Introduce the function $u = T - T_0$ so that we measure the temperature relative to T_0 . Then $g_1(y) = (T_3 - T_1) \frac{y}{l_y}$, etc., and by evaluating the integrals in Equations (7.55) and (7.58), we obtain

$$A_n = \frac{2}{\pi n} [T_3 - T_0 - (-1)^n T_2], \quad B_n = -2 \frac{(-1)^n}{\pi n} (T_1 - T_0), \quad (7.61)$$

$$C_n = \frac{2}{\pi n} [T_1 - T_0 - (-1)^n T_2], \quad D_n = -2 \frac{(-1)^n}{\pi n} (T_3 - T_0). \quad (7.62)$$

These coefficients decay only as $1/n$; thus, the series in Equations (7.52) and (7.57) converge rather slowly.

From the discussion in the Section 7.1, we know the solution to this problem can also be obtained as a combination of harmonic functions x , y , and xy . Searching for the solution for $T = T_0 + u$ in the form

$$T = T_0 + c_1 x + c_2 y + c_3 xy,$$

and taking into account the first boundary conditions in Equations (7.59) and (7.60), we have:

$$c_1 = \frac{T_1 - T_0}{l_x}, \quad c_2 = \frac{T_3 - T_0}{l_y}.$$

Then, from the second condition in Equation (7.59), we obtain

$$T(x, y)|_{x=l_x} = T_1 + (T_2 - T_1) \frac{y}{l_y} = c_1 l_x + c_2 y + c_3 l_x y = T_1 + \left(\frac{T_3 - T_0}{l_y} + c_3 l_x \right) y,$$

which leads to

$$c_3 = \frac{T_0 + T_2 - T_1 - T_3}{l_x l_y}.$$

Let us check whether the second condition in Equation (7.60) is satisfied:

$$T(x, y)|_{y=l_y} = c_1 x + c_2 l_y + c_3 l_y x = \frac{T_1 - T_0}{l_x} x + T_3 + \frac{T_0 + T_2 - T_1 - T_3}{l_x} x, \quad (7.63)$$

which is equal the value $T_3 + (T_2 - T_3) x / x l_x l_x$, following from Equation (7.60). Thus, the analytical solution of the problem (not in the form of a Fourier series) is

$$T = T_0 + \frac{T_1 - T_0}{l_x} x + \frac{T_3 - T_0}{l_y} y + \frac{T_0 + T_2 - T_1 - T_3}{l_x l_y} xy. \quad (7.64)$$

Let us plot the isothermal lines (lines of equal temperature) using partial sums of the series (7.52) and (7.57) with coefficients given by Equations (7.61) and (7.62), using for simplicity $l_y = 2l_x$, $T_0 = T_2$, and $T_1 = T_3$. Moving to the dimensionless variables l_x for the unit of length, and $T_1 - T_0$ for the unit of temperature, and shifting the reference level of temperature by T_0 , we obtain:

$$l_x = 1, \quad l_y = 2, \quad T_0 = T_2 = 0, \quad T_1 = T_3 = 1.$$

Notice that the boundary conditions for functions $u_{1,2}$ in Equation (7.37) do not match at $x = l_x$, $y = 0$. We have

$$u_1(x = 0, y \rightarrow l_y) = 0, \quad u_1(x \rightarrow 0, y = l_y) = g_4(0) = T_3 = 1,$$

and

$$u_2(x = 0, y \rightarrow l_y) = g_1(l_y) = T_3 = 1, \quad u_2(x \rightarrow 0, y = l_y) = 0.$$

Also the boundary conditions for $u_{1,2}$ do not agree at $x = 0$, $y = l_y$ (the conditions at $x = l_x$, $y = l_y$ correspond to each other because we have

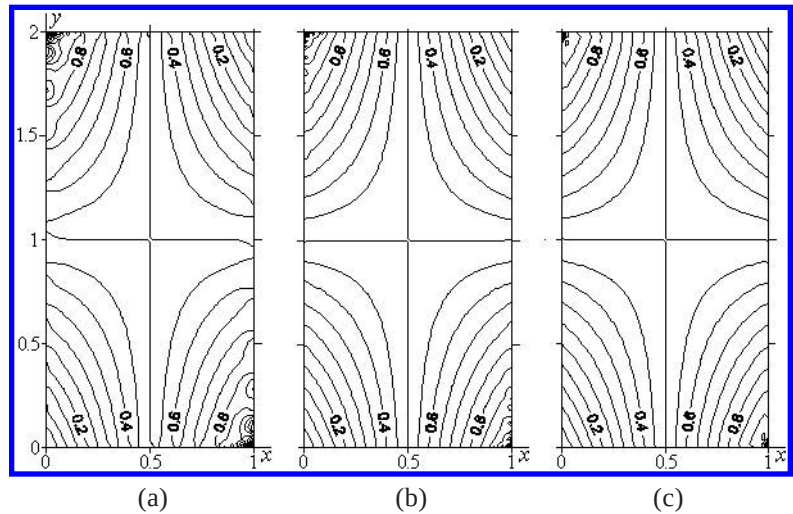


Figure 7.3. Temperature field keeping different numbers of terms in the partial sum: (a) $N = 20$, (b) $N = 50$, (c) $N = 100$.

chosen $T_2 = T_0$). As we saw, this makes the speed of convergence of the series substantially worse.

In Figure 7.3 we show the result of partial sums with the number of terms, N , equal to (a) 20; (b) 50; and (c) 100. We see that if we keep 20 terms, the results cannot be considered as satisfactory; unphysical oscillations seriously distort the temperature field. Keeping more terms makes the results much better, but near the point where the boundary conditions disagree with each other, the results are still not reasonable.

7.2.1 A Method to Improve the Series Convergence when Boundary Conditions Match Each Other

Notice that slow convergence of the series in Example 7.2 is related not to the problem itself (the boundary conditions in the original problem in Equations (7.59) and (7.60) do not disagree with each other), but to the method that was used. We also saw in this problem, however, that it was easy to construct a solution by using Equation (7.63). This observation suggests an approach that will cause the series to converge substantially faster when boundary conditions match each other. We develop this method in the following discussion.

Let us present the function we seek as the sum of two harmonic functions:

$$u = T + w, \quad (7.65)$$

where $T(x, y)$ is the simplest function which satisfies the following conditions at the corners of the rectangle:

$$\begin{aligned} T(x=0, y=0) &= g_1(0) = g_3(0) \equiv T_0, \\ T(x=l_x, y=0) &= g_2(0) = g_3(l_x) \equiv T_1, \\ T(x=l_x, y=l_y) &= g_2(l_y) = g_4(l_x) \equiv T_2, \\ T(x=0, y=l_y) &= g_1(l_y) = g_4(0) \equiv T_3. \end{aligned} \quad (7.66)$$

Clearly, this function can be chosen, for instance, in the form of Equation (7.64). Then for the function $w(x, y)$, we obtain the following boundary value problem:

$$\nabla^2 w = 0, \quad (7.67)$$

$$w(x, y)|_{x=0} = \tilde{g}_1(y), \quad w(x, y)|_{x=l_x} = \tilde{g}_2(y), \quad (7.68)$$

$$w(x, y)|_{y=0} = \tilde{g}_3(x), \quad w(x, y)|_{y=l_y} = \tilde{g}_4(x), \quad (7.69)$$

where $\tilde{g}_1(y) = g_1(y) - T(x=0, y)$, and similar obvious substitutions are made for the other functions.

The obtained boundary value problem in Equations (7.67) through (7.69) coincides with the original problem in Equations (7.34) through (7.36), but because

$$\tilde{g}_{1,2}(0) = \tilde{g}_{1,2}(l_y) = \tilde{g}_{3,4}(0) = \tilde{g}_{3,4}(l_x) = 0,$$

the corresponding series (7.52) and (7.57) for $w(x, y)$ converge considerably faster. Example 7.3 demonstrates the advantage of this approach.

Example 7.3. Find the temperature in a rectangle with the boundary conditions

$$u(x, y)|_{x=0} = 0, \quad u(x, y)|_{x=l_x} = T_0 \frac{y^2}{l_y^2}, \quad (7.70)$$

$$u(x, y)|_{y=0} = 0, \quad u(x, y)|_{y=l_y} = T_0 \frac{x^2}{l_x^2}. \quad (7.71)$$

Solution. Clearly, these boundary conditions are self-consistent at all corners. However, if we separate the solution into two functions, u_1 and u_2 , as in Equation (7.37), the boundary conditions for each problem are not in agreement.

Solving the problem in this way, we obtain Fourier coefficients of the series:

$$A_n = C_n = -2T_0 \frac{(-1)^n}{\pi n} - \frac{4T_0}{\pi^3 n^3} [1 - (-1)^n], \quad B_n = D_n = 0. \quad (7.72)$$

To increase the speed of convergence, let us present the solution in the form

$$u(x, y) = T_0 \frac{xy}{l_x l_y} + w(x, y), \quad (7.73)$$

where the first term is the solution in Equation (7.64) with T_0, T_1, T_3 omitted, and T_2 replaced by T_0 (compare the boundary conditions (7.70) and (7.71) with those in Equation (7.66)). For the auxiliary function $w(x, y)$ that satisfies Laplace's equation, we obtain the following boundary conditions:

$$\begin{aligned} w(x, y)|_{x=0} &= 0, & w(x, y)|_{x=l_x} &= T_0 \frac{y^2 - l_y y}{l_y^2}, \\ w(x, y)|_{y=0} &= 0, & w(x, y)|_{y=l_y} &= T_0 \frac{x^2 - l_x x}{l_x^2}. \end{aligned}$$

Expanding the functions $\tilde{g}_{2,4}$ in Fourier series, we obtain the coefficients of expansion of the function $w(x, y)$ in a manner similar to that for Equations (7.52) and (7.57):

$$\tilde{A}_n = \tilde{C}_n = -\frac{4T_0}{\pi^3 n^3} [1 - (-1)^n], \quad \tilde{B}_n = \tilde{D}_n = 0. \quad (7.74)$$

Comparing the Fourier coefficients in Equations (7.72) and (7.74), we see that explicitly separating out the term $T_0 xy/(l_x l_y)$ in Equation (7.73) allows us to remove the slowly converging part from the series.

The results of a numeric summation of partial sums when $l_y = 2l_x$ are presented on Figure 7.4. The solution is presented in dimensionless form with length in units of l_x and temperature in units of T_0 . It is seen that the direct solution in Equations (7.52) and (7.57) with the coefficients given by Equation (7.72) exhibit oscillations in the vicinity of the corner $x = l_x, y = l_y$ even if 50 terms in the series are kept. Contrary to this, the solution in Equation (7.73) looks smooth even if we keep only 10 (nonzero) terms in the partial sum ($N = 20$).

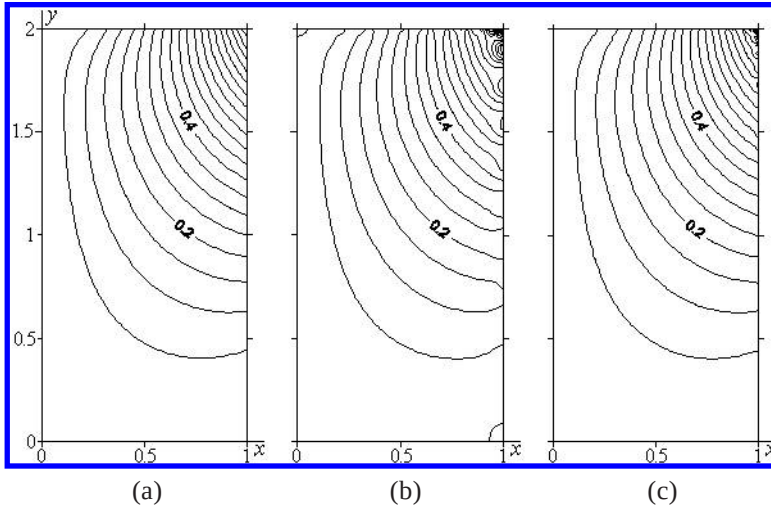


Figure 7.4. Temperature field given by Equation (7.73) for (a) $N = 20$; and direct solution with coefficients from Equation (7.72) for (b) $N = 20$ and (c) $N = 50$.

7.2.2 Other Types of Boundary Conditions

For other types of boundary conditions, Laplace's equation in a rectangular domain can be solved in the same way. However, the basis functions that serve as eigenfunctions in the series expansions may be different. Clearly, we need to discuss these different cases for only one of the functions, let us say u_1 , since the procedure for another function, u_2 , will be the same. Let us consider the problem in its most general form:

$$\nabla^2 u_1 = 0, \quad (7.75)$$

$$P_1[u_1] \equiv \alpha_1 \frac{\partial u_1}{\partial x} + \beta_1 u_1 \Big|_{x=0} = 0, \quad P_2[u_1] \equiv \alpha_2 \frac{\partial u_1}{\partial x} + \beta_2 u_1 \Big|_{x=l_x} = 0, \quad (7.76)$$

$$P_3[u_1] \equiv \alpha_3 \frac{\partial u_1}{\partial x} + \beta_3 u_1 \Big|_{y=0} = g_3(x), \quad P_4[u_1] \equiv \alpha_4 \frac{\partial u_1}{\partial x} + \beta_4 u_1 \Big|_{y=b} = g_4(x). \quad (7.77)$$

Separating the variables, we present the solution of the boundary value problem in Equations (7.75) through (7.77) as a superposition of functions

$$u_n(x, y) = X_n(x) Y_n(y).$$

Here, $X_n(x)$ is the solution of a Sturm-Liouville problem:

$$X_n'' + \lambda_{xn} X_n = 0, \quad (7.78)$$

$$P_1[X_n] = P_2[X_n] = 0. \quad (7.79)$$

Assuming $X_n(x)$ is not equal to zero identically, it is convenient to represent $Y_n(y)$ as a sum (for the moment with unidentified coefficients) of two particular solutions, $Y_n^{(1)}$ and $Y_n^{(2)}$, of the equation

$$Y_n'' - \lambda_{xn} Y_n = 0, \quad (7.80)$$

satisfying the following boundary conditions:

$$P_3[Y_n^{(1)}] = 0, \quad P_4[Y_n^{(1)}] \equiv 1, \quad (7.81)$$

$$P_3[Y_n^{(2)}] = 1, \quad P_4[Y_n^{(2)}] \equiv 0. \quad (7.82)$$

The form of functions $X_n(x)$ and $Y_n^{(1,2)}(y)$ depends on the boundary conditions, as discussed next.

Following the general idea of the method of separation of variables, we write the solution of the problem in Equations (7.75) through (7.77) as

$$u_1(x, y) = \sum_{n=1}^{\infty} \left[A_n Y_n^{(1)}(y) + B_n Y_n^{(2)}(y) \right] X_n(x).$$

(For some variants of boundary conditions, it is more convenient to number solutions starting with $n = 0$. In such cases in all formulas of this section, it is necessary only to replace the low limit of summation by zero to effect this alternative formulation.)

When we satisfy the boundary conditions at $y = 0$, we obtain, by using Equations (7.81) and (7.82),

$$P_3[u_1] = \sum_{n=1}^{\infty} \left\{ A_n P_3[Y_n^{(2)}] + B_n P_3[Y_n^{(2)}] \right\} X_n(x) = \sum_{n=1}^{\infty} B_n X_n(x) = g_3(x).$$

Similarly, at $y = l_y$ we obtain

$$\sum_{n=1}^{\infty} A_n X_n(x) = g_4(x).$$

Likewise, A_n and B_n are the coefficients of expansion of the functions $g_4(x)$ and $g_3(x)$ in the basis $\{X_n(x)\}$. The completeness of the set of functions gives:

$$A_n = \frac{1}{\|X_n^2\|} \int_0^{l_x} g_4(x) X_n(x) dx \text{ and } B_n = \frac{1}{\|X_n^2\|} \int_0^{l_x} g_3(x) X_n(x) dx,$$

where

$$\|X_n\|^2 = \int_0^{l_x} X_n^2(x) dx$$

is norm squared of the function $X_n(x)$.

The solutions of the Sturm-Liouville problem in Equations (7.78) and (7.79) for different boundary conditions are presented in Appendix A. The reader may also obtain these results as a reading exercise. The solutions of the problem in Equations (7.80) through (7.82) (with λ_{xn} found in the problem for $X_n(x)$) for different types of boundary conditions are presented in Appendix C. Again, we present the solutions and leave the proofs to the reader as a reading exercise.

7.2.3 Example of the Temperature Distribution in a Rectangular Domain

We will not discuss here the theoretical detail of the approach that affords an increase in the speed of convergence of the series when boundary conditions match each other on each side of the rectangle. Instead, we present an example showing how this approach works. An appropriate auxiliary function, $T(x, y)$, can be found in other problems by following a similar approach.

Let us find the solution of Laplace's equation in a rectangle under the following boundary conditions:

$$u(x, y)|_{x=0} = 0, \quad u(x, y)|_{x=l_x} = T_0 \frac{y}{l_y}, \quad (7.83)$$

$$u(x, y)|_{y=0} = 0, \quad \left. \frac{\partial u}{\partial y}(x, y) \right|_{y=l_y} = q(x) = q_0 (e^{\alpha x} - 1). \quad (7.84)$$

In this problem, the heat flux, $-\kappa q(x)$, is given on one boundary, and on the adjacent boundary a linear distribution of temperature is maintained. The boundary conditions at $x = l_x$, $y = l_y$ match each other if we choose

$$\frac{T_0}{l_y} = q(x = l_x) = q_0 (e^{\alpha l_x} - 1).$$

Using the formulas from Appendix C (Cases 1 and 2), we will search for the solution of this problem in the form

$$u = \sum_{n=1}^{\infty} A_n \frac{\sinh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y} \sin \sqrt{\lambda_{xn}} x + \sum_{n=0}^{\infty} B_n \frac{\sinh \sqrt{\lambda_{yn}} x}{\sinh \sqrt{\lambda_{yn}} l_x} \sin \sqrt{\lambda_{yn}} y, \quad (7.85)$$

where $\sqrt{\lambda_{xn}} = \pi n / l_x$ and $\sqrt{\lambda_{yn}} = \pi(2n + 1) / (2l_y)$. The coefficients of expansion in Equation (7.85) are

$$A_n = \frac{2}{l_x} \int_0^{l_x} q(x) \sin \sqrt{\lambda_{xn}} x dx = \frac{2q_0}{l_x} \left[\frac{\sqrt{\lambda_{xn}}}{\lambda_{xn} + \alpha^2} (1 - (-1)^n e^{\alpha l_x}) - \frac{1 - (-1)^n}{\sqrt{\lambda_{xn}}} \right], \quad (7.86)$$

$$B_n = \frac{2T_0}{l_y^2} \int_0^{l_y} y \sin \sqrt{\lambda_{yn}} y dy = 2T_0 \frac{(-1)^n}{\lambda_{yn} l_y^2}. \quad (7.87)$$

Let us plot the temperature distribution by using dimensionless parameters with l_x as the unit of length and T_0 as the unit of temperature. For illustrative purposes, consider $l_y = 2l_x$ and $\alpha = 1/l_x$. The series in Equation (7.85) with coefficients (7.87) and (7.88) converge sufficiently fast, as can be seen in Figure 7.5(a), where the temperature field is shown keeping only 20 terms in the partial sum. In Figure 7.5(a), this distribution cannot be distinguished from the exact solution. In contrast, a series for heat flow q when $q \sim u_y$ converges much slower. In Figures 7.5(b) and 7.5(c), field u_y is shown in the vicinity of the point $(x = l_x, y = l_y)$ where the convergence of the series is slow. In these figures, the partial sums of the series in Equation (7.85) with Equation (7.87) is limited to 20 and 50 terms.

We can avoid these undesirable oscillations in u_y by searching for a new function

$$u(x, y) = w(x, y) + T(x, y),$$

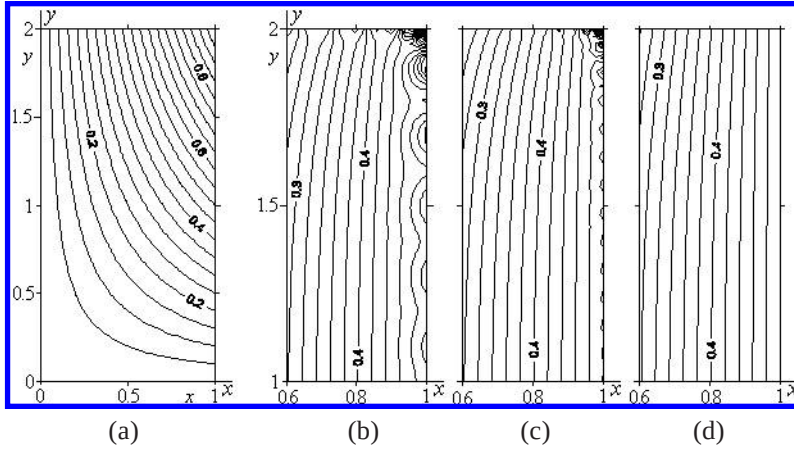


Figure 7.5. (a) Temperature field $u(x, y)$. Field $u_y(x, y)$ in the vicinity of $x = l_x$, $y = l_y$ obtained with Equations (7.85) through (7.87) for (b) $N = 20$ and (c) $N = 50$. (d) Field $u_y(x, y)$ obtained using Equation (7.88) for $N = 20$.

where $T(x, y)$ is a harmonic function defined from conditions:

$$T(x = 0, y = 0) = T(x = 0, y = l_y) = T(x = l_x, y = 0) = 0,$$

$$T(x = l_x, y = l_y) = -T_0, \quad T_y(x = l_x, y = l_y) = -q(x = l_x) = -\frac{T_0}{l_y}.$$

Such a function is easy to choose; for example, we may use

$$T = -T_0 \frac{xy}{l_x l_y},$$

which gives the following boundary value problem for $w(x, y)$:

$$\nabla^2 w = 0,$$

$$\left. \frac{\partial w}{\partial y} \right|_{x=0} = 0, \quad w(x, y)|_{x=l_x} = 0,$$

$$w(x, y)|_{y=0} = 0, \quad \left. \frac{\partial w}{\partial y} \right|_{y=l_y} = \tilde{q}(x) = q(x) - \frac{T_0 x}{l_x l_y}.$$

The solution to this problem is now evident:

$$w(x, y) = \sum_{n=1}^{\infty} \tilde{A}_n \frac{\sinh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y} \sin \sqrt{\lambda_{xn}} x.$$

The coefficient of expansion is

$$\tilde{A}_n = \frac{2}{l_x} \int_0^{l_x} \tilde{q}(x) \sin \sqrt{\lambda_{xn}} x dx = -\frac{2q_0 \alpha^2}{l_x \sqrt{\lambda_{xn}} (\lambda_{xn} + \alpha^2)}.$$

Thus the complete solution to Laplace's equation with boundary conditions given in Equations (7.83) and (7.84) is

$$u(x, y) = T_0 \frac{xy}{l_x l_y} + \sum_{n=1}^{\infty} \tilde{A}_n \frac{\sinh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y} \sin \sqrt{\lambda_{xn}} x. \quad (7.88)$$

For this solution, the convergence of the series for u_y is substantially better. In Figure 7.5(d), the solution has only 20 terms in a partial sum.

7.2.4 The Poisson Equation in a Rectangular Domain

Consider the first boundary value problem for the Poisson equation in a rectangular domain:

$$\begin{aligned} \nabla^2 u &= -f(x, y), \quad 0 < x < l_x, \quad 0 < y < l_y, \\ u(x, y)|_{x=0} &= g_1(y), \quad u(x, y)|_{x=l_x} = g_2(y), \\ u(x, y)|_{y=0} &= g_3(x), \quad u(x, y)|_{y=l_y} = g_4(x). \end{aligned} \quad (7.89)$$

Let us present a solution of this problem as a sum of the solution of Poisson's equation $u_p(x, y)$ satisfying the boundary value problem,

$$\nabla^2 u_p = -f(x, y), \quad (7.90)$$

$$u_p(x, y)|_{x=0} = u_p(x, y)|_{x=l_x} = 0, \quad (7.91)$$

$$u_p(x, y)|_{y=0} = u_p(x, y)|_{y=l_y} = 0 \quad (7.92)$$

and a partial solution of Laplace's equation $u_0(x, y)$:

$$\nabla^2 u_0 = 0, \quad (7.93)$$

$$u_0(x, y)|_{x=0} = g_1(y), \quad u_0(x, y)|_{x=l_x} = g_2(y), \quad (7.94)$$

$$u_0(x, y)|_{y=0} = g_3(x), \quad u_0(x, y)|_{y=l_y} = g_4(x). \quad (7.95)$$

A solution of boundary value problem in Equations (7.93) through (7.95) has already been discussed in the beginning of Section 7.2. Notice that for u_p we can choose any solution of the nonhomogeneous Equation (7.90) without taking into account the boundary conditions (7.91) and (7.92). In this case, we can change only the boundary conditions for function u_0 :

$$u_0(x, y)|_{x=0} = g_1(y) - u_p(0, y), \quad u_0(x, y)|_{x=l_x} = g_2(y) - u_p(l_x, y),$$

$$u_0(x, y)|_{y=0} = g_3(x) - u_p(x, 0), \quad u_0(x, y)|_{y=l_y} = g_4(x) - u_p(x, l_y).$$

In some cases, the above provides a convenient method for solving this problem (see Section 7.3 on circular domains), but here we present another approach to find a particular solution for u_p satisfying the boundary value problem in Equations (7.90) through (7.92). This solution can be composed as an expansion in eigenfunctions of the corresponding Sturm-Liouville problem. Let $\{\lambda_{nm}\}_{n,m=1}^{\infty}$ and $\{V_{nm}(x, y)\}_{n,m=1}^{\infty}$ be the solution of the following Sturm-Liouville problem:

$$\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right] + \lambda V = 0, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad (7.96)$$

$$V(x, y)|_{x=0} = 0, \quad V(x, y)|_{x=l_x} = 0, \quad (7.97)$$

$$V(x, y)|_{y=0} = 0, \quad V(x, y)|_{y=l_y} = 0. \quad (7.98)$$

The discussion in Section 7.2 allows us to conclude (and we leave it to the reader to check as a reading exercise) that the solution of this problem is

$$V_{nm} = \sin \sqrt{\lambda_{xn}} x \sin \sqrt{\lambda_{ym}} y, \quad \lambda_{xn} = \left(\frac{\pi n}{l_x} \right)^2, \quad \lambda_{ym} = \left(\frac{\pi m}{l_y} \right)^2, \quad (7.99)$$

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym}.$$

The eigenfunctions are orthogonal, and

$$\int_0^{l_x} dx \int_0^{l_y} dy V_{nm}(x, y) V_{pq}(x, y) = \frac{l_x l_y}{4} \delta_{np} \delta_{mq}.$$

A solution to the problem in Equations (7.90) through (7.92) can be expanded in a series by using these eigenfunctions:

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} V_{nm}(x, y), \quad (7.100)$$

where A_{nm} are coefficients to be determined. It is clear that because of the conditions in Equations (7.97) and (7.98), the homogeneous boundary conditions in Equation (7.92) and (7.93) are satisfied for each term in Equation (7.100).

Substitute the expansion in Equation (7.100) into Equation (7.90) and, because each function V_{nm} satisfies Equation (7.96), we obtain:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \left(\frac{\partial^2 V_{nm}}{\partial x^2} + \frac{\partial^2 V_{nm}}{\partial y^2} \right) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{nm} A_{nm} V_{nm} = -f. \quad (7.101)$$

Expand the right side of this equation in functions V_{nm} :

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm} V_{nm}(x, y), \quad (7.102)$$

where

$$f_{nm} = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} f(x, y) V_{nm}(x, y) dx dy. \quad (7.103)$$

By substituting this expansion into Equation (7.101) and equating the coefficients of each basis function V_{nm} , we find

$$A_{nm} = \frac{f_{nm}}{\lambda_{nm}}. \quad (7.104)$$

Because of Equation (7.99), the inequality $\lambda_{nm} > 0$ is valid for all n and m , which is why all the values A_{nm} are bounded. Equation (7.104) along with Equations (7.100) and (7.103) give the solution to the problem.

Solutions of the second and third type of boundary value problem for Poisson's equation can be obtained in a similar manner and we leave them to the reader as a reading exercise. Notice that for the Neumann problem, $\lambda_{00} = 0$ ($V_{00} = 1$), but the problem still has a solution. Indeed, for the

Neumann problem, the condition of

$$f_{00} = \frac{1}{l_x l_y} \int_0^{l_x} \int_0^{l_y} f(x, y) dx dy = 0 \quad (7.105)$$

should hold; thus, A_{00} cannot be determined from Equation (7.104). This is a consequence of the fact that an arbitrary constant can be added to a solution of the Neumann problem.

The condition in Equation (7.105) has a clear physical meaning: for the inner problem, there is no stationary solution of the heat equation if the average value of heat creation (or absorption) in the domain is not zero when a heat flux through the boundary is absent. As an illustration of this approach, let us consider Example 7.4.

Example 7.4. Find the heat distribution inside a rectangular domain if heat flow through the boundaries at $x = 0$, $x = l_x$ is absent, and at the boundaries $y = 0$, $y = l_y$ the same constant temperature is maintained. Inside the domain, a source and sink of heat exists such that the density of heat is Q_0 when $x < l_x/2$ and $-Q_0$ when $x > l_x/2$.

Solution. The relevant equations for the problem can be written as follows:

$$\begin{aligned} \nabla^2 T &= -f(x, y), \quad f = \frac{Q_0}{\kappa} \begin{cases} 1, & x < l_x/2 \\ -1, & x > l_x/2 \end{cases} \\ \frac{\partial T}{\partial x}(x, y) \Big|_{x=0} &= \frac{\partial T}{\partial x}(x, y) \Big|_{x=l_x} = 0, \\ T(x, y) \Big|_{y=0} &= T(x, y) \Big|_{y=l_y} = 0. \end{aligned}$$

Here, $\kappa = \rho c \chi$ is the coefficient of heat conductivity.

First we move to dimensionless variables, taking l_x for the unit of length and $2Q_0 l_x^2 / \kappa$ for the unit of temperature. Keeping the same notation for the variables, the problem with dimensionless variables becomes:

$$\nabla^2 T = -f(x, y), \quad f = \frac{1}{2} \begin{cases} 1, & x < 1/2 \\ -1, & x > 1/2 \end{cases} \quad (7.106)$$

$$\frac{\partial T}{\partial x}(x, y) \Big|_{x=0} = \frac{\partial T}{\partial x}(x, y) \Big|_{x=1} = 0, \quad (7.107)$$

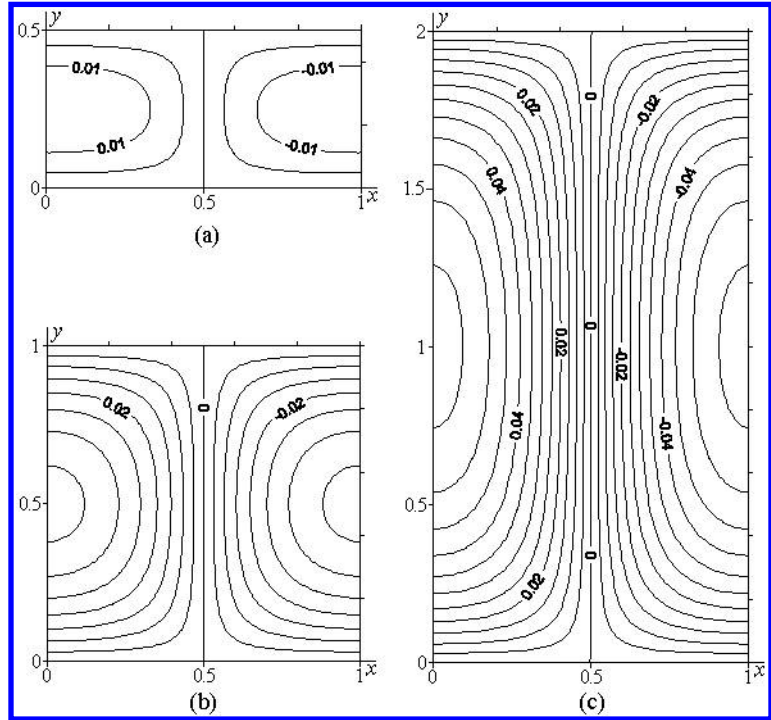


Figure 7.6. Temperature distribution in a rectangular domain, with (a), (b), and (c) corresponding to values $L = 0.5, 1$, and 2 , respectively. Isotherms are drawn at unit-less intervals of 0.005 for all three cases.

$$T(x, y)|_{y=0} = T(x, y)|_{y=L} = 0 \quad (7.108)$$

(here $L = l_y/l_x$). Let us search for the solution as

$$T = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} V_{nm}(x, y),$$

where $V_{nm}(x, y)$ are the solutions of the corresponding Sturm-Liouville problem:

$$V_{nm} = \cos \sqrt{\lambda_{xn}} x \sin \sqrt{\lambda_{ym}} y, \quad \lambda_{xn} = (\pi n)^2, \quad \lambda_{ym} = \left(\frac{\pi m}{L}\right)^2, \quad n = 0, 1, \dots, m = 1, 2, \dots$$

Determining the coefficients of the expansion of function $f(x, y)$ in a series with functions $V_{nm}(x, y)$ using Equation (7.103), we obtain that

$f_{nm} = 0$ when at least one of the indices n or m is even. For $n = 2k + 1$ and $m = 2j + 1$, we obtain

$$f_{2k+1,2j+1} = \frac{8(-1)^k}{\pi^2 (2k+1)(2j+1)}.$$

Thus, the solution to the problem in Equations (7.106) through (7.108) is the series

$$T = \frac{8}{\pi^4} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k \cos \pi (2k+1) x \sin \pi (2j+1) y / L}{(2k+1)(2j+1) [(2k+1)^2 + (2j+1)^2 / L^2]}. \quad (7.109)$$

Numerical sums of the series in Equation (7.109) are shown in Figure 7.6 for three values of $L = l_y / l_x$. As we see, as the value of L increases (i.e., the distance between bottom and top sides having the same value of temperature increases), the deviation of temperature from this value gets larger.

7.3 Laplace's and Poisson's Equations for Two-Dimensional Domains with Circular Symmetry

In this section, we consider two-dimensional problems that have a symmetry that allows the use of polar coordinates. Solutions to these problems contain simple trigonometric functions of the polar angle as well as power and logarithmic functions of the radius.

7.3.1 The Fourier Method for Laplace's Equation in Polar Coordinates

In polar coordinates, (r, φ) , the Laplacian has the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \equiv \nabla_r^2 + \frac{1}{r^2} \nabla_\varphi^2,$$

where we have introduced notations to indicate the radial, ∇_r^2 , and angular, ∇_φ^2 , parts of the Laplacian derivative, ∇^2 . We begin by solving the homogeneous equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0 \quad (7.110)$$

by using the Fourier method of separation of variables. First, we represent $u(r, \varphi)$ in the form

$$u(r, \varphi) = R(r)\Phi(\varphi). \quad (7.111)$$

by substituting Equation (7.111) into Laplace's equation and separating the variables, we have

$$\frac{r^2 \nabla_r R}{R} = -\frac{\nabla_\varphi^2 \Phi}{\Phi}.$$

Because the first term does not depend on the angular variable φ and the second does not depend on r , each term must equal a constant. Denoting this constant as p , we have

$$\frac{r^2 \nabla_r R}{R} = -\frac{\nabla_\varphi^2 \Phi}{\Phi} \equiv p.$$

From here we obtain two separate equations for $R(r)$ and $\Phi(\varphi)$:

$$\Phi'' + p\Phi = 0,$$

$$r^2 R'' + rR' - pR = 0.$$

Consider first the equation for $\Phi(\varphi)$. For the following discussion, we assume that the variable φ varies from 0 to 2π . The case when φ varies over a smaller domain, $0 \leq \varphi < \alpha < 2\pi$, corresponds to the solution of Laplace's equation in a sector, which will be considered in Section 7.3.5. For the case $0 \leq \varphi < 2\pi$, a single-valued solution has to be periodic in φ with period 2π . Therefore, to find function $\Phi(\varphi)$ we have the one-dimensional Sturm-Liouville problem:

$$\Phi'' + p\Phi = 0, \quad 0 \leq \varphi < 2\pi,$$

with the periodicity condition

$$\Phi(\varphi + 2\pi) = \Phi(\varphi) \quad \text{for any } \varphi.$$

This periodicity is possible only when $n = \sqrt{p}$ is an integer; thus, for $\Phi(\varphi)$, the solutions will have a discrete spectrum of eigenvalues

$$p_n = n^2, \quad n = 0, 1, 2, \dots$$

with eigenfunctions

$$\Phi = \Phi_n(\varphi) = \begin{cases} \cos n\varphi, \\ \sin n\varphi. \end{cases}$$

Note that negative values of n correspond to the same eigenfunctions and therefore need not be included in the list of eigenvalues.

The eigenvalues, p_n , imply the following form for the equation for $R(r)$:

$$r^2 R'' + r R' - n^2 R = 0,$$

which is known as the Euler equation. The general solution to this equation is

$$\begin{aligned} R &= R_n(r) = C_1 r^n + C_2 r^{-n}, \quad n \neq 0, \\ R_0(r) &= C_1 + C_2 \ln r, \quad n = 0. \end{aligned} \quad (7.112)$$

Combining the above results, we obtain the following particular and general solutions of Laplace's equation:

1. Under the condition that the solution be finite at $r = 0$ and infinite at $r \rightarrow \infty$, we have

$$u_n(r, \varphi) = r^n \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases}, \quad n = 0, 1, \dots$$

By using Equation (7.112), we can write a general solution for Laplace's problem for a disk, $0 \leq r \leq a$, as the expansion with these particular solutions:

$$u(r, \varphi) = \sum_{n=0}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi).$$

The term with $n = 0$ is more conveniently written as $A_0/2$; thus, we have

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.113)$$

2. For the case in which the solution is infinite at $r = 0$ and finite at $r \rightarrow \infty$, we have

$$u_n(r, \varphi) = \frac{1}{r^n} \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases}, \quad n = 0, 1, \dots$$

These functions may be used as solutions to Laplace's problem for regions outside of a circle. The general solution of Laplace's equation for such an *exterior boundary value problem* ($r \geq a$), limited (i.e., bounded) at infinity, can be written as

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{1}{r^n} (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.114)$$

3. We also have a third set of solutions, for the cases in which the solution is unbounded as $r \rightarrow 0$, as well as $r \rightarrow \infty$:

$$1, \ln r, r^n \begin{Bmatrix} \cos n\varphi \\ \sin n\varphi \end{Bmatrix}, \frac{1}{r^n} \begin{Bmatrix} \cos n\varphi \\ \sin n\varphi \end{Bmatrix}, \quad n = 1, 2, \dots \quad (7.115)$$

This set is used to solve Laplace's equation for regions that form a circular ring or annulus, $a \leq r \leq b$.

7.3.2 Laplace's Equation for Interior Boundary Value Problems for a Circle

In this section, we look at specific solutions for the first of the three cases presented in Section 7.3.1. To begin, let us solve the boundary value problem for a circle or disk:

$$\nabla^2 u = 0 \text{ in } 0 \leq r < l, \quad (7.116)$$

with boundary conditions

$$u(r, \varphi)|_{r=l} = f(\varphi). \quad (7.117)$$

Applying the boundary condition (7.116) to formula (7.113), we obtain

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} l^n (A_n \cos n\varphi + B_n \sin n\varphi) = f(\varphi). \quad (7.118)$$

We thus see that $l^n A_n$ and $l^n B_n$ are the Fourier coefficients of expansion of the function $f(\varphi)$ in the system (or basis) of trigonometric functions $\{\cos n\varphi, \sin n\varphi\}$. We may evaluate the coefficients by using the formulas

$$A_n l^n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos n\varphi d\varphi, \quad B_n l^n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin n\varphi d\varphi, \quad n = 0, 1, 2, \dots \quad (7.119)$$

Thus, the solution of the *Dirichlet problem* for Laplace's equation is

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{l}\right)^n (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.120)$$

It is obvious from both the linearity of the boundary value problem in Equations (7.116) and (7.117) and from the form of solution given by Equations (7.119) and (7.120) that for $f(\varphi) = f_1(\varphi) + f_2(\varphi)$ the solution is the sum of the partial solutions corresponding to $f_1(\varphi)$ and $f_2(\varphi)$ *separately*. This method can obviously be applied for all boundary value problems under consideration.

Example 7.5. Find the temperature distribution inside a circle if the boundary is kept at the temperature $T_0 = C_1 + C_2 \cos \varphi + C_3 \sin 2\varphi$.

Solution. It is obvious that for this particular case the series given by Equation (7.118) reduces to three nonzero terms:

$$A_0 = 2C_1, \quad lA_1 = C_2, \quad l^2B_2 = C_3.$$

In this case, the solution given by Equation (7.120) is

$$T = C_1 + C_2 \frac{r}{l} \cos \varphi + C_3 \left(\frac{r}{l}\right)^2 \sin 2\varphi.$$

Similarly, we can obtain solutions of the second and third boundary value problems for Laplace's equation in a circle. We leave it to the reader as reading exercises to check that the resulting formulas are correct for the following two cases of Neumann and mixed boundary conditions.

The *Neumann problem* for Laplace's equation with boundary condition

$$\left. \frac{\partial u}{\partial r} \right|_{r=l} = f(\varphi) \quad (7.121)$$

has the solution

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{r^n}{nl^{n-1}} (A_n \cos n\varphi + B_n \sin n\varphi) + C, \quad (7.122)$$

where C is an arbitrary function. Note that a solution of the interior Neumann problem can exist only under the condition

$$\int_{C_l} f dl = \int_0^{2\pi} f(\varphi) d\varphi = 0.$$

This condition is necessary and sufficient; the function $u(r, \varphi)$ in such a case is not unique because any arbitrary constant C can be added to this solution.

The *mixed problem* for Laplace's equation with boundary condition

$$\left. \frac{\partial u}{\partial r} + hu \right|_{r=l} = f(\varphi), \quad h = \text{const} \quad (7.123)$$

has the solution

$$u(r, \varphi) = \frac{A_0}{2h} + \sum_{n=1}^{\infty} \frac{r^n}{(n + lh)l^{n-1}} (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.124)$$

Coefficients in the expansions in Equations (7.120), (7.122), and (7.124) are determined by using Equation (7.119).

As can be seen from Equation (7.119), for the case of homogeneous boundary conditions ($f(\varphi) = 0$), all three problems have only trivial solutions (i.e., equal to zero, or for the Neumann problem, equal to any constant). This result is clear from a physical point of view and follows from the uniqueness of the solution. That is, it is obvious that $u = 0$ is a solution for the homogeneous boundary conditions, and the solution $u = C$ is the solution of the Neumann problem.

Let us briefly discuss the convergence of the series (7.120) (or Equations (7.122) and (7.124)). If the function $f(\varphi)$ defining the boundary condition can be integrated absolutely, its Fourier coefficients are bounded and, as can be seen from the structure of these series, they converge in any interior point of the circle ($r < l$). The convergence will not be worse than a geometric progression with the ratio $q = r/l$, as we show in the following discussion. The smoother the function $f(\varphi)$ is, the faster these series converge. The series can be differentiated term by term any number of times, and the sums satisfy Laplace's equation (i.e., they are harmonic functions). The same can be said for the problems discussed in the following sections of this chapter.

An approximate value of $u(r, \varphi)$ can be obtained by keeping the first N terms in Equation (7.120) (or Equations (7.122) or (7.124)). The absolute error of this approximation is

$$\varepsilon = \left| \sum_{n=N+1}^{\infty} \left(\frac{r}{l} \right)^n (A_n \cos n\varphi + B_n \sin n\varphi) \right| \leq \sum_{n=N+1}^{\infty} \left(\frac{r}{l} \right)^n (|A_n| + |B_n|).$$

Because the terms in the series monotonically decrease, at $r < a$ the last expression is

$$\sum_{n=N+1}^{\infty} \left(\frac{r}{l}\right)^n (|A_n| + |B_n|) \leq (|A_{N+1}| + |B_{N+1}|) \sum_{n=N+1}^{\infty} \left(\frac{r}{l}\right)^n = (|A_{N+1}| + |B_{N+1}|) \left(\frac{r}{l}\right)^{N+1} \frac{1}{1 - \frac{r}{l}}.$$

The last expression was obtained as a sum of the geometric progression. From this we see that the values of ε are smaller for smaller values of r . Therefore, these series converge *faster* in any internal point of the domain than on the boundary.

Example 7.6. Let $u(r, \varphi)|_{r=a} = \sin(\varphi/2)$ at $0 \leq \varphi < 2\pi$, $l = 10$. Keeping six terms in Equation (7.120), find the temperature at several points of the circle: $P_1(0, \varphi)$, $P_2(2, \pi/18)$, and $P_3(3, \pi/18)$ in polar coordinates (i.e., $P_n(r, \varphi)$ where r and φ are polar coordinates).

Solution. The coefficients in Equation (7.118) are

$$A_0 = \frac{4}{\pi}, \quad A_n = \frac{4}{\pi(1 - 4n^2)}$$

(for $n=1, 2, \dots$), and $B_n = 0$. Thus, we have the expansion

$$u(r, \varphi) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left(\frac{r}{l}\right)^n \frac{4}{\pi(1 - 4n^2)} \cos n\varphi. \quad (7.125)$$

At point $P_1(0, 0)$, the temperature is $u(0, 0) = 2/\pi$ and the error is zero because $r = 0$. At point $P_2(2, \pi/18)$, keeping six terms in the partial sum and rounding off the result with accuracy, $\varepsilon_1 = 10^{-4}$, we obtain $u(2, \pi/18) = 0.5496$ and

$$\varepsilon \leq \left| \left(\frac{2}{10}\right)^7 \frac{4}{\pi(1 - 4 \cdot 7^2)} \frac{1}{1 - \frac{2}{10}} \right| \approx 0.5 \cdot 10^{-5}.$$

The total error is $\varepsilon + \varepsilon_1 \approx 10^{-4}$.

Similarly, at point $P_3(3, \pi/18)$, $u(3, \pi/18) = 0.5031$, $\varepsilon \approx 10^{-4}$, and $\varepsilon + \varepsilon_1 \approx 2 \cdot 10^{-4}$.

Example 7.7. Find the temperature at the point $P_2(2, \pi/18)$ in Example 7.6 with an accuracy of 10^{-5} .

Solution. This is the inverse question to Example 7.6. First, we have to decide how many terms it is necessary to keep in the partial sum. Keeping only two, we obtain

$$\varepsilon \leq \left| \left(\frac{2}{10} \right)^3 \frac{4}{\pi(1 - 4 \cdot 3^2)} \frac{1}{1 - \frac{2}{10}} \right| \approx 10^{-3}.$$

This is insufficient accuracy. By including a few more terms, we find that $N = 6$ gives $\varepsilon \approx 0.5 \cdot 10^{-5}$, which is sufficient. Thus, we have to keep six terms and to round off the result with an accuracy of 10^{-5} . The result is $u(2, \pi/18) = 0.54956$.

It is important to highlight that in the case of a *piecewise continuous boundary function* $f(\varphi)$, the solutions of the boundary value problems exist and approach the values of $f(\varphi)$ continuously at points of continuity of this function. At a discontinuity point of the function $f(\varphi)$ the Fourier series, as was discussed in Chapter 1, converges to one half the sum of the left and right limits.

Example 7.8. Consider the following boundary problem. Let an infinite homogenous cylinder with a circular surface of radius a be kept at a constant temperature

$$u(r, \varphi)|_{r=l} = \begin{cases} T_0, & 0 \leq \varphi < \pi, \\ -T_0, & \pi \leq \varphi < 2\pi \end{cases} \quad (7.126)$$

for any z . After a long period of time, the temperature inside the cylinder will become constant (i.e., the system reaches equilibrium). Find the temperature distribution inside the cylinder when this occurs.

Solution. Clearly, we need to solve the Dirichlet interior problem for a circle. The solution to this problem is given by Equation (7.120), and from the conditions in Equation (7.119), it is clear that

$$A_n = 0, \quad B_{2k} = 0, \quad B_{2k+1} = \frac{4T_0}{\pi(2k+1)}, \quad k = 0, 1, 2, \dots \quad (7.127)$$

The results of numerical summation of the series in Equation (7.120), with N terms are presented in Figures 7.7, 7.8 and 7.9 by using the dimension-

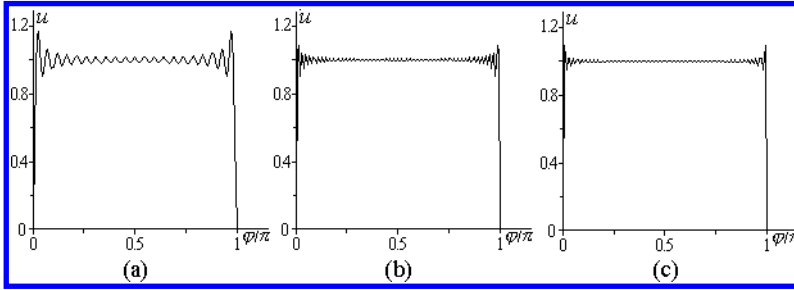


Figure 7.7. Temperature distribution on the circle boundary ($r = 1$) obtained with partial sums of the series in Equation (7.120) for (a) $N = 20$, (b) $N = 50$, and (c) $N = 100$ terms.

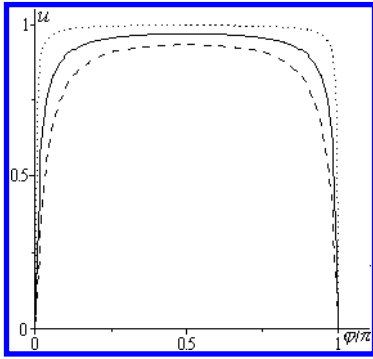


Figure 7.8. Temperature, $u(\varphi)$, at $r = 0.9, 0.95$, and 0.99 (dashed, solid, and dotted lines, respectively).

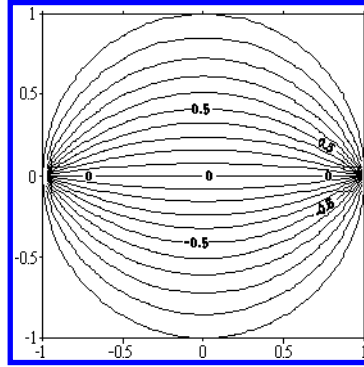


Figure 7.9. Temperature distribution, u , inside the cylinder.

less variables of length in units of the radius l and temperature in units of T_0 . Due to the symmetry of the problem, we have

$$u(r, \pi - \varphi) = -u(r, \pi + \varphi),$$

and we can search for the solution in the half-domain ($\varphi \leq \pi$).

Figure 7.7 shows the temperature distribution on the boundary of the cylinder for different values of N . As can be seen from the graphs, when $\varphi \neq 0, \pi$ the series converges uniformly, but closer to these points we have to keep more terms to keep the same precision. It is clear that at the discontinuity points $\varphi = 0, \pi$ the series gives a value of zero for the temperature.

Close to the surface, the series in Equation (7.120) converges for any φ . As shown in Figure 7.8, the functions are smooth, including as $r \rightarrow 1$, where the function $u(r, \varphi)$ approaches the discontinuous function defined by the expression in Equation (7.126). Temperature fields inside the body are presented in Figure 7.9.

Note that for the piecewise continuous boundary function, $f(\varphi)$, the series converges more slowly than for the continuous case. It should be obvious that $|B_n| \sim n^{-1}$ at large n for Example 7.8 (see Equation (7.127)), whereas $|B_n| \sim n^{-2}$ in Equation (7.125), where there is a jump discontinuity in the first derivative at $\varphi = 0$. We also see that $|B_n| \sim n^{-3}$ or smaller for the continuously differentiable function $f(\varphi)$. Thus, given Equation (7.125), we see we need more terms in a series for a piecewise continuous function, $f(\varphi)$, to ensure the same accuracy of the solution.

Reading Exercise. Solve Example 7.7 again but with discontinuous boundary conditions $u(r, \varphi)|_{r=a} = \sin(\varphi/2)$ for $0 < \varphi < \pi$ and $u(r, \varphi)|_{r=a} = \cos(\varphi/2)$ for $\pi < \varphi < 2\pi$. Find the temperature at the point $P_2(2, \pi/18)$ with an accuracy of 10^{-5} . How many terms should you keep to satisfy this accuracy?

Reading Exercise. Consider examples of discontinuous boundary conditions with big and small jumps at discontinuity points—for instance $u(r, \varphi)|_{r=a} = A$ for $0 < \varphi < \pi$ and $u(r, \varphi)|_{r=a} = B$ for $\pi < \varphi < 2\pi$. Using different constant values of A and B , check that the number of terms necessary to keep the same accuracy at some given point P depends on the value of the difference $|A - B|$.

Hint. This problem can be easily reduced to the problem solved in the previous Example 7.8. See also Example 7.11 in Section 7.3.9 for help.

Along with analytical solutions, you may use the program Laplace to solve these and other problems and to make useful estimations. Directions for using the program Laplace are found in Appendix E.

7.3.3 Laplace's Equation for Exterior Boundary Value Problems for a Circle

Here we continue with the solution of the second boundary value problem presented at the beginning of this chapter. This problem is formulated as

$$\nabla^2 u = 0 \text{ for } r > l.$$

Applying the boundary condition $u|_{r=l} = f(\varphi)$ for Equation (7.114), we directly obtain the solution of the Dirichlet problem as

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{l}{r}\right)^n (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.128)$$

The Neumann problem with boundary condition

$$\left. \frac{\partial u}{\partial r} \right|_{r=l} = f(\varphi) \quad (7.129)$$

has the solution

$$u(r, \varphi) = - \sum_{n=1}^{\infty} \frac{1}{n} \frac{l^{n+1}}{r^n} (A_n \cos n\varphi + B_n \sin n\varphi) + C, \quad (7.130)$$

where C is an arbitrary constant. Recall that the exterior Neumann problem for a plane has a solution only under the condition

$$\int_{C_l} f dl = \int_0^{2\pi} f(\varphi) d\varphi = 0$$

and its solution has an arbitrary additive constant.

The mixed problem, with a boundary condition

$$\left. \frac{\partial u}{\partial r} - hu \right|_{r=l} = f(\varphi), \quad (7.131)$$

has the solution

$$u(r, \varphi) = -\frac{A_0}{2h} - \sum_{n=1}^{\infty} \frac{l^{n+1}}{(n+lh)r^n} (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.132)$$

Notice that different signs for h (which was assumed to be positive, $h > 0$, in the discussion in Chapter 2) in Equation (7.123) as compared to Equation (7.131) are due to different directions of the vector normal to the boundary. For the interior problem, this vector is directed outward, for the exterior problem it is inward.

Coefficients A_n and B_n in the series (7.128), (7.130) and (7.132) are the Fourier coefficients of function $f(\varphi)$ and are calculated with Equations (7.119). We leave it to the reader as useful reading exercises to prove the results in Equations (7.128), (7.130) and (7.132).

7.3.4 Laplace's Equation for Boundary Value Problems for an Annulus

Let us work now on the boundary value problem for Laplace's equation for a ring or annular region. First, we consider the Dirichlet problem:

$$\nabla^2 u = 0 \text{ on an annulus } a < r < b, \quad (7.133)$$

$$u|_{r=a} = f_1(\varphi), \quad u|_{r=b} = f_2(\varphi). \quad (7.134)$$

The solution of this problem can be obtained by using the eigenfunction expansion of Equation (7.115). Calculations can be made substantially simpler, however, if, for each n , we consider the set of fundamental solutions $\{R_n^{(a)}(r), R_n^{(b)}(r)\}$ of the "radial" equation

$$r^2 R'' + r R' - n^2 R = 0, \quad (7.135)$$

obeying homogeneous boundary conditions

$$R_n^{(a)}(a) = 0, \quad R_n^{(b)}(b) = 0.$$

A general solution of Equation (7.135) is

$$R = C_1 + C_2 \ln r \text{ when } n = 0,$$

and

$$R = C_1 r^n + C_2 r^{-n} \text{ when } n \neq 0,$$

and by taking appropriate values for the coefficients C_1 and C_2 , we obtain the necessary solutions. For $n = 0$, we can choose

$$R_0^{(a)}(r) = \ln \frac{r}{a}, \quad R_0^{(b)}(r) = \ln \frac{b}{r},$$

and for $n \neq 0$,

$$R_n^{(a)}(r) = \frac{r^{2n} - a^{2n}}{r^n}, \quad R_n^{(b)}(r) = \frac{b^{2n} - r^{2n}}{r^n}.$$

With these functions, $R_n^{(a)}$ and $R_n^{(b)}$, we have particular solutions of Laplace's equation, given by

$$u_n^{(a)}(r, \varphi) = R_n^{(a)}(r) \begin{Bmatrix} \cos n\varphi \\ \sin n\varphi \end{Bmatrix}, \quad u_n^{(b)}(r, \varphi) = R_n^{(b)}(r) \begin{Bmatrix} \cos n\varphi \\ \sin n\varphi \end{Bmatrix}, \quad n \neq 0, \quad (7.136)$$

$$u_0^{(a)}(r, \varphi) = \ln \frac{r}{a}, \quad u_0^{(b)}(r, \varphi) = \ln \frac{b}{r}, \quad n = 0.$$

These solutions are finite inside the annulus and satisfy the boundary conditions

$$u_n^{(a)} \Big|_{r=a} = 0, \quad u_n^{(b)} \Big|_{r=b} = 0. \quad (7.137)$$

Notice that

$$u_n^{(a)} \Big|_{r=b} \neq 0, \quad u_n^{(b)} \Big|_{r=a} \neq 0.$$

Now we can obtain the solution of the boundary value problem of Equations (7.133) and (7.134) as a series in these particular solutions:

$$\begin{aligned} u(r, \varphi) = & \frac{A_0 \ln(r/a)}{2 \ln(b/a)} + \frac{C_0 \ln(b/r)}{2 \ln(b/a)} + \sum_{n=1}^{\infty} \frac{(r^{2n} - a^{2n}) b^n}{(b^{2n} - a^{2n}) r^n} (A_n \cos n\varphi + B_n \sin n\varphi) \\ & + \sum_{n=1}^{\infty} \frac{(b^{2n} - r^{2n}) a^n}{(b^{2n} - a^{2n}) r^n} (C_n \cos n\varphi + D_n \sin n\varphi). \end{aligned} \quad (7.138)$$

Substituting Equation (7.138) into the boundary condition $u|_{r=a} = f_1(\varphi)$ with Equation (7.137), we obtain

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} (C_n \cos n\varphi + D_n \sin n\varphi) = f_1(\varphi).$$

We can then find coefficients C_n and D_n :

$$C_n = \frac{1}{\pi} \int_0^{2\pi} f_1(\varphi) \cos n\varphi d\varphi, \quad D_n = \frac{1}{\pi} \int_0^{2\pi} f_1(\varphi) \sin n\varphi d\varphi. \quad (7.139)$$

Similarly, substituting Equation (7.138) into the boundary condition $u|_{r=b} = f_2(\varphi)$, we can find coefficients A_n and B_n :

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f_2(\varphi) \cos n\varphi d\varphi, \quad B_n = \frac{1}{\pi} \int_0^{2\pi} f_2(\varphi) \sin n\varphi d\varphi. \quad (7.140)$$

As we can see, obtaining the radial functions $R_n^{(a)}$ and $R_n^{(b)}$, each satisfying separately the necessary homogeneous boundary conditions at $r = a$ and $r = b$, was a useful first step to solve the boundary value problem.

We may follow a similar method to solve other types of boundary value problems for Laplace's equation for an annulus. Consider, for instance, the Neumann problem. For $n \neq 0$, the necessary pair of fundamental solutions is

$$R_n^{(a)}(r) = \frac{r^{2n} + a^{2n}}{r^n}, \quad R_n^{(b)}(r) = \frac{r^{2n} + b^{2n}}{r^n}, \quad n \neq 0.$$

When $n = 0$, we may assume two linearly independent solutions of Equation (7.135), one of which satisfies the condition

$$\left. \frac{dR_0^{(a)}}{dr} \right|_{r=a} = 0,$$

and the other the condition

$$\left. \frac{dR_0^{(b)}}{dr} \right|_{r=b} = 0.$$

The same solution, $R_0(r) \equiv 1$, satisfies both these conditions. Therefore, the Neumann problem for an annulus, $a \leq r \leq b$:

$$\nabla^2 u = 0, \quad a < r < b,$$

$$\left. \frac{\partial u}{\partial r} \right|_{r=a} = f_1(\varphi), \quad \left. \frac{\partial u}{\partial r} \right|_{r=b} = f_2(\varphi)$$

has a solution that is conveniently written as the series

$$\begin{aligned} u(r, \varphi) = & \frac{C_0}{2} \ln r + \sum_{n=1}^{\infty} \frac{R_n^{(a)}(r)}{R_n^{(a)'}(b)} (A_n \cos n\varphi + B_n \sin n\varphi) \\ & + \sum_{n=1}^{\infty} \frac{R_n^{(b)}(r)}{R_n^{(b)'}(a)} (C_n \cos n\varphi + D_n \sin n\varphi) + \text{const} \end{aligned} \quad (7.141)$$

containing an arbitrary constant. The coefficients for $n \neq 0$ are determined from Equations (7.139) and (7.140). The coefficient C_0 is

$$C_0 = \frac{a}{\pi} \int_0^{2\pi} f_1(\varphi) d\varphi = \frac{b}{\pi} \int_0^{2\pi} f_2(\varphi) d\varphi. \quad (7.142)$$

In spite of the fact that the functions $f_1(\varphi)$ and $f_2(\varphi)$ are arbitrary, there is no disagreement in the equality in Equation (7.142) because it satisfies the condition under which the Neumann problem can be solved, namely,

$$\int_0^{2\pi} [b f_2(\varphi) - a f_1(\varphi)] d\varphi = 0.$$

Next consider the convergence of the obtained series. Take, for example, the series in Equation (7.138). For $a < r < b$, we have

$$\frac{(r^{2n} - a^{2n}) b^n}{(b^{2n} - a^{2n}) r^n} = \frac{1 - \left(\frac{a}{r}\right)^{2n}}{1 - \left(\frac{a}{b}\right)^{2n}} \left(\frac{r}{b}\right)^n \leq \frac{1}{1 - \frac{a}{b}} \left(\frac{r}{b}\right)^n = \frac{b}{b-a} \left(\frac{r}{b}\right)^n$$

and

$$\frac{(b^{2n} - r^{2n}) a^n}{(b^{2n} - a^{2n}) r^n} \leq \left(\frac{a}{r}\right)^n$$

so the series (7.138) converges inside an annulus, $a < r < b$, at a rate not worse than a geometric progression. The smoother the boundary functions $f_1(\varphi)$ and $f_2(\varphi)$, the faster these series converge.

7.3.5 Laplace's Equation for Boundary Value Problems for a Circular Region

Building on the previous discussion, we consider problems for a circular sector ($0 \leq r \leq l, 0 \leq \varphi \leq \alpha$) and the sector of an annulus ($a \leq r \leq b, 0 \leq \varphi \leq \alpha$) when the boundary conditions are homogeneous on the radial lines $\varphi = 0$ and $\varphi = \alpha$. Specifically, consider the boundary value problem for a circular sector

$$\nabla^2 u = 0, \quad 0 \leq r < l, \quad 0 < \varphi < \alpha \quad (7.143)$$

with the boundary conditions

$$u|_{r=l} = f(\varphi), \quad (7.144)$$

$$\alpha_1 \frac{\partial u}{\partial \varphi} + \beta_1 u \Big|_{\varphi=0} = 0, \quad \alpha_2 \frac{\partial u}{\partial \varphi} + \beta_2 u \Big|_{\varphi=\alpha} = 0, \quad (7.145)$$

$$|\alpha_i| + |\beta_i| \neq 0, \quad \alpha_1/\beta_1 < 0, \quad \alpha_2/\beta_2 > 0.$$

First, let us find particular solutions by using separation of variables:

$$u(r, \varphi) = R(r)\Phi(\varphi). \quad (7.146)$$

By substituting Equation (7.146) into Laplace's equation and separating the variables, we obtain the Sturm-Liouville problem for the function $\Phi(\varphi)$ on the interval $0 \leq \varphi \leq \alpha$:

$$\begin{aligned} \Phi'' + \lambda\Phi &= 0, \\ \alpha_1\Phi' + \beta_1\Phi|_{\varphi=0} &= 0, \\ \alpha_2\Phi' + \beta_2\Phi|_{\varphi=\alpha} &= 0 \end{aligned} \quad (7.147)$$

as well as the problem for $R(r)$ given by

$$r^2 R'' + rR' - \lambda R = 0, \quad 0 < r < l, \quad |R(0)| < \infty. \quad (7.148)$$

The problem expressed by Equations (7.147) is a standard Sturm-Liouville problem on a limited interval $0 \leq \varphi \leq \alpha$, the solution to which was discussed in Chapter 2. A finite solution of Equation (7.148) at $r = 0$ is

$$R(r) = Cr^{\sqrt{\lambda}}, \quad C = \text{const.}$$

Thus, we have obtained a set of particular solutions of Laplace's equation bounded at $r = 0$ given by

$$u_n(r, \varphi) = r^{\sqrt{\lambda_n}} \Phi_n(\varphi), \quad n = 1, 2, \dots,$$

where $\Phi_n(\varphi)$ and $\lambda_n \geq 0$ are the eigenfunctions and eigenvalues of the problem in Equation (7.147) for the interval $0 \leq \varphi \leq \alpha$. Therefore, a general solution of Laplace's equation for a circular sector can be written as

$$u(r, \varphi) = \sum_{n=1}^{\infty} C_n \left(\frac{r}{l}\right)^{\sqrt{\lambda_n}} \Phi_n(\varphi). \quad (7.149)$$

To obtain the coefficients C_n , we first use the boundary condition (7.144):

$$\sum_{n=1}^{\infty} C_n \Phi_n(\varphi) = f(\varphi),$$

from which we obtain

$$C_n = \frac{1}{\|\Phi_n\|^2} \int_0^\alpha f(\varphi) \Phi_n(\varphi) d\varphi. \quad (7.150)$$

Note that Laplace's equation with nonhomogeneous boundary conditions on the rays $\varphi = 0$ and $\varphi = \alpha$ should be reduced to Poisson's equation with homogeneous boundary conditions, following the discussion in Section 7.4.5 (for the method of solution, see the problem after the last reading exercise in Section 7.4.5, and also Section 7.4.6).

7.3.6 Example of Fluid Motion in a Cylinder as a Neumann Problem in a Plane

Let us examine irrotational (i.e., generally speaking, time-dependent; see Section 7.1.3) flow of an incompressible fluid inside a cylinder with radius l and axis parallel to z . In this case, fluid velocity, $\vec{v}$, is a gradient of a scalar potential u , which satisfies Laplace's equation:

$$\nabla^2 u = 0.$$

A boundary condition for function u on the surface S of the cylinder is:

$$\left. \frac{\partial u}{\partial n} \right|_S = f,$$

where the function $f(\varphi, z, t)$ is defined at all points on S and is the projection of $\vec{v}$ onto an outside normal to the cylinder vector $\vec{n}$.

In the following discussion, we consider only plane flow so that f does not depend on the z -coordinate. In this case, u and $\vec{v}$ are also independent of z , and Laplace's equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0. \quad (7.151)$$

The boundary condition can be written as

$$\left. \frac{\partial u}{\partial n} \right|_{r=l} = f(\varphi, t), \quad (7.152)$$

where $f(\varphi, t)$ is a known, piecewise continuous function limited by $0 \leq \varphi < 2\pi$. As we know, this problem can be solved only if $f(\varphi, t)$ satisfies the condition

$$\int_{\Gamma} f(\varphi, t) d\Gamma = 0,$$

where the integration is performed along the circle, Γ , formed by the cylinder where it crosses the x - y plane. The arc $d\Gamma$ is related to $d\varphi$ by

$$d\Gamma = l d\varphi,$$

and the previous condition can be rewritten as

$$\int_0^{2\pi} f(\varphi, t) d\varphi = 0. \quad (7.153)$$

The function u is finite and periodic in φ :

$$u(r, \varphi + 2\pi) = u(r, \varphi);$$

thus, it is clear that the function $u(r, \varphi)$ can be written as an infinite series:

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi). \quad (7.154)$$

To apply the boundary condition in Equation (7.152), notice that in cylindrical coordinates the vector of the outward normal is $\vec{n} = (1, 0, 0)$ and $\partial u / \partial n = \partial u / \partial r$. Then we differentiate both sides of Equation (7.154) by r and set $r = l$ to get

$$f(\varphi, t) = \frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} (A_n n l^{n-1} \cos n\varphi + B_n n l^{n-1} \sin n\varphi). \quad (7.155)$$

This is the Fourier expansion of function $f(\varphi, t)$ on the interval $[0, 2\pi]$. The free term of this expansion cannot be determined since, due to Equation (7.153),

$$\frac{1}{\pi} \int_0^{2\pi} f(\varphi, t) d\varphi = 0.$$

For other coefficients, we obtain

$$A_n(t) = \frac{1}{n\pi l^{n-1}} \int_0^{2\pi} f(\varphi, t) \cos n\varphi d\varphi, \quad (7.156)$$

$$B_n(t) = \frac{1}{n\pi l^{n-1}} \int_0^{2\pi} f(\varphi, t) \sin n \varphi d\varphi. \quad (7.157)$$

The fact that the coefficient A_0 cannot be defined and is left as an arbitrary number is in agreement with the problem; the potential $u(r, \varphi)$ can be only defined up to a constant.

The series in Equation (7.154) with coefficients given in Equations (7.156) and (7.157) gives the solution to the Neumann problem for a circle with planar fluid flow (i.e., parallel plane flow within a cylinder). Taking the gradient of $u(r, \varphi)$ yields $\vec{v}$; the velocity at each point in the cylinder.

Example 7.9. Define the fluid motion for small-amplitude transversal oscillations of a cylinder along the x -axis:

$$x = x_0 + \varepsilon \cos \omega t, \quad (7.158)$$

with ε and ω being the amplitude of displacement and the frequency, respectively.

Solution. For oscillations given by Equation (7.158), the velocity of the boundary is

$$\vec{v}|_S = -\varepsilon \omega \sin \omega t \vec{e}_x. \quad (7.159)$$

For small-amplitude oscillations, the boundary conditions can be shifted at the time-averaged position of the cylinder, $r = l$, so that we have

$$u_r = -\varepsilon \omega \sin \omega t \vec{n} \cdot \vec{e}_x = -\varepsilon \omega \sin \omega t \cos \varphi = f(\varphi, t).$$

It is obvious that the series (7.155) contains only the term $aA_1 = -\varepsilon \omega \times \sin \omega t$; that is, the solution to the problem under consideration is

$$u = -\varepsilon \omega r \cos \varphi \sin \omega t = -\varepsilon \omega x \sin \omega t.$$

Taking the gradient of this field, one can readily see that the fluid velocity coincides with the velocity of the boundary, given by Equation (7.159). Physically, this means that the fluid moves as a solid body.

7.3.7 Poisson's Equation: General Notes and a Simple Case

Let us briefly discuss how to find a solution of Poisson's equation

$$\nabla^2 u = -f \quad (7.160)$$

with the nonhomogeneous boundary condition

$$u(r, \varphi)|_{r=l} = g(\varphi). \quad (7.161)$$

Either the interior or exterior problem can be considered. Note that, contrary to the solution of Laplace's equation, the solution of Poisson's equation is not zero, even for homogeneous boundary conditions $g(\varphi) \equiv 0$.

We will not develop a general formalism here but just indicate an approach that is helpful in many practical situations. The main idea is to find a particular solution of the nonhomogeneous equation (Poisson's equation in this case) without taking into consideration boundary conditions, and then to add a solution of Laplace's equation to this particular solution in a way that the sum satisfies the necessary boundary conditions. In other words, the function u is presented as the sum of two functions, $u = u_p + u_0$, where u_p is a particular solution of Poisson's equation

$$\nabla^2 u_p = -f, \quad (7.162)$$

and u_0 is a solution of Laplace's equation

$$\nabla^2 u_0 = 0. \quad (7.163)$$

The function u should satisfy the necessary boundary conditions, from which follow the boundary conditions for u_0 :

$$u_0(r, \varphi)|_{r=l} = g(\varphi) - u_p. \quad (7.164)$$

The boundary value problem defined by Equations (7.163) and (7.164) was considered in Sections 7.3.2 and 7.3.3 in detail; thus, we now turn to the solution of Equation (7.162). The question is how to find a particular solution of Poisson's equation that is finite in the center (for the interior problem) or at infinity (for the exterior one) irrespective of the boundary condition at $r = l$. Note that to change the type of boundary condition for

Poisson's equation (7.160), we need only to change the boundary conditions for Laplace's equation (7.163) (see Sections 7.3.2 and 7.3.3).

Let us start from a simple particular case. Very often, the nonhomogeneous term $f(r, \varphi)$ has the form

$$f(r, \varphi) = r^m \cos n\varphi \quad (7.165)$$

(or perhaps $r^m \sin n\varphi$). Here, m is an arbitrary real number, and n is an integer since the function $f(r, \varphi)$ should be periodic in φ , or $f(r, \varphi + 2\pi) = f(r, \varphi)$. In this case, a particular solution of Poisson's equation can be obtained by using the method of unknown coefficients. Note that $m > -2$, corresponds to an interior problem, whereas $m < -2$ to an exterior one. Indeed, the function $f(r, \varphi)$ can be infinite at $r = 0$ for the interior problem; we should only ensure that the integral

$$\int_S |f(r, \varphi)| dS$$

remains finite (e.g., in electrostatics it means that the full charge inside the domain is finite). It is obvious that at $m > -2$ we have a finite value of

$$\int_0^l f(r, \varphi) r dr.$$

In contrast, at $m < -2$ both $f(r \rightarrow \infty, \varphi)$ and the integral

$$\int_l^\infty f(r, \varphi) r dr$$

remain finite. The value $m = -2$ can be used only for a boundary problem involving solutions inside an annulus.

In polar coordinates, Equation (7.160) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = -f(r, \varphi)$$

and because

$$\nabla^2 r^{m+2} \cos n\varphi = [(m+2)^2 - n^2] r^m \cos n\varphi,$$

the particular solution of the equation

$$\nabla^2 u = -r^m \cos n\varphi$$

is

$$u_p = -\frac{r^{m+2} \cos n\varphi}{(m+2)^2 - n^2}. \quad (7.166)$$

A problem occurs if $m+2 = \pm n$, in which case we cannot apply Equation (7.166). In this case, we may seek the solution in the form

$$u_p = R(r) \cos n\varphi,$$

and obtain for $R(r)$ the equation

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = -r^{\pm n-2},$$

where the derivatives with respect to r are denoted by primes. The particular solution of this equation is

$$\mp \frac{r^{\pm n} \ln r}{2n} \text{ at } n \neq 0 \quad (7.167)$$

and

$$-\frac{\ln^2 r}{2} \text{ at } n = 0. \quad (7.168)$$

Recall that $m > -2$ corresponds to an interior problem, whereas n stays in the argument of $\cos n\varphi$ (i.e., there is no need to consider negative values of n). The result is the solution in Equation (7.167) with the upper sign used for the interior problem, $r \leq l$, and the lower sign for the exterior problem, $r \geq l$. The solution of Equation (7.168) with $n = 0$ is the particular solution for a boundary value problem inside an annulus.

Example 7.10. Solve the boundary value problem for a disk given by

$$\nabla^2 u = -Axy, \quad r \leq l, \quad (7.169)$$

$$u|_{r=l} = 0.$$

Solution. The function, f , on the right side of Poisson's equation is $f = Axy = Ar^2 \sin 2\varphi/2$; thus, we have the situation described by Equation (7.165) with $m = n = 2$. Using the result of Equation (7.166), we obtain a particular solution of Equation (7.169) as

$$u_p(r, \varphi) = -\frac{A}{2 \cdot 12} r^4 \sin 2\varphi. \quad (7.170)$$

Taking the solution given by Equation (7.170) into account, the boundary value problem of Equations (7.163) and (7.164) takes the following form:

$$\nabla^2 u_0 = 0,$$

$$r = l : \quad u_0 = -u_p = \frac{A}{24} l^4 \sin 2\varphi. \quad (7.171)$$

Since the boundary condition in Equation (7.171) contains only one Fourier harmonic, we can conclude that $u_0(r, \varphi) = Cr^2 \sin 2\varphi$ (see Example 7.5 in Section 7.3.2) with $C = -Al^2/24$. Thus the function $u = u_p + u_0$ that satisfies the given boundary condition is

$$u(r, \varphi) = \frac{A}{24} r^2 (l^2 - r^2) \sin 2\varphi. \quad (7.172)$$

Reading Exercise. Develop the method of undetermined coefficients for the Poisson's equation for $f = AJ_m(\gamma r) \cos m\varphi + BJ_m(\gamma r) \sin m\varphi$, where γ is an arbitrary number (generally speaking, complex-valued).

7.3.8 Poisson's Equation: The General Case

If the nonhomogeneous term $f(r, \varphi)$ is not as simple as in Equation (7.165), we can resolve $f(r, \varphi)$ in Fourier series in the polar angle φ :

$$f(r, \varphi) = \frac{f_0^{(1)}}{2} + \sum_{n=1}^{\infty} \left[f_n^{(1)}(r) \cos n\varphi + f_n^{(2)}(r) \sin n\varphi \right], \quad (7.173)$$

where

$$f_n^{(1)}(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \alpha) \cos n\alpha d\alpha, \quad f_n^{(2)}(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \alpha) \sin n\alpha d\alpha. \quad (7.174)$$

Representing the solution u_p in a similar form,

$$u_p(r, \varphi) = \frac{u_0^{(1)}(r)}{2} + \sum_{n=1}^{\infty} \left[u_n^{(1)}(r) \cos n\varphi + u_n^{(2)}(r) \sin n\varphi \right],$$

we obtain

$$\begin{aligned} \nabla^2 u_p(r, \varphi) = & \sum_{n=1}^{\infty} \left(\frac{d^2 u_n^{(1)}}{dr^2} + \frac{1}{r} \frac{du_n^{(1)}}{dr} - \frac{n^2}{r^2} u_n^{(1)} \right) \cos n\varphi \\ & + \sum_{n=1}^{\infty} \left(\frac{d^2 u_n^{(2)}}{dr^2} + \frac{1}{r} \frac{du_n^{(2)}}{dr} - \frac{n^2}{r^2} u_n^{(2)} \right) \sin n\varphi + \frac{1}{2} \left(\frac{d^2 u_0^{(1)}}{dr^2} + \frac{1}{r} \frac{du_0^{(1)}}{dr} \right) = -f(r, \varphi). \end{aligned}$$

Accounting for Equation (7.173) and collecting the terms with the same trigonometric functions, we arrive at the nonhomogeneous Euler equation for the Fourier harmonics depending on r :

$$\frac{d^2 u_n^{(1,2)}}{dr^2} + \frac{1}{r} \frac{du_n^{(1,2)}}{dr} - \frac{n^2}{r^2} u_n^{(1,2)} = -f_n^{(1,2)}. \quad (7.175)$$

General solutions of the homogeneous equation corresponding to Equation (7.175) with $f_n^{(1,2)} = 0$ were considered in Section 7.3.2. This solution is

$$u_{n0}^{(1,2)} = A_n^{(1,2)} r^n + B_n^{(1,2)} r^{-n}, \quad n \neq 0, \quad (7.176)$$

$$u_{00}^{(1,2)} = A_0^{(1,2)} + B_0^{(1,2)} \ln r, \quad (7.177)$$

where the subscript “0” indicates that Equations (7.176) and (7.177) determine only the solution of the homogeneous version of Equation (7.175).

Let us seek a particular solution of the nonhomogeneous equations by means of the method of variation of constants; that is, the structure of the solutions remains as in Equations (7.176) and (7.177) but the constants are replaced by functions of r .

Let us review here the general outline the method of variation of constants. If $y_1(x)$ and $y_2(x)$ are two fundamental solutions of a second-order linear homogeneous differential equation, and $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution, then the particular solution of the corresponding nonhomogeneous equation with the right side equal to the function $f(x)$

can be found as $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, where the derivatives of functions $c_1(x)$ and $c_2(x)$ satisfy two equations:

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0, \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= f(x). \end{aligned}$$

Applying this method for our case, we have

$$u_n^{(1,2)} = A_n^{(1,2)}(r)r^n + B_n^{(1,2)}(r)r^{-n}, \quad n \neq 0, \quad (7.178)$$

$$u_0^{(1,2)} = A_0^{(1,2)}(r) + B_0^{(1,2)}(r) \ln r, \quad (7.179)$$

assuming that

$$\dot{A}_n^{(1,2)}(r)r^n + \dot{B}_n^{(1,2)}(r)r^{-n} = 0, \quad n \neq 0, \quad (7.180)$$

$$\dot{A}_0^{(1,2)}(r) + \dot{B}_0^{(1,2)}(r) \ln r = 0, \quad (7.181)$$

where we denote derivatives with respect to r by dots. For the first and the second derivatives of $u_n^{(1,2)}$, we obtain

$$\begin{aligned} \dot{u}_n^{(1,2)} &= n \left[A_n^{(1,2)}(r)r^{n-1} - B_n^{(1,2)}(r)r^{-n-1} \right], \\ \ddot{u}_n^{(1,2)} &= n \left[\dot{A}_n^{(1,2)}(r)r^{n-1} - \dot{B}_n^{(1,2)}(r)r^{-n-1} \right] + n \left[(n-1) A_n^{(1,2)}(r)r^{n-2} + (n+1) B_n^{(1,2)}(r)r^{-n-2} \right], \end{aligned}$$

for $n \neq 0$ and

$$\begin{aligned} \dot{u}_0^{(1,2)} &= \frac{B_0^{(1,2)}(r)}{r}, \\ \ddot{u}_0^{(1,2)} &= \frac{\dot{B}_0^{(1,2)}(r)}{r} - \frac{B_0^{(1,2)}(r)}{r^2} \end{aligned}$$

for $n = 0$. Therefore, the right-hand side of Equation (7.175) is

$$\begin{aligned} \frac{d^2 u_n^{(1,2)}}{dr^2} + \frac{1}{r} \frac{du_n^{(1,2)}}{dr} - \frac{n^2}{r^2} u_n^{(1,2)} &= n \left[\dot{A}_n^{(1,2)}(r)r^{n-1} - \dot{B}_n^{(1,2)}(r)r^{-n-1} \right] \\ &\quad + A_n^{(1,2)}(r)r^{n-2} [n(n-1) + n - n^2] \\ &\quad + B_n^{(1,2)}(r)r^{-n-2} [n(n+1) - n - n^2] \\ &= n \left[\dot{A}_n^{(1,2)}(r)r^{n-1} - \dot{B}_n^{(1,2)}(r)r^{-n-1} \right] \end{aligned} \quad (7.182)$$

for $n \neq 0$, and

$$\frac{d^2 u_0^{(1,2)}}{dr^2} + \frac{1}{r} \frac{du_0^{(1,2)}}{dr} = \frac{\dot{B}_0^{(1,2)}(r)}{r} - \frac{B_0^{(1,2)}(r)}{r^2} + \frac{B_0^{(1,2)}(r)}{r^2} = \frac{\dot{B}_0^{(1,2)}(r)}{r} \quad (7.183)$$

for $n = 0$.

Substituting Equations (7.182) and (7.183) into Equation (7.175), we obtain

$$n \left[\dot{A}_n^{(1,2)}(r) r^{n-1} - \dot{B}_n^{(1,2)}(r) r^{-n-1} \right] = -f_n^{(1,2)}, \quad n \neq 0, \quad (7.184)$$

$$\dot{B}_0^{(1,2)}(r) r^{-1} = -f_0^{(1,2)}. \quad (7.185)$$

The solution to the set of algebraic Equations (7.180) and (7.184) has the following form:

$$\dot{A}_n^{(1,2)} = -\frac{r^{1-n}}{2n} f_n^{(1,2)}, \quad \dot{B}_n^{(1,2)} = \frac{r^{n+1}}{2n} f_n^{(1,2)}, \quad n \neq 0. \quad (7.186)$$

At $n = 0$, we solve Equations (7.181) and (7.185) to get

$$\dot{A}_0^{(1,2)}(r) = r \ln r f_0^{(1,2)}, \quad \dot{B}_0^{(1,2)}(r) = -r f_0^{(1,2)}. \quad (7.187)$$

Let us consider the solution of an *interior* problem. Integration of Equations (7.186) and (7.187) should show the result that all particular solutions of these equations, as well as the corresponding solutions of the form in Equations (7.178) and (7.179), are finite at the point of origin. This means that $B_n^{(1,2)}(0) = 0$, or the second expressions in the sets of Equations (7.186) and (7.187) should be integrated from 0 to r , whereas the first ones should be integrated from r to l (note, that the upper limit of the integral is arbitrary). We thus have

$$A_n^{(1,2)} = \frac{1}{2n} \int_r^l \rho^{1-n} f_n^{(1,2)}(\rho) d\rho, \quad B_n^{(1,2)} = \frac{1}{2n} \int_0^r \rho^{n+1} f_n^{(1,2)}(\rho) d\rho,$$

$$A_0^{(1,2)} = - \int_r^l \rho \ln \rho f_0^{(1,2)}(\rho) d\rho, \quad B_0^{(1,2)} = - \int_0^r \rho f_0^{(1,2)}(\rho) d\rho.$$

Because $f(r=0)$ is finite, all functions $f_n^{(1,2)}(r)$ tend to finite values as $r \rightarrow 0$. Therefore, the following relations are valid at small r :

$$A_n^{(1,2)} \sim r^{2-n}, \quad B_n^{(1,2)} \sim r^{n+2},$$

$$A_0^{(1,2)} \sim r^2 \ln r, \quad B_0^{(1,2)} \sim -r^2$$

which, in view of Equations (7.178) and (7.179), shows u_p to be finite at $r=0$. It is clear that for an *exterior* problem, it is convenient to integrate $A_n^{(1,2)}$ from r to infinity whereas $B_n^{(1,2)}$ should be integrated from l to r .

Thus, the solution of the nonhomogeneous Equation (7.162) can be represented in the form

$$u_p = -\frac{1}{2} \int_0^r \left\{ f_0^{(1)}(\rho) \ln r - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{r} \right)^n \left[\cos n\varphi f_n^{(1)}(\rho) + \sin n\varphi f_n^{(2)}(\rho) \right] \right\} \rho d\rho$$

$$- \frac{1}{2} \int_r^l \left\{ f_0^{(1)}(\rho) \ln \rho - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{\rho} \right)^n \left[\cos n\varphi f_n^{(1)}(\rho) + \sin n\varphi f_n^{(2)}(\rho) \right] \right\} \rho d\rho.$$

Or, accounting for the coefficients $f_n^{(1,2)}(r)$ given by Equations (7.174), we obtain:

$$u_p = \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{r} \right)^n \cos n(\varphi - \alpha) - \ln r \right] f(\rho, \alpha) \rho d\rho d\alpha$$

$$+ \frac{1}{2\pi} \int_r^l \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{\rho} \right)^n \cos n(\varphi - \alpha) - \ln \rho \right] f(\rho, \alpha) \rho d\rho d\alpha.$$

(7.188)

To evaluate the series, we notice that for any $|t| < 1$ the expression

$$\sum_{n=1}^{\infty} \frac{1}{n} t^n \cos n\theta = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} (te^{i\theta})^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} (te^{-i\theta})^n$$

$$= -\frac{1}{2} \ln(1 - te^{i\theta}) - \frac{1}{2} \ln(1 - te^{-i\theta}) = -\frac{1}{2} \ln(1 - 2t \cos \theta + t^2)$$

(7.189)

is valid. Therefore, we obtain

$$u_p = -\frac{1}{2\pi} \int_0^r \int_0^{2\pi} \left[\frac{1}{2} \ln \left(1 + \frac{\rho^2}{r^2} - 2\frac{\rho}{r} \cos(\varphi - \alpha) \right) + \ln r \right] f(\rho, \alpha) \rho d\rho d\alpha$$

$$- \frac{1}{2\pi} \int_r^l \int_0^{2\pi} \left[\frac{1}{2} \ln \left(1 + \frac{r^2}{\rho^2} - 2\frac{r}{\rho} \cos(\varphi - \alpha) \right) + \ln \rho \right] f(\rho, \alpha) \rho d\rho d\alpha$$

or

$$u_p = -\frac{1}{2\pi} \int_0^l \int_0^{2\pi} \ln \sqrt{r^2 + \rho^2 - 2\rho r \cos(\varphi - \alpha)} f(\rho, \alpha) \rho d\rho d\alpha. \quad (7.190)$$

The formula in Equation (7.190) is a particular case of a well-known solution to Poisson's equation in electrostatics (or gravity field theory). Indeed, it is obvious that the potential satisfying Poisson's equation

$$\nabla^2 \varphi = -4\pi \rho(\vec{r})$$

inside a body is given by the expression

$$\varphi(\vec{r}) = \int_{V'} \frac{\rho(\vec{r}')}{|\vec{r}' - \vec{r}|} dV'.$$

The physical meaning of such a solution is quite obvious: We represent a potential, $\varphi(\vec{r})$, as a sum of potentials produced by the “point” charge $\rho(\vec{r}') dV$ located in each elementary volume dV of a body at point $\vec{r}'$. Thus, a particular solution of Equation (7.160) can be written as

$$u(\vec{r}) = \frac{1}{4\pi} \int_{V'} \frac{f(\vec{r}')}{|\vec{r}' - \vec{r}|} dV'. \quad (7.191)$$

For the two-dimensional case, the situation is quite similar. We present a solution of Poisson's equation

$$\nabla^2 \varphi = -2\pi \tau(\vec{r}) \quad (7.192)$$

as a superposition of the potentials produced by each elementary filament located at a point $\vec{r}'$ as follows:

$$\varphi(\vec{r}) = - \int_S \tau(\vec{r}') \ln |\vec{r}' - \vec{r}| dS'. \quad (7.193)$$

Thus, a particular solution of Equation (7.160) in the two-dimensional case is

$$u(\vec{r}) = - \frac{1}{2\pi} \int_S f(\vec{r}') \ln |\vec{r}' - \vec{r}| dS'. \quad (7.194)$$

Equations (7.191) and (7.194) are the particular solutions to Poisson's equation in three- and two-dimensional domains, respectively. They are valid in any coordinate system for an arbitrary physically "reasonable" function $f(\vec{r})$. It should be emphasized that the usual way to obtain these formulas is based on Green's method; however, here we prefer to present the application of Fourier's method, which is appropriate for circular domains only.

For a number of interesting cases, the integrals in Equations (7.191) or (7.194) are relatively easily evaluated using the symmetry of the problem. Such examples are found in many standard electrostatic and gravity problems. Note that sometimes it is more convenient to use the series expansion for a logarithm, integrating Equation (7.188) instead of Equation (7.190).

Reading Exercise. Starting from Equation (7.188), obtain the solution to Poisson's equation given by Equation (7.166) (with the replacement of n by l) for $f = r^m \cos l\varphi$. Why is there a difference between the two solutions?

Reading Exercise. Starting from Equations (7.186) and (7.187), write out the solution to Poisson's equation for the exterior problem.

Reading Exercise. Obtain the solution of Poisson's equation in an annulus $a \leq r \leq b$.

Hint. One can use the solution of either the interior problem or the exterior problem as a partial solution to Poisson's equation inside the annulus.

7.3.9 Poisson's Integral

It is possible to present a solution of the Dirichlet problem for Laplace's equation as an integral formula. Let us do this first for the interior problem for a circle. By substituting the formulas for Fourier coefficients in Equation (7.119) into Equation (7.120) and switching the order of summation and integration, we obtain

$$\begin{aligned} u(r, \varphi) &= \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{l} \right)^n (\cos n\phi \cos n\varphi + \sin n\phi \sin n\varphi) \right\} d\phi \\ &= \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{l} \right)^n \cos n(\phi - \varphi) \right\} d\phi. \end{aligned} \quad (7.195)$$

Since $t \equiv r/l < 1$, the expression in the parentheses can be transformed as follows (see Equation (7.189)):

$$\begin{aligned} Z \equiv \frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n(\phi - \varphi) &= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} t^n [e^{in(\varphi-\phi)} + e^{-in(\varphi-\phi)}] \\ &= \frac{1}{2} \left\{ 1 + \sum_{n=1}^{\infty} [(te^{i(\varphi-\phi)})^n + (te^{-i(\varphi-\phi)})^n] \right\}. \end{aligned}$$

Using

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - 1 = \frac{x}{1-x},$$

we have

$$Z = \frac{1}{2} \left[1 + \frac{te^{i(\varphi-\phi)}}{1-te^{i(\varphi-\phi)}} + \frac{te^{-i(\varphi-\phi)}}{1-te^{-i(\varphi-\phi)}} \right] = \frac{1}{2} \cdot \frac{1-t^2}{1-2t \cos(\varphi-\phi) + t^2}.$$

Therefore, Equation (7.195) becomes

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{l^2 - r^2}{r^2 - 2lr \cos(\varphi - \phi) + l^2} d\phi. \quad (7.196)$$

This formula gives the solution to the first boundary value problem inside a circle and is called the Poisson integral. The expression

$$u(r, \varphi, l, \phi) = \frac{l^2 - r^2}{r^2 - 2lr \cos(\varphi - \phi) + l^2} \quad (7.197)$$

is called the Poisson kernel.

In spite of the fact that expression (7.196) is not applicable at $r = l$, its limit as $r \rightarrow l$ for any fixed value of φ is equal to $f(\varphi)$ because the series we used to obtain Equation (7.196) is a continuous function in the closed region $r \leq l$. Thus, the function defined by the formula

$$u(r, \varphi) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{l^2 - r^2}{r^2 - 2lr \cos(\varphi - \phi) + l^2} d\phi & \text{if } r < l, \\ f(\varphi) & \text{if } r = l \end{cases}$$

is a harmonic function satisfying the equation $\nabla^2 u = 0$ for $r < l$ and continuous in the closed region $r \leq l$. Similarly, we obtain the solution to the exterior boundary value problem for a circle as

$$u(r, \varphi) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{r^2 - l^2}{r^2 - 2lr \cos(\varphi - \phi) + l^2} d\phi & \text{if } r > l, \\ f(\varphi) & \text{if } r = l. \end{cases} \quad (7.198)$$

Poisson's integrals cannot be evaluated analytically in many cases; however, they are often very useful in certain applications. In particular, they are more useful for numerical calculations than the infinite series solution.

Example 7.11. As an example of the integral approach, let us solve the problem considered in Example 7.8 in Section 7.3.2. Find the temperature distribution inside an infinite homogenous cylinder of radius l , kept under constant temperature

$$u(r, \varphi)|_{r=l} = \begin{cases} T_0, & 0 \leq \varphi < \pi, \\ -T_0, & \pi \leq \varphi < 2\pi. \end{cases}$$

Solution. First, for the sake of simplicity, let us modify the solution as follows:

$$u(r, \varphi) = -T_0 + w(r, \varphi)$$

so that we have

$$w(r, \varphi)|_{r=l} = \begin{cases} 2T_0, & 0 \leq \varphi < \pi, \\ 0, & \pi \leq \varphi < 2\pi. \end{cases}$$

The solution is given by Poisson's integral with

$$f(\phi) = \begin{cases} 2T_0, & 0 \leq \phi < \pi, \\ 0, & \pi \leq \phi < 2\pi \end{cases}$$

and we have

$$w(r, \varphi) = \frac{1}{2\pi} \int_0^\pi 2T_0 \frac{l^2 - r^2}{r^2 - 2lr \cos(\phi - \varphi) + l^2} d\phi. \quad (7.199)$$

Let us evaluate this integral for positive values of φ ($0 < \varphi < \pi$). To do this, we will make the substitution $\tan[(\phi - \varphi)/2] = \zeta$. When ϕ takes on values from 0 to π , $(\phi - \varphi)/2$ takes on values from $-\varphi/2$ to $(\pi - \varphi)/2$. On this interval the tangent is a continuous and monotonous function (in Equation (7.199), φ is a fixed number between 0 and π ; thus, such a substitution is possible).

With this substitution, we have

$$w(r, \varphi) = \frac{T_0}{\pi} \int_{-tg \frac{\varphi}{2}}^{tg \frac{\pi - \varphi}{2}} \frac{2(l^2 - r^2)}{(l - r)^2 + \zeta^2(l + r)^2} d\zeta.$$

Evaluating this integral, we obtain

$$u(r, \varphi) = -T_0 + \frac{2T_0}{\pi} \left[\arctan \frac{(l + r) \cot \frac{\varphi}{2}}{l - r} + \arctan \frac{(l + r) \tan \frac{\varphi}{2}}{l - r} \right]. \quad (7.200)$$

This formula gives the temperature distribution in the cylinder for the angles $0 < \varphi < \pi$. For instance, at the point $r = 1/2$, $\varphi = \pi/2$, we have

$$u\left(\frac{1}{2}, \frac{\pi}{2}\right) = -T_0 + \frac{2T_0}{\pi} \cdot 2 \cdot \arctan(3) \approx -T_0 + \frac{2T_0}{\pi} 2 \cdot 1.25 \approx 0.6T_0.$$

Reading Exercise. Check that when $r \rightarrow l$, Equation (7.200) gives the temperature $\pm T_0$:

$$\lim_{r \rightarrow l-0} u(r, \varphi) = \begin{cases} T_0, & 0 \leq \varphi < \pi, \\ -T_0, & \pi \leq \varphi < 2\pi. \end{cases}$$

Reading Exercise. Evaluate the integral in Equation (7.199) for $0 < \varphi < 2\pi$ to find an expression for $u(r, \varphi)$ for this interval of the values for φ .

Example 7.12. Consider a stationary membrane's deflection from the equilibrium position. For the stationary case, the membrane surface is described by the function $u = u(x, y)$, which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If the membrane contour projection onto the x - y plane is a circle of radius l , we can consider this problem as an interior Dirichlet problem for a circle.

Let the equation of the contour be given by function $u = f(\varphi)$, where f is a z -coordinate of the contour at angle φ . We can express $u(r, \varphi)$ by using Poisson's integral as

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{l^2 - r^2}{r^2 - 2lr \cos(\phi - \varphi) + l^2} d\phi.$$

As an example, consider a film fixed on a firm frame that has a circular projection onto the x - y plane with radius l and center at point O . The equation of the film contour in cylindrical coordinates is $u = C \cos 2\varphi$ ($0 \leq \varphi \leq 2\pi$), $r = l$. Find the shape, $u(r, \varphi)$, of the film.

Solution. The solution is given by Poisson's integral:

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} C \cos 2\phi \frac{l^2 - r^2}{r^2 - 2lr \cos(\phi - \varphi) + l^2} d\phi.$$

To evaluate this integral, we use the substitution $\phi - \varphi = \zeta$. The limits of integration will not change because the integrand is periodic with period 2π , which means an integral with limits from $-\varphi$ to $2\pi - \varphi$ is equal to the same integral with limits from 0 to 2π). Thus, we have

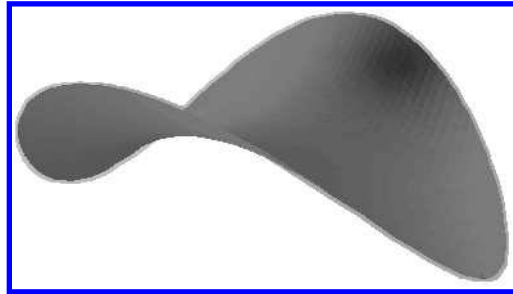


Figure 7.10. Shape of the film in Example 7.12.

$$\begin{aligned}
 u(r, \varphi) &= \frac{C(l^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{\cos(2\zeta + 2\varphi)}{r^2 - 2lr \cos \zeta + l^2} d\zeta \\
 &= \frac{C(l^2 - r^2)}{2\pi} \left[\cos 2\varphi \int_0^{2\pi} \frac{\cos 2\zeta d\zeta}{r^2 - 2lr \cos \zeta + l^2} - \sin 2\varphi \int_0^{2\pi} \frac{\sin 2\zeta d\zeta}{r^2 - 2lr \cos \zeta + l^2} \right].
 \end{aligned}$$

The second of these integrals is equal to zero because, if we change the interval to $(-\pi, \pi)$, it becomes the integral of the odd function on this interval. So for this case,

$$u(r, \varphi) = \frac{C(l^2 - r^2)}{2\pi} \cos 2\varphi \int_{-\pi}^{\pi} \frac{\cos 2\zeta}{r^2 - 2lr \cos \zeta + l^2} d\zeta.$$

With the substitution $\tan(\zeta/2) = v$, we finally obtain

$$u(r, \varphi) = \frac{C(l^2 - r^2)}{2\pi} \cos 2\varphi \frac{2\pi r^2}{l^2(l^2 - r^2)} = \frac{Cr^2}{l^2} \cos 2\varphi.$$

Thus, we have found the displacement of every membrane point (r, φ) from the x - y plane.

The function $C(r/l)^2 \cos 2\varphi$ is harmonic and takes the values $C \cos 2\varphi$ on the contour of the circle. The surface $u = C(r/l)^2 \cos 2\varphi$ is a hyperbolic paraboloid; consequently, the film that is fixed on a given contour will have the form a hyperbolic paraboloid (a saddle), shown in Figure 7.10.

7.4 Laplace's Equation in Cylindrical Coordinates

In this section, we extend the previous concepts to include problems involving cylindrical symmetry. We start by finding a general solution of Laplace's equation inside a circular cylinder of radius l and height H . The equation and boundary conditions for this problem are given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad (7.201)$$

$$0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad 0 < z < H.$$

The following boundary condition on the lateral surface is

$$\alpha u_r(r, \varphi, z) + \beta u(r, \varphi, z)|_{r=l} = g(\varphi, z), \quad (7.202)$$

and on the bases, the boundary conditions are

$$\alpha_0 u_z(r, \varphi, z) + \beta_0 u(r, \varphi, z)|_{z=0} = f_0(r, \varphi) \quad (7.203)$$

and

$$\alpha_H u_z(r, \varphi, z) + \beta_H u(r, \varphi, z)|_{z=H} = f_H(r, \varphi). \quad (7.204)$$

Functions $g(\varphi, z)$ and $f_{0,H}(r, \varphi)$ are defined at $0 \leq z \leq H$ and at $0 \leq r \leq l$, respectively. These functions are obviously periodic with respect to the polar angle:

$$g(\varphi, z) = g(\varphi + 2\pi, z),$$

$$f_{0,H}(r, \varphi) = f_{0,H}(r, \varphi + 2\pi).$$

A similar condition is valid for the solution we are seeking:

$$u(r, \varphi, z) = u(r, \varphi + 2\pi, z).$$

The three pairs of real constants, α, β ; α_0, β_0 ; and α_H, β_H (with $|\alpha| + |\beta| \neq 0$ and $|\alpha_{0,H}| + |\beta_{0,H}| \neq 0$) characterize the type of corresponding boundary conditions, and the signs of the ratios are $\beta/\alpha > 0$, $\beta_0/\alpha_0 < 0$, and $\beta_H/\alpha_H > 0$. To begin, we consider three main types of boundary conditions for the particular case of the lateral surface, Equation (7.202):

1. For boundary conditions of the first type with $\alpha = 0$ and $\beta = 1$ we have

$$u(r, \varphi, z)|_{r=l} = g(\varphi, r),$$

which is the Dirichlet condition.

2. For boundary conditions of the second type with $\alpha = 1$ and $\beta = 0$ we have

$$u_r(r, \varphi, z)|_{r=l} = g(\varphi, z),$$

which is the Neumann condition.

3. For boundary conditions of the third type with $\alpha = 1$ and $\beta = h = \text{const}$, we have

$$u_r(r, \varphi, z) + hu(r, \varphi, z)|_{r=l} = g(\varphi, z),$$

which is the mixed condition.

The same three types of boundary conditions can be prescribed for each base of the cylinder.

In accordance with the general approach to solving Laplace's equation with nonhomogeneous boundary conditions, we represent the solution u as a superposition of two functions:

$$u = u_1 + u_2,$$

each of which is a solution of the following boundary value problem:

$$\nabla^2 u_1 = 0, \quad (7.205)$$

$$u_1(r, \varphi, z) = u_1(r, \varphi + 2\pi, z), \quad (7.206)$$

$$\alpha u_{1r}(r, \varphi, z) + \beta u_1(r, \varphi, z)|_{r=l} = 0, \quad (7.207)$$

$$\alpha_0 u_{1z}(r, \varphi, z) + \beta_0 u_1(r, \varphi, z)|_{z=0} = f_0(r, \varphi), \quad (7.208)$$

$$\alpha_H u_{1z}(r, \varphi, z) + \beta_H u_1(r, \varphi, z)|_{z=H} = f_H(r, \varphi) \quad (7.209)$$

and

$$\nabla^2 u_2 = 0, \quad (7.210)$$

$$u_2(r, \varphi, z) = u_2(r, \varphi + 2\pi, z), \quad (7.211)$$

$$\alpha u_{2r}(r, \varphi, z) + \beta u_2(r, \varphi, z)|_{r=l} = g(\varphi, z), \quad (7.212)$$

$$\alpha_0 u_{2z}(r, \varphi, z) + \beta_0 u_2(r, \varphi, z)|_{z=0} = 0, \quad (7.213)$$

$$\alpha_H u_{2z}(r, \varphi, z) + \beta_H u_2(r, \varphi, z)|_{z=H} = 0. \quad (7.214)$$

One can see that u_1 satisfies the boundary value problem with *homogeneous boundary conditions at the lateral surface*, whereas for u_2 we have *homogeneous boundary conditions at the bases*. Note that for each boundary value problem, however, we have nonhomogeneous sets of boundary conditions. Next we consider each problem separately, starting from Equations (7.205) through (7.209).

7.4.1 Homogeneous Boundary Conditions at the Lateral Surface

First we perform a separation of variables:

$$u_1(r, \varphi, z) = R(r)\Phi(\varphi)Z(z), \quad (7.215)$$

substitute Equation (7.215) into Equation (7.205), and divide by u to yield

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (7.216)$$

Because the last term in Equation (7.216) does not depend on r and φ , it must be a constant, which we denote as λ . Thus, we may write

$$\frac{d^2 Z}{dz^2} - \lambda Z = 0 \quad (7.217)$$

and

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{r}{R} \frac{dR}{dr} + \lambda r^2 + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0. \quad (7.218)$$

Similarly, the last term in Equation (7.218) must be a constant, which we denote as $-p$. Thus, we have

$$\frac{d^2 \Phi}{d\varphi^2} + p\Phi = 0 \quad (7.219)$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda - \frac{p}{r^2} \right) R = 0. \quad (7.220)$$

Therefore, we obtain three ordinary differential equations, one for each of the functions $Z(z)$, $\Phi(\varphi)$ and $R(r)$. Homogeneous boundary conditions in Equations (7.206) and (7.207) provide the necessary boundary conditions for Equation (7.219):

$$\Phi(\varphi) = \Phi(\varphi + 2\pi), \quad (7.221)$$

and for Equation (7.220):

$$\alpha R'(r) + \beta R(r) \Big|_{r=l} = 0, \quad (7.222)$$

respectively. The boundary conditions on the bases given in Equations (7.208) and (7.209) will be satisfied when we reach the final step of the solution.

Note that the function $V(r, \varphi) = R(r)\Phi(\varphi)$ is the eigenfunction of the two-dimensional Laplace operator in Equation (7.201) with homogeneous boundary conditions similar to Equations (7.206) and (7.207). In other words,

$$\nabla^2 V + \lambda V = 0, \quad (7.223)$$

$$V(r, \varphi) = V(r, \varphi + 2\pi), \quad (7.224)$$

$$\alpha V_r(r, \varphi) + \beta V(r, \varphi) \Big|_{r=l} = 0. \quad (7.225)$$

Note that $V(r, \varphi)$ is not a solution of Laplace's equation except for the case $\lambda = 0$ (it is obvious from the principle of the maximum that there is no such a solution except for $V = \text{const}$ at $\beta = 0$).

Before proceeding, we first determine R and Φ . From the boundary value problem in Equations (7.219) and (7.221), it follows that $p = n^2$, $n = 0, 1, 2, \dots$. The general solution of this problem is then

$$\Phi_n(\varphi) = C \cos n\varphi + D \sin n\varphi. \quad (7.226)$$

(We reserve symbols A and B for the coefficients in the function $Z(z)$; see Equation (7.237) below.)

With the above stipulations, Equation (7.218) now becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda - \frac{n^2}{r^2} \right) R = 0, \quad (7.227)$$

with the boundary conditions (7.222). Introducing the new independent variable $x = kr$, $k = \sqrt{\lambda}$, Equation (7.227) becomes Bessel's equation, given by

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{n^2}{x^2} \right) R = 0, \quad (7.228)$$

which has a general solution that is a linear combination of Bessel's and Neumann's functions:

$$R_n(r) = EJ_n(kr) + FN_n(kr).$$

Bessel's function of the n th order, $J_n(kr)$, is regular at $r = 0$; Neumann's function of the n th order, $N_n(kr)$, is infinite as $r \rightarrow 0$. Thus, for the interior problem under consideration, the coefficient F should be set equal to zero. The extension to the case of problems in a cylindrical layer defined by $a \leq r \leq b$, $0 \leq z \leq H$ is left to reader as a reading exercise.

Upon satisfying the boundary conditions for the function $R_n(r)$, we arrive at the following equation:

$$\alpha k J'_n(kl) + \beta J_n(kl) = 0.$$

Thus, the solution of the Sturm-Liouville interior boundary value problem yields eigenvalues and eigenfunctions for the radial function $R(r)$ given by

$$\lambda_{nm} = k_{nm}^2 = \left(\frac{\mu_m^{(n)}}{l} \right)^2, \quad R_{nm}(r) = J_n \left(\sqrt{\lambda_{nm}} r \right), \quad n, m = 0, 1, 2, \dots,$$

where $\mu_m^{(n)}$ is the m th root of the equation

$$\alpha \mu J'_n(\mu) + \beta l J_n(\mu) = 0. \quad (7.229)$$

The roots of Equation (7.229) can be found numerically. Examples of a graphical solution for this equation can be found in the program Laplace (see Appendix E).

From the previous discussion, we see that we have obtained the solution of the boundary value problem in Equations (7.223) through (7.225). The eigenfunctions of the Laplacian for a circle satisfying the homogeneous boundary conditions given by Equations (7.224) and (7.225) are

$$V_{nm}^{(1)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi. \quad (7.230)$$

Functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are orthogonal in the domain D : $\{0 \leq \varphi < 2\pi, 0 \leq r \leq l\}$, in which case we have

$$\begin{aligned}
\int_D V_{nm}^{(p)}(r, \varphi) V_{n'm'}^{(q)}(r, \varphi) dS &= \int_0^{2\pi} \int_0^l V_{nm}^{(p)}(r, \varphi) V_{n'm'}^{(q)}(r, \varphi) r dr d\varphi \\
&= \|V_{nm}^{(p)}\|^2 \delta_{nn'} \delta_{mm'} \delta_{pq}, \quad p, q = 1, 2
\end{aligned} \tag{7.231}$$

and the norms are

$$\|V_{nm}^{(p)}\|^2 = \int_0^{2\pi} \int_0^l r V_{nm}^{(p)2}(r, \varphi) dr d\varphi = \begin{cases} 2\pi \|R_n(r)\|^2 & \text{if } n = 0, \\ \pi \|R_n(r)\|^2 & \text{if } n \neq 0. \end{cases} \tag{7.232}$$

Expressions for the squared norms of eigenfunctions depend on the coefficients of the relevant boundary condition. We have the following cases.

1. If $\alpha = 0$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[J_n'(\mu_m^{(n)}) \right]^2. \tag{7.233}$$

2. If $\beta = 0$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2(\mu_m^{(n)})^2} \left[(\mu_m^{(n)})^2 - n^2 \right] J_n^2(\mu_m^{(n)}). \tag{7.234}$$

3. If $\alpha < \beta$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[1 + \frac{(\mu_m^{(n)})^2 - n^2}{l^2} \cdot \frac{\alpha^2}{\beta^2} \right] \left[J_n'(\mu_m^{(n)}) \right]^2. \tag{7.235}$$

4. If $\alpha > \beta$, we have

$$\|R_{nm}\|^2 = \frac{l^2}{2(\mu_m^{(n)})^2} \left[l^2 \frac{\beta^2}{\alpha^2} + (\mu_m^{(n)})^2 - n^2 \right] J_n^2(\mu_m^{(n)}). \tag{7.236}$$

Next, having a solution for the function $V(r, \varphi)$, we can consider the function $Z(z)$. The general solution of Equation (7.217) is

$$Z_{nm}(z) = A \cosh \sqrt{\lambda_{nm}} z + B \sinh \sqrt{\lambda_{nm}} z. \quad (7.237)$$

Note that for the Dirichlet or Neumann conditions, it is more convenient to choose a different form of the general solution (see the example in Section 7.4.2). Thus, we can represent the solution to the boundary value problem in Equations (7.205) through (7.209) in the following form:

$$\begin{aligned} u_1 = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(1)}(r, \varphi) \left[A_{nm}^{(1)} \cosh \sqrt{\lambda_{nm}} z + B_{nm}^{(1)} \sinh \sqrt{\lambda_{nm}} z \right] \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(2)}(r, \varphi) \left[A_{nm}^{(2)} \cosh \sqrt{\lambda_{nm}} z + B_{nm}^{(2)} \sinh \sqrt{\lambda_{nm}} z \right]. \end{aligned} \quad (7.238)$$

This series satisfies Laplace's equation and the boundary conditions at the lateral surface in Equation (7.207). It is obvious that

$$\begin{aligned} u_{1z} = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(1)}(r, \varphi) \sqrt{\lambda_{nm}} \left[A_{nm}^{(1)} \sinh \sqrt{\lambda_{nm}} z + B_{nm}^{(1)} \cosh \sqrt{\lambda_{nm}} z \right] \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(2)}(r, \varphi) \sqrt{\lambda_{nm}} \left[A_{nm}^{(2)} \sinh \sqrt{\lambda_{nm}} z + B_{nm}^{(2)} \cosh \sqrt{\lambda_{nm}} z \right], \end{aligned}$$

and we have for the particular cases

$$\begin{aligned} u_1|_{z=0} &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[A_{nm}^{(1)} V_{nm}^{(1)}(r, \varphi) + A_{nm}^{(2)} V_{nm}^{(2)}(r, \varphi) \right], \\ u_{1z}|_{z=0} &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sqrt{\lambda_{nm}} \left[B_{nm}^{(1)} V_{nm}^{(1)}(r, \varphi) + B_{nm}^{(2)} V_{nm}^{(2)}(r, \varphi) \right]. \end{aligned}$$

We should also satisfy the boundary conditions for the bases given in Equations (7.208) and (7.209). We have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[V_{nm}^{(1)}(r, \varphi) \left(\alpha_0 \sqrt{\lambda_{nm}} B_{nm}^{(1)} + \beta_0 A_{nm}^{(1)} \right) \right. \\ \left. + V_{nm}^{(2)}(r, \varphi) \left(\alpha_0 \sqrt{\lambda_{nm}} B_{nm}^{(2)} + \beta_0 A_{nm}^{(2)} \right) \right] = f_0(r, \varphi), \end{aligned} \quad (7.239)$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(1)}(r, \varphi) \left[\alpha_H \sqrt{\lambda_{nm}} \left(A_{nm}^{(1)} \sinh \sqrt{\lambda_{nm}} H + B_{nm}^{(1)} \cosh \sqrt{\lambda_{nm}} H \right) \right. \\
& \quad \left. + \beta_H \left(A_{nm}^{(1)} \cosh \sqrt{\lambda_{nm}} H + B_{nm}^{(1)} \sinh \sqrt{\lambda_{nm}} H \right) \right] \\
& + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm}^{(2)}(r, \varphi) \left[\alpha_H \sqrt{\lambda_{nm}} \left(A_{nm}^{(2)} \sinh \sqrt{\lambda_{nm}} H + B_{nm}^{(2)} \cosh \sqrt{\lambda_{nm}} H \right) \right. \\
& \quad \left. + \beta_H \left(A_{nm}^{(2)} \cosh \sqrt{\lambda_{nm}} H + B_{nm}^{(2)} \sinh \sqrt{\lambda_{nm}} H \right) \right] = f_H(r, \varphi).
\end{aligned} \tag{7.240}$$

Expanding the functions $f_{0,H}(r, \varphi)$ in a Fourier-Bessel series yields

$$f_{0,H}(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[f_{nm,0,H}^{(1)} V_{nm}^{(1)}(r, \varphi) + f_{nm,0,H}^{(2)} V_{nm}^{(2)}(r, \varphi) \right], \tag{7.241}$$

which allows us, in view of Equation (7.231), to write

$$\begin{aligned}
f_{nm,0}^{(1,2)} &= \frac{1}{\|V_{nm}^{(1,2)}\|^2} \int_0^{2\pi} \int_0^l f_0(r, \varphi) V_{nm}^{(1,2)} r dr d\varphi, \\
f_{nm,H}^{(1,2)} &= \frac{1}{\|V_{nm}^{(1,2)}\|^2} \int_0^{2\pi} \int_0^l f_H(r, \varphi) V_{nm}^{(1,2)} r dr d\varphi.
\end{aligned} \tag{7.242}$$

Comparing Equations (7.239) through (7.241), we obtain a simple linear nonhomogeneous algebraic set of equations for the coefficients $A_{nm}^{(1,2)}$ and $B_{nm}^{(1,2)}$. Solving this set of equations allows us to find the solution of the first boundary value problem, given by Equations (7.205) through (7.209). In the following section, we give an example to clarify the details of the above calculations.

7.4.2 Example of Homogeneous Boundary Conditions at the Lateral Surface

Consider the boundary value problem for a circular cylinder:

$$\nabla^2 T = 0 \tag{7.243}$$

with boundary conditions

$$T|_{r=l} = T_0, \tag{7.244}$$

$$T|_{z=0} = T_0 + T_1 \frac{l^2 - r^2}{l^2}, \quad (7.245)$$

$$T_z|_{z=H} = 0. \quad (7.246)$$

A physical example of such a problem is a barrel, with heat being applied to the bottom and open at the top. Equation (7.243) implies that this heat source should not be very powerful so that the heat is steadily distributed throughout the barrel, and the problem remains at steady state (i.e., there is no time dependence). The temperature distribution in Equation (7.245) might be from a cigarette lighter or other small heat source located at the center of the bottom of the barrel, while the lateral surface of the cylinder (the barrel walls) stays at a constant temperature equal to the temperature of the environment. The parameter T_1 characterizes the intensity of the heat source and depends on the power of the source and the efficiency of heat conductivity through the bottom surface. The thermal conductivity of air is much less than for water, thus the heat flux at the top, at the water surface, is approximately zero.

We may exclude the constant temperature term, T_0 , and search for the solution of this problem as

$$T = T_0 + u, \quad (7.247)$$

where

$$\nabla^2 u = 0, \quad (7.248)$$

$$u|_{r=l} = 0, \quad (7.249)$$

$$u|_{z=0} = G(r) = T_1 \frac{l^2 - r^2}{l^2}, \quad (7.250)$$

$$u_z|_{z=H} = 0. \quad (7.251)$$

According to the general method discussed above, the solution of this problem is the series

$$\begin{aligned} u(r, \varphi, z) = & \sum_{m=1} \sum_{n=0} \left[A_{nm}^{(1)} \cosh \sqrt{\lambda_{nm}} (H - z) + B_{nm}^{(1)} \sinh \sqrt{\lambda_{nm}} z \right] \\ & \times \cos n\varphi J_n \left(\sqrt{\lambda_{nm}} r \right) + \sum_{m=1} \sum_{n=1} \left[A_{nm}^{(2)} \cosh \sqrt{\lambda_{nm}} (H - z) \right. \\ & \left. + B_{nm}^{(2)} \sinh \sqrt{\lambda_{nm}} z \right] \sin n\varphi J_n \left(\sqrt{\lambda_{nm}} r \right). \end{aligned} \quad (7.252)$$

Here, k_{nm} is m th root of the equation $J_n \left(\sqrt{\lambda_{nm}} l \right) = 0$, which guarantees that the boundary condition of Equation (7.249) is fulfilled.

Notice that the general solution to Equation (7.217) is chosen in a slightly different way than it was for Equation (7.237). Here, we use $\cosh \sqrt{\lambda_{nm}} (H - z)$ instead of $\cosh \sqrt{\lambda_{nm}} z$, which makes the calculations shorter, since this particular solution has a derivative equal to zero at $z = H$. Obviously, the two particular solutions, $\sinh \sqrt{\lambda_{nm}} z$ and $\cosh \sqrt{\lambda_{nm}} (H - z)$, are linearly independent.

Now we need to find coefficients $A_{nm}^{(1,2)}$ and $B_{nm}^{(1,2)}$ to satisfy the boundary conditions at the bottom and top, Equations (7.250) and (7.251), respectively. We begin with the equation for the top.

$$\begin{aligned} u_z(r, \varphi, z)|_{z=H} &= \sum_{m=1} \sum_{n=0} \sqrt{\lambda_{nm}} \cosh \sqrt{\lambda_{nm}} H \left[B_{nm}^{(1)} \cos n\varphi + B_{nm}^{(2)} \sin n\varphi \right] \\ &\quad \times J_n \left(\sqrt{\lambda_{nm}} r \right) \\ &= 0. \end{aligned} \quad (7.253)$$

Clearly, this function can be zero for all values of r and φ only when the coefficients in Equation (7.253) are equal to zero, in which case we have

$$B_{nm}^{(1,2)} = 0. \quad (7.254)$$

Including Equation (7.254) and using the condition in Equation (7.250) results in the equation for the bottom:

$$\begin{aligned} u(r, \varphi, z)|_{z=0} &= \sum_{m=1} \sum_{n=0} \cosh \sqrt{\lambda_{nm}} H \left[A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right] \\ &\quad \times J_n \left(\sqrt{\lambda_{nm}} r \right) \\ &= G(r). \end{aligned} \quad (7.255)$$

We may expand the function $G(r)$ in a Fourier-Bessel series as

$$G(r) = \sum_{m=1} \sum_{n=0} \left[G_{nm}^{(1)} \cos n\varphi + G_{nm}^{(2)} \sin n\varphi \right] J_n \left(\sqrt{\lambda_{nm}} r \right),$$

where

$$G_{nm}^{(1)} = \frac{1}{\|V_{nm}\|^2} \int_0^{2\pi} \int_0^l G(r) J_n \left(\sqrt{\lambda_{nm}} r \right) \cos n\varphi r dr d\varphi,$$

$$G_{nm}^{(2)} = \frac{1}{\|V_{nm}\|^2} \int_0^{2\pi} \int_0^l G(r) J_n(\sqrt{\lambda_{nm}} r) \sin n\varphi r dr d\varphi.$$

Due to axial symmetry, the function is independent of angle φ , and we have

$$G_{nm}^{(1)} = G_{nm}^{(2)} = 0, \quad n > 0.$$

It is obvious that for $n = 0$ only coefficients $G_{0m}^{(1)}$ are actually part of the solution and we need to find only one set of coefficients:

$$G_{0m}^{(1)} = \frac{1}{\|R_{0m}\|^2} \int_0^l G(r) J_0(\sqrt{\lambda_{0m}} r) r dr, \quad (7.256)$$

where $\|R_{0m}\|^2$ is given by Equation (7.233). Evaluating the integral in Equation (7.256), we obtain

$$G_{0m}^{(1)} = -\frac{T_1}{J'_0(\sqrt{\lambda_{0m}} l)} \left(\frac{2}{\sqrt{\lambda_{0m}} l} \right)^3.$$

Returning to Equation (7.255), we can obtain the coefficients of the series:

$$A_{nm}^{(1)} = \frac{G_{nm}^{(1)}}{\cosh \sqrt{\lambda_{nm}} H} = -\frac{T_1}{J'_0(\sqrt{\lambda_{0m}} l) \cosh \sqrt{\lambda_{0m}} H} \left(\frac{2}{\sqrt{\lambda_{0m}} l} \right)^3 \delta_{n0}, \quad A_{nm}^{(2)} = 0. \quad (7.257)$$

Combining Equations (7.252), (7.254), and (7.257) gives the solution of the problem:

$$T = T_0 - T_1 \sum_{m=1}^{\infty} \left(\frac{2}{\sqrt{\lambda_{0m}} l} \right)^3 \frac{J_0(\sqrt{\lambda_{0m}} r) \cosh \sqrt{\lambda_{0m}} z}{J'_0(\sqrt{\lambda_{0m}} l) \cosh \sqrt{\lambda_{0m}} H}. \quad (7.258)$$

Figure 7.11 presents graphs obtained by tabulating the series (7.258) for three values of H : (a) 0.1, (b) 1.0, and (c) 2.0. The following values of the parameters were used: $l = T_1 = 1$, $T_0 = 0$, where the parameters are made dimensionless by putting the temperature in units of $T_1 - T_0$ and length in units of l . The partial sum for Figure 7.11 includes 20 terms, which

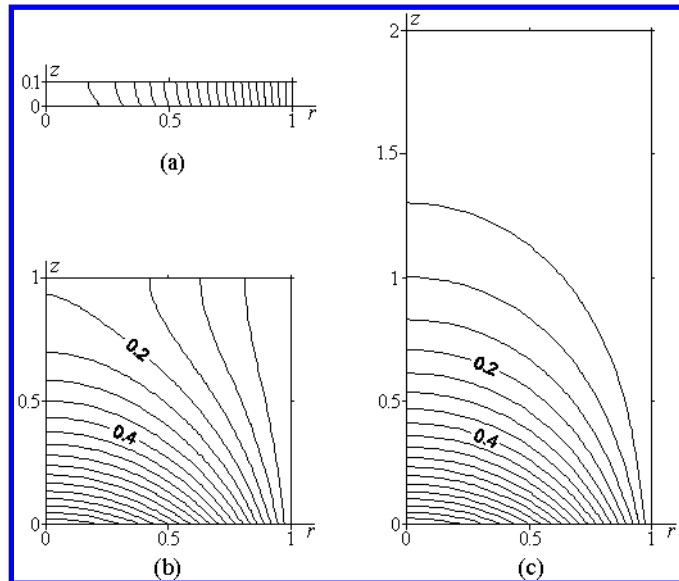


Figure 7.11. Isotherms for three values of the height of the cylinder, H : (a) 0.1, (b) 1.0, and (c) 2.0.

guarantees a precision up to $3 \cdot 10^{-4}$ for the boundary condition (7.255). Isothermal lines are plotted at intervals of 0.05 in temperature.

As we see in Figure 7.11, in the first case the temperature barely depends on z and the isotherms are practically vertical, especially near the cylinder's axis. In the third case, beginning with $z = 1.5$, the inside temperature practically coincides with the temperature of the environment and we have a maximum temperature on the free surface of about 0.018. The temperature distribution for larger values of H is practically the same as seen in Figure 7.11(c).

To conclude, a cigarette lighter will be able to warm up only the portion of the water in the barrel close to the center of the base. If the height of the cylinder is more than $1.5l$, the upper part has a temperature close to the temperature of the environment.

Reading Exercise. Find the solution if the heat source is shifted to the point $x = x_0 < l$; use

$$G(r, \varphi) = l^{-4} (r \cos \varphi - x_0)^2 (l^2 - r^2).$$

Hint. In this case, the solution also depends on the angle φ , but the solution contains only terms with $A_{nm}^{(1)}$, where $n = 0, 1, 2$.

7.4.3 Homogeneous Boundary Conditions at the Bases

Now let us consider the boundary value problem in Equations (7.210) through (7.214). All the Equations (7.215) through (7.220) remain valid. Moreover, the periodic conditions for $\Phi(\varphi)$ hold and, consequently, solutions will be in the form given in Equation (7.226) with the same values, $p = n^2$. However, in this case we should take the boundary conditions (7.213) and (7.214) into account, which yields

$$\alpha_0 Z'(0) + \beta_0 Z(0) = 0 \quad (7.259)$$

and

$$\alpha_H Z'(H) + \beta_H Z(H) = 0. \quad (7.260)$$

Since the z -coordinate is Cartesian, the Sturm-Liouville boundary value problem given by Equations (7.217), (7.259), and (7.260) is the same as the one discussed in detail for Laplace's equation inside a rectangular domain (see Section 7.2). Here we briefly repeat the results:

$$Z_m = -\alpha_0 k_m \cos k_m z - \beta_0 \sin k_m z, \quad (7.261)$$

$$\lambda_m = -k_m^2, \quad (7.262)$$

where k_m is the m th root of the algebraic equation:

$$\alpha_H k_m (\alpha_0 k_m \sin k_m H + \beta_0 \cos k_m H) - \beta_H (\alpha_0 k_m \cos k_m H - \beta_0 \sin k_m H) = 0.$$

Functions $Z_m(z)$ are orthogonal, and their norms are given in Appendix C:

$$\int_0^H Z_m(z) Z_j(z) dz = \|Z_m\|^2 \delta_{jm}.$$

In view of Equation (7.262), we arrive at the following equation for the radial part of the solution:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \left(k_m^2 + \frac{n^2}{r^2} \right) R = 0. \quad (7.263)$$

Introducing the new, independent variable $x = k_m r$, this equation becomes the so-called *modified Bessel's equation*:

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 + \frac{n^2}{x^2}\right) R = 0. \quad (7.264)$$

The solution of Equation (7.264) is finite in the center of the cylinder and is the modified Bessel's function of n th order:

$$R_{nm} = I_n(k_m r), \quad (7.265)$$

which is proportional to the Bessel function of n th order for a pure imaginary argument:¹

$$I_n(x) = i^{-n} J_n(ix).$$

One special case should be considered separately. Indeed, when Neumann conditions are prescribed at both bases (i.e., $\beta_0 = \beta_H = 0$, $\alpha_0 = \alpha_H = 1$), there exists a nontrivial solution of the Sturm-Liouville boundary value problem in Equations (7.217), (7.259), and (7.260) given by

$$Z_0(z) = 1$$

corresponding to $\lambda_0 = 0$. This means that the corresponding particular solution does not depend on z , and we return to the two-dimensional problem considered in Section 7.3.2. In this case, Equation (7.263) reduces to Euler's equation

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{n^2}{r^2} R = 0,$$

whose solution, when $n \neq 0$, is

$$R(r) = Er^n + Fr^{-n}, \quad (7.266)$$

and when $n = 0$, is

$$R(r) = E + F \ln r. \quad (7.267)$$

Since the point $r = 0$ belongs to the domain of the solution, the coefficient F should be set equal to zero in Equations (7.266) and (7.267).

¹For details, see, for example, G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Company, New York, 1968.

Thus, we have that the general solution to the boundary-value problem in Equations (7.210) through (7.214) can be represented as the series

$$u_2 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n(k_m r) \left[A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right] Z_m(z).$$

Note that hereafter we do not write a separate term corresponding to $n = 0$ (as was done in Section 7.3.1 through 7.3.8) for the sake of brevity.

We also may write

$$\begin{aligned} u_2 = & \sum_{n=0}^{\infty} r^n \left[A_{n0}^{(1)} \cos n\varphi + A_{n0}^{(2)} \sin n\varphi \right] \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I_n(k_m r) \left[A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right] Z_m(z) \end{aligned}$$

for $\beta_0 = \beta_H = 0$, $\alpha_0 = \alpha_H = 1$. Next, we consider the first case; we leave the second to the reader as a reading exercise.

The coefficients A_{nm} and B_{nm} are unknown as yet, but they can be found from the boundary conditions at the lateral surface, by using Equation (7.212):

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[\alpha k_m I_n'(k_m a) + \beta I_n(k_m a) \right] \left(A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right) Z_m(z) = g(\varphi, z). \quad (7.268)$$

Expanding $g(\varphi, z)$ in a Fourier series with respect to both z and φ yields

$$g(\varphi, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(g_{nm}^{(1)} \cos n\varphi + g_{nm}^{(2)} \sin n\varphi \right) Z_m(z),$$

where

$$\begin{aligned} g_{nm}^{(1)} &= \frac{1}{\pi \|Z_m\|^2} \int_0^{2\pi} \int_0^H g(\varphi, z) \cos n\varphi Z_m(z) d\varphi dz, \quad n > 0, \\ g_{nm}^{(2)} &= \frac{1}{\pi \|Z_m\|^2} \int_0^{2\pi} \int_0^H g(\varphi, z) \sin n\varphi Z_m(z) d\varphi dz, \quad n > 0, \end{aligned}$$

and

$$g_{0m}^{(1)} = \frac{1}{2\pi \|Z_m\|^2} \int_0^{2\pi} \int_0^H g(\varphi, z) Z_m(z) d\varphi dz, \quad g_{0m}^{(2)} = 0.$$

Comparing this with Equation (7.268), we obtain the coefficients

$$A_{nm}^{(1,2)} = \frac{g_{nm}^{(1,2)}}{\alpha k_m I'_n(k_m a) + \beta I_n(k_m a)}.$$

Thus, we have obtained the solution of the boundary value problem in Equations (7.210) through (7.214). Together with the solution of Equations (7.205) through (7.209) obtained earlier, we have, finally, completely solved the boundary value problem expressed in Equations (7.201) through (7.204).

Example 7.13. Consider a solid cylinder of radius l and height H . The top and bottom are maintained at zero temperature; the temperature on the lateral surface does not depend on φ and is kept constant:

$$u(r, \varphi, z)|_{r=l} = \frac{T_0}{H^2}(H - z)z. \quad (7.269)$$

Find the temperature within the cylinder.

Solution. Due to the symmetry in φ , we have the equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

with the boundary conditions

$$u(r, z, \varphi)|_{z=0} = u(r, z, \varphi)|_{z=H} = 0,$$

plus the condition in Equation (7.269). For $Z(z)$, we have, from the development above,

$$Z_m(z) = \sin k_m z, \quad k_m = \frac{\pi m}{H}, \quad m = 1, 2, \dots$$

and the equation for $R(r)$ becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - k_m^2 R = 0.$$

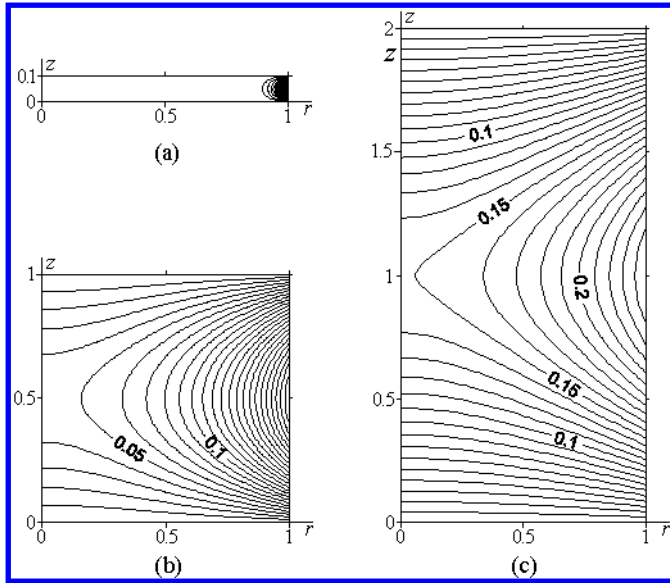


Figure 7.12. Temperature for three values of H : (a) 0.1, (b) 1.0, and (c) 2.0.

This is a modified Bessel equation, Equation (7.265), which has solutions that are modified Bessel functions of the first and second kind, $I_0(x)$ and $K_0(x)$. Thus, the general solution for $R(r)$ is

$$R_m(r) = c_1 I_0(k_m r) + c_2 K_0(k_m r).$$

Because $K_0(x)$ diverges at $x = 0$, the coefficient $c_2 = 0$.

Collecting the above results we may write the final solution as

$$u(r, z) = \sum_{m=1}^{\infty} A_m I_0(k_m r) \sin(k_m z). \quad (7.270)$$

At $r = a$, the series in Equation (7.270) is a trigonometric Fourier sine series with coefficients

$$\begin{aligned} A_m I_0(k_m a) &= \frac{2}{H} \int_0^H u(a, z) \sin k_m z dz \\ &= \frac{2T_0}{H^3} \int_0^H z(H - z) \sin k_m z dz = \frac{4T_0}{(\pi m)^3} [1 - (-1)^m], \end{aligned}$$

and we have

$$A_{2l} = 0, \quad A_{2l+1} = \frac{8T_0}{\pi^3(2l+1)^3 I_0(k_{2l+1}a)}, \quad l = 0, 1, \dots$$

Finally, the solution to the problem is

$$u(r, z) = \frac{8T_0}{\pi^3} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^3} \frac{I_0(k_{2l+1}r)}{I_0(k_{2l+1}a)} \operatorname{sink}_{2l+1}z.$$

The dimensionless temperature field is given in Figure 7.12 for three different values of H . The parameters are made dimensionless by plotting temperature in units of $T_1 - T_0$ and length in units of a . One can readily see that in layers at a small height, H , the temperature varies only near lateral side.

7.4.4 Example of Homogeneous Boundary Conditions at the Bases

As an example of homogeneous boundary conditions at the bases, consider the following problem for a circular cylinder:

$$\nabla^2 T = 0,$$

$$T|_{r=a} = T_0 + T_1 (1 + \cos \varphi),$$

$$T|_{z=0} = T_0, \quad T_z|_{z=H} = 0.$$

This problem is similar to the problem of the water barrel considered in Section 7.4.2, but in this case the barrel sits on a surface for which the temperature, T_0 , does not change. Also, the barrel is placed in the sun and

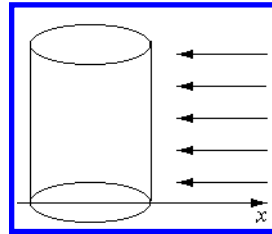


Figure 7.13. Heated barrel.

we assume that the sun's rays are parallel to the x -axis (see Figure 7.13). Thus, the temperature has a maximum value $T_0 + 2T_1$ when $\varphi = 0$ and is equal to $T_0 + T_1$ when $\varphi = \pi/2$. When $\varphi = \pi$ (for the part of the barrel in the shade), the barrel's surface temperature, is equal to the temperature of the environment, T_0 , (assumed to be air). The top of the barrel is opened and, due to the small thermal conductivity of air, the heat flux at the top is equal to zero.

As previously, we measure the temperature relative to T_0 , which means introducing a function u according to Equation (7.247). For u , we obtain the expressions

$$\nabla^2 u = 0,$$

$$u|_{r=l} = T_1 (1 + \cos \varphi),$$

$$u|_{z=0} = 0, \quad u_z|_{z=H} = 0.$$

Separating the variables, we obtain Equations (7.217) through (7.219). The equation for $Z(z)$ is limited by the conditions

$$Z(0) = 0, \quad Z'(H) = 0. \quad (7.271)$$

The Sturm-Liouville problem of Equations (7.217) and (7.271) has the obvious solution

$$Z_m(z) = \sin k_m z, \quad k_m = \frac{\pi}{2H} (2m + 1), \quad m = 0, 1, \dots,$$

with a squared norm equal to

$$\|Z_m\|^2 = \frac{H}{2}.$$

A general solution of Laplace's equation satisfying the boundary conditions at the bottom of the barrel is thus

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} I_n(k_m r) \left[A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right] \sin k_m z.$$

The boundary conditions at the walls give

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} I_n(k_m l) \left[A_{nm}^{(1)} \cos n\varphi + A_{nm}^{(2)} \sin n\varphi \right] \sin k_m z = T_1 (1 + \cos \varphi).$$

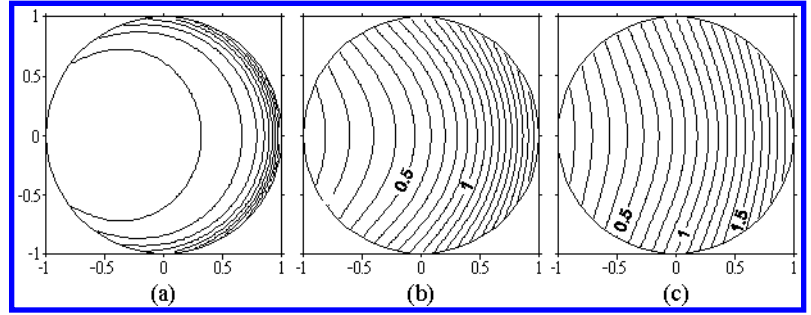


Figure 7.14. Isotherms for three different values of z : (a) 0.1, (b) 0.5, and (c) 1.0.

As previously, we expand the right side of this equation in a Fourier series in both coordinates:

$$T_1 (1 + \cos \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[T_{nm}^{(1)} \cos n\varphi + T_{nm}^{(2)} \sin n\varphi \right] \sin k_m z.$$

Upon examination, we have

$$T_{nm}^{(1)} = 0 \quad (n > 1, m \geq 0), \quad T_{nm}^{(2)} = 0 \quad (n \geq 0, m \geq 0).$$

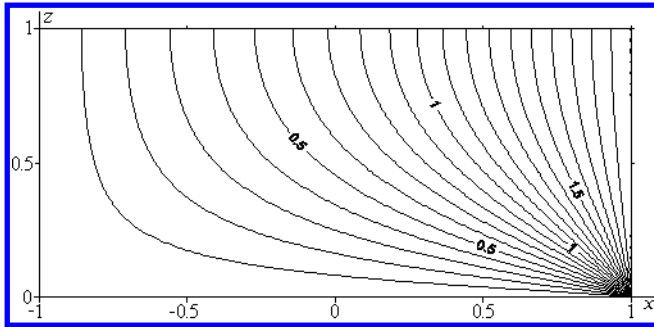
For the remaining coefficients, we obtain

$$T_{0m}^{(1)} = T_{1m}^{(1)} = \frac{2T_1}{H} \int_0^H \sin k_m z dz = \frac{2T_1}{k_m H}.$$

Thus, the water temperature in the barrel is

$$T = T_0 + 2T_1 \sum_{m=0}^{\infty} \frac{1}{k_m H} \left[\frac{I_0(k_m r)}{I_0(k_m l)} + \frac{I_1(k_m r)}{I_1(k_m l)} \cos \varphi \right] \sin k_m z. \quad (7.272)$$

Figures 7.14 and 7.15 demonstrate the results obtained by keeping 20 terms in the series (7.272) and for the values $H = l = T_1 = 1$, $T_0 = 0$. Figure 7.14 gives the temperature distribution for three values of z : (a) 0.1, (b) 0.5, and (c) 1.0. It is seen that near the bottom, the fluid temperature is close to the temperature of the environment almost everywhere. As z increases the temperature distribution becomes close to linear.

Figure 7.15. Isotherms in x - z plane.

The temperature in the x - z plane is shown in Figure 7.15. Because the boundary conditions on the walls and on the bottom do not match, the series in Equation (7.273) does not converge uniformly near the point $(1, 0)$, where the temperature has a discontinuity. In real situations, such a discontinuity does not occur because of a thin intermediate layer in which temperature smoothly changes from T_0 on the bottom to $T_0 + T_1 (1 + \cos \varphi)$ at the walls.

7.4.5 Application of Bessel Functions for the Solution of Laplace's and Poisson's Equations in a Circle

We have studied two-dimensional boundary value problems for Laplace's equation in Sections 7.3.2 through 7.3.6 and obtained the solutions of the two-dimensional Laplace's equation in polar coordinates in the form of a power series with respect to two linearly independent solutions, $r^{\pm n} \sin n\varphi$ (for $n = 0$ the second solution, linearly independent of r , is $\ln r$). However, in some cases, such series converge too slowly and another method can be more efficient. Note that this method is realized in the program Laplace (see Appendix E).

As was found in Section 7.4.1, the eigenfunctions of the Laplacian in the two-dimensional case can be also expressed in terms of the Bessel functions

$$V_{nm}^{(1)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi. \quad (7.273)$$

Therefore, we may also apply them to the solution of Laplace's and Poisson's equations for domains with circular symmetry. This method is very similar to that used for a solution of hyperbolic and parabolic equations.

Let us consider the following interior boundary value problem:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = -f(r, \varphi), \quad 0 \leq r < l, \quad 0 \leq \varphi < 2\pi, \quad (7.274)$$

$$\alpha \frac{\partial u}{\partial r}(r, \varphi) + \beta u(r, \varphi) \Big|_{r=l} = g(\varphi), \quad (7.275)$$

$$u(r, \varphi) = u(r, \varphi + 2\pi).$$

Here, α and β are real constants, $|\alpha| + |\beta| \neq 0$, and l is the radius of the circle. We consider the same three types of boundary conditions as were presented in Section 7.3.2.

The difficulty in solving for such nonhomogeneous boundary conditions is that it is impossible to separate variables to obtain an ordinary differential equation with respect to the independent variable r , as we did in Section 7.3.1. However, we can split the function $u(r, \varphi)$ into two functions:

$$u(r, \varphi) = v(r, \varphi) + w(r, \varphi), \quad (7.276)$$

where the introduced auxiliary function $w(r, \varphi)$ must satisfy the nonhomogeneous boundary condition (i.e., it includes the function $g(\varphi)$ and leaves the boundary condition for the function $v(r, \varphi)$ homogeneous). In this case, we write

$$\alpha \frac{\partial w}{\partial r}(a, \varphi) + \beta w(a, \varphi) = g(\varphi), \quad (7.277)$$

and a new, unknown function $v(r, \varphi)$ satisfies the Poisson equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} = -f^*(r, \varphi),$$

where the right-hand side becomes

$$f^*(r, \varphi) = f(r, \varphi) + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2}.$$

The boundary condition for function $v(r, \varphi)$ is homogeneous:

$$\alpha \frac{\partial v}{\partial r}(a, \varphi) + \beta v(a, \varphi) = 0.$$

The function $w(r, \varphi)$ can be chosen in different ways; the only restriction is that it should be continuous and finite. We will see next that, chosen in the form

$$w(r, \varphi) = (c_0 + c_1 r^2)g(\varphi),$$

we can accommodate Equation (7.220).

It is easy to verify by direct substitution that the following expressions for $w(r, \varphi)$ satisfy Equation (7.220).

1. For $\alpha = 0, \beta = 1$, we have:

(a) Boundary condition $u(r, \varphi)|_{r=l} = g(\varphi)$ with auxiliary function

$$w(r, \varphi) = \frac{r^2}{l^2} g(\varphi). \quad (7.278)$$

(b) Boundary condition $u(r, \varphi)|_{r=l} = g_0 = \text{const}$ with auxiliary function

$$w(r, \varphi) = g_0. \quad (7.279)$$

2. For $\alpha = 1, \beta = 0$, we have the boundary condition

$$\left. \frac{\partial u}{\partial r}(r, \varphi) \right|_{r=l} = g(\varphi),$$

with auxiliary function

$$w(r, \varphi) = \frac{r^2}{2l} g(\varphi) + C, \quad (7.280)$$

where C is an arbitrary constant.

3. For $\alpha = 1, \beta = h > 0$, we have the boundary condition

$$\left. \frac{\partial u}{\partial r}(r, \varphi) + hu(r, \varphi) \right|_{r=l} = g(\varphi),$$

with auxiliary function

$$w(r, \varphi) = \frac{r^2}{l(2 + hl)} g(\varphi). \quad (7.281)$$

Thus, the expression in Equation (7.276) with $w(r, \varphi)$ given by one of the Equations (7.278) through (7.281) allows us to solve Poisson's equation with homogeneous boundary conditions instead of solving Poisson's equation (or Laplace's equation at $f \equiv 0$) with nonhomogeneous conditions, given in Equation (7.275). The solution of Poisson's equation with homogeneous boundary conditions is discussed next.

Reading Exercise. Solve the above problem inside a ring: $a \leq r \leq b$. Boundary conditions are given by Equation (7.275) (with the opposite sign for β to ensure that the normal vector is external to the domain) and

$$\alpha_b u_r(r, \varphi) + \beta_b u(r, \varphi)|_{r=b} = g_b(\varphi).$$

Hint. Chose a quadratic function with respect to r for w if $\alpha \neq 0$ and $\alpha_b \neq 0$ and a linear function in other cases.

Reading Exercise. Reduce the solution of Laplace's equation inside a sector with nonhomogeneous boundary conditions at the boundaries of the sector given by

$$\nabla^2 u = 0,$$

$$u(r, \varphi)|_{r=l} = g(\varphi),$$

$$u(r, \varphi, \alpha)|_{\varphi=0} = f_{0,\alpha}(r)$$

to the solution of Poisson's equation with homogeneous boundary conditions at $\varphi = 0, \alpha$.

Hint. Introduce the auxiliary function $w(r, \varphi)$ as

$$w(r, \varphi) = u(r, \varphi) + \frac{\alpha - \varphi}{\alpha} [f_0(r) - f_\alpha(r)] + f_\alpha(r).$$

7.4.6 Poisson's Equation in Polar Coordinates with Homogeneous Boundary Conditions

Consider a boundary value problem for Poisson's equation,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} = -f^*(r, \varphi), \quad (7.282)$$

with zero boundary condition

$$\alpha \frac{\partial v}{\partial r}(r, \varphi) + \beta v(r, \varphi) \Big|_{r=l} = 0. \quad (7.283)$$

Recall that the solution of Poisson's equation is not zero for the case of homogeneous boundary conditions of Equation (7.283). We have discussed the method for solving this boundary value problem in Sections 7.3.7 and 7.3.8. However, for some cases, another method, using a series of the eigenfunctions of the Laplacian, is quite helpful. This method is also realized in the program Laplace. We investigate this method in the following discussion.

The solution to the problem of Equations (7.282) and (7.283) can be expanded in a series by eigenfunctions of the Sturm-Liouville problem for the Laplace operator over the same domain as

$$v(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[A_{nm} V_{nm}^{(1)}(r, \varphi) + B_{nm} V_{nm}^{(2)}(r, \varphi) \right], \quad (7.284)$$

where $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are the eigenfunctions of the Laplacian satisfying the corresponding boundary conditions. It is clear that, due to the orthogonality, the coefficients in Equation (7.284) can be represented via the (unknown) function $v(r, \varphi)$:

$$\begin{aligned} A_{nm} &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \int_0^l v(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \\ B_{nm} &= \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \int_0^l v(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi. \end{aligned} \quad (7.285)$$

To find a final form for these functions, we first multiply Equation (7.282) by $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ and integrate over the circular domain of radius l :

$$\int_0^{2\pi} \int_0^l \nabla^2 v V_{nm}^{(1)}(r, \varphi) r dr d\varphi = - \int_0^{2\pi} \int_0^l f^*(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \quad (7.286)$$

$$\int_0^{2\pi} \int_0^l \nabla^2 v V_{nm}^{(2)}(r, \varphi) r dr d\varphi = - \int_0^{2\pi} \int_0^l f^*(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi. \quad (7.287)$$

Now substitute Equation (7.284) into Equations (7.286) and (7.287). Because $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$ are the eigenfunctions of the Laplacian, we have

$$\nabla^2 V_{nm}^{(1,2)}(r, \varphi) = -\lambda_{nm} V_{nm}^{(1,2)}(r, \varphi)$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \equiv \nabla_r^2 + \frac{1}{r^2} \nabla_\varphi^2,$$

and the left sides of Equations (7.286) and (7.287) become

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda_{ij} \int_0^{2\pi} \int_0^l \left[A_{ij} V_{ij}^{(1)}(r, \varphi) + B_{ij} V_{ij}^{(2)}(r, \varphi) \right] V_{ij}^{(p)}(r, \varphi) r dr d\varphi,$$

Due to the orthogonality relation in Equation (7.231) for the functions $V_{nm}^{(p)}(r, \varphi)$, the only term in the sums that differs from zero is

$$\lambda_{nm} A_{nm} \|V_{nm}^{(p)}\|^2 \delta_{p1} \text{ or } \lambda_{nm} B_{nm} \|V_{nm}^{(p)}\|^2 \delta_{p2}.$$

Comparing with the right sides in Equations (7.286) and (7.287), we obtain

$$\lambda_{nm} A_{nm} = f_{nm}^{(1)},$$

$$\lambda_{nm} B_{nm} = f_{nm}^{(2)}, \quad n, m = 0, 1, 2, \dots,$$

where

$$\begin{aligned} f_{nm}^{(1)} &= \frac{1}{\|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \int_0^a f^*(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi, \\ f_{nm}^{(2)} &= \frac{1}{\|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \int_0^a f^*(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi. \end{aligned} \tag{7.288}$$

In the case of boundary conditions of the first or third type (Dirichlet condition or mixed condition), the eigenvalues $\lambda_{nm} \neq 0$ for all $n, m = 0, 1, 2, \dots$; in this case, the solution is defined uniquely. The coefficients are

$$\|R_{nm}\|^2 = \frac{l^2}{2} \left[J_n' \left(\mu_m^{(n)} \right) \right]^2,$$

and the solution has the form

$$v(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\lambda_{nm}} \left[f_{nm}^{(1)} V_{nm}^{(1)}(r, \varphi) + f_{nm}^{(2)} V_{nm}^{(2)}(r, \varphi) \right]. \quad (7.289)$$

In the case of boundary conditions of the second type (Neumann condition), the eigenvalue $\lambda_{00} = 0$ ($V_{00}^{(1)} = 1$, $V_{00}^{(2)} = 0$) and all other eigenvalues are nonzero.

If

$$f_{00}^{(1)} = \int_0^{2\pi} \int_0^l f^*(r, \varphi) r dr d\varphi = 0,$$

then the coefficient A_{00} is undefined. A solution to the given problem exists but is determined only up to an arbitrary additive constant. The other coefficients are defined uniquely. The solution in this case is

$$v(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[A_{nm} V_{nm}^{(1)}(r, \varphi) + B_{nm} V_{nm}^{(2)}(r, \varphi) \right] + \text{const.}$$

If

$$\int_0^{2\pi} \int_0^l f^*(r, \varphi) r dr d\varphi \neq 0,$$

then the solution to the given problem does not exist.

Thus, the *general solution* of Poisson's problem over a circular domain with a nonzero boundary condition has the form

$$\begin{aligned} u(r, \varphi) &= w(r, \varphi) + v(r, \varphi) \\ &= w(r, \varphi) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[A_{nm} V_{nm}^{(1)}(r, \varphi) + B_{nm} V_{nm}^{(2)}(r, \varphi) \right], \end{aligned}$$

or

$$u(r, \varphi) = w(r, \varphi) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [A_{nm} \cos n\varphi + B_{nm} \sin n\varphi] J_n \left(\frac{\mu_m^{(n)}}{l} r \right), \quad (7.290)$$

where

$$A_{nm} = \frac{f_{nm}^{(1)}}{\lambda_{nm}} = \frac{1}{\lambda_{nm} \|V_{nm}^{(1)}\|^2} \int_0^{2\pi} \int_0^l f^*(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi \quad (7.291)$$

and

$$B_{nm} = \frac{f_{nm}^{(2)}}{\lambda_{nm}} = \frac{1}{\lambda_{nm} \|V_{nm}^{(2)}\|^2} \int_0^{2\pi} \int_0^l f^*(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi, \quad (7.292)$$

with $V_{nm}^{(p)}(r, \varphi)$ defined by Equation (7.273) and the auxiliary function $w(r, \varphi)$ satisfying the boundary condition of Equation (7.277).

Reading Exercise. Explain why (1) in the case of boundary conditions of the first or third type, the eigenvalues $\lambda_{nm} \neq 0$ for all $n, m = 0, 1, 2, \dots$, and (2) in the case of boundary conditions of the second type, the eigenvalue $\lambda_{00} = 0$ and all other eigenvalues are nonzero.

Example 7.14. We return to the boundary value problem solved in Section 7.3.7:

$$\nabla^2 u = -Axy = -\frac{A}{2}r^2 \sin 2\varphi, \quad 0 \leq r < l, \quad (7.293)$$

$$u|_{r=l} = 0. \quad (7.294)$$

Verify that the solution derived in Section 7.3.7 is equivalent to a solution arrived at by using the method of the present section.

Solution. The solution to the boundary value problem in Equations (7.293) and (7.294) is given by the series in Equation (7.290) with coefficients $f_{nm}^{(1,2)}$ defined by Equations (7.288). It is obvious that

$$f(r, \varphi) = \frac{A}{2}r^2 \sin 2\varphi = \sin 2\varphi \sum_{m=1}^{\infty} f_{2m}^{(2)} J_2(k_{2m}r),$$

where k_{2m} is defined by the equation $J_2(k_{2m}l) = 0$ and

$$f_{2m}^{(2)} = \frac{A}{2 \|V_{2m}\|^2} \int_0^l r^3 J_2(k_{2m}r) dr, \quad \|V_{2m}\|^2 = \frac{l^2}{2} [J_2'(k_{2m}l)]^2. \quad (7.295)$$

The other coefficients, $f_{nm}^{(1,2)}$, equal zero. Evaluating the integral in Equation (7.295), we obtain

$$f_{2m}^{(2)} = -\frac{Al^2 J_0(k_{2m}l)}{2 [J_2'(k_{2m}l)]^2}.$$

Keeping in mind that $\lambda_{nm} = k_{nm}^2$, we arrive at the following form of solution to the boundary value problem:

$$u(r, \varphi) = -\frac{Al^2}{2} \sin 2\varphi \sum_{m=1}^{\infty} \frac{J_0(k_{2m}l) J_2(k_{2m}r)}{k_{2m}^2 [J_2'(k_{2m}l)]^2}. \quad (7.296)$$

On the other hand, this solution has to coincide with the function (see Example 7.10):

$$u(r, \varphi) = \frac{A}{24} r^2 (l^2 - r^2) \sin 2\varphi. \quad (7.297)$$

Expanding Equation (7.297) in the Fourier-Bessel series, we obtain

$$u(r, \varphi) = \frac{A}{24} r^2 (l^2 - r^2) \sin 2\varphi = \sin 2\varphi \sum_{m=1}^{\infty} u_m J_m(k_{2m}r),$$

where

$$u_m = \frac{A}{12a^2 [J_2'(k_{2m}l)]^2} \int_0^l r^3 (l^2 - r^2) J_2(k_{2m}r) dr. \quad (7.298)$$

Integrating Equation (7.298) twice by parts and taking the formulas

$$x^3 J_2(x) = [x^3 J_3(x)]' \text{ and } x^{-2} J_3(x) = -[x^{-2} J_2(x)]'$$

into account, we arrive at the expression

$$u_m = \frac{A}{k_{2m}^2 l^2 [J_2'(k_{2m}l)]^2} \int_0^l r^3 J_2(k_{2m}r) dr.$$

It is readily seen that $f_{2m}^{(2)} = -k_{2m}^2 u_m$. That is, solutions given by Equations (7.296) and (7.297) coincide with each other.

A surface plot of the solution $u(r, \varphi)$ to Example 7.14 for the values of the parameters $A = 10$, $l = 2$, $n_{\max} = m_{\max} = 5$ is presented in Figure 7.16.

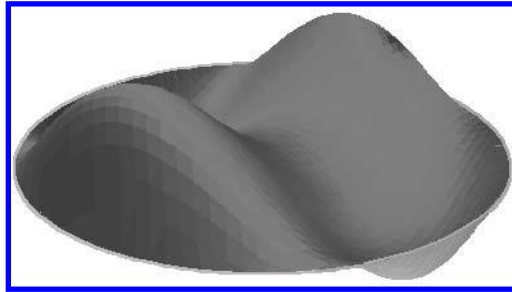


Figure 7.16. Surface plot of the solution $u(r, \varphi)$ to Example 7.14.

Problems

First, we present problems for elliptic equations for a rectangular domain.

In Problems 7.1 through 7.5, a thin homogeneous square plate with sides of length π lies in the x - y plane, and the center of the plate is located at the point O . The bottom and top surfaces are insulated, and the edges of the plate are kept at the temperature described by functions, $u(x, y)|_{\Gamma}$, given below. For each problem:

1. Find the temperature distribution in the plate.
2. Find the temperature at the points

$$P_1 \left(x = \frac{\pi}{2} - 1, y = 0 \right) \text{ and } P_2 \left(x = \frac{\pi}{2} + 1, y = 0 \right)$$

with a precision of 10^{-4} .

- 7.1.** $u|_{y=\pi/2} = x^2$, $u|_{y=-\pi/2} = u|_{x=0} = u|_{x=\pi} = 0$.
- 7.2.** $u|_{y=\pi/2} = u|_{y=-\pi/2} = \cos 2x$, $u|_{x=0} = u|_{x=\pi} = 0$.
- 7.3.** $u|_{y=\pi/2} = x^2$, $u|_{y=-\pi/2} = 0.8x^2$, $u|_{x=0} = u|_{x=\pi} = 0$.
- 7.4.** $u|_{y=\pi/2} = x^2 + \cos 2x$, $u|_{y=-\pi/2} = u|_{x=0} = u|_{x=\pi} = 0$.
- 7.5.** $u|_{x=0} = u|_{x=\pi} = (y + \pi/2)^2$, $u|_{y=\pi/2} = u|_{y=-\pi/2} = 0$.

In Problems 7.6 and 7.7, a thin homogeneous metal plate with sides of length π lies in the x - y plane, and the center of the plate is located at point O . The bottom and top surfaces are electrically insulated, and the edges of the plate are kept at an electric potential described by the functions, $u(x, y)|_{\Gamma}$, given below. For each problem:

1. Find the electric potential in the plate.

2. Find the potential at the points

$$P_1 \left(x = \frac{\pi}{2} - 1, y = 0 \right) \text{ and } P_2 \left(x = \frac{\pi}{2} + 1, y = 0 \right)$$

with a precision of 10^{-4} .

7.6. $u|_{y=\pi/2} = u|_{y=-\pi/2} = x^2 - x$, $u|_{x=0} = u|_{x=\pi} = 0$.

7.7. $u|_{y=-\pi/2} = x^2$, $u|_{y=\pi/2} = 0.7x$, $u|_{x=0} = u|_{x=\pi} = 0$.

In problems 7.8 and 7.9, an infinitely long rectangular cylinder has its central axis along the z -axis, and its cross section is a rectangle with sides of length π . The sides of the cylinder are kept at an electric potential described by the functions $u(x, y)|_{\Gamma}$ given below. For each problem:

1. Find the electric potential within the cylinder.
2. Find the potential at the points

$$P_1 \left(x = \frac{\pi}{2} - 1, y = 0 \right) \text{ and } P_2 \left(x = \frac{\pi}{2} + 1, y = 0 \right)$$

with a precision of 10^{-4} .

7.8. $u|_{y=\pi/2} = u|_{y=-\pi/2} = u|_{y=-\pi/2} = 3x^2$, $u|_{x=0} = u|_{x=\pi} = 0$.

7.9. $u|_{y=\pi/2} = \cos 2x - 2x$, $u|_{y=-\pi/2} = u|_{x=0} = u|_{x=\pi} = 0$.

7.10. Find a steady-state temperature distribution in a very long heat-conducting bar (the axis is along the z -axis) of rectangular cross section ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), with the following restrictions:

1. The face at $x = l_x$ is kept at a constant temperature T_0 , and the other three faces are kept at zero temperature.

Answer.

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)} \frac{\sinh[(2n+1)\pi x/l_y]}{\sinh[(2n+1)\pi l_x/l_y]} \sin \frac{n\pi y}{l_y}.$$

2. Write in a closed form (without solving the equation with the Fourier method) the solution if the face $y = l_y$ is kept at temperature T_0 , and the remaining three faces are kept at zero temperature.
3. Repeat when the faces $x = l_x$ and $y = l_y$ are kept at temperature T_0 , and the faces $x = 0$ and $y = 0$ are kept at zero temperature.

7.11. Find a steady-state temperature distribution in a cube of side l if the face at $z = 0$ is kept at temperature T_0 and the other five faces are kept at zero temperature.

Hint. While solving the three-dimensional Laplace's equation with the Fourier method, expand T_0 in the double Fourier series using $\sin(n\pi x/l) \sin(n\pi y/l)$.

7.12. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), if the sides $x = 0$ and $x = l_x$ are kept at the temperature $u = u_1 + (u_2 - u_1)y/l_y$, side $y = 0$ is kept at temperature $u = u_1$, and side $y = l_y$ is kept at temperature $u = u_2$.

Hint. Solve the boundary value problem for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$u(0, y) = u(l_x, y) = u_1 + \frac{u_2 - u_1}{l_y} y, \quad u(x, 0) = u_1, \quad u(x, l_y) = u_2.$$

Answer. $u(x, y) = w(x, y) = u_1 + \frac{u_2 - u_1}{l_y} y$.

7.13. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), if the sides $x = 0$ and $y = 0$ are kept at zero temperature, side $x = l_x$ is insulated, and side $y = l_y$ is kept at temperature $u(x, l_y) = \sin(5\pi x/2l_x)$.

Hint. Solve the boundary value problem for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$u(0, y) = u(x, 0) = 0, \quad \frac{\partial u}{\partial x}(l_x, y) = 0, \quad u(x, l_y) = \sin \frac{5\pi x}{2l_x}.$$

Answer.

$$\begin{aligned} u(x, y) &= w(x, y) + v(x, y) \\ &= \frac{y}{l_y} \sin \frac{5\pi x}{2l_x} + \sum_{m=1}^{\infty} (-1)^m \frac{50l_y^2}{m\pi [25l_y^2 + 4m^2l_x^2]} \sin \frac{5\pi x}{2l_x} \sin \frac{m\pi y}{l_y}. \end{aligned}$$

7.14. Solve the boundary value problem for Laplace's equation in a rectangle ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) with boundary conditions $u(l_x, y) = \sin^2(\pi y/l_y)$ and $u = 0$ on the other sides.

7.15. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq \pi$, $0 \leq y \leq \pi$), if through the sides $x = 0$, $x = \pi$ there are heat flows $u_x(0, y) = \sin y$ and $u_x(\pi, y) = \sin 5y$, respectively, and the two other sides are kept at temperatures $u(x, 0) = \cos x$, $u(x, \pi) = \cos 3x$.

Answer.

$$u(x, y) = \left(x - \frac{x^2}{2\pi}\right) \sin y + \frac{x^2}{2\pi} \sin 5y + \left(1 - \frac{y}{\pi}\right) \cos x + \frac{y}{\pi} \cos 3x \\ + \sum_{n=0}^{\infty} [C_{n1} \sin y + C_{n5} \sin 5y] \cos nx + \sum_{m=1}^{\infty} [C_{1m} \cos x + C_{3m} \cos 3x] \sin my,$$

where

$$C_{01} = -\frac{1}{\pi} \left(\frac{\pi^2}{3} + 1\right), \quad C_{n1} = \frac{2}{\pi n^2(n^2 + 1)}, \quad C_{05} = \frac{1}{\pi} \left(\frac{1}{25} - \frac{\pi^2}{6}\right), \\ C_{n5} = (-1)^{n+1} \frac{50}{\pi n^2(n^2 + 25)}, \quad C_{1m} = -\frac{2}{\pi m(m^2 + 1)}, \quad C_{3m} = (-1)^m \frac{18}{\pi m(m^2 + 9)}.$$

7.16. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) if the side $y = l_y$ is kept at zero temperature, side $y = 0$ is thermally insulated, side $x = 0$ is kept at zero temperature, and side $x = l_x$ is kept at temperature $u(l_x, y) = \cos(3\pi y/2l_y)$.

Answer.

$$u(x, y) = 1 - \frac{x}{l_x} \left(1 - \cos \frac{3\pi y}{2l_y}\right) - \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y} \sin \frac{n\pi x}{l_x} \\ + \sum_{n=1}^{\infty} (-1)^n \frac{18l_x^2}{n\pi [4n^2 l_y^2 + 9l_x^2]} \sin \frac{n\pi x}{l_x} \cos \frac{3\pi y}{2l_y}.$$

7.17. Solve the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$u(0, y) = u(x, 0) = u_1, \quad u(l_x, y) = u(x, l_y) = u_2.$$

Answer.

$$u(x, y) = u_1 + (u_2 - u_1) \frac{x}{l_x} + 2(u_2 - u_1) \sum_{n=1}^{\infty} \frac{1}{n\pi \sinh \sqrt{\lambda_n} l_y} \left\{ \sinh \sqrt{\lambda_n} y + (-1)^n \sinh \sqrt{\lambda_n} (l_y - y) \right\} \sin \frac{n\pi x}{l_x}.$$

7.18. Solve the boundary value problem for Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Q\delta(x - x_0)\delta(y - y_0), \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial y}(x, 0) = 0, \quad u(l_x, y) = u(x, l_y) = u_0$$

when the source of heat is acting at the point (x_0, y_0) ($0 < x_0 < l_x, 0 < y_0 < l_y$).

Answer.

$$u(x, y) = u_0 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} \cos \frac{(2n+1)\pi x}{2l_x} \cos \frac{(2m+1)\pi y}{2l_y},$$

where

$$C_{nm} = \frac{16Ql_x l_y}{\pi^2 [(2n+1)^2 l_y^2 + (2m+1)^2 l_x^2]} \cos \frac{(2n+1)\pi x_0}{2l_x} \cos \frac{(2m+1)\pi y_0}{2l_y}.$$

7.19. Find the steady-state temperature distribution in a very long heat-conducting bar (the axis is along the z -axis) of rectangular cross section ($0 \leq x \leq l_x, 0 \leq y \leq l_y$) if there are two uniformly distributed heat flows: one entering through the face $y = 0$ and the other leaving through the face $x = 0$. The two other faces are thermally insulated.

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$\frac{\partial u}{\partial x}(0, y) = -\frac{Q}{\kappa l_y}, \quad \frac{\partial u}{\partial x}(l_x, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{Q}{\kappa l_x}, \quad \frac{\partial u}{\partial y}(x, l_y) = 0.$$

A solution exists because

$$\oint_C \frac{\partial u}{\partial n} ds = \int_0^{l_x} [g_3(x) + g_4(x)] dx + \int_0^{l_y} [g_1(x) + g_2(x)] dx = 0.$$

Answer. $u(x, y) = \frac{Q}{2\kappa l_x l_y} [-x(2l_x - x) + y(2l_y - y)].$

7.20. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x, 0 \leq y \leq l_y$) if a constant heat flow Q is applied to one of its sides, and the other three sides are kept at constant temperature u_0 (which can be taken to be zero).

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$u(0, y) = u(l_x, y) = u(x, 0) = u_0, \quad \frac{\partial u}{\partial y}(x, l_y) = \frac{Q}{\kappa}.$$

Answer.

$$u(x, y) = u_0 + \frac{4Ql_x}{\kappa\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{\sinh \frac{(2k+1)\pi y}{l_x}}{\cosh \frac{(2k+1)\pi l_y}{l_x}} \sin \frac{(2k+1)\pi x}{l_x}.$$

7.21. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) if there is a heat exchange with a medium with temperature $u_{md} = \text{const}$ through the sides of the plate. A constant uniform source of heat Q is acting in the plate.

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Q, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$\left. \frac{\partial u}{\partial x} - h(u - u_{md}) \right|_{x=0} = 0, \quad \left. \frac{\partial u}{\partial x} + h(u - u_{md}) \right|_{x=l_x} = 0,$$

$$\left. \frac{\partial u}{\partial y} - h(u - u_{md}) \right|_{y=0} = 0, \quad \left. \frac{\partial u}{\partial y} + h(u - u_{md}) \right|_{y=l_y} = 0.$$

Answer.

$$u(x, y) = u_{md} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} X_n(x) Y_m(y),$$

where

$$C_{nm} = -\frac{Q}{\|V_{nm}\|^2 \lambda_{nm} \sqrt{\lambda_{xn} \lambda_{ym} (\lambda_{xn} + h^2) (\lambda_{ym} + h^2)}} \times \left[\sqrt{\lambda_{xn}} \sin \sqrt{\lambda_{xn}} l_x + h \left(1 - \cos \sqrt{\lambda_{xn}} l_x \right) \right] \left[\sqrt{\lambda_{ym}} \sin \sqrt{\lambda_{ym}} l_y + h \left(1 - \sin \sqrt{\lambda_{ym}} l_y \right) \right].$$

7.22. Find a steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) if there is a heat exchange with a medium with temperature $u_{md} = \text{const}$ through to the sides $x = 0$ and $x = l_x$. The two other sides, $y = 0$ and $y = l_y$, are thermally insulated. A source of heat

$$Q(x, y) = A \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y}$$

is acting in the plate.

Hint. The boundary value problem is

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= -A \cos \frac{\pi x}{l_x} \cos \frac{\pi y}{l_y}, \quad 0 < x < l_x, \quad 0 < y < l_y, \\ \frac{\partial u}{\partial x} - h(u - u_{md}) \Big|_{x=0} &= 0, \quad \frac{\partial u}{\partial x} + h(u - u_{md}) \Big|_{x=l_x} = 0, \quad \frac{\partial u}{\partial y} \Big|_{y=0} = \frac{\partial u}{\partial y} \Big|_{y=l_y} = 0. \end{aligned}$$

Answer.

$$u(x, y) = u_{md} + \sum_{n=0}^{\infty} C_{n1} \frac{1}{\sqrt{\lambda_{xn} + h^2}} \left[\sqrt{\lambda_{xn}} \cos \sqrt{\lambda_{xn}} x + h \sin \sqrt{\lambda_{xn}} x \right] \cos \frac{\pi y}{l_y},$$

where

$$\begin{aligned} C_{n1} &= \frac{2Al_y^2 l_x \sqrt{\lambda_{xn} (\lambda_{xn} + h^2)}}{[l_x (\lambda_{xn} + h^2) + 2h] (\pi^2 + \lambda_{xn} l_y^2) (\pi^2 - \lambda_{xn} l_x^2)} \\ &\quad \times \left[\sin \sqrt{\lambda_{xn}} l_x - h l_x \left(1 + \cos \sqrt{\lambda_{xn}} l_x \right) \right]. \end{aligned}$$

7.23. Find the steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$). There is a heat exchange with a medium with temperature $u_{md} = \text{const}$ through the side $x = 0$, the side $x = l_x$ is thermally insulated, and the sides $y = 0$ and $y = l_y$ are kept at temperatures $u(x, 0) = u_1$ and $u(x, l_y) = \sin(5\pi x/2l_x)$, respectively. A source of heat

$$Q(x, y) = Ax \sin \frac{2\pi y}{l_y}$$

is acting in the plate.

Hint. The boundary value problem is

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= -Ax \sin \frac{2\pi y}{l_y}, \quad 0 < x < l_x, \quad 0 < y < l_y, \\ \frac{\partial u}{\partial x} - h(u - u_{md}) \Big|_{x=0} &= 0, \quad \frac{\partial u}{\partial x} \Big|_{x=l_x} = 0, \quad u|_{y=0} = u_1, \quad u|_{y=l_y} = \sin \frac{5\pi x}{2l_x}. \end{aligned}$$

Answer.

$$\begin{aligned}
 u(x, y) = & \left[u_1 - h(u_1 - u_{md}) \left(1 - \frac{x}{l_x} \right) \right] \left(1 - \frac{y}{l_y} \right) \\
 & + \left[Ax \sin \frac{2\pi y}{l_y} - \left(hu_{md} + \frac{5\pi}{2l_x} \right) \left(1 - \frac{x}{l_x} \right) \right] \frac{y}{l_y} \\
 & + \sum_{n=1}^{\infty} \frac{A_n \sinh \sqrt{\lambda_n} y + B_n \sinh \sqrt{\lambda_n} (l_y - y)}{\sinh \sqrt{\lambda_n} l_y} \cos \sqrt{\lambda_{xn}} (l_x - x) \\
 & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \cos \sqrt{\lambda_{xn}} (l_x - x) \sin \frac{m\pi y}{l_y},
 \end{aligned}$$

where

$$A_n = \frac{1}{\|X_n\|^2} \left(hu_{md} + \frac{5\pi}{2l_x} \right) \int_0^{l_x} \left(1 - \frac{x}{l_x} \right) \cos \sqrt{\lambda_n} (l - x) dx,$$

$$B_n = \frac{1}{\|X_n\|^2} h(u_1 - u_{md}) \int_0^{l_x} \left(1 - \frac{x}{l_x} \right) \cos \sqrt{\lambda_n} (l - x) dx,$$

$$C_{nm} = \frac{1}{\lambda_{nm} \|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} Ax \sin \frac{2\pi y}{l_y} \cdot \cos \sqrt{\lambda_{xn}} (l_x - x) \sin \frac{m\pi y}{l_y} dx dy,$$

$$\|X_n\|^2 = \frac{1}{2} \left(l_x + \frac{h}{\lambda_{xn} + h^2} \right), \quad \|V_{nm}\|^2 = \frac{l_y}{4} \left(l_x + \frac{h}{\lambda_{xn} + h^2} \right).$$

7.24. Find the steady-state temperature distribution in a thin homogeneous rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$). There is a heat exchange with a medium with temperature $u_{md} = \text{const}$ through the side $x = l_x$, the side $y = 0$ is thermally insulated, the side $x = 0$ is kept at temperature $u(0, x) = u_1$, and the side $y = l_y$ is kept at zero temperature. A source of heat $Q(x, y) = Axy$ is acting in the plate.

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Axy, \quad 0 < x < l_x, \quad 0 < y < l_y,$$

$$u|_{x=0} = u_1, \quad \frac{\partial u}{\partial x} + h(u - u_{md}) \Big|_{x=l_x} = 0, \quad \frac{\partial u}{\partial y} \Big|_{y=0} = 0, \quad u|_{y=l_y} = 0.$$

Answer.

$$u(x, y) = u_1 - (u_1 - hu_{md}) \frac{x}{l_x} + \sum_{n=1}^{\infty} A_n \frac{\cosh \sqrt{\lambda_n} y}{\cosh \sqrt{\lambda_n} l_y} \sin \sqrt{\lambda_n} x \\ + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \sqrt{\lambda_n} x \cos \frac{\pi(2m+1)y}{2l_y},$$

where

$$A_n = \frac{1}{\|X_n\|^2} \int_0^{l_x} \left[-u_1 + (u_1 - hu_{md}) \frac{x}{l_x} \right] \sin \sqrt{\lambda_n} x dx, \\ C_{nm} = \frac{1}{\lambda_{nm} \|V_{nm}\|^2} \int_0^{l_x} \int_0^{l_y} Axy \sin \sqrt{\lambda_n} x \cos \frac{\pi(2m+1)y}{2l_y} dx dy, \\ \|X_n\|^2 = \frac{1}{2} \left(l_x + \frac{h}{\lambda_{xn} + h^2} \right), \quad \|V_{nm}\|^2 = \frac{l_y}{4} \left(l_x + \frac{h}{\lambda_{xn} + h^2} \right).$$

In the following problems, we provide further applications of the theory presented in this chapter. The answer to each problem can be easily verified by making sure that each term in the answer is a harmonic function and checking that the boundary conditions are satisfied. In some problems, the method described in Section 7.3.2 should be used. For these cases, represent f as the sum of two terms, $f = f_1(\varphi) + f_2(\varphi)$, and search for the particular solutions corresponding to $f_1(\varphi)$ and $f_2(\varphi)$ separately. In many cases, the Fourier series contains a finite number of nonzero harmonics, and the coefficients of harmonics can be found without integration (see also Example 7.5 in Section 7.3.2). To model the problems, you may use the program Laplace. Directions for the use of this program are in Appendix E.

The Interior Dirichlet Problem for a Circle

7.25. Find the electrostatic potential of an infinitely long cylinder of radius $r = l$ if the potential on the surface is given by the function

$$f(\varphi) = \sin 5\varphi.$$

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < l, \quad 0 \leq \varphi < 2\pi,$$

$$u(r, \varphi)|_{r=l} = \sin 5\varphi.$$

For an infinite cylinder, function u does not depend on z .

Answer. $u(r, \varphi) = \left(\frac{r}{l}\right)^5 \sin 5\varphi$.

7.26. Solve Problem 7.25 for the condition $u|_{r=l} = A$.

Answer. $u(r, \varphi) = A$.

7.27. Solve Problem 7.25 for the condition $u|_{r=l} = A \cos \varphi$.

Answer. $u(r, \varphi) = \frac{A}{l} r \cos \varphi$ or $u(x, y) = \frac{A}{l} x$.

7.28. Find the steady-state distribution of the temperature in a thin homogeneous plate of radius l if half of the plate contour ($0 \leq \varphi < \pi/2$) is kept at temperature $A = \text{const}$, and the other half ($\pi/2 \leq \varphi < 2\pi$) is kept at temperature $B = \text{const}$. Heat exchange between the plate and environment is absent.

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < l, \quad 0 \leq \varphi < 2\pi,$$

and the boundary condition is

$$u(l, \varphi) = \begin{cases} A, & 0 \leq \varphi < \frac{\pi}{2}, \\ B, & \frac{\pi}{2} \leq \varphi < 2\pi. \end{cases}$$

Answer.

$$u = \frac{A+3B}{4} + \frac{A-B}{\pi} \sum_{n=1}^{\infty} \left[\left(\frac{r}{l}\right)^n \frac{\sin n\varphi}{n} - (-1)^n \left(\frac{r}{l}\right)^{2n} \left(\frac{l}{r} \frac{\cos(2n-1)\varphi}{2n-1} + \frac{\sin 2n\varphi}{2n} \right) \right].$$

For Problems 7.29 through 7.31, consider a circle of radius l with the center at the origin of the coordinate system. Let (r, φ) be polar coordinates and (x, y) be Cartesian coordinates. Find the solution of the first interior boundary value problem for Laplace's equation for the indicated boundary conditions.

7.29. $u|_{r=l} = A + By$.

Answer. $u(r, \varphi) = A + Br \sin \varphi$ or $u(x, y) = A + By$.

7.30. $u|_{r=l} = Axy$.

Answer. $u(r, \varphi) = \frac{A}{2l^2} r^2 \sin 2\varphi$ or $u(x, y) = Axy$.

$$7.31. \quad u|_{r=l} = A \sin^2 \varphi + B \cos^2 \varphi.$$

Hint. While obtaining a solution, take into account that x , y , xy , $x^2 - y^2$, and their linear combinations are harmonic functions. You can justify the correctness of the answer by substituting it into the equation and applying the boundary conditions.

Answer.

$$u(r, \varphi) = \frac{A}{2} \left(1 - \frac{r^2}{l^2} \cos 2\varphi \right) + \frac{B}{2} \left(1 + \frac{r^2}{l^2} \cos 2\varphi \right)$$

or

$$u(x, y) = \frac{A+B}{2} + \frac{B-A}{2l^2} (x^2 - y^2).$$

Comment. The Fourier series representing the solution actually contains several nonzero terms because the integrals in expressions for the coefficients A_n and B_n are not zero for only a few values of n when the functions representing the boundary conditions are sine or cosine functions.

As an illustration, consider Problem 7.28. When we change from variables (r, φ) to variables (x, y) , the boundary condition becomes $u(x, y) = (A/l)x$. Thus, the solution is the harmonic function $u(x, y) = (A/l)x$. But it is also clear that an expansion of $u|_{r=l} = A \cos \varphi$ into Fourier series contains only one term, $\cos \varphi$, with coefficients A . Therefore, the solution of the interior problem is

$$u(r, \varphi) = \frac{A}{l} r \cos \varphi.$$

In Problems 7.32 through 7.36, a thin homogeneous circular plate of radius $l = 10$ lies in the x - y plane, and the center of the plate is located at point O . The bottom and top surfaces are insulated, and the edge boundary of the plate is kept at the temperature described by functions of the polar angle $\tilde{u}(\varphi)$, given in each problem.

1. Find the temperature distribution in the plate.
2. Find the temperature at the points $P_1(r = 1, \varphi = \pi/6)$ and $P_2(r = 5, \varphi = \pi/6)$ with a precision of 10^{-4} .

$$7.32. \quad \tilde{u}(\varphi) = \cos \varphi/2.$$

$$7.33. \quad \tilde{u}(\varphi) = 0.5 + \sin \varphi/2.$$

$$7.34. \quad \tilde{u}(\varphi) = \cos \varphi/2 + \varphi/\pi.$$

$$7.35. \quad \tilde{u}(\varphi) = \cos \varphi/2 + \sin \varphi/2.$$

7.36. $\tilde{u}(\varphi) = 0.02\varphi^2 - 0.04\pi\varphi$.

In Problems 7.37 and 7.38, a thin homogeneous circular plate of radius $l = 10$ lies in the x - y plane, and the center of the plate is located at point O . The bottom and top surfaces are electrically insulated, and the edge boundary of the plate is kept at an electric potential described by the functions of the polar angle $\tilde{u}(\varphi)$ given in each problem.

1. Find the electric potential in the plate.
2. Find the potential at the points $P_1(r = 1, \varphi = \pi/6)$ and $P_2(r = 5, \varphi = \pi/6)$ with a precision of 10^{-4} .

7.37. $\tilde{u}(\varphi) = 1/\pi + 0.1 \sin \varphi/2$.

7.38. $\tilde{u}(\varphi) = 0.2 \sin \varphi/2 + 1/10\pi$.

In Problems 7.39 and 7.40, an infinitely long circular cylinder has the central axis along the z -axis and a radius of $l = 10$. The surface of the cylinder is kept at an electric potential described by functions of the polar angle $\tilde{u}(\varphi)$, given in each problem.

1. Find the electric potential within the cylinder.
2. Find the potential at the points $P_1(r = 1, \varphi = \pi/6)$ and $P_2(r = 5, \varphi = \pi/6)$ with a precision of 10^{-4} .

7.39. $\tilde{u}(\varphi) = \cos \varphi/2 - \sin \varphi/2$.

7.40. $\tilde{u}(\varphi) = \sin \varphi/2 - \varphi/\pi^2$.

The Interior Neumann Problem for a Circle

Remember that a solution of the second boundary value problem contains an arbitrary constant, so the answers to the following problems all contain the constant C .

7.41. Find the steady-state temperature distribution in an infinitely long circular cylinder of radius l if a steady heat flow supplied to its surface is

$$\left. \frac{\partial u}{\partial r}(l, \varphi) \right|_{r=l} = f(\varphi) = A \sin \varphi + B \sin^3 \varphi.$$

Hint. The boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < l, \quad 0 \leq \varphi < 2\pi,$$

with the boundary condition

$$\left. \frac{\partial u}{\partial r}(r, \varphi) \right|_{r=l} = A \sin \varphi + B \sin^3 \varphi.$$

The problem is correctly posed because

$$\int_C f(l, \varphi) ds = \frac{2}{l} \left[3(A + 2B) \int_0^{2\pi} \sin \varphi d\varphi - 5B \int_0^{2\pi} \sin^3 \varphi d\varphi \right] = 0.$$

Answer.

$$u(r, \varphi) = \left(A + \frac{3}{4}B \right) r \sin \varphi - \frac{B}{12l^2} r^3 \sin 3\varphi + C$$

or

$$u(x, y) = \left(A + \frac{3}{4}B \right) y - \frac{B}{12l^2} [3(x^2 + y^2)y - 4y^3] + C.$$

In Problems 7.42 through 7.46, solve the second boundary value problem for Laplace's equation

$$\nabla^2 u = 0, \quad \left. \frac{\partial u}{\partial n} \right|_C = f(\varphi),$$

for a circle C with radius l centered in the point $r = 0$ for the given cases.

7.42. $f = A$.

Answer. The problem is set incorrectly because $\int_C f ds \neq 0$.

7.43. $f = Ax$.

Hint. The function $u(x, y) = Dx$ or $u(r, \varphi) = Dr \cos \varphi$ is harmonic. Differentiation along the normal to the circle line coincides with the differentiation with r . From the boundary condition at $r = l$, we find $D = lA$; thus, $u(x, y) = Alx$ or $u(r, \varphi) = Alr \cos \varphi$.

Answer. $u(r, \varphi) = Alr \cos \varphi + C$ or $u(x, y) = Alx + C$.

7.44. $f = A(x^2 - y^2)$.

Answer. $u(r, \varphi) = Alr \cos 2\varphi + C$ or $u(x, y) = \frac{A}{2}l(x^2 - y^2) + B$.

7.45. $f = A \cos \varphi + B$.

Answer. $u(r, \varphi) = Alr \cos \varphi + C$ if $B = 0$. If $B \neq 0$ the problem is set incorrectly, because $\int_C f ds \neq 0$.

7.46. $f = A \sin 3\varphi$.

Various Interior Problems for a Circle

7.47. Find the steady-state temperature distribution in an infinitely long circular cylinder of radius l if, at its surface, there is a heat exchange with the medium. The medium temperature is

$$u_{md}(\varphi) = \sin \varphi + \cos 4\varphi.$$

Hint. The boundary value problem consists of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < l, \quad 0 \leq \varphi < 2\pi,$$

and the boundary condition is

$$\frac{\partial u}{\partial r}(l, \varphi) + h [u(l, \varphi) - u_{md}(\varphi)] = 0,$$

or

$$\frac{\partial u}{\partial r}(l, \varphi) + hu(l, \varphi) = h(\sin \varphi + \cos 4\varphi).$$

Answer. $u(r, \varphi) = \frac{hl}{hl+1} \frac{r}{l} \sin \varphi + \left(\frac{r}{l}\right)^4 \frac{hl}{hl+4} \cos 4\varphi.$

In Problems 7.48 through 7.51, solve the boundary value problem for Laplace's equation inside a circle ($0 \leq r \leq l$, $0 \leq \varphi \leq 2\pi$), for the given boundary conditions.

7.48. $u|_{r=l} = 2 \sin^2 \varphi + 4 \cos^3 \varphi.$

Hint. $u(r, \varphi) = 1 + 3\frac{r}{l} \cos \varphi - \left(\frac{r}{l}\right)^2 \cos 2\varphi + \left(\frac{r}{l}\right)^3 \cos 3\varphi.$

7.49. $\left. \frac{\partial u}{\partial r} \right|_{r=l} = 4 \sin^4 \varphi - \frac{3}{2}.$

Hint. $u(r, \varphi) = \frac{l}{8} \left(\frac{r}{l}\right)^4 \cos 4\varphi - l \left(\frac{r}{l}\right)^2 \cos 2\varphi + \text{const.}$

7.50. $u|_{r=l} = 32 (\sin^6 \varphi + \cos^6 \varphi).$

Hint. $u(r, \varphi) = 20 + 12 \left(\frac{r}{l}\right)^4 \cos 4\varphi.$

7.51. $\left. \frac{\partial u}{\partial r} + hu \right|_{r=l} = \sin \varphi + \cos 4\varphi.$

Hint. $u(r, \varphi) = \frac{r}{hl+1} \sin \varphi + \frac{1}{l^3} \frac{r^4}{hl+4} \cos 4\varphi.$

Various Exterior Problems for a Circle

For Problems 7.52 through 7.54, solve the boundary value problem for Laplace's equation outside a circle ($r \geq l$, $0 \leq \varphi \leq 2\pi$) with the given boundary conditions.

7.52. $u|_{r=l} = 8 \cos^4 \varphi$.

Answer. $u(r, \varphi) = 3 + 4 \left(\frac{l}{r}\right)^2 \cos 2\varphi + \left(\frac{l}{r}\right)^4 \cos 4\varphi$.

7.53. $\left. \frac{\partial u}{\partial r} \right|_{r=l} = \sin \varphi + 4 \sin^3 \varphi$.

Answer. $u(r, \varphi) = -4 \frac{l^2}{r} \sin \varphi + \frac{l^4}{3r^3} \sin 3\varphi + \text{const.}$

7.54. $\left. \frac{\partial u}{\partial r} - hu \right|_{r=l} = 1 + \cos 2\varphi$.

Answer. $u(r, \varphi) = -\frac{1}{h} - \frac{l}{hl+2} \left(\frac{l}{r}\right)^2 \cos 2\varphi$.

Various Problems for a Pie-Shaped Sector

In Problems 7.55 through 7.58, solve the boundary value problem for Laplace's equation inside a pie-shaped sector ($0 \leq r \leq l$, $0 \leq \varphi \leq \alpha$) for the given boundary conditions.

7.55. $\alpha = \pi$, $u|_{r=l} = A \sin^2 \varphi + 2B \cos^2 \varphi$, $u|_{\varphi=0} = u|_{\varphi=\pi} = 0$.

Answer.

$$u(r, \varphi) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(n^2 - 4)} (Bn^2 - A) \left(\frac{r}{l}\right)^n \sin n\varphi,$$

or

$$u(r, \varphi) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{B(2k+1)^2 - A}{(2k+1)(2k-1)(2k-3)} \left(\frac{r}{l}\right)^{2k+1} \sin(2k+1)\varphi.$$

Notice that the boundary conditions along the sector boundaries and on the circular boundary do not match each other when $B \neq 0$, which leads to slow convergence of the series. Problems 7.56 and 7.58 are two such cases.

$$7.56. \alpha = \frac{\pi}{4}, u|_{r=l} = 1, u|_{\varphi=0} = 0, \left. \frac{\partial u}{\partial \varphi} \right|_{\varphi=\pi/4} = 0.$$

$$\text{Answer. } u(r, \varphi) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{r}{l} \right)^{4n+2} \sin(4n+2)\varphi.$$

$$7.57. \alpha = \frac{\pi}{2}, u|_{r=l} = 1 + \cos 4\varphi, \left. \frac{\partial u}{\partial \varphi} \right|_{\varphi=0} = \left. \frac{\partial u}{\partial \varphi} \right|_{\varphi=\pi/2} = 0.$$

$$\text{Answer. } u(r, \varphi) = 1 + \left(\frac{r}{l} \right)^4 \cos 4\varphi.$$

$$7.58. \alpha = \pi, \left. \frac{\partial u}{\partial r} \right|_{r=l} = \sin \varphi, \left. \frac{\partial u}{\partial \varphi} \right|_{\varphi=0} = u|_{\varphi=\pi} = 0.$$

$$\text{Answer. } u(r, \varphi) = -\frac{16l}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+3)(4n^2-1)} \left(\frac{r}{l} \right)^{n+1/2} \cos \left(n + \frac{1}{2} \right) \varphi.$$

Laplace's Equation on a Ring

In Problems 7.59 through 7.62, solve the boundary value problem for Laplace's equation in an annulus ($a \leq r \leq b$, $0 \leq \varphi \leq 2\pi$) with the given boundary conditions.

$$7.59. u|_{r=a} = 0, u|_{r=b} = \sin \varphi + 2 \cos^2 \varphi.$$

$$\text{Answer. } u(r, \varphi) = \frac{\ln(r/a)}{\ln(b/a)} + \frac{r^2 - a^2}{b^2 - a^2} \frac{b}{r} \sin \varphi + \frac{r^4 - a^4}{b^4 - a^4} \left(\frac{b}{r} \right)^2 \cos 2\varphi.$$

$$7.60. \left. \frac{\partial u}{\partial r} \right|_{r=a} = 4 \sin^3 \varphi, u|_{r=b} = 0.$$

$$\text{Answer. } u(r, \varphi) = -3 \frac{b^2 - r^2}{b^2 + a^2} \frac{a^2}{r} \sin \varphi + \frac{1}{3} \frac{b^6 - r^6}{b^6 + a^6} \frac{a^4}{r^3} \sin 3\varphi.$$

$$7.61. u|_{r=a} = 1, \left. \frac{\partial u}{\partial r} \right|_{r=b} = 2 \sin^2 \varphi.$$

$$\text{Answer. } u(r, \varphi) = 1 - \frac{r^4 - a^4}{2(b^4 + a^4)} \frac{b^3}{r^2} \cos 2\varphi + b \ln \frac{r}{a}.$$

$$7.62. \left. \frac{\partial u}{\partial r} \right|_{r=a} = \sin \varphi, \left. \frac{\partial u}{\partial r} \right|_{r=b} = \cos \varphi.$$

$$\text{Answer. } u(r, \varphi) = \frac{r^2 + a^2}{b^2 - a^2} \frac{b^2}{r} \cos \varphi - \frac{r^2 + b^2}{b^2 - a^2} \frac{a^2}{r} \sin \varphi.$$

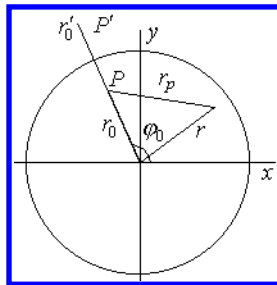


Figure 7.17. Domain for Problem 7.63.

Two-Dimensional Poisson's Problems

7.63. Find the steady-state temperature distribution in a thin uniform plate of radius l if the plate contour is maintained at zero temperature. A source of heat supplies a constant power Q and is placed at the point $P(r_0, \varphi_0)$, where $0 \leq r_0 < l$, $0 \leq \varphi_0 < 2\pi$, as shown in Figure 7.17. The plate is insulated and there is no heat exchange with the environment.

Answer. The problem is described by Poisson's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = -Q\delta(r - r_0)\delta(\varphi - \varphi_0)$$

under the zero boundary condition $u(r, \varphi)|_{r=l} = 0$.

Consider two ways to solve this problem:

1. First solve the problem with eigenfunctions of the Laplace operator in a circle. Eigenvalues and eigenfunctions of the problem are

$$\lambda_{nm} = \left(\mu_m^{(n)} / l \right)^2$$

where $\mu_m^{(n)}$ are positive roots of the equation $J_n(\mu) = 0$,

$$V_{nm}^{(1)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)}(r, \varphi) = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi.$$

$$\|V_{nm}^{(1)}\|^2 = \|V_{nm}^{(2)}\|^2 = \sigma_n \pi \frac{a^2}{2} \left[J_n' \left(\mu_m^{(n)} \right) \right]^2, \quad \sigma_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n \neq 0. \end{cases}$$

Find $f_{nm}^{(1)}$ and $f_{nm}^{(2)}$:

$$f_{nm}^{(1)} = \frac{2Q}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \cos(n\varphi_0) J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right).$$

$$f_{nm}^{(2)} = \frac{2Q}{\sigma_n \pi l^2 \left[J_n' \left(\mu_m^{(n)} \right) \right]^2} \sin(n\varphi_0) J_n \left(\frac{\mu_m^{(n)}}{l} r_0 \right).$$

Thus, the temperature distribution inside the plate is given by the series

$$u(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{l^2}{\left(\mu_m^{(n)} \right)^2} \left[f_{nm}^{(1)} \cos(n\varphi) + f_{nm}^{(2)} \sin(n\varphi) \right] J_n \left(\frac{\mu_m^{(n)}}{l} r \right).$$

2. Obtain the solution to this problem another way by finding some particular solution of Poisson's equation by using an electrostatics analogue (see Section 7.3.8, Equations (7.192) through (7.194)):

$$u_p = Q \ln \frac{l}{r_p} = -\frac{Q}{2} \ln \frac{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}{l^2}.$$

This solution is the electric potential of an infinitely thin charged line placed at the point (r_0, φ_0) . Obviously, this solution does not satisfy the boundary conditions, and we have to add to it a solution of Laplace's equation:

$$\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r} \frac{\partial u_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_0}{\partial \varphi^2} = 0$$

with the boundary condition

$$u_0(r, \varphi)|_{r=l} = -u_p(l, \varphi) = \frac{Q}{2} \ln [1 + t^2 - 2t \cos(\varphi - \varphi_0)],$$

where $t = r_0/l < 1$. With the help of Equation (7.189) from Section 7.3.8,

$$\sum_{n=1}^{\infty} \frac{1}{n} t^n \cos n\theta = -\frac{1}{2} \ln (1 - 2t \cos \theta + t^2),$$

we obtain the boundary conditions for u_0 :

$$u_0(l, \varphi) = -Q \sum_{n=1}^{\infty} \frac{1}{n} t^n \cos n(\varphi - \varphi_0).$$

Gathering terms, the solution of Laplace's equation with the given boundary conditions may be written as

$$u_0(r, \varphi) = -Q \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{rr_0}{l^2} \right)^n \cos n(\varphi - \varphi_0) = \frac{Q}{2} \ln \left[1 + \left(\frac{rr_0}{l^2} \right)^2 - 2 \frac{rr_0}{l^2} \cos(\varphi - \varphi_0) \right].$$

Thus, a general solution of the problem is

$$u(r, \varphi) = u_0 + u_p = \frac{Q}{2} \ln \frac{l^4 + (rr_0)^2 - 2l^2 rr_0 \cos(\varphi - \varphi_0)}{l^2 [r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)]}.$$

7.64. Find the steady-state temperature distribution in a thin homogeneous circular plate of radius l if the contour of the plate is maintained at zero temperature and a uniformly distributed source of heat with power

$$Q(r, \varphi) = r^2 \cos 2\varphi$$

is acting in the plate. The plate is insulated and there is no heat exchange with the environment.

Answer. $u = \frac{1}{12} \left[2 \left(\frac{r}{l} \right)^2 - \left(\frac{r}{l} \right)^4 \right] \cos 2\varphi.$

7.65. Find the solution of Poisson's equation $\nabla^2 u = -A$ in an annulus ($a \leq r \leq b$) if $u|_{r=a} = u_1$, $u|_{r=b} = u_2$, where A , u_1 , u_2 are constants.

Answer.

$$u(r, \varphi) = u_2 - \frac{A}{4} (r^2 - b^2) + \frac{u_1 - u_2 - A(b^2 - a^2)/4}{\ln b - \ln a} \ln \frac{b}{r}$$

7.66. Solve the boundary value problem for a disk given by

$$\nabla^2 u = 1, \quad r < l, \quad u|_{r=l} = 0.$$

Answer. $u = \frac{1}{4}(r^2 - l^2).$

7.67. Solve Poisson's equation,

$$\nabla^2 u = A + B(x^2 - y^2)$$

for an annulus ($a \leq r \leq b$) with boundary conditions

$$u|_{r=a} = c, \quad \frac{\partial u}{\partial r} \Big|_{r=b} = 0.$$

Answer.

$$u = c + \frac{A}{4} \left(r^2 - a^2 - 2b^2 \ln \frac{r}{a} \right) + \frac{B}{12} \left[r^2 (r^2 - a^2) - b^4 \frac{2b^2 - a^2}{a^4 + b^4} \frac{r^4 - a^4}{r^2} \right] \cos 2\varphi.$$

Truncation of a Series with a Given Accuracy

For Problems 7.68 through 7.71, the side surface of an infinitely long cylinder is kept at a steady temperature. Find the temperature distribution inside the cylinder, first analytically and then by using the program Laplace. The radius of the cylinder is $l = 10$, and the angle φ is in the range $[0, 2\pi)$. Find the temperature at the points $P_1(1, \pi/6)$, $P_2(5, \pi/6)$, and $P_3(8, \pi/6)$, assuming the necessary accuracy is 10^{-5} for the given boundary conditions.

7.68. $u(r, \varphi)|_{r=l} = \cos(\varphi/2) - \sin(\varphi/2).$

7.69. $u(r, \varphi)|_{r=l} = 0.5 \cos(\varphi/2) + 1/2.$

7.70. $u(r, \varphi)|_{r=l} = \varphi/\pi + \cos \varphi.$

7.71. $u(r, \varphi)|_{r=l} = \varphi/\pi^2 - \sin(\varphi/4).$

Laplace's and Poisson's Equations for a Cylinder

7.72. Find the heat distribution in a homogeneous cylinder of radius l and height H if the side surface and one of the bases are kept at constant temperature T_0 , and on the other base there is a heat flux given by $T_z|_{z=H} = Q_0\delta(r)$.

Answer.

$$T = T_0 + \frac{2Q_0}{l^2} \sum_{m=1}^{\infty} \frac{J_0(k_m r) \sinh k_m z}{J_1^2(k_m l) k_m \cosh k_m H},$$

where $J_0(k_m l) = 0$. Here we have used $J'_0(x) = -J_1(x)$ and $\delta(r) = \delta(x)\delta(y)$ for the two-dimensional delta function.

7.73. Solve Problem 7.72 for the case $T_z|_{z=H} = Q_0\delta(x - x_0)\delta(y - y_0)$, where $x_0 = r_0 \cos \theta_0$, $y_0 = r_0 \sin \theta_0$.

Answer.

$$T = T_0 + \frac{Q_0}{\pi a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_n \frac{J_n(k_{mn} r_0) J_n(k_{mn} r) \cos n(\varphi - \varphi_0) \sinh k_{mn} z}{[J'_n(k_{mn} a)]^2 k_{mn} \cosh k_{mn} H},$$

where $J_n(k_{mn} l) = 0$ and $A_n = 1$ for $n = 0$ and 2 for $n \neq 0$.

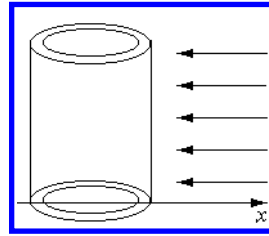


Figure 7.18. Tube for Problem 7.74.

7.74. Solve the problem

$$\nabla^2 T = 0, \quad a < r < b, \quad 0 \leq \varphi < 2\pi, \quad 0 < z < H,$$

$$T|_{r=a} = T_0, \quad T|_{r=b} = T_0 + T_1 (1 + \cos \varphi),$$

$$T|_{z=0} = T_0, \quad T_z|_{z=H} = 0.$$

This problem is similar to the problem considered in Section 7.4.4, but in the present case we consider heating a tube, not a barrel. The tube remains on the ground (at constant temperature T_0) and is heated by rays of the sun that are parallel to the x -axis (see Figure 7.18). It is clear that the temperature has the maximum value $T_0 + 2T_1$ when $\varphi = 0$, and has the value $T_0 + T_1$ at $\varphi = \pi/2$. At $\varphi = \pi$, the surface temperature coincides with the temperature of the environment, T_0 . In the shade, the tube temperature is in equilibrium with the surrounding air. The top end of the tube is open, and, due to the low heat conductivity of air heat flux at this surface, is approximately zero.

Answer.

$$T = T_0 + \frac{4T_1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^1 \frac{A_n(r) \cos n\varphi}{2m+1} \sin k_m z,$$

where $k_m a = \pi(2m+1)/2$,

$$A_n(r) = \frac{I_n(k_m r) K_n(k_m a) - I_n(k_m a) K_n(k_m r)}{I_n(k_m b) K_n(k_m a) - I_n(k_m a) K_n(k_m b)},$$

$K_n(r)$ is Macdonald's function, or any particular solution of the modified Bessel equation (see Equation (7.264), Section 7.4.3) and is linearly independent from $I_n(z)$ —see Chapter 8 and the *Mathematical Handbook* by Korn and Korn, cited in Section 7.4.3).

7.75. Find the temperature field inside a cylinder of height H with a quarter of a circle as a base (l being the radius of the circle). The lower base as well as two

lateral sides (at $r = l$ and at $\varphi = 0$) are kept at the fixed temperature T_0 , the upper base is kept at the fixed temperature T_1 , and the third lateral side (at $\varphi = \pi/2$) is heated. Do not evaluate the integral over r .

Answer.

$$T = T_0 + (T_1 - T_0) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{\sinh k_{mn} z}{\sinh k_{mn} H} J_{2m+1}(k_{mn} r) \sin(2m+1)\varphi,$$

where $J_{2m+1}(k_{mn}l) = 0$,

$$A_{mn} = \frac{8}{\pi l^2 [J'_{2m+1}(k_{mn}l)]^2 (2m+1)} \int_0^l J_{2m+1}(k_{mn}r) r dr.$$

7.76. Find the temperature distribution in a rod if heat flow through its side surface is absent and the bases are kept at constant temperature, which can be chosen to be zero. In the rod, there is a heat source Q with power depending on the value of z only:

$$Q = Q_0 \exp[-\gamma^2(z - z_0)^2].$$

Hint. Notice that in fact the problem is one-dimensional; that is, Poisson's equation reduces to the simple ordinary differential equation (ODE) $T_{zz} = Q(z)$ with zero boundary conditions.

Answer.

$$\begin{aligned} T = & \frac{Q_0}{2\gamma^2} \left[\left(1 - \frac{z}{H}\right) \exp\{-\gamma^2 z_0^2\} + \frac{z}{H} \exp\{-\gamma^2(H - z_0)^2\} - \exp\{-\gamma^2(z - z_0)^2\} \right] \\ & - \frac{Q_0}{2\gamma} \sqrt{\pi} \left[(z - z_0) \operatorname{erf}(\gamma(z - z_0)) - \frac{z}{H} (H - z_0) \operatorname{erf}(\gamma(H - z_0)) \right. \\ & \left. - \left(1 - \frac{z}{H}\right) z_0 \operatorname{erf}(\gamma z_0) \right]. \end{aligned}$$

Consider the case of a source of heat, $Q = q_0 \delta(z - z_0)$, located at z_0 , and compare the result with the case when $\gamma \rightarrow \infty$, $Q_0 \rightarrow \infty$ (the full heat flux remains constant: $q_0 = \sqrt{\pi} Q_0 / \gamma$). For the local source of heat, taking into account that $\operatorname{erf} \gamma x = \operatorname{sign} x$ for $\gamma \gg 1$, we obtain

$$T = -\frac{q_0}{2} \left[|z - z_0| - (H - z_0) \frac{z}{H} - z_0 \left(1 - \frac{z}{H}\right) \right];$$

that is, a piecewise linear temperature distribution with maximum $q_0 z_0 (1 - z_0/H)$ at the point z_0 :

$$T = q_0 \begin{cases} \left(1 - \frac{z_0}{H}\right) z, & z < z_0, \\ \left(1 - \frac{z}{H}\right) z_0, & z > z_0. \end{cases} \quad (7.299)$$

7.77. For $\gamma \gg 1$, consider Problem 7.76 for constant zero temperature at the side surface, which represents ideal conductivity of the surrounding environment.

Answer.

$$T = T_p - \frac{2q_0 H}{\pi^2} \sum_{m=1}^{\infty} \frac{\sin k_m z_0}{m^2} \frac{I_0(k_m r)}{I_0(k_m l)} \sin k_m z,$$

where $k_m = \pi m/H$, and T_p is a particular solution of Poisson's equation given by Equation (7.300).

Hint. Represent the solution in the form $T = T_p + T_0$, where T_0 is a harmonic function determined from the boundary condition at the lateral side.

8

Bessel Functions

As we have seen previously in this book, Bessel functions can be used as a set of basis functions for the Fourier expansion of solutions to certain types of physical problems. In this chapter, we discuss in detail the origins of the Bessel functions and their properties. As we will see, in addition to being of general interest, Bessel functions are a convenient set of basis functions for problems with specific types of symmetries, such as circular or cylindrical symmetry. Bessel functions have also been used in practical applications such as signal processing.

8.1 Boundary Value Problems Leading to Bessel Functions

In applications, one often encounters an eigenvalue problem containing a differential equation with the generic form

$$r^2 y''(r) + r y'(r) + (\lambda r^2 - p^2) y(r) = 0, \quad (8.1)$$

where p is a given fixed value and λ is an additional parameter. Equation (8.1) is called the *Bessel equation of order p* . Such equations frequently arise after a separation of variables procedure in cylindrical or polar coordinates in many problems of mathematical physics. In particular, problems related to heat conduction inside a circular cylinder, vibrations of a circular membrane, and boundary value problems for the electric potential of a

cylindrical area involve solving a Bessel equation of some order. Generally Equation (8.1) has nontrivial solutions that correspond to a given set of boundary conditions only for certain values of the parameter λ , which are called *eigenvalues*. The goal then becomes finding these eigenvalues for a given set of boundary conditions and the corresponding solutions, $y(r)$, which are called *eigenfunctions*.

In any physical application of the Bessel equation we will encounter, in addition to Equation (8.1), boundary conditions. For example, the function $y(r)$, defined on a closed interval $[0, l]$, could be restricted to a specified behavior at the points $r = 0$ and $r = l$. For instance, at $r = l$, the value of the solution $y(l)$, or its derivative $y'(l)$, or their linear combination $\alpha y'(l) + \beta y(l)$, could be prescribed. As we have seen previously this leads, at $r = l$, to one of the three conditions referred to as Dirichlet, Neumann, or mixed boundary conditions.

Other restrictions, which are not formally boundary conditions but limit the behavior of the solutions, include the case in which the variable r can take any value on the finite interval $0 < r \leq l$, perhaps excluding point $r = 0$, where function $y(r)$ might have a singularity. Another common case occurs when the variable r is limited to a finite interval and the function $y(r)$ is bounded at $r = 0$ and takes a given value at $r = l$, for instance:

$$\begin{cases} |y(r \rightarrow +0)| < \infty, \\ y(l) = 0. \end{cases} \quad (8.2)$$

Later we will also see cases in which the solutions are limited over an infinite interval, for instance for $0 \leq r < \infty$, or $l \leq r < \infty$.

Let us begin with a detailed consideration of the boundary conditions (8.2), which lead to a discrete spectrum of λ . Obviously, Equation (8.1) and boundary conditions (8.2) form a Sturm-Liouville problem, as discussed in Chapter 2. First, we make the change of variables $x = \sqrt{\lambda}r$ in Equation (8.1) to yield

$$x^2 y''(x) + x y'(x) + (x^2 - p^2) y(x) = 0. \quad (8.3)$$

By dividing Equation (8.3) by x , we then obtain

$$(xy')' + \left(x - \frac{p^2}{x}\right) y = 0. \quad (8.4)$$

The structure of Equation (8.4) is reminiscent of the Euler equation. Recall that the Euler equation of second order is an equation of the form

$$c_0x^2y'' + c_1xy' + c_2y = f(x), \quad (8.5)$$

where c_0 , c_1 , and c_2 are constants. This equation can be reduced to a linear equation with constant coefficients by using the change of the independent variable $x = e^t$.

To solve Equation (8.4), we first consider integer values of the parameter p . Let us try to find the solution in the form of a power series in x given by

$$y = b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots \quad (8.6)$$

This choice of the form of the solution assumes that $y(x)$ is bounded as $x \rightarrow 0$. Next, we change the notation for convenience of calculations in the following way. We let a_0 be the first nonzero coefficient of the series in Equation (8.6), and we let $a_1, a_2, a_3, \dots$ be subsequent coefficients. It may be the case that $b_0 \neq 0$, so $b_0 = a_0, b_1 = a_1, \dots$. If, however, $b_0 = b_1 = \dots = b_{m-1} = 0$ but $b_m \neq 0$, we then let $b_m = a_0, b_{m+1} = a_1, b_{m+2} = a_2, \dots$ and so on. Thus, in the general case, we may write

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots + a_nx^{m+n} + \dots, \quad (8.7)$$

where $m \geq 0, a_0 \neq 0$, and, by construction, m is an integer.

We may differentiate Equation (8.7) to find an expression for y' and, by multiplying the series for y' by x and differentiating once more, we obtain

$$(xy')' = a_0m^2x^{m-1} + a_1(m+1)^2x^m + a_2(m+2)^2x^{m+1} + \dots + a_n(m+n)^2x^{m+n-1} + \dots \quad (8.8)$$

Substitution of this Equation (8.8) for $(xy')'$ and Equation (8.7) for y into Equation (8.4) results in the following equality:

$$\begin{aligned} & a_0m^2x^{m-1} + a_1(m+1)^2x^m + a_2(m+2)^2x^{m+1} + \dots + a_n(m+n)^2x^{m+n-1} + \dots \\ & - a_0p^2x^{m-1} - a_1p^2x^m - a_2p^2x^{m+1} - \dots - a_np^2x^{m+n-1} - \dots \\ & + a_0x^{m+1} + a_1x^{m+2} + \dots + a_{n-2}x^{m+n-1} + \dots \equiv 0. \end{aligned}$$

By setting coefficients of each power of x to zero, we obtain an infinite system of equations for the coefficients a_n , the first few of which are given

by

$$\begin{aligned}
 a_0(m^2 - p^2) &= 0, \\
 a_1[(m+1)^2 - p^2] &= 0, \\
 a_2[(m+2)^2 - p^2] + a_0 &= 0, \\
 a_3[(m+3)^2 - p^2] + a_1 &= 0, \\
 &\vdots \\
 a_n[(m+n)^2 - p^2] + a_{n-2} &= 0, \\
 &\vdots
 \end{aligned} \tag{8.9}$$

From the first of Equations (8.9), we obtain $m^2 - p^2 = 0$ (since $a_0 \neq 0$). Thus $m = p$ (recall that m is nonnegative; thus, $m \neq -p$). Obviously, this method for solving Equation (8.4) with a power series works for $p \geq 0$ only. We next generalize the method to arbitrary values of p .

From the other Equations (8.9), we obtain the coefficients $a_1, a_2, a_3, \dots, a_n, \dots$. However, to simplify the calculations, let us transform the expressions in the square brackets in these equations, taking into account that $m = p$, in the following way:

$$\begin{aligned}
 a_1[(p+1)^2 - p^2] &= 0, \\
 a_2[(p+2)^2 - p^2] + a_0 &= 0, \\
 a_3[(p+3)^2 - p^2] + a_1 &= 0, \\
 &\vdots \\
 a_n[(p+n)^2 - p^2] + a_{n-2} &= 0, \\
 &\vdots
 \end{aligned}$$

or

$$\begin{aligned}
 a_1(2p+1) &= 0, \\
 a_2 2(2p+2) &= -a_0, \\
 a_3 3(2p+3) &= -a_1, \\
 a_4 4(2p+4) &= -a_2, \\
 &\vdots \\
 a_n \cdot n \cdot (2p+n) &= -a_{n-2}, \\
 &\vdots
 \end{aligned}$$

Thus we see that $a_1 = a_3 = a_5 = \dots = a_{2k+1} = 0$, and

$$\begin{aligned} a_2 &= \frac{-a_0}{2(2p+2)}, \\ a_4 &= -\frac{a_2}{4(2p+4)} = \frac{a_0}{2 \cdot 4 \cdot (2p+2)(2p+4)}, \\ a_6 &= \frac{-a_4}{6(2p+6)} = \frac{-a_0}{2 \cdot 4 \cdot 6 \cdot (2p+2)(2p+4)(2p+6)}, \text{ and so on.} \end{aligned}$$

Finally, on inspection, we obtain a recurrence relation that will generate any term in the series:

$$\begin{aligned} a_{2k} &= \frac{a_0(-1)^k}{2 \cdot 4 \cdot \dots \cdot 2k \cdot (2p+2)(2p+4) \dots (2p+2k)} \\ &= \frac{a_0(-1)^k}{2^k \cdot k! \cdot 2^k \cdot (p+1)(p+2) \dots (p+k)} = \frac{a_0 p! (-1)^k}{2^{2k} k! (p+k)!}. \end{aligned}$$

By substituting the coefficients that we have found in the series in Equation (8.7), and taking into account that $m = p$, we obtain

$$y = a_0 \left[x^p - \frac{p!}{2^2 1! (p+1)!} x^{p+2} + \frac{p!}{2^4 2! (p+2)!} x^{p+4} - \dots + \frac{p! (-1)^k}{2^{2k} k! (p+k)!} x^{p+2k} + \dots \right],$$

which may be written in a simpler form as

$$y = a_0 2^p p! \cdot \left[\frac{\left(\frac{x}{2}\right)^p}{p!} - \frac{\left(\frac{x}{2}\right)^{p+2}}{1!(p+1)!} + \frac{\left(\frac{x}{2}\right)^{p+4}}{2!(p+2)!} - \dots + (-1)^k \frac{\left(\frac{x}{2}\right)^{p+2k}}{k!(p+k)!} + \dots \right]. \quad (8.10)$$

The series in the square brackets of Equation (8.10) is absolutely convergent for all values of x , which is easy to confirm by using the D'Alembert criterion: $\lim_{k \rightarrow \infty} |a_{k+1}/a_k| = 0$. Due to the presence of factorials in the denominator the series converges very fast. The sum of this series is called the Bessel function of order p , and it is denoted as $J_p(x)$:

$$J_p(x) = \frac{\left(\frac{x}{2}\right)^p}{p!} - \frac{\left(\frac{x}{2}\right)^{p+2}}{1!(p+1)!} + \frac{\left(\frac{x}{2}\right)^{p+4}}{2!(p+2)!} - \dots + (-1)^k \frac{\left(\frac{x}{2}\right)^{p+2k}}{k!(p+k)!} + \dots; \quad (8.11)$$

that is,

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (p+k)!} \left(\frac{x}{2}\right)^{p+2k}. \quad (8.12)$$

Thus, we have obtained the solution of Equation (8.4) in the case that the function $y(x)$ is finite at $x = 0$. Note that this solution was obtained for *integer, nonnegative values of p only*. A solution of Equation (8.4) is any function in the form $AJ_p(x)$, where A denotes the number $2^p p!$, which results from the fact that a_0 can take any value and because multiplication of the solution by a constant also yields a solution due to the linearity and homogeneity of the equation.

Now let us generalize the above case for p as a positive integer to the case of *arbitrary real p* . In this case, it is necessary to replace the integer-valued function $p!$ by the Gamma function $\Gamma(p)$, which is defined for arbitrary real values of p . The definition and main properties of the Gamma function are listed in Section 8.8. By using the Gamma function, we can define the Bessel function of order p , where p is real, by a series that is built analogously to the series in Equation (8.12):

$$\begin{aligned} a_2 &= -\frac{a_0}{2(2+2p)} = -\frac{a_0}{2^2(1+p)} = -\frac{a_0\Gamma(p+1)}{2^2\Gamma(p+2)}, \\ a_4 &= -\frac{a_2}{2^3(p+2)} = \frac{a_0}{2!2^4(p+1)(p+2)} = \frac{a_0\Gamma(p+1)}{2!2^4\Gamma(p+3)}, \\ a_6 &= -\frac{a_4}{3!2(p+3)} = -\frac{a_0}{3!2^6(p+1)(p+2)(p+3)} = -\frac{a_0\Gamma(p+1)}{3!2^6\Gamma(p+4)}, \\ &\vdots \\ a_{2k} &= \frac{(-1)^k a_0\Gamma(p+1)}{2^{2k} k! \Gamma(p+k+1)!}, \\ &\vdots \end{aligned}$$

which gives

$$\begin{aligned} J_p(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{p+2k} \\ &= \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k}, \end{aligned} \quad (8.13)$$

which converges for any p . In particular, replacing p by $-p$, we obtain

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k}. \quad (8.14)$$

Since Bessel Equation (8.4) contains p^2 , functions $J_p(x)$ and $J_{-p}(x)$ are solutions of the equation for the same p . If p is *noninteger* (the integer case $p = n$ will be considered below), these solutions are linearly independent, since the first terms in Equations (8.13) and (8.14) contain different powers of x : x^p and x^{-p} , respectively. Point $x = 0$ must be excluded from the domain of definition of the function (8.14), since x^{-p} for $p > 0$ diverges at this point.

The functions, $J_p(x)$ are bounded as $x \rightarrow 0$. In fact, the functions $J_p(x)$ are continuous for all x since they are the sum of a converging power series (8.12). For this reason, they are bounded in the neighborhood of any point, in particular in the neighborhood of zero. For noninteger values of p this follows from the properties of the Gamma function and the series (8.13).

The solutions $J_p(x)$ are not the only solutions of Equation (8.4). All solutions of Equation (8.4) have a common name and are referred to as the *cylindrical functions* or *Bessel functions*. They are comprised of the cylindrical functions of the first kind $J_p(x)$ (called *Bessel functions*), which we have just obtained; the cylindrical functions of the second kind, $Y_p(x)$ (called *Weber functions*; an alternative terminology is *Neumann functions* denoted as $N_p(x)$); and the cylindrical functions of the third kind, $H_p^{(1)}(x)$ and $H_p^{(2)}(x)$ (called *Hankel functions*). Descriptions and properties of these functions are given in the following sections. In Chapter 9, we also discuss *spherical Bessel functions* related to the solutions of some boundary value problems in spherical coordinates.

The Bessel equation is of second order, and its general solution must contain two linearly independent particular solutions. If p is an *integer*, then the functions $J_p(x)$ and $J_{-p}(x)$ are not linearly independent and they cannot be used to form a general solution.

Reading Exercise.

1. Show that $J_{-n}(x) = (-1)^n J_n(x)$ for $n = 1, 2, 3 \dots$
2. Show that $\lim_{x \rightarrow 0} \frac{J_1(x)}{x} = \frac{1}{2}$.

In the case that $J_p(x)$ and $J_{-p}(x)$ are not linearly independent, the general solution can be formed from the linear combination of the functions $J_p(x)$ and $Y_p(x)$ as

$$y = C_1 J_p(x) + C_2 Y_p(x). \quad (8.15)$$

The functions $Y_p(x)$ are singular at $x = 0$; thus, if the physical formulation of the problem requires regularity of the solution at zero, the coefficient C_2 in the solution in Equation (8.15) must be zero.

If the *index p is not an integer*, then the general solution of Equation (8.4) can be written in the form

$$y = C_1 J_p(x) + C_2 J_{-p}(x), \quad (8.16)$$

where C_1 and C_2 are arbitrary constants. When the index is arbitrary, $J_p(x)$, $Y_p(x)$, $H_p^{(1)}(x)$, and $H_p^{(2)}(x)$ are linearly independent and may serve as a fundamental system of solutions for Equation (8.4). For instance, for some problems, the solution of Equation (8.4) is conveniently written as

$$y = C_1 H_p^{(1)}(x) + C_2 H_p^{(2)}(x). \quad (8.17)$$

8.2 Bessel Functions of the First Kind

In this section, we consider Bessel functions of the first kind of integer order $p = n$. We emphasize this case in the following discussion since in many applications p is a nonnegative integer.

The first few terms of the expansion in Equation (8.11) near zero for the first three Bessel functions of the first kind are

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \cdot 2! \cdot 2!} - \dots, \\ J_1(x) &= \frac{x}{2} - \frac{x^3}{2^3 \cdot 2!} + \frac{x^5}{2^5 \cdot 2! \cdot 3!} - \dots, \\ J_2(x) &= \frac{x^2}{2^2 \cdot 2!} - \frac{x^4}{2^4 \cdot 3!} + \frac{x^6}{2^6 \cdot 2! \cdot 4!} - \dots \end{aligned}$$

Note that the $J_n(x)$ are even if n is an integer and odd if n is a noninteger (although usually, in physical problems $x \geq 0$, and we do not need to consider the behavior of the function for $x < 0$). For future reference, we present the useful fact that at $x = 0$ we have $J_0(0) = 1$ and $J_n(0) = 0$

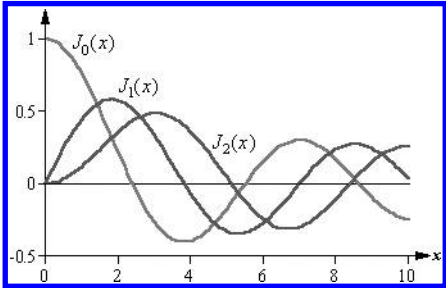


Figure 8.1. Graphs of functions $J_0(x)$, $J_1(x)$ and $J_2(x)$.

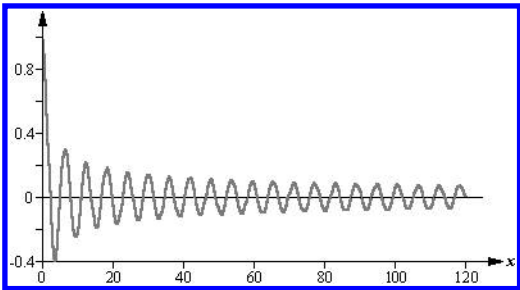


Figure 8.2. Graph of the Bessel function, $J_0(x)$.

for $n \geq 1$. Figure 8.1 shows graphs of functions $J_0(x)$, $J_1(x)$, and $J_2(x)$. From the plot of the function $J_0(x)$ (where about 30 zeros of this function are computed), it can be seen (see [Figure 8.2](#) and [Table 8.1](#)) that the difference of two adjacent roots tends to π as the root number increases:

$$\lim_{k \rightarrow \infty} \left(\mu_0^{(k)} - \mu_0^{(k-1)} \right) = \pi.$$

Here, $\mu_0^{(k)}$ is the k th positive root of the equation $J_0(\mu_0) = 0$.

Table 8.2 lists a few first roots of Bessel functions of orders 0, 1, and 2.

Root number	Value	Distance from adjacent root	Root number	Value	Distance from adjacent root
25	77.75602563	—	28	87.18062984	3.14153906
26	80.89755587	3.14153024	29	90.32217264	3.14154280
27	84.03909078	3.14153491	30	93.46371878	3.14154614

Table 8.1. Positive roots of $J_0(x)$ and difference of values of adjacent roots.

	Roots				
Function	μ_1	μ_2	μ_3	μ_4	μ_5
$J_0(x)$	2.4048	5.5201	8.6537	11.7915	14.9309
$J_1(x)$	3.8317	7.0156	10.1735	13.3237	16.4706
$J_2(x)$	5.136	8.417	11.620	14.796	17.960

Table 8.2. Positive roots of $J_0(x)$, $J_1(x)$, $J_2(x)$.

In Table 8.2, roots of $J_n(x) = 0$ are denoted by μ_k . Another convenient notation is $\mu_k^{(n)}$, where n stands for the order of the Bessel function and k stands for the root number. In cases when the value of n is clear, we will omit the upper index in μ_k .

8.3 Properties of Bessel Functions of the First Kind: $J_n(x)$

All of the following properties come from the expansion in Equation (8.11).

1. All Bessel functions are defined and continuous on the real axis and have derivatives of all orders. This is because any Bessel function can be expanded in a power series that converges for all x , and the sum of the power series is a continuous function that has derivatives of all orders.
2. Bessel functions of even orders are even functions (since their expansion contains only even powers of the argument). Bessel functions of odd orders are odd functions.
3. The zeros of Bessel functions (roots of the equation $J_n(x) = 0$) are nondegenerate (i.e., there are no duplicates). Additionally, zeros of $J_n(x) = 0$ fall between the zeros of $J_{n+1}(x) = 0$.

Each Bessel function has an infinite number of real roots. Roots located on the positive semiaxis can be marked by integer numbers in an increasing order.

4. The behavior of Bessel functions in the vicinity of zero is given by the first terms of the series in Equation (8.12); for large x the asymp-

otic formula may be used:

$$J_p(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right). \quad (8.18)$$

As x grows, the accuracy of this formula quickly increases: the deviation of the values of $J_p(x)$ given by (8.18) from the exact values is very small for large x and has the same order as $x^{-3/2}$.

From Equation (8.18) it follows, in particular, that the function $J_p(x)$ has roots that are close (for large x) to the roots of the equation

$$\cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right) = 0.$$

Thus, the difference between two adjacent roots of the function $J_p(x)$ tends to π when roots tend to infinity. A graph of $J_p(x)$ has the shape of a curve that depicts decaying oscillation; it is almost periodic, with the period close to 2π , and the amplitude decays at a rate inversely proportional to the square root of x . In fact, we have $\lim_{x \rightarrow \infty} J_p(x) = 0$.

5. The differentiation formulas are

$$\frac{d}{dx} [x^{-p} J_p(x)] \equiv -x^{-p} J_{p+1}(x), \quad (8.19)$$

and

$$\frac{d}{dx} [x^p J_p(x)] \equiv x^p J_{p-1}(x). \quad (8.20)$$

These formulas can be easily proved by the direct differentiation of the corresponding series (that is, the series that are the result of the multiplication or division of the power series for $J_p(x)$ by x^p).

In particular (prove this as a reading exercise):

$$\frac{d}{dx} J_0(x) = -J_1(x), \quad \frac{d}{dx} [x J_1(x)] = x J_0(x).$$

6. Recurrence formulas are

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x),$$

$$J_{p+1}(x) = \frac{p}{x} J_p(x) - J'_p(x), \quad (8.21)$$

$$J'_p(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x).$$

In particular (prove this as a reading exercise):

$$J'_1(x) = J_0(x) - \frac{J_1(x)}{x}, \quad J_0(x) - J_2(x) = 2J'_1(x),$$

$$J_1(x) + J_3(x) = \frac{4}{x} J_2(x).$$

Equations (8.19) through (8.21) are valid for any value of p , not only integer values. These identities are easily established by operating on the series that define the function. For instance, let us prove Equation (8.20). We multiply the series in Equation (8.13) by x^p and differentiate with respect to x to yield

$$\begin{aligned} \frac{d}{dx} [x^p J_p(x)] &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)} \frac{x^{2p+2k}}{2^{p+2k}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (2p+2k)}{\Gamma(k+1)\Gamma(k+p+1)} \frac{x^{2p+2k-1}}{2^{p+2k}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2(p+k)}{\Gamma(k+1)(k+p)\Gamma(k+p)} \frac{x^{2p+2k-1}}{2^{p+2k}} \\ &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p)} \left(\frac{x}{2}\right)^{p+2k-1} \\ &= x^p J_{p-1}(x). \end{aligned}$$

By using the asymptotic Equation (8.18) and the recurrence relation in Equation (8.21), it is easy to show that

$$\int_0^{\infty} J_1(x) dx = -J_0(x)|_0^{\infty} = 1,$$

and also that

$$\int_0^{\infty} J_0(x) dx = 1.$$

Reading Exercise.

1. Prove Equations (8.21) for integer p .
2. Using the recurrence Equations (8.21), show that $J_0(x_*) = J_2(x_*)$, where x_* is the point of a maximum or minimum of $J_1(x)$, and $J_0(x) = -J_2(x) = J_1'(x)$ for any positive root of the equation $J_1(x) = 0$. Using a computer, plot the graphs of $J_0(x)$, $J_1(x)$, $J_1'(x)$, and $J_2(x)$, and show that the obtained results are correct.

Up until now, we have considered in detail only the case in which p is an integer. In many physical problems with spherical symmetry, we encounter Bessel functions of half-integer order, where $p = (2n + 1)/2$ for $n = 0, 1, 2, \dots$. For instance, solving Equation (8.4) with $p = 1/2$ and $p = -1/2$ by using the series expansion

$$y(x) = x^{1/2} \sum_{k=0}^{\infty} a_k x^k, \quad (a_0 \neq 0), \quad (8.22)$$

we obtain

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (8.23)$$

and

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \quad (8.24)$$

Clearly these two expansions are linearly independent because the first one starts with $x^{1/2}$, and second one with $x^{-1/2}$. Then, for this case, the general solution of Equation (8.4) is

$$y(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x). \quad (8.25)$$

By comparing expansions in Equations (8.23) and (8.24) to the Maclaurin series expansions in $\sin x$ and $\cos x$, we obtain

$$J_{1/2} = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad (8.26)$$

$$J_{-1/2} = \left(\frac{2}{\pi x}\right)^{1/2} \cos x. \quad (8.27)$$

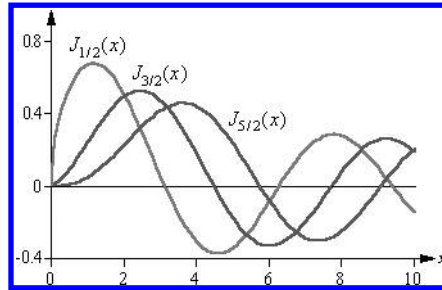


Figure 8.3. Graphs of functions $J_{1/2}(x)$, $J_{3/2}(x)$, and $J_{5/2}(x)$.

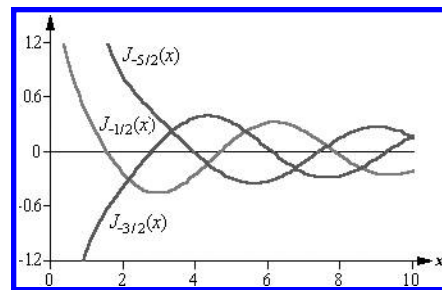


Figure 8.4. Graphs of $J_{-1/2}(x)$, $J_{-3/2}(x)$, and $J_{-5/2}(x)$.

Note that the solution given by Equation (8.26) is bounded for all x and the function in Equation (8.27) diverges at $x = 0$. Recall that Equation (8.13) gives an expansion of $J_p(x)$ that is valid for any value of p . Figure 8.3 shows graphs of functions $J_{1/2}(x)$, $J_{3/2}(x)$ and $J_{5/2}(x)$; Figure 8.4 shows graphs for $J_{-1/2}(x)$, $J_{-3/2}(x)$, and $J_{-5/2}(x)$.

Reading Exercise.

- Using Equation (8.13), obtain functions $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$, and $J_{-5/2}(x)$. Show that $J_{3/2}(x)$ and $J_{-3/2}(x)$ are linearly independent; that is, the general solution of Equation (8.4) for $p = 3/2$ is

$$y(x) = C_1 J_{3/2}(x) + C_2 J_{-3/2}(x). \quad (8.28)$$

- Using the recurrence equations (8.21) and the expression for $J_{1/2}(x)$, obtain the functions $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$, and $J_{-5/2}(x)$. For

instance, the answer for $J_{3/2}(x)$ is

$$J_{3/2} = \left(\frac{2}{\pi x}\right)^{1/2} \left(\frac{\sin x}{x} - \cos x\right). \quad (8.29)$$

8.4 Bessel Functions of the Second Kind

Let us now consider *Bessel functions of the second kind*, or *Neumann functions* $N_p(x)$ (or the alternate notation $Y_p(x)$). Neumann functions can be defined as

$$N_p(x) = \frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi}. \quad (8.30)$$

If $p \neq 0, 1, 2, \dots$ (i.e., p is not an integer), then Equation (8.30) gives a linear combination of functions $J_p(x)$ and $J_{-p}(x)$, and the solution given by Equation (8.16) can be written as

$$y(x) = C_1 J_p(x) + C_2 N_p(x). \quad (8.31)$$

For cases when p is an integer, the solution given by Equation (8.30) is indeterminate since it involves the ratio $0/0$; however, this may be dealt with by using L'Hôpital's rule. Thus, Equation (8.31) for $x > 0$ is the general solution of the Bessel equation for any real p .

Substituting into Equation (8.30) the expressions for functions $J_p(x)$ and $J_{-p}(x)$ in the form of the power series obtained from Equation (8.13) and (8.14), we obtain, for instance, in the limit $p = 0$

$$N_0(x) = \frac{2}{\pi} \left[J_0(x) \left(\ln \frac{x}{2} + \gamma \right) + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k}}{2^{2k} (k!)^2} h_k \right], \quad (8.32)$$

where $x > 0$, and $\gamma = 0.5772 \dots$ is the Euler constant. The Euler constant may be defined as the limit of the series

$$\gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln n \right). \quad (8.33)$$

In Equation (8.32), h_k is defined as

$$h_1 = 1, \quad h_k = 1 + \frac{1}{2} + \dots + \frac{1}{k} \quad \text{for } k = 2, 3, 4, \dots \quad (8.34)$$

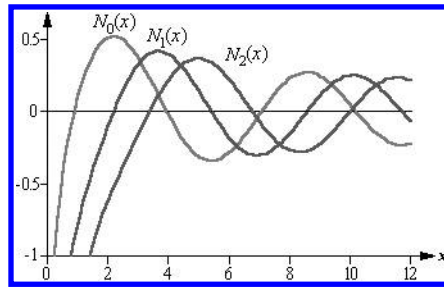


Figure 8.5. Graphs of $N_0(x)$, $N_1(x)$, and $N_2(x)$.

For the result for a positive integer, p (recall that for noninteger p one can use the general Equation (8.30)), we have

$$N_n(x) = \frac{2}{\pi} J_n(x) \left(\ln \frac{x}{2} + \gamma \right) + \frac{x^n}{2} \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^{2k} (h_k + h_{k+n})}{2^{2k+n} (k!) (k+n)!} - \frac{x^{-n}}{\pi} \sum_{k=0}^{\infty} \frac{(n-k-1)! x^{2k}}{2^{2k-n} k!}, \quad (8.35)$$

where $x > 0$, $n = 0, 1, 2, \dots$, $h_0 = 0$, $h_1 = 1$, and $h_{k+n} = 1 + \frac{1}{2} + \dots + \frac{1}{n+k}$.

Graphs of $N_0(x)$, $N_1(x)$, and $N_2(x)$ are shown in Figure 8.5, and the roots of these functions are given in Table 8.3.

Reading Exercise. Show that the series in Equation (8.35) for $N_n(x)$ converge for all x .

It can be shown that

$$N_{-n}(x) = (-1)^n N_n(x). \quad (8.36)$$

Another useful property is that as $x \rightarrow 0$, the functions $N_n(x)$ diverge logarithmically, and as $x \rightarrow \infty$, $N_n(x) \rightarrow 0$, oscillating with decaying amplitude. At large x , we have the asymptotic form

$$N_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right). \quad (8.37)$$

Notice that Equation (8.37) implies that $\lim_{x \rightarrow \infty} N_p(x) = 0$.

	Roots				
Function	μ_1	μ_2	μ_3	μ_4	μ_5
$N_0(x)$	0.8936	3.9577	7.0861	10.2223	13.3611
$N_1(x)$	2.1971	5.4297	8.5960	11.7492	14.8974
$N_2(x)$	3.3842	6.7938	10.0235	13.2199	16.3789

Table 8.3. Positive roots of the functions $N_0(x)$, $N_1(x)$, and $N_2(x)$.

8.5 Bessel Functions of the Third Kind

The *cylindrical functions of the third kind*, or *Hankel functions*, can be expressed by using functions of the first and second kind:

$$H_p^{(1)}(x) = J_p(x) + iN_p(x), \quad H_p^{(2)}(x) = J_p(x) - iN_p(x). \quad (8.38)$$

Hankel functions are linearly independent and are often used, for instance, in electrodynamics in the problems involving the theory of radiation. Their convenience follows primarily from the fact that at large x they have formulas in the form of plane waves:

$$\begin{aligned} H_n^{(1)}(x) &\approx \sqrt{\frac{2}{\pi x}} \exp \left\{ i \left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right\}, \\ H_n^{(2)}(x) &\approx \sqrt{\frac{2}{\pi x}} \exp \left\{ -i \left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right\}. \end{aligned} \quad (8.39)$$

Note that Equations (8.19) through (8.21) are also valid for Bessel functions of the second and third kind, $N_p(x)$ and $H_p^{(1,2)}(x)$, respectively.

Also, an important result used in the theory of radiation is the following:

$$e^{ix \cos \theta} = J_0(x) + 2 \sum_{n=1}^{\infty} i^n J_n(x) \cos nx. \quad (8.40)$$

Reading Exercise.

1. Find expressions for the functions $N_{1/2}(x)$ and $H_{1/2}^{(1,2)}(x)$.
2. Using the recurrence relations and the expressions for $N_{1/2}(x)$ and $H_{1/2}^{(1,2)}(x)$, find expressions for the corresponding functions for $p = \pm 3/2, \pm 5/2$.

Reading Exercise. There exist many relations connecting Bessel functions of different types.

1. Prove the formula

$$J_p(x)N_{p+1}(x) - J_{p+1}(x)N_p(x) = -\frac{2}{\pi x}. \quad (8.41)$$

2. Assuming that the equality in Equation (8.40) can be integrated term by term (explain why this is valid), obtain the integral form for the function $J_0(x)$:

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta. \quad (8.42)$$

3. Show that

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta}. \quad (8.43)$$

8.6 Modified Bessel Functions

Next we discuss the frequently encountered *modified Bessel equation* given by

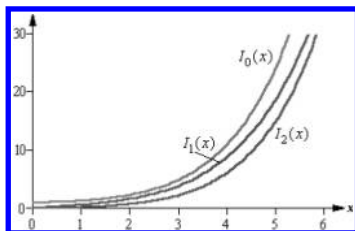
$$x^2 y''(x) + x y'(x) - (x^2 + p^2) y(x) = 0. \quad (8.44)$$

Using the variable transformation $x = iz$, this equation can be reduced to the ordinary Bessel Equation (8.4):

$$z^2 y''(z) + z y'(z) + (z^2 - p^2) y(z) = 0. \quad (8.45)$$

Solving Equation (8.45) by using the power series in an analogous way to the solution of Equation (8.3), we obtain an expression that is very similar to Equation (8.13) but without the oscillating sign:

$$\begin{aligned} I_p(x) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{p+2k} \\ &= \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k} \end{aligned} \quad (8.46)$$

Figure 8.6. Graph of $I_0(x)$, $I_1(x)$, and $I_2(x)$.

Note that if we make the change of a variable $x = iz$ in Equation (8.44), it becomes Equation (8.45); thus, the expansion (8.46) can be obtained simply by making the change of a variable $x = iz$ in Equation (8.13). This formula is correct for arbitrary real p ; in particular, to obtain I_{-p} , we only need to replace p by $-p$ in Equation (8.46).

Reading Exercise. Show that the series (8.46) for the case $I_0(x)$ converges for all x .

Function $I_p(x)$ is called the *modified Bessel function of the first kind*. It is obvious that the relation between this function and the ordinary Bessel function of the first kind is given by

$$I_p(x) = i^p J_p(ix). \quad (8.47)$$

Since the terms in the series (8.47) have constant signs, $I_p(x)$ increases monotonically with increasing x . Figure 8.6 shows graphs of the functions $I_0(x)$, $I_1(x)$, and $I_2(x)$. Note that $I_0(0) = 1$ and $I_n(0) = 0$ when $n = 1, 2, 3, \dots$

Reading Exercise. Prove that, analogous to formulas (8.26) and (8.27) for functions $J_{\pm 1/2}(x)$, the functions $I_{\pm 1/2}(x)$ admit simple representations given by

$$I_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sinh(x), \quad I_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{-1/2} \cosh(x). \quad (8.48)$$

Another solution of Equation (8.44) can be defined as follows:

$$K_p(x) = \frac{\pi}{2} \frac{I_{-p}(x) - I_p(x)}{\sin p\pi}. \quad (8.49)$$

The function $K_p(x)$ is called the *modified Bessel function of the second kind*, or *Macdonald's function*. In analogy to the relation in Equation

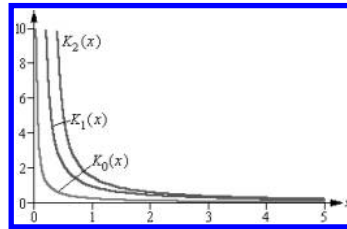


Figure 8.7. Graphs of functions $K_0(x)$, $K_1(x)$, and $K_2(x)$.

(8.47), there exists a connection between this function and the ordinary Bessel function:

$$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(x). \quad (8.50)$$

Reading Exercise. Obtain the relation in Equation (8.50).

The general solution of Equation (8.44) is

$$y(x) = C_1 I_p(x) + C_2 K_p(x). \quad (8.51)$$

If $p \neq 0, 1, 2, \dots$, then the solution given by Equation (8.51) is the combination of the linearly independent functions $I_p(x)$ and $I_{-p}(x)$. If p is a nonnegative integer, the second term in Equation (8.51) contains an indeterminate ratio, $0/0$, which can be resolved with L'Hopital's rule. Therefore, we can use Equation (8.51) as a general solution of the Bessel equation for any real p . The expression for $K_p(x)$ is analogous to the expression for $N_p(x)$. In particular, function $K_0(x)$ may be written as

$$K_0(x) = -I_0(x) \left(\ln \frac{x}{2} + \gamma \right) + \sum_{k=1}^{\infty} \frac{x^{2k}}{2^{2k} (k!)^2} h_k. \quad (8.52)$$

Figure 8.7 shows graphs of functions $K_0(x)$, $K_1(x)$, and $K_2(x)$. For integer p , the functions $K_n(x)$ diverge as $x \rightarrow 0$ and monotonically decrease with increasing x .

8.7 The Effect of Boundaries on Bessel Functions

Now that we have defined various solutions to Equation (8.1) and its variants, we return to a discussion of the effect of the boundaries on these

solutions. Because, in general, Bessel functions arise in problems of circular or spherical symmetry, we may specify boundary value problems as of two kinds: the outer type and the inner type. Boundary value problems are referred to as an outer type if, in addition to a boundary condition on the inner or central boundary, the physical behavior of the solution at infinity is known. For example, when the distribution of temperature is known inside a sphere, we say the boundary value problem is of the inner type. If, in addition we know the temperature is constant at distances far from the sphere, we have a condition on temperature values at infinity leading to a boundary value problem of the outer type. These conditions ensure the uniqueness of the solution in the same way that boundary conditions in other boundary value problems do.

Now let us return to Equation (8.1) with the boundary conditions of Equation (8.2) and define the eigenvalues of the boundary value problem (i.e., the spectrum of values, λ) and the eigenfunctions. Noninteger values of p are allowed as well as integer values, but negative values of p (which are allowed for negative integer values of p only since $J_{-|p|}$ diverges at $r \rightarrow 0$ for noninteger negative p) do not give new solutions. The fact that the solutions are bound in the limit as $r \rightarrow 0$ leads to $C_2 = 0$ in the solution given in Equation (8.31); the opposite choice would result in the function y being unbounded due to the behavior of the function $N_p(x)$.

The condition $y(l) = 0$ leads to the equality $C_1 J_p(\sqrt{\lambda}l) = 0$. Since $C_1 \neq 0$ for nontrivial solutions, we have $J_p(\sqrt{\lambda}l) = 0$. This means that $\sqrt{\lambda}l$ is the root of the equation $J_p(x) = 0$, in which case $\sqrt{\lambda}l = \mu_k^{(p)}$, where $\mu_1^{(p)}, \mu_2^{(p)}, \dots$ are positive roots of the Bessel function $J_p(x)$. We thus have $\sqrt{\lambda_k} = \mu_k/l$, in which case the eigenvalues are $\lambda_k = \mu_k^2/l^2$. Therefore, the eigenvalues of Equation (8.1) for the boundary conditions in Equation (8.2) are

$$\lambda_1 = \frac{\mu_1^2}{l^2}, \quad \lambda_2 = \frac{\mu_2^2}{l^2}, \quad \dots, \quad \lambda_k = \frac{\mu_k^2}{l^2}, \quad \dots \quad (8.53)$$

This result could have been anticipated since this problem is an example of the Sturm-Liouville problem, which has the nontrivial solutions only for nonnegative eigenvalues λ and, as we saw in Chapter 2, these values can be arranged in an increasing sequence:

$$0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots \quad (8.54)$$

The eigenfunctions are

$$y_1 = J_p\left(\frac{\mu_1}{l}r\right), \quad y_2 = J_p\left(\frac{\mu_2}{l}r\right), \quad \dots, \quad y_k = J_p\left(\frac{\mu_k}{l}r\right). \quad (8.55)$$

These eigenfunctions are obtained from the equality in Equation (8.15) while taking into the account the fact that $C_2 = 0$ and $\sqrt{\lambda_k} = \mu_k/l$. The coefficient C_1 can assume any nonzero value; here, we took $C_1 = 1$. It follows from this that the function $J_p(x)$ has an infinite number of roots. Indeed, each root μ satisfies $\mu = \sqrt{\lambda}$, and there are an infinite number of eigenvalues, λ .

The following useful properties of these eigenfunctions are true for arbitrary values of p , not necessarily integer values:

1. To any eigenvalue $\lambda_k = \mu_k^2/l^2$, there corresponds (up to a multiplicative constant) *only one* eigenfunction, in this case, $J_p(\mu_k r/l)$.
2. Eigenfunctions from the sequence

$$J_p\left(\frac{\mu_1}{l}r\right), \quad J_p\left(\frac{\mu_2}{l}r\right), \quad \dots, \quad J_p\left(\frac{\mu_k}{l}r\right), \quad \dots$$

are pair-wise orthogonal with weight r on a segment $(0, l)$. This means that for any k and j ($k \neq j$), the following equality holds:

$$\int_0^l J_p\left(\frac{\mu_k}{l}r\right) J_p\left(\frac{\mu_j}{l}r\right) r dr = 0. \quad (8.56)$$

The property expressed in Equation (8.56) will be proven in Section 8.8.

8.8 Orthogonality and Normalization of Bessel Functions

It is natural to ask what value the integral in Equation (8.56) will have in the case $k = j$. Before pursuing this question, we verify the following formula (in the following, we change the notations, denoting the variable r as x):

$$\int_0^l \left[J_p\left(\frac{\mu_k}{l}x\right) \right]^2 x dx = \frac{l^2}{2} [J'_p(\mu_k)]^2. \quad (8.57)$$

Equations (8.56) and (8.57) will be needed for the theory of the Fourier-Bessel series discussed in Section 8.9. Let us first derive Equation (8.57) for the particular case $l = 1$; that is, we first prove the equality

$$\int_0^1 [J_p(\mu_k x)]^2 x dx = \frac{1}{2} [J'_p(\mu_k)]^2. \quad (8.58)$$

For the proof, first consider a slightly modified form of Equation (8.4):

$$(xy')' + \left(v^2 x - \frac{p^2}{x}\right) y = 0, \quad (8.59)$$

which is satisfied by any function $J_p(vx)$, where $v = \sqrt{\lambda}$. Substitution of the function $J_p(vx)$ into Equation (8.59) results in

$$\left(x \frac{d}{dx} J_p(vx)\right)' + \left(v^2 x - \frac{p^2}{x}\right) J_p(vx) \equiv 0. \quad (8.60)$$

In particular, setting $v = \mu_k$, we have

$$\left(x \frac{d}{dx} J_p(\mu_k x)\right)' + \left(\mu_k^2 x - \frac{p^2}{x}\right) J_p(\mu_k x) \equiv 0. \quad (8.61)$$

By multiplying all terms of Equation (8.61) by $J_p(vx)$ and Equation (8.60) by $J_p(\mu_k x)$, and canceling like terms, we obtain

$$\begin{aligned} J_p(vx) \left(x \frac{d}{dx} J_p(\mu_k x)\right)' - J_p(\mu_k x) \cdot \left(x \frac{d}{dx} J_p(vx)\right)' \\ + (\mu_k^2 - v^2) x J_p(\mu_k x) J_p(vx) \equiv 0. \end{aligned}$$

Let us calculate the integral of this equation in the limits from 0 to 1, integrating the first two terms by parts. After simplifications, we obtain

$$\left[J_p(vx) x \frac{d}{dx} J_p(\mu_k x) - J_p(\mu_k x) x \frac{d}{dx} J_p(vx) \right]_0^1 = (v^2 - \mu_k^2) \int_0^1 x J_p(\mu_k x) J_p(vx) dx,$$

or

$$\left[\mu_k J_p(\nu x) x J'_p(\mu_k x) - \nu J_p(\mu_k x) x J'_p(\nu x) \right]_0^1 = (\nu^2 - \mu_k^2) \int_0^1 x J_p(\mu_k x) J_p(\nu x) dx. \quad (8.62)$$

Substituting the limits 0 and 1 into the left-hand side of Equation (8.62), and taking into account that μ_k is the root of the function $J_p(x)$, we can rewrite this equation as

$$\mu_k J_p(\nu) J'_p(\mu_k) = (\nu^2 - \mu_k^2) \int_0^1 x J_p(\mu_k x) J_p(\nu x) dx,$$

or

$$\int_0^1 x J_p(\mu_k x) J_p(\nu x) dx = \frac{\mu_k J_p(\nu) J'_p(\mu_k)}{\nu^2 - \mu_k^2}. \quad (8.63)$$

Now, if we let ν go to μ_k , we obtain the desired integral

$$\begin{aligned} \int_0^1 x [J_p(\mu_k x)]^2 dx &= \lim_{\nu \rightarrow \mu_k} \int_0^1 x J_p(\mu_k x) J_p(\nu x) dx \\ &= \lim_{\nu \rightarrow \mu_k} \frac{\mu_k J_p(\nu) J'_p(\mu_k)}{\nu^2 - \mu_k^2} = \lim_{\nu \rightarrow \mu_k} \frac{\mu_k J'_p(\nu) J'_p(\mu_k)}{2\nu} = \frac{1}{2} [J'_p(\mu_k)]^2, \end{aligned}$$

where we have used L'Hopital's rule on the right-hand side of Equation (8.63). Thus,

$$\int_0^1 [J_p(\mu_k x)]^2 x dx = \frac{1}{2} [J'_p(\mu_k)]^2,$$

and Equation (8.58) has been proved.

To derive Equation (8.57) for any $l > 0$, first make the change of variable $x/l = z$ in the integral

$$\int_0^l \left[J_p\left(\frac{\mu_k}{l}x\right) \right]^2 x dx.$$

Then

$$\int_0^l \left[J_p \left(\frac{\mu_k}{l} x \right) \right]^2 x dx = l^2 \int_0^1 [J_p(\mu_k z)]^2 z dz = \frac{l^2}{2} [J'_p(\mu_k)]^2,$$

in which case the derivation of Equation (8.57) is completed. Note that if $\nu = \mu_i$ ($i \neq k$) is substituted into Equation (8.63), we obtain

$$\int_0^1 x J_p(\mu_k x) J_p(\mu_i x) dx = 0. \quad (8.64)$$

From Equation (8.64) follows the formulated earlier orthogonality property of Bessel functions in Equation (8.56).

Finally, we point out an analogy between the set of eigenfunctions of the boundary value problem for the equation $y'' + \lambda y = 0$ with the boundary conditions $y(0) = 0$ and $y(l) = 0$, which yield a set of trigonometric eigenfunctions, and the eigenfunctions of the boundary value problem given in Equations (8.1) and (8.2). Denote the positive roots of the equation $\sin x = 0$ as $\mu_1 < \mu_2 < \dots$, where $\mu = k\pi$. Then the eigenfunctions of the first problem can be written as

$$\sin \frac{\mu_1}{l} x, \quad \sin \frac{\mu_2}{l} x, \quad \dots, \quad \sin \frac{\mu_k}{l} x, \quad \dots$$

Graphs of these functions can be obtained from the graph of the function $y = \sin x$ by compression along the x -axis so that the line segments $(0; \mu_1)$, $(0; \mu_2)$, $\dots$, $(0; \mu_k)$ $\dots$ are mapped onto the segment $[0, l]$ of the

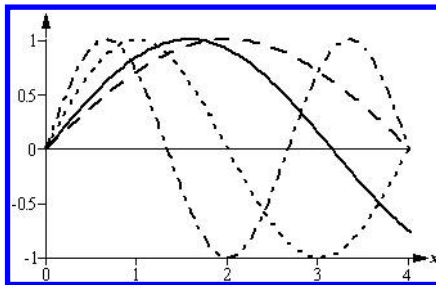


Figure 8.8. Graph of $\sin x$ (solid line) and of the three basis functions: $\sin(\pi x/4)$ (dashed line), $\sin(\pi x/2)$ (dotted line), and $\sin(3\pi x/4)$ (dashed-dotted line).

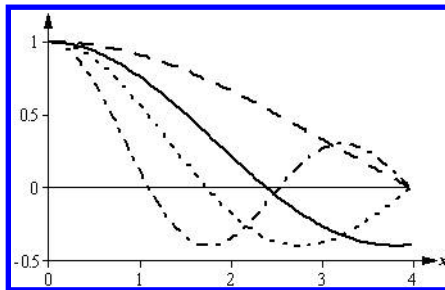


Figure 8.9. Graph of $J_0(x)$ (solid line) and of the three basis functions: $J_0(\mu_1 x/4)$ (dashed line), $J_0(\mu_2 x/4)$ (dotted line), and $J_0(\mu_3 x/4)$ (dashed-dotted line).

x -axis. The graph of $y = \sin x$ and the graphs of the first three eigenfunctions on the segment $[0, l]$ are shown in Figure 8.8.

Now we plot a few consecutive eigenfunctions of the Bessel equation

$$(xy')' - \frac{p^2}{x}y + \lambda xy = 0.$$

Their graphs also can be obtained from the graph of the function $y = J_p(x)$ by compression along the x -axis so that the segments $(0; \mu_1)$, $(0; \mu_2)$, $\dots$, $(0; \mu_k)$ $\dots$ are mapped onto a segment $[0, l]$. Here, μ_k are the positive roots of the function $J_p(x)$. Figure 8.9 shows the graph of $y = J_p(x)$ and graphs of the first three eigenfunctions—the Bessel equation boundary value problem. The analogy between solutions of these two equations is obvious.

8.9 The Fourier-Bessel Series

Section 8.8 showed that there exists infinitely many orthogonal sets of Bessel functions, one set for any given value of p . Thus, for fixed p , the set of functions

$$J_p\left(\frac{\mu_1}{l}x\right), \quad J_p\left(\frac{\mu_2}{l}x\right), \quad \dots, \quad J_p\left(\frac{\mu_k}{l}x\right), \quad \dots \quad (8.65)$$

is orthogonal with weight x on the interval $(0, l)$. Here we assume $p \geq 0$ is a fixed real number and $\mu_1 < \mu_2 < \dots < \mu_k < \dots$ are positive roots of the function $J_p(x)$. This orthogonality is an obvious manifestation of the

fact that the Bessel functions are the eigenfunctions of a Hermitian (self-adjoint) Sturm-Liouville operator. Because of that, these eigenfunctions form a *complete set* of functions on the interval $(0, l)$.

A Fourier series expansion (or generalized Fourier series, as it is sometimes called) of an arbitrary function $f(x)$ using this set of functions is called the *Fourier-Bessel series* of the function $f(x)$ and is given by the expression

$$f(x) = \sum_{k=0}^{\infty} c_k J_p \left(\frac{\mu_k}{l} x \right). \quad (8.66)$$

As usual, the orthogonality property of the basis functions allows us to find the coefficients of this series. We multiply Equation (8.66) by $J_p(\mu_k x/l)$ and integrate term by term with weight x . This gives an expression for the coefficient as

$$c_k = \frac{\int_0^l f(x) J_p \left(\frac{\mu_k}{l} x \right) x dx}{\int_0^l x \left[J_p \left(\frac{\mu_k}{l} x \right) \right]^2 dx}. \quad (8.67)$$

The integral in the denominator of Equation (8.67) has been calculated previously and is

$$\int_0^l x \left[J_p \left(\frac{\mu_k}{l} x \right) \right]^2 dx = \frac{l^2}{2} [J'_p(\mu_k)]^2.$$

Thus, formulas for the coefficients of the Fourier-Bessel series can be reduced to the following form:

$$c_k = \frac{2}{l^2 [J'_p(\mu_k)]^2} \int_0^l f(x) J_p \left(\frac{\mu_k}{l} x \right) x dx. \quad (8.68)$$

The completeness of the set of functions $J_p(\mu_k x/l)$ on the interval $(0, l)$ means that for any square integrable function $f(x)$, the following is true:

$$\int_0^l x f^2(x) dx = \sum_k \left\| J_p \left(\frac{\mu_k}{l} x \right) \right\|^2 c_k^2.$$

This is Parseval's equality for the Fourier-Bessel series. It has the same significance of completeness as in the case of the trigonometric Fourier series with sines and cosines as the basis functions, where the weight function equals 1 instead of x as in the Bessel series.

Regarding the convergence of the series (8.66), we note that the sequence of the partial sums of the series, $S_n(x)$, converges on the interval $(0, l)$ on average (i.e., in the mean) to $f(x)$ (with weight x), which may be written as

$$\int_0^l [f(x) - S_n(x)]^2 x dx \rightarrow 0 \quad \text{if } n \rightarrow \infty.$$

This property is true for any function $f(x)$ from the class of piecewise continuous functions because the orthogonal set of functions in Equation (8.65) are complete on the interval $(0, l)$. For such functions $f(x)$, the series in Equation (8.66) converges absolutely and uniformly. We present without proof the following theorem, which states a somewhat stronger result about the convergence of the series in Equation (8.66) than convergence in the mean:

Theorem 8.1. *If the function $f(x)$ is piecewise continuous on the interval $(0, l)$, then the Fourier-Bessel series converges to $f(x)$ at the points where the function $f(x)$ is continuous, and to*

$$\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)],$$

if x_0 is a point of finite discontinuity of the function $f(x)$.

The rate of convergence of the series in Equation (8.66) depends on matching values of $f(x)$ and values of the function $J_p(\mu_k x/l)$ at the endpoints, $[0, l]$. If $f(x) = a J_p(\mu_k x/l)$ (where a is a constant) at both endpoints of the interval $[0, l]$, then the series is said to be well defined.

Because the other Bessel functions we have seen—for example, $N_p(x)$ —also constitute complete sets of orthogonal functions, an arbitrary function (satisfying “reasonable” restrictions) can be resolved in a series of these sets. Neumann functions can be used to expand a function defined within a ring $a \leq x \leq b$ or in an infinite interval $a \leq x < \infty$. Series expansions in Hankel functions, $H_p(x)$, or Macdonald functions, $K_p(x)$, are useful for solving problems in the theory of radiation, distant from a source.

The following examples use the expansion of functions into the Fourier-Bessel series using the functions $J_p(x)$. In some cases, the coefficients can be found analytically; otherwise, we may calculate them numerically, for example, by using the program `FourierSeries` included with this book. This program also allows the user to change the number of terms in the partial sum and investigate individual terms. All the calculations and the related figures below are generated with this program, which performs the calculations by using the methods described in this chapter. Instructions for the use of the program `FourierSeries` is found in Appendix E.

Example 8.1. Expand the function $f(x) = A$, $A = \text{const}$, in a series using the Bessel functions of the first kind, $X_k(x) = J_0(\mu_k^{(0)}x/l)$, on the interval $[0, l]$, where $\mu_k^{(0)}$ are the positive roots of the equation $J_0(\mu) = 0$.

Solution. First, we calculate the norm, $\|X_k\|^2 = \|J_0(\mu_k^{(0)}x/l)\|^2$, by using the relation $J_0'(x) = -J_1(x)$ to obtain

$$\|J_0(\mu_k^{(0)}x/l)\|^2 = \frac{l^2}{2} [J_0'(\mu_k^{(0)})]^2 = \frac{l^2}{2} J_1^2(\mu_k^{(0)}).$$

To calculate the coefficients, c_k , of the expansion in Equation (8.66), we use the following relations for Bessel functions of the first kind:

$$\int x^n J_{n-1}(x) dx = x^n J_n(x),$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

Using the substitution $z = \mu_k^{(0)}x/l$, we may calculate the integral

$$\int_0^l J_0\left(\frac{\mu_k^{(0)}}{l}x\right) x dx = \frac{l^2}{(\mu_k^{(0)})^2} \int_0^{\mu_k^{(0)}} J_0(z) z dz = \frac{l^2}{(\mu_k^{(0)})^2} [z J_1(z)]_0^{\mu_k^{(0)}} = \frac{l^2}{\mu_k^{(0)}} J_1(\mu_k^{(0)}). \quad (8.69)$$

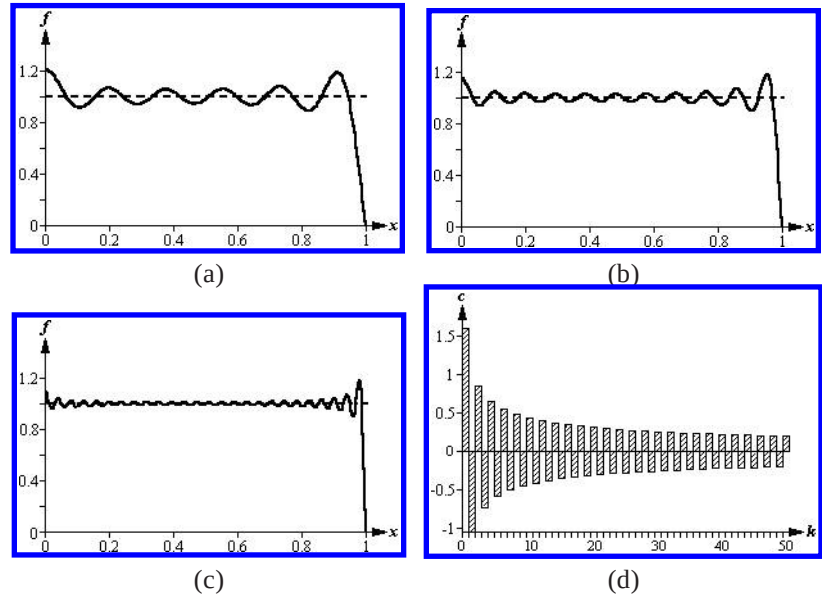


Figure 8.10. The function $f(x) = 1$ and the partial sum of its Fourier-Bessel series. The graph of $f(x)$ is shown by the dashed line, and the graph of the series is shown by the solid line. $(N + 1)$ terms are kept in the series. (a) $N = 10$. (b) $N = 20$. (c) $N = 50$. (d) Values of the coefficients c_k of the series.

Using Equation (8.69), we have

$$\begin{aligned}
 c_k &= \frac{1}{\left\| J_0 \left(\mu_k^{(0)} x / l \right) \right\|^2} \int_0^l A J_0 \left(\mu_k^{(0)} x / l \right) x dx \\
 &= \frac{2A}{l^2 \left[J_1 \left(\mu_k^{(0)} \right) \right]^2} \frac{l^2}{\mu_k^{(0)}} J_1 \left(\mu_k^{(0)} \right) = \frac{2A}{\mu_k^{(0)} J_1 \left(\mu_k^{(0)} \right)}.
 \end{aligned}$$

Thus, the expansion is

$$f(x) = 2A \sum_{k=0}^{\infty} \frac{1}{\mu_k^{(0)} J_1 \left(\mu_k^{(0)} \right)} J_0 \left(\frac{\mu_k^{(0)}}{l} x \right).$$

Figure 8.10 shows the function $f(x) = 1$ and the partial sum of its Fourier-Bessel series when $l = 1$. From this figure, it is seen that the series converges very slowly (see Figure 8.10(d)) and even when 50 terms are kept

in the expansion (Figure 8.10(c)), the difference from $f(x) = 1$ can easily be seen. This is because at the endpoints of the interval, the value of the function $f(x) = 1$ and the functions $J_0(\mu_k^{(0)}x)$ are different. The obtained expansion does not converge well near the endpoints.

Example 8.2. Modify the boundary condition in Example 8.1. Expand the function $f(x) = A$, $A = \text{const}$, given on the interval $[0, l]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_0(\mu_k^{(0)}x/l)$, where $\mu_k^{(0)}$ are now the positive roots of the equation $J'_0(\mu) = 0$.

Solution. The norms of eigenfunctions for Bessel functions of zeroth order and Neumann boundary condition are

$$\mu_k^{(0)} = 0, \quad X_0(x) = J_0(0) = 1, \quad \|X_0\|^2 = \|J_0(0)\|^2 = \frac{l^2}{2},$$

$$\|X_k\|^2 = \frac{l^2}{2} J_0^2(\mu_k^{(0)}), \quad k > 0.$$

The first coefficient of the expansion is

$$c_0 = \frac{A}{\|J_0(0)\|^2} \int_0^l J_0(0)x dx = \frac{2A}{l^2} \int_0^l x dx = A.$$

The next term can be evaluated by using the substitution $z = \mu_k^{(0)}x/l$ and using the integration formula

$$\int_0^l J_0\left(\frac{\mu_k^{(0)}}{l}x\right) x dx = \frac{l^2}{\left(\mu_k^{(0)}\right)^2} [z J_1(z)]_0^{\mu_k^{(0)}} = \frac{l^2}{\mu_k^{(0)}} J_1(\mu_k^{(0)}).$$

By applying the relation $J'_0(x) = -J_1(x)$ and then recalling that $J'_0(\mu_k^{(0)}) = 0$, we find

$$c_k = \frac{2A}{\mu_k^{(0)}} J_1(\mu_k^{(0)}) = 0 \quad \text{when } k > 0. \quad (8.70)$$

Thus, we obtain the simple expansion, $f(x) = c_0 J_0(\mu_0^{(0)}x/l) = A$. In fact, this means that the given function is actually one of the functions from the set of eigenfunctions used for eigenfunction expansion.

Figure 8.11 shows the graph of the function $f(x) = 1$ and its expansion into a Fourier-Bessel series. It is seen from the graph that only one term is nonzero in the expansion in Equation (8.66).

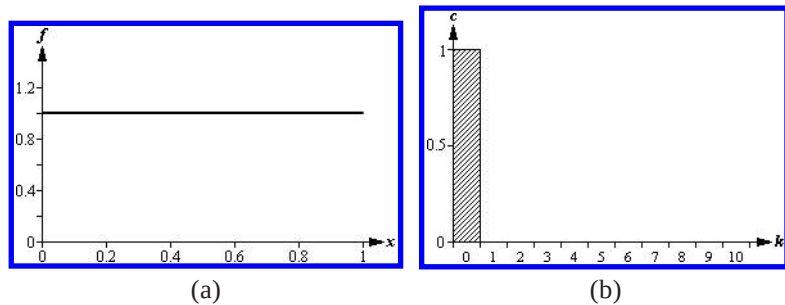


Figure 8.11. The function $f(x) = 1$ and the partial sum of the Fourier-Bessel series (Equation 8.66). (a) The graph of $f(x)$ and the graph of the partial sum for the first term in the series completely coincide. (b) Values of the coefficients c_k of the series.

In applications, it is often necessary to solve Bessel equations of some order n accompanied by a boundary condition containing a Bessel functions of order different than n . To solve such boundary value problems, we need to resolve some function $f(x)$ (describing the boundary condition) in a Fourier-Bessel series on an interval $0 \leq x \leq l$ in functions $J_n(\mu_k^{(n)} x/l)$, where $\mu_k^{(n)}$ are solutions of some algebraic equation, $F(J_m(x), J'_m(x), \dots) = 0$, which follows from the boundary condition.

Example 8.3. Now consider the case of mixed boundary conditions. Expand the function $f(x) = A$, $A = \text{const}$, given on the interval $[0, l]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_0(\mu_k^{(0)} x/l)$, where now $\mu_k^{(0)}$ are the positive roots of the equation $\mu J'_0(\mu) + hl J_0(\mu) = 0$.

Solution. First, we calculate the norm, $\|X_k\|^2 = \|J_0(\mu_k^{(0)} x/l)\|^2$:

$$\|X_k\|^2 = \frac{l^2}{2} \left[1 + \frac{(\mu_k^{(0)})^2}{l^2 h^2} \right] \left[J'_0(\mu_k^{(0)}) \right]^2.$$

Or, taking into account $\mu_k^{(0)} J'_0(\mu_k^{(0)}) + hl J_0(\mu_k^{(0)}) = 0$, we have

$$\|X_k\|^2 = \left[l^2 h^2 + (\mu_k^{(0)})^2 \right] \frac{l^2}{2 (\mu_k^{(0)})^2} J_0^2(\mu_k^{(0)}). \quad (8.71)$$

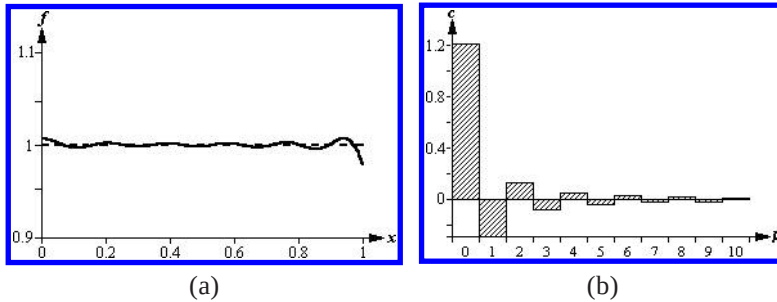


Figure 8.12. The function $f(x) = 1$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

From Equation (8.71), we obtain

$$c_k = \frac{1}{\|X_k\|^2} \int_0^l J_0\left(\frac{\mu_k^{(0)}}{l}x\right) x dx = \frac{2A\mu_k^{(0)}}{\left[\left(\mu_k^{(0)}\right)^2 + h^2l^2\right] J_0^2\left(\mu_k^{(0)}\right)} J_1\left(\mu_k^{(0)}\right).$$

Therefore, the expansion is

$$f(x) = 2A \sum_{k=0}^{\infty} \frac{\mu_k^{(0)} J_1\left(\mu_k^{(0)}\right)}{\left[\left(\mu_k^{(0)}\right)^2 + h^2l^2\right] J_0^2\left(\mu_k^{(0)}\right)} J_0\left(\frac{\mu_k^{(0)}}{l}x\right).$$

Figure 8.12 shows the graph of the function $f(x) = 1$ and its expansion into a Fourier-Bessel series when $l = 1$ and $h = 1$. Unlike Example 8.2, in which only one coefficient of the expansion was nonzero, here we obtain a spectrum of nonzero coefficients c_k , and the graph of the partial sum differs from the graph of the given function. We emphasize again that in this example the boundary conditions do not match function values at the interval endpoints.

Example 8.4. Next, expand the function $f(x) = A$, $A = \text{const}$, given on the interval $[0, l]$, in a series using the functions $X_k(x) = J_2(\mu_k^{(2)}x/l)$, where $\mu_k^{(2)}$ are the positive roots of the equation $J_2(\mu) = 0$.

Solution. We calculate the norm $\|X_k\|^2 = \|J_2(\mu_k^{(2)}x/l)\|^2$ by using the recurrence formulas (8.21) and the boundary condition $J_2(\mu_k^{(2)}) = 0$:

$$\|X_k\|^2 = \frac{l^2}{2} \left[J_2' \left(\mu_k^{(2)} \right) \right]^2 = \frac{l^2}{2} \left[-\frac{2}{\mu_k^{(2)}} J_2 \left(\mu_k^{(2)} \right) + J_1 \left(\mu_k^{(2)} \right) \right]^2 = \frac{l^2}{2} J_1^2 \left(\mu_k^{(2)} \right).$$

Next, we calculate the integral

$$\begin{aligned} \int_0^l J_2 \left(\frac{\mu_k^{(2)}}{l} x \right) x dx &= \frac{l^2}{\left(\mu_k^{(2)} \right)^2} \int_0^{\mu_k^{(2)}} [2J_1(z) - zJ_0(z)] dz \\ &= \frac{l^2}{\left(\mu_k^{(2)} \right)^2} \int_0^{\mu_k^{(2)}} [-2J_0'(z) - zJ_0(z)] dz \\ &= \frac{l^2}{\left(\mu_k^{(2)} \right)^2} \left[2 - 2J_0 \left(\mu_k^{(2)} \right) - \mu_k^{(2)} J_0 \left(\mu_k^{(2)} \right) \right]. \end{aligned}$$

We thus obtain the coefficients of the series

$$c_k = \frac{A}{\|X_k\|^2} \int_0^l J_2 \left(\frac{\mu_k^{(2)}}{l} x \right) x dx = \frac{2A \left[2 - 2J_0 \left(\mu_k^{(2)} \right) - \mu_k^{(2)} J_0 \left(\mu_k^{(2)} \right) \right]}{\left(\mu_k^{(2)} \right)^2 J_1^2 \left(\mu_k^{(2)} \right)},$$

and the expansion is

$$f(x) = 2A \sum_{k=0}^{\infty} \frac{2 - 2J_0 \left(\mu_k^{(2)} \right) - \mu_k^{(2)} J_0 \left(\mu_k^{(2)} \right)}{\left(\mu_k^{(2)} \right)^2 J_1^2 \left(\mu_k^{(2)} \right)} J_2 \left(\frac{\mu_k^{(2)}}{l} x \right).$$

A graph of the function $f(x) = 1$ and its Fourier-Bessel series on the interval $[0, 1]$ is shown in Figure 8.13. The series converges poorly at the interval endpoints because at the endpoints, the function $f(x)$ and the basis functions $J_2(\mu_k^{(2)}x)$ have the very different values: $f(0) = 1$, $J_2(0) = 0$, $f(1) = 1$, and $J_2(\mu_k^{(2)}) = 0$. We thus see that the basis functions are zero at the endpoints in contrast to the values of $f(x)$, which are nonzero there.

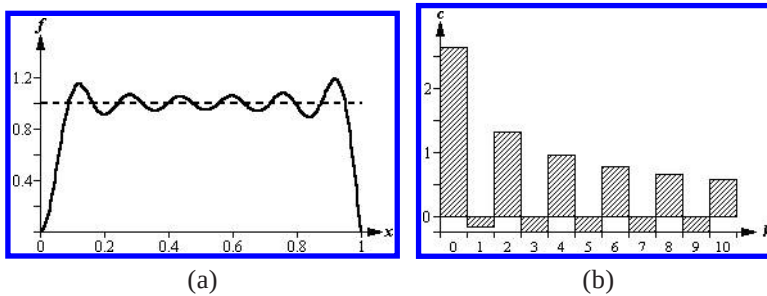


Figure 8.13. The function $f(x) = 1$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

Example 8.5. Next, expand the function $f(x) = A$, $A = \text{const}$, given on the interval $[0, l]$, in a series using functions $X_k(x) = J_2(\mu_k^{(2)}x/l)$, where $\mu_k^{(2)}$ are the positive roots of the equation $J_2'(\mu) = 0$.

Solution. The coefficients of the series are calculated by using the formulas

$$c_k = \frac{A}{\|X_k\|^2} \int_0^l J_2\left(\frac{\mu_k^{(2)}}{l}x\right) x dx,$$

or, after substituting the norm

$$\|X_k\|^2 = \frac{l^2}{2} \left[1 - \frac{4}{(\mu_k^{(2)})^2} \right] J_2^2(\mu_k^{(2)}),$$

we have

$$c_k = \frac{2A \left[2 - 2J_0(\mu_k^{(2)}) - \mu_k^{(2)} J_1(\mu_k^{(2)}) \right]}{\left[(\mu_k^{(2)})^2 - 4 \right] J_2^2(\mu_k^{(2)})}.$$

Graphs of the function $f(x) = 1$ and its Fourier-Bessel series expansion on the interval $[0, 1]$ are shown in Figure 8.14. As in the previously considered case, the expansion is poorly matched to the true function. This

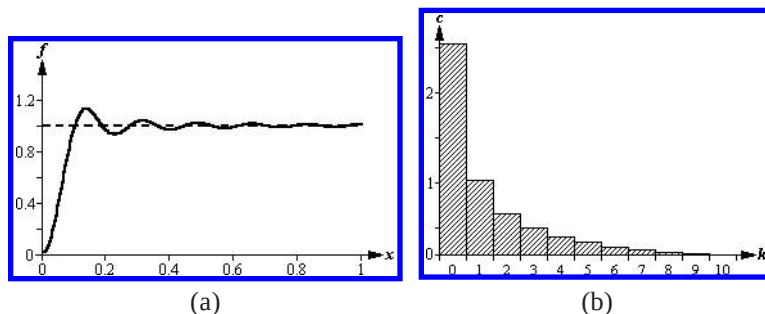


Figure 8.14. The function $f(x) = 1$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

is due to a mismatch of values of the function and the basis functions $J_2(\mu_k^{(2)}x)$ at the left endpoint, $x = 0$, where $f(0) = 1$ but $J_2(0) = 0$. However, values do match at $x = 1$, where $f'(1) = J_2'(\mu_k^{(2)}) = 0$; therefore, the series converges more quickly than in Example 8.4 (excluding the neighborhood of $x = 0$, where this series as well as the previous one converge very slow because of the mismatch of the expanded function and the basis functions).

Example 8.6. Expand the function $f(x) = 1 - x$ in a series using the functions $J_0(\mu_k^{(0)}x)$ on the interval $[0, 1]$, where $\mu_k^{(0)}$ are the roots of the equation $J_0(\mu) = 0$.

Solution. The coefficients of the series are calculated by using the formulas

$$c_k = \frac{2}{\left[J_0'(\mu_k^{(0)})\right]^2} \int_0^1 x(1-x) J_0(\mu_k^{(0)}x) dx,$$

or, after the change of variables $\mu_k^{(0)}x = z$,

$$c_k = \frac{2}{\left[J_0'(\mu_k^{(0)})\right]^2} \left[\frac{1}{(\mu_k^{(0)})^2} \int_0^{\mu_k^{(0)}} z J_0(z) dz - \frac{1}{(\mu_k^{(0)})^2} \int_0^{\mu_k^{(0)}} z^2 J_0(z) dz \right].$$

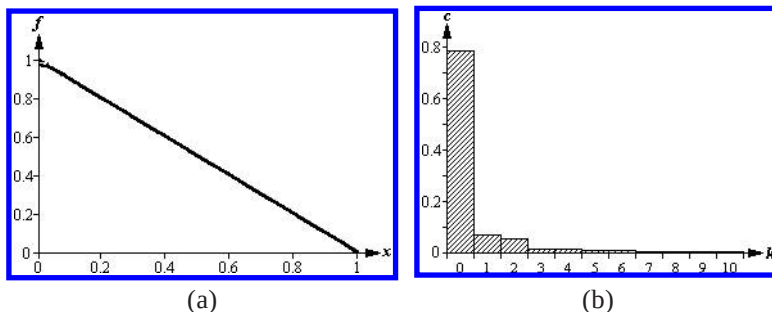


Figure 8.15. The function $f(x) = 1 - x$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

To simplify this expression, we use differentiation formulas for the functions $J_0(x)$ and $xJ_1(x)$, and also the equality $\int zJ_0(z)dz = zJ_1(z) + C$, which is implied from them. By using these formulas and integrating the second term twice by parts, we obtain, after simplifications,

$$c_k = \frac{2}{\left[J'_0\left(\mu_k^{(0)}\right)\right]^2 \left(\mu_k^{(0)}\right)^2} \int_0^{\mu_k^{(0)}} J_0(z) dz.$$

The coefficients c_k can be found by using the program FourierSeries.

Example 8.7. Expand the function $f(x) = Ax^2$ in a series by using the functions $J_0(\mu_k^{(0)}x/l)$ on the interval $[0, l]$, where $\mu_k^{(0)}$ are positive roots of the equation $J_0(\mu) = 0$.

Solution. We calculate the norm $\|X_k\|^2 = \|J_0(\mu_k^{(0)}x/l)\|^2$ by using the relation $J'_0(x) = -J_1(x)$:

$$\left\|J_0\left(\mu_k^{(0)}x/l\right)\right\|^2 = \frac{l^2}{2} \left[J'_0\left(\mu_k^{(0)}\right)\right]^2 = \frac{l^2}{2} \left[J_1\left(\mu_k^{(0)}\right)\right]^2.$$

To calculate coefficients, c_k , we use the following relations for the Bessel functions of the first kind:

$$\begin{aligned} \int_0^l x^n J_{n-1}(x) dx &= x^n J_n(x), \\ J_{n+1}(x) &= \frac{2n}{x} J_n(x) - J_{n-1}(x). \end{aligned} \quad (8.72)$$

By using Equations (8.72) and the fact that $J_0(\mu_k^{(0)}) = 0$, we have

$$\begin{aligned} \int_0^l x^2 J_0\left(\frac{\mu_k^{(0)}}{l}x\right) dx &= \frac{l^4}{\left(\mu_k^{(0)}\right)^4} \int_0^{\mu_k^{(0)}} x^2 [2J_1(x) - xJ_2(x)] dx \\ &= \frac{l^4}{\left(\mu_k^{(0)}\right)^4} x [(x^2 - 4)J_1(x) + 2xJ_0(x)]_0^{\mu_k^{(0)}} \\ &= \frac{l^4}{\mu_k^{(0)}} \left[1 - \frac{4}{\left(\mu_k^{(0)}\right)^2} \right] J_1\left(\mu_k^{(0)}\right). \end{aligned}$$

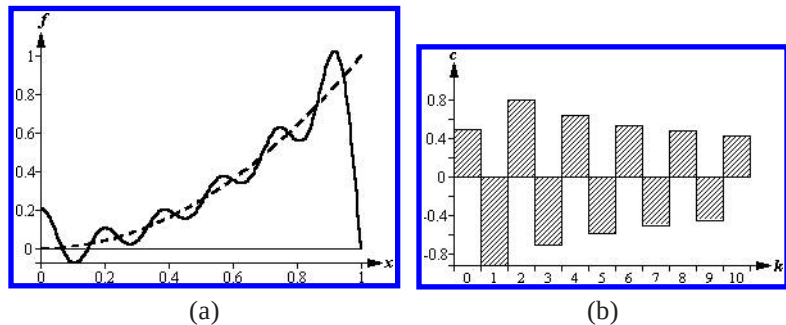


Figure 8.16. The function $f(x) = x^2$ and the partial sum of its Fourier-Bessel series when $l = 1$. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

Therefore, we have

$$c_k = \frac{1}{\|X_k\|^2} \int_0^l A x^2 J_0 \left(\mu_k^{(0)} x / l \right) x dx = \frac{2Al^2}{\mu_k^{(0)} J_1 \left(\mu_k^{(0)} \right)} \left[1 - \frac{4}{\left(\mu_k^{(0)} \right)^2} \right].$$

The expansion is

$$f(x) = 2Al^2 \sum_{k=0}^{\infty} \frac{1}{\mu_k^{(0)} J_1 \left(\mu_k^{(0)} \right)} \left[1 - \frac{4}{\left(\mu_k^{(0)} \right)^2} \right] J_0 \left(\frac{\mu_k^{(0)}}{l} x \right).$$

Example 8.8. Expand the function $f(x) = Ax^2$, $A = \text{const}$, in a series by using functions $X_k(x) = J_0(\mu_k^{(0)} x / l)$ on the interval $[0, l]$, where $\mu_k^{(0)}$ are positive roots of the equation $J'_0(\mu) = 0$.

Solution. We calculate the norm $\|X_k\|^2 = \|J_0(\mu_k^{(0)} x / l)\|^2$:

$$\mu_0^{(0)} = 0, \quad J_0 \left(\mu_0^{(0)} x / l \right) = J_0(0) = 1, \quad \|J_0(0)\|^2 = \frac{l^2}{2},$$

$$\left\| J_0 \left(\mu_k^{(0)} x / l \right) \right\|^2 = \frac{l^2}{2} J_0^2 \left(\mu_k^{(0)} \right), \quad k > 0.$$

We then calculate the integral

$$\begin{aligned} \int_0^l x^2 J_0 \left(\frac{\mu_m^{(0)}}{l} x \right) x dx &= \frac{l^4}{\left(\mu_m^{(0)} \right)^4} \int_0^{\mu_m^{(0)}} x^2 [2J_1(x) - xJ_2(x)] dx \\ &= \frac{l^4}{\left(\mu_m^{(0)} \right)^4} [2x^2 J_2(x) - x^3 J_3(x)]_0^{\mu_m^{(0)}} \\ &= \frac{l^4}{\left(\mu_m^{(0)} \right)^3} \left[\left(\left(\mu_m^{(0)} \right)^2 - 4 \right) J_1 \left(\mu_m^{(0)} \right) + 2\mu_m^{(0)} J_0 \left(\mu_m^{(0)} \right) \right]. \end{aligned}$$

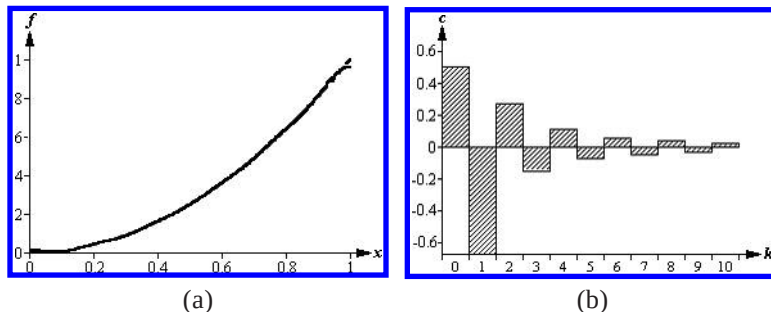


Figure 8.17. The function $f(x) = x^2$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

We thus have

$$\begin{aligned} c_k &= \frac{A}{\|X_k\|^2} \int_0^l x^2 J_0 \left(\frac{\mu_k^{(0)}}{l} x \right) x dx \\ &= \frac{2Al^2}{\left(\mu_k^{(0)}\right)^3 J_0^2 \left(\mu_k^{(0)}\right)} \left\{ \left[\left(\mu_k^{(0)}\right)^2 - 4 \right] J_1 \left(\mu_k^{(0)}\right) + 2\mu_k^{(0)} J_0 \left(\mu_k^{(0)}\right) \right\}. \end{aligned}$$

The final expression for the expansion is

$$f(x) = 2Al^2 \sum_{k=0}^{\infty} \frac{\left[\left(\mu_k^{(0)}\right)^2 - 4 \right] J_1 \left(\mu_k^{(0)}\right) + 2\mu_k^{(0)} J_0 \left(\mu_k^{(0)}\right)}{\left(\mu_k^{(0)}\right)^3 J_0^2 \left(\mu_k^{(0)}\right)} J_0 \left(\frac{\mu_k^{(0)}}{l} x \right).$$

Example 8.9. Expand the function $f(x) = Ax^n$, $A = \text{const}$, given on the interval $[0, l]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_n(\mu_k^{(n)} x/l)$, where $\mu_k^{(n)}$ are positive roots of the equation $J'_n(\mu) = 0$.

Solution. The norm of the basis function is

$$\|X_k\|^2 = \frac{l^2}{2} \left[1 - \frac{n^2}{\left(\mu_k^{(n)}\right)^2} \right] J_n^2 \left(\mu_k^{(n)}\right).$$

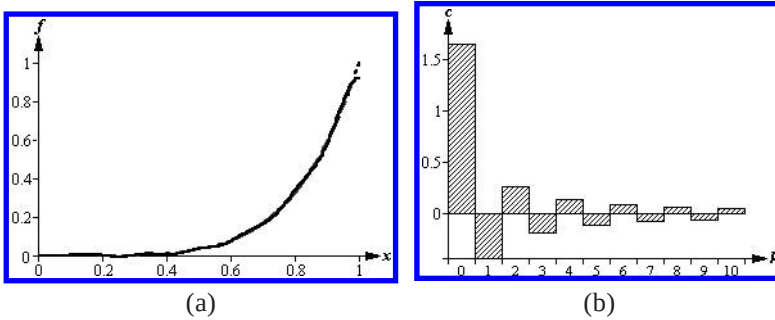


Figure 8.18. The function $f(x) = x^5$ and the partial sum of its Fourier-Bessel series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

We calculate the integral

$$\begin{aligned}
 \int_0^l x^n J_n \left(\frac{\mu_k^{(n)}}{l} x \right) x dx &= \frac{l^{n+2}}{\left(\mu_k^{(n)} \right)^{n+2}} \int_0^{\mu_k^{(n)}} z^{n+1} J_n(z) dz \\
 &= \frac{l^{n+2}}{\left(\mu_k^{(n)} \right)^{n+2}} \left[z^{n+1} J_{n+1}(z) \right]_0^{\mu_k^{(n)}} \\
 &= \frac{l^{n+2}}{\mu_k^{(n)}} J_{n+1} \left(\mu_k^{(n)} \right).
 \end{aligned}$$

We then calculate coefficients of the solution to obtain

$$c_k = \frac{2A \left(\mu_k^{(n)} \right)^2}{l^2 \left[\left(\mu_k^{(n)} \right)^2 - n^2 \right] J_n^2 \left(\mu_k^{(n)} \right)} \int_0^l x^n J_n \left(\frac{\mu_k^{(n)}}{l} x \right) x dx = \frac{2Al^n \mu_k^{(n)} J_{n+1} \left(\mu_k^{(n)} \right)}{\left[\left(\mu_k^{(n)} \right)^2 - n^2 \right] J_n^2 \left(\mu_k^{(n)} \right)}.$$

The expansion is

$$f(x) = 2Al^n \sum_{k=0}^{\infty} \frac{\mu_k^{(n)} J_{n+1} \left(\mu_k^{(n)} \right)}{\left[\left(\mu_k^{(n)} \right)^2 - n^2 \right] J_n^2 \left(\mu_k^{(n)} \right)} J_n \left(\frac{\mu_k^{(n)}}{l} x \right).$$

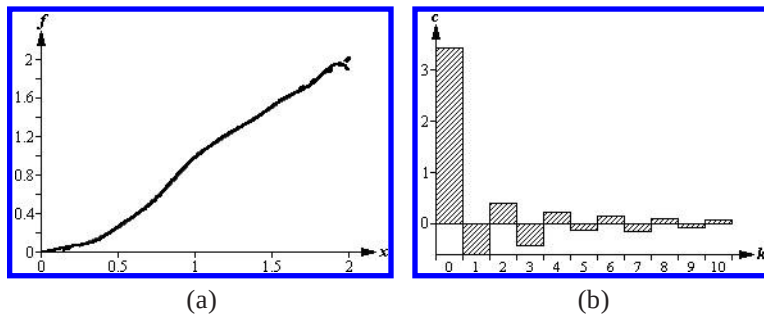


Figure 8.19. The function $f(x)$ and the partial sum of its Fourier-Bessel series when $h = 1$. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with 10 terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

Example 8.10. Expand the function

$$f(x) = \begin{cases} x^2, & 0 \leq x < 1, \\ x, & 1 \leq x < 2, \end{cases}$$

given on the interval $[0, 2]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_2(\mu_k^{(2)}x/2)$, where $\mu_k^{(2)}$ are positive roots of the equation $\mu J_2'(\mu_k^{(2)}) + 2hJ_2(\mu_k^{(2)}) = 0$.

Solution. The norm of the basis function is

$$\|X_k\|^2 = 2 \left[1 + \frac{4h^2 - 4}{(\mu_k^{(2)})^2} \right] J_2^2(\mu_k^{(2)}).$$

The coefficients of the solution are

$$\begin{aligned} c_k &= \frac{1}{\|X_k\|^2} \int_0^2 f(x) J_2(\mu_k^{(2)}x/2) x dx \\ &= \frac{1}{\|X_k\|^2} \left\{ \int_0^1 x^2 J_2(\mu_k^{(2)}x/2) x dx + \int_1^2 x J_2(\mu_k^{(2)}x/2) x dx \right\}. \end{aligned}$$

The expansion is

$$f(x) = \sum_{k=0}^{\infty} c_k J_2 \left(\frac{\mu_k^{(2)}}{2} x \right).$$

8.10 Further Examples of Fourier-Bessel Series Expansions

In this section, we derive analytical solutions for a few more useful examples involving Bessel series expansions.

Example 8.11. Solve the Laplace equation

$$\nabla^2 u = 0, \quad (8.73)$$

inside a circular bounded cylinder, $r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$, with zero boundary condition at the lateral surface and boundary conditions at the bottom and top surfaces given by

$$u|_{r=a} = 0, \quad u|_{z=0} = f(r, \varphi), \quad u|_{z=l} = F(r, \varphi). \quad (8.74)$$

Solution. We solve this problem (a Dirichlet boundary value problem defined by Equations (8.73) and (8.74)) by representing the unknown function, u , in the following form:

$$u(r, \varphi, z) = V(r, \varphi) Z(z). \quad (8.75)$$

Substituting Equation (8.75) into Equation (8.73), we get, after the separation of variables,

$$\frac{1}{Vr} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{Vr^2} \frac{\partial^2 V}{\partial \varphi^2} = -\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -\lambda,$$

where $\lambda > 0$ is the separation constant, which will be determined from the conditions of existence of a nontrivial solution of the problem. As a result, we obtain the equation and the boundary condition

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \lambda V = 0, \quad (8.76)$$

$$V(a, \varphi) = 0 \quad (8.77)$$

for the function $V(r, \varphi)$. For $Z(z)$, we have the equation

$$Z'' - \lambda Z = 0,$$

with a solution that can be written in the form

$$Z(z) = d_1 \cosh \sqrt{\lambda} z + d_2 \sinh \sqrt{\lambda} z.$$

By using the substitution

$$V(r, \varphi) = R(r)\Phi(\varphi),$$

in analogy with Equation (8.76), we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \left(\lambda - \frac{\eta}{r^2} \right) R = 0, \quad (8.78)$$

$$\Phi'' + \eta \Phi = 0, \quad (8.79)$$

where η is a separation constant. From the periodicity of the function $\Phi(\varphi)$ in the angle φ (i.e., $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ and $\Phi'(\varphi) = \Phi'(\varphi + 2\pi)$), we find $\eta = n^2$, where $n = 0, 1, 2, \dots$ and

$$\Phi(\varphi) = c_1 \cos n\varphi + c_2 \sin n\varphi. \quad (8.80)$$

For the function $R(r)$, we have the Bessel Equation (8.78) along with the boundary conditions

$$R(a) = 0, \quad |R(0)| < \infty.$$

We thus see that

$$R(r) = J_n \left(\sqrt{\lambda} r \right). \quad (8.81)$$

The boundary condition gives, at $r = a$,

$$J_n \left(\mu_n^{(m)} \right) = 0,$$

where $\mu_n^{(m)} = \sqrt{\lambda} a$, and $m = 0, 1, 2, \dots$ is the number of the root of this equation.

Therefore, the boundary value problem in Equations (8.76) and (8.77) for $V(r, \varphi)$ has the eigenvalues

$$\lambda_{mn} = \left(\frac{\mu_n^{(m)}}{a} \right)^2,$$

and the corresponding eigenfunctions are obtained from the solutions in Equations (8.80) and (8.81):

$$V_{mn}^{(1)} = J_n \left(\frac{\mu_n^{(m)}}{a} r \right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_n^{(m)}}{a} r \right) \sin n\varphi.$$

These eigenfunctions form two orthogonal systems of functions, which yield the norms

$$\|V_{nm}^{(1)}\|^2 = \frac{a^2}{2} \left| J_n'(\mu_n^{(m)}) \right|^2 \pi \varepsilon_n \quad \text{and} \quad \|V_{nm}^{(2)}\|^2 = \frac{a^2}{2} \left| J_n'(\mu_n^{(m)}) \right|^2 \pi,$$

where

$$\varepsilon_n = \begin{cases} 2, & n = 0, \\ 1, & n \neq 0. \end{cases}$$

Thus, the general solution of the first boundary value problem for Equations (8.73) and (8.74), using the above results, is represented as the series

$$u(r, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \left[a_{1mn} V_{nm}^{(1)}(r, \varphi) + b_{1mn} V_{nm}^{(2)}(r, \varphi) \right] \cosh \left(\sqrt{\lambda_{mn}} z \right) + \left[a_{2mn} V_{nm}^{(1)}(r, \varphi) + b_{2mn} V_{nm}^{(2)}(r, \varphi) \right] \sinh \left(\sqrt{\lambda_{mn}} z \right) \right\}.$$

From the boundary condition at $z = 0$, we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[a_{1mn} V_{nm}^{(1)}(r, \varphi) + b_{1mn} V_{nm}^{(2)}(r, \varphi) \right] = f(r, \varphi),$$

where the coefficients a_{1mn} and b_{1mn} may be determined by expanding the function $f(r, \varphi)$ in a Fourier series in basis functions $V_{nm}^{(1)}(r, \varphi)$ and $V_{nm}^{(2)}(r, \varphi)$:

$$a_{1mn} = \frac{1}{\|V_{nm}^{(1)}\|} \int_0^a \int_0^{2\pi} f(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi,$$

$$b_{1mn} = \frac{1}{\|V_{nm}^{(2)}\|} \int_0^a \int_0^{2\pi} f(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi.$$

Analogously, we find the coefficients a_{2mn} and b_{2mn} by using the boundary condition at $z = l$:

$$a_{2mn} = \frac{1}{\|V_{nm}^{(1)}\|} \int_0^a \int_0^{2\pi} F(r, \varphi) V_{nm}^{(1)}(r, \varphi) r dr d\varphi,$$

$$b_{2mn} = \frac{1}{\|V_{nm}^{(2)}\|} \int_0^a \int_0^{2\pi} F(r, \varphi) V_{nm}^{(2)}(r, \varphi) r dr d\varphi.$$

Example 8.12 (Electrostatic Field Inside a Cylinder). Find the expression for the potential of the electrostatic field inside a cylindrical box having a circular cross section, $r \leq r_0$, $0 \leq z \leq l$. Both ends of the box are grounded, and the side surface is charged so that its potential is u_0 . Find the strength of the field on the axis of symmetry.

Solution. The problem is conveniently solved in the cylindrical coordinate system using variables (r, φ, z) . Here we let the z -axis be along the cylinder axis. First, we formulate the boundary value problem. The electrostatic potential $u(r, \varphi, z)$ satisfies the Laplace equation,

$$\nabla^2 u = 0. \quad (8.82)$$

Boundary conditions at the ends of the cylindrical box are

$$u|_{z=0} = 0, \quad u|_{z=l} = 0.$$

The boundary condition on the side surface of the box is

$$u|_{r=r_0} = u_0. \quad (8.83)$$

Let us solve the boundary value problem in Equations (8.82) through (8.83) by using the method of separation of variables, in analogy to the problem in Example 8.11. The form of the potential $u(r, \varphi, z)$ is

$$u(r, \varphi, z) = R(r)\Phi(\varphi)Z(z).$$

Substituting this into Equation (8.82) gives

$$\frac{1}{Rr} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{\Phi r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0. \quad (8.84)$$

By separating variables in Equation (8.84), we obtain three equations:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) - \left(\lambda + \frac{\eta}{r^2} \right) R = 0, \quad (8.85)$$

$$\Phi'' + \eta \Phi = 0, \quad (8.86)$$

$$Z'' + \lambda Z = 0, \quad (8.87)$$

where $\lambda > 0$ and $\eta > 0$ are separation constants. For the function $\Phi(\varphi)$, we have periodic boundary conditions

$$\Phi(\varphi) = \Phi(\varphi + 2\pi), \quad \Phi'(\varphi) = \Phi'(\varphi + 2\pi),$$

and for the function $Z(z)$, boundary conditions given by

$$Z(0) = 0, \quad Z(l) = 0. \quad (8.88)$$

The problem for $\Phi(\varphi)$ has two linearly independent solutions,

$$\Phi_1(\varphi) = \cos n\varphi \quad \text{and} \quad \Phi_2(\varphi) = \sin n\varphi,$$

where $\eta = n^2$, $n = 0, 1, 2, \dots$. Equations (8.87) and (8.88) yield as solutions

$$Z(z) = \sin \left(\sqrt{\lambda_m} z \right),$$

where $\lambda_m = (\pi m / l)^2$, $m = 1, 2, 3, \dots$. Thus, the two systems of orthogonal functions

$$\sin \left(\sqrt{\lambda_m} z \right) \cos n\varphi \quad \text{and} \quad \sin \left(\sqrt{\lambda_m} z \right) \sin n\varphi$$

are the eigenfunctions for the problem.

Equation (8.85) for $R(r)$ is the modified Bessel equation. Taking into account the boundedness of $R(r)$ at $r = 0$, the solution to Equation (8.85) is written in the form

$$R(r) = I_n \left(\sqrt{\lambda_m} r \right).$$

The general solution of the boundary value problem in Equations (8.82) through (8.83) may thus be represented in the form

$$u(r, \varphi, z) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (a_{mn} \cos n\varphi + b_{mn} \sin n\varphi) I_n \left(\sqrt{\lambda_m} r \right) \sin \left(\sqrt{\lambda_m} z \right). \quad (8.89)$$

Coefficients a_{mn} and b_{mn} are found from the boundary value problem at $r = a$:

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [a_{mn} \cos n\varphi + b_{mn} \sin n\varphi] I_n(\sqrt{\lambda_m} r_0) \sin(\sqrt{\lambda_m} z) = u_0. \quad (8.90)$$

Expanding the right-hand side of the Equation (8.90) in a Fourier series in basis functions, we get

$$u_0 = \sum_{m=1}^{\infty} \left[\frac{1}{2} f_{1m0} + \sum_{n=1}^{\infty} f_{1mn} \cos n\varphi + \sum_{n=1}^{\infty} f_{2mn} \sin n\varphi \right] \sin \frac{\pi m}{l} z,$$

and we find the coefficients of expansion,

$$f_{1m0} = \frac{2u_0}{\pi l} \int_0^l \int_0^{2\pi} \sin \frac{\pi m}{l} z dz d\varphi = \frac{4[1 - (-1)^m] u_0}{\pi m},$$

$$f_{1mn} = \frac{2u_0}{\pi l} \int_0^l \int_0^{2\pi} \sin \frac{\pi m}{l} z \cdot \cos n\varphi dz d\varphi = 0, \quad n \neq 0,$$

$$f_{2mn} = \frac{2u_0}{\pi l} \int_0^l \int_0^{2\pi} \sin \frac{\pi m}{l} z \cdot \sin n\varphi dz d\varphi = 0.$$

Therefore, using these expressions for a_{mn} and b_{mn} , from the condition in Equation (8.90), we obtain

$$a_{m0} = \frac{2u_0 [1 - (-1)^m]}{\pi m I_0\left(\frac{\pi m}{l} r_0\right)},$$

$$a_{mn} = 0, \quad n \neq 0,$$

$$b_{mn} = 0.$$

Finally, we write the potential in Equation (8.89) as

$$u(r, \varphi, z) = \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \frac{[1 - (-1)^m] I_0\left(\frac{\pi m}{l} r\right)}{m I_0\left(\frac{\pi m}{l} r_0\right)} \sin \frac{\pi m}{l} z.$$

Let us simplify this expression by using a change of variables to get

$$u(r, \varphi, z) = \frac{2u_0}{\pi} \sum_{k=0}^{\infty} \frac{I_0 \left[\frac{\pi(2k+1)}{l} r \right]}{(2k+1) I_0 \left[\frac{\pi(2k+1)}{l} r_0 \right]} \sin \frac{\pi(2k+1)z}{l}. \quad (8.91)$$

The solution in Equation (8.91) does not depend on the azimuth angle φ . This is intuitively clear since the boundary conditions do not depend on φ and we are solving a problem with an axis of symmetry.

The field on the cylinder axis is thus

$$E_z(0, z) = - \left(\frac{\partial u}{\partial z} \right)_{r=0} = - \frac{4u_0}{l} \sum_{k=0}^{\infty} \frac{\cos \frac{(2k+1)\pi z}{l}}{I_0 \left[\frac{(2k+1)\pi}{l} r_0 \right]},$$

$$E_z(0, 0) = - \frac{4u_0}{l} \sum_{k=0}^{\infty} \frac{1}{I_0 \left[\frac{(2k+1)\pi}{l} r_0 \right]}.$$

Example 8.13. Find the oscillations of the free surface of a liquid in a circular vertical cylinder with a horizontal bottom for the case of radially symmetric initial conditions and constant pressure on the free surface.

Solution. The potential, u , of the horizontal velocities of fluid particles is the solution of the boundary value problem given by

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right), \quad 0 \leq r \leq a, \quad 0 < t < \infty, \quad (8.92)$$

$$u|_{t=0} = \varphi(r), \quad u_t|_{t=0} = \psi(r), \quad 0 \leq r \leq a, \quad (8.93)$$

$$u_r|_{r=a} = 0, \quad |u|_{r=0} < \infty, \quad 0 < t < \infty. \quad (8.94)$$

Here, c is the wave speed. To solve the problem, we use separation of variables. First, we represent the unknown function $u(r, t)$ in the form

$$u(r, t) = T(t)R(r)$$

and separate variables by substituting this expression into Equation (8.92) to yield

$$\frac{T''}{c^2 T} = \frac{R''}{R} + \frac{R'}{r R} = -\lambda,$$

where $\lambda > 0$ is the separation constant. Taking into account boundary conditions (8.94), we obtain, for the radial part, the eigenvalue problem

$$R'' + \frac{1}{r} R' + \lambda R = 0, \quad (8.95)$$

$$R(a) = 1, \quad |R(0)| < \infty. \quad (8.96)$$

Equation (8.95) is the Bessel equation. Using the boundary conditions (8.94), we can represent the solutions in the form

$$R(r) = J_0 \left(\sqrt{\lambda_n} r \right),$$

where $\lambda_n = (\mu_n^{(0)}/a)^2$ are eigenvalues of the problem in Equations (8.95) and (8.96), and $\mu_n^{(0)}$ are positive roots of the equation $J_0(\mu) = 0$. Taking into account the solution for the function $T(t)$, we can represent the function $u(r, t)$ in series form as

$$u(r, t) = \sum_{n=0}^{\infty} [A_n \cos \omega_n t + B_n \sin \omega_n t] J_0 \left(\sqrt{\lambda_n} r \right),$$

where $\omega_n = c \sqrt{\lambda_n}$. In order to find the unknown integration constants, A_n and B_n , we must substitute the obtained solution into the initial conditions in Equation (8.93):

$$\sum_{n=0}^{\infty} A_n J_0 \left(\sqrt{\lambda_n} r \right) = \varphi(r), \quad (8.97)$$

$$\sum_{n=0}^{\infty} \omega_n B_n J_0 \left(\sqrt{\lambda_n} r \right) = \psi(r). \quad (8.98)$$

We expand the right-hand side of Equations (8.97) and (8.98) in terms of eigenfunctions, that is, in a series of Bessel functions $J_0(\sqrt{\lambda_n} r)$. Equating

the coefficients of the corresponding linearly independent solutions, we obtain

$$A_n = \frac{2}{a^2 \left| J'_0(\mu_n^{(0)}) \right|^2} \int_0^a \varphi(r) J_0(\sqrt{\lambda_n} r) r dr,$$

$$B_n = \frac{2}{\omega_n a^2 \left| J'_0(\mu_n^{(0)}) \right|^2} \int_0^a \psi(r) J_0(\sqrt{\lambda_n} r) r dr.$$

8.11 Spherical Bessel Functions

In this section, we consider *spherical Bessel functions*, which are related to the solutions of certain boundary value problems in *spherical coordinates* (a definition of spherical coordinates can be found in Appendix D). In Chapter 9, we discuss several physical problems in which such functions are used. In all these problems, the unknown function $\psi(r, \theta, \varphi)$, which is the solution to a boundary value problem, depends on the spherical coordinates (r, θ, φ) , and in the case of spherical symmetry of a given problem, this function can be represented as $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$. Then the equations for the radial and angular parts, $R(r)$ and $Y(\theta, \varphi)$, respectively, can be separated, in which case the equation for R becomes

$$\frac{d^2 R(x)}{dx^2} + \frac{2}{x} \frac{dR(x)}{dx} + \left[1 - \frac{l(l+1)}{x^2} \right] R(x) = 0. \quad (8.99)$$

Here $x = kr$ is a dimensionless variable, where k is a number with dimension $1/r$ that depends on the physical nature of the problem. The value of the parameter l is the result of the solution of the boundary value problem for the function $Y(\theta, \varphi)$, and, as we shall see in Chapter 9, l takes discrete nonnegative integer values: $l = 0, 1, 2, \dots$. Equation (8.99) is called the *spherical Bessel equation*. It differs from the cylindrical Bessel equation, Equation (8.4), that we studied earlier when solving the boundary value problems in *cylindrical coordinates* by the coefficient 2 in the second term. Equation (8.99) can be transformed to a Bessel cylindrical equation by the substitution

$$R(x) = y(x)/\sqrt{x}. \quad (8.100)$$

Reading Exercise. Check that equation for $y(x)$ defined in Equation (8.100) is

$$\frac{d^2 y(x)}{dx^2} + \frac{1}{x} \frac{dy(x)}{dx} + \left[1 - \frac{(l + 1/2)^2}{x^2} \right] y(x) = 0. \quad (8.101)$$

If we introduce $s = l + 1/2$ in Equation (8.101), we recognize this equation as the Bessel equation that has the general solution

$$y(x) = c_1 J_s(x) + c_2 N_s(x), \quad (8.102)$$

where $J_s(x)$ and $N_s(x)$ are (cylindrical) Bessel and Neumann functions, respectively. Because $s = l + 1/2$, these functions are of half-integer order. Inverting the transformation, we have that the solution, $R(x)$, to Equation (8.99) is

$$R(x) = c_1 \frac{J_{l+1/2}(x)}{\sqrt{x}} + c_2 \frac{N_{l+1/2}(x)}{\sqrt{x}}. \quad (8.103)$$

The *spherical Bessel function* $j_l(x)$ is defined to be a solution finite at $x = 0$ thus it is a multiple of $J_{l+1/2}(x)/\sqrt{x}$. The coefficient of proportionality is usually chosen to be $\sqrt{\pi/2}$, so that

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x). \quad (8.104)$$

For $l = 0$, $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, and thus

$$j_0(x) = \frac{\sin x}{x}. \quad (8.105)$$

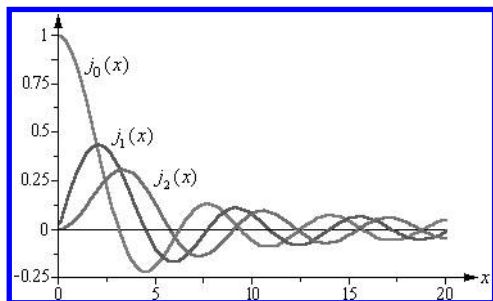
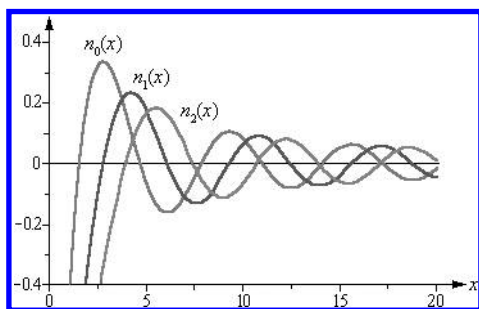
By using Equation (8.19), it is easy to obtain a similar formula for $j_l(x)$:

$$j_l(x) = x^l \left(-\frac{1}{x} \frac{d}{dx} \right)^l j_0(x), \quad l = 1, 2, 3, \dots \quad (8.106)$$

Analogously, we may define the *spherical Neumann functions* as

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+1/2}(x), \quad (8.107)$$

$$n_l(x) = x^l \left(-\frac{1}{x} \frac{d}{dx} \right)^l n_0(x), \quad l = 1, 2, 3, \dots, \quad (8.108)$$

Figure 8.20. Graphs of functions $j_0(x)$, $j_1(x)$, and $j_2(x)$.Figure 8.21. Graphs of functions $n_0(x)$, $n_1(x)$, and $n_2(x)$.

where

$$n_0(x) = \sqrt{\frac{\pi}{2x}} J_{-1/2}(x) = -\frac{\cos x}{x}. \quad (8.109)$$

Expressions for the first few terms of the functions $j_l(x)$ and $n_l(x)$ are

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x, \quad (8.110)$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x} \right) \cos x - \frac{3}{x^2} \sin x. \quad (8.111)$$

The spherical Bessel functions with $l = 0, 1, 2$ are graphed in Figures 8.20 and 8.21.

Below, we list a few more useful properties of spherical Bessel functions, which the reader may verify as reading exercises.

- *Recurrence relations* (here the symbol f is written for j or n):

$$f_{l-1}(x) + f_{l+1}(x) = (2l+1)x^{-1}f_l(x), \quad (8.112)$$

$$lf_{l-1}(x) - (l+1)f_{l+1}(x) = (2l+1)\frac{d}{dx}f_l(x), \quad (8.113)$$

$$\frac{d}{dx} [x^{l+1}j_l(x)] = x^{l+1}j_{l-1}(x), \quad (8.114)$$

$$\frac{d}{dx} [x^{-l}j_l(x)] = -x^{-l}j_{l+1}(x). \quad (8.115)$$

- *Orthogonality and normalization:*

$$\int_0^\infty j_l(kr)j_l(k'r)r^2dr = \frac{\pi}{2k^2}\delta(k-k'), \quad (8.116)$$

$$\int_0^\infty f_l^2(x)x^2dx = \frac{x^3}{2} [f_l^2(x) - f_{l-1}(x)f_{l+1}(x)]. \quad (8.117)$$

- *Asymptotic values:*

$$j_l(x) \sim \frac{x^l}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2l+1)}, \quad (8.118)$$

$$n_l(x) \sim \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2l-1)}{x^{l+1}} \text{ as } x \rightarrow 0,$$

$$j_l(x) \sim \frac{1}{x} \cos \left[x - \frac{\pi}{2}(l+1) \right], \quad (8.119)$$

$$n_l(x) \sim \frac{1}{x} \sin \left[x - \frac{\pi}{2}(l+1) \right] \text{ as } x \rightarrow \infty.$$

(The last expression has good precision for $x \gg l(l+1)$).

In some applications (for example in scattering theory), spherical Hankel functions of the first and second kind, defined as

$$h_l^{(\pm)}(x) = n_l(x) \pm i j_l(x), \quad (8.120)$$

are useful. The asymptotic value of $h_l^{(\pm)}(x)$ as $x \rightarrow \infty$ clearly is

$$h_l^{(\pm)}(x) \sim \frac{i}{x} \exp \left[\pm i \left(x - \frac{\pi}{2}(l+1) \right) \right]. \quad (8.121)$$

8.12 The Gamma Function

In this section, we develop the essential properties of the Gamma function. One of the most important applications of the Gamma function is that it allows us to find factorials of positive numbers that are not integers. The Gamma function is defined by the integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0. \quad (8.122)$$

Here, x is an arbitrary, real, nonnegative number. In the case that x is an integer and $x \geq 2$, the integral in Equation (8.122) can be evaluated by parts and we have, after the substitution $t^{x-1} = u$, $e^{-t} dt = dv$:

$$\Gamma(x) = t^{x-1} e^{-t} \Big|_0^{\infty} + (x-1) \int_0^{\infty} t^{x-2} e^{-t} dt = (x-1) \int_0^{\infty} t^{x-2} e^{-t} dt. \quad (8.123)$$

The obtained integral is equal to $\Gamma(x-1)$; that is, we may write

$$\Gamma(x) = (x-1)\Gamma(x-1). \quad (8.124)$$

By substituting Equation (8.124) repeatedly into Equation (8.124), we have

$$\Gamma(x-1) = (x-2)\Gamma(x-2),$$

$$\Gamma(x-2) = (x-3)\Gamma(x-3),$$

and so forth, and thus we obtain the general expression

$$\Gamma(x) = (x-1)(x-2) \dots \Gamma(1), \quad (8.125)$$

where

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1. \quad (8.126)$$

By substituting Equation (8.126) into Equation (8.125) we have

$$\Gamma(x) = (x-1)! \quad \text{for } x = 2, 3, \dots \quad (8.127)$$

We derived Equation (8.127) for integer values $x \geq 2$, but it is possible to generalize it to define *factorials of any numbers*. First, we verify that

Equation (8.127) is valid for $x = 1$. Let $x = 1$ in Equation (8.127), in which case we have $\Gamma(1) = 0! = 1$, which does agree with Equation (8.126). Thus, for integer values of the argument, $n = 1, 2, 3, \dots$,

$$\Gamma(1) = 1, \quad \Gamma(2) = 1, \quad \Gamma(n) = (n-1)!, \quad (8.128)$$

Now we consider noninteger values of x . For $x = 1/2$, taking into account definition (8.122) and with the substitution $t = z^2$, we obtain

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = 2 \int_0^\infty e^{-z^2} dz = \sqrt{\pi}. \quad (8.129)$$

Now, using Equation (8.124), we find

$$\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2. \quad (8.130)$$

Reading Exercise. Show that for any integer $n \geq 1$,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \Gamma\left(\frac{1}{2}\right). \quad (8.131)$$

We can also generalize definition (8.122) to negative values of x by using Equation (8.123). First, we replace x by $x+1$, which gives

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}. \quad (8.132)$$

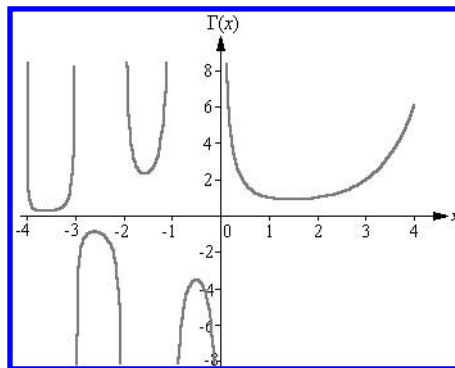


Figure 8.22. Graph of the Gamma function, $\Gamma(x)$.

We may use Equation (8.132) to find, for example, $\Gamma(-1/2)$ in the following way:

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}. \quad (8.133)$$

It is clear then, that by using Equations (8.122) and (8.133), we can find a value of $\Gamma(x)$ for all values of x except 0 and negative integers.

The function $\Gamma(x)$ diverges at $x = 0$, as is seen from Equation (8.122). Then, from Equation (8.132), we see that $\Gamma(-1)$ is not defined because it involves $\Gamma(0)$. Thus, $\Gamma(x)$ does not exist for negative integer values of x . From Equation (8.132), it is obvious (taking into account that $\Gamma(1) = 1$) that at all of these values of x , the function $\Gamma(x)$ has simple poles. A graph of $\Gamma(x)$ is plotted in Figure 8.22.

Equation (8.132) allows us to find the value of $\Gamma(x)$ for any nonnegative integer x by using the value of $\Gamma(x)$ on the interval $1 \leq x \leq 2$. For example, $\Gamma(3.4) = 3.4 \cdot \Gamma(2.4) = 3.4 \cdot 2.4 \cdot \Gamma(1.4)$. Based on this fact, a table of values, such as Table 8.4, for $\Gamma(x)$ need include only the interval $[1, 2]$ as values of x . The minimum value of $\Gamma(x)$ is reached at $x = 1.46116321 \dots$

Equation (8.128), which is now valid for all nonnegative integer values of x , can be written as

$$\Gamma(x + 1) = x! \quad (8.134)$$

In contrast, from Equation (8.122), we have

$$\Gamma(x + 1) = \int_0^{\infty} t^x e^{-t} dt. \quad (8.135)$$

x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$
1	1	1.35	0.8911514420	1.7	0.9086387329
1.05	0.9735042656	1.4	0.8872638175	1.75	0.9190625268
1.1	0.9513507699	1.45	0.8856613803	1.8	0.9313837710
1.15	0.9330409311	1.5	0.8862269255	1.85	0.9456111764
1.2	0.9181687424	1.55	0.8888683478	1.9	0.9617658319
1.25	0.9064024771	1.6	0.8935153493	1.95	0.9798806513
1.3	0.8974706963	1.65	0.9001168163	2	1

Table 8.4. Values of $\Gamma(x)$ for $x \in [1, 2]$.

The integrand $t^x e^{-t}$ has a sharp peak at $t = x$ that allows us to obtain a famous approximation formula for $x!$, known as *Stirling's approximation*, which works very well for large x :

$$x! \sim (2\pi x)^{1/2} x^x e^{-x}. \quad (8.136)$$

This formula agrees very well with the precise value of $x!$ even for values of x that are not very large. For instance, for $x = 10$, the relative error of Equation (8.136) is less than 0.8%. Most of the applications of Equation (8.136) belong to statistical physics, where it is often necessary to evaluate factorials of very large numbers.

We present without proof two other useful formulas for the Gamma function:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad (8.137)$$

$$\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x)\Gamma(x+1/2). \quad (8.138)$$

Problems

8.1. Find general solutions in the form of Bessel functions of the following ODEs:

1. $y'' + 4x^2 y = 0$;
2. $x^2 y'' + 5xy' + x^2 y = 0$;
3. $y'' + (1/x)y' - (1/16x^2)y = 0$;
4. $y'' - (1/x)y' + (1 - 3/x^2)y = 0$;
5. $y'' - (3/x)y' + (1 + 4/x^2)y = 0$;
6. $x^2 y'' + xy' + 1/4(x^2 - 1)y = 0$;
7. $4x^2 y'' + 4xy' + (x - 9)y = 0$, by using the transformation $u = x^{1/2}$;
8. $x^2 y'' + 4xy' + (9x^3 - 32)y = 0$, by using the transformation $u = x^{3/2}$;
9. $4x^2 y'' + 8xy' + (4x^2 - 1)y = 0$, by using the transformation $u = yx^{1/2}$;
10. $x^2 y'' + xy' + 4(x^4 - p^2)y = 0$, by using the transformation $u = x^2$;
11. $x^2 y'' + xy' + 9(x^6 - p^2)y = 0$, by using the transformation $u = x^3$;
12. $y'' + (e^{2x} - 1/9)y = 0$, by using the transformation $u = e^x$.

8.2. Find eigenvalues and eigenfunctions of the following Sturm-Liouville problems:

1. $x^2 y'' + xy' + (\lambda x^2 - 1)y = 0, \quad y(0) = 0, \quad y(1) = 0;$
2. $x^2 y'' + xy' + (\lambda x^2 - 4)y = 0, \quad y(0) = 0, \quad y(1) = 0;$
3. $x^2 y'' + xy' + (\lambda x^2 - 1)y = 0, \quad y'(0) = 0, \quad y(1) = 0;$
4. $x^2 y'' + xy' + \lambda x^2 y = 0, \quad y(0) = 0, \quad y'(1) = 0;$
5. $x^2 y'' + xy' + (\lambda x^2 - 9)y = 0, \quad y'(0) = 0, \quad y'(1) = 0;$
6. $x^2 y'' + xy' + (\lambda x^2 - 1)y = 0, \quad y(1) = 0, \quad y(2) = 0;$
7. $x^2 y'' + xy' - (\lambda x^2 + 1)y = 0, \quad y(0) = 0, \quad y(1) = 0;$
8. $x^2 y'' + xy' - \lambda x^2 y = 0, \quad y(0) = 0, \quad y(1) = 0.$

8.3. Prove the following integral formulas:

$$\begin{aligned} \int x^p J_{p-1}(x) dx &= x^p J_p(x) + c; \\ \int x^{-p} J_{p+1}(x) dx &= -x^{-p} J_p(x) + c; \\ \int J_{p+1}(x) dx &= \int J_{p-1}(x) dx - 2J_p(x). \end{aligned}$$

8.4. Show that $\int J_0(x) J_1(x) dx = -(1/2) J_0^2(x), \int_0^1 x J_0(ax) dx = (1/a) J_1(a).$

8.5. Show that $\int_0^1 x J_0^2(ax) dx = [J_0^2(a) + J_1^2(a)] / 2.$

8.6. Show that $J_{p-1}(x) = J_{p+1}(x)$ at every value of the extremes of the function $J_p(x)$, and $J_{p-1}(x) = -J_{p+1}(x) = J'_p(x)$ for every positive root of the equation $J_p(x) = 0$. Using the computer program FourierSeries, plot graphs of functions $J_2(x)$, $J_3(x)$, and $J_4(x)$ on the same screen and verify that the above equalities at the extremes are correct.

8.7. Show that

$$\int_0^\infty J_1(x) dx = \int_0^\infty J_3(x) dx = \dots = \int_0^\infty J_{2n+1}(x) dx$$

and

$$\int_0^\infty J_0(x) dx = \int_0^\infty J_2(x) dx = \dots = \int_0^\infty J_{2n}(x) dx.$$

8.8. Show that $N'_0(x) = -N_1(x)$.

8.9. Show that $\int x^{n+1} J_n(x) dx = x^{n+1} J_{n+1}(x) + C$ for integer $n \geq 1$.

8.10. Show that $I_0(x) = (1/2\pi) \int_0^{2\pi} e^{\pm x \sin \theta} d\theta$.

8.11. Using a series expansion of the function $J_0(x)$, show that

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} \cdots$$

8.12. Prove that $\int x I_0(ax) dx = (x/a) I'_0(ax) + C$.

8.13. Prove that $\int j_l(x) dx = -j_0(x)$, $\int j_0(x) x^2 dx = x^2 j_l(x)$.

8.14. Establish the power series

$$j_l(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{l+2k}}{k! \cdot 2^k \cdot (2l+2k+1)!!},$$

where

$$(2l+2k+1)!! = (2l+2k+1) \cdot (2l+2k-1) \cdots 1.$$

8.15. Expand the function $f(x)$, given on the interval $[0, 1]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_0(\mu_k x)$, where μ_k are positive roots of the equation $J_0(\mu) = 0$, if:

1. $f(x) = 1 - \frac{J_0(x)}{J_0(1)};$

2. $f(x) = 1 - \frac{J_0(2x)}{J_0(2)};$

3. $f(x) = \sin \pi x;$

4. $f(x) = x^2;$

5. $f(x) = 1 - \frac{J_0(3x)}{J_0(3)};$

6. $f(x) = \frac{J_0(2x)}{J_0(2)};$

7. $f(x) = \sin^2(\pi x);$

8. $f(x) = 1 - x^2;$

9. $f(x) = \cos \frac{\pi x}{2}$

8.16. Expand the function $f(x)$, given on the interval $[0, 1]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_1(\mu_k x)$, where μ_k are positive roots of the equation $J_1(\mu) = 0$, if:

1. $f(x) = 1 - \frac{J_1(x)}{J_1(1)}$;
2. $f(x) = 1 - \frac{J_1(2x)}{J_1(2)}$;
3. $f(x) = x$;
4. $f(x) = \sin \pi x$;
5. $f(x) = \sin^2(\pi x)$;
6. $f(x) = x(1 - x)$;
7. $f(x) = x(1 - x^2)$.

8.17. Expand the function $f(x)$, given on the interval $[0, 1]$, in a Fourier series in Bessel functions of the first kind, $X_k(x) = J_0(\mu_k x)$, where μ_k are positive roots of the equation $J'_0(\mu) = 0$, if:

1. $f(x) = x(1 - x)$;
2. $f(x) = x(1 - x^3)$;
3. $f(x) = x(1 - x^2)$;
4. $f(x) = x^3$.

8.18. Expand the function

$$f(x) = A \left(1 - \frac{x^2}{l^2} \right), \quad A = \text{const},$$

given on the interval $[0, l]$, in a Fourier series in Bessel functions of the first kind,

$$X_k(x) = J_0 \left(\frac{\mu_k^{(0)} x}{l} \right),$$

where $\mu_k^{(0)}$ are positive roots of the equation

$$J_0 \left(\mu_k^{(0)} \right) = 0.$$

8.19. Expand the function

$$f(x) = Ax, \quad A = \text{const},$$

given on the interval $[0, l]$, in a Fourier series in Bessel functions of the first kind using the functions

$$X_k(x) = J_1\left(\frac{\mu_k^{(1)}x}{l}\right),$$

where $\mu_k^{(1)}$ are positive roots of the equation

$$J_1'(\mu_k^{(1)}) = 0.$$

For the theory associated with the following physical applications, see Chapter 7.

8.20. Find the distribution of temperature inside a rigid finite cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) if the surface $z = 0$ is at constant temperature u_0 , the side surface, $r = a$, is at temperature $f(z) = u_0(1 - z/l)$, and the surface $z = l$ is at zero temperature. The initial temperature of the cylinder is u_0 .

8.21. Solve the problem of the cooling of an infinite cylindrical pipe with radius r ($r_1 \leq r \leq r_2$) filled with a liquid, if the temperature of the liquid always equals the temperature of the inner surface of the pipe, and the outer surface is thermally insulated. The initial temperature of the pipe is $u|_{t=0} = f(r)$, $r_1 < r < r_2$.

8.22. Solve Problem 8.21 assuming convective heat exchange with the external medium at zero temperature.

8.23. Find the temperature of a finite circular cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) whose surface undergoes convective heat exchange with the surrounding medium, which is held at zero temperature. The initial temperature of the cylinder is $u|_{t=0} = f(r, \varphi, z)$, $0 \leq r < a$, $0 \leq \varphi < 2\pi$, $0 < z < l$.

8.24. Find an expression for the potential of the electrostatic field inside an infinite circular cylinder with radius r ($0 \leq r \leq 1$), if its outside surface has constant potential φ_0 .

8.25. A long cylinder is cut into quarter cylinders that are insulated from each other, and alternate quarter cylinders held at potentials V_0 and $-V_0$. Find the electrostatic potential inside the cylinder.

8.26. Two conducting hemicylindrical shells of radius a are connected at their boundaries by a thin insulating material to form a cylinder. One shell is kept at the potential $+V$, the other at the potential $-V$. Find the potential inside the cylinder.

Hint. The boundary condition is

$$u(r = a, z, \varphi) = \begin{cases} +V, & 0 \leq \varphi < \pi/2, \\ -V, & \pi/2 < \varphi \leq \pi. \end{cases}$$

Answer.

$$u(r, z) = \frac{2V}{\pi} \int_0^{\infty} \frac{I_0(kr) \sin kz}{I_0(ka) k} dk.$$

8.27. Find an expression for the potential of the electrostatic field inside a cylindrical box with circular cross section $r \leq 1$, $0 \leq z \leq 1$, both ends of which are grounded, and the outside surface is charged to a potential of $z(1 - z)$. Find the strength of the field on the axis.

8.28. Find an expression for the potential of the electrostatic field inside a cylindrical box with circular cross section $r \leq a$, $0 \leq z \leq l$, the upper end and outside surfaces of which are grounded, and the lower end is held at constant potential φ_0 . Find the strength of the field on the axis.

8.29. Find an expression for the potential of the electrostatic field inside a rigid cylindrical box ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) if the lower end at $z = 0$ is grounded, the outside surface at $r = a$ has potential $f(z) = u_0 z/l$, and the upper end at $z = l$ has potential u_0 .

8.30. Find the equilibrium distribution of temperature inside a finite rigid cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) if thermal flux q is transmitted to the lower end at $z = 0$, the outside surface at $r = a$ participates in heat exchange with the outer medium (which has zero temperature), and the upper end at $z = l$ is at zero temperature.

8.31. Solve Problem 8.30 if the upper end at $z = l$ is thermally insulated.

8.32. Find the equilibrium distribution of temperature inside a rigid finite cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) if thermal flux q is transmitted to the lower end at $z = 0$, and the outside surface at $r = a$, and upper end at $z = l$ participate in heat exchange with an external medium held at a constant temperature of zero.

8.33. Find the equilibrium distribution of temperature inside a rigid finite cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) if the lower end at $z = 0$ is at constant temperature u_0 , the outside surface at $r = a$ is at temperature $f(z) = u_0(1 - z/l)$, and the upper end at $z = l$ is at zero temperature.

8.34. Solve the boundary value problem for the Laplace equation inside a finite-length cylinder ($0 \leq r \leq a$, $0 \leq \varphi < 2\pi$, $0 \leq z \leq l$) for the case $u|_{r=a} = 0$, $u|_{z=0} = f_0(r, \varphi)$, $u|_{z=l} = f_1(r, \varphi)$, where $f_0(r, \varphi)$ and $f_1(r, \varphi)$ are known functions.

8.35. Solve Problem 8.34 for the case $u|_{z=0} = f_0(r)$, $u|_{z=l} = f_1(r)$.

8.36. Calculate $\Gamma(-5/2)$, $\Gamma(-7/2)$, $\Gamma(9/4)$, $\Gamma(11/4)$, and $\Gamma(11/2)$.

8.37. Prove that $\Gamma(x+n) = x(x+1)\dots(x+n-1)\Gamma(x)$, $n = 1, 2, \dots$.

8.38. Prove that

$$\Gamma(x-n) = \frac{\Gamma(x)}{(x-1)(x-2)\dots(x-n)}, \quad n = 1, 2, \dots$$

8.39. Evaluate $\int_0^\infty t^5 e^{-t^4} dt$ in terms of the Gamma function.

8.40. Evaluate $\int_0^\infty t^m e^{-t^n} dt$ in terms of the Gamma function, where m and n are positive integers.

8.41. Show that

$$\int_0^\infty t^{2n} e^{-at^2} dt = \frac{\Gamma(n+1/2)}{2a^{n+1/2}}.$$

8.42. Show that $\Gamma(n+1/2) \cdot \Gamma(1/2-n) = \pi/(\cos n\pi)$.

Legendre Functions

The famous Legendre functions are widely used in the sciences; in particular, they serve as a set of basis functions for the Fourier expansion of solutions of many physical problems in spherical coordinates. As we see in this chapter, problems with spherical symmetry, such as vibrating spheres, spherical charge distributions, and the quantum probability distribution of an electron trapped in the spherical potential of an atom, in general lead to Legendre functions. We discuss the origin of the Legendre functions, their properties, and several applications.

9.1 Boundary Value Problems Leading to Legendre Polynomials

In applications, one often encounters an eigenvalue problem containing a second-order linear homogeneous differential equation with the generic form

$$(1 - x^2) y'' - 2xy' + \lambda y = 0, \quad -1 \leq x \leq 1, \quad (9.1)$$

where λ is a real parameter. Equation (9.1) can be rewritten in *self-adjoint Sturm-Liouville form* given by

$$\frac{d}{dx} \left[(1 - x^2) \frac{dy}{dx} \right] + \lambda y = 0, \quad -1 \leq x \leq 1 \quad (9.2)$$

and is called the *Legendre equation*. Such an equation frequently arises after a separation-of-variables procedure in *spherical coordinates* in many

problems of mathematical physics. Prominent examples include heat conduction in a spherical domain, vibrations of spherical solids and shells, as well as boundary value problems for the electric potential in spherical coordinates. As in previous examples with differential equations for a physical application we need to consider boundary conditions to arrive at useful solutions to Equation (9.2). As we see in this chapter, Equation (9.2) has nontrivial solutions that correspond to given boundary conditions only for certain values of the parameter λ , called the *eigenvalues*. The goal is to find these eigenvalues for a given set of boundary conditions and the corresponding solutions, $y(x)$, known as the *eigenfunctions*. Because this problem is a particular example of a Sturm-Liouville problem, we can expect that the eigenvalues take *nonnegative real discrete* values λ_n , and the eigenfunctions corresponding to different eigenvalues are *orthogonal* on the interval $[-1, 1]$ with weight $r(x) = 1$:

$$\int_{-1}^1 y_n(x)y_m(x)dx = 0 \quad \text{if } m \neq n. \quad (9.3)$$

In spherical coordinates for problems leading to Equation (9.2), the variable x is $x = \cos \theta$, where θ is a meridian angle (a definition of the angles used in spherical coordinate systems is given in Appendix D). If this angle belongs to the full interval, $\theta \in [0, \pi]$, we have an entire sphere (assuming that for the azimuth angle φ , $0 \leq \varphi \leq 2\pi$). In this case, the function $y(x)$ is defined on the closed interval $[-1, 1]$, and the only reasonable boundary condition to impose on function $y(x)$ is that it is bounded at the points $x = -1$ and $x = 1$. We will see that this allows us to determine the eigenvalues, λ , and they form a discrete spectrum. For a hemisphere where $\theta \in [0, \pi/2]$, or $x \in [0, 1]$, different types of boundary conditions are possible; the most common physical problems in this case are of the Dirichlet and Neumann type.

Let us solve Equation (9.2) on the interval $x \in [-1, 1]$ assuming the following boundary condition: *the function $y(x)$ is finite at the points $x = -1$ and $x = 1$* . Let us search for a solution in the form of a power series in x :

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \quad (9.4)$$

Substituting Equation (9.4) into Equation (9.2) results in the following equality:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 2na_n x^n + \lambda \sum_{n=2}^{\infty} a_n x^n = 0.$$

Changing the index of summations in the first term from n to $n+2$ yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

and allows us to group all the terms with $n \geq 2$, leaving the terms with $n = 0$ and $n = 1$, which we write separately. We thus have

$$(6a_3 - 2a_1 + \lambda a_1)x + 2a_2 + \lambda a_0 + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - (n^2 + n - \lambda)a_n]x^n = 0.$$

By setting coefficients of each power of x to zero, we obtain an infinite system of equations for the coefficients a_n :

$$n = 0 \quad 2a_2 + \lambda a_0 = 0 \quad (9.5)$$

$$n = 1 \quad 6a_3 - 2a_1 + \lambda a_1 = 0 \quad (9.6)$$

$$n \geq 2 \quad (n+2)(n+1)a_{n+2} - (n^2 + n - \lambda)a_n = 0. \quad (9.7)$$

Equation (9.7) is the *recurrence formula* for coefficients. From Equation (9.5), we have

$$a_2 = -\frac{\lambda}{2}a_0. \quad (9.8)$$

By using Equations (9.8) and (9.7), we obtain

$$a_4 = \frac{6-\lambda}{3 \cdot 4}a_2 = \frac{-\lambda(6-\lambda)}{4!}a_0,$$

$$a_6 = \frac{20-\lambda}{5 \cdot 6}a_4 = \frac{-\lambda(6-\lambda)(20-\lambda)}{6!}a_0,$$

and so on. Each coefficient, a_{2n} , with an *even index* is multiplied by a_0 and depends on n and the parameter λ .

Similarly, let us proceed with the odd terms, a_{2k+1} . From Equation (9.6), we have

$$a_3 = \frac{2-\lambda}{6}a_1 = \frac{2-\lambda}{3!}a_1.$$

Then, from the recurrence formula (9.7), we obtain

$$a_5 = \frac{12 - \lambda}{4 \cdot 5} a_3 = \frac{(2 - \lambda)(12 - \lambda)}{5!} a_1,$$

$$a_7 = \frac{30 - \lambda}{6 \cdot 7} a_5 = \frac{(2 - \lambda)(12 - \lambda)(30 - \lambda)}{7!} a_1,$$

and so on. Each coefficient, a_{2k+1} , with an *odd index* is multiplied by a_1 and depends on n and λ .

Substituting the obtained coefficients in Equation (9.4), we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 \left[1 - \frac{\lambda}{2} x^2 - \frac{\lambda(6 - \lambda)}{4!} x^4 - \dots \right] \\ &\quad + a_1 \left[x + \frac{2 - \lambda}{3!} x^3 + \frac{(2 - \lambda)(12 - \lambda)}{5!} x^5 + \dots \right]. \end{aligned} \quad (9.9)$$

Equation (9.9) has two sums, one of which contains coefficients with even indexes and the other with odd indexes. As a result, we obtain two *linearly independent* solutions of Equation (9.2); one contains even powers of x , and the other contains odd powers of x :

$$y^{(1)}(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 \left[1 - \frac{\lambda}{2} x^2 - \frac{\lambda(6 - \lambda)}{4!} x^4 - \dots \right], \quad (9.10)$$

$$y^{(2)}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = a_1 \left[x + \frac{2 - \lambda}{3!} x^3 + \frac{(2 - \lambda)(12 - \lambda)}{5!} x^5 + \dots \right]. \quad (9.11)$$

Now, from the recurrence relation (9.7) for the coefficients,

$$a_{n+2} = \frac{(n^2 + n - \lambda)}{(n + 2)(n + 1)} a_n, \quad (9.12)$$

we can state an important fact. The series in Equation (9.4) converges on an open interval, $-1 < x < 1$, as can be seen from a ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = x^2,$$

but diverges at the points $x = \pm 1$. Therefore this series cannot be used as an acceptable solution of the differential equation on the entire interval $-1 \leq x \leq 1$ unless it terminates as a polynomial with a finite number of terms. This can occur if the numerator in Equation (9.12) is zero for some index value, $n_{\max}$, such that

$$\lambda = n_{\max}(n_{\max} + 1).$$

This gives $a_{n_{\max}+2} = 0$, and consequently $a_{n_{\max}+4} = 0$, $a_{n_{\max}+6} = 0$, $\dots$. Thus, $y(x)$ will contain a finite number of terms and thus turn out to be a polynomial of degree $n_{\max}$. In order not to overcomplicate the notation, from here on we denote $n_{\max}$ as n . We may conclude from this discussion that λ can take only nonnegative integer values:

$$\lambda = n(n + 1). \quad (9.13)$$

Let us consider several particular cases.

- If $n = 0$ (which means that the highest degree of the polynomial is 0), $a_0 \neq 0$ and $a_2 = 0$, $a_4 = 0$, and so forth, then the value of λ is $\lambda = 0$, and we have $y^{(1)}(x) = a_0$.
- If $n = 1$, $a_1 \neq 0$ and $a_3 = a_5 = \dots = 0$, then $\lambda = 2$ and $y^{(2)}(x) = a_1 x$.
- If $n = 2$, the highest degree of the polynomial is 2, and we have $a_2 \neq 0$, $a_4 = a_6 = \dots = 0$, $\lambda = 6$, and from the recurrence relation we obtain $a_2 = -3a_0$. This results in $y^{(1)}(x) = a_0(1 - 3x^2)$.
- If $n = 3$, $a_3 \neq 0$ and $a_5 = a_7 = \dots = 0$, $\lambda = 12$, and from the recurrence relation we obtain $a_3 = -5/3 a_1$. This results in $y^{(2)}(x) = a_1(1 - 5x^3/3)$.

Constants a_0 and a_1 remain arbitrary unless we impose some additional requirement. A convenient requirement is that the solutions (the polynomials) obtained in this way should have the value 1 when $x = 1$.

The polynomials obtained above are denoted as $P_n(x)$ and called the *Legendre polynomials*. The first few, which we derived above, are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x). \quad (9.14)$$

Let us list two more (which the reader may derive as a reading exercise by using the above relationships):

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x). \quad (9.15)$$

As we see, $P_n(x)$ are even functions for even values of n , and odd functions for odd n . Notice that in many applications the letter l instead of n is usually used.

The functions $y^{(1)}(x)$ and $y^{(2)}(x)$, bounded on the closed interval $-1 \leq x \leq 1$, are

$$y^{(1)}(x) = P_{2n}(x), \quad y^{(2)}(x) = P_{2n+1}(x). \quad (9.16)$$

Reading Exercise. Obtain $P_n(x)$ for $n = 6, 7$.

Reading Exercise. Show by direct substitution that $P_2(x)$ and $P_3(x)$ satisfy Equation (9.2).

An explicit formula for the Legendre polynomials, $P_n(x)$, is

$$P_n = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}, \quad (9.17)$$

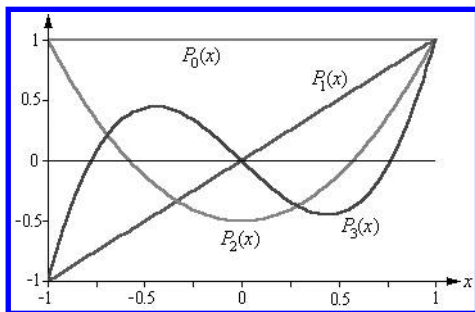
where $[n/2]$ is the greatest integer less than or equal to $n/2$. In other words, $[n/2] = n/2$ if n is even (i.e., $k \leq n/2$) and $[n/2] = (n-1)/2$ if n is odd (i.e., $k \leq (n-1)/2$). Thus, the denominator contains the factorials of nonnegative numbers only.

An alternative formula for calculating the Legendre polynomials is Rodrigues's formula, which is a useful tool to study the properties of Legendre polynomials. Rodrigues's formula is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (9.18)$$

(here, a zero-order derivative means the function itself). This formula can be proved by expanding $(x^2 - 1)^n$ by using the binomial theorem:

$$(x^2 - 1)^n = \sum_{k=0}^n \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}.$$

Figure 9.1. The first four Legendre polynomials, $P_n(x)$.

Differentiation n times gives

$$\frac{d^n}{dx^n} (x^2 - 1)^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

which, after multiplication by $1/(2^n n!)$, gives Equation (9.18) for $P_n(x)$.

Reading Exercise. Use Rodrigues's formula to show that

$$P_n(-x) = (-1)^n P_n(x), \text{ and} \quad (9.19)$$

$$P_n(-1) = (-1)^n. \quad (9.20)$$

Let us state a few more useful properties of Legendre polynomials:

$$P_n(1) = 1, \quad P_{2n+1}(0) = 0, \quad P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n}. \quad (9.21)$$

The Legendre polynomial $P_n(x)$ has n *real* and *simple* (i.e., not repeated) roots, all lying in the interval $-1 < x < 1$. Zeros of the polynomials $P_n(x)$ and $P_{n+1}(x)$ alternate as x increases. Figure 9.1 shows graphs of the first four polynomials, $P_n(x)$, and their properties, as listed in Equations (9.19) through (9.21).

To summarize, the solution of the Sturm-Liouville problem for Equation (9.2) with boundary conditions stating that the solution is bounded on the closed interval $-1 \leq x \leq 1$ is a set of Legendre polynomials, $P_n(x)$, which are the eigenfunctions of the Sturm-Liouville operator. The

eigenvalues are $\lambda = n(n + 1)$, $n = 0, 1, 2, \dots$. As a solution of the Sturm-Liouville problem, the Legendre polynomials, $P_n(x)$, form a *complete orthogonal* set of functions on the closed interval $[-1, 1]$ —a property we will find very useful in the applications considered in this chapter.

If the points $x = \pm 1$ are excluded from a domain, the solution in the form of an infinite series is also acceptable. As previously mentioned, when $\lambda = 0$ ($n = 0$), from Equation (9.10), we have $y^{(1)}(x) = a_0$ or $P_0(x) = 1$ with the choice of a_0 used above. However, for $n = 0$ there is also the solution $y^{(2)}(x)$. From Equation (9.11), we have

$$y^{(2)}(x) = a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right), \quad (9.22)$$

which clearly converges for $-1 < x < 1$ (but diverges at the points $x = \pm 1$).

Reading Exercise. Apply the ratio test to investigate the convergence of the series in Equation (9.22).

The series (9.22) is essentially the Maclaurin expansion of the function $Q_0(x) = (1/2) \ln [(1+x)/(1-x)]$:

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \quad (9.23)$$

Thus, $y^{(2)}(x) = a_1 Q_0(x)$. The function $Q_0(x)$ logarithmically diverges at $x = \pm 1$ but is well behaved on the interval $-1 < x < 1$. Now we can present a general solution of Equation (9.2) for $\lambda = 0$ ($n = 0$) as a linear combination of the functions $P_0(x)$ and $Q_0(x)$:

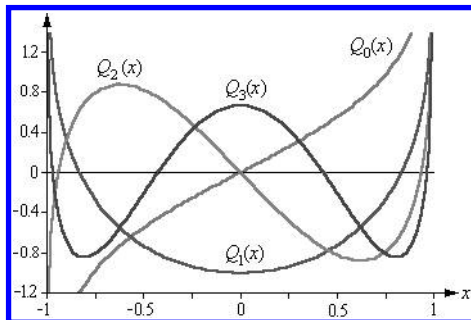
$$y(x) = c_1 P_0(x) + c_2 Q_0(x). \quad (9.24)$$

When $\lambda = 2$ ($n = 1$), a general solution of Equation (9.2) can be written as

$$y(x) = c_1 P_1(x) + c_2 Q_1(x), \quad (9.25)$$

where $Q_1(x) = xQ_0(x) - 1$. In this case,

$$y^{(1)}(x) = a_0 \left(1 + x^2 + \frac{x^4}{3} + \frac{x^6}{5} + \frac{x^8}{7} + \dots \right) \sim Q_1(x) \text{ and } y^{(2)}(x) \sim P_1(x).$$

Figure 9.2. First four functions, $Q_n(x)$.

Thus, at the boundaries $x = \pm 1$, solution $y^{(1)}(x)$ diverges. Generally, for even values of n at the points $x = \pm 1$, solution $y^{(2)}(x)$ diverges; for odd n , solution $y^{(1)}(x)$ diverges.

Reading Exercise. Check that for $\lambda = 6$ ($n = 2$), a general solution of Equation (9.2) is

$$y(x) = c_1 P_2(x) + c_2 Q_2(x), \quad (9.26)$$

where $Q_2(x) = P_2(x)Q_0(x) - D_1(x)$, and $D_1(x)$ is a polynomial of first degree (to be determined by the reader).

For arbitrary values of n , the recurrence formula for $Q_n(x)$ can be presented as

$$Q_n(x) = P_n(x)Q_0(x) - D_{n-1}(x), \quad (9.27)$$

where $D_{n-1}(x)$ is some polynomial of degree $n - 1$. Figure 9.2 presents a plot of the first four functions, $Q_n(x)$.

Therefore, a general solution, Equation (9.9), of the Legendre Equation (9.2) for *nonnegative integer* λ is

$$y(x) = c_1 P_n(x) + c_2 Q_n(x), \quad n = 0, 1, 2, \dots \quad (9.28)$$

The logarithmic functions $Q_n(x)$ given by Equation (9.27) diverge at $x = \pm 1$. From a physical point of view, if we need a finite solution, $y(x)$, for all $x \in [-1, 1]$ in a problem, we must set $c_2 = 0$. But when the ends at $x = \pm 1$ of the interval are excluded (for example, for a problem involving a sphere with the angles $\theta = 0$ and $\theta = \pi$ excluded), the function $Q_n(x)$ should be considered as part of a general solution.

9.2 Generating Function for Legendre Polynomials

Consider a spherical coordinate system in which $\vec{l}$ and $\vec{r}$ are the position vectors of points L and M , respectively, and θ is the angle between these vectors (see Figure 9.3). The distance, R , between these two points can be obtained with the theorem of cosines, which yields

$$\frac{1}{R} = \frac{1}{\sqrt{l^2 + r^2 - 2lr \cos \theta}} = \begin{cases} \frac{1}{l} \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}}, & r < l, \\ \frac{1}{r} \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}}, & r > l, \end{cases} \quad (9.29)$$

where $x = \cos \theta$, $-1 \leq x \leq 1$, and $\rho = r/l < 1$ for $r < l$ and $\rho = l/r < 1$ for $r > l$. Notice that ρ is always less than 1.

The function

$$G(\rho, x) = \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}}, \quad 0 < \rho < 1, \quad -1 \leq x \leq 1 \quad (9.30)$$

is called the *generating function* for Legendre polynomials. Let us expand $G(\rho, x)$ in a Maclaurin series in powers of ρ . Starting with the expansion

$$\frac{1}{\sqrt{1 - z}} = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{15}{48}z^3 + \dots, \quad -1 < z < 1,$$

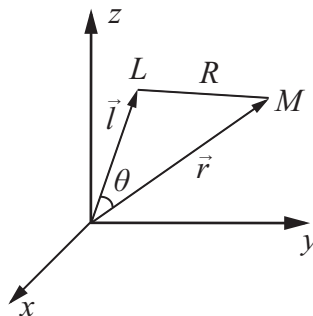


Figure 9.3. Illustration of parameters in a spherical coordinate system.

and setting $z = 2\rho x - x^2$, we obtain

$$\begin{aligned}
 G(\rho, x) &= \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}} \\
 &= 1 + \frac{1}{2}(2\rho x - \rho^2) + \frac{3}{8}(2\rho x - \rho^2)^2 + \frac{15}{48}(2\rho x - \rho^2)^3 + \dots \\
 &= 1 + x\rho + \frac{1}{2}(3x^2 - 1)\rho^2 + \frac{1}{2}(5x^3 - 3x)\rho^3 + \dots \\
 &= P_0(x) + P_1(x)\rho + P_2(x)\rho^2 + P_3(x)\rho^3 + \dots = \sum_{n=0}^{\infty} P_n(x)\rho^n.
 \end{aligned} \tag{9.31}$$

From Equation (9.31), we see that the coefficients of expansion of the function $G(\rho, x)$ are Legendre polynomials $P_n(x)$:

$$P_n(x) = \frac{1}{n!} \left. \frac{\partial^n G(\rho, x)}{\partial \rho^n} \right|_{\rho=0}. \tag{9.32}$$

Thus, $G(\rho, x)$ “generates” the Legendre polynomials. Obviously, the function $G(\rho, x)$ satisfies the Legendre equation with respect to the variable x .

Notice that from Equations (9.29) and (9.31), we obtain a very useful formula for the distance between two points in spherical coordinates as an expansion in Legendre polynomials:

$$\frac{1}{R} = \begin{cases} \frac{1}{l} \sum_{n=0}^{\infty} \left(\frac{r}{l}\right)^n P_n(\cos \theta), & r < l, \\ \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{l}{r}\right)^n P_n(\cos \theta), & r > l. \end{cases} \tag{9.33}$$

The generating function in Equation (9.30) provides an efficient way to derive many useful properties and applications of Legendre polynomials. In the following sections, we present mathematical and physical applications of the generating function.

9.3 Recurrence Relations

With the help of the generating function $G(\rho, x)$ given by Equation (9.30), it is easy to obtain the recurrence formulas for Legendre polynomials. Dif-

differentiating $G(\rho, x)$ with respect to the parameter ρ , we have

$$\frac{\partial G(\rho, x)}{\partial \rho} = -\frac{1}{2} (1 + \rho^2 - 2\rho x)^{-3/2} (2\rho - 2x) = \frac{x - \rho}{(1 + \rho^2 - 2\rho x)^{3/2}}.$$

Let us rearrange this formula as

$$(1 + \rho^2 - 2\rho x) G_\rho - (x - \rho) G = 0, \quad (9.34)$$

where $G_\rho = \partial G(\rho, x)/\partial \rho$. Substituting Equation (9.31) into Equation (9.34) yields

$$(1 + \rho^2 - 2\rho x) \sum_{n=0}^{\infty} n P_n(x) \rho^{n-1} - (x - \rho) \sum_{n=0}^{\infty} P_n(x) \rho^n = 0.$$

Let us now multiply the sums by the quantity in brackets and change the index of summation in such a way that the degree of ρ becomes the same in each sum:

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) \rho^n + \sum_{n=2}^{\infty} (n-1) P_{n-1}(x) \rho^n - \\ & \sum_{n=1}^{\infty} 2nx P_n(x) \rho^n - \sum_{n=0}^{\infty} x P_n(x) \rho^n + \sum_{n=1}^{\infty} P_{n-1}(x) \rho^n = 0. \end{aligned}$$

We finally group all the terms with $n \geq 2$ in one sum and write the terms with $n = 0$ and $n = 1$ separately, which results in

$$\begin{aligned} & P_1(x) + 2P_2(x)\rho - 3xP_1(x)\rho - xP_0(x) + P_0(x)\rho \\ & + \sum_{n=2}^{\infty} [(n+1)P_{n+1}(x) + (n-1)P_{n-1}(x) - 2nxP_n(x) - xP_n(x) + P_{n-1}(x)] \rho^n = 0. \end{aligned} \quad (9.35)$$

The functions ρ^n are linearly independent; thus, coefficients of each power of ρ should be zero for each n . Therefore,

$$\begin{aligned} n = 0 & \quad P_1(x) - xP_0(x) = 0, \\ n = 1 & \quad 2P_2(x) - 3xP_1(x) + P_0(x) = 0, \\ n \geq 2 & \quad (n+1)P_{n+1}(x) + (n-1)P_{n-1}(x) - 2nxP_n(x) - xP_n(x) + P_{n-1}(x) = 0. \end{aligned}$$

We rewrite these formulas as

$$P_1(x) = xP_0(x) = x, \quad (9.36)$$

$$P_2(x) = \frac{1}{2} [3xP_1(x) - P_0(x)] = \frac{1}{2} (3x^2 - 1), \quad (9.37)$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \quad (9.38)$$

Equation (9.38) is a recurrence formula which relates three polynomials. This formula gives a simple (and the most practical) way to obtain the Legendre polynomials of any order, one by one, starting with $P_0(x) = 1$ and $P_1(x) = x$.

Reading Exercise. Differentiate the function $G(\rho, x)$ with respect to x and use Equations (9.36) and (9.38) to obtain the following useful recurrence relations for Legendre polynomials:

$$P'_{n-1}(x) - xP'_n(x) + nP_n(x) = 0, \quad (9.39)$$

$$P'_n(x) - xP'_{n-1}(x) - nP_{n-1}(x) = 0. \quad (9.40)$$

9.4 Orthogonality of Legendre Polynomials

The Legendre polynomials are the eigenfunctions of a boundary value problem that is a particular case of a Sturm-Liouville problem; thus, the eigenfunctions should be orthogonal on the interval $[-1, 1]$:

$$\int_{-1}^1 P_n(x)P_m(x)dx = 0, \quad m \neq n. \quad (9.41)$$

It is useful to prove directly that the relation in Equation (9.41) is satisfied; this will also help to find the norms of the Legendre polynomials. Let us write the Legendre equation for two indexes, n and m :

$$[(1-x^2)P'_n(x)]' + n(n+1)P_n(x) = 0, \quad (9.42)$$

$$[(1-x^2)P'_m(x)]' + m(m+1)P_m(x) = 0. \quad (9.43)$$

Next, we multiply Equation (9.42) by $P_m(x)$, and Equation (9.43) by $P_n(x)$, subtract the second result from the first, and integrate the difference on the interval $[-1, 1]$ to get

$$\begin{aligned}
& \int_{-1}^1 [(1-x^2) P'_n(x)]' P_m(x) dx - \int_{-1}^1 [(1-x^2) P'_m(x)]' P_n(x) dx \\
&= [m(m+1) - n(n+1)] \int_{-1}^1 P_n(x) P_m(x) dx.
\end{aligned} \tag{9.44}$$

On the left side of Equation (9.44), we integrate by parts:

$$\begin{aligned}
& \int_{-1}^1 [(1-x^2) P'_n(x)]' P_m(x) dx - \int_{-1}^1 [(1-x^2) P'_m(x)]' P_n(x) dx = \\
& (1-x^2) P'_n(x) P_m(x) \Big|_{-1}^1 - \int_{-1}^1 (1-x^2) P'_n(x) P'_m(x) dx \\
& - (1-x^2) P'_m(x) P_n(x) \Big|_{-1}^1 + \int_{-1}^1 (1-x^2) P'_n(x) P'_m(x) dx = 0.
\end{aligned}$$

From this, we see that the left side of Equation (9.44) is equal to zero for any n and m . Therefore, the right side also equals zero when $m \neq n$ (when $m = n$, the result is trivial), and this proves the orthogonality property in Equation (9.41) of Legendre polynomials.

Next, let us find the norm, $\|P_n\|$, of the Legendre polynomials $P_n(x)$:

$$\|P_n\|^2 = \int_{-1}^1 P_n^2(x) dx. \tag{9.45}$$

We will need this norm for Fourier-Legendre series expansions in the examples in Section 9.8. First, by using the recurrence formula given in Equation (9.38), we reduce $P_n(x)$ to a combination of $xP_{n-1}(x)$ and $P_{n-2}(x)$, and then reduce $xP_n(x)$ to a combination of $P_{n+1}(x)$ and $P_{n-1}(x)$:

$$\begin{aligned}
P_n(x) &= \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x), \\
x P_n(x) &= \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x).
\end{aligned}$$

By substituting these relations into Equation (9.45) and using the orthogonality condition in Equation (9.41), we obtain

$$\begin{aligned}
 \|P_n\|^2 &= \frac{1}{n} \int_{-1}^1 [(2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)] P_n(x) dx \\
 &= \frac{2n-1}{n} \int_{-1}^1 xP_n(x)P_{n-1}(x) dx \\
 &= \frac{2n-1}{n(2n+1)} \int_{-1}^1 [(n+1)P_{n+1}(x) + nP_{n-1}(x)] P_{n-1}(x) dx \\
 &= \frac{2n-1}{2n+1} \int_{-1}^1 P_{n-1}^2(x) dx \\
 &= \frac{2n-1}{2n+1} \|P_{n-1}\|^2.
 \end{aligned}$$

By applying this recurrence relation, we obtain

$$\|P_n\|^2 = \frac{(2n-1)(2n-3)(2n-5)\dots 1}{(2n+1)(2n-1)(2n-3)\dots 3} \|P_0(x)\|^2 = \frac{1}{2n+1} \|P_0\|^2.$$

Since $\|P_0\|^2 = \int_{-1}^1 P_0^2(x) dx = \int_{-1}^1 dx = 2$, we finally obtain the norm squared of Legendre polynomials as

$$\|P_n\|^2 = \frac{2}{2n+1}. \quad (9.46)$$

Equations (9.41) and (9.46) can be combined and written as

$$\int_{-1}^1 P_n(x)P_m(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{2}{2n+1}, & m = n. \end{cases} \quad (9.47)$$

Another useful result follows from the completeness and orthogonality of Legendre polynomials. Let $M_m(x)$ be a polynomial of degree m , and let $m < n$. Then,

$$\int_{-1}^1 M_m(x) P_n(x) dx = 0 \quad (m < n). \quad (9.48)$$

This is easy to prove if we first expand $M_m(x)$ as a series in the polynomials $P_n(x)$:

$$M_m(x) = \sum_{k=0}^m c_k P_k(x).$$

Then

$$\int_{-1}^1 M_m(x) P_n(x) dx = \sum_{k=0}^m c_k \int_{-1}^1 P_k(x) P_n(x) dx = 0,$$

since $\int_{-1}^1 P_k(x) P_n(x) dx = 0$ for $0 \leq k \leq m < n$. In particular from this result follows that for all $n > 0$ we have

$$\int_{-1}^1 P_n(x) dx = 0 \quad (9.49)$$

(which is also obvious from Equation (9.41)).

9.5 The Multipole Expansion in Electrostatics

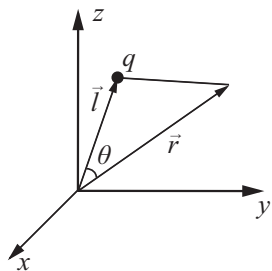
We now describe a very useful application of the generating function defined above: the multipole expansion in electrostatics. Suppose an electric charge, q , is located at a point given by the vector $\vec{l}$ (see [Figure 9.4](#)). The electrostatic potential at a point given by the radial vector $\vec{r}$ is

$$\varphi = \frac{kq}{|\vec{r} - \vec{l}|}, \quad (9.50)$$

where k is Coulomb's constant. With

$$|\vec{r} - \vec{l}| = \sqrt{l^2 + r^2 - 2rl \cos \theta},$$

the potential in Equation (9.50) is

Figure 9.4. Electric potential at $\vec{r}$ due to charge q .

$$\varphi = \frac{kq}{r\sqrt{1 + \left(\frac{l}{r}\right)^2 - 2\frac{l}{r}\cos\theta}}. \quad (9.51)$$

Comparing Equation (9.51) with Equation (9.30) we see that Equation (9.51) contains the generating function, $G(\rho, x)$, for Legendre polynomials. Thus, with the substitutions $x = \cos\theta$ and $\rho = l/r$, Equation (9.51) becomes

$$\varphi = \frac{kq}{r\sqrt{1 + \rho^2 - 2\rho x}} = \frac{kq}{r}G(\rho, x), \quad (9.52)$$

which is the electric potential due to charge q at point $\vec{r}$.

Reading Exercise. Argue that φ given by Equation (9.51) satisfies Laplace's equation, $\nabla^2\varphi = 0$.

Expanding $G(\rho, x)$ in a series in powers of ρ , as in Equation (9.31), we obtain the expression for electrostatic potential φ of charge q in terms of Legendre polynomials, $P_n(x)$:

$$\varphi = \frac{kq}{r} \sum_{n=0}^{\infty} \rho^n P_n(x) = kq \sum_{n=0}^{\infty} \frac{l^n}{r^{n+1}} P_n(\cos\theta).$$

Similarly, since the equation for electrostatic potential is linear, we may extend this to an electrostatic potential of N charges as

$$\varphi = k \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \left[\sum_{k=1}^N q_k l_k^n P_n(\cos\theta_k) \right], \quad (9.53)$$

where l_k is the distance from the origin to the charge q_k .

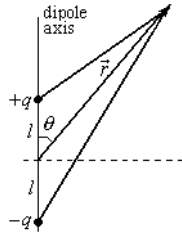


Figure 9.5. Electric dipole.

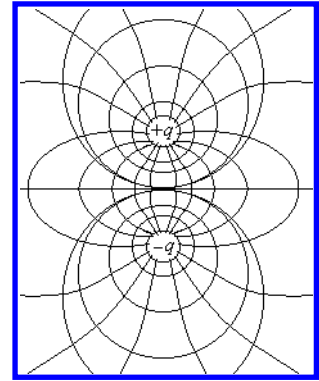


Figure 9.6. The equipotential lines and the corresponding electric field due to the electric dipole shown in Figure 9.5

Equation (9.53) has an important physical interpretation. If we denote

$$M_n = \sum_{k=1}^N q_k l_k^n P_n(\cos \theta_k), \quad (9.54)$$

then, from Equation (9.53), the electrostatic potential becomes

$$\varphi = k \sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}}. \quad (9.55)$$

Consider a system of two charges, opposite in sign but having the same magnitude, which defines an *electric dipole* (see Figure 9.5). With the definition in Equation (9.54), the first two M_n are

$$M_0 = q - q = 0, \quad M_1 = \frac{ql}{2} \cos \theta - \frac{ql}{2} \cos(\pi - \theta) = ql \cos \theta = \mu \cos \theta,$$

where $\mu = ql$ is called the *dipole moment* of a dipole. The next term for this charge configuration is

$$M_2 = \frac{ql^2}{2} P_2(\cos \theta) - \frac{ql^2}{2} P_2(\cos(\pi - \theta)) = 0.$$

If we want to find the potential at a point very distant from the dipole (i.e., for $r \gg l$) the main contribution is due to the first term in Equation (9.55).

As the result, Equation (9.55) for the electric potential of a dipole becomes

$$\varphi = k \frac{\mu \cos \theta}{r^2} + O\left(\frac{1}{r^4}\right). \quad (9.56)$$

From Equation (9.56), we see that the potential due to a dipole is zero at points along a line perpendicular to the dipole axis ($\theta = \pi/2$), a result that should also be obvious from physical intuition.

The electric field, $\vec{E} = -\nabla\varphi$, for a dipole corresponding to the potential of Equation (9.56) is

$$\vec{E} = \frac{\partial\varphi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial\varphi}{\partial\theta} \vec{e}_\theta = -k \frac{\mu \cos \theta}{r^3} \vec{e}_r - k \frac{\mu \sin \theta}{r^3} \vec{e}_\theta, \quad (9.57)$$

where $\vec{e}_r$ and $\vec{e}_\theta$ are radial and meridian unit vectors in spherical coordinates. Figure 9.6 shows equipotential lines (lines on which $\varphi = \text{const}$) and the electric field lines, which are perpendicular to equipotential lines.

Reading Exercise. Find an expression for φ for a dipole that includes the contributions M_3 and M_4 .

Consider one more example: a system of three charges lying along the same line. Two charges, q , lie at the ends, and one charge, $-2q$, is located at the center (shown in [Figure 9.7](#))—an arrangement known as a linear *quadrupole*. The first few M_n for this system are

$$M_0 = q - 2q + q = 0,$$

$$M_1 = \frac{ql}{2} \cos \theta + \frac{ql}{2} \cos(\pi - \theta) = 0,$$

$$M_2 = \frac{ql^2}{4} P_2(\cos \theta) + \frac{ql^2}{4} P_2(\cos(\pi - \theta)) = \frac{ql^2}{2} P_2(\cos \theta) = \frac{ql^2}{2} \frac{3 \cos^2 \theta - 1}{2},$$

$$M_3 = \frac{ql^3}{8} P_3(\cos \theta) + \frac{ql^3}{8} P_3(\cos(\pi - \theta)) = 0.$$

Here, while calculating these M_n , we used the properties of Legendre polynomials, $P_2(-\cos \theta) = P_2(\cos \theta)$ and $P_3(-\cos \theta) = -P_3(\cos \theta)$. Putting these moments into Equation (9.55) for the electrostatic potential, φ , of a quadrupole we have

$$\varphi = k \frac{ql^2}{2} \frac{P_2(\cos \theta)}{r^3} + O\left(\frac{1}{r^5}\right) = kQ \frac{3 \cos^2 \theta - 1}{2r^3} + O\left(\frac{1}{r^5}\right), \quad (9.58)$$

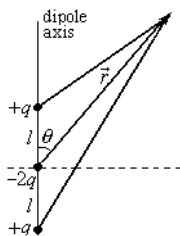


Figure 9.7. The linear quadrupole.

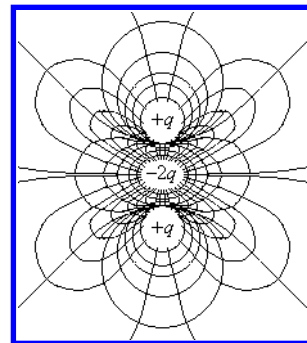


Figure 9.8. The equipotential lines and the corresponding electric field due to the linear quadrupole shown in Figure 9.7

where $Q = ql^2/2$ is called the quadrupole moment. Figure 9.8 shows the equipotential lines $\varphi = \text{const}$, given by Equations (9.58), and the electric field lines, which are perpendicular to equipotential lines.

Reading Exercise. Find an expression for φ for a quadrupole that includes the contributions M_4 and M_5 .

The quantities M_n are called *multipole moments*. M_1 is related to the dipole moment μ , M_2 to the quadrupole moment, Q , and so forth. Equation (9.55) is called the multipole expansion for a collection of charges. It is very important in electrodynamic applications such as in magnetic resonance and molecule interactions.

9.6 Associated Legendre Functions $P_n^m(x)$

In this section, we consider a generalization of Equation (9.1):

$$(1 - x^2) y'' - 2xy' + \left(\lambda - \frac{m^2}{1 - x^2} \right) y = 0, \quad -1 \leq x \leq 1, \quad (9.59)$$

where m is a specified number. Like Equation (9.1), Equation (9.59) has nontrivial solutions bounded at $x = \pm 1$ only for the values of $\lambda = n(n+1)$. In mathematical physics problems, the values of m are *integers*, and we see examples of such problems in Section 9.10; we also show that the values

of m and n are related by inequality $|m| \leq n$. Equation (9.59) is called the *associated Legendre equation of order m* . In Sturm-Liouville form, this equation can be written as

$$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \left(\lambda - \frac{m^2}{1-x^2} \right) y = 0, \quad -1 \leq x \leq 1. \quad (9.60)$$

To solve Equation (9.60), we can use a solution of Equation (9.2). First we discuss positive values of m . Let us introduce a new function, $z(x)$, to replace $y(x)$ in Equation (9.60), using the definition

$$y(x) = (1-x^2)^{\frac{m}{2}} z(x). \quad (9.61)$$

By substituting Equation (9.61) into Equation (9.60), we obtain

$$(1-x^2) z'' - 2(m+1)xz' + [\lambda - m(m+1)] z = 0. \quad (9.62)$$

If $m = 0$, Equation (9.62) reduces to Equation (9.1); thus, its solutions are Legendre polynomials $P_n(x)$.

Let us solve Equation (9.62) by expanding $z(x)$ in a power series:

$$z = \sum_{k=0}^{\infty} a_k x^k. \quad (9.63)$$

We thus have

$$z' = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} k a_k x^{k-1}, \quad (9.64)$$

$$z'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k, \quad (9.65)$$

$$x^2 z'' = \sum_{k=2}^{\infty} k(k-1) a_k x^k = \sum_{k=0}^{\infty} k(k-1) a_k x^k. \quad (9.66)$$

By substituting Equations (9.64) through (9.66) into Equation (9.62), we obtain

$$\sum_{k=0}^{\infty} \{ (k+2)(k+1) a_{k+2} + [\lambda - (k+m)(k+m+1)] a_k \} x^k = 0.$$

Because the functions x^k are linearly independent, the coefficients of each power of x^k must be zero, which leads to a recurrence relation for the coefficients a_k :

$$a_{k+2} = -\frac{\lambda - (k+m)(k+m+1)}{(k+2)(k+1)}a_k. \quad (9.67)$$

Reading Exercise. Using this recurrence relation, check that the series in Equation (9.63) converges for $-1 < x < 1$ and diverges at the ends of the intervals, $x = \pm 1$.

Here, we discuss only solutions that are regular on the closed interval, $-1 \leq x \leq 1$. This means that the series in Equation (9.63) should terminate as a polynomial of some maximum degree. Denoting this degree as q , we obtain $a_q \neq 0$ and $a_{q+2} = 0$, so that if $\lambda = (q+m)(q+m+1)$, $q = 0, 1, \dots$, then $a_{q+2} = a_{q+4} = \dots = 0$. Introducing $n = q+m$ (note again that in applications, the letter l is often used instead of n), because q and m are nonnegative integers, we have $n = 0, 1, 2, \dots$, and $n \geq m$. Thus, we see that $\lambda = n(n+1)$, as in the case of Legendre polynomials. Clearly, if $n = 0$, the value of $m = 0$; thus, $\lambda = 0$ and the functions $z(x) = a_0$ and $y(x) = P_0(x)$.

We therefore obtain that $z(x)$ is an even or odd polynomial of degree $(n-m)$:

$$z(x) = a_{n-m}x^{n-m} + a_{n-m-2}x^{n-m-2} + \dots + \begin{cases} a_0 \\ a_1x. \end{cases} \quad (9.68)$$

We now present several examples for $m = 1$. If $n = 1$, then $q = 0$, and thus $z(x) = a_0$. If $n = 2$, then $q = 1$, and thus $z(x) = a_1x$. If $n = 3$, then $q = 2$, and thus $z(x) = a_0 + a_2x^2$, and from the recurrence formula, we have $a_2 = -5a_0$.

Reading Exercise.

1. Find $z(x)$ for $m = 1$ and $n = 4$.
2. Find $z(x)$ for $m = 2$ and $n = 4$.
3. For all above examples check that (keeping the lowest coefficients arbitrary) $z(x) = (d^m/dx^m)P_n(x)$.

Given that $\lambda = n(n + 1)$, we can obtain a solution of Equation (9.62) by using the solution of the Legendre equation, Equation (9.2). Let us differentiate equation (9.63) with respect to the variable x :

$$(1 - x^2) (z')'' - 2[(m + 1) + 1]x (z')' + [n(n + 1) - (m + 1)(m + 2)]z' = 0. \quad (9.69)$$

It is seen that if in this equation we replace z' by z and $(m + 1)$ by m , the resulting equation becomes Equation (9.62). In other words, if $P_n(x)$ is a solution of Equation (9.62) for $m = 0$, then $P_n'(x)$ is a solution of Equation (9.69) for $m = 1$. Repeating this, we obtain that $P_n''(x)$ is a solution for $m = 2$, $P_n'''(x)$ is a solution for $m = 3$, and so forth. For an arbitrary integer m , where $0 \leq m \leq n$, a solution of Equation (9.62) is the function $(d^m/dx^m)P_n(x)$; thus,

$$z(x) = \frac{d^m}{dx^m} P_n(x), \quad 0 \leq m \leq n. \quad (9.70)$$

With Equations (9.70) and (9.61), we have a solution of Equation (9.60) given by

$$y(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \quad 0 \leq m \leq n. \quad (9.71)$$

The functions defined in Equation (9.71) are called the *associated Legendre functions* and denoted as $P_n^m(x)$:

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x). \quad (9.72)$$

(Note that in the literature the upper index is often put in parentheses and so appears as $P_n^{(m)}(x)$.) Notice that $(d^m/dx^m)P_n(x)$ is a polynomial of degree $n - m$; thus,

$$P_n^m(-x) = (-1)^{n-m} P_n^m(x), \quad (9.73)$$

which is referred to as the *parity property*. From Equation (9.73), we directly see that $P_n^m(x)$, for $|m| > n$, are equal to zero because in this case, the m th-order derivatives of polynomial $P_n(x)$ of degree n are equal to zero. The graphs of several $P_n^m(x)$ are plotted in Figure 9.9.

Thus, from the above discussion, we see that Equation (9.59) has eigenvalues λ :

$$m(m + 1), \quad (m + 1)(m + 2), \quad (m + 2)(m + 3), \quad \dots \quad (9.74)$$

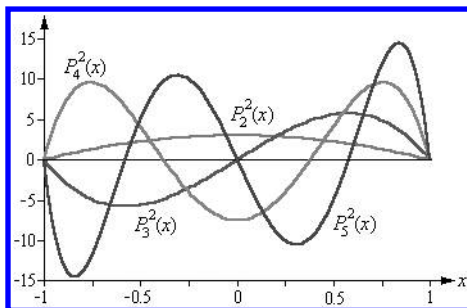


Figure 9.9. Graphs of $P_2^2(x)$, $P_3^2(x)$, $P_4^2(x)$, and $P_5^2(x)$.

and the corresponding eigenfunctions, bounded on $[-1, 1]$, are

$$P_m^m(x), \quad P_{m+1}^m(x), \quad P_{m+2}^m(x), \quad \dots \quad (9.75)$$

Equation (9.59) (or (9.60)) does not change when the sign of m changes. Therefore, a solution of Equation (9.59) for positive m is also a solution for negative values $-|m|$. Thus, we can define $P_n^m(x)$ as equal to $P_n^{|m|}(x)$ for $-n \leq m \leq n$:

$$P_n^{-|m|}(x) = P_n^{|m|}(x). \quad (9.76)$$

The first several associated Legendre functions, $P_n^1(x)$, for $m = 1$ are

$$\begin{aligned} P_1^1(x) &= \sqrt{1-x^2} \cdot [P_1(x)]' = \sqrt{1-x^2}, \\ P_2^1(x) &= \sqrt{1-x^2} \cdot [P_2(x)]' = \sqrt{1-x^2} \cdot 3x, \\ P_3^1(x) &= \sqrt{1-x^2} \cdot [P_3(x)]' = \sqrt{1-x^2} \cdot \frac{3}{2} (5x^2 - 1); \end{aligned}$$

and the first several associated Legendre functions, $P_n^2(x)$, for $m = 2$ are

$$\begin{aligned} P_2^2(x) &= (1-x^2) \cdot [P_2(x)]'' = (1-x^2) \cdot 3, \\ P_3^2(x) &= (1-x^2) \cdot [P_3(x)]'' = (1-x^2) \cdot 15x, \\ P_4^2(x) &= (1-x^2) \cdot [P_4(x)]'' = (1-x^2) \cdot \frac{15}{2} (7x^2 - 1). \end{aligned}$$

Associated Legendre function $P_n^m(x)$ has $(n - m)$ simple (not repeating) real roots on the interval $-1 < x < 1$.

The following *recurrence formulas* are valid, and stated without proof:

$$P_n^{m+2}(x) - 2(m+1)\frac{x}{\sqrt{1-x^2}}P_n^{m+1}(x) + (n-m)(n+m+1)P_n^m(x) = 0, \quad (9.77)$$

$$(2n+1)xP_n^m(x) - (n-m+1)P_{n+1}^m(x) - (n+m)P_{n-1}^m(x) = 0. \quad (9.78)$$

A differential equation that relates associated Legendre functions with the same upper index m but different lower index also exists:

$$(x^2-1)\frac{d}{dx}P_n^m(x) - (n-m+1)P_{n+1}^m(x) + (n+1)xP_{n-1}^m(x) = 0. \quad (9.79)$$

9.7 Orthogonality and the Norm of Associated Legendre Functions

According to Sturm-Liouville theory, associated Legendre functions $P_n^m(x)$ of order m form an orthogonal set with the weight function $r(x) = 1$. Let us show by a direct calculation that these functions are orthogonal and also find their norm. We first multiply Equation (9.62) by $(1-x^2)^m$ and replace $(m+1)$ by m . By taking into account Equation (9.70), we thus obtain

$$\frac{d}{dx} \left[(1-x^2)^m \frac{d^m P_n(x)}{dx^m} \right] + [n(n+1) - m(m+1)] (1-x^2)^{m-1} \frac{d^{m-1} P_n(x)}{dx^{m-1}} = 0. \quad (9.80)$$

Let us define

$$N_{nk}^m = \int_{-1}^1 P_n^m(x) P_k^m(x) dx = \int_{-1}^1 (1-x^2)^m \frac{d^m P_n(x)}{dx^m} \frac{d^m P_k(x)}{dx^m} dx. \quad (9.81)$$

Integrating by parts, we have

$$N_{nk}^m = \left[\frac{d^{m-1} P_k}{dx^{m-1}} \frac{d^m P_n}{dx^m} (1-x^2)^m \right]_{-1}^1 - \int_{-1}^1 \frac{d^{m-1} P_k(x)}{dx^{m-1}} \frac{d}{dx} \left[(1-x^2)^m \frac{d^m P_n(x)}{dx^m} \right] dx. \quad (9.82)$$

The first term in Equation (9.82) is equal to zero, and the integrand in the second term, by using Equation (9.80), can be transformed to

$$N_{nk}^m = [n(n+1) - m(m-1)] N_{nk}^{m-1} = (n+m)(n-m+1) N_{nk}^{m-1}.$$

This is a recurrence formula that allows us to relate N_{nk}^m to N_{nk}^0 :

$$\begin{aligned} N_{nk}^m &= (n+m)(n+m-1) \dots (n+1)n \dots (n-m+1) N_{nk}^0 \\ &= \frac{(n+m)!}{n!} \frac{n!}{(n-m)!} N_{nk}^0 \\ &= \frac{(n+m)!}{(n-m)!} N_{nk}^0. \end{aligned}$$

The functions N_{nk}^0 are known, since $P_n^{(0)}(x) = P_n(x)$. Finally, for N_{nk}^m , we obtain

$$\int_{-1}^1 P_n^m(x) P_k^m(x) dx = \begin{cases} 0, & n \neq k, \\ \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, & n = k. \end{cases} \quad (9.83)$$

From Equation (9.83), it follows that the associated Legendre functions of order m are mutually orthogonal, and the square norm of P_n^m is

$$\|P_n^m\|^2 = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}. \quad (9.84)$$

9.8 Fourier-Legendre Series in Legendre Polynomials

As we have seen, the Legendre polynomials, $P_n(x)$, are the solutions of a Sturm-Liouville problem; thus, they form an *orthogonal* (with respect to the weight function $r(x) = 1$) and *complete* set of functions on the closed interval $[-1, 1]$. This, of course, means that the set $\{P_n(x)\}$, $n = 0, 1, 2, \dots$, provides a basis for an eigenfunction expansion for functions $f(x)$, bounded on the interval $[-1, 1]$:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x). \quad (9.85)$$

Obviously, due to the orthogonality of the functions $P_n(x)$ with different indexes, the coefficients c_n are

$$c_n = \frac{1}{\|P_n\|^2} \int_{-1}^1 f(x) P_n(x) dx = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx. \quad (9.86)$$

As we know from the general theory discussed previously in this book, the sequence of the partial sums of this series, $S_N(x)$, converges on the interval $(-1, 1)$ on average (i.e., in the mean) to $f(x)$, which may be written as

$$\int_{-1}^1 [f(x) - S_N(x)]^2 dx \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (9.87)$$

The real function $f(x)$ should be square integrable; that is, we require that the integral $\int_{-1}^1 f^2(x) dx$ exists.

For an important class of piecewise continuous functions, the series in Equation (9.85) converges absolutely and uniformly. The following theorem (presented without proof) states a stronger result about the convergence of the series (9.85) than convergence in the mean.

Theorem 9.1. *If the function $f(x)$ is piecewise continuous on the interval $(-1, 1)$, then the Fourier-Legendre series converges to $f(x)$ at the points where the function $f(x)$ is continuous and to*

$$\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)], \quad (9.88)$$

if x_0 is a point of finite discontinuity of the function $f(x)$.

Because the Legendre polynomials form a complete set, for any square integrable function, $f(x)$, we have

$$\int_{-1}^1 f^2(x) dx = \sum_{n=0}^{\infty} \|P_n\|^2 c_n^2 = \sum_{n=0}^{\infty} \frac{2}{2n+1} c_n^2. \quad (9.89)$$

This is Parseval's equality (sometimes referred to as the completeness equation) for the Fourier-Legendre series. Clearly, for a partial sum on

the right we have Bessel's inequality

$$\int_{-1}^1 f^2(x) dx \geq \sum_{n=0}^N \frac{2}{2n+1} c_n^2. \quad (9.90)$$

We next consider several examples of the expansion of functions into the Fourier-Legendre series. In some cases, the coefficients can be found analytically; otherwise, we may calculate them numerically, for example, by using the program FourierSeries. A description of how to use this program is found in Appendix E.

Example 9.1. Expand the function $f(x) = A$, $A = \text{const}$, in a Fourier-Legendre series in $P_n(x)$ on the interval $-1 \leq x \leq 1$.

Solution. The series is

$$A = c_0 P_0(x) + c_1 P_1(x) + \dots,$$

where the coefficients c_n are

$$c_n = \frac{1}{\|P_n^2(x)\|} \int_{-1}^1 A P_n(x) dx = \frac{(2n+1)A}{2} \int_{-1}^1 P_n(x) dx.$$

From this formula, it is clear that the only nonzero coefficient is $c_0 = A$.

Example 9.2. Expand the function $f(x) = x$ in a Fourier-Legendre series in $P_n(x)$ on the interval $-1 \leq x \leq 1$.

Solution. The series is

$$x = c_0 P_0(x) + c_1 P_1(x) + \dots,$$

where the coefficients c_n are

$$c_n = \frac{1}{\|P_n^2(x)\|} \int_{-1}^1 x P_n(x) dx = \frac{2n+1}{2} \int_{-1}^1 x P_n(x) dx.$$

Clearly, the only nonzero coefficient is

$$c_1 = \frac{3}{2} \int_{-1}^1 x P_1(x) dx = \frac{3}{2} \int_{-1}^1 x^2 dx = 1.$$

As in Example 9.1, this result is apparent because one of the polynomials, $P_1(x)$ in this example, coincides with the given function, $f(x) = x$.

Example 9.3. Expand the function $f(x)$ given by

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 1 \end{cases}$$

in a Fourier-Legendre series.

Solution. The expansion $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ has coefficients

$$c_n = \frac{1}{\|P_n\|^2} \int_{-1}^1 f(x) P_n(x) dx = \frac{2n+1}{2} \int_0^1 P_n(x) dx.$$

The first several are

$$c_0 = \frac{1}{2} \int_0^1 dx = \frac{1}{2}, \quad c_1 = \frac{3}{2} \int_0^1 x dx = \frac{3}{4}, \quad c_2 = \frac{5}{2} \int_0^1 \frac{1}{2} (3x^2 - 1) dx = 0.$$

Continuing, we find for the given function $f(x)$,

$$f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \frac{11}{32} P_5(x) + \dots$$

The series converges slowly due to discontinuity of the given function $f(x)$ at point $x = 0$, as seen in Figure 9.10.

Example 9.4. Let $f(x) = \cos(\pi x/2)$ for $-1 \leq x \leq 1$. Functions $f(x)$ and $f'(x)$ are continuous on $[-1, 1]$. Show that the Fourier-Legendre expansion of $f(x)$ rapidly converges to $\cos(\pi x/2)$ for $-1 < x < 1$.

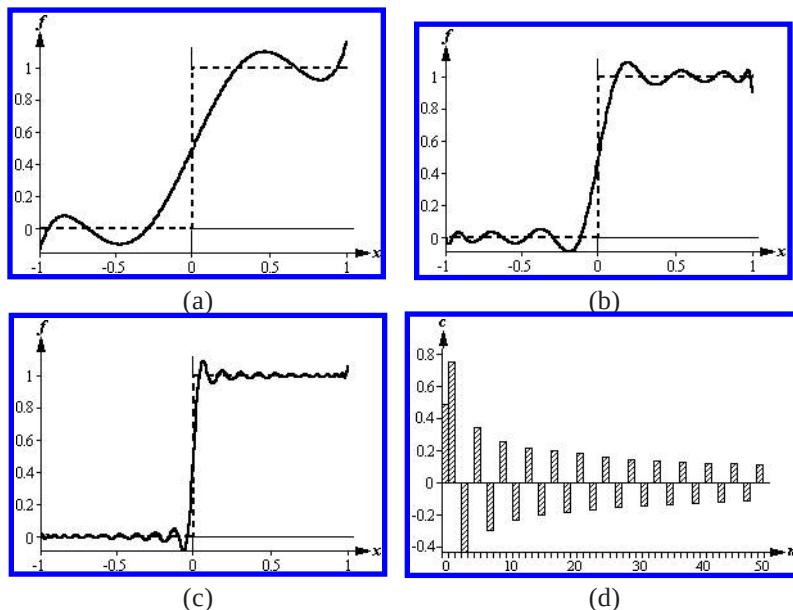


Figure 9.10. The function $f(x)$ and the partial sum of its Fourier-Legendre series. The graph of $f(x)$ is shown by the dashed line, and the graph of the series is shown by the solid line. $(N + 1)$ terms are kept in the series. (a) $(N = 5)$. (b) $N = 15$. (c) $N = 50$. (d) Values of the coefficients c_n of the series.

Solution. The coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^1 \cos \frac{\pi x}{2} P_n(x) dx.$$

The integrand is an odd function for n odd, and the corresponding $c_n = 0$. Several first coefficients with even index are:

$$c_0 = \frac{1}{2} \int_{-1}^1 \cos \frac{\pi x}{2} dx = \frac{2}{\pi},$$

$$c_2 = \frac{5}{2} \int_{-1}^1 \cos \frac{\pi x}{2} \frac{1}{2} (3x^2 - 1) dx = 10 \frac{\pi^2 - 12}{\pi^3},$$

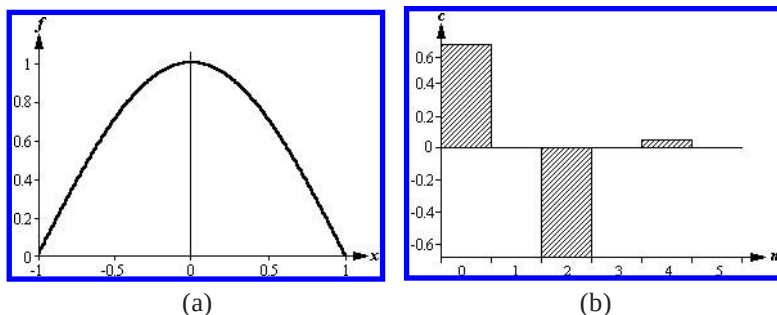


Figure 9.11. The function $f(x) = \cos(\pi x/2)$ and the partial sum of its Fourier-Legendre series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with $N = 5$ terms of the series is shown by the solid line. (b) Values of the coefficients c_n of the series.

$$c_4 = \frac{9}{2} \int_{-1}^1 \cos \frac{\pi x}{2} \frac{1}{8} (35x^4 - 30x^2 + 3) dx = 18 \frac{\pi^4 + 1680 - 180\pi^2}{\pi^5}.$$

Then, for $-1 < x < 1$

$$\cos \frac{\pi x}{2} = c_0 + c_2 P_2(x) + c_4 P_4(x) + \dots$$

Figure 9.11 shows a graph of $\cos(\pi x/2)$ and the partial sum with first three nonzero terms of its Fourier-Legendre expansion. This series agrees very well with $\cos(\pi x/2)$ for $-1 < x < 1$.

Example 9.5. Expand the function

$$f(x) = \begin{cases} 1+x, & -1 \leq x < 0, \\ 1-x, & 0 \leq x \leq 1 \end{cases}$$

in terms of Legendre polynomials.

Solution. The first coefficient is

$$c_0 = \frac{1}{2} \left[\int_{-1}^0 (1+x) dx + \int_0^1 (1-x) dx \right] = \frac{1}{2}.$$

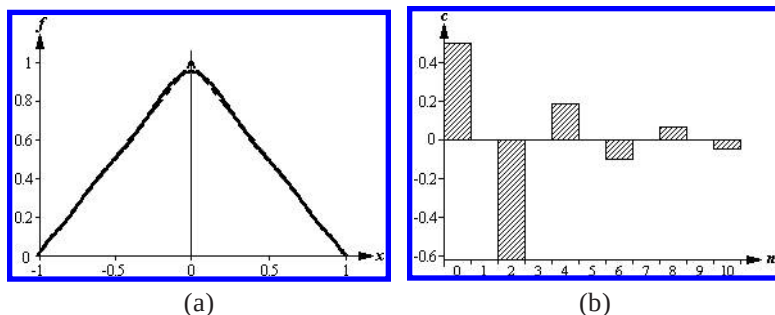


Figure 9.12. The function $f(x)$ and the partial sum of its Fourier-Legendre series. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with $N = 10$ terms of the series is shown by the solid line. (b) Values of the coefficients c_n of the series.

Because $f(x)$ is an even function of x , the odd coefficients are

$$c_1 = \frac{3}{2} \left[\int_{-1}^0 x(1+x)dx + \int_0^1 x(1-x)dx \right] = 0, \quad c_3 = c_5 = \dots = 0.$$

The first several coefficients with even index are

$$\begin{aligned} c_2 &= \frac{5}{2} \left[\int_{-1}^0 (1+x)P_2(x)dx + \int_0^1 (1-x)P_2(x)dx \right] \\ &= 5 \int_0^1 (1-x)P_2(x)dx = -\frac{5}{8}, \\ c_4 &= 9 \int_0^1 (1-x)P_4(x)dx = \frac{3}{16}. \end{aligned}$$

Figure 9.12 shows a graph of $f(x)$ and the partial sum with first six nonzero terms of its Fourier-Legendre expansion. The figures in Section 9.8 were generated with the program FourierSeries as solutions of the corresponding problems.

9.9 Fourier-Legendre Series in Associated Legendre Functions

Functions $P_n^m(x)$ for a fixed value of $|m|$ (the upper index of the associated Legendre functions) and all possible values of the lower index,

$$P_m^m(x), \quad P_{m+1}^m(x), \quad P_{m+2}^m(x), \quad \dots \quad (9.91)$$

form an *orthogonal* (with respect to the weight function $r(x) = 1$) and *complete* set of functions on the interval $[-1, 1]$. In other words, for each value of m , there is an orthogonal and complete set of functions (9.91). This follows from the fact that these functions are also the solutions of a Sturm-Liouville problem. Thus, the set of Equations (9.91) for any given m is a basis for an eigenfunction expansion for real functions bounded on $[-1, 1]$, and we may write

$$f(x) = \sum_{k=0}^{\infty} c_k P_{m+k}^m(x). \quad (9.92)$$

The formula for the coefficients c_k ($n = 0, 1, 2, \dots$) follows from the orthogonality of the functions (9.91):

$$c_k = \frac{1}{\|P_{m+k}^m\|^2} \int_{-1}^1 f(x) P_{m+k}^m(x) dx = \frac{2(m+k)+1}{2} \frac{k!}{(2m+k)!} \int_{-1}^1 f(x) P_{m+k}^m(x) dx. \quad (9.93)$$

As previously, the sequence of the partial sums $S_N(x)$ of series in Equation (9.92) converges on the interval $(-1, 1)$ in the mean to the square integrable function $f(x)$. That is,

$$\int_{-1}^1 [f(x) - S_N(x)]^2 dx \rightarrow 0 \text{ as } N \rightarrow \infty.$$

For piecewise-continuous functions, Theorem 9.1 is valid.

Reading Exercise. Formulate Parseval's equality for associated Legendre functions.

In applications, the associated Legendre functions are used to constitute expansions in spherical harmonics (which we consider in the following sections), and the Fourier series in Legendre functions themselves are not often used in practice. Simple examples of such series are presented next.

Example 9.6. Expand the function $f(x) = x^2$ in a Fourier-Legendre series on $-1 \leq x \leq 1$ in terms of associated Legendre functions $P_n^m(x)$ of order $m = 1$.

Solution. The series is

$$x^2 = c_0 P_1^1(x) + c_1 P_2^1(x) + c_2 P_3^1(x) + c_3 P_4^1(x) \dots,$$

where the coefficients c_k are

$$c_k = \frac{1}{\|P_{k+1}^1\|^2} \int_{-1}^1 x^2 P_{k+1}^1(x) dx = \frac{2k+3}{2} \frac{k!}{(k+2)!} \int_{-1}^1 x^2 P_{k+1}^1(x) dx.$$

The function $x^2 P_{k+1}^1(x)$ is an odd function for k odd, and the corresponding coefficients $c_k = 0$. The first three coefficients with even index are

$$\begin{aligned} c_0 &= \frac{1}{\|P_1^1\|^2} \int_{-1}^1 x^2 P_1^1(x) dx = \frac{3}{4} \int_{-1}^1 x^2 \sqrt{1-x^2} dx = \frac{3\pi}{32}, \\ c_2 &= \frac{1}{\|P_3^1\|^2} \int_{-1}^1 x^2 P_3^1(x) dx = \frac{7}{24} \int_{-1}^1 x^2 \frac{3}{2} (5x^2 - 1) \sqrt{1-x^2} dx = \frac{21\pi}{256}, \\ c_4 &= \frac{1}{\|P_5^1\|^2} \int_{-1}^1 x^2 P_5^1(x) dx = \frac{11}{60} \int_{-1}^1 x^2 \frac{15}{8} (21x^4 - 14x^2 + 1) \sqrt{1-x^2} dx = \frac{99\pi}{4096}. \end{aligned}$$

Figure 9.13 shows a graph of $f(x) = x^2$ for this example.

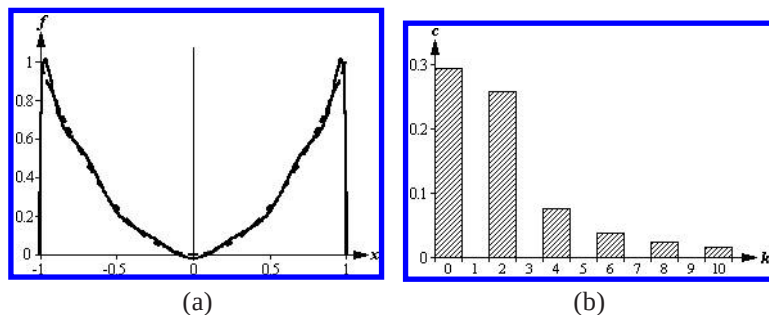


Figure 9.13. The function $f(x) = x^2$ and the partial sum of its Fourier-Legendre series in terms of associated Legendre functions $P_n^1(x)$. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with $N = 10$ terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

Example 9.7. Expand the function

$$f(x) = \begin{cases} 1 + x, & -1 \leq x < 0, \\ 1 - x, & 0 \leq x \leq 1 \end{cases}$$

in terms of associated Legendre functions $P_n^m(x)$ of order $m = 2$.

Solution. The series is

$$f(x) = c_0 P_2^2(x) + c_1 P_3^2(x) + c_2 P_4^2(x) + c_3 P_5^2(x) \dots,$$

where the coefficients c_k are

$$c_k = \frac{2k+5}{2} \frac{k!}{(k+4)!} \left[\int_{-1}^0 (1+x) P_{k+2}^2(x) dx + \int_0^1 (1-x) P_{k+2}^2(x) dx \right].$$

Because $f(x)$ is an even function of x ,

$$c_1 = c_3 = c_5 = \dots = 0.$$

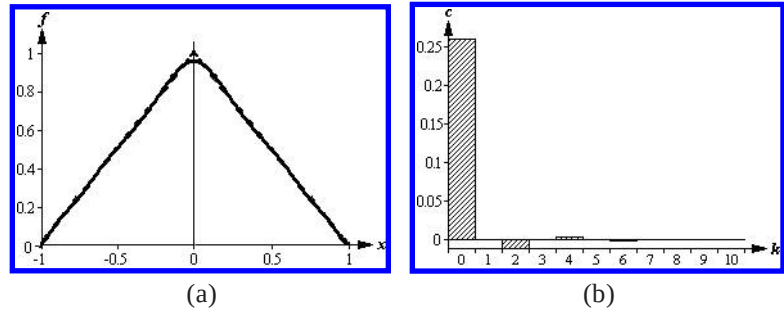


Figure 9.14. The function $f(x)$ and the partial sum of its Fourier-Legendre series in terms of associated Legendre functions $P_n^2(x)$. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with $N = 10$ terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

The first three coefficients with even index are

$$c_0 = \frac{5}{48} \left[\int_{-1}^0 (1+x)3(1-x^2) dx + \int_0^1 (1-x)3(1-x^2) dx \right] = \frac{25}{96},$$

$$c_2 = \frac{1}{80} \left[\int_{-1}^0 (1+x)\frac{15}{2}(1-x^2)(7x^2-1) dx + \int_0^1 (1-x)\frac{15}{2}(1-x^2)(7x^2-1) dx \right] = \frac{1}{80},$$

$$c_4 = \frac{13}{3360} \left[\int_{-1}^0 (1+x)\frac{105}{8}(1-x^2)(33x^4-18x^2+1) dx + \int_0^1 (1-x)\frac{105}{8}(1-x^2)(33x^4-18x^2+1) dx \right] = \frac{13}{3360} \cdot \frac{23}{32} = \frac{299}{107520} \approx 0.0028.$$

Figure 9.14 shows a graph of $f(x)$ for this example.

Example 9.8. Expand the function

$$f(x) = \frac{1}{50x^2 - 10x + 2}$$

defined on the interval $(-1, 1)$ using the system of associated Legendre functions $P_n^m(x)$ of order $m = 4$.

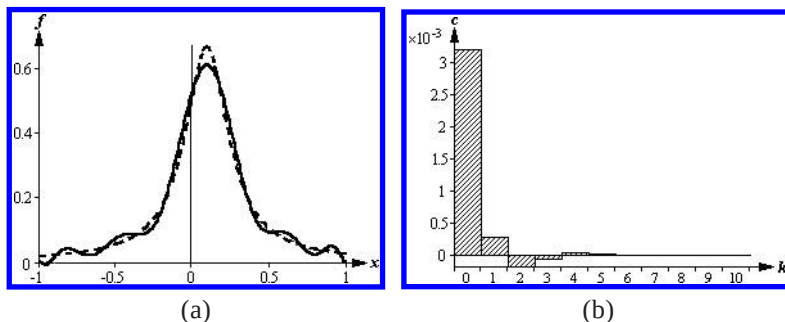


Figure 9.15. The function $f(x)$ and the partial sum of its Fourier-Legendre series in terms of associated Legendre functions $P_{4+k}^4(x)$. (a) The graph of $f(x)$ is shown by the dashed line, and the graph of the partial sum with $N = 10$ terms of the series is shown by the solid line. (b) Values of the coefficients c_k of the series.

Solution. The series is

$$f(x) = \sum_{k=0}^{\infty} c_k P_{4+k}^4(x),$$

where the coefficients c_k are

$$c_k = \frac{2k+9}{2} \frac{k!}{(k+8)!} \int_{-1}^1 \frac{1}{50x^2 - 10x + 2} P_{k+4}^4(x) dx.$$

The reader can find the values of these coefficients using the program FourierSeries.

Figure 9.15 shows a graph of $f(x)$ for this example.

9.10 Laplace's Equation in Spherical Coordinates and Spherical Functions

In this section we discuss solutions of Laplace's equation in spherical coordinates. This will bring us to very important spherical functions that require Legendre and associated Legendre functions. Notice that all these functions appear as a result of the inherent spherical symmetry of the problem being treated. To demonstrate the main concepts, we consider three classical applications in mathematical physics. For readers interested only

in mathematical content, the material of this section provides solutions of Laplace's equation (and the more general Helmholtz equation) in spherical coordinates and several classical mathematical boundary value problems.

9.10.1 Schrödinger Equation for a Central Potential

First, we discuss solutions to Schrödinger's equation that describe the wave-like behavior of an electron in an externally applied potential. As we shall see, for physical applications that are spherically symmetric in nature, the solutions will involve Legendre functions. The time-independent Schrödinger equation for the wave function $\psi(r, \theta, \varphi)$ is

$$\nabla^2 \psi + k^2 \psi = 0. \quad (9.94)$$

Here,

$$k^2 = \frac{2m_e}{\hbar^2}(E - V), \quad (9.95)$$

where m_e is the electron mass, $\hbar$ is Planck's constant, E is the electron's energy, and V is any applied potential felt by the electron. Equation (9.94) is known as the Helmholtz equation. In many important problems, the potential V depends only on distance from some central point. Physical applications include atoms with one electron (e.g., hydrogen; see [Figure 9.16](#)), once-ionized helium He^+ ($Z = 2$), doubly ionized lithium Li^{2+} ($Z = 3$), and so on. In these cases, $V = -Ze^2/r$ is the potential energy of the electron in an external electric potential $-Ze/r$ due to a nucleus of charge Ze , where Z is the atomic number (here and below we use the CGS system of units, in which the Coulomb constant is equal to 1).

For such problems, a spherical coordinate system using r , θ , and φ , where $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$, is the most convenient system

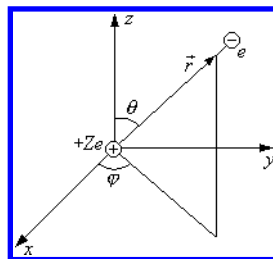


Figure 9.16. One electron atom.

in which to work. The potential energy, V , does not depend on the angles θ or φ , but the wave function $\psi(r, \theta, \varphi)$, generally speaking, still depends on all three variables.

In such a central potential where $V = V(r)$, a separation of variables is possible. Let us represent the wave function in the form

$$\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi). \quad (9.96)$$

In spherical coordinates, the Laplacian is

$$\nabla^2 = \nabla_r^2 + \frac{1}{r^2} \nabla_{\theta\varphi}^2 \quad (9.97)$$

where

$$\begin{aligned} \nabla_r^2 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}, & \nabla_{\theta\varphi}^2 &= \nabla_\theta^2 + \frac{1}{\sin^2 \theta} \nabla_\varphi^2, \\ \nabla_\theta^2 &= \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right), & \nabla_\varphi^2 &= \frac{\partial^2}{\partial \varphi^2} \end{aligned}$$

(see Appendix D for a discussion of the Laplacian in spherical coordinates). By substituting Equation (9.97) into Equation (9.94), we have

$$\left(\nabla_r^2 + \frac{1}{r^2} \nabla_{\theta\varphi}^2 \right) RY + k^2 RY = 0,$$

which may be rewritten as

$$Y \nabla_r^2 R + \frac{R}{r^2} \nabla_{\theta\varphi}^2 Y + k^2 RY = 0.$$

We then divide this by RY , and, because $k = k^2(r)$, we can separate the functions depending on r and θ and φ :

$$r^2 \frac{\nabla_r^2 R}{R} + k^2 r^2 = - \frac{\nabla_{\theta\varphi}^2 Y}{Y} = \lambda, \quad (9.98)$$

where λ is the separation of variables constant. Thus, we have two equations, one equation for the radial part $R(r)$ of the wave function,

$$\nabla_r^2 R + \left(k^2 - \frac{\lambda}{r^2} \right) R = 0, \quad (9.99)$$

and one equation for the angular function, $Y(\theta, \varphi)$, given by

$$\nabla_{\theta\varphi}^2 Y + \lambda Y = 0. \quad (9.100)$$

Equation (9.100) does not contain the potential $V(r)$; that is, it is the same for any central potential. The function Y depends on the angles only and can be thought of as a function on a sphere of unit radius.

We now continue the separation of variables and represent $Y(\theta, \varphi)$ in the form

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi). \quad (9.101)$$

Substituting Equation (9.101) into Equation (9.100) gives

$$\Phi \nabla_{\theta}^2 \Theta + \frac{\Theta}{\sin^2 \theta} \nabla_{\varphi}^2 \Phi + \lambda \Theta \Phi = 0.$$

By dividing by $\Theta\Phi$ and separating the functions of θ and φ , we obtain

$$\left(\frac{\nabla_{\theta}^2 \Theta}{\Theta} + \lambda \right) \sin^2 \theta = -\frac{\nabla_{\varphi}^2 \Phi}{\Phi} = m^2, \quad (9.102)$$

where m^2 is the separation of variables constant. Thus,

$$\nabla_{\theta}^2 \Theta + \left(\lambda - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0 \quad (9.103)$$

and

$$\nabla_{\varphi}^2 \Phi + m^2 \Phi = 0. \quad (9.104)$$

The last equation is the same as

$$\frac{d^2}{d\varphi^2} \Phi(\varphi) + m^2 \Phi(\varphi) = 0,$$

which has as a solution $\Phi = Ce^{im\varphi}$. Clearly, this solution must obey a periodicity condition in the angle φ ; in other words, $\Phi(\varphi + 2\pi) = \Phi(\varphi) = e^{2im\pi} = 1$, and we see that m is an *integer*: $m = 0, \pm 1, \pm 2, \dots$. Now the choice of sign in the right-hand side of Equation (9.102) is clear because we need the periodic function $\Phi(\varphi)$. For physical reasons, m is called the

magnetic quantum number. The functions $e^{im\varphi}$ are orthogonal on $[0, 2\pi]$ and, by using the normalization condition, we have

$$\int_0^{2\pi} \Phi_m^* \Phi_m d\varphi = \delta_{m'm}, \quad (9.105)$$

which gives $C = 1/\sqrt{2\pi}$, and we may write

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \quad m = 0, \pm 1, \pm 2, \dots \quad (9.106)$$

Clearly, these functions form a complete set of complex exponential functions on the interval $[0, 2\pi]$ and thus form the basis of a complex trigonometric Fourier series (see Chapter 1).

Now, we consider the equation for $\Theta(\theta)$. We may introduce a new variable, $x = \cos \theta$, with the range $-1 \leq x \leq 1$, which follows from the range $0 \leq \theta \leq \pi$, and change the notation to the conventional one: $\Theta(\theta) \Rightarrow P(x)$.

Reading Exercise. Show that $\nabla_\theta^2 \Theta = \frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} P(x) \right]$.

The equation for $\Theta(\theta)$ thus becomes the second-order differential equation for $P(x)$:

$$\frac{d}{dx} \left[(1 - x^2) \frac{dP}{dx} \right] + \left(\lambda - \frac{m^2}{1 - x^2} \right) P = 0. \quad (9.107)$$

As discussed earlier, this equation has the *associated Legendre functions* as its solutions. We previously obtained the values of the parameter m , which are $m = 0, \pm 1, \pm 2, \dots$. Regarding the parameter λ , we can use the results obtained for the associated Legendre functions, which, as we have seen, gives $\lambda = n(n + 1)$ with $n = 0, 1, 2, \dots$. In physical problems, the usual notation for n is l ; thus, $\lambda = l(l + 1)$, where l , in quantum mechanics, is called the angular (or orbital) quantum number because in classical physics it is related to the angular momentum. With those notations the solutions to the generalized Legendre equation are $P_l^m(x)$, or $P_l^m(\cos \theta)$. From this analysis, we have the important results for the allowed values of l and m :

$$l = 0, 1, 2, \dots, \quad m = 0, \pm 1, \pm 2, \dots, \quad |m| \leq l.$$

Therefore, for $l = 0$ the only possible value of m is 0; for $l = 1$, three values of m are possible; $m = 0, \pm 1$, and so on. When $l = 0$, we have $\lambda = 0$ and $P_0^0(x) = 1$, which is why the solution for $l = 0$ of the Schrödinger equation does not depend on the angles θ and φ ; in other words, it is spherically isotropic. Thus, applying the mathematics that we already know from the preceding examples, unrelated to quantum mechanics, we directly obtain a fundamental result: the angular momentum (and the magnetic moment related with it) are *quantized*; that is, only certain values of these quantities are allowed).

As we know, the associated Legendre functions are mutually orthogonal on the fundamental interval $-1 \leq x \leq 1$, so that

$$\int_{-1}^{+1} P_l^m(x) P_{l'}^m(x) dx = 0 \quad l \neq l', \quad (9.108)$$

and their squared norms are

$$\int_{-1}^{+1} \{P_l^m(x)\}^2 dx = \left[\frac{2}{2l+1} \frac{(l+|m|)!}{(l-|m|)!} \right]. \quad (9.109)$$

For a fixed value of m , the associated Legendre functions form a complete orthogonal sequence for the interval $-1 \leq x \leq 1$.

Combining the above results, it follows that the set of *spherical harmonics*, defined by

$$Y_l^m(\theta, \varphi) = N_l^m P_l^m(\cos \theta) e^{im\varphi}, \quad |m| < l, \quad l = 0, 1, \dots, \quad (9.110)$$

forms a complete orthogonal system of eigenfunctions. The corresponding eigenvalues $\lambda = l(l+1)$ with $l = 0, 1, 2, \dots$ are $(2l+1)$ -fold degenerate, corresponding to the $(2l+1)$ values that m can take: $m = -l, -l+1, \dots, l-1, l$.

It is most convenient to define the coefficients N_l^m in such a way that the spherical harmonics $Y_l^m(\theta, \varphi)$ have norms equal to 1, so that the radial and spatial wave functions are normalized:

$$\int |\psi(r, \theta, \varphi)|^2 dV = \int_0^\infty |R|^2 r^2 dr \cdot \int_0^\pi |\Theta|^2 \sin \theta d\theta \cdot \int_0^{2\pi} |\Phi|^2 d\varphi = 1. \quad (9.111)$$

Here, each integral equals 1 independently ($dV = r^2 dr \sin \theta d\theta d\varphi$ is the volume element in spherical coordinates). This leads to (apart from an arbitrary phase factor)

$$N_l^m = (-1)^{\frac{1}{2}(m+|m|)} \left[\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right]^{\frac{1}{2}}. \quad (9.112)$$

The reason the phase factor is chosen to be $(-1)^{(m+|m|)/2}$ in Equation (9.112) is that spherical harmonics are related to the angular momentum of a spherically symmetric system. In the study of these angular properties, the spherical harmonics occur naturally with these phase factors.

From Equation (9.112), it follows that

$$N_l^{-m} = (-1)^m N_l^m,$$

so that from Equation (9.110), we have

$$(Y_l^m(\theta, \varphi))^* = (-1)^m Y_l^{-m}(\theta, \varphi).$$

By using Equations (9.105), (9.108), (9.109), and (9.112), we can verify the orthogonality of the spherical harmonics:

$$\begin{aligned} (Y_{l'}^{m'}, Y_l^m) &= \int d\Omega (Y_{l'}^{m'}(\theta, \varphi))^* Y_l^m(\theta, \varphi) \\ &= N_{l'}^{m'} N_l^m \int_{-1}^{+1} dx P_{l'}^{m'}(x) P_l^m(x) \int_0^{2\pi} d\varphi e^{i(m-m')\varphi} = \delta_{ll'} \delta_{mm'}. \end{aligned} \quad (9.113)$$

Here, $d\Omega = \sin \theta d\theta d\varphi$ is the solid angle in steradians. Note that the coefficients N_l^m have been chosen in such a way that the spherical functions, $Y_l^m(\theta, \varphi)$, are normalized:

$$\|Y_l^m(\theta, \varphi)\|^2 = 1.$$

This result is reflected in Equation (9.113).

Because the spherical harmonics, $Y_l^m(\theta, \varphi)$, form a complete set of functions, any function $f(\theta, \varphi)$ for which the integral

$$\int d\Omega |f(\theta, \varphi)|^2$$

is finite can be expressed in a series given by

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l C_l^m Y_l^m(\theta, \varphi), \quad (9.114)$$

where the coefficients C_l^m are determined as

$$C_l^m = (Y_l^m, f) = \int d\Omega (Y_l^m(\theta, \varphi))^* f(\theta, \varphi). \quad (9.115)$$

The parity of spherical harmonics is easily found from the fact that an inversion that transforms $\vec{r}$ into $-\vec{r}$ is expressed in polar coordinates by the transformation

$$\theta \rightarrow \pi - \theta, \quad \varphi \rightarrow \varphi + \pi. \quad (9.116)$$

Because $\cos(\pi - \theta) = -\cos \theta$, we have

$$P_l^m(\cos[\pi - \theta]) = (-1)^{l-|m|} P_l^m(\cos \theta),$$

and

$$e^{im(\varphi+\pi)} = (-1)^{|m|} e^{im\varphi},$$

so that

$$Y_l^m(\pi - \theta, \varphi + \pi) = (-1)^l Y_l^m(\theta, \varphi); \quad (9.117)$$

that is, Y_l^m has parity $(-1)^l$.

Here, we list the first few spherical harmonics explicitly:

$$\begin{aligned} Y_0^0 &= \frac{1}{\sqrt{4\pi}}, \\ Y_1^0 &= \left(\frac{3}{4\pi}\right)^{1/2} \cos \theta, \\ Y_1^{\pm 1} &= \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin \theta e^{\pm i\varphi}, \\ Y_2^0 &= \left(\frac{5}{16\pi}\right)^{1/2} (3 \cos^2 \theta - 1), \\ Y_2^{\pm 1} &= \mp \left(\frac{15}{32\pi}\right)^{1/2} \sin 2\theta e^{\pm i\varphi}, \\ Y_2^{\pm 2} &= \left(\frac{15}{32\pi}\right)^{1/2} \sin^2 \theta e^{\pm 2i\varphi}. \end{aligned}$$

Polar plots of these functions are sketched in Figure 9.17.

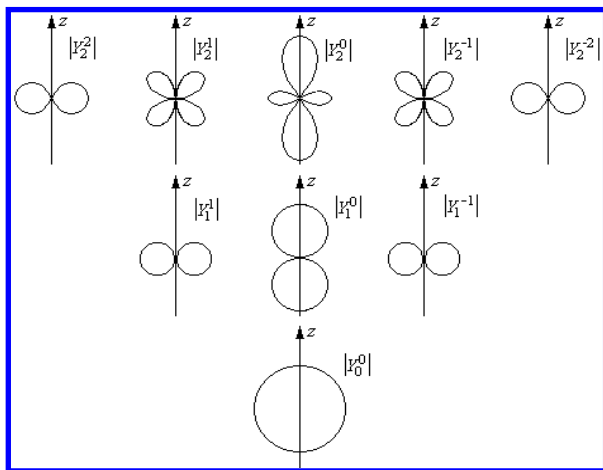


Figure 9.17. Polar plots of $|Y_l^m|$ versus θ in any plane throughout the z -axis for $l = 0, 1, 2$. The equality $|Y_l^m| = |Y_l^{-m}|$ is exhibited.

Now we return to the equation for the radial function $R(r)$:

$$\nabla_r^2 R + \left(k^2 - \frac{\lambda}{r^2} \right) R = 0. \quad (9.118)$$

To obtain the complete solution of the Schrödinger equation, we need to combine the spherical harmonics with the radial function. For a given value of l , Equation (9.118) possesses a series of radial eigenfunctions, $R_{kl}(r)$. The index k corresponds to the value of k in equation (9.118). For finite motion, such as that of an electron within an atom, the value of k becomes discrete; that is, the energy of an electron trapped in the potential energy well around an atom is quantized. In this case, instead of k , another index, n , which is the principal quantum number, is generally used. The finiteness of motion may be stated as $R(r \rightarrow \infty) = 0$, in the case of electrons in an atom, or $R(r = a) = 0$ for a spherical impenetrable potential well of radius a . These relations represent the boundary conditions for equation (9.99) and the quantized values of n appear in the process of solving the corresponding boundary value problem. As the result, the eigenvalues of the problem for function $R(r)$ give the discrete energy levels of the system, E_{kl} , or E_{nl} . The wave function thus depends on the quantum numbers n, l, m or k, l, m :

$$\psi_{nlm}(r, \theta, \varphi) = R_{nl}(r) Y_l^m(\theta, \varphi). \quad (9.119)$$

For $V = -Ze^2/r$, Equation (9.118) for $R(r)$ becomes the Laguerre equation, which has solutions known as Laguerre's polynomials. This is a bit beyond the scope of the present chapter, so here we consider the simpler case when $V = \text{const}$ on the interval $(0, a)$. With the substitution $x = kr$ we have $d/dr = k d/dx$ and $d^2/dr^2 = k^2 d^2/dx^2$, in which case Equation (9.118) for $R(x)$ becomes

$$\frac{d^2 R(x)}{dx^2} + \frac{2}{x} \frac{dR(x)}{dx} + \left[1 - \frac{l(l+1)}{x^2} \right] R(x) = 0. \quad (9.120)$$

This is the *spherical Bessel equation* we studied in Chapter 8. Equation (9.120) can be transformed to the Bessel cylindrical equation by the substitution $R(x) = y(x)/\sqrt{x}$, which gives the equation

$$\frac{d^2 y(x)}{dx^2} + \frac{1}{x} \frac{dy(x)}{dx} + \left[1 - \frac{(l+1/2)^2}{x^2} \right] y(x) = 0. \quad (9.121)$$

If we introduce the parameter $s = l + 1/2$, Equation (9.121) is the Bessel equation, which has the general solution

$$y(x) = C_1 J_s(x) + C_2 N_s(x) \quad (9.122)$$

where $J_s(x)$ and $N_s(x)$ are (cylindrical) Bessel and Neumann functions. Because $s = l + 1/2$, these functions are of half-integer order. Thus, we may write the solution of the equation for $R(x)$ as

$$R(x) = C_1 \frac{J_{l+1/2}(x)}{\sqrt{x}} + C_2 \frac{N_{l+1/2}(x)}{\sqrt{x}}. \quad (9.123)$$

If we are searching for a solution that is finite at $x = 0$, as it should be for the case of an electron in an atom, only the first term in Equation (9.123) is appropriate. Using the *spherical Bessel function*, $j_l(x)$, defined as

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) \quad (9.124)$$

we have

$$R(x) = C j_l(x),$$

or

$$R_{kl}(kr) = C j_l(kr). \quad (9.125)$$

The coefficient C can be found from the normalizing condition

$$\int_0^{\infty} |R_{kl}|^2 r^2 dr = 1. \quad (9.126)$$

As an example, consider a particle (for instance, an electron) captured in a spherically symmetrical potential well of radius a . On the interval $r < a$, we have $V = \text{const}$ (which we can take to be zero, $V = 0$), and we require that the particle cannot leave this region (i.e., $V = \infty$ for $r = a$). This can be considered as a simplified model of an atom with radius a and an impenetrable surface.

Inside the well, $r < a$, the electron moves freely; thus,

$$\psi_{klm}(r, \theta, \varphi) = C j_l(kr) Y_l^m(\theta, \varphi). \quad (9.127)$$

The wave function vanishes for $r > a$ because the particle cannot penetrate into a region where $V = \infty$. The continuity of the wave function leads to the boundary condition at $r = a$

$$j_l(ka) = 0. \quad (9.128)$$

Denoting the roots of the Bessel spherical function $j_l(x) = 0$ of order l as x_{nl} , Equation (9.128) gives

$$x_{nl} = a k_{nl} \quad (\text{where } n = 1, 2, 3 \dots \text{ numbers the roots}), \quad (9.129)$$

so

$$k_{nl} = \frac{x_{nl}}{a}. \quad (9.130)$$

With Equation (9.130) for k , we obtain the spectrum of energy of an electron inside the well:

$$E_{nl} = \frac{\hbar^2 k_{nl}^2}{2m_e} = \frac{\hbar^2}{2m_e a^2} x_{nl}^2. \quad (9.131)$$

Equation (9.131) shows that energy does not depend on the value of m , which is why each eigenvalue of the boundary value problem is $(2l + 1)$ -fold degenerated; that is, the same energy E_{nl} corresponds to $(2l + 1)$ different functions ψ_{nlm} . For $l = 0$, the roots of the equation $j_0(x) = 0$ are $x = n\pi$, so

$$E_{n0} = \frac{1}{2m_e} \left(\frac{n\pi\hbar}{a} \right)^2. \quad (9.132)$$

For the *ground state* $n = 1$, the lowest energy level is $E_{10} = (\pi\hbar/a)^2/2m_e$. The wave functions for the states with $l = 0$ (due to $|m| \leq l$, $m = 0$) are

$$\psi_{n00}(r, \theta, \varphi) = C j_0(kr) Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{2\pi a}} \frac{1}{\sqrt{4\pi}} \frac{\sin(n\pi r/a)}{n\pi r/a}, \quad (9.133)$$

where the coefficient $C = 1/\sqrt{2\pi a}$ was obtained from the normalization condition

$$\int_0^a R_{n0} r^2 dr = 1. \quad (9.134)$$

The quantum mechanical interpretation of the wave function, $\psi_{nlm}(r, \theta, \varphi)$, is that its modulus squared, $|\psi_{nlm}(r, \theta, \varphi)|^2$, determines the probability of finding a particle located in space at the point with coordinates r, θ, φ . Equation (9.133) for ψ_{n00} shows that this probability changes periodically with the distance from the center and there are some distances where the particle cannot be located at all, such as $r = a/2$ for the state with $n = 2$. Such “nodes” are separated by the maxima where a particle is the most likely can be located. These maxima correspond to the “orbits” in a classical interpretation. For the states with $l \neq 0$, such orbits have more sophisticated geometry, which is determined by the shape of $|Y_l^m(\theta, \varphi)|^2$.

Reading Exercise. For $l = 1$, plot three-dimensional graphs of this function for three possible values of $m = 0, \pm 1$.

The superposition of the wave functions $\psi_{nlm}(r, \theta, \varphi)$ with all possible values of quantum numbers, that is,

$$\sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{nlm} \psi_{nlm}(r, \theta, \varphi),$$

can also be considered, in which case the values $|c_{nlm}|^2$ give the probability of finding a particle in the state with the particular values of the quantum numbers n, l, m .

9.10.2 Oscillations of a Sphere: Sound Waves in a Spherical Cavity

It is a remarkable property of nature that the same mathematics often describe very different problems. For readers who feel that the development

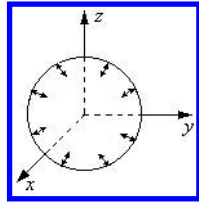


Figure 9.18. Sound waves inside a spherical drop.

of a quantum mechanical system here is too abstract, we now present a classical physics problem of spherical oscillations; for instance, gas oscillations inside a spherical cavity. This problem has different applications, one of which is acoustic waves propagating in a spherical cavity, which we discuss here; see Figure 9.18.

A deviation of density from the equilibrium value, described by the function $u(r, \theta, \varphi, t)$, obeys the wave equation (to simplify the problem, we neglect dissipation)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \quad (9.135)$$

where c is the speed of sound. The boundary condition on the cavity surface depends on the physics of the problem. Let us consider two situations.

- **Case 1.** First, we consider a vibrating “drop” of liquid surrounded by a low density gas (e.g., the atmosphere). In this case, the density at the surface of the drop does not change, thus its deviation from equilibrium is zero. Because the deviation of pressure and the deviation of density are proportional to each other, the boundary condition at the surface is

$$u(a, \theta, \varphi, t) = 0, \quad (9.136)$$

where a is the drop's radius in an equilibrium position and $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$.

The other boundary conditions are obvious:

$$u(r, \theta, \varphi + 2\pi, t) = u(r, \theta, \varphi, t) \quad (9.137)$$

(i.e., periodicity); and the values $u(0, \theta, \varphi, t)$, $u(r, 0, \varphi, t)$, and $u(r, \pi, \varphi, t)$ are finite. The initial conditions are

$$u(r, \theta, \varphi, 0) = f_1(r, \theta, \varphi), \quad u'_t(r, \theta, \varphi, 0) = f_2(r, \theta, \varphi), \quad (9.138)$$

where $f_{1,2}(r, \theta, \varphi)$ are given functions.

We already have experience in solving these kinds of problems. Clearly, we can separate the variables in three steps:

$$1. \quad u(r, \theta, \varphi, t) = \psi(r, \theta, \varphi)T(t), \quad (9.139)$$

$$2. \quad \psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi), \quad (9.142)$$

$$3. \quad Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi). \quad (9.141)$$

Completing the first step, we obtain

$$\frac{T''}{c^2 T} = \frac{\nabla^2 \psi}{\psi} = -k^2, \quad (9.142)$$

and thus,

$$T'' + k^2 c^2 T = 0, \quad (9.143)$$

$$\nabla^2 \psi + k^2 \psi = 0. \quad (9.144)$$

The choice of sign in the right side of Equation (9.142) is obvious because we want to obtain periodic motion. The only thing new compared to the problem in Section 9.10.1 is the equation for the function $T(t)$. The eigenfunctions of the boundary value problem for function ψ are

$$\psi_{nlm}(r, \theta, \varphi) = C_{nlm} j_l(k_{nl} r) Y_l^m(\theta, \varphi), \quad (9.145)$$

and the eigenvalues are

$$k_{nl} = x_{nl}/a, \quad (9.146)$$

where x_{nl} is the n th root of the equation $j_l(x) = 0$. Thus,

$$T_{nl}(t) = a_{nl} \cos \omega_{nl} t + b_{nl} \sin \omega_{nl} t, \quad \omega_{nl} = c k_{nl}. \quad (9.147)$$

(Note that $T_{nl}(t)$ also can be written in a complex form as $T_{nl}(t) = c_{nl} e^{-i\omega_{nl} t}$.) Finally, the solution with the values l , n , and m is

$$\psi_{nlm}(r, \theta, \varphi) = (A_{nlm} \cos \omega_{nl} t + B_{nlm} \sin \omega_{nl} t) j_l(k_{nl} r) Y_l^m(\theta, \varphi) \quad (9.148)$$

(where coefficients A_{nlm} and B_{nlm} include all other coefficients). Here $l = 0, 1, 2, \dots$; $m = -l, -l+1, \dots, l-1, l$; and $n = 1, 2, 3, \dots$

This finishes the investigation of the patterns of standing waves inside a spherical cavity (a “drop”) and their frequencies. Most often, the frequencies are the subject of experimental study. Because the roots of the equation $j_l(x) = 0$ are known (for $l = 0$ the positive roots of $j_0(x) = \sin x/x = 0$ are simply $x_{nl} = n\pi$), we can immediately obtain these frequencies as $\omega_{nl} = cx_{nl}/a$.

Reading Exercise. Tabulate the frequencies, ω_{nl} , for several values of l and n and discuss the degeneracy (the number of eigenfunctions corresponding to the same frequency) of several of the lowest frequencies.

In some cases, the function

$$u(r, \theta, \varphi, t) = \sum_n \sum_l \sum_m \psi_{nlm}$$

might also be of interest in spite of the fact that it is not easy to set up an experiment to measure it. If, for instance, two gases of different density are initially separated by a diaphragm that is suddenly removed, the problem of finding $u(\vec{r}, t)$ can be modeled by taking $f_2 = 0$ (zero velocity) and f_1 as a piecewise constant function.

Generally, the coefficients A_{nlm} and B_{nlm} can be determined by using the initial conditions. For these purposes the functions $f_1(r, \theta, \varphi)$ and $f_2(r, \theta, \varphi)$ should be expanded in a Fourier-Legendre-Bessel series. Clearly $A_{nlm} = 0$ if $f_1(r, \theta, \varphi) = 0$, and $B_{nlm} = 0$ if $f_2(r, \theta, \varphi) = 0$. Technically a problem appears simpler when the functions $f_{1,2}(r)$ do not depend on the angles.

- **Case 2.** Now let us study sound waves generated inside a spherical cavity with a rigid surface of radius a . The problem can be formulated in terms of the velocity potential ($\mathbf{v} = -\nabla u$), which satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \quad (9.149)$$

where c is the speed of sound. The boundary condition at the surface is

$$u'_r(a, \theta, \varphi, t) = 0, \quad (9.150)$$

which follows from the fact that at the surface the normal component of velocity is equal to zero (molecules cannot leave the cavity). The other boundary conditions are the same as Case 1: the initial conditions should specify the initial distribution of velocity potential and its time derivative (which is proportional to change in density).

Because this problem is very similar to that of Case 1, its general solution is given by Equation (9.148). The difference is that the allowed values of frequencies are determined from the boundary condition

$$\left. \frac{d}{dr} j_l(kr) \right|_{r=a} = 0. \quad (9.151)$$

We leave it to the reader as a simple reading exercise to repeat all the steps of Case 1. A complete development involves tabulating the roots of the equation $j'_l(x) = 0$ for $l = 0, 1, 2$ and finding the lowest frequencies. Show that the case $l = 0, n = 1$ leads to $\psi_{nlm} = \text{const}$ which, because $v = -\nabla u$, does not result in a physical sound wave. The mode with the lowest frequency, $l = 1, n = 1$, has a triple degeneracy with the values of $m = 0, \pm 1$.

Physical applications involving vibrations are often formulated to investigate eigenfrequencies (the normal mode vibrations) of a cavity (or body). These frequencies can often be easily measured, which allows a comparison between theory and experiment. Applications are numerous, from designing sound emitters so that they respond correctly at the necessary frequencies, to designing rocket fuel tanks to avoid resonance.

9.10.3 Cooling of a Solid Sphere

Consider a solid sphere with an initial temperature distribution depending only on r :

$$u(r, t)|_{t=0} = u_0(r). \quad (9.152)$$

As shown in Figure 9.19, a sphere is placed in the surrounding medium with constant temperature u_1 , and the surface cooling obeys Newton's law:

$$\left. \frac{\partial u}{\partial r} + h(u - u_1) \right|_{r=b} = 0 \quad (9.153)$$

(here we use the letter b for the radius to not confuse it with the constant a in the heat equation). From a physical point of view, the temperature of

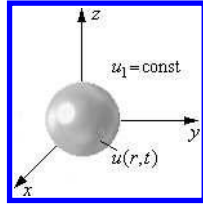


Figure 9.19. Heat conduction in a spherical body.

the sphere does not depend on the angles, in which case the temperature distribution, $u(r, t)$, will be a solution to the heat equation

$$\frac{\partial u(r, t)}{\partial t} = a^2 \nabla_r^2 u(r, t). \quad (9.154)$$

We separate the variables by using the expression

$$u(r, t) = R(r)T(t), \quad (9.155)$$

and with

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr},$$

we obtain

$$T' + a^2 \lambda T = 0, \quad (9.156)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \lambda R = 0. \quad (9.157)$$

From a physical point of view, the separation constant $\lambda > 0$ because if $\lambda = 0$, both T and R are constants (which is not interesting), and when $\lambda < 0$, T grows exponentially. For $\lambda > 0$, we have

$$T(t) = e^{-a^2 k^2 t}, \quad R(r) = j_0(kr) = \frac{\sin kr}{kr}, \quad \text{where } k = \sqrt{\lambda}. \quad (9.158)$$

The eigenvalues, k , are obtained by applying the boundary condition. To avoid dealing with a nonhomogeneous boundary condition in Equation (9.153), we replace $u - u_1$ by u , in which case the boundary condition becomes

$$\left. \frac{dj_0(kr)}{dr} + h j_0(kr) \right|_{r=b} = 0.$$

From here, it follows that

$$-\frac{\sin kb}{kb^2} + \frac{\cos kb}{b} + h \frac{\sin kb}{kb} = 0,$$

which leads to the equation

$$(\tan kb)/kb = 1/(1 - hb). \quad (9.159)$$

Setting $x = kb$ and $\mu = 1/(1 - hb)$, and plotting the curves $y = \tan x$ and $y = \mu x$, we can find the roots of Equation (9.159) from the intersections of these curves (we need only find positive roots). This procedure gives the values k_n ($n = 1, 2, 3, \dots$). The solution to the problem (eigenfunction) is

$$u(r, t) = u_1 + \sum_{n=1}^{\infty} A_n \frac{\sin k_n r}{k_n r} e^{-a^2 k_n^2 t}. \quad (9.160)$$

The coefficients A_n can be found from the initial condition in Equation (9.152):

$$u_0(r) - u_1 = \sum_{n=1}^{\infty} A_n j_0(k_n r). \quad (9.161)$$

Using the orthogonality (with the weight function r^2) of Bessel functions $j_0(k_n r)$ on the interval $(0, b)$, we can obtain A_n by multiplying Equation (9.161) by $r^2 j_0(k_n r)$ and integrating over $(0, b)$:

$$A_n = \frac{\int_0^b [u_0(r) - u_1] r^2 j_0(k_n r) dr}{\int_0^b [j_0(k_n r)] r^2 dr}.$$

The denominator (the norm of $j_0(k_n r)$ squared) can be obtained from formulas in Chapter 8, or by evaluating the integral directly:

$$\int_0^b [j_0(k_n r)] r^2 dr = \frac{1}{k_n^2} \int_0^b \sin^2 k_n r dr = \frac{1}{4k_n^2} (2k_n b - \sin 2k_n b).$$

With the obtained coefficients A_n , Equation (9.160) gives the solution to the problem.

Problems

9.1. Using the recurrence formula in Equation (9.12), show that the coefficient of x^n (the highest degree term) in $P_n(x)$ is

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!}.$$

9.2. Prove Rodrigues's formula by using the solution of Problem 9.1.

Hint. Show that $z = (x^2 - 1)^n$ satisfies the equation

$$(1 - x^2) \frac{d^{n+2}z}{dx^{n+2}} - 2x \frac{d^{n+1}z}{dx^{n+1}} + n(n+1) \frac{d^n z}{dx^n} = 0,$$

which means that $d^n z/dx^n$ is a solution of the Legendre equation. Therefore, this solution must be a constant multiple of $P_n(x)$: $d^n z/dx^n = c P_n(x)$. Then, comparing the coefficients of highest degree on both sides of this relation, one can obtain $(d^n/dx^n)(x^2 - 1)^n = 2^n n! P_n(x)$, which is Rodrigues's formula.

9.3. Prove the useful recurrence formulas

$$P_{n-1}(x) = x P_n(x) + \frac{1-x^2}{n} P'_n(x), \quad n \geq 1,$$

$$P_{n+1}(x) = x P_n(x) - \frac{1-x^2}{n+1} P'_n(x), \quad n \geq 0.$$

9.4. The integral $I_{nm} = \int_{-1}^1 x P_n(x) P_m(x) dx$ is important in applications (for example, in quantum mechanics problems). Show that

$$I_{nm} = \frac{2(n+1)}{(2n+1)(2n+3)} \delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)} \delta_{m,n-1}.$$

9.5. Prove that

$$\int_{-1}^1 x P_n(x) P'_n(x) dx = \frac{2n}{2n+1}.$$

9.6. Integrate the recurrence relation in Equation (9.39) by parts to show that

$$I_n = \int_0^1 P_n(x) dx = \frac{P_{n-1}(0)}{n+1}, \quad n \geq 1.$$

This integral is useful in applications. With Equation (9.21), we can find that for odd values $n = 1, 3, 5, \dots$:

$$I_n = (-1)^{(n-1)/2} \frac{(n-1)!}{2^n \left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)!} = \frac{\Gamma(3/2)}{\Gamma(3/2 + n/2)\Gamma(1 - n/2)}$$

(it is clear that $I_1 = 1$), $I_n = 0$ if n is even.

9.7. Let us present the Legendre equation as the eigenvalue problem $Ly(x) = \lambda y(x)$, where

$$L = -\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right].$$

Check that operator L is Hermitian on $[-1, 1]$, or

$$\int_{-1}^1 P_m(x) L P_n(x) dx = \int_{-1}^1 P_n(x) L P_m(x) dx.$$

In each of Problems 9.8 through 9.18, expand the function $f(x)$ in the Fourier-Legendre series. Graph the function and the first few partial sums of this expansion (you may use the program FourierSeries, directions for which are found in Appendix E). Note that (and explain why) for some functions, just a few terms provide a good approximation to the function on $(-1, 1)$, whereas for other functions, more terms are needed. Verify Parseval's equality.

9.8. $f(x) = 2x - 1$.

9.9. $f(x) = 1 - x^2$.

9.10. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

9.11. $f(x) = \begin{cases} 0, & -1 < x < 0, \\ x, & 0 < x < 1. \end{cases}$

9.12. $f(x) = \arcsin x$.

9.13. $f(x) = \begin{cases} 0, & -1 < x < 0, \\ \sqrt{1-x}, & 0 < x < 1. \end{cases}$

9.14. $f(x) = \sin \pi x$.

9.15. $f(x) = e^x$.

9.16. $f(x) = \sin^2 x$.

9.17. $f(x) = \cos \pi x - \sin \pi x$.

9.18. $f(x) = \cos ax$, $a = \text{const}$.

9.19. If we need to expand a function $f(x)$ defined on the interval (a, b) in terms of Legendre polynomials, show that the transformation $z = (2x - a - b)/(b - a)$ maps $f(x)$ in the function $f(z)$ onto the interval $(-1, 1)$. As an example, expand the function $f(x) = \sin \pi x$, $x \in [-0.5, 1]$, in terms of Legendre polynomials.

9.20. Expand the function $f(x) = 1 - x^2/4$, $-2 \leq x \leq 2$, in terms of Legendre polynomials.

9.21 through 9.32. For each of Problems 9.8 through 9.18, expand the function $f(x)$ in the Fourier-Legendre functions $P_n^m(x)$ on $-1 \leq x \leq 1$ for $m = 1$ and $m = 2$. For each problem, graph the given function and the first few partial sums of its expansion.

In each of Problems 9.33 through 9.35, expand the function $f(\theta, \varphi)$ in spherical harmonics $Y_l^m(\theta, \varphi)$. Find coefficient A and the eigenvalues l and m that contribute to the expansion.

9.33. $f(\theta, \varphi) = A \sin \theta \cos \varphi$.

Answer. $A = \sqrt{3/4\pi}$; explain why only the value $l = 1$ contributes to the expansion.

9.34. $f(\theta, \varphi) = A(3 \cos^2 \theta - 1 + \sin 2\theta \cdot \cos \varphi)$. Explain why $l = 2$.

9.35. $f(\theta, \varphi) = A \sin \theta \cos 2\varphi$. Explain why in this case the convergence is slower than in Problems 9.33 and 9.34.

Hint. Compare with the expansion of $\sin x$ in cosine functions on $[0, 2\pi]$.

The following problems contain material from several preceding chapters and present a good conclusion to the book. From these examples, we hope that you can see how far we have come in mathematical sophistication from the beginning of the book.

9.36. Show that a general solution of the Laplace equation

$$\nabla^2 u = 0 \quad (9.162)$$

inside a sphere, $r \leq a$, can be expressed in the form

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^n (A_{nm} \cos m\varphi + B_{nm} \sin m\varphi) r^n P_n^m(\cos \theta). \quad (9.163)$$

Hint. Separate the variables $u(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ and show that a general solution for function $R(r)$ is

$$R(r) = C_1 r^n + C_2 \frac{1}{r^{n+1}}. \quad (9.164)$$

Notice that the function $\Phi_m(\varphi)$ can be used in two forms: complex, as in Equation (9.106), $\Phi_m(\varphi) \sim e^{im\varphi}$, $m = 0, \pm 1, \pm 2, \dots$; and real, in which case the fundamental solutions are $\cos m\varphi$ and $\sin m\varphi$. Clearly, in the latter case, m can be chosen to be nonnegative, $m = 0, 1, 2, \dots$. In the solution shown in Equation (9.163), we present this real form for the function $\Phi_m(\varphi)$.

Another useful way to write the solution is

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n Y_n(\theta, \varphi), \quad r \leq a, \quad (9.165)$$

where

$$Y_n(\theta, \varphi) = \sum_{m=0}^n (A_{nm} \cos m\varphi + B_{nm} \sin m\varphi) P_n^m(\theta) \quad (9.166)$$

is called the *spherical function*.

9.37. Find a general solution of the Laplace equation outside a sphere, $r \geq a$.

Answer.

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} Y_n(\theta, \varphi), \quad r \geq a. \quad (9.167)$$

9.38. Solve the Dirichlet problem with boundary condition $u(a, \theta, \varphi) = f(\theta, \varphi)$ for the Laplace equation inside and outside a sphere (find the formulas for A_{lm} and B_{lm}).

Answer.

$$A_{mn} = \frac{1}{\|Y_n^m\|^2} \int_0^{2\pi} \int_0^\pi f(\varphi, \theta) P_n^m(\cos \theta) \cos m\varphi \sin \theta d\varphi d\theta,$$

$$B_{mn} = \frac{1}{\|Y_n^m\|^2} \int_0^{2\pi} \int_0^\pi f(\varphi, \theta) P_n^m(\cos \theta) \sin m \varphi \sin \theta d\varphi d\theta,$$

where the coefficients are the same for the inner and outer problems.

9.39. As an application of Problem 9.38, solve the following problem: Two conducting hemispherical shells of radius a are attached by a circle of insulating material. One hemisphere is kept at the potential $+V$, and the other at the potential $-V$. Find the potential inside the sphere.

Answer. The electric potential satisfies the Laplace equation $\nabla^2 u = 0$, and it is clear that our problem does not depend on the azimuth angle φ . In this case, the only possible value of m is zero, $m = 0$, and the function $\Phi_0(\varphi) = \text{const}$. Thus, using the result of Problem 9.38, we can write

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta), \quad (9.168)$$

where the coefficients are obtained by using the boundary condition $u(a, \theta) = f(\theta)$:

$$A_n = \frac{1}{a^n} \frac{2n+1}{2} \int_0^\pi f(\theta) P_n(\cos \theta) \sin \theta d\theta. \quad (9.169)$$

Function

$$f(\theta) = \begin{cases} +V, & 0 \leq \theta < \frac{\pi}{2}, \\ -V, & \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

is antisymmetric with respect to the interchange $\theta \leftrightarrow \pi - \theta$. Since $\sin(\pi - \theta) = \sin \theta$, $\cos(\pi - \theta) = -\cos \theta$, and $P_n(\cos \theta)$ contains only odd powers of $\cos \theta$ for odd values of l and even powers for even values of l , it follows that the coefficients $A_n = 0$ for even values of l . For odd values of l ,

$$\int_0^\pi f(\theta) P_n(\cos \theta) \sin \theta d\theta = 2V \int_0^{\pi/2} P_n(\cos \theta) \sin \theta d\theta = 2V \int_0^1 P_n(x) dx.$$

The integral $I_n = \int_0^1 P_n(x) dx$ was evaluated in Problem 9.6; thus, we obtain the coefficients A_n :

$$A_n = \frac{V}{a^n} (2n+1) \frac{\Gamma(3/2)}{\Gamma(3/2 + n/2) \Gamma(1 - n/2)}.$$

The resultant series for $u(r, \theta)$,

$$u(r, \theta) = V \sum_{n=1,3,5,\dots}^{\infty} \left(\frac{r}{a}\right)^n \frac{(2n+1)\sqrt{\pi/2}}{\Gamma[(3+n)/2]\Gamma[(2-n)/2]} P_n(\cos \theta), \quad (9.170)$$

satisfies all conditions of the problem and uniformly converges for all $0 \leq r < a$, although when r is close to a the convergence is slow. With the help of the program FourierSeries, plot the potential distribution within the sphere and the equipotential lines, and investigate the convergence of the series at different values of r .

9.40. Similar to Problem 9.39, consider two conducting spheres, each of which is a combination of two hemispheres, insulated from each other. The spheres have radii a and b and a common center. One hemisphere of the outer sphere is kept at the potential $+V$, the other hemisphere at the potential $-V$. The potential is reversed for the inner sphere. Find the potential between the spheres, inside and outside.

Hint. The equations are linear, so apply the superposition principle.

9.41. Solve the Neumann problem for the Laplace equation inside and outside a sphere. The boundary condition is

$$\left. \frac{\partial u(r, \theta, \varphi)}{\partial r} \right|_{r=a} = f(\theta, \varphi).$$

Answer.

1. For the inner problem,

$$u(r, \varphi, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{r^n}{n r_0^{n-1}} (A_{mn} \cos m\varphi + B_{mn} \sin m\varphi) P_n^m(\cos \theta) + C, \quad (9.171)$$

2. For the outer problem,

$$u(r, \varphi, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{r_0^{n+2}}{(n+1) r^{n+1}} (A_{mn} \cos m\varphi + B_{mn} \sin m\varphi) P_n^m(\cos \theta) + C. \quad (9.172)$$

Here, C is an arbitrary constant and the function $f(\theta, \varphi)$ should satisfy the restriction (see Chapter 7)

$$\int_0^{2\pi} \int_0^{\pi} f(\varphi, \theta) \sin \theta d\varphi d\theta = 0. \quad (9.173)$$

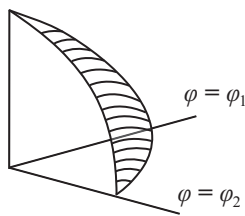


Figure 9.20. Part of a sphere bounded by azimuth angles φ_1 and φ_2 .

9.42. Solve the Laplace equation for a quarter of a sphere ($0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq \pi$) for the following Dirichlet boundary conditions: $u(a, \theta, \varphi) = f(\theta, \varphi)$, $u|_{\varphi=0, \pi} = 0$, $u|_{\theta=\pi/2} = 0$, and $u|_{\theta=0}$ is finite. (See Figure 9.20 for this and the following problems.)

Answer.

$$\begin{aligned}
 u(r, \varphi, \theta) = & \sum_{n=1}^{\infty} \sum_{m=1}^n A_{mn} \left(\frac{r}{r_0} \right)^{2n+1} P_{2n+1}^{2m}(\cos \theta) \sin 2m \varphi \\
 & + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} B_{mn} \left(\frac{r}{r_0} \right)^{2n} P_{2n}^{2m+1}(\cos \theta) \sin(2m+1)\varphi,
 \end{aligned} \tag{9.174}$$

where the coefficients are

$$A_{mn} = \frac{1}{\|P_{2n+1}^{2m}(\cos \theta) \sin 2m \varphi\|^2} \int_0^{\pi} \int_0^{\pi/2} f(\varphi, \theta) P_{2n+1}^{2m}(\cos \theta) \sin 2m \varphi \sin \theta d\varphi d\theta.$$

The value of m should be an integer to satisfy the periodicity condition in the azimuth angle φ (i.e., $\sin m \varphi$ is 0 at the ends of the interval, $0 \leq \varphi \leq \pi$). Two sums with sines of even ($2m$) and odd ($2m+1$) factors in the arguments in Equation (9.174) appear in order that the functions $P_n^m(\cos \theta)$ be zero at both ends of the interval $0 \leq \theta \leq \pi/2$ for arbitrary φ . This is possible only if we have different parity of the indices in the functions P_n^m . Note that the parity of $P_n^m(\cos \theta)$ relative to the point $\cos \theta = 0$ (i.e., the equatorial plane, $\theta = \pi/2$) coincides with the parity of $n - m$.

Hint. Divide by a factor of 4 when you calculate the norm in the coefficients above.

9.43. Solve Problem 9.42 for a quarter of a sphere ($0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq \pi/2$) with similar boundary conditions.

Answer.

$$u(r, \varphi, \theta) = \sum_{n=1}^{\infty} \sum_{m=1}^n A_{mn} \left(\frac{r}{r_0} \right)^{2n} P_n^{2m}(\cos \theta) \sin 2m\varphi, \quad (9.175)$$

where the coefficients are

$$A_{mn} = \frac{1}{\|P_n^{2m}(\cos \theta) \sin 2m\varphi\|^2} \int_0^{\pi/2} \int_0^{\pi} f(\varphi, \theta) P_n^{2m}(\cos \theta) \sin 2m\varphi \sin \theta d\varphi d\theta.$$

In this problem, we need to keep only $\sin 2m\varphi$ to satisfy the boundary conditions.

9.44. Solve the Dirichlet problem for an eighth of a sphere ($0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq \pi/2$) with the following boundary conditions: $u(a, \theta, \varphi) = f(\theta, \varphi)$, $u|_{\varphi=0, \pi/2} = 0$, $u|_{\theta=\pi/2} = 0$, and $u|_{\theta=0}$ is finite.

Answer.

$$u(r, \varphi, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^n A_{mn} \left(\frac{r}{r_0} \right)^{2n+1} \sin 2m\varphi P_{2n+1}^{2m}(\cos \theta), \quad (9.176)$$

where the coefficients are

$$A_{mn} = \frac{1}{\|P_{2n+1}^{2m}(\cos \theta) \sin 2m\varphi\|^2} \int_0^{\pi/2} \int_0^{\pi/2} f(\varphi, \theta) P_{2n+1}^{2m}(\cos \theta) \sin 2m\varphi \sin \theta d\varphi d\theta.$$

9.45. Solve the Laplace equation in a hemisphere ($0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$) for the following Dirichlet boundary conditions: $u(a, \theta, \varphi) = f(\theta, \varphi)$, $u|_{\theta=\pi/2} = 0$, and $u|_{\theta=0}$ is finite, on the equatorial plane $u(a, \theta = \pi/2, \varphi) = \text{const}$.

9.46 through 9.49. Solve problems 9.42 through 9.45 for the heat equation. Two major differences compared to the solution of the Laplace equation above are: the solution of the equation for $R(r)$ is a spherical Bessel function $j_n(k_{nl}r)$, and the solution of the equation for $T(t)$ is $T_{nlm}(t) = A_{nlm} e^{-\omega_{nl}t}$. Coefficients A_{nlm} can be found from the initial condition.

9.50. Find the charge distribution on a grounded conducting spherical surface induced by a point-like charge q located (a) inside the sphere and (b) outside the sphere.

Answer. Place the origin of the spherical coordinate system in the center of the sphere and denote the sphere radius as a . Put the charge q at the point $r = r_0$,

$\theta = 0$ on the z -axis; obviously there is symmetry in the azimuth angle φ and thus this variable can be omitted. If R is the distance between some point in space (r, θ) and the charge at $(r_0, 0)$, the electric potential due to this charge at the point (r, θ) is $V = q/R = q/|\vec{r} - \vec{r}_0|$ (the Coulomb constant equals 1 in the CGS system). As we know, $1/R = 1/|\vec{r} - \vec{r}_0|$ can be resolved in a Fourier-Legendre series:

$$\frac{1}{R} = \begin{cases} \frac{1}{r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos \theta), & r < r_0, \\ \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(\cos \theta), & r > r_0. \end{cases} \quad (9.177)$$

As we also know, the potential of a charged sphere is described by the Laplace equation and can be presented as a series in Legendre polynomials (see Problems 9.36 and 9.37). Thus, the total potential of the system consisting of the point-like charge q and the sphere is

$$u(r, \theta) = \frac{q}{R} + \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos \theta), \quad r < a, \quad (9.178)$$

$$u(r, \theta) = \frac{q}{R} + \sum_{n=0}^{\infty} B_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos \theta), \quad r > a. \quad (9.179)$$

Note that the potential is not zero inside the (conducting) sphere because the charge distribution on the surface is not uniform. Because the sphere is grounded, the total potential on the surface is equal to zero (the potential due to surface charges only is not zero); thus, the coefficients A_n and B_n can be determined from the boundary condition, $u|_{r=a} = 0$. This gives

$$A_n = -q \frac{a^n}{r_0^{n+1}}, \quad B_n = -q \frac{r_0^n}{a^{n+1}}.$$

Surface charge density can be found from

$$\sigma = -\frac{1}{4\pi} \left(\frac{\partial u}{\partial r} \right)_{r=a}. \quad (9.180)$$

Finally, we obtain the following:

(a) if $r_0 < a$, the potential

$$u(r, \theta) = \begin{cases} q \sum_{n=0}^{\infty} \left(\frac{r^n}{r_0^{n+1}} - \frac{r_0^n r^n}{a^{2n+1}} \right) P_n(\cos \theta), & r < r_0, \\ q \sum_{n=0}^{\infty} \left(\frac{r_0^n}{r^{n+1}} - \frac{r_0^n r^n}{a^{2n+1}} \right) P_n(\cos \theta), & r > r_0, \end{cases} \quad (9.181)$$

and surface charge density

$$\sigma = -\frac{q}{4\pi} \sum_{n=0}^{\infty} (2n+1) \frac{r_0^n}{a^{n+2}} P_n(\cos \theta); \quad (9.182)$$

(b) if $r_0 > a$, the potential

$$u(r, \theta) = \begin{cases} q \sum_{n=0}^{\infty} \left(\frac{r^n}{r_0^{n+1}} - \frac{a^{2n+1}}{r_0^{n+1} r^{n+1}} \right) P_n(\cos \theta), & r < r_0, \\ q \sum_{n=0}^{\infty} \left(\frac{r_0^n}{r^{n+1}} - \frac{a^{2n+1}}{r_0^{n+1} r^{n+1}} \right) P_n(\cos \theta), & r > r_0, \end{cases} \quad (9.183)$$

and surface charge density

$$\sigma = -\frac{q}{4\pi} \sum_{n=0}^{\infty} (2n+1) \frac{a^{n-1}}{r_0^{n+1}} P_n(\cos \theta). \quad (9.184)$$

Note that a less formal and more physical solution can be obtained with the image method.

Reading Exercise. Check that the total charge on the sphere (i.e., the surface integral of $\sigma(r, \theta, \varphi)$ over the sphere with radius a) is $Q = -q$ when $r_0 < a$ and $Q = -qa/r_0$ when $r_0 > a$.

9.51. Solve the problem of cooling of a sphere of radius b if there is heat exchange with the environment at zero temperature. The initial temperature of the sphere is

$$u|_{t=0} = f(r, \theta, \varphi), \quad 0 \leq r < b, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi.$$

Answer. The solution to the boundary value problem

$$\frac{\partial u(r, \theta, \varphi, t)}{\partial t} = a^2 \nabla^2 u(r, \theta, \varphi, t),$$

$$\left[\frac{\partial u}{\partial r} + hu \right]_{r=b} = 0, \quad u|_{t=0} = f(r, \theta, \varphi)$$

is

$$u(r, \theta, \varphi, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^n \exp \left[- \left(\frac{a \mu_m^{(n)}}{b} \right)^2 t \right] \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}} \left(\frac{\mu_m^{(n)} r}{b} \right) \quad (9.185)$$

$$\times P_n^k(\cos \theta) (A_{mnk} \cos k\varphi + B_{mnk} \sin k\varphi),$$

where $\mu_m^{(n)}$ are positive roots of equation

$$\mu_m^{(n)} J'_{n+\frac{1}{2}} \left(\mu_m^{(n)} \right) + \left(r_0 h - \frac{1}{2} \right) J_{n+\frac{1}{2}} \left(\mu_m^{(n)} \right) = 0.$$

The coefficients are

$$A_{mnk} = \frac{\int_0^{r_0} \int_0^\pi \int_0^{2\pi} f(r, \theta, \varphi) r^{\frac{3}{2}} J_{n+\frac{1}{2}} \left(\frac{\mu_m^{(n)} r}{r_0} \right) \sin \theta P_n^k(\cos \theta) \cos k\varphi dr d\theta d\varphi}{\varepsilon_k \frac{\pi b^2 (n+k)!}{(2n+1)(n-k)!} \left[1 + \frac{(bh+n)(bh-n-1)}{\mu_m^{(n)2}} \right] J_{n+\frac{1}{2}}^2 \left(\mu_m^{(n)} \right)},$$

$$B_{mnk} = \frac{\int_0^{r_0} \int_0^\pi \int_0^{2\pi} f(r, \theta, \varphi) r^{\frac{3}{2}} J_{n+\frac{1}{2}} \left(\frac{\mu_m^{(n)} r}{r_0} \right) \sin \theta P_n^k(\cos \theta) \sin k\varphi dr d\theta d\varphi}{\varepsilon_k \frac{\pi b (n+k)!}{(2n+1)(n-k)!} \left[1 + \frac{(bh+n)(bh-n-1)}{\mu_m^{(n)2}} \right] J_{n+\frac{1}{2}}^2 \left(\mu_m^{(n)} \right)}.$$

Here $\varepsilon_k = 2$ for $k = 0$ and $\varepsilon_k = 1$ for $k \neq 0$.

9.52. Describe a potential flow of incompressible inviscid fluid around the stationary sphere of radius b .

Answer. Let us put the origin of the spherical coordinate system (r, θ, φ) in the sphere's center and direct the z -axis along the fluid flow. Far from the sphere, the fluid velocity is $v_0 e_z$; thus, when the influence of the sphere appears, the flow velocity can be presented as $\vec{v} = v_0 e_z + \nabla u$, where u is the velocity potential. It is clear that there is no dependence on the azimuth angle φ . Thus, for a stationary flow, the potential depends only on r and θ .

Because $\text{div} \vec{v} = 0$, the potential $u(r, \theta)$ satisfies the Laplace equation: $\nabla \cdot \nabla(v_0 e_z + \nabla u) = \nabla^2 u(r, \theta) = 0$. As we already know, its solution for $r > b$ can be presented as

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n \left(\frac{b}{r} \right)^{n+1} P_n(\cos \theta). \quad (9.186)$$

To find coefficients A_n we have to apply the boundary condition on the spherical surface. The continuity of the normal component of speed on the interface means that $\vec{n} \cdot \vec{v} = 0$ at $r = b$, or

$$v_0 \vec{e}_z \vec{n} + \nabla_n u = v_0 \cos \theta + \frac{\partial u}{\partial r} = 0, \quad \text{for } r = b;$$

that is,

$$\left. \frac{\partial u}{\partial r} \right|_{r=b} = -v_0 \cos \theta. \quad (9.187)$$

Substituting the boundary condition (9.187) into Equation (9.186), we have

$$-\frac{1}{b} \sum_{n=0}^{\infty} (n+1) A_n P_n(\cos \theta) = -v_0 \cos \theta. \quad (9.188)$$

Clearly, this equality can be satisfied only for the following coefficients A_n (recall that $P_1(x) = x$):

$$A_0 = 0, A_1 = -\frac{1}{2} b v_0, \quad A_2 = A_3 = \dots = 0. \quad (9.189)$$

Substituting these coefficients into Equation (9.186), we finally find the potential

$$u = \frac{v_0 b^3}{2r^2} \cos \theta. \quad (9.190)$$

A

Eigenvalues and Eigenfunctions of the Sturm-Liouville Problem

The one-dimensional Sturm-Liouville boundary value problem for eigenvalues and eigenfunctions is formulated as:

Theorem A.1. *Find values of parameter λ for which there exist nontrivial (not identically equal to zero) solutions of the boundary value problem:*

$$X'' + \lambda X = 0, \quad 0 < x < l,$$

$$P_1[X] \equiv \alpha_1 X' + \beta_1 X \Big|_{x=0} = 0, \quad |\alpha_1| + |\beta_1| \neq 0,$$

$$P_2[X] \equiv \alpha_2 X' + \beta_2 X \Big|_{x=l} = 0, \quad |\alpha_2| + |\beta_2| \neq 0.$$

The eigenfunctions of this Sturm-Liouville problem are

$$X_n(x) = \frac{1}{\sqrt{\alpha_1^2 \lambda_n + \beta_1^2}} \left[\alpha_1 \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x - \beta_1 \sin \sqrt{\lambda_n} x \right].$$

These eigenfunctions are orthogonal. Their square norms are

$$\|X_n\|^2 = \int_0^l X_n^2(x) dx = \frac{1}{2} \left[l + \frac{(\beta_2 \alpha_1 - \beta_1 \alpha_2)(\lambda_n \alpha_1 \alpha_2 - \beta_1 \beta_2)}{(\lambda_n \alpha_1^2 + \beta_1^2)(\lambda_n \alpha_2^2 + \beta_2^2)} \right].$$

The eigenvalues are

$$\lambda_n = \left(\frac{\mu_n}{l} \right)^2,$$

where μ_n is the n th root of the equation

$$\tan \mu = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) l \mu}{\mu^2 \alpha_1 \alpha_2 + l^2 \beta_1 \beta_2}.$$

Next, we consider all possible cases of boundary conditions.

1. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} X(0) = 0 & \text{(Dirichlet condition),} \\ X(l) = 0 & \text{(Dirichlet condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\pi n / l)^2, \quad n = 1, 2, 3, \dots$

Eigenfunctions: $X_n(x) = \sin(\pi n x / l), \quad \|X_n\|^2 = l/2.$

2. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} X(0) = 0 & \text{(Dirichlet condition),} \\ X'(l) = 0 & \text{(Neumann condition).} \end{cases}$$

Eigenvalues: $\lambda_n = [\pi(2n + 1)/2l]^2, \quad n = 0, 1, 2, \dots$

Eigenfunctions: $X_n(x) = \sin[\pi(2n + 1)x/2l], \quad \|X_n\|^2 = l/2.$

3. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} X(0) = 0 & \text{(Dirichlet condition),} \\ X'(l) + h_2 X(l) = 0 & \text{(mixed condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\mu_n / l)^2, \quad n = 0, 1, 2, \dots$, where μ_n is the n th root of the equation $\tan \mu = -\mu / h_2 l$.

Eigenfunctions:

$$X_n(x) = \sin \sqrt{\lambda_n} x, \quad \|X_n\|^2 = \frac{1}{2} \left(l + \frac{h_2}{\lambda_n + h_2^2} \right).$$

4. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} X'(0) = 0 & \text{(Neumann condition),} \\ X(l) = 0 & \text{(Dirichlet condition).} \end{cases}$$

Eigenvalues: $\lambda_n = [\pi(2n+1)/2l]^2, \quad n = 0, 1, 2, \dots$

Eigenfunctions: $X_n(x) = \cos[\pi(2n+1)x/2l], \quad \|X_n\|^2 = l/2.$

5. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} X'(0) = 0 & \text{(Neumann condition),} \\ X'(l) = 0 & \text{(Neumann condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\pi n/l)^2, \quad n = 0, 1, 2, \dots$

Eigenfunctions: $X_n(x) = \cos(\pi n x/l), \quad \|X_n\|^2 = \begin{cases} l, & n = 0, \\ l/2, & n > 0. \end{cases}$

6. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} X'(0) = 0 & \text{(Neumann condition),} \\ X'(l) + h_2 X(l) = 0 & \text{(mixed condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\mu_n/l)^2, n = 0, 1, 2, \dots$, where μ_n is the n th root of the equation $\tan \mu = h_2 l / \mu$.

Eigenfunctions:

$$X_n(x) = \cos \sqrt{\lambda_n} x, \quad \|X_n\|^2 = \frac{1}{2} \left(l + \frac{h_2}{\lambda_n + h_2^2} \right).$$

7. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} X'(0) - h_1 X(0) = 0 & \text{(mixed condition),} \\ X(l) = 0 & \text{(Dirichlet condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\mu_n/l)^2, n = 0, 1, 2, \dots$, where μ_n is the n th root of the equation $\tan \mu = -\mu/h_1 l$.

Eigenfunctions:

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h_1^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h_1 \sin \sqrt{\lambda_n} x \right],$$

$$\|X_n\|^2 = \frac{1}{2} \left(l + \frac{h_1}{\lambda_n + h_1^2} \right).$$

8. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} X'(0) - h_1 X(0) = 0 & \text{(mixed condition),} \\ X'(l) = 0 & \text{(Neumann condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\mu_n/l)^2, n = 0, 1, 2, \dots$, where μ_n is the n th root of the equation $\tan \mu = h_1 l / \mu$.

Eigenfunctions:

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h_1^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h_1 \sin \sqrt{\lambda_n} x \right],$$

$$\|X_n\|^2 = \frac{1}{2} \left(l + \frac{h_1}{\lambda_n + h_1^2} \right).$$

9. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} X'(0) - h_1 X(0) = 0 & \text{(mixed condition),} \\ X'(l) + h_2 X(l) = 0 & \text{(mixed condition).} \end{cases}$$

Eigenvalues: $\lambda_n = (\mu_n/l)^2, n = 0, 1, 2, \dots$, where μ_n is the n th root of the equation

$$\tan \mu = \frac{(h_1 + h_2)l\mu}{\mu^2 - h_1 h_2 l^2}.$$

Eigenfunctions:

$$X_n(x) = \frac{1}{\sqrt{\lambda_n + h_1^2}} \left[\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + h_1 \sin \sqrt{\lambda_n} x \right],$$

$$\|X_n\|^2 = \frac{1}{2} \left(l + \frac{(h_1 + h_2)(\lambda_n + h_1 h_2)}{(\lambda_n + h_1^2)(\lambda_n + h_2^2)} \right).$$

B

Auxiliary Functions for Different Types of Boundary Conditions

In the case of nonhomogeneous boundary conditions

$$P_1[u] \equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = g_1(t), \quad |\alpha_1| + |\beta_1| \neq 0,$$

$$P_2[u] \equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=l} = g_2(t), \quad |\alpha_2| + |\beta_2| \neq 0,$$

the solution to the boundary value problem can be expressed as the sum of two functions

$$u(x, t) = v(x, t) + w(x, t),$$

where $w(x, t)$ is an auxiliary function satisfying the boundary conditions and $v(x, t)$ is a solution of the boundary value problem with zero boundary conditions.

We seek an auxiliary function $w(x, t)$ in a form

$$w(x, t) = g_1(t)\overline{X}(x) + g_2(t)\overline{\overline{X}}(x),$$

where $\overline{X}(x)$ and $\overline{\overline{X}}(x)$ are polynomials of first or second order, respectively. The coefficients of these polynomials are adjusted to satisfy the boundary conditions.

Functions $\overline{X}(x)$ and $\overline{\overline{X}}(x)$ should be chosen in such a way that

$$P_1 \left[\overline{X}(0) \right] = 1, \quad P_2 \left[\overline{X}(l) \right] = 0,$$

$$P_1 \left[\overline{\overline{X}}(0) \right] = 0, \quad P_2 \left[\overline{\overline{X}}(l) \right] = 1.$$

If $\beta_1 \neq 0$ or $\beta_2 \neq 0$, then functions $\overline{X}(x)$ and $\overline{\overline{X}}(x)$ are polynomials of first order,

$$\overline{X}(x) = \gamma_1 + \delta_1 x, \quad \overline{\overline{X}}(x) = \gamma_2 + \delta_2 x.$$

Coefficients γ_1 , δ_1 , γ_2 , and δ_2 of these polynomials are defined uniquely and depend on the types of boundary conditions:

$$\gamma_1 = \frac{\alpha_2 + \beta_2 l}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \quad \delta_1 = \frac{-\beta_2}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1},$$

$$\gamma_2 = \frac{-\alpha_1}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}, \quad \delta_2 = \frac{\beta_1}{\beta_1 \beta_2 l + \beta_1 \alpha_2 - \beta_2 \alpha_1}.$$

If $\beta_1 = \beta_2 = 0$, then functions $\overline{X}(x)$ and $\overline{\overline{X}}(x)$ are polynomials of second order,

$$\overline{X}(x) = x - \frac{x^2}{2l}, \quad \overline{\overline{X}}(x) = \frac{x^2}{2l}.$$

1. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} u(0, t) = g_1(t) & \text{(Dirichlet condition),} \\ u(l, t) = g_2(t) & \text{(Dirichlet condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \left[1 - \frac{x}{l}\right] \cdot g_1(t) + \frac{x}{l} \cdot g_2(t).$$

2. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} u(0, t) = g_1(t) & \text{(Dirichlet condition),} \\ u_x(l, t) = g_2(t) & \text{(Neumann condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = g_1(t) + x g_2(t).$$

3. *Boundary conditions* ($\alpha_1 = 0, \beta_1 = -1, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} u(0, t) = g_1(t) & \text{(Dirichlet condition),} \\ u_x(l, t) + h_2 u(l, t) = g_2(t) & \text{(mixed condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \left[1 - \frac{h_2}{1 + h_2 l} x \right] \cdot g_1(t) + \frac{x}{1 + h_2 l} \cdot g_2(t).$$

4. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} u_x(0, t) = g_1(t) & \text{(Neumann condition),} \\ u(l, t) = g_2(t) & \text{(Dirichlet condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = (x - l)g_1(t) + g_2(t).$$

5. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} u_x(0, t) = g_1(t) & \text{(Neumann condition),} \\ u_x(l, t) = g_2(t) & \text{(Neumann condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \left[x - \frac{x^2}{2l} \right] \cdot g_1(t) + \frac{x^2}{2l} \cdot g_2(t).$$

6. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} u_x(0, t) = g_1(t) & \text{(Neumann condition),} \\ u_x(l, t) + h_2 u(l, t) = g_2(t) & \text{(mixed condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \left[x - \frac{1 + h_2 l}{h_2} \right] \cdot g_1(t) + \frac{1}{h_2} \cdot g_2(t).$$

7. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 0, \beta_2 = 1$):

$$\begin{cases} u_x(0, t) - h_1 u(0, t) = g_1(t) & \text{(mixed condition),} \\ u(l, t) = g_2(t) & \text{(Dirichlet condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \frac{x - l}{1 + h_1 l} \cdot g_1(t) + \frac{1 + h_1 x}{1 + h_1 l} \cdot g_2(t).$$

8. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 1, \beta_2 = 0$):

$$\begin{cases} u_x(0, t) - h_1 u(0, t) = g_1(t) & \text{(mixed condition),} \\ u_x(l, t) = g_2(t) & \text{(Neumann condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = -\frac{1}{h_1} \cdot g_1(t) + \left[x + \frac{1}{h_1} \right] \cdot g_2(t).$$

9. *Boundary conditions* ($\alpha_1 = 1, \beta_1 = -h_1, \alpha_2 = 1, \beta_2 = h_2$):

$$\begin{cases} u_x(0, t) - h_1 u(0, t) = g_1(t) & \text{(mixed condition),} \\ u_x(l, t) + h_2 u(l, t) = g_2(t) & \text{(mixed condition).} \end{cases}$$

Auxiliary function:

$$w(x, t) = \frac{h_2(x - l) - 1}{h_1 + h_2 + h_1 h_2 l} \cdot g_1(t) + \frac{1 + h_1 x}{h_1 + h_2 + h_1 h_2 l} \cdot g_2(t).$$



The Sturm-Liouville Problem and the Laplace Equation

Here we calculate eigenfunctions of the Sturm-Liouville boundary value problem for the Laplace equation in a rectangular domain for different types of boundary conditions. Let the function $u(x, y)$ be a particular solution of the Laplace problem with the following boundary conditions

$$\begin{aligned} P_1[u]_{x=0} &= 0, & P_2[u]_{x=l_x} &= 0, \\ P_3[u]_{y=0} &= g_3(x), & P_4[u]_{y=l_y} &= g_4(x). \end{aligned}$$

The solution of this problem has the form

$$u(x, y) = \sum_{n=1}^{\infty} \{A_n Y_{1n}(y) + B_n Y_{2n}(y)\} X_n(x),$$

where λ_{xn} and $X_n(x)$ are the eigenvalues and eigenfunctions, respectively, of the Sturm-Liouville problem for an interval:

$$X'' + \lambda X = 0, \quad 0 < x < l_x$$

$$P_1[X]_{x=0} = P_2[X]_{x=l_x} = 0.$$

(The solution of this boundary value problem depends on the types of boundary conditions $P_1[u]$ and $P_2[u]$.)

Taking into account the eigenvalues λ_{xn} , we obtain an equation for $Y(y)$:

$$Y'' - \lambda_{xn} Y = 0, \quad 0 < y < l_y.$$

The functions $Y_{1n}(y)$ and $Y_{2n}(y)$ are sought in the form

$$Y_{1n}(y) = a \sinh \sqrt{\lambda_{xn}} y + b \cosh \sqrt{\lambda_{xn}} y,$$

$$Y_{2n}(y) = c \sinh \sqrt{\lambda_{xn}} (l_y - y) + d \cosh \sqrt{\lambda_{xn}} (l_y - y).$$

The values of coefficients a , b , c , and d are adjusted to satisfy the boundary conditions $P_3[u]_{y=0}$ and $P_4[u]_{y=l_y}$ and should be chosen in such a way that

$$P_3[Y_1(0)] = 0, \quad P_3[Y_1(l_y)] = 1,$$

$$P_4[Y_2(0)] = 1, \quad P_4[Y_2(l_y)] = 0.$$

1. Boundary conditions

$$\begin{cases} P_3[u] \equiv u|_{y=0} = g_3(x) & \text{(Dirichlet condition),} \\ P_4[u] \equiv u|_{y=l_y} = g_4(x) & \text{(Dirichlet condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\sinh \sqrt{\lambda_{xn}} y}{\sinh \sqrt{\lambda_{xn}} l_y}, \quad Y_{2n}(y) = \frac{\sinh \sqrt{\lambda_{xn}} (l_y - y)}{\sinh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{y}{l_y}, \quad Y_{2n}(y) = 1 - \frac{y}{l_y}.$$

2. Boundary conditions

$$\begin{cases} P_3[u] \equiv u|_{y=0} = g_3(x) & \text{(Dirichlet condition),} \\ P_4[u] \equiv u_y|_{y=l_y} = g_4(x) & \text{(Neumann condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\sinh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y}, \quad Y_{2n}(y) = \frac{\cosh \sqrt{\lambda_{xn}} (l_y - y)}{\cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = y, \quad Y_{2n}(y) = 1.$$

3. *Boundary conditions*

$$\begin{cases} P_3[u] \equiv u|_{y=0} = g_3(x) & \text{(Dirichlet condition),} \\ P_4[u] \equiv u_y + h_4 u|_{y=l_y} = g_4(x) & \text{(mixed condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\sinh \sqrt{\lambda_{xn}} y}{h_4 \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y},$$

$$Y_{2n}(y) = \frac{h_4 \sinh \sqrt{\lambda_{xn}} (l_y - y) + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} (l_y - y)}{h_4 \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{y}{1 + h_4 l_y}, \quad Y_{2n}(y) = 1 - \frac{h_4}{1 + h_4 l_y} y.$$

4. *Boundary conditions*

$$\begin{cases} P_3[u] \equiv u_y|_{y=0} = g_3(x) & \text{(Neumann condition),} \\ P_4[u] \equiv u|_{y=l_y} = g_4(x) & \text{(Dirichlet condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\cosh \sqrt{\lambda_{xn}} y}{\cosh \sqrt{\lambda_{xn}} l_y}, \quad Y_{2n}(y) = -\frac{\sinh \sqrt{\lambda_{xn}} (l_y - y)}{\sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = 1, \quad Y_{2n}(y) = y - l_y.$$

5. *Boundary conditions*

$$\begin{cases} P_3[u] \equiv u_y|_{y=0} = g_3(x) & \text{(Neumann condition),} \\ P_4[u] \equiv u_y|_{y=l_y} = g_4(x) & \text{(Neumann condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\cosh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y}, \quad Y_{2n}(y) = -\frac{\cosh \sqrt{\lambda_{xn}} (l_y - y)}{\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{1}{2l_y} y^2, \quad Y_{2n}(y) = y - \frac{1}{2l_y} y^2.$$

6. Boundary conditions

$$\begin{cases} P_3[u] \equiv u_y \big|_{y=0} = g_3(x) & \text{(Neumann condition),} \\ P_4[u] \equiv u_y + h_4 u \big|_{y=l_y} = g_4(x) & \text{(mixed condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{\cosh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y + h_4 \cosh \sqrt{\lambda_{xn}} l_y},$$

$$Y_{2n}(y) = -\frac{h_4 \sinh \sqrt{\lambda_{xn}} (l_y - y) + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} (l_y - y)}{\sqrt{\lambda_{xn}} [\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y + h_4 \cosh \sqrt{\lambda_{xn}} l_y]}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{y}{h_4}, \quad Y_{2n}(y) = y - \frac{1 + h_4 l_y}{h_4}.$$

7. Boundary conditions

$$\begin{cases} P_3[u] \equiv u_y - h_3 u \big|_{y=0} = g_3(x) & \text{(mixed condition),} \\ P_4[u] \equiv u \big|_{y=l_y} = g_4(x) & \text{(Dirichlet condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{h_3 \sinh \sqrt{\lambda_{xn}} y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} y}{h_3 \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y},$$

$$Y_{2n}(y) = -\frac{\sinh \sqrt{\lambda_{xn}} (l_y - y)}{h_3 \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{1 + h_3 y}{1 + h_3 l_y}, \quad Y_{2n}(y) = \frac{y - l_y}{1 + h_3 l_y}.$$

8. Boundary conditions

$$\begin{cases} P_3[u] \equiv u_y - h_3 u \big|_{y=0} = g_3(x) & \text{(mixed condition),} \\ P_4[u] \equiv u_y \big|_{y=l_y} = g_4(x) & \text{(Neumann condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{h_3 \sinh \sqrt{\lambda_{xn}} y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} y}{\sqrt{\lambda_{xn}} \left[\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y + h_3 \cosh \sqrt{\lambda_{xn}} l_y \right]},$$

$$Y_{2n}(y) = -\frac{\cosh \sqrt{\lambda_{xn}} (l_y - y)}{\sqrt{\lambda_{xn}} \sinh \sqrt{\lambda_{xn}} l_y + h_3 \cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_{1n}(y) = \frac{1 + h_3 y}{1 + h_3 l_y}, \quad Y_{2n}(y) = \frac{y - l_y}{1 + h_3 l_y}.$$

9. Boundary conditions

$$\begin{cases} P_3[u] \equiv u_y - h_3 u \big|_{y=0} = g_3(x) & \text{(mixed condition),} \\ P_4[u] \equiv u_y + h_4 u \big|_{y=l_y} = g_4(x) & \text{(mixed condition).} \end{cases}$$

Fundamental system:

$$Y_{1n}(y) = \frac{h_3 \sinh \sqrt{\lambda_{xn}} y + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} y}{(\lambda_{xn} + h_3 h_4) \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} (h_3 + h_4) \cosh \sqrt{\lambda_{xn}} l_y},$$

$$Y_{2n}(y) = -\frac{h_4 \sinh \sqrt{\lambda_{xn}} (l_y - y) + \sqrt{\lambda_{xn}} \cosh \sqrt{\lambda_{xn}} (l_y - y)}{(\lambda_{xn} + h_3 h_4) \sinh \sqrt{\lambda_{xn}} l_y + \sqrt{\lambda_{xn}} (h_3 + h_4) \cosh \sqrt{\lambda_{xn}} l_y}.$$

If $\lambda_{x0} = 0$, $X_0(x) \equiv 1$, then

$$Y_1(y) = \frac{1 + h_3 y}{h_3 + h_4 + h_3 h_4 l_y}, \quad Y_2(y) = \frac{h_4 (y - l_y) - 1}{h_3 + h_4 + h_3 h_4 l_y}.$$

D

Vector Calculus

In this appendix, we give a brief review of vector calculus. The intent is not to introduce these concepts to a reader for the first time, but rather to remind the reader of useful concepts and relations that are used in the text.

We begin with a definition of scalars and vectors. *Scalars* are mathematical entities used to represent physical quantities that have only a magnitude. Examples include temperature, speed, distance, etc. In some cases, we need a mathematical entity that contains more information—for example, magnitude and direction—in which case we may use a *vector*. Velocity, displacement, and force are examples of physical quantities represented by vectors. The mathematical objects known as tensors may be required to represent other physical quantities that encompass even more than two pieces of information.

We may represent vectors in various ways. For example, in Cartesian coordinates, a displacement vector might be represented as $\vec{A} \equiv A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are vectors of unit length in the x , y , and z directions, respectively, and A_x , A_y , and A_z represent the magnitude, or length, of the components in each of these directions. Note that in some texts boldface type is used to denote vector quantities. In the following discussion, we extend the concept of a vector to *vector fields*, by which we mean vectors whose components are functions of location: $A_x = A_x(x, y, z)$, $A_y = A_y(x, y, z)$, and $A_z = A_z(x, y, z)$. The magnitudes of the components may also be written as

$$A_x = |\vec{A}| \cos \theta_1, \quad A_y = |\vec{A}| \cos \theta_2, \quad \text{and} \quad A_z = |\vec{A}| \cos \theta_3,$$

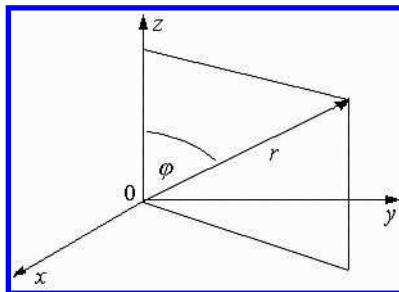


Figure D.1. Definition of cylindrical polar coordinates.

where θ_1 , θ_2 , and θ_3 are the angles the vector makes with the respective axis. Here, the length of the vector is denoted as $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ in Cartesian coordinates.

For problems that are cylindrically symmetric, we may wish to use *cylindrical polar coordinates*, defined in Figure D.1, in which case we represent the vector $\vec{A}$ as $\vec{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{k}$. Here, the unit vector $\hat{r}$ is along a radial direction from the coordinate origin, and $\hat{\phi}$ represents the direction of an infinitesimal rotation away from the z -axis. The relation between the Cartesian coordinates and cylindrical coordinates are

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z,$$

and the inverses:

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}(y/x), \quad z = z.$$

Note that for vector fields, the components of the vector may be functions of r , ϕ , and z .

For problems that involve spherical symmetry, we may use *spherical polar coordinates* defined in Figure D.2, where vectors are represented as $\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$. Here, θ is measured from the z -axis, and ϕ is the angular location measured from the x -axis in the x - y plane. The relation between the Cartesian coordinates and spherical coordinates is

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

and the inverses:

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right), \quad \phi = \tan^{-1}(y/x).$$

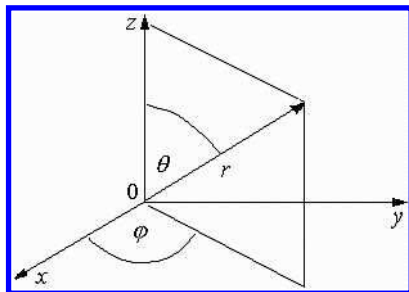


Figure D.2. Definition of spherical polar coordinates.

Vectors may be added, obeying the usual rules of commutability and associativity that ordinary numbers do. If $\vec{C}$ is the sum of the vectors $\vec{A}$ and $\vec{B}$, we may write $\vec{C} = \vec{A} + \vec{B}$, by which we mean that the components of $\vec{C}$ are given by $C_x = A_x + B_x$, $C_y = A_y + B_y$, and $C_z = A_z + B_z$ in Cartesian coordinates.

We may multiply vectors by scalars, in which case we have a new vector that is longer or shorter than the original but has the same direction. For example, we may multiply the vector $\vec{C}$ by the scalar a (which may be greater or less than 1) so that we have $a\vec{C} = aC_x\hat{i} + aC_y\hat{j} + aC_z\hat{k}$. Scalar multiplication also enables us to create the negative of a vector and thus subtract vectors. In this case, we have $-\vec{A} = (-1)\vec{A} = -A_x\hat{i} - A_y\hat{j} - A_z\hat{k}$. We may thus write $\vec{C} = \vec{A} - \vec{B}$, which is shorthand notation for $C_x = A_x - B_x$, $C_y = A_y - B_y$, and $C_z = A_z - B_z$ in Cartesian coordinates.

There are two possibilities for multiplying two vectors. In the case of the *scalar* or *inner product* (also called the *dot product*), the final object is a scalar quantity. The inner product is defined as $\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}| \cos \theta \equiv A_x B_x + A_y B_y + A_z B_z$ in Cartesian coordinates, where θ is the angle between the two vectors. From the definition, we can see that the dot product multiplies the component of a vector lying along the direction of the second vector times the length of the second vector. The reader may verify that the dot product of two vectors that are perpendicular is zero; they have no components in a common direction.

We may also multiply two vectors and obtain a new vector using the *vector* or *outer product* (also known as the *cross product*). The outer product is defined as $\vec{C} = \vec{A} \times \vec{B}$, where $C_x = A_y B_z - A_z B_y$, $C_y =$

$A_z B_x - A_x B_z$, and $C_z = A_x B_y - A_y B_x$. This may be written more conveniently as the determinate,

$$\vec{C} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

The cross product of two vectors that are parallel is zero, as may be verified by the reader. The magnitude or length of the cross product is given by $|\vec{C}| = |\vec{A} \times \vec{B}| \equiv |\vec{A}||\vec{B}|\sin\theta$, and its direction is given by a right-hand rule: If the thumb of the right-hand points in the direction of the first vector and the fingers point in the direction of the second vector, the palm will point in the direction of the cross product. Note that changing the order of the vectors in the cross product introduces a minus sign: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

As an example of the use of the two types of vector multiplication, the physical quantity work, which is a scalar quantity, is represented as a scalar product of force and distance: $W = \vec{F} \cdot \vec{r} \equiv |\vec{F}||\vec{r}|\cos\theta$. The same two vectors may be multiplied by using the vector product to represent a torque, $\vec{\tau} = \vec{r} \times \vec{F}$, which is a vector quantity that has magnitude and direction. Note that in simple cases the vectors $\vec{F}$ and $\vec{r}$ may be constants, but in general they may be functions of location and time.

Two useful relations for vector multiplication, which may be easily verified by using the definitions of the cross and dot products, are $\vec{D} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C}$ and $\vec{D} = \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$. As should be obvious from the definitions, the vector product takes precedence over the scalar product.

In general, vector division is not uniquely defined; however, we may formally solve the equation $a = \vec{A} \cdot \vec{B}$ for the vector $\vec{B}$ and write

$$\vec{B} = \frac{a\vec{A}}{\vec{A} \cdot \vec{A}} + \vec{C},$$

where $\vec{C}$ is an arbitrary vector that is perpendicular to $\vec{A}$. We may also write a formal solution for $\vec{B}$ in the equation $\vec{C} = \vec{A} \times \vec{B}$ as

$$\vec{B} = \frac{\vec{A} \times \vec{C}}{\vec{A} \cdot \vec{A}} + k\vec{A},$$

where k is an arbitrary scalar.

We may continue now with definitions from calculus and apply these concepts to scalar and vector fields, by which we mean scalars and vectors whose components are functions of location. First, we define the *gradient*, which is a vector derivative of a scalar field. For the scalar function $f(x, y, z)$, which is a field of magnitudes at every location (x, y, z) , we may define the following derivative:

$$\text{grad}f(x, y, z) \equiv \nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Physically, the components of the gradient vector represent the change of the function $f(x, y, z)$ in each of the Cartesian directions, where the direction of the vector indicates the direction of maximum change. As a two-dimensional example, if the function $f(x, y)$ were to represent the contour of a local terrain (the altitude at location x, y), then the gradient vector would indicate the direction of maximum slope and give the steepness in the x and y directions. In the case that $f(r, \varphi, z)$ is a function of cylindrical polar coordinates, the gradient is given by

$$\nabla f(r, \varphi, z) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{k}$$

by using the unit vectors and definitions previously shown in Figure D.1.

In spherical polar coordinates, the gradient is given by

$$\nabla f(r, \theta, \varphi) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi},$$

where unit vectors and angles are as defined in Figure D.2.

From this definition for the gradient in Cartesian coordinates, it is convenient to define the *del operator*, ∇ , as the vector operator,

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

The del operator provides a useful mnemonic for defining derivatives of vectors that result in either a scalar or a vector. The scalar derivative of a vector field, $\vec{F}(x, y, z)$, is known as the *divergence* and is given by the scalar product of the del operator with the vector field:

$$\text{div}\vec{F} \equiv \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

where the components $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$ of the vector field $\vec{F}$ may be functions of x , y , and z . In cylindrical polar coordinates, the del operator is

$$\nabla \equiv \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \varphi} \hat{\varphi} + \frac{\partial}{\partial z} \hat{k},$$

so that the divergence becomes

$$\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial (r F_r)}{\partial r} + \frac{1}{r} \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}.$$

In spherical polar coordinates,

$$\nabla \equiv \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \hat{\varphi},$$

and the divergence is given by

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta F_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial F_\varphi}{\partial \varphi}.$$

The vector derivative of a vector field is called the *curl* and may be written in Cartesian coordinates as the cross product of the del operator with a vector field:

$$\text{curl } \vec{F} \equiv \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix},$$

or in cylindrical polar coordinates:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{r} & \hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_r & r F_\varphi & F_z \end{vmatrix},$$

or in spherical polar coordinates:

$$\nabla \times \vec{F} = \begin{vmatrix} \frac{\hat{r}}{r^2 \sin \theta} & \frac{\hat{\theta}}{r \sin \theta} & \frac{\hat{\varphi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & r F_\theta & r \sin \theta F_\varphi \end{vmatrix}.$$

We state the following relations which the reader may easily verify in Cartesian coordinates from the given definitions.

The curl of the gradient of any scalar field is zero:

$$\nabla \times (\nabla f) \equiv 0.$$

The divergence of the curl of any vector field is zero:

$$\nabla \cdot (\nabla \times \vec{F}) \equiv 0.$$

The curl of the curl of a vector field results in a vector given by

$$\nabla \times (\nabla \times \vec{F}) \equiv \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F},$$

where the final term is referred to as the *Laplacian operator* of the vector field $\vec{F}$. The Laplacian is a scalar operator, which we may write in Cartesian coordinates as

$$\nabla \cdot \nabla \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Since it is a scalar operator, it may be applied to either a scalar field, f , in which case the result is a scalar, or to a vector field, $\vec{F}$, in which case the result is a vector. In cylindrical polar coordinates the Laplacian is

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2},$$

and in spherical polar coordinates it is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

For two scalar fields, f and g , and two vector fields, $\vec{F}$ and $\vec{G}$, we may combine the various operations defined above in a variety of ways. The following useful identities may be proved by the reader using the given

properties of the various operators and are independent of the coordinate system being used:

$$\begin{aligned}
 \nabla(f + g) &= \nabla f + \nabla g. \\
 \nabla \cdot (\vec{F} + \vec{G}) &= \nabla \cdot \vec{F} + \nabla \cdot \vec{G}. \\
 \nabla \times (\vec{F} + \vec{G}) &= \nabla \times \vec{F} + \nabla \times \vec{G}. \\
 \nabla(fg) &= g\nabla f + f\nabla g. \\
 \nabla \cdot (f\vec{F}) &= f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f. \\
 \nabla \times (f\vec{F}) &= f\nabla \times \vec{F} + \nabla f \times \vec{F}. \\
 \nabla \cdot (\vec{F} \times \vec{G}) &= \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}. \\
 \nabla \times (\vec{F} \times \vec{G}) &= \vec{F}\nabla \cdot \vec{G} - \vec{G}\nabla \cdot \vec{F} + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}.
 \end{aligned}$$

Here, for convenience, we have written ∇ in place of grad; $\nabla \cdot$ in place of div; and $\nabla \times$ in place of curl; the alternate notation is used in the text.

Strictly speaking, integration is defined for scalars only; however, in many cases we are interested in scalars formed from vectors, such as the scalar product, and we may also integrate the components of a vector individually. In this way, we may integrate scalar or vector fields and we may integrate in one dimension, in two dimensions, or in three dimensions. Generally speaking, path and surface integrals involving vectors and volume integrals of both scalars and vectors are found to be of most use in physical applications.

We start by defining the line or path integral of a vector. First, imagine a path C , which connects two points a and b where an infinitesimal vector $d\vec{l}$ of unit length defines the direction of the path at any location. The scalar product of the infinitesimal vector $d\vec{l}$ with the vector field $\vec{F}$ selects the component of the field, which lies in the direction of the path, since the scalar product of vector components perpendicular to each other is zero, as shown earlier. The one-dimensional integral of the vector $\vec{F}$ along the path C that joins the points a and b thus sums the components of the vector field that lie along the path. Formally, we may divide the path into N infinitesimal steps and write

$$\int_a^b \vec{F} \cdot d\vec{l} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \vec{F}_i \cdot \Delta\vec{l}_i,$$

where $\Delta \vec{l}_i$ is a vanishingly small segment of the path with a direction along the path. Note that this integral depends on the path, C , in addition to the endpoints, a and b . If the two endpoints are the same point, we have an integral over a closed path, which may be denoted as $\oint \vec{F} \cdot d\vec{l}$. The integral over a closed path may or may not equal zero, depending on the physical field being integrated and the actual path of integration. In most cases, in order to actually perform the integration, we must have enough information about the path so that the integral can be written. For example, the sum of three integrals in the three infinitesimal spatial directions dx , dy , and dz is

$$\int_a^b \vec{F} \cdot d\vec{l} = \int_a^b (F_x dx + F_y dy + F_z dz).$$

In cylindrical coordinates, the infinitesimal displacement vector is $d\vec{l} = dr\hat{r} + r d\varphi\hat{\phi} + dz\hat{k}$, whereas it is $d\vec{l} = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\varphi\hat{\phi}$ for spherical coordinates.

In a similar fashion, we may define the integral of a vector over a surface area. First, we denote an infinitesimal surface element vector as $d\vec{A}$, which has a direction perpendicular to the surface and units of length squared. For open surfaces there is an arbitrary sign for the direction of $d\vec{A}$, since there are two equivalent sides to the surface, but for closed surfaces the convention is to have the infinitesimal area vector point out of the enclosed volume. The scalar product of this infinitesimal area with a vector field $\vec{F}$ measures the amount of field that passes through the surface element, and is called the *flux*. In this case, we may write the flux as

$$\int_S \vec{F} \cdot d\vec{A},$$

where S is the total surface in question. The integral is actually a double integral, which in Cartesian coordinates may be written as

$$\int_S \vec{F} \cdot d\vec{A} = \iint_S (F_x dydz + F_y dx dz + F_z dx dy).$$

For cylindrical polar coordinates, the infinitesimal surface element is $d\vec{A} = r d\varphi dz\hat{r} + r dr dz\hat{\phi} + r dr d\varphi\hat{k}$, and in spherical polar coordinates, it is $d\vec{A} = r^2 \sin\theta d\theta d\varphi\hat{r} + r \sin\theta dr d\varphi\hat{\theta} + r dr d\theta\hat{\phi}$.

There are several interesting physical cases in which the flux of a particular field through a closed surface is proportional to the sources of the field enclosed by the surface, which is known as *Gauss's law*. For example, in the case of static electric fields, the electric field pointing through a closed surface is proportional to the charge inside the surface:

$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0},$$

where ϵ_0 is a constant and q is the charge enclosed. In the magnetic case, the magnetic flux pointing into any closed volume equals the flux pointing out (i.e., there are no magnetic monopoles) so that Gauss's law for static magnetic fields is

$$\oiint_S \vec{B} \cdot d\vec{A} = 0.$$

Again, to solve the integral in a particular application, we must know the parameters that determine the surface so that the integral can be reduced to a form that can be integrated.

For the surface integral of a closed volume, we find interesting connections to the divergence and the curl of a vector, which may be used to define those quantities. They are given here without proof:

$$\lim_{V \rightarrow 0} \frac{1}{V} \oiint_S \vec{F} \cdot d\vec{A} = \nabla \cdot \vec{F} \equiv \text{div } \vec{F}$$

$$\lim_{V \rightarrow 0} \frac{1}{V} \oiint_S d\vec{A} \times \vec{F} = \nabla \times \vec{F} \equiv \text{curl } \vec{F}.$$

We may define volume integrals of both scalars and vectors, respectively, as

$$\int_V f dV = \iiint_V f dV \text{ and } \int_V \vec{F} dV = \hat{i} \iiint_V F_x dV + \hat{j} \iiint_V F_y dV + \hat{k} \iiint_V F_z dV,$$

where it is understood that the volume element, dV , is $dx dy dz$ in Cartesian coordinates, and the integrals are triple integrals. In cylindrical polar coordinates, the volume element is $dV = r dr d\phi dz$, and in spherical polar coordinates, we have $dV = r^2 \sin \theta dr d\theta d\phi$. Note that the components

of $\vec{F}$ may be functions of all of the variables—for example x , y , and z in Cartesian coordinates.

Two useful theorems may be derived from the above definitions, stated here without proof.

The *divergence theorem* relates the volume integral of the divergence of a vector field to its surface integral:

$$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} dV.$$

As a physical example, the theorem relates the flow of a fluid crossing a surface with the strength of the source (or sink, if the gradient is negative) of the fluid inside the volume bounded by the surface. If the gradient on the right-hand side represents a charge density, the theorem reduces to Gauss's law for static electric fields, discussed previously. The theorem may also be applied to a gravity field, in which case the gradient on the right-hand side represents a mass density.

Stokes's theorem relates the path integral of a vector field to the curl of the surface integral bounded by the path:

$$\oint_C \vec{F} \cdot d\vec{l} = \iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{A}.$$

Examples in physics include Faraday's law of induction,

$$\oint_C \vec{E} \cdot d\vec{l} = \iint_S (\operatorname{curl} \vec{E}) \cdot d\vec{A} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{A},$$

and Ampere's law,

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S (\operatorname{curl} \vec{B}) \cdot d\vec{A} = \mu_o \iint_S \vec{J} \cdot d\vec{A} - \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{A}.$$

If only one component of the vector quantities on both sides is considered—for example,

$$\oint_C F_x dx = \iint_S \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dy dz$$

—we have what is referred to as *Green's theorem*.

From the above theorems and definitions, we may also derive the following useful relations, stated without proof:

$$\oint_S f d\vec{A} = \iiint_V \operatorname{div} f dV,$$

$$\oint_S d\vec{A} \times \vec{F} = \iiint_V \operatorname{curl} \vec{F} dV,$$

$$\oint_C f d\vec{l} = \iint_S d\vec{A} \times \nabla f.$$

E

How to Use the Software Associated with this Book

The purpose of this appendix is to explain how to use the software associated with the book. The included programs are TrigSeries, Waves, Heat, Laplace, and FourierSeries. The program TrigSeries may be used for the trigonometric Fourier series problems discussed in Chapter 1. The program Waves accompanies Chapters 3 and 4, and is designed to solve the following types of problems: vibrations of infinite or semi-infinite strings, vibrations of finite strings, and vibrations of flexible rectangular or circular membranes. The Heat program is designed to solve problems discussed in Chapters 5 and 6—heat conduction within an infinite and semi-infinite rod, heat conduction within a finite uniform rod, and heat conduction within a thin, uniform, rectangular or circular membrane. The Laplace program is designed to solve boundary value problems for elliptic equations over rectangular and circular domains, discussed in Chapter 7. Such equations describe problems involving electrostatic or gravitational potentials, steady temperature distributions, and other phenomena. The FourierSeries program is associated with Chapters 8 and 9 and provides solutions to three kinds of problems: (1) expansion in a generalized Fourier series in terms of classical orthogonal polynomials (Legendre, Chebyshev of the first and second kind, Jacoby, Laguerre, Hermite), (2) expansion in Fourier-Bessel series, and (3) expansion in Fourier series in terms of associated Legendre functions.

In Section E.1, we discuss the common features of all the programs. The five sections that follow the overview provide examples of how to use each of the programs for solving problems, some of which were taken

from the book problem sets without any changes (and in many cases the same or related examples are provided as library examples), and others of which were modified to show how to use different program options. The programs are also sufficiently flexible to be able to solve many other problems not found in the text. The results of computations can be saved in the users' files to be examined again. The library sets can be extended by the reader. The programs use the same analytical formulas and methods which are presented in the main text. Thus, the programs follow the same steps the reader is supposed to take in solving the problems analytically. The only numeric calculation the program performs is the evaluation of the coefficients of the Fourier series (with Gauss's method and its modifications) and partial sums.

It is always useful to solve a problem analytically, and if this solution is correct it should give the same results as those that the software provides. The two solutions can be compared in detail and the analytical solution can be expressed in graphical form, which can be done with the software. Also, from the analytical solution one can often determine general behavior, for instance the zeroes (nodes) of harmonics. Agreement between the analytical solution and the solution obtained with the software can indicate the correctness of both solutions.

Note that one of the advantages of using the software is that a range of parameters may be explored very quickly. This opens the way to exploring a wide variety of different variants of basically the same problem.

Each program has an integrated help system, including a detailed explanation of theoretical background. Help is available by clicking the "Help" button from the main menu of the program. The help is context-dependent, meaning that the help provided is associated with the window currently being used.

E.1 Program Overview

In the following discussion, we use the Heat program to demonstrate various program menus and screens, and the other programs function in basically the same way. After you start any of the applications, you will see a main window ([Figure E.1](#) shows the Heat program main window). From this window, you may choose the type of the problem you wish to solve (for some programs, such as TrigSeries, there is only one choice).

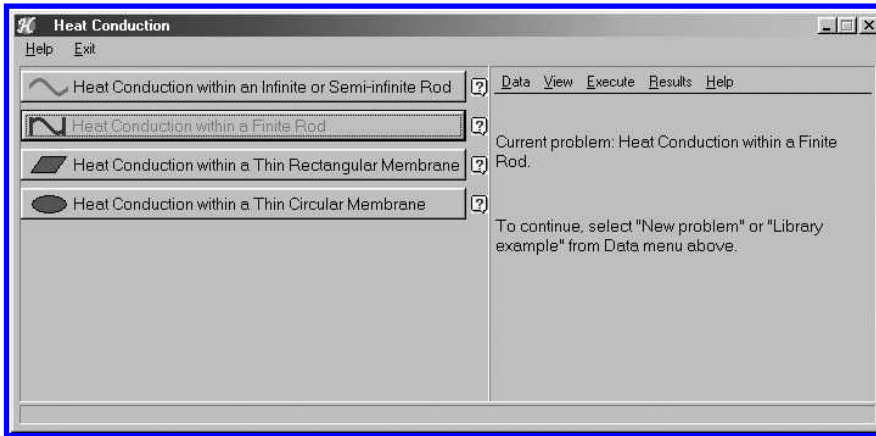


Figure E.1. The Heat program main window.

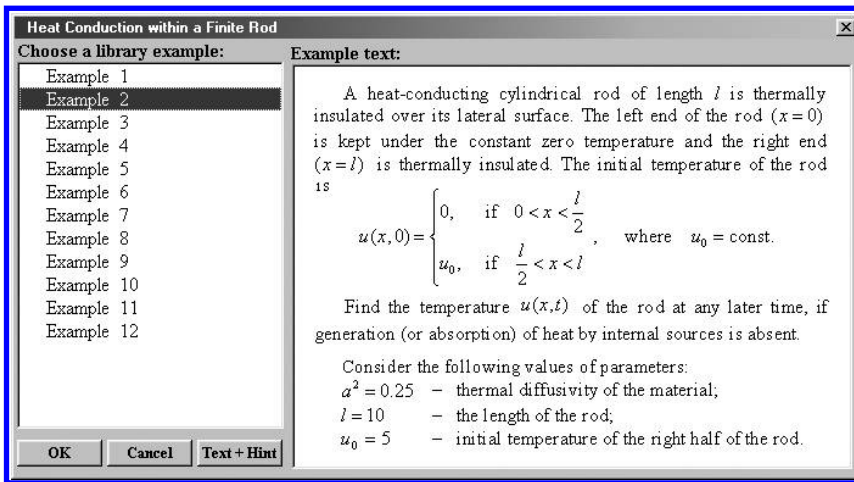


Figure E.2. Library example selection for the Heat program.

Suppose, for example, you chose the second option, “Heat Conduction within a Finite Rod.” To solve a particular problem, you must next enter the values of the parameters. Starting from the “Data” menu, there are two basic ways to proceed: by choosing “Library example” or “New problem.” If you choose to load a library example, you will be asked to select one of the available problems (shown in Figure E.2 for the Heat program).

A description of each of the examples can be read by clicking the “Text + Hint” button. Once you select an example, you will see the parameters

Enter Parameters and Functions of the Problem

Heat Conduction Equation: $u_t' = a^2 u_{xx}'' + \xi u_x' - \gamma u + f(x, t)$

Temperature Source Function:
 $f(x, t) = 0$

Initial Distribution of Temperature within the Rod:
 $\phi(x) = 5 \text{imp}(x, 5, 10)$

Boundary Values:

at the left end ($x = 0$):
 0 (Zero temperature) Edit...

at the right end ($x = l$):
 2 (Given heat flow) Edit...

Parameters of the Equation:

Thermal diffusivity (> 0):
 $a^2 = 0.25$

Convection coefficient:
 $\xi = 0$

Heat exchange coefficient:
 $\gamma = 0$

Number of Terms in the Solution ($0 < N \leq 30$):
 $N = 10$

Length of the Rod (> 0):
 $l = 10$

OK Cancel Help

Figure E.3. Dialog box with problem parameters for the Heat program.

Enter Boundary Value Function

The boundary value at $x = l$ (heat exchange):
 $u_x(l, t) + h_2 u(l, t) = g_2(t) \quad (h_2 > 0)$

$h_2 = 0.5$

$g_2(t) = 2$

OK Cancel Help

Figure E.4. Boundary values dialog window for the Heat program.

of the problem in a dialog box with parameters already entered according to the selected example (Figure E.3 for the Heat program).

If you chose “New problem” in the previous step, parameters of the problem are not defined and the spaces for input functions and parameters on the previous screen (Figure E.3) will be left blank. To define boundary values, use the drop-down dialog and select one of the available options. For example, if the “Heat exchange” option is selected, parameters of the boundary value may be chosen by pressing the “Edit” button and entering values in the dialog box (Figure E.4):

As in the other programs associated with this text, the delta function $\delta(x)$ is denoted as $\text{delta}(x, x_0)$. (The reader may wish to review the properties of the delta function given in Chapter 1.) Here we remind the reader

that graphs plotted by the program using this function are limited to a maximum value equal to 1:

$$\text{delta}(x, x_0) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{if } x \neq x_0, \end{cases}$$

$$\text{delta}(x, x_0) * \text{delta}(y, y_0) = \begin{cases} 1 & \text{if } x = x_0, y = y_0, \\ 0 & \text{if } x \neq x_0 \text{ or } y \neq y_0. \end{cases}$$

but in actual calculations $\text{delta}(x, x_0)$ has the same properties as the “correct” $\delta(x)$.

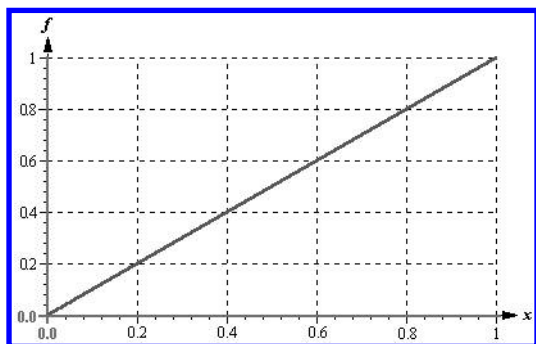
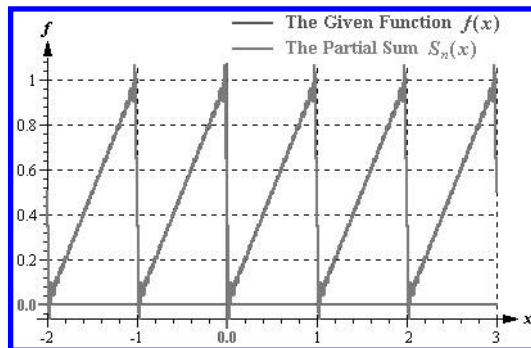
To solve cases in which functions are defined only on intervals, the following function is also helpful:

$$\text{Imp}(x, a, b) = \begin{cases} 1 & \text{if } a \leq x < b, \\ 0 & \text{otherwise.} \end{cases}$$

E.2 Examples Using the Program TrigSeries

The interactive TrigSeries program is a convenient instrument for working with trigonometric Fourier series. The function to be expanded as a Fourier series can be given either analytically or as a table of values. In the latter case, the table can be entered directly from the keyboard or as an ASCII file prepared in advance. The program also includes a formula analyzer with a wide set of functions, allowing analytical input of initial functions and real parameters (see the help topic “Mathematical Functions, Operations and Constants”). The graphical features of the program make it possible to draw graphs of the initial functions, individual members of the series, or certain partial sums of the expansions. A number of examples of how to apply the TrigSeries program to solve different problems follow.

Example E.1. Let the function $f(x) = x$ be defined in the interval $[0, l]$. Expand $f(x)$ in a trigonometric Fourier series by using the general (sine and cosine) expansion. Calculate the coefficients of the expansion. Examine the behavior of the individual harmonics of this expansion and of its partial sum $S_n(x)$ by using the option “Choose Terms...” Draw the bar chart of squared amplitudes A_n^2 and the graph of the periodic extensions in the interval $[-2l, 3l]$.

Figure E.5. The given function $f(x)$.Figure E.6. Graph of the partial sum $S_{20}(x)$, $\delta^2 = 0.314$.

Return to the dialog box for entering parameters and solve this problem by using the even- (and then odd-) terms-only method of expansion. Compare the results of these expansions.

Solution. The program TrigSeries has a set of predefined examples of expansions into trigonometric Fourier series, and the given problem is from this set. To load the problem, select “Data” from the main menu, then click “Library example.” From the given list of problems, select “Example 1.” The problem text will appear on the right side of the screen (analogous to Figure E.2). To proceed with the selected problem, click the “OK” button; in the dialog window parameters for the current problem will be displayed (analogous to Figure E.3). Since all parameters for the problem have been entered, proceed by clicking “OK.”

At this point, the program is ready to solve the problem with the selected parameters. To see the graph of the given function (Figure E.5) select the command “View” from the main menu, and then click “Return” to get back to the problem solution screen.

To run the solver, choose the “Execute” command from the main menu. Results can be seen by selecting “Results” → “Graph of the Partial Sum” (Figure E.6 shows these results). The graph range for displaying the partial sum $S_n(x)$ and the limits for the graph can be adjusted with the help of the “Set Attributes” command.

To examine the behavior of the individual harmonics of the expansion and of their partial sums, use the option “Choose Terms...” (Figure E.7).

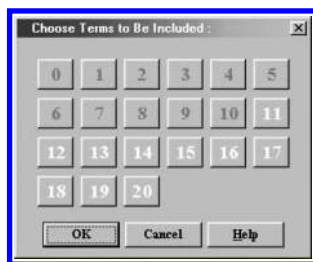
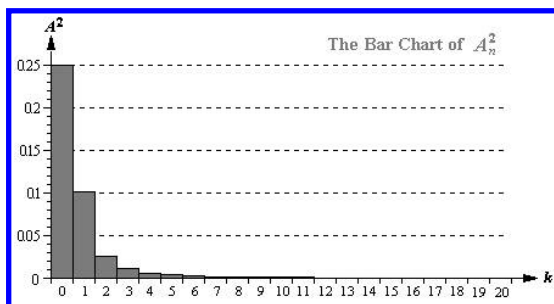


Figure E.7. Dialog window "Choose Terms to Be Included."

Figure E.8. The bar chart of squared amplitudes A_n^2 .

To select (or unselect) the individual term, click on the corresponding button. The terms whose numbers are selected (red) are included in the partial sum, and the ones not selected (white) are excluded. The buttons associated with terms that have practically zero Fourier coefficients are inactive. Click "OK" to save your selection and return to the graph.

More results can be displayed with the "Results" → "Bar Chart of Squared Amplitudes" option. Figure E.8 depicts the bar chart of squared amplitudes A_k^2 . The bar chart gives a graphic picture of the contribution of individual terms to the Fourier expansion of $f(x)$. A rectangle of a height equal to the squared amplitude, A_k^2 (measured along the vertical axis), for the harmonic with frequency $\omega_k = \pi k/T$ at each k value appears in the graph.

To investigate other properties of the solution, try these menus:

- "Results" → "Fourier Coefficients,"
- "Results" → "Graphs of Orthogonal Functions,"
- "Results" → "Tabulate Expansion."

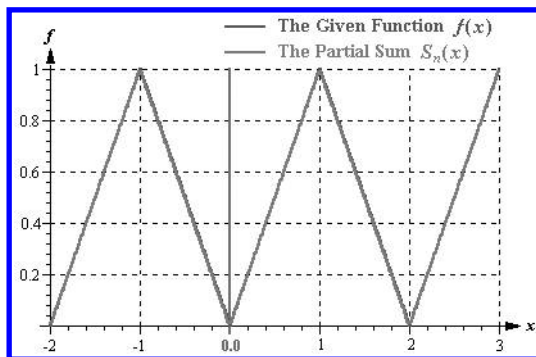


Figure E.9. Graph of the partial sum at $[-2, 3]$ (even method), $\delta^2 = 1.7 \cdot 10^{-6}$.

More information on every option can be found by selecting the “Help” menu on the corresponding screen.

To solve this problem by using only even terms in the expansion, select “Data” from the main menu, then click “Change Current Problem.” Change the method of expansion on the “Enter Parameters and Functions of the Problem” dialog by selecting the option “Even (cosines only).” Proceed by clicking the “OK” button, and then running the problem solver by selecting “Execute” from the main menu. Results of the solution can be displayed by selecting menu “Results” → “Graph of the Partial Sum” (Figure E.9).

Example E.2. Let the function

$$f(x) = \begin{cases} -\pi/2 & \text{if } -\pi \leq x < 0, \\ \pi/2 & \text{if } 0 \leq x \leq \pi, \end{cases}$$

be defined in the interval $[-\pi, \pi]$. Expand this function in a trigonometric Fourier series by using the general method of expansion ($N = 10, 20, 30$). Note that the Gibbs phenomenon (see Section 1.12) may be observed in the neighborhood of the points $x = k\pi$ ($k = 0, \pm 1, \pm 2, \dots$).

Draw the graph of the partial sum of the expansion in the interval $[0, \pi]$ and try to evaluate the coordinates of its extremes with the help of the cursor (use the option “Graph of the Partial Sum”).

Solution. The given problem is also from the library set. To load it, select “Data” → “Library example” → “Example 5.” To proceed with the

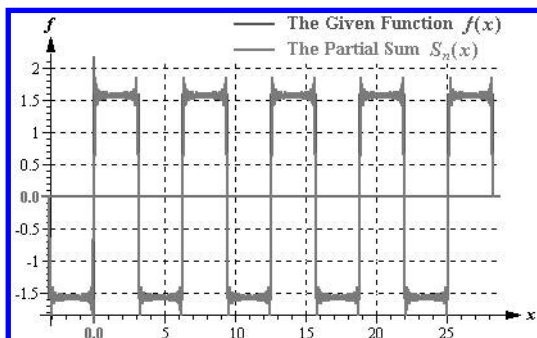


Figure E.10. Graph of the partial sum $S_{30}(x)$ at $[-\pi, 9\pi]$, $\delta^2 = 0.209$.

selected problem, click the “OK” button; the dialog window will display parameters for the current problem. The given function $f(x)$ can be entered by using the *impulse function*, $\text{Imp}(x, a, b) = 1$ if $a \leq x < b$ and 0 otherwise. Thus, the dialog box shows

$$f(x) = (-\pi/2) * \text{Imp}(x, -\pi, 0) + (\pi/2) * \text{Imp}(x, 0, \pi).$$

Since all parameters for the problem have been entered, we can proceed by clicking “OK.” To see the given function, select “View” from the main menu. To run the solver, click the “Execution” option.

Figure E.10 depicts the partial sum $S_N(x)$ ($N = 30$) on the interval $[-\pi, 9\pi]$ (use the command “Graph of the Partial Sum”).

The expansion gives (an odd) continuation of the given function $f(x)$ on the interval $(-\pi/2, \pi/2)$ to the entire x -axis. Because of the periodicity we can restrict the analysis to the interval $(0, \pi)$. To adjust the parameters of the graph, use the “Set Attributes” menu item (Figure E.11).

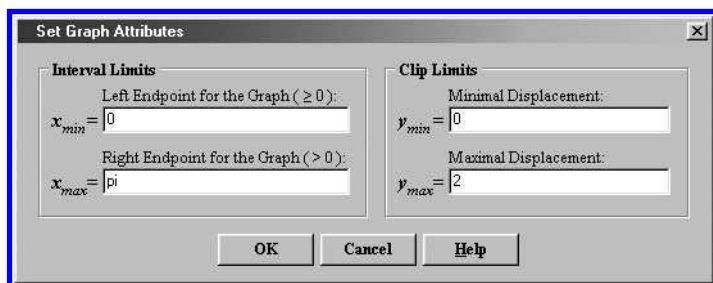


Figure E.11. “Set Attributes” dialog.

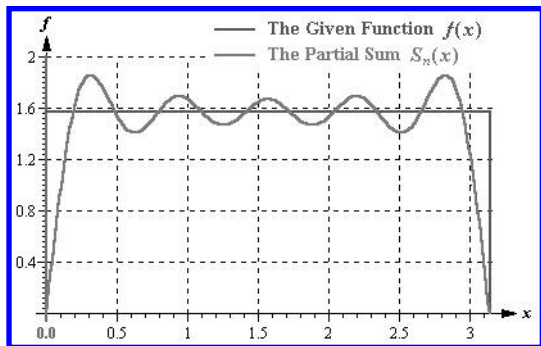


Figure E.12. Graph of the partial sum $S_{10}(x)$, $\delta^2 = 0.626$.

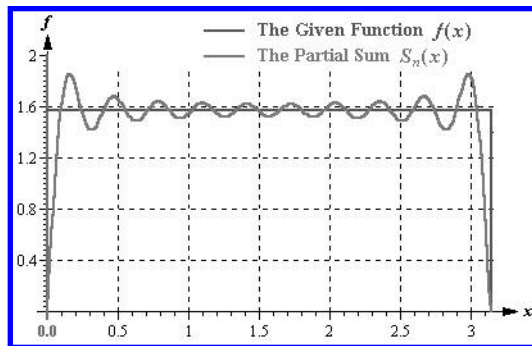


Figure E.13. Graph of the partial sum $S_{20}(x)$, $\delta^2 = 0.314$.

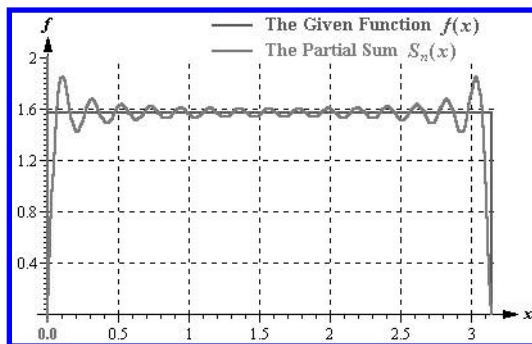


Figure E.14. Graph of the partial sum $S_{30}(x)$, $\delta^2 = 0.209$.

Figures E.12, E.13, and E.14 show the plots of the partial sum $S_N(x)$ on the interval $[0, \pi]$ for $N = 10, 20$, and 30 , respectively. An interesting phenomenon can be observed near points $x = 0$ and $x = \pi$: the graphs of the partial sums near these points oscillate about the straight line $y = \pi/2$. A significant result that can be seen in these graphs is that the amplitudes of these oscillations do not diminish to zero as N increases. To the contrary, the height of the first bump (closest to $x = 0$) approaches the value of $\Delta = 0.281$ above the $y = \pi/2$ line. The situation is similar as x approaches the value π from the left. Recall that such a defect of the convergence is the Gibbs phenomenon.

The height of the first peak of $S_N(x)$ for $N = 10, 20$, and 30 may be evaluated with the help of the screen cursor. We can also evaluate the x -

coordinates of the extremes of $S_N(x)$, and verify that the points of maxima and minima are

$$x_k = k \frac{\pi}{N}$$

(maxima for odd k and minima for even k).

Example E.3. Let the function $f(x) = (x - 1)^2$ be given at the discrete set of points in the interval $[0, 2]$. For example,

$$x_k = (k-1)\frac{2}{M}, \quad y_k = \frac{(2k - M - 2)^2}{M^2}, \quad M = 20, \quad k = 1, 2, \dots, M+1.$$

Solution. This problem is not in the library set. To solve it with the program select “New problem” from the “Data” menu on the problem screen.

The function $f(x)$ that is to be expanded as a Fourier series is given as a table. The table of values can be input directly from the keyboard (using the command “Input Data via Keyboard”) or from an ASCII [.dat] file prepared in advance (using the command “Read from File”). The length of the data file cannot exceed 1000 lines (a program limitation). Each line of the file must contain values of x_k and $y_k = f(x_k)$, separated by one or more spaces. The program allows editing of the data and provides some operations for altering the data.

After selecting the type of function, you will see the dialog window “Enter Parameters of the Problem (Table Function)” (Figure E.15). Click “General (Sines and Cosines)” to specify the properties of the expansion.

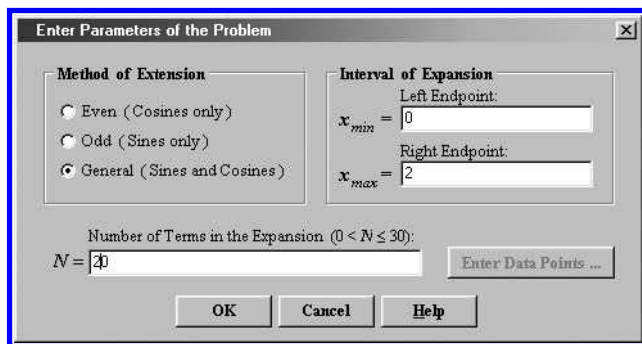


Figure E.15. “Parameters of the Problem” dialog box for Example E.3.

i	x_i	y_i
1	0	1
2	0.1	0.81
3	0.2	0.64
4	0.3	0.49
5	0.4	0.36
6	0.5	0.25
7	0.6	0.16
8	0.7	0.09
9	0.8	0.04

Number of Dots (≤ 1000):
M = 21

Transformations

Transform X
Transform Y
Exchange X and Y

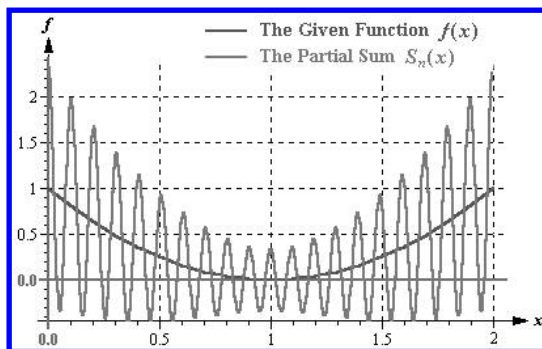
Insert Delete Save in File

OK Cancel Help

Figure E.16. Dialog window “Enter Coordinates of Data Points.”

Then enter the interval of expansion ($x_{min} = 0$, $x_{max} = 2$) and the number of terms in the expansion ($N = 20$).

The function $f(x)$ is given at the discrete set of points. To look through the table function (or to enter it), click the “Enter Data Points...” button; the dialog window with coordinates of data points will be displayed (Figure E.16). Note that, after editing, your data are put in *ascending order* of the variable x .

Figure E.17. Graph of the partial sum $S_{20}(x)$, $\delta^2 = 1.08$.

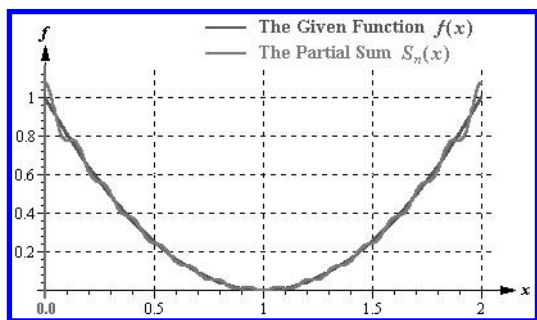


Figure E.18. Graph of the partial sum $S_{20}(x)$, $\delta^2 = 1.08$.

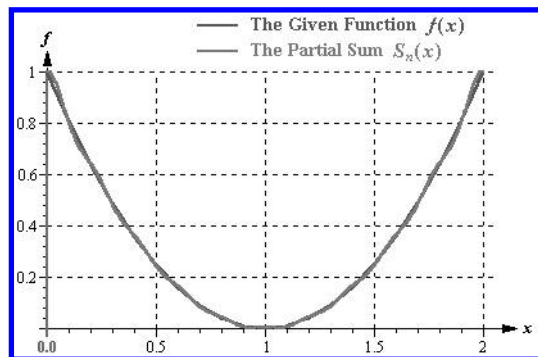


Figure E.19. Graph of the partial sum $S_{15}(x)$, $\delta^2 = 0.0011$.

When all parameters are entered, follow the same steps as in Example E.1, starting with the “Execute” command from the main menu. The partial sum of the Fourier expansion can be displayed by selecting the menu option “Results” → “Graph of the Partial Sum.”

Draw the graphs of the partial sum $S_N(x)$ for $N = 20, 15$, and 10 (by using the menu option “Choose Terms. . .”). Figures E.17, E.18, and E.19 show the resulting expansions. You can see that adding terms whose numbers are larger than $M/2$ (where M is a number of table function points) only makes the precision of the approximation worse.

E.3 Examples Using the Program Waves

A number of examples of how to apply the Waves program to solve different problems follow.

Example E.4. An elastic uniform string is fixed at the ends $x = 0$ and $x = l$. The string initially has the form of a quadratic parabola, symmetric with respect to the center of the string (h is a maximal initial deflection):

$$\varphi(x) = -\frac{4h}{l} \left(\frac{x^2}{l} - x \right).$$

At time $t = 0$, the string is released without initial velocity. Find the vibrations of the string if the resistance of a medium is absent. Choose

tension T (or the wave speed, a) so that the period of vibrations decrease by (1) twice, (2) four times, and (3) five times. Obtain the same results by changing the length, l , of the string.

To begin, the following values of the parameters are assigned: $\rho = 1$ (the mass per unit length of the string), $l = 100$ (the length of the string), $h = 5$ (the deflection of the string in $x = l/2$ at $t = 0$), $a^2 = T/\rho = 1$ (the wave speed squared).

Solution. First, select the type of problem from the main menu by clicking “Vibrations of a Finite String.” The program has a set of predefined examples on the selected subject and the given problem is from this set. To load the problem, select “Data” from the main menu and click “Library example.” From the given list of problems, select “Example 1.” A description of the problem is found on the right side of the screen. The problem description, solution hints, and theoretical solution can be accessed by clicking the “Text + Hint” button. To proceed with the selected example, click “OK” button, and parameters for the current problem will be displayed in the dialog window.

This problem involves the solution of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad a = \sqrt{\frac{T}{\rho}}.$$

Since all parameters for the problem have been entered, proceed by clicking “OK.” At this point, the program is ready to solve the problem with the selected parameters. To see the initial deviation of the string

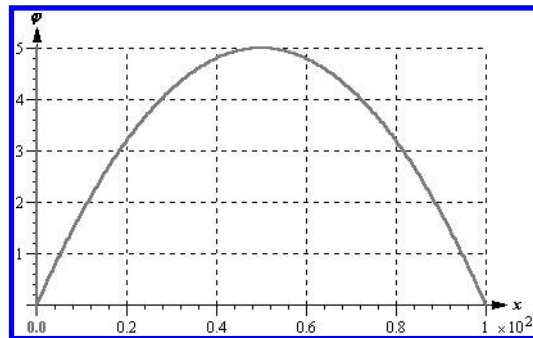


Figure E.20. Initial displacement $\varphi(x)$ for Example E.4.

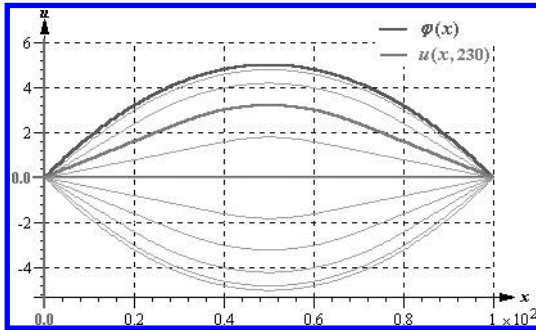


Figure E.21. Evolution of string profile for Example E.4.

shown in Figure E.20, select “View” → “First Initial Condition” from the main menu; click “Return” to go back to the problem solution screen.

To run the solver, choose the “Execute” command from the main menu. The results shown in Figure E.21 can be seen by selecting “Results” → “Evolution of String Profile.”

Graph attributes and parameters of the numerical solver can be adjusted in the “Set Attributes” menu. The parameters, initial time, length of time step, number of time steps, graph range, and initial delay may be adjusted (see Figure E.22).

The following menu options may be used to investigate other properties of the solution:

- “Results” → “Free Vibrations,”
- “Results” → “Graphs of EigenFunctions,”
- “Results” → “Time Traces of String Points,”

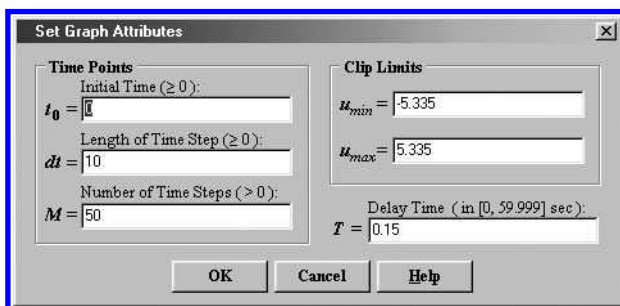


Figure E.22. “Set Graph Attributes” dialog window.

- “Results” → “Bar Charts of $|V_n(x)|$,”
- “Results” → “Energy of the System” → “Graphs of the Energies,”
- “Results” → “Energy of the System” → “Bar Charts of the Energies.”

More information on each option can be found by selecting the “Help” menu on the corresponding screen.

As derived in Chapter 3, eigenvalues and eigenfunctions of the given boundary value problem have the form

$$\lambda_n = \left[\frac{n\pi}{l} \right]^2, \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad \|X_n\|^2 = \frac{l}{2}, \quad n = 1, 2, 3, \dots$$

The frequency of the fundamental (lowest) tone is

$$\omega_1 = a \sqrt{\lambda_1} = \frac{\pi}{l} \sqrt{\frac{T}{\rho}}.$$

The period of vibrations is defined by the fundamental tone:

$$\tau_1 = \frac{2\pi}{\omega_1} = 2l \sqrt{\frac{\rho}{T}} = 200.$$

To decrease the period of vibrations τ_1 by a factor of two, choose the value of wave speed $a = \sqrt{T/\rho} = 2$ or halve the length of the string. To accomplish this, select menu option “Data” → “Change Current Problem,” and in the parameters of the problem dialog box (similar to [Figure E.3](#)), enter the value $a^2 = T/\rho = 4$ in the edit window “Wave speed squared a^2 ,” or the value $l = 50$ in the edit window “Length of the string.” Now run the solver again.

Example E.5. An elastic uniform string is fixed at the ends $x = 0$ and $x = l$. At the point $x = x_0$, the string is moved a small distance h from the equilibrium position, and at time $t = 0$ it is released with zero speed. The string is oscillating in a medium with a resistance proportional to speed.

Find the vibrations of the string. Choose a coefficient value corresponding to medium resistance so that oscillations decay (with a precision of about 5%) during (1) two periods, (2) three periods, and (3) four periods of the main mode.

Find a point x_0 that will eliminate all even harmonics. To start, the following values of the parameters are assigned: $l = 100$, $h = 6$, $\rho = 1$, $\kappa = R/2\rho = 0.001$ (damping coefficient), $a^2 = T/\rho = 1$, and $x_0 = 35$.

Solution. To load the problem, select “Data” from the main menu for Waves, and click “Library example.” From the given list of problems, select “Example 4.”

This problem involves the solution of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad a = \sqrt{\frac{T}{\rho}}.$$

$$u(x, 0) = \begin{cases} \frac{h}{x_0}x & \text{if } 0 \leq x \leq x_0, \\ \frac{h}{l-x_0}(l-x) & \text{if } x_0 < x \leq l, \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = u(l, t) = 0.$$

The initial deflection of the string can be entered using by the impulse function defined by $\text{Imp}(x, a, b) = 1$ if $a \leq x < b$ and 0 otherwise. For the given parameters, the following should be entered in the dialog box:

$$\varphi(x) = \text{Imp}(x, 0, 35) * 6 * x / 35 + \text{Imp}(x, 35, 100) * 6 * (100 - x) / (100 - 35).$$

Since all parameters for the problem have been entered, you may run the solver by choosing the “Execute” command from the main menu and then selecting “Results” → “Evolution of String Profile.” The animation sequence in Figure E.23 shows the profile of the string at times $t_k = k\Delta t$.

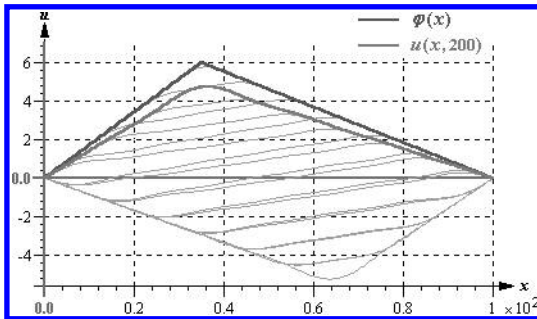


Figure E.23. Evolution of string profile for Example E.5.

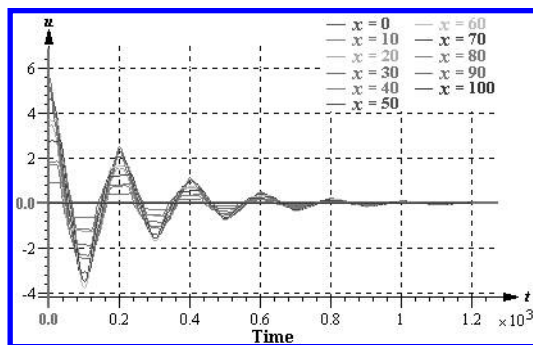
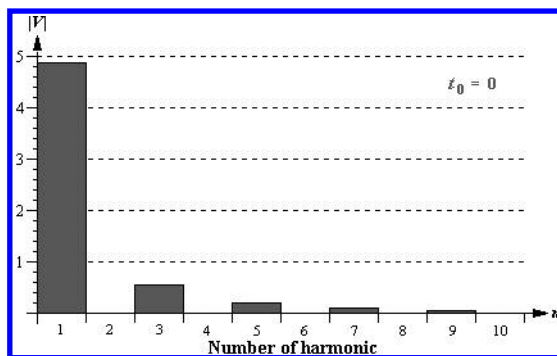


Figure E.24. Time traces of string points for Example E.5.

To change the decay time of oscillations due to friction, use the menu option “Results” → “Time Traces of String Points.” Choosing different values of the damping coefficient κ , trace the time evolution of the string until the free oscillations have decayed (with some reasonable precision).

For example, for a damping coefficient of $\kappa = 0.004$, the amplitude of oscillations fades to approximately zero during five periods of the main mode, as shown in Figure E.24.

In the expansion of the solution $u(x, t)$, the terms for which $\sin(n\pi x_0/l) = 0$ vanish; that is, the solution does not contain overtones for which the point $x = x_0$ is a node. If x_0 is at the middle of the string, the solution does not contain harmonics with even numbers. To eliminate all even harmonics, select “Data” → “Change Current Problem” and enter

Figure E.25. Bar charts of $|V_n(x)|$ for Example E.5.

the new initial function $\varphi(x)$ in the parameters of the problem dialog box:

$$\varphi(x) = \text{Imp}(x, 0, 50) * 6 * x / 50 + \text{Imp}(x, 50, 100) * 6 * (100 - x) / (100 - 50).$$

Run the problem solver and then select “Results” → “Free Vibrations” or “Results” → “Bar Charts of $|V_n(x)|$ ” to ensure that the even harmonics have been eliminated (Figure E.25).

Example E.6. An elastic uniform string is fixed at the ends $x = 0$ and $x = l$. At time $t = 0$, it is excited by a sharp blow from a hammer that transmits an impulse I to the string at point x_0 . The initial displacement is zero and the string is oscillating in a medium with a resistance proportional to speed. No external forces are present.

Find the vibrations of the string. Find the point where the hammer should strike to minimize the energies of seventh and eighth harmonics. To start, the following values of the parameters are assigned: $\rho = 1$, $a^2 = 1$, $\kappa = 0.001$, $l = 100$, $I = 10$, and $x_0 = 30$.

Solution. This problem is also from the library set. To load the problem, select “Data” → “Library example” → “Example 5.”

The boundary value problem modeling this process is given by the equations

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0, \\ u(x, 0) = 0 \quad \frac{\partial u}{\partial t}(x, 0) &= \frac{I}{\rho} \delta(x - x_0), \\ u(0, t) = u(l, t) &= 0. \end{aligned}$$

The initial distribution of velocities can be entered by using the *delta function*. Accounting for the given parameters, type $\psi(x) = 10 * \text{delta}(x, 30)$ for the initial condition in the dialog box.

To run the solver, click “Execute” from the main menu; the results can be seen by selecting “Results” → “Evolution of String Profile” (Figure E.26).

It should be obvious that if a hammer blow occurs at the node of n th harmonic the energy of this harmonic will be zero. Therefore, to decrease the energies of seventh and eighth overtones the string should be struck between the nodes of the seventh and eighth harmonics, not far from the

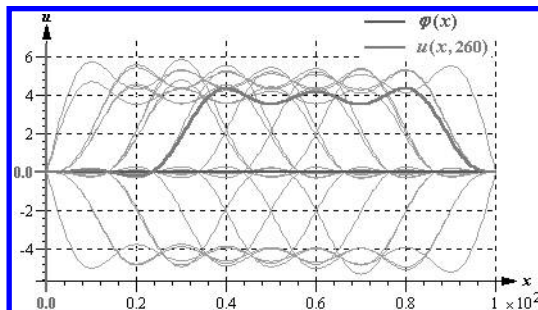


Figure E.26. Evolution of string profile with $x_0 = 30$ for Example E.6.

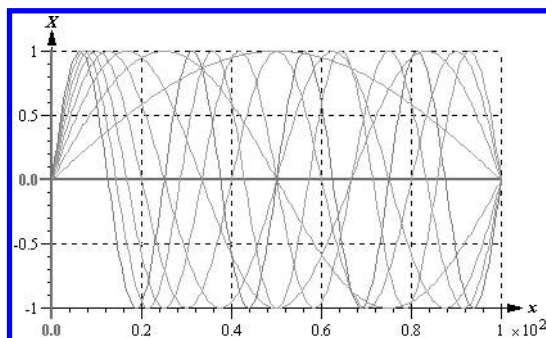


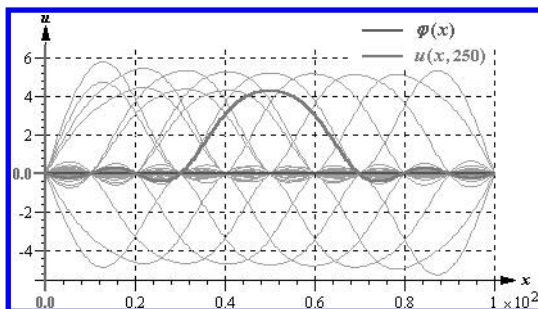
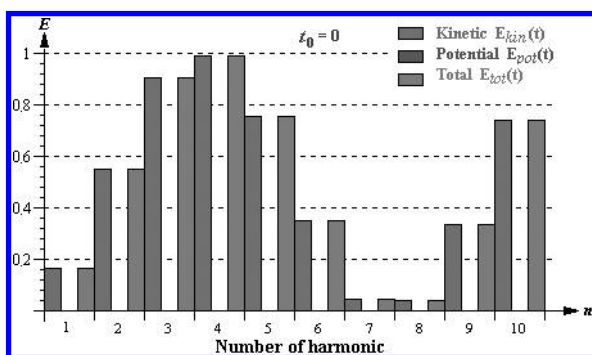
Figure E.27. Eigenfunctions for Example E.6. The red line (seen here as a solid thin curve) represents $X_8(x)$.

fixed end. Use the menu option “Graphs of Eigenfunctions” to find an adequate point for the hammer blow.

Figure E.27 shows the first eight eigenfunctions, $X_n(x)$, for Example E.6. Moving the screen pointer to some point between the first nodes of the seventh and eighth harmonics node location, we can read the value of coordinate x in the status bar and choose $x \approx 13.3$ as the location for the hammer blow.

Select menu option “Data” $\rightarrow$ “Change Current Problem,” and in the parameters of the problem dialog enter the new coordinate x_0 in the initial function: $\psi(x, 0) = 10 * \text{delta}(x, 13.3)$. The graph shown in Figure E.28 depicts the animated solution $u(x, t)$, and the red line (seen here as a solid bold curve) represents $u(x, t = 250)$.

By using menu options “Bar Charts of $|V_n(x)|$ ” and “Bar Charts of the Energies,” we can ensure that the energy in the seventh and eighth har-

Figure E.28. Evolution of string profile with $x_0 = 13.3$ for Example E.6.Figure E.29. Bar charts of energies with $x_0 = 13.3$ for Example E.6.

monics have been substantially decreased compared to other modes (see Figure E.29).

By studying the energy in each harmonic and choosing different values of the impulse, I , it can be seen that the energy of the string is proportional to I^2 .

Two-Dimensional Problems. All the preceding examples in this appendix were one-dimensional problems. The Waves program can also solve two-dimensional problems, such as those involving vibrations of thin rectangular or circular membranes. Next, we consider several examples showing the capacity of the program to deal with these types of problems. The theory related to this topic can be found in Chapter 4 of this book and in the “Help” system of the Waves program.

Example E.7. A flexible rectangular membrane is clamped at the edges, $x = 0$, $y = 0$; the edge $y = l_y$ is attached elastically; and the edge $x = l_x$

is free. Initially, the membrane is at rest in a horizontal plane. Starting at time $t = 0$, a uniformly distributed transversal force with density

$$f(x, y, t) = Ae^{-0.5t}x \sin \frac{4\pi y}{l_y}$$

acts on the membrane.

Find the transverse vibrations of the membrane if the resistance of the surrounding medium is proportional to the speed of the membrane. Choose a coefficient of medium resistance so that oscillations decay to zero (with precision of about 5%) during times (1) $t = 20$, (2) $t = 30$, and (3) $t = 40$. The assigned parameter values are $A = 0.5$, $a^2 = 1$, $\kappa = 0.01$, $l_x = 4$, and $l_y = 6$.

Solution. First, select the type of problem from the main menu by clicking “Vibrations of a Rectangular Membrane.” The given problem is not from the library set. To solve it, instead of selecting a library example as we did before, select “New problem” from the “Data” menu on the problem screen.

The problem involves solving the wave equation,

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} - a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] = Ae^{-0.5t}x \sin \frac{4\pi y}{l_y}$$

under the conditions

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 0, \quad u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(l_x, y, t) = 0,$$

$$u(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, l_y, t) + h_4 u(x, l_y, t) = 0.$$

To specify the boundary condition at the edge $y = l_y$, choose the option “3 (elastic fixing),” by selecting this item from the dialog box. Enter the parameters by clicking the “Edit” button and typing $h_4 = 0.1$ and $g(x, t) = 0$.

To see the animated surface plot of the external force, select “View” → “External force $f(x, y, t)$ ” from the main menu (Figure E.30) and click “Return” to go back to the problem solution screen.

To solve the problem with the help of the program, follow the same steps as in Example E.6. Results of the solution can be displayed by

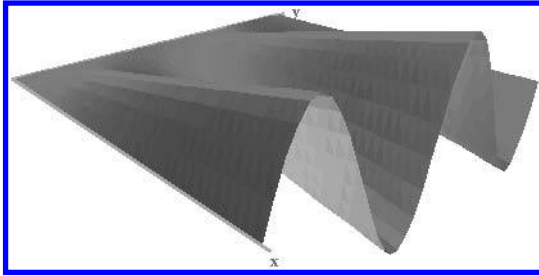


Figure E.30. External force $f(x, y, t)$ at $t = 1.5$ for Example E.7.

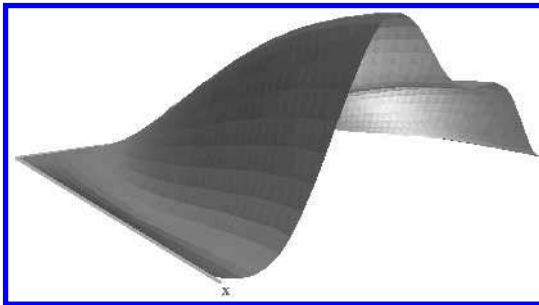


Figure E.31. Surface graph of membrane at $t = 10.5$ for Example E.7.

selecting the menu option “Surface Graphs of Membrane.” The three-dimensional surface shown in Figure E.31 depicts the animated solution $u(x, y, t)$ at $t = 10.5$. Note that the edges of the surface adhere to the given boundary conditions.

The following menus may be used to investigate other properties of the solution:

- “Results” → “Free Vibrations,”
- “Results” → “Surface Graphs of EigenFunctions,”
- “Results” → “Evolution of Membrane Profile at $y = \text{const}$,”
- “Results” → “Evolution of Membrane Profile at $x = \text{const}$.”

More information on each option can be found by selecting the “Help” menu on the corresponding screen.

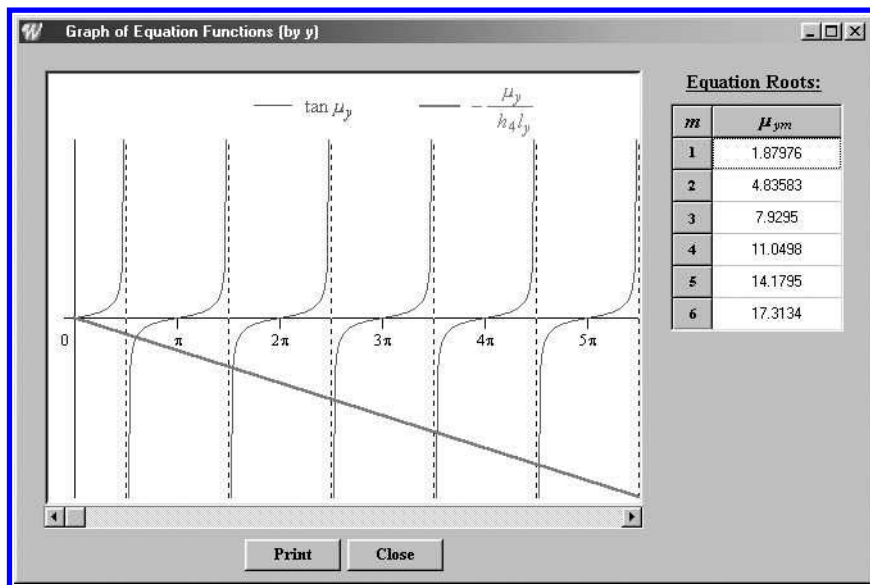


Figure E.32. Graphs of functions for evaluating eigenvalues and the table of roots μ_{ym} for Example E.7.

Eigenvalues of the given boundary value problem have the form

$$\lambda_{nm} = \lambda_{xn} + \lambda_{ym} \text{ where } \lambda_{xn} = \left[\frac{\pi(2n-1)}{2l_x} \right]^2,$$

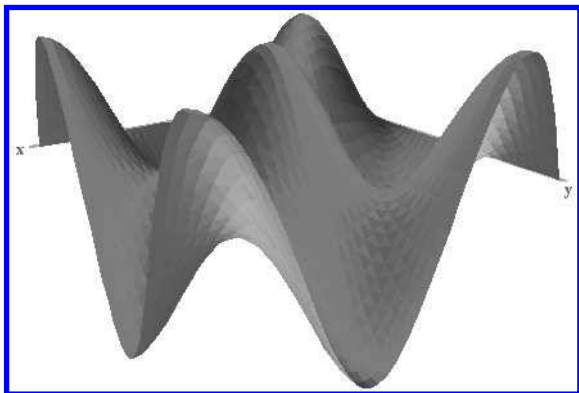
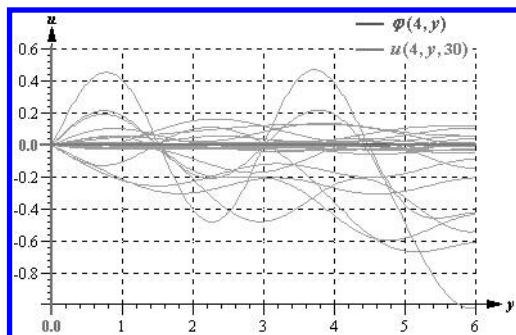
$$\lambda_{ym} = \left[\frac{\mu_{ym}}{l_y} \right]^2, \quad n, m = 1, 2, 3, \dots$$

and μ_{ym} is the m th root of the equation $\tan \mu_y = -\mu_y / (h_4 l_y)$. Figure E.32 shows graphs of the functions $\tan \mu_y$ and $-\frac{\mu_y}{h_4 l_y}$ used for evaluating eigenvalues. Eigenfunctions of the given boundary value problem are

$$V_{nm}(x, y) = X_n(x)Y_m(y) = \sin \frac{(2n-1)\pi x}{2l_x} \sin \sqrt{\lambda_{ym}} y.$$

The three-dimensional picture shown in Figure E.33 depicts one of the eigenfunctions for the problem

$$\lambda_{33} = 5.6019, \quad V_{33}(x, y) = \sin \frac{5\pi x}{2l_x} \sin \sqrt{\lambda_{33}} y, \quad \|V_{33}\|^2 = 6.05693.$$

Figure E.33. Eigenfunction $V_{33}(x, y)$ for Example E.7.Figure E.34. Evolution of membrane profile at $x = l_x$, $\kappa = 0.175$ for Example E.7.

To change the decay time of oscillations due to friction, use one of the following menu options: “Surface Graphs of Membrane,” “Evolution of Membrane Profile at $y = \text{const.}$ ” or “Evolution of Membrane Profile at $x = \text{const.}$ ” Choosing different values of the damping coefficient, κ , trace the time evolution of the membrane until the oscillations decay (with some reasonable precision). For example, for damping coefficient $\kappa = 0.175$, the amplitude of oscillations fades away during time $t = 30$ (Figure E.34).

Example E.8. Find the transversal vibrations of a flexible circular membrane caused by the motion of its periphery under the constraint

$$g(\varphi, t) = A \sin \omega t.$$

The initial displacement and initial velocity are zero. Assume the surrounding medium offers no resistance. The assigned parameter values are $A = 0.2$, $\omega = 3$, $a^2 = 1$, $l = 2$.

Solution. Select the type of problem from the main menu by clicking “Vibrations of a Circular Membrane.” This problem is from the library set. To load the problem, select “Data” $\rightarrow$ “Library example” $\rightarrow$ “Example 6.” To proceed with the selected problem, click the “OK” button, and the parameters for the current problem will be displayed in the dialog window.

The problem involves the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] = 0$$

under the conditions

$$u(r, \varphi, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = A \sin \omega t.$$

Eigenvalues of the given boundary value problem have the form

$$\lambda_{nm} = \left[\frac{\mu_m^{(n)}}{l} \right]^2, \quad n, m = 0, 1, 2, \dots,$$

where $\mu_m^{(n)}$ are positive roots of the equation $J_n(\mu) = 0$. Figure E.35 shows the graph of function $J_n(\mu)$ for evaluating eigenvalues for the selection $n = 0$.

The eigenfunctions of the given boundary value problem are

$$V_{nm}^{(1)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi.$$

The three-dimensional picture shown in Figure E.36 depicts one of the eigenfunctions for this problem:

$$\lambda_{11} = 12.3046, \quad V_{11}^{(1)} = J_1 \left(\mu_1^{(1)} r / 2 \right) \cos \varphi, \quad \left\| V_{11}^{(1)} \right\|^2 = 0.565923.$$

To solve the problem with the help of the program, follow the same steps as in Problem E.7. Results of the solution can be displayed by

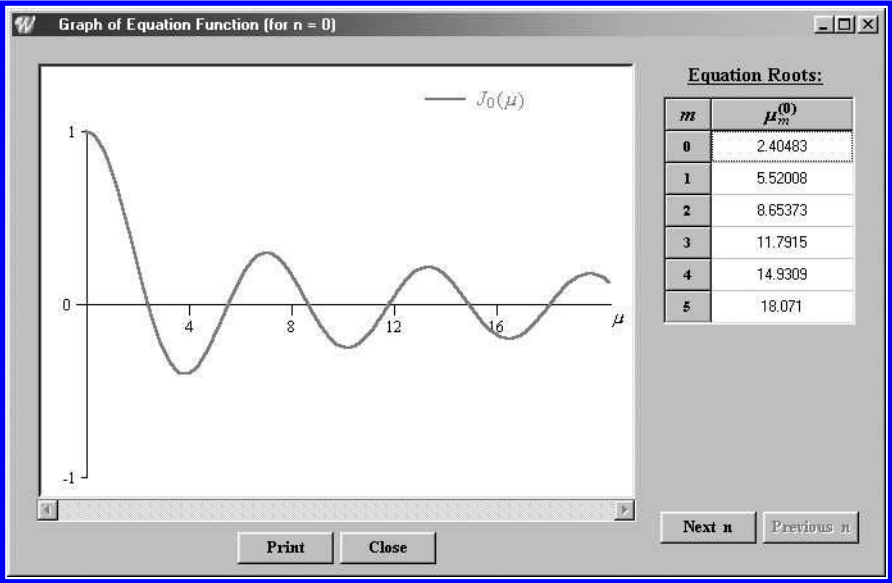


Figure E.35. Graph of equation for evaluating eigenvalues for $n = 0$ and the table of roots $\mu_m^{(0)}$ for Example E.8.

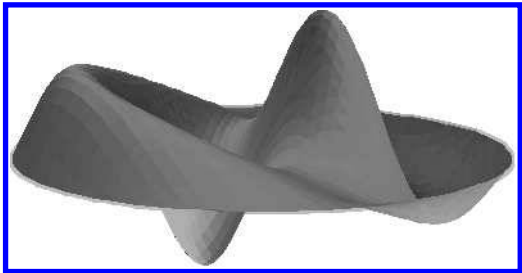


Figure E.36. Eigenfunction $V_{11}^{(1)}$ for Example E.8.

selecting the menu option “Surface Graphs of Membrane.” The three-dimensional surface shown in Figure E.37 depicts the animated solution $u(r, \varphi, t)$ at $t = 13$.

The solution to the given problem can be expressed as the sum of two functions

$$u(r, \varphi, t) = w(r, \varphi, t) + v(r, \varphi, t),$$

where $w(r, \varphi, t) = A \sin \omega t$ is an auxiliary function satisfying the boundary value function and $v(r, \varphi, t)$ is the solution of the boundary value prob-

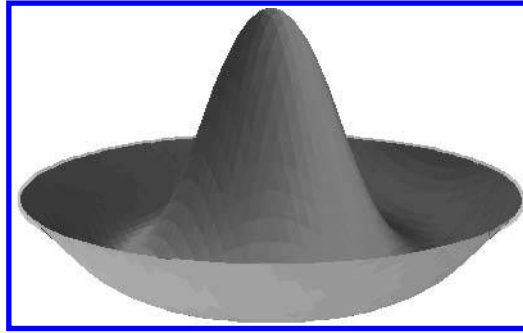


Figure E.37. Surface graph of membrane at $t = 13$ for Example E.8.

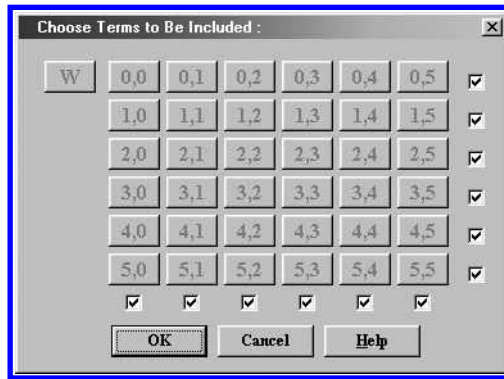


Figure E.38. Selection of terms in the Fourier expansion of the solution.

lem with zero boundary conditions, where

$$f^*(r, \varphi, t) = A\omega^2 \sin \omega t,$$

$$\phi^*(r, \varphi) = 0, \quad \psi^*(r, \varphi) = -A\omega.$$

The program Waves allows you to study the behavior of the auxiliary function $w(r, \varphi, t)$ and of any single term of the partial sum of the solution,

$$u_{nm}(r, \varphi, t) = w(r, \varphi, t) + \sum_{i=0}^n \sum_{j=0}^m \left\{ \left[a_{ij} y_{ij}^{(1)}(t) + b_{ij} y_{ij}^{(2)}(t) \right] V_{ij}^{(1)}(r, \varphi) + \left[c_{ij} y_{ij}^{(1)}(t) + d_{ij} y_{ij}^{(2)}(t) \right] V_{ij}^{(2)}(r, \varphi) \right\}.$$

Select the “Choose Terms...” option from the graph menu to display the “Choose Terms to Be Included” dialog box (Figure E.38). To select (or

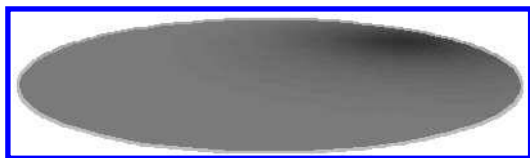


Figure E.39. Surface graph of the function $w(r, \varphi, t)$ at $t = 13$ for Example E.8.

unselect) the individual term, click on the corresponding button. The terms whose numbers are selected (red) are included in the partial sum, and the ones not selected (white) are excluded. The buttons associated with terms that have practically zero Fourier coefficients are inactive.

The button “W” corresponds to the auxiliary function $w(r, \varphi, t)$. Select the button “W” and unselect all other buttons to see the surface graph of the auxiliary function $w(r, \varphi, t) = A \sin \omega t$. Figure E.39 shows this graph for $t = 13$.

E.4 Examples Using the Program Heat

Examples of how to use the Heat program to solve different problems follow.

Example E.9. A heat-conducting cylindrical rod of length l is thermally insulated over its lateral surface. The left end of the rod ($x = 0$) is kept at a constant temperature of zero and the right end ($x = l$) is thermally insulated. The initial temperature of the rod is

$$\varphi(x) = \begin{cases} 0 & \text{if } 0 < x < l/2, \\ u_0 & \text{if } l/2 < x < l, \end{cases} \text{ where } u_0 = \text{const.}$$

Find the distribution of temperature, $u(x, t)$, in the rod at later times if generation (or absorption) of heat by internal sources is absent. To start, the following values of the parameters may be assigned: $l = 10$, $u_0 = 5$, and $a^2 = 0.25$.

Solution. First, select the type of problem from the main menu by clicking “Heat Conduction within a Finite Uniform Rod.” The program has a set of predefined examples on the selected subject, and Problem E.9 is from this set. To load the problem, select “Data” from the main menu and click “Library example.” From the given list of problems, select “Example 2.”

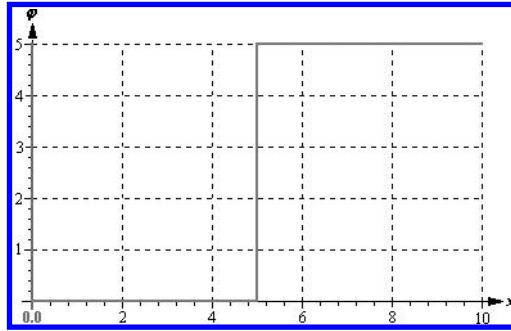


Figure E.40. Initial condition $u(x, 0) = \varphi(x)$ for Example E.9.

The problem description may be read on the right side of the screen (see [Figure E.2](#)). The problem description, solution hints, and theoretical solution can be accessed by clicking the “Text + Hint” button. To proceed with the selected problem, click the “OK” button to display parameters for the current example in the dialog box.

Since all parameters for the problem have been entered, we can proceed by clicking “OK.” At this point, the program is ready to solve the problem with selected parameters. To see the initial temperature distribution, select “View” → “Initial condition $u(x, 0)$ ” from the main menu, then click “Return” to get back to the problem solution screen.

To run the solver, choose the “Execute” command from the main menu and then select “Results” → “Evolution of Rod Temperature.” The animation sequence in [Figure E.41](#) shows the profile of the rod temperature at times $t_k = k\Delta t$. As can be seen from [Figure E.41](#), the temperature in the rod is approaching zero along its entire length as time goes to infinity.

Graph attributes and parameters of the numerical solver can be adjusted in the “Set Graph Attributes” menu. Available parameters are shown in [Figure E.42](#).

Other properties of the solution may be investigated by accessing the following menus:

- “Results” → “Free Heat Exchange,”
- “Results” → “Graphs of EigenFunctions,”
- “Results” → “Time Traces of Temperature at Rod Points,”
- “Results” → “Bar Charts of $|V_n(x)|$.”

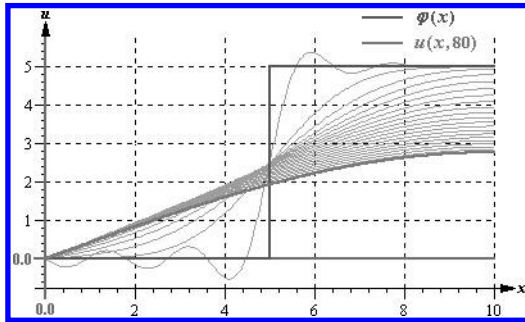


Figure E.41. The animated solution $u(x, t)$ for Example E.9.

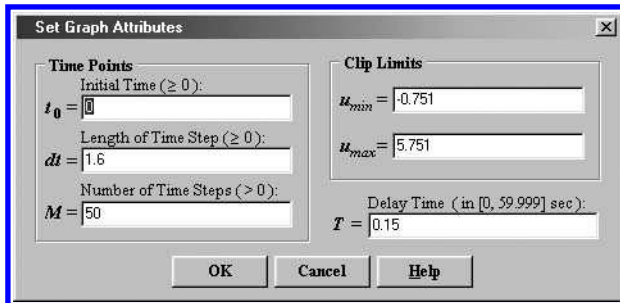


Figure E.42. “Set Graph Attributes” dialog window.

More information on each option can be found by selecting the “Help” menu on the corresponding screen.

Next, we consider several problems connected to Example E.9. They can be solved by changing the initial conditions, boundary conditions, and constants of the problem in the “Enter Parameters and Functions of the Problem” dialog box. Available sets of analytic functions and mathematical operators can be found in the program “Help” menu under the topic “Mathematical Functions, Operations and Constants.”

Example E.10. Repeat Example E.9, but with both ends of the rod thermally insulated.

Solution. This problem is not in the library set. To solve it with the help of the program we will need to modify parameters of the existing library

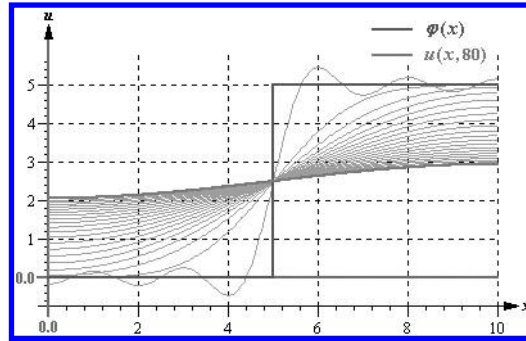


Figure E.43. The animated solution $u(x, t)$ for Example E.10.

problem. First, follow the same steps as in Example E.9, as far as the “Enter Parameters and Functions of the Problem” screen. Here, we need to change the boundary condition at the left end of the rod ($x = 0$). To make boundary conditions appropriate to the given problem, for the left end of the rod, select, “2 (Given heat flow)” from the dialog box (see Figure E.3). To assign a value to a heat flow, click “Edit.” For this case, the value should be zero.

At this point, we may proceed by clicking the “OK” button and run the problem solver by selecting “Execute” from the main menu. Results of the solution can be displayed by selecting the menu option “Evolution of Rod Temperature.” In this case, the temperature tends to a uniform distribution after a significant period of time, as shown in Figure E.43.

Example E.11. The initial temperature of a slender wire of length l lying along the x -axis is $u(x, 0) = u_0$, where $u_0 = \text{const}$, and the ends of the wire are kept under the constant temperatures $u(0, t) = u_1 = \text{const}$, $u(l, t) = u_2 = \text{const}$. The wire is experiencing a heat loss through the lateral surface proportional to the difference between the wire temperature and the surrounding temperature. The temperature of the surrounding medium is $u_{md} = \text{const}$, and we introduce the coefficient γ as a heat loss coefficient.

Find the distribution of temperature $u(x, t)$ in the wire for $t > 0$ if it is free of internal sources of heat. The assigned values of the parameters are $a^2 = 0.25$ (thermal diffusivity of the material), $l = 10$, $\gamma = h/c\rho = 0.05$, $u_{md} = 15$ (the temperature of the medium), and $u_0 = 5$ (initial temperature of the wire).

Solution. This problem is also given in the library set. To load it, select “Data” → “Library example” → “Example 6.”

The problem is an example of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - \gamma(u - u_{md})$$

under the conditions

$$u(x, 0) = u_0, \quad u(0, t) = u_1, \quad u(l, t) = u_2.$$

Since all parameters for the problem have been entered, you may run the solver by choosing the “Execute” command from the main menu and then selecting “Results” → “Evolution of Rod Temperature.” The animation sequence in Figure E.44 shows the solution $u(x, t_k)$ at times $t_k = k \Delta t$.

The menu option “Time Traces of Temperature at Rod Points” allows you to investigate the significance of the changes in magnitudes of the

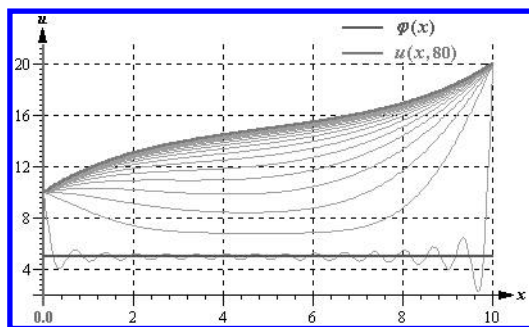


Figure E.44. The animated solution $u(x, t)$ for Example E.11.

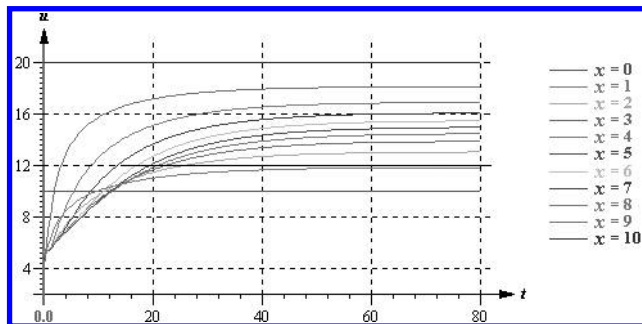


Figure E.45. Time traces of temperature at rod points for Example E.11.

diffusivity, a^2 , and the coefficient of lateral heat exchange with a medium, γ , by modifying them and noting their effect on the solution to the problem (Figure E.45).

More results can be displayed with the “Bar charts of $|V_n(t)|$ ” command. This command produces a sequence of bars of the values

$$V_n(t) = T_n(t) + C_n e^{-(a^2 \lambda_n + \gamma)t}$$

at successive time points. Each bar chart gives a graphic picture of the contribution of individual terms to the solution $u(x,t)$ at a certain time. Bar charts display data by drawing a rectangle of the height equal to $|V_n(t)|$ (measured along the vertical axis) at each n value along the graph. Therefore, the height of the n th rectangle corresponds to the absolute value of the amplitude of the n th harmonic at the respective time t .

The sequence of bars (Figure E.46) shows that the higher-order terms decay more rapidly.

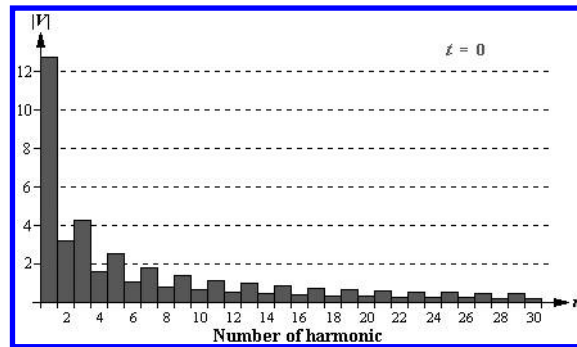


Figure E.46. Bar charts of $|V_n(t)|$ at $t = 0$ for Example E.11.

Two-Dimensional Problems. All the above examples E.9 through E.11 are one-dimensional problems, since we neglected the width of the rod. The Heat program can also solve two-dimensional problems, such as heat conduction in thin rectangular or circular membranes. Several examples showing the capacity of the program to deal with these types of problems are considered next. The theory related to this topic can be found in Chapter 6 of the book and in the “Help” system of the program.

Example E.12. A heat-conducting, thin, uniform, rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. One part of the boundary (at $x = 0$ and $y = 0$) is thermally insulated, and the other part (at $x = l_x$ and $y = l_y$) is held at a constant zero temperature. The initial temperature distribution within the membrane is $u(x, y, 0) = u_0 = \text{const}$. A constant internal source of heat acts at the point (x_0, y_0) of the membrane. The magnitude of this source is $Q = \text{const}$.

Find the temperature distribution $u(x, y, t)$ in the membrane at $t > 0$. Find the time, τ , when the temperature distribution in the membrane arrives at the steady-state mode. Study the influence of the thermal diffusivity coefficient, a^2 , on this time. To start, the following values of the parameters are assigned: $a^2 = 0.25$ (thermal diffusivity of the material), $l_x = 4$, $l_y = 6$, $u_0 = 10$, $Q = 50$, $x_0 = 2$, and $y_0 = 3$.

Solution. First, select the type of problem from the main menu by clicking “Heat Conduction within a Thin Rectangular Membrane.” The present problem is not in the library set. To solve it, instead of selecting a library example, as we did before, select “New problem” from the “Data” menu on the problem screen.

The problem involves the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + Q\delta(x - x_0)\delta(y - y_0)$$

under the conditions

$$u(x, y, 0) = u_0,$$

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \quad u(l_x, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad u(x, l_y, t) = 0.$$

When all parameters are entered, click the “OK” button. To see the source function, select “View” → “Temperature Source Function $f(x, y, t)$ ” from the main menu. Figure E.47 shows this function.

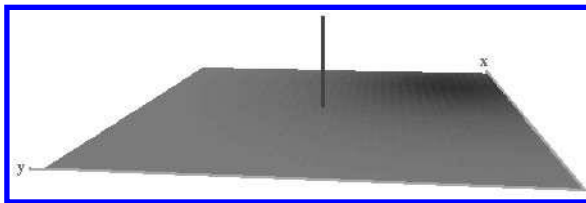


Figure E.47. Temperature source function $f(x, y, t)$ for Example E.12.

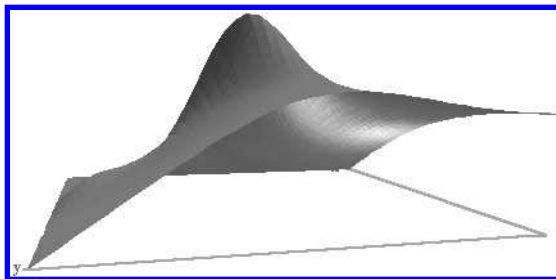


Figure E.48. Surface plot of the solution $u(x, y, t)$ at $t = 60$ for Example E.12.

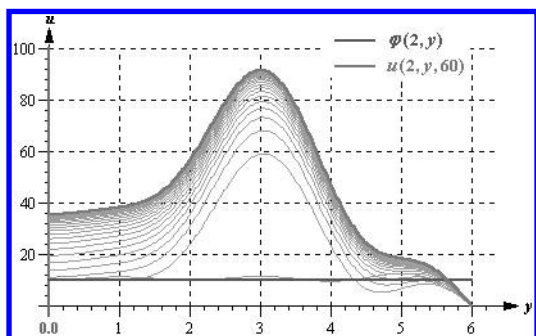


Figure E.49. Animated sequence of the solution profile $u(2, y, t)$; $a^2 = 0.25$.

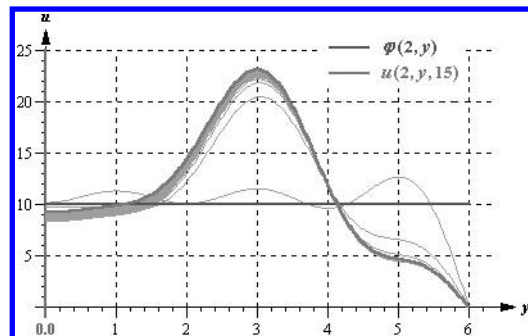


Figure E.50. Animated sequence of the solution profile $u(2, y, t)$; $a^2 = 1$.

To run the solver, click “Execute” from the main menu. Results can be seen by selecting “Results” → “Evolution of the Membrane Temperature.” Figure E.48 shows a snapshot at time $t = 60$ from an animation sequence.

It is convenient to use the options “Evolution of Temperature Profile at $x = \text{const}$ ” and “Evolution of Temperature Profile at $y = \text{const}$ ” to find the time, τ , when the temperature distribution in the membrane arrives at the steady-state mode and to study the influence of the thermal diffusivity coefficient, a^2 , on this time. Figures E.49 and E.50 show the animation sequences for two different values of thermal diffusivity; $a^2 = 0.25$ and $a^2 = 1$. You can see that if a^2 is quadrupled, then the time τ (and the maximal value of temperature) decreases by a factor of 4. By changing the dimensions of the membrane, the same result can be obtained and studied.

Example E.13. A heat-conducting, thin, uniform, rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. One part of the boundary ($x = 0$, $y = 0$) is kept under a constant temperature $u = u_1$, and the other part ($x = l_x$, $y = l_y$) under a constant temperature $u = u_2$. The initial temperature distribution within the membrane is $u(x, y, 0) = u_0 = \text{const}$.

Find the temperature $u(x, y, t)$ of the membrane at any later time if generation (or absorption) of heat by internal sources is absent. The assigned parameter values are $a^2 = 0.25$ (thermal diffusivity of the material), $l_x = 4$, $l_y = 6$, $u_0 = 10$, $u_1 = 20$, and $u_2 = 50$.

Solution. This problem is from the library set. To load it, select “Data” from the main menu, and then click “Library example.” From the given list of problems, select “Example 7.”

The problem depends upon the solution to the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

under the conditions

$$u(x, y, 0) = u_0, \quad u(0, y, t) = u(x, 0, t) = u_1, \quad u(l_x, y, t) = u(x, l_y, t) = u_2.$$

In this problem, the boundary value functions don’t satisfy the conforming conditions at points $(0, l_y)$ and $(l_x, 0)$:

$$P_1[g_4]_{x=0} = u_2 \neq P_4[g_1]_{y=l_y} = u_1, \quad P_2[g_3]_{x=l_x} = u_1 \neq P_3[g_2]_{y=0} = u_2.$$

As a result, when you run the solver by clicking “Execute” from the main menu, the following message will be displayed ([Figure E.51](#)):

Although the boundary conditions do not match, we may still solve the problem with the help of the program by searching for a generalized solution. Start by following the same steps as in Example E.12. Results of the solution can be displayed by selecting menu option “Evolution of Membrane Temperature.” The three-dimensional surface shown in [Figure E.52](#) depicts the frame of the animated solution $u(x, y, t)$ at $t = 0.1$. Note that the edges of the surface adhere to the given boundary conditions.

The solution to the given problem can be expressed as the sum of three functions

$$u(x, y) = w_1(x, y) + w_2(x, y) + v(x, y),$$

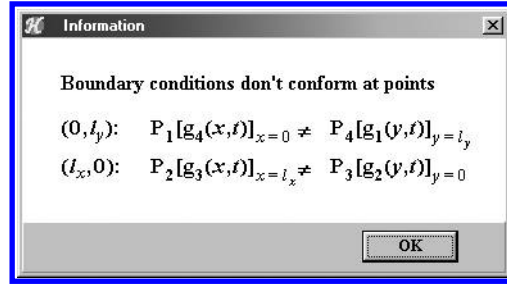
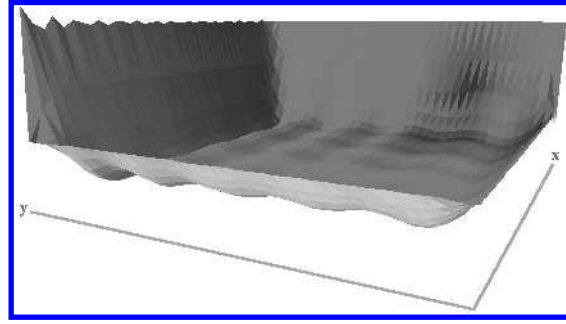


Figure E.51. Information window for Example E.13.

Figure E.52. Distribution of temperature $u(x, y, t)$ at $t = 0.1$ for Example E.13.

where $w_1(x, y)$ is an auxiliary function satisfying the boundary value functions

$$P_1[w_1]_{x=0} = g_1(y, t) = u_1, \quad P_2[w_1]_{x=l_x} = g_2(y, t) = u_2,$$

$$P_3[w_1]_{y=0} = u_1 + (u_2 - u_1) \frac{x}{l_x}, \quad P_4[w_1]_{y=l_y} = u_1 + (u_2 - u_1) \frac{x}{l_x};$$

$w_2(x, y)$ is a particular solution of Laplace problem with the following boundary conditions

$$P_1[w_2]_{x=0} = 0, \quad P_2[w_2]_{x=l_x} = 0,$$

$$P_3[w_2]_{y=0} = \frac{x}{l_x} (u_1 - u_2), \quad P_4[w_2]_{y=l_y} = (u_2 - u_1) \left(1 - \frac{x}{l_x}\right).$$

and $v(x, y)$ is a solution of a diffusion equation with zero boundary conditions:

$$v(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} e^{-\lambda_{nm} a^2 t} V_{nm}(x, y).$$

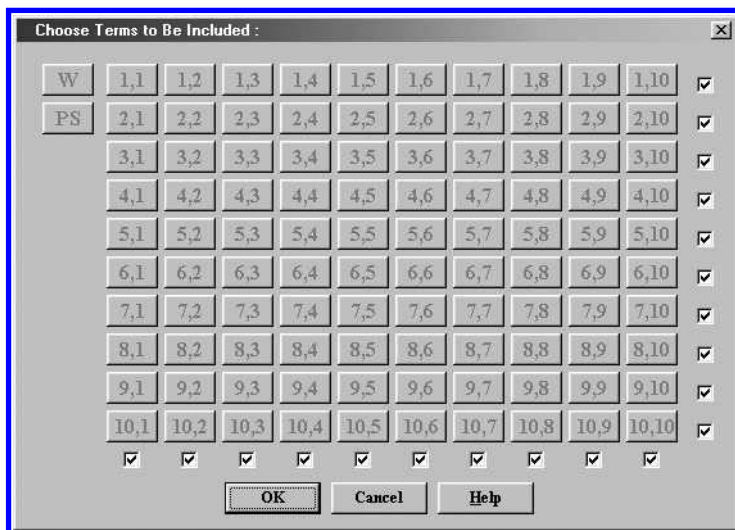


Figure E.53. Selection of terms in the expansion of the solution.

The program Heat allows you to study the behavior of the functions $w_1(x, y, t)$, $w_2(x, y, t)$ and of any single term of the partial sum of the solution

$$u_{NM}(x, y, t) = w_1(x, y, t) + w_2(x, y, t) + \sum_{n=1}^N \sum_{m=1}^M C_{nm} e^{-\lambda_{nm} a^2 t} V_{nm}(x, y).$$

Select the “Choose Terms...” option from the graph menu to display the “Choose Terms to Be Included” dialog box (Figure E.53). To select (or unselect) the individual term, click on the corresponding button. The terms whose numbers are selected (red) are included in the partial sum, and the ones not selected (white) are excluded. The buttons associated with terms that have practically zero Fourier coefficients are inactive.

The button “W” corresponds to the auxiliary function $w_1(x, y, t)$. Select the button “W” and unselect all other buttons to see the surface graph of the auxiliary function

$$w_1(x, y, t) = u_1 + (u_2 - u_1) \frac{x}{l_x},$$

shown in Figure E.54.

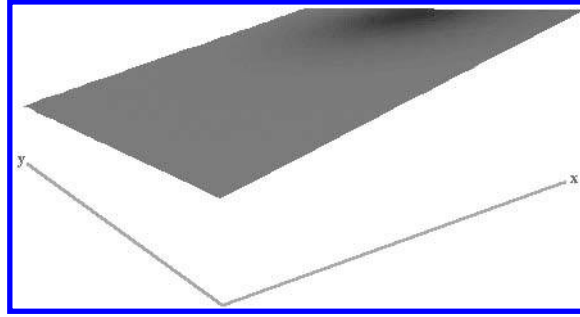


Figure E.54. Surface plot of the auxiliary function $w_1(x, y, t)$ for Example E.13.

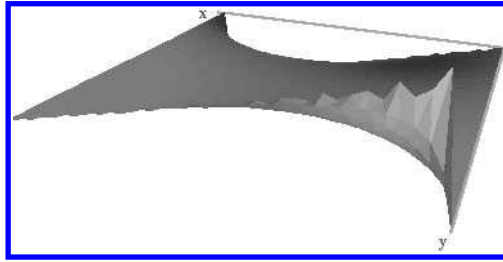


Figure E.55. Surface plot of the particular solution $w_2(x, y, t)$ for Example E.13.

The button “PS” corresponds to the particular solution $w_2(x, y, t)$. This button is active because the boundary value functions don’t satisfy the conforming conditions. Select the button “PS” and unselect all other buttons to see the surface graph of the particular solution

$$w_2(x, y, t) = \frac{2}{\pi} (u_2 - u_1) \sum_{n=1}^{\infty} \frac{1}{n \cdot \sinh \sqrt{\lambda_n} l_y} \left\{ \sinh \sqrt{\lambda_n} y + (-1)^n \sinh \sqrt{\lambda_n} (l_y - y) \right\} \sin \frac{n\pi x}{l_x},$$

shown in Figure E.55.

Unselect the buttons “W” and “PS,” and the partial sum

$$v_{NM}(x, y, t) = \sum_{n=1}^N \sum_{m=1}^M C_{nm} e^{-\lambda_{nm} a^2 t} V_{nm}(x, y)$$

will be displayed (Figure E.56).

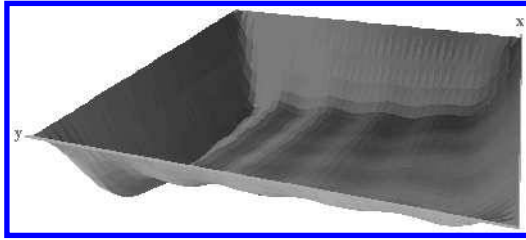


Figure E.56. Surface plot of the partial sum $v_{NM}(x, y, t)$ for Example E.13.

Example E.14. Consider a very long (infinite) cylinder of radius l . At time $t = 0$, let a magnetic field, parallel to the cylinder axis, be instantly established outside of the cylinder. The external magnetic field strength is

$$H = H_0 \sin \omega t, \quad H_0 = \text{const}, \quad 0 < t < \infty.$$

Find the magnetic field strength within the cylinder if the initial field is zero. To start, take the following values of the parameters: $a^2 = 0.25$, $l = 4$, $H_0 = 5$, and $\omega = 2$.

Solution. This problem is mathematically identical to a heat conduction problem found in the program library of problems. Select the type of the problem from the main menu by clicking “Heat Conduction within a Thin Circular Membrane.” To load the appropriate problem, select “Data” → “Library example” → “Example 6.”

The problem depends on the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right]$$

under the conditions

$$u(r, \varphi, 0) = 0, \quad u(l, \varphi, t) = H_0 \sin \omega t.$$

Eigenvalues of the given boundary value problem have the form

$$\lambda_{nm} = \left[\frac{\mu_m^{(n)}}{l} \right]^2, \quad n, m = 0, 1, 2, \dots,$$

where $\mu_m^{(n)}$ are positive roots of the equation $J_n(\mu) = 0$. Figure E.57 shows the graph of the function $J_n(\mu)$ and associated eigenvalues for $n = 2$.

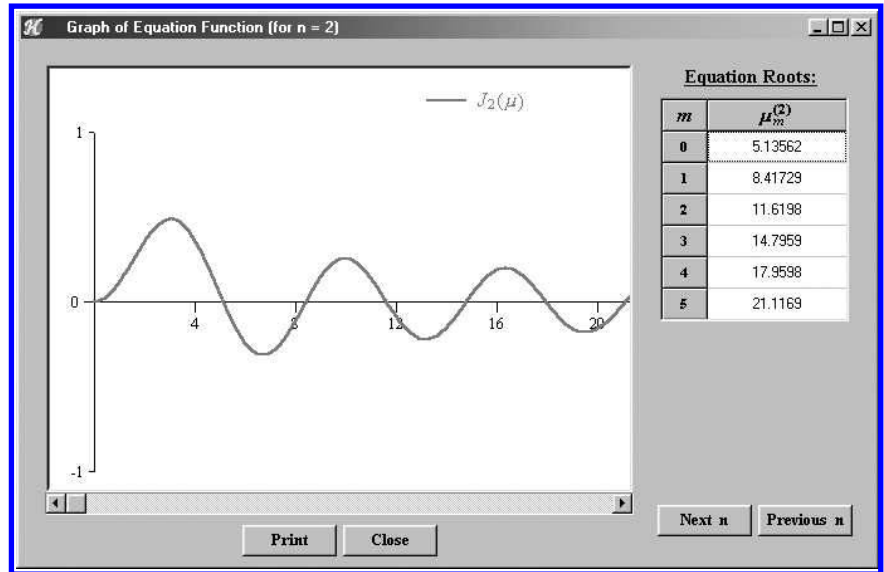


Figure E.57. Graph of equation for evaluating eigenvalues for $n = 2$ and the table of roots $\mu_m^{(2)}$ for Example E.14.

Eigenfunctions of the given boundary value problem are

$$V_{nm}^{(1)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n \left(\frac{\mu_m^{(n)}}{l} r \right) \sin n\varphi.$$

The three-dimensional picture shown in Figure E.58 depicts one of the eigenfunctions for the given problem

$$(\lambda_{21} = 4.42817, \quad V_{21}^{(1)} = J_2 \left(\frac{\mu_1^{(2)}}{4} r \right) \cos 2\varphi, \quad \|V_{21}^{(1)}\|^2 = 1.85061).$$

To solve the problem with the help of the program, follow the same steps as in Example E.12. Results of the solution can be displayed by selecting menu option “Evolution of Membrane Temperature.” The three-dimensional surface shown in Figure E.59 depicts the animated solution $u(r, \varphi, t)$ at time $t = 2$.

By changing the values of thermal diffusivity a^2 , radius l , and parameters H_0 and ω of the boundary value function, you can obtain and study different pictures of magnetic field distribution.

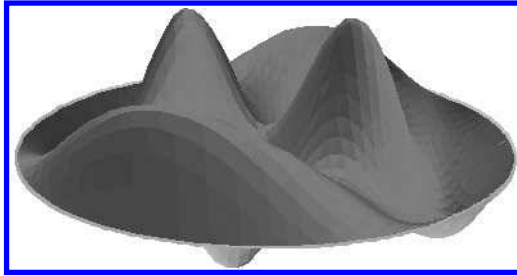


Figure E.58. Eigenfunction $V_{21}^{(1)}(r, \varphi)$ for Example E.14.

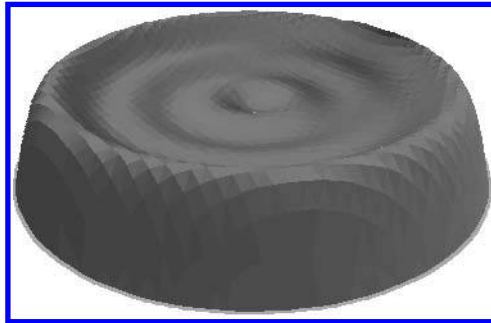


Figure E.59. Distribution of magnetic field $u(r, \varphi, t)$ at $t = 2$ for Example E.14.

E.5 Examples Using the Program Laplace

A number of examples of how to apply the Laplace program to solve different problems follow.

Example E.15. A heat-conducting, thin, uniform, rectangular membrane ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$) is thermally insulated over its lateral faces. One part of the boundary (at $x = 0$ and $y = 0$) is thermally insulated, and the other part (at $x = l_x$ and $y = l_y$) is held at a fixed temperature $u = u_0$. One constant internal source of heat acts at the point (x_0, y_0) of the membrane. The value of this source is $Q = \text{const}$.

Find the steady-state temperature distribution in the membrane. To start, the following values of the parameters are assigned: $l_x = 4$, $l_y = 6$, $u_0 = 10$, $Q = 100$, $x_0 = 2$, and $y_0 = 3$.

Solution. First, select the type of problem from the main menu by clicking “Elliptic Equations over Rectangular Domains.” The program has a set of

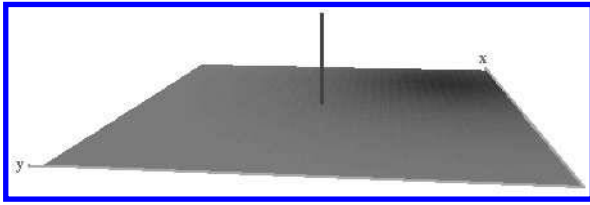


Figure E.60. Source function $f(x, y)$ for Example E.15.

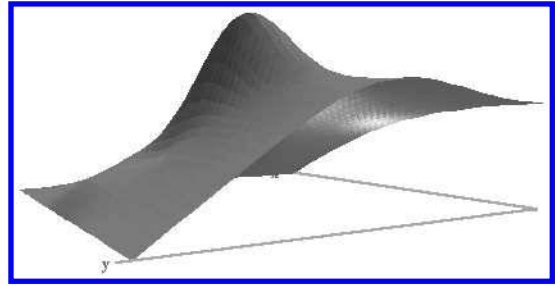


Figure E.61. Surface plot of the solution $u(x, y)$ for Example E.15.

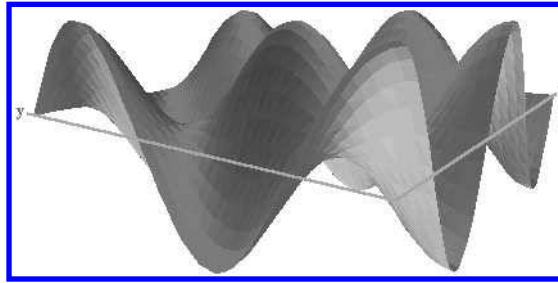


Figure E.62. Eigenfunction $V_{43}(x, y)$; $\|V_{43}\|^2 = 6$, $\lambda_{43} = 9.26989$ for Example E.15.

predefined examples on the selected subject, and the given problem is from this set. To load the problem, select “Data” from the main menu and click “Library example.” From the given list of problems, select “Example 6.” A description of the problem can be read on the right side of the screen. The problem description, solution hints, and theoretical solution can be accessed by clicking the “Text + Hint” button. To proceed with the selected example, click the “OK” button to display the parameters for the current problem in the dialog window.

The given problem involves the solution of the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Q\delta(x - x_0)\delta(y - y_0)$$

under the conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad u(l_x, y) = u_0, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad u(x, l_y) = u_0.$$

Since all parameters for the problem have been entered, proceed by clicking “OK.” At this point, the program is ready to solve the problem with the selected parameters. To see the source function, select “View” → “Source function $f(x,y)$ ” (Figure E.60). Click “Return” to return to the problem solution screen.

To run the solver, choose the “Execute” command from the main menu; results can be seen (Figure E.61) by selecting “Results” → “Surface Plot of the Solution.”

The following menus allow the user to investigate other properties of the solution:

- “Results” → “Fourier Coefficients,”
- “Results” → “Surface Plots of EigenFunctions,”
- “Results” → “Contour Map of the Solution,”
- “Results” → “Solution Profile at $y = \text{const.}$,”
- “Results” → “Solution Profile at $x = \text{const.}$,”

More information for each option can be found by selecting the “Help” menu on the corresponding screen.

The three-dimensional surface shown in Figure E.62 depicts one of the eigenfunctions for this problem. The contour map of the solution $u(x, y)$ is shown in Figure E.63.

Example E.16. Consider a heat-conducting, uniform, rectangular plate ($0 \leq x \leq l_x$, $0 \leq y \leq l_y$), thermally insulated over its lateral faces. Let heat be generated throughout the plate; the intensity of internal sources (per unit mass of the plate) is $Q(x, y) = Axy$. The side $y = 0$ of the plate is thermally insulated, the side $x = 0$ is held at a fixed temperature $u = u_1$, the side $y = l_y$ has a constant temperature of zero, and the side $x = l_x$ is subjected to convective heat transfer with a medium according to Newton’s law of cooling. The temperature of the medium is $u_{md} = \text{const}$ and the coefficient of thermal conductivity is h .

Find the steady-state temperature distribution in the plate. To start, the following values of the parameters are assigned: $A = 5$, $u_1 = 10$, $u_{md} = 40$, $h = 0.5$, $l_x = 4$, and $l_y = 6$.

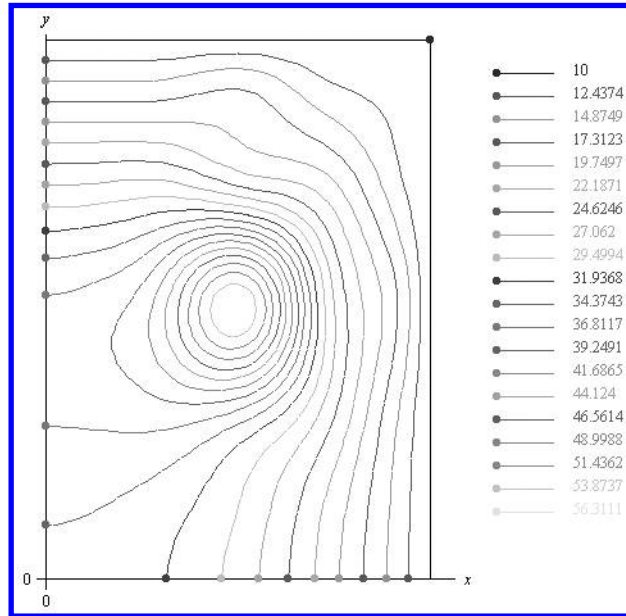


Figure E.63. Contour map of the solution $u(x, y)$ for Example E.15.

Solution. This problem is also from the library set. To load the problem, select “Data” from the main menu, and then click “Library example.” From the given list of examples, select “Example 12.”

The problem involves the solution of the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Axy$$

under the conditions

$$u|_{x=0} = u_1, \quad \left. \frac{\partial u}{\partial x} + h(u - u_{md}) \right|_{x=l_x} = 0, \quad \left. \frac{\partial u}{\partial y} \right|_{y=0} = 0, \quad u|_{y=l_y} = 0.$$

Figure E.64 shows the source function $f(x, y)$ for the problem.

The boundary value functions of the given problem do not satisfy the conforming conditions at points $(0, l_y)$ and (l_x, l_y) :

$$P_1[g_4]_{x=0} = 0 \neq P_4[g_1]_{y=l_y} = u_1,$$

$$P_2[g_4]_{x=l_x} = hu_{md} \neq P_4[g_2]_{y=l_y} = 0.$$

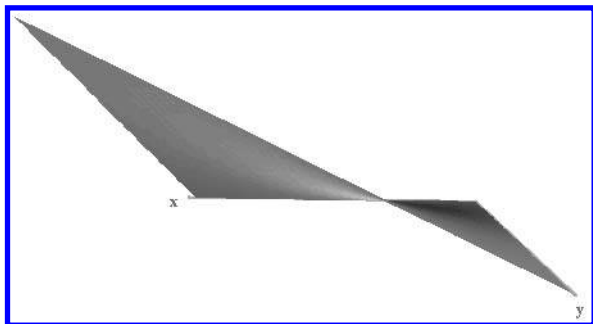


Figure E.64. Source function $f(x, y)$ for Example E.16.

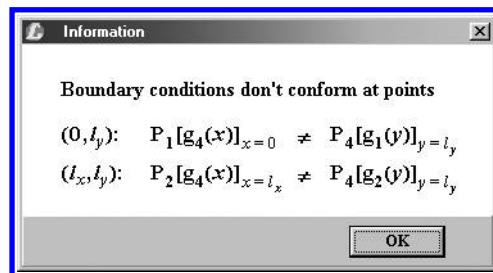


Figure E.65. Information window for Example E.16.

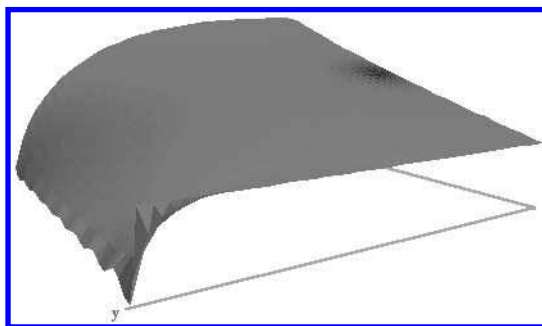


Figure E.66. Stationary distribution of temperature $u(x, y)$ for Example E.16.

As a result, when you run the solver by selecting the “Execute” command from the main menu, the message shown in Figure E.65 is displayed.

Although the boundary conditions do not match, you may still solve the problem with the help of the program by searching for a generalized solution, following the same steps as in Example E.15. Results of the solution can be displayed by selecting the menu “Results” → “Surface Plot of the Solution.” The three-dimensional surface shown in Figure E.66 depicts the steady-state temperature distribution $u(x, y)$ in the rectangular plate. Note that the edges of the surface do adhere to the given boundary conditions.

The solution to the given problem can be expressed as the sum of three functions

$$u(x, y) = w_1(x, y) + w_2(x, y) + v(x, y),$$

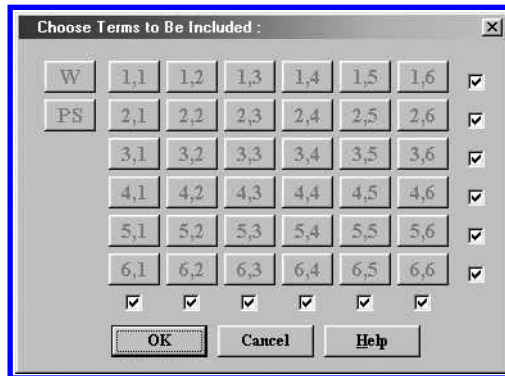


Figure E.67. Selection of terms in the expansion of the solution.

where $w_1(x, y)$ is an auxiliary function satisfying the conforming boundary value functions

$$P_1[w_1]_{x=0} = u_1, \quad P_2[w_1]_{x=l_x} = hu_{md},$$

$$P_3[w_1]_{y=0} = 0, \quad P_4[w_1]_{y=l_y} = u_1 - (u_1 - hu_{md}) \frac{x}{l_x};$$

$w_2(x, y)$ is a particular solution of the Laplace problem with boundary conditions

$$P_1[w_2]_{x=0} = 0, \quad P_2[w_2]_{x=l_x} = 0,$$

$$P_3[w_2]_{y=0} = 0, \quad P_4[w_2]_{y=l_y} = -u_1 + (u_1 - hu_{md}) \frac{x}{l_x};$$

and $v(x, y)$ is a solution of a Poisson equation with zero boundary conditions:

$$v(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} V_{nm}(x, y).$$

The program Laplace allows you to study the behavior of the functions $w_1(x, y)$, $w_2(x, y)$ and of any single term of the partial sum of the solution

$$u_{nm}(x, y) = w_1(x, y) + w_2(x, y) + \sum_{i=1}^n \sum_{j=1}^m C_{ij} V_{ij}(x, y).$$

Select the “Choose Terms...” option from the graph menu to display the “Choose Terms to Be Included” dialog box (Figure E.67).

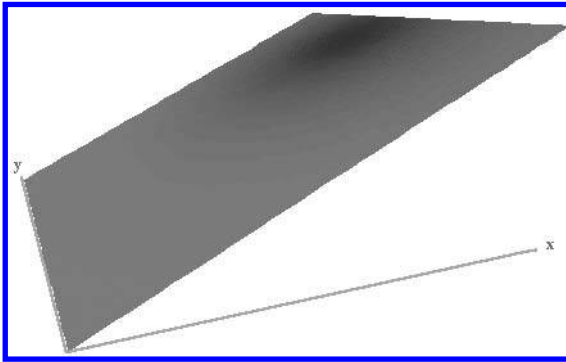


Figure E.68. Surface plot of the auxiliary function $w_1(x, y)$ for Example E.16.

The button “W” corresponds to the auxiliary function $w_1(x, y)$. Select the button “W” and unselect all other buttons to see the surface graph of the auxiliary function

$$w_1(x, y) = u_1 - (u_1 - hu_{md}) \frac{x}{l_x}$$

(shown in [Figure E.68](#)).

The button “PS” corresponds to the particular solution $w_2(x, y)$. This button is active because the boundary value functions do not satisfy the conforming conditions. Select the button “PS” and unselect all other buttons to see the surface graph of the particular solution

$$w_2(x, y) = \sum_{k=1}^N A_k \frac{\cosh \sqrt{\lambda_k} y}{\cosh \sqrt{\lambda_k} l_y} \sin \sqrt{\lambda_k} x$$

(see [Figure E.69](#)).

Unselect the buttons “W” and “PS” to display the partial sum

$$v_{nm}(x, y) = \sum_{i=1}^n \sum_{j=1}^m C_{ij} V_{ij}(x, y)$$

(see [Figure E.70](#)).

All of these figures can be rotated to view the shape of the surfaces from different directions.

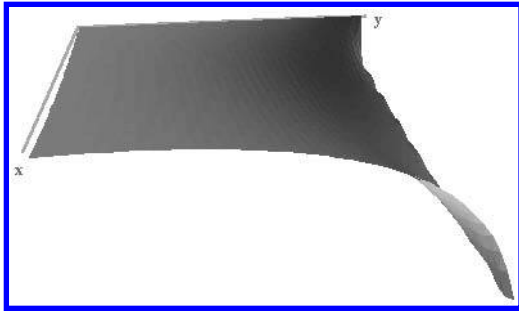


Figure E.69. Surface plot of the particular solution $w_2(x, y)$ for Example E.16.

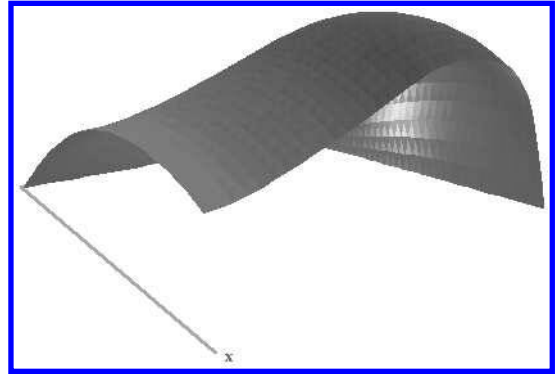


Figure E.70. Surface plot of the partial sum $v_{nm}(x, y)$ for Example E.16.

Example E.17. A heat-conducting, thin, uniform, circular membrane of radius l is thermally insulated over its lateral faces. The membrane is subjected to convective heat transfer according to Newton's law at its boundary. The temperature of the medium is $u_{md} = \text{const}$ and the coefficient of thermal conductivity is h .

Let heat be generated throughout the membrane, where the intensity of internal sources (per unit mass of the membrane) is

$$f(r, \varphi) = 5r^2 \cos 5\varphi.$$

Find the stationary distribution of temperature within the membrane. To start, take the following values of the parameters: $l = 2$, $u_{md} = 15$, and $h = 3$.

Solution. First, select the type of problem from the main menu by clicking "Elliptic Equations over Circular Domains." This problem is not from the library set. To solve it, instead of selecting a library example as we did before, select "New problem" from the "Data" menu on the problem screen.

The problem involves a solution to the Poisson equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = -5r^2 \cos 5\varphi$$

under the condition

$$\frac{\partial u}{\partial r}(l, \varphi) + h [u(l, \varphi) - u_{md}] = 0 \quad \text{or} \quad \frac{\partial u}{\partial r}(l, \varphi) + hu(l, \varphi) = hu_{md}.$$

Enter Boundary Value Function

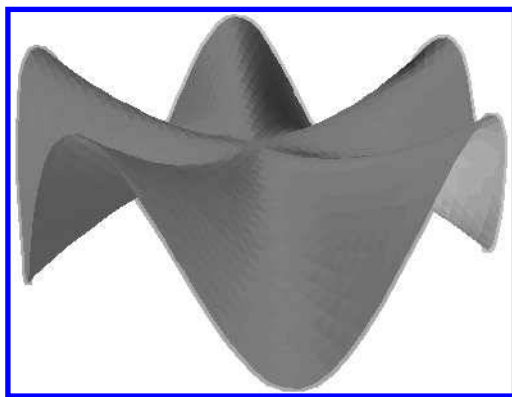
The boundary value at $r = l$ (mixed condition):

$$u_r(l, \varphi) + h u(l, \varphi) = g(\varphi) \quad (h > 0)$$

$h =$

$g(\varphi) =$

Note: use letter f for polar angle φ in function $g(\varphi, l)$

Figure E.71. Parameters of heat exchange on the boundary $r = l$.Figure E.72. Source function $f(r, \varphi)$ for Example E.17.

To specify heat conductivity according to Newton's law at the circular periphery of the membrane, choose option "3 (Mixed condition)" for boundary values by selecting this item from the dialog box. Enter the parameters of heat exchange by clicking the "Edit" button. Since $u_{md} = 15$, the value of $g(\varphi) = h \cdot u_{md} = 45$ (Figure E.71).

The intensity of the internal sources is determined by the function $f(r, \varphi) = 5r^2 \cos 5\varphi$.

When all parameters are entered, click the "OK" button. Figure E.72 shows the source function for the problem.

Eigenvalues of the given boundary value problem have the form

$$\lambda_{nm} = \left[\frac{\mu_m^{(n)}}{l} \right]^2, \quad n, m = 0, 1, 2, \dots,$$

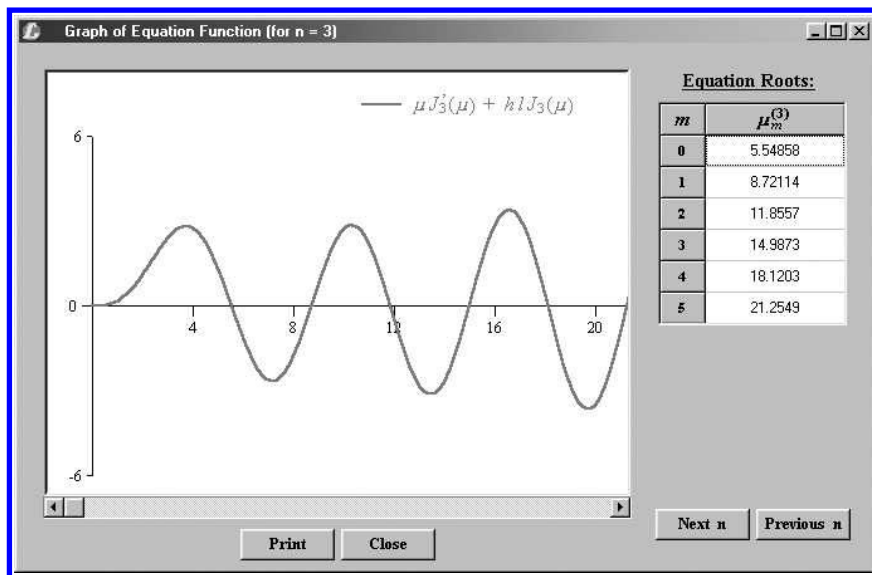


Figure E.73. Graph of the equation for evaluating eigenvalues for $n = 3$ and the table of roots $\mu_m^{(3)}$ for Example E.17.

where $\mu_m^{(n)}$ are positive roots of the equation

$$\mu J'_n(\mu) + hlJ_n(\mu) = 0.$$

Figure E.73 shows the graph of function $\mu J'_n(\mu) + hlJ_n(\mu)$ used for evaluating eigenvalues for the choice $n = 3$.

Eigenfunctions of the given boundary value problem are

$$V_{nm}^{(1)} = J_n\left(\frac{\mu_m^{(n)}}{l}r\right) \cos n\varphi, \quad V_{nm}^{(2)} = J_n\left(\frac{\mu_m^{(n)}}{l}r\right) \sin n\varphi.$$

The three-dimensional picture shown in Figure E.74 depicts the eigenfunction $V_{32}^{(1)}$ for the current problem:

$$\lambda_{32} = 35.1394, \quad V_{32}^{(1)} = J_3\left(\mu_2^{(3)}r/2\right) \cos 3\varphi, \quad \left\|V_{32}^{(1)}\right\|^2 = 0.338524.$$

To solve the problem by using the program Laplace, follow the same steps as in Example E.15. Results of the solution can be displayed by selecting menu “Results” → “Surface Plot of the Solution.” The stationary

temperature distribution $u(r, \varphi)$ in the circular membrane is represented in the three-dimensional plot shown in Figure E.75.

More results can be displayed with the “Results” → “Contour Map of the Solution” option (Figure E.76).

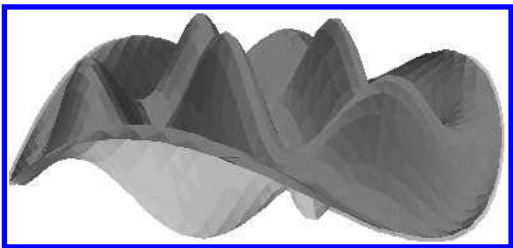


Figure E.74. Eigenfunction $V_{32}^{(1)}(r, \varphi)$ for Example E.17.

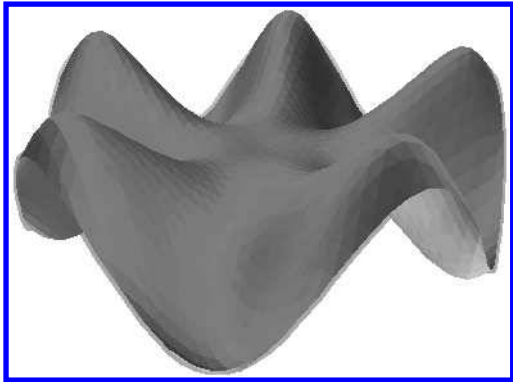


Figure E.75. Stationary distribution of temperature $u(r, \varphi)$ for Example E.17.

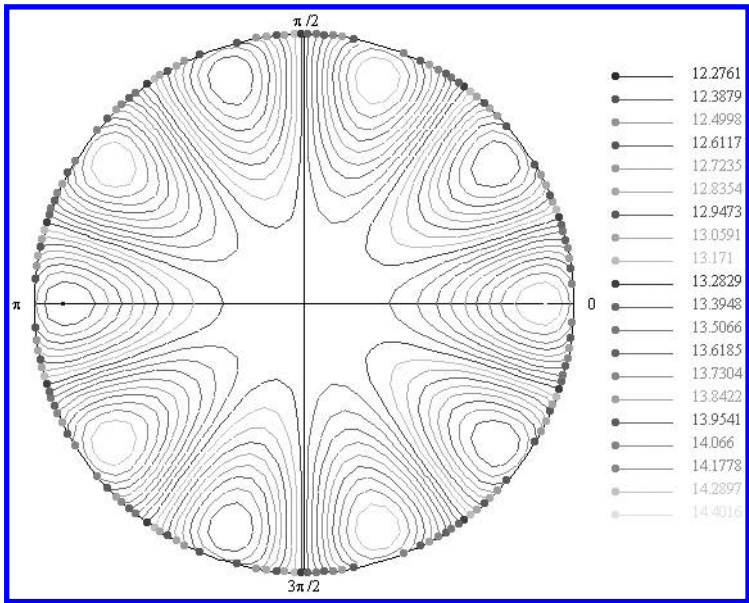


Figure E.76. Contour map of the solution $u(r, \varphi)$ for Example E.17.

E.6 Examples Using the Program FourierSeries

A number of examples of how to apply the FourierSeries program to solve different problems follow. Some of the problems were taken from the book without any changes; others were modified to show how to use different program options.

Example E.18 (Expansion into Classical Orthogonal Polynomials). Expand the trigonometric function

$$f(x) = \sin(x)$$

defined on the interval $[0, \pi/4]$ by using the system of Chebyshev polynomials of the first kind.

Find the partial sum of the expansion obtained in powers of the variable x . Find the number of terms in the power series sufficient to approximate the given function to a precision of $\varepsilon = 10^{-5}$.

Solution. First, select the type of problem from the main menu by clicking on “Expansion into Classical Orthogonal Polynomials.” The program has a set of predefined examples on the selected subject, and the given problem is from this set. To load the problem, select “Data” from the main menu, and then click “Library example.” From the given list of examples, select “Example 1.” You can read the problem text on the right side of the screen. To proceed with the selected example, click the “OK” button to display parameters for the current problem in the dialog window. Since all parameters for the problem have been entered, proceed by clicking “OK.”

At this point, the program is ready to solve the problem with the selected parameters. To see the graph of the initial function, select “View” from the main menu ([Figure E.77](#)), then click “Return” to go back to the problem solution screen.

To run the solver, choose the “Execute” command from the main menu. The results ([Figure E.78](#)) can be seen by selecting the option “Graph of the Partial Sum.” The graph range for displaying the partial sum $S_n(x)$ and the limits for the graph can be adjusted with the help of the “Set Attributes” command.

To examine the behavior of the individual harmonics of the expansion and of their partial sums, use the option “Choose Terms to Be Included”

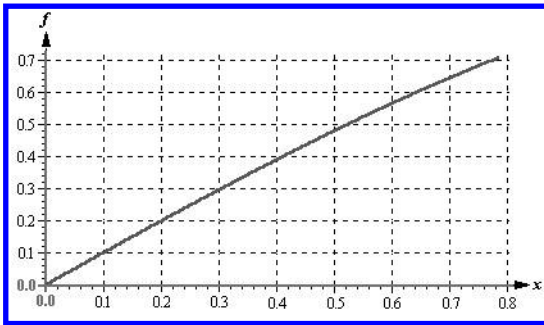


Figure E.77. The given function $f(x)$ for Example E.18.

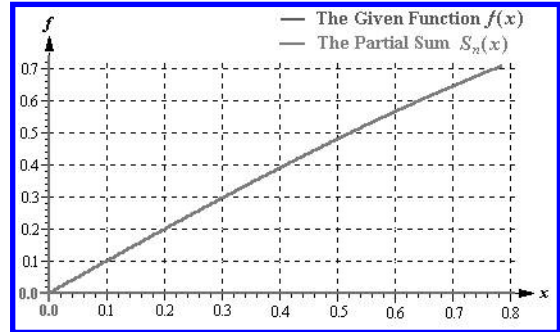


Figure E.78. Graph of the partial sum $S_{10}(x)$, $\delta^2 = 1.04 \cdot 10^{-17}$ for Example E.18.

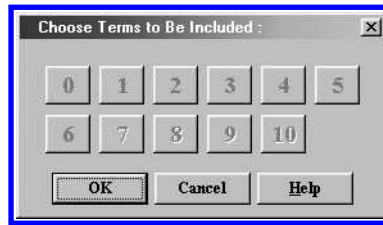


Figure E.79. Dialog window “Choose Terms to Be Included.”

(Figure E.79). To select (or unselect) the individual term, click on the corresponding button. The terms whose numbers are selected (red) are included in the partial sum, and the ones not selected (white) are excluded. The buttons associated with terms that have practically zero Fourier coefficients are inactive. Click “OK” to mark your selection and get back to drawing the graph.

The option “Choose Terms...” allows you to find the number of terms in the expansion sufficient to approximate the given function to a precision of $\varepsilon = 10^{-5}$. By unselecting higher terms of the expansion, you can see that the first three terms are enough to obtain the requisite precision.

To find the respective partial sum in powers of variable x , select “Data” from the main menu, then click “Change Current Problem.” Change the number of terms in the expansion in the parameters of the problem dialog box and enter “ $N=2$.” Now you can proceed by clicking the “OK” button, then running the problem solver by selecting the “Execute” option from

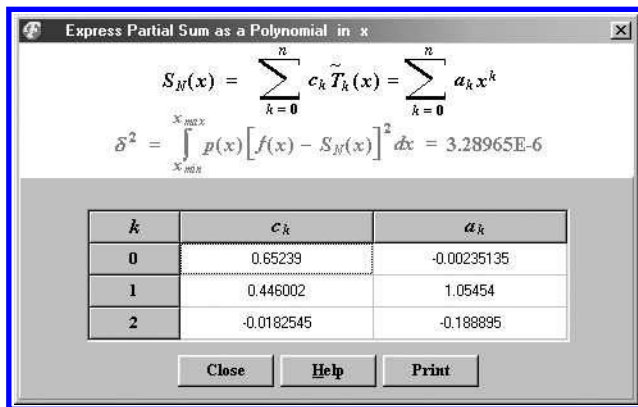


Figure E.80. Partial sum of the expansion obtained in powers of the variable x for Example E.18.

the main menu. To obtain the required polynomial, select “Results” → “Express Partial Sum as a Polynomial in x ” (Figure E.80).

The polynomial approximating the function $f(x) = \sin(x)$ to a precision of $\varepsilon = 10^{-5}$ is

$$f(x) = -0.188895x^2 + 1.05454x - 0.00235135.$$

This expansion is valid only in the interval $[0, \pi/4]$. However you can use the polynomial you obtain for calculating the trigonometric functions of an arbitrary argument if you take advantage of the trigonometric identities connecting the trigonometric functions with $\sin(-x)$, $\sin(x + \pi/2)$, $\sin(x + \pi)$, and so forth.

Example E.19 (Expansion into Classical Orthogonal Polynomials). Let the function $f(x) = x(2\pi - x)/\pi^2$ be given at the discrete set of points in the interval $[0, 2\pi]$. For example,

$$x_k = (k-1)\frac{2\pi}{M}, \quad y_k = \frac{4(k-1)(M-k+1)}{M^2}, \quad M = 12, \quad k = 1, 2, \dots, M+1.$$

Approximate this discrete function by polynomials of third, tenth and fifteenth order. Find such polynomials with the help of the Fourier expansion by the system of Legendre polynomials, and then find the corresponding partial sums in powers of the variable x . Compare the results of these expansions.

Solution. This problem is not in the library set. To solve it with the program, select “New problem” from the “Data” menu on the problem screen. The function $f(x)$ that is to be expanded as a Fourier series is given as a table. You can input your table function directly from the keyboard (use the command “Input Data via Keyboard”) or from an ASCII [.dat] file prepared in advance (using the command “Read from File”). The length of your data file must not exceed the 1000 lines (a program limitation). Each line of the file must contain values of x_k and $y_k = f(x_k)$, separated by one or more spaces. The program allows editing of the data and provides some operations for altering the data.

After selecting the type of function, you will see the dialog window “Enter Parameters of the Problem” (Figure E.81) for the table function. To specify the type of polynomial, choose the option “Legendre” by selecting this item from the combo box. Then enter the interval of expansion ($x_{min} = 0$, $x_{max} = 2\pi$) and the number of terms in the expansion ($N = 15$).

The function $f(x)$ is given at the discrete set of points. To look through the table of points (or to enter them), click the “Enter Data Points...” button to display the dialog window with coordinates of data points (Figure E.82). Note that after editing the data is put in *ascending order* of the variable x .

When all parameters are entered, follow the same steps as in Example E.18, and select the “Execute” command from the main menu. A partial sum of the Fourier expansion can be displayed by selecting the menu option “Results” → “Graph of the Partial Sum.”

Figure E.81. “Parameters of the Problem” dialog for Example E.19.

Enter Coordinates of Data Points

i	x_i	y_i
1	0	0
2	0.524	0.306
3	1.047	0.556
4	1.57	0.75
5	2.094	0.889
6	2.618	0.972
7	3.1416	1
8	3.665	0.972
9	4.189	0.889

Number of Dots (≤ 1000):
M = 13

Transformations

Transform X

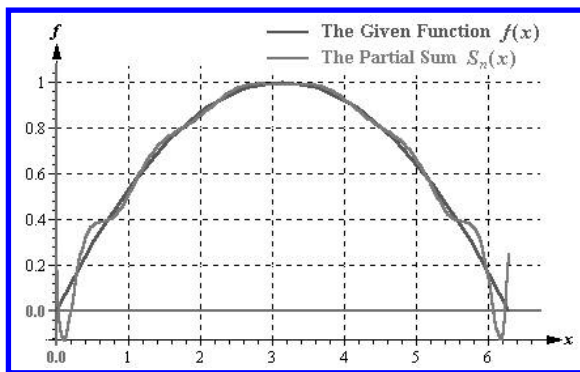
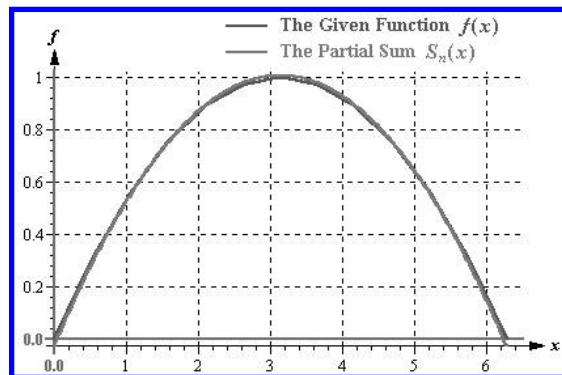
Transform Y

Exchange X and Y

Insert Delete Save in File

OK Cancel Help

Figure E.82. Dialog window “Enter Coordinates of Data Points.”

Figure E.83. Graph of the partial sum $S_{15}(x)$, $\delta^2 = 0.039$ for Example E.19.Figure E.84. Graph of the partial sum $S_3(x)$, $\delta^2 = 0.00085$ for Example E.19.

Draw the graphs of the partial sum $S_N(x)$ for $N = 15, 10$, and 3 (use the menu option “Choose Terms...”). Figures E.83 and E.84 show two of the calculated expansions. You can see that adding terms makes the precision of approximation only worse in this example.

Example E.20 shows that the generalized Fourier series of orthogonal polynomials have some advantage over the Taylor series expansion. To

expand the function $f(x)$ defined in the interval $[a, b]$ in a Taylor series, it is necessary that the function have derivatives of all orders in the interval (a, b) . Moreover, the Fourier series converges faster for many functions than does the Taylor series.

Example E.20 (Expansion into Fourier-Bessel Series). Expand the function

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1, \\ x & \text{if } 1 \leq x \leq 2, \end{cases}$$

defined at the interval $[0, 2]$ by using the system of Bessel functions of the first kind of order $m = 2$:

$$X_k(x) = J_2\left(\frac{\mu_k^{(2)}}{l}x\right),$$

where $\mu_k^{(2)}$ are positive roots of the equation $\mu J_2'(\mu) + h l J_2(\mu) = 0$.

Search for this problem when $h = 0.1, 1, 10$.

Solution. Select the type of the problem from the main menu by clicking “Expansion into Fourier-Bessel Series.” The given problem is from the library set. To load it select “Data” → “Library problem” → “Problem 6.” To proceed with the selected problem, click the “OK” button; parameters for the current problem will be displayed in the dialog window (Figure E.85).

In the expression for the given function

$$f(x) = x^2 * \text{Imp}(x, 0, 1) + x * \text{Imp}(x, 1, 2),$$

$\text{Imp}(x, a, b)$ is an impulse function, where $\text{Imp}(x, a, b) = 1$ if $a \leq x < b$ and 0 otherwise.

Since all parameters for the problem have been entered, we can proceed by clicking “OK.” To see the given function, select “View” from the main menu, then click “Return” to go back to the problem solution screen. To look over the values of coefficients c_k in the expansion, select “Results” → “Fourier Coefficients.”

To run the solver, click the “Execution” option; results can be seen by selecting “Results” → “Graph of the Partial Sum” (Figure E.86).

Figure E.87 depicts the bar chart of Fourier coefficients, c_k (option “Bar Chart of Fourier Coefficients”). The bar chart gives a graphic picture of the contribution of individual terms to the Fourier-Bessel expansion.

Enter Parameters of the Problem

The Given Function:
 $f(x) = [x^2 \text{Imp}(x, 0, 1) + x \text{Imp}(x, 1, 2)]$

Boundary Condition
 Boundary condition for orthogonal system of Bessel functions
 Mixed condition: $X(l) + h * X(l) = 0$
 Coefficient for mixed condition (>0):
 $h = 1$

The Order of Bessel Functions ($0 \leq m \leq 10$):
 $m = 2$

The Length of the Interval (>0):
 $l = 2$

Number of Terms in the Expansion ($0 < N \leq 30$):
 $N = 15$

OK Cancel Help

Figure E.85. “Parameters of the Problem” dialog box for Example E.20.

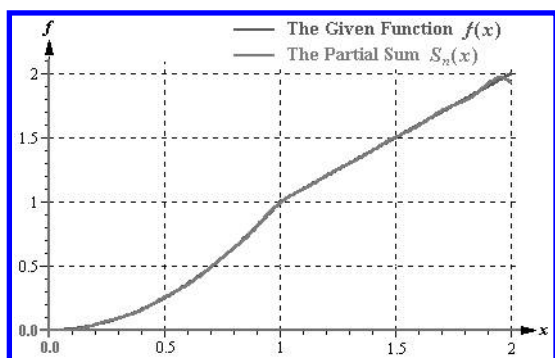


Figure E.86. Graph of the partial sum $S_{15}(x)$, $\delta^2 = 0.00021$ for Example E.20.

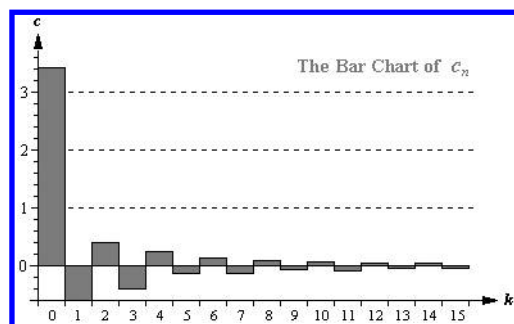
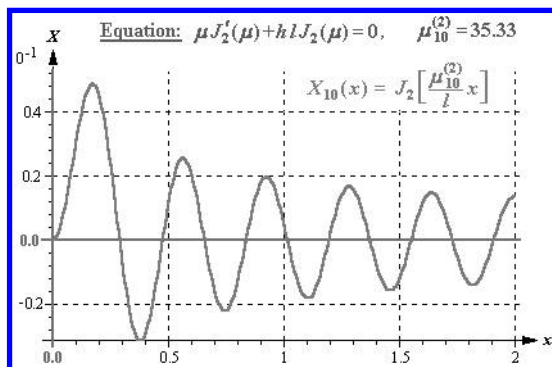


Figure E.87. Bar chart of Fourier coefficients, c_k , for Example E.20.

sion of $f(x)$. The bar chart displays data by drawing a rectangle of the height equal to the k th Fourier coefficient (measured along the vertical axis) at each k value along the graph. The height of the k th rectangle corresponds to the value of the Fourier coefficient c_k of the general expansion of $f(x)$.

The menu option “Results” → “Graphs of Bessel Functions” allows you to look through the plots of Bessel functions of the first kind of the given order $m = 2$ (functions of the orthonormal system) for the successive values of $k = 0, 1, \dots, N$. One of the Bessel functions for the problem

Figure E.88. Graph of Bessel function $X_{10}(x)$ for Example E.20.

represented in Figure E.88 is

$$X_{10}(x) = J_2 \left(\mu_{10}^{(2)} x / l \right),$$

where $\mu_{10}^{(2)}$ is the tenth root of the equation $\mu J_2'(\mu) + h l J_2(\mu) = 0$.

Example E.21 (Expansion into Associated Legendre Functions). Expand the function

$$f(x) = \frac{30}{50x^2 - 5x + 7}$$

defined in the interval $(-1,1)$ by using the system of associated Legendre functions of order $m = 4$, which in normalized form are:

$$\hat{X}_k(x) = \hat{P}_{4+k}^4(x) = \sqrt{\frac{2k+9}{2} \cdot \frac{k!}{(8+k)!}} P_{4+k}^4(x).$$

Solution. This problem is not in the library set. To solve it with the program FourierSeries, select “New problem” → “Formula for $f(x)$ ” from the “Data” menu on the problem screen, then enter the given function and parameters (Figure E.89).

To run the solver, select the “Execute” command. Figure E.90 depicts the partial sum of the expansion.

Use the menu option “Results” → “Graphs of Associated Legendre Functions” to look through the plots of associated Legendre functions of the first kind of the given order $m = 4$ (functions of the orthonormal

Figure E.89. “Parameters of the Problem” dialog box for Example E.21.

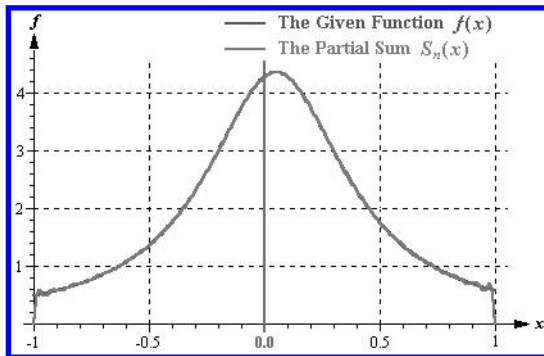


Figure E.90. Graph of the partial sum $S_{30}(x)$, $\delta^2 = 0.0036$, for Example E.21.

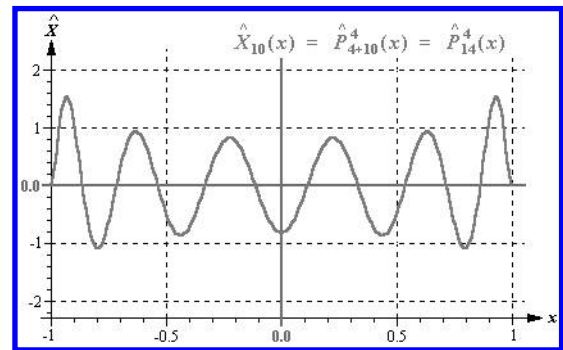


Figure E.91. Graph of associated Legendre function, $X_{10}(x)$, for Example E.21.

system) for the successive values of $k = 0, 1, \dots, N$. One of the associated Legendre functions for the problem, $\hat{X}_{10}(x) = \hat{P}_{14}^4(x)$, is represented in Figure E.91.

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