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Maurizio Gasperini

Theory of Gravitational Interactions

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Maurizio Gasperini

Theory of Gravitational Interactions

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To my parents

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Preface

This book grew out of lectures given by the author at the University of Turin and at the University of Bari. It is primarily intended for undergraduate students taking classes in gravitational theory, as prescribed by modern academic plans to graduate in Physics with a theoretical/high-energy physics or astrophysics curriculum. The challenge is to provide students with a textbook which, on one hand, can represent a self-contained reference for a semester cycle of lectures and, on the other hand, may be accessible and of profitable use also for students having different interests and following different academic tracks.

To this aim the book includes a first, conventional part introducing general relativity as a geometric theory of the macroscopic gravitational field, and a second, more advanced part, connecting general relativity to the gauge theories of fundamental interactions. A discussion of the deep analogies (and of the physical differences) existing between gravity and the other standard-model interactions fills a gap which is present within the traditional geometric approach to general relativity, and which usually puzzles students about the role of gravity in the context of a unified model of all interactions.

In this spirit, the formalism of differential geometry has been reduced to the necessary minimum, leaving more room to current interesting aspects of gravitational physics of both applicative/observational type (such as the phenomenology of gravitational waves) and theoretical/fundamental type (such as the gravitational interactions of spinors, supergravity and higher-dimensional gravity). We have included, however, a final appendix introducing the so-called "Cartan calculus" of exterior (or differential) forms, in view of the important applications of this formalism not only to the gravitational theory but also to many other fields of theoretical physics. A second appendix introduces various possible approaches to the problem of embedding a four-dimensional theory of gravity in the context of a higher-dimensional space–time manifold.

For most profitable use of this book the reader is expected to have a basic knowledge of special relativity, electromagnetic theory and classical theory of fields. Except for the above input, however, the book aims at being self-contained as much as possible, following the informal style of class lectures where all the required no-

tions and techniques are explicitly recalled and/or introduced whenever necessary. Also, for a better pedagogic efficiency, all computations are explicitly carried out in the main text (leaving no "voids" to be filled by the readers), or presented as solved exercises at the end of each chapter.

The present book is certainly not intended to represent a complete reference for a rigorous and comprehensive study of all theoretical aspects of the gravitational interaction. Its main purpose is to provide students with the basic starting notions, enabling them to do further independent work and subsequent deeper studies on more professional textbooks and papers. The readers interested in advanced discussions of some specific topic are strongly advised to refer to the list of specialistic books presented in the bibliography.

Finally, it should be noted that this book deliberately avoids any gravitational application to cosmology and large-scale astrophysics, because—according to modern academic plans of studies—they are a matter of specific courses and lectures, well separated from a course on the theory gravity. The field of relativistic cosmology is today so extended, with so many branches and applications, as to deserve by itself a dedicated book. We refer, for this purpose, to the excellent books quoted in the bibliography, as well as to an introduction to theoretical cosmology which represents the natural continuation of this book, and which currently exists as a Springer Italian edition [20].

Acknowledgements

It is a pleasure, as well as a duty, to thank all my colleagues and students for their comments, suggestions, and criticism that contributed over the years to correct and improve these lecture notes. Listing all of them would be an impossible task, so let me thank them jointly for their important help.

A warm acknowledgement is also due to Venzo De Sabbata, who was one of my Professors when (many years ago!) I was a student of Physics at the University of Bologna. Professor De Sabbata introduced me to the study of gravitation and cosmology, and the interest he was able to stimulate towards those branches of physics was so intense as to be still alive, and fully effective, even today in my present scientific activity.

Finally, I wish to thank Marina Forlizzi, Executive Editor for Springer-Verlag, for her kind encouragement, advice, and many useful suggestions.

Cesena, Italy Maurizio Gasperini

Notations, Units and Conventions

Throughout this book we will use the index 0 for the time-like components of vector and tensor objects, while the indices 1, 2, 3 will refer to the space-like components. For the space–time metric $g_{\mu\nu}$ we will adopt the signature with a positive time-like eigenvalue, namely:

$$g_{\mu\nu} = \mathrm{diag}(+, -, -, -).$$

Our conventions for the curvature and the covariant derivatives will be as follows. Riemann tensor:

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - \{\mu \leftrightarrow \nu\},$$

where the symbol $\{\mu \leftrightarrow \nu\}$ means that we must insert all the preceding terms with the indices μ and ν interchanged between themselves. Ricci tensor:

$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}.$$

Covariant derivative:

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma_{\mu\beta}{}^{\alpha} V^\beta; \qquad \nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma_{\mu\alpha}{}^{\beta} V_\beta;$$

Lorentz covariant derivative:

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b; \qquad D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b.$$

Also, the symbol $\Box$ will denote the usual d'Alembert operator of the flat Minkowski space–time, i.e.:

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

where η is the Minkowski metric and $\nabla^2 = \delta^{ij}\partial_i\partial_j$ the Laplacian operator of the Euclidean three-dimensional space.

Unless otherwise stated, we will use small Latin letters $i, j, k, \ldots$ for the spatial indices 1, 2, 3; small Greek letters $\mu, \nu, \alpha, \ldots$ for the space–time indices 0, 1, 2, 3. In higher-dimensional space–times, with a number $d > 3$ of spatial dimensions, the space–time indices will be denoted instead by capital Latin letters: $A, B, C, \ldots = 0, 1, 2, 3, \ldots, d$.

Two (or more) indices, when enclosed in round or square brackets, will satisfy, respectively, the symmetry or antisymmetry property defined by:

$$T_{(\alpha\beta)} \equiv \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}), \qquad T_{[\alpha\beta]} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}).$$

In the presence of more than two indices, two symmetric (or antisymmetric) indices which are not contiguous will be separated from the others by a vertical bar. We write, for instance:

$$T_{(\mu|\alpha|\nu)} \equiv \frac{1}{2}(T_{\mu\alpha\nu} + T_{\nu\alpha\mu}),$$
$$T_{[\mu|\alpha|\nu]\beta} \equiv \frac{1}{2}(T_{\mu\alpha\nu\beta} - T_{\nu\alpha\mu\beta}),$$

to mean that the first object is symmetric with respect to μ and ν (at fixed α), while the second object is antisymmetric with respect to μ and ν (at fixed α and β). An so on for a higher number of indices.

Such a symmetrization/antisymmetrization procedure can be easily extended to an arbitrary number of indices $n \geq 2$, by including all their possible permutations and then dividing by the total number of permutations, $n!$. For the symmetrization procedure all permutations are to be added with the same sign, while, for the antisymmetrization procedure, even permutations are to be added with the plus sign and odd permutations with the minus sign. For instance:

$$T_{(\mu\nu\alpha)} = \frac{1}{3!}(T_{\mu\nu\alpha} + T_{\nu\alpha\mu} + T_{\alpha\mu\nu} + T_{\mu\alpha\nu} + T_{\nu\mu\alpha} + T_{\alpha\nu\mu}),$$
$$T_{[\mu\nu\alpha]} = \frac{1}{3!}(T_{\mu\nu\alpha} + T_{\nu\alpha\mu} + T_{\alpha\mu\nu} - T_{\mu\alpha\nu} - T_{\nu\mu\alpha} - T_{\alpha\nu\mu}).$$

An so on for a higher number of indices. Finally, the fully antisymmetric tensor (also called the Levi-Civita symbol) of the Minkowski space–time, $\epsilon^{\mu\nu\alpha\beta}$, will be defined according to the following conventions:

$$\epsilon^{0123} = +1, \qquad \epsilon_{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\alpha\beta}.$$

Its generalization to the case of an arbitrary Riemann manifold will be introduced in Chap. 3, Sect. 3.2.

The units that will be adopted for our numerical estimates and for the electromagnetic variables are the so-called (unrationalized) c.g.s. units, where the Maxwell equations take the form:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad A^\mu = (\phi, \boldsymbol{A}).$$

When dealing with scalar and spinor fields we will instead refer to the more convenient "natural" units, where both the light velocity c and the Planck constant $\hbar$ are set to one, $c = \hbar = 1$. In such units the Newton constant G acquires length-squared

(or inverse mass-squared) dimensions, and is related to the Planck mass $M_{\rm P}$ and to the Planck length $\lambda_{\rm P}$ by:

$$G^{-1} = M_{\rm P}^2 = \lambda_{\rm P}^{-2}.$$

In c.g.s. units:

$$M_{\rm P} = \left(\frac{\hbar c}{G}\right)^{1/2} \simeq 2 \times 10^{-5}\ \text{g},$$
$$\lambda_{\rm P} = \left(\frac{G\hbar}{c^3}\right)^{1/2} \simeq 1.6 \times 10^{-33}\ \text{cm}.$$

The energy associated to the Planck mass is $E_{\rm P} = M_{\rm P}c^2 \simeq 10^{19}$ GeV, where $1\ \text{GeV} = 10^9$ eV is the energy scale associated to the mass of the proton. As evident from the above definitions, the Planck energy scale characterizes the strength of the gravitational coupling, and also controls the importance of the quantum corrections to the equations of the classical gravitational theory.

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Chapter 1
Elementary Notions of Relativistic Field Theory

The aim of this chapter is to introduce a few basic aspects of classic field theory, concerning in particular the covariant variational formalism and its main relativistic applications. The notions we will present here will be used for the subsequent formulation of a relativistic theory of the gravitational field.

Our discussion will be focused on the symmetries of the Minkowski space–time, with the main purpose of showing how the definitions of the canonical energy-momentum and angular momentum tensors arise, respectively, from the invariance of the action under global space–time translations and global Lorentz rotations. We will also provide explicit examples of energy-momentum tensors for simple systems of physical interest, such as scalar and vector fields, point particles and perfect fluids.

We will always implicitly assume, throughout this chapter, that the gravitational interaction is absent (or negligible), and that the physical systems we are considering can be correctly described in a special-relativistic context using the appropriate representations of the Lorentz group in four-dimensional Minkowski space–time. For a useful reference to special relativity and to the related formalism we refer the readers to the excellent books [2, 4, 32, 45, 46] listed in the bibliography.

1.1 Symmetries and Conservation Laws

We start considering a generic physical system represented by a field $\psi(x)$, whose a classical dynamics is controlled by the action functional

$$S = \int_{\Omega} d^4x \, \mathcal{L}(\psi, \partial\psi, x), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density (depending ψ and on its gradients), and Ω is an appropriate four-dimensional integration domain on the Minkowski space–time. Here and in what follows we will collectively denote with x a generic dependence on all the space–time coordinates. Note that $\mathcal{L}$ has dimensions of energy density, so that

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the above action has dimensions $[S] = [\text{energy}] \times [\text{length}]$, because of the factor c contained inside $d^4x = dx^0\, d^3x = c dt\, d^3x$. The canonical dimensions of the action ($[\text{energy}] \times [\text{time}]$) can be easily restored by multiplying the integral (1.1) by the factor $1/c$. Such a factor, however, is irrelevant for all the topics discussed in this chapter, and will be omitted.

Let us first recall that the classical evolution of our physical system is described by the so-called Euler–Lagrange equations of motion. They are obtained by imposing on the action to be stationary with respect to local variations of the field ψ, with the constraint that such variations are vanishing on the border $\partial\Omega$ of the integration domain.

We may consider, in particular, an infinitesimal transformation of the field ψ, at a fixed space–time position x, such that

$$\psi(x) \to \psi'(x) = \psi(x) + \delta\psi(x). \tag{1.2}$$

The corresponding infinitesimal variation of the action is given by

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta(\partial_\mu \psi) \right], \tag{1.3}$$

where, for simplicity, we have assumed that $\mathcal{L}$ depends on the first derivatives of ψ but not on its higher-order derivatives (our computation, however, can be easily extended to Lagrangian densities containing derivatives of arbitrarily high order, $\mathcal{L} = \mathcal{L}(\psi, \partial^n \psi)$).

Since we are varying the field ψ at fixed x, the variation $\delta\psi$ commutes with the partial derivative of the field, i.e.:

$$\delta(\partial_\mu \psi) = \partial_\mu \psi' - \partial_\mu \psi = \partial_\mu(\delta\psi). \tag{1.4}$$

Using this property, and integrating by parts the second term of Eq. (1.3), we obtain

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] \delta\psi + \int_\Omega d^4x\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \right]. \tag{1.5}$$

By applying the Gauss theorem we can transform the last integral (containing a four-divergence) into a flux integral (of the argument of the divergence) over the border $\partial\Omega$ of the four-volume Ω:

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] \delta\psi + \int_{\partial\Omega} dS_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \right] \tag{1.6}$$

(we have denoted by dS_μ the volume element on the hypersurface $\partial\Omega$, oriented outwards along the normal direction). If we impose on the field variation to be vanishing on the border $\partial\Omega$,

$$\delta\psi|_{\partial\Omega} = 0, \tag{1.7}$$

we finally obtain the result that the last term of Eq. (1.6) is identically vanishing. It follows that the condition of stationary action, $\delta S = 0$, is automatically satisfied, for all variations $\delta\psi$, provided that the field ψ satisfies the Euler–Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial \mathcal{L}}{\partial \psi} \tag{1.8}$$

(the condition $\delta S = 0$ is also called "minimal-action principle").

Let us now consider an infinitesimal transformation where both the field and the space–time coordinates are independently varied:

$$\psi(x) \to \psi'(x) = \psi(x) + \delta\psi(x), \qquad x^\mu \to x'^\mu = x^\mu + \delta x^\mu(x) \tag{1.9}$$

(note that, as before, the field variation is defined by comparing ψ' and ψ *at the same space–time position* x). As we shall see below, in many cases of physical interest the variations $\delta\psi$ and δx^μ are parametrized by a set of coefficients $\epsilon^1, \ldots, \epsilon^n$, which can be treated as first-order infinitesimal quantities, and which are typical of the considered group of transformations. Such parameters can be constant or position-dependent ($\epsilon = \epsilon(x)$), and the corresponding transformations (1.9) are called, respectively, *global* or *local*.

We will now compute the infinitesimal variation of the action induced by the transformations (1.9) (without imposing, for the moment, any boundary condition). Since we are varying also the coordinates we can write, in general,

$$\begin{aligned}\delta S &= \int_\Omega \left[d^4x\, \delta\mathcal{L} + \mathcal{L}\,\delta\big(d^4x\big)\right] \\ &= \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta(\partial_\mu \psi) + (\partial_\mu \mathcal{L})\,\delta x^\mu\right] + \int_\Omega \mathcal{L}\,\delta\big(d^4x\big).\end{aligned} \tag{1.10}$$

Let us explicitly evaluate the last contribution, using the fact that the transformation of the four-volume element is controlled by the Jacobian determinant $|\partial x'/\partial x|$ of the associated coordinate transformation:

$$d^4x \to d^4x' = d^4x \left|\frac{\partial x'}{\partial x}\right|. \tag{1.11}$$

For the infinitesimal transformation (1.9) we have, to first order in δx^μ,

$$\begin{aligned}\left|\frac{\partial x'}{\partial x}\right| &\equiv \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det\big(\delta^\mu_\nu + \partial_\nu \delta x^\mu\big) \\ &= 1 + \partial_\mu \delta x^\mu + \mathcal{O}\big(\delta x^2\big),\end{aligned} \tag{1.12}$$

so that

$$\mathcal{L}\,\delta\big(d^4x\big) = \mathcal{L}\big(d^4x' - d^4x\big) = \mathcal{L}\, d^4x\, \partial_\mu\big(\delta x^\mu\big). \tag{1.13}$$

Inserting this result into Eq. (1.10), adding up all terms, using Eq. (1.4) and integrating by parts we finally obtain

$$\delta S = \int_{\Omega} d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \delta\psi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \mathcal{L}\, \delta x^\mu \right) \right]. \quad (1.14)$$

This expression represents, to first order, the full variation of the action under the given infinitesimal transformation (1.9).

It may be useful to recall, at this point, the precise definition of symmetry transformation in a field theory context. A transformation (of the fields and/or of the coordinates) is called a *symmetry* of the given physical system if (and only if) the equations of motion of the system are left unchanged by the given transformation. We can say, in particular, that if ψ is a solution of the equations of motion, then the transformation $\psi \to \psi'$ represents a symmetry if and only if ψ' is also a solution of the same equations.

By applying the variational formalism, on the other hand, it can be easily shown that the equations of motion are left unchanged by an infinitesimal transformation provided that the corresponding variation of the action can be written as the integral of a total divergence,

$$\delta S = \int_{\Omega} d^4x\, \partial_\mu K^\mu, \quad (1.15)$$

where K^μ is a four-vector determined by the infinitesimal variation of the field ψ. We can immediately check in particular that, for any given Lagrangian $\mathcal{L}$, the two Lagrangians $\mathcal{L}$ and $\mathcal{L} + \partial_\mu f^\mu(\psi)$ lead to the same equations of motion: in fact, the application of the Euler–Lagrange operator to $\partial_\mu f^\mu(\psi)$ always gives identically zero (see Exercise 1.1).

More generally, we can also note that the variational contribution (1.15) (using the Gauss theorem) can be put in the form

$$\delta S = \int_{\partial\Omega} dS_\mu\, K^\mu. \quad (1.16)$$

If $K \sim \delta\psi$ then this contribution is vanishing because, as already stressed, the equations of motion are obtained by imposing $\delta\psi = 0$ on the border hypersurface $\partial\Omega$. Such a conclusion is no longer valid, in general, if K^μ depends not only on $\delta\psi$ but also on the variation of the field derivatives (see in particular Chap. 7, Sect. 7.1). However, even in that case the field equations are unchanged provided the field ψ and its derivatives are localized on a finite portion of space, in such a way that the vector K^μ is identically vanishing on the border $\partial\Omega$ of the considered space–time domain.

By imposing on the action variation δS, given by (1.14), the condition required for the invariance of the equations of motion, Eq. (1.15), and using the above definition of symmetry, we can thus conclude that the transformation (1.9) represents a

symmetry of our physical system provided that

$$\left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right]\delta\psi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \mathcal{L}\,\delta x^\mu\right] = \partial_\mu K^\mu. \tag{1.17}$$

The invariance of the action ($\delta S = 0$) requires instead a stronger condition, namely:

$$\left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right]\delta\psi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \mathcal{L}\,\delta x^\mu\right] = 0. \tag{1.18}$$

However, if K^μ is vanishing on the border $\partial\Omega$ of the integration domain, then the contribution of $\partial_\mu K^\mu$ to δS disappears thanks to the Gauss theorem (see Eq. (1.16)). In that case the symmetry condition (1.17), besides preserving the equations of motion, is also able to ensure the exact invariance of the action.

We are now in the position of introducing the result—universally known as Nöther's theorem—expressing in a quantitative way the close connection existing between symmetries and conservation laws (announced by the title of this section). In fact, from the condition (1.17) it follows that to any symmetry transformation $\{\delta\psi, \delta x\}$, and to any field configuration satisfying the equations of motion (1.8), we can always associate a vector current J^μ, defined by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \mathcal{L}\,\delta x^\mu - K^\mu, \tag{1.19}$$

which is conserved—namely which has zero divergence—as a consequence of the transformation properties of the given physical system:

$$\partial_\mu J^\mu = 0. \tag{1.20}$$

It should be stressed, however, that the definition of the conserved current associated to a symmetry is not unique, in general. Given a Lagrangian density, in fact, we can always add to that Lagrangian the covariant divergence of a four-vector without changing the equations of motion, hence without breaking the symmetries of the given system. The modified Lagrangian then leads to define a new Nöther current which is still conserved, even if it is different, in general, from the current associated to the original Lagrangian (see Sect. 1.2 for an explicit example).

Finally, it should be noted that if the symmetry transformation depends on n independent parameters $\epsilon^1, \ldots, \epsilon^n$, then there are in general n vector currents which are separately conserved.

Let us suppose, for instance, that the infinitesimal variation (1.9) of the field and of the coordinates can be expressed in terms of the n constant parameters ϵ^A as follows:

$$\delta\psi = \epsilon^A \delta_A \psi, \qquad \delta x^\mu = \epsilon^A \delta_A x^\mu, \quad A = 1, 2, \ldots, n. \tag{1.21}$$

Let us repeat the same procedure as before, defining $K^\mu = \epsilon^A K^\mu_A$, and imposing that the equations of motion of the field ψ are satisfied. By factorizing the constant

parameters ϵ^A we find that, according to the symmetry condition (1.17), to each independent parameter is associated a specific conserved current J^μ_A, $A = 1, 2, \ldots, n$, such that

$$J^\mu_A = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta_A \psi + \mathcal{L}\, \delta_A x^\mu - K^\mu_A, \quad \partial_\mu J^\mu_A = 0. \tag{1.22}$$

Examples of this type will be explicitly presented in the following sections.

1.2 Global Translations and Canonical Energy-Momentum Tensor

A simple but important example of symmetry, typical of Minkowski space–time, is represented by the invariance under *global* space–time translations, and is associated to the transformation

$$x^\mu \to x'^\mu(x) = x^\mu + \epsilon^\mu, \tag{1.23}$$

where ϵ^μ are four constant (and independent) infinitesimal parameters. The inverse transformation is given by

$$x^\mu\big(x'\big) = x'^\mu - \epsilon^\mu, \tag{1.24}$$

and the Jacobian matrix of this transformation reduces to the identity matrix,

$$\delta x^\mu = x'^\mu - x^\mu = \epsilon^\mu = \text{const} \quad \Longrightarrow \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu, \tag{1.25}$$

since we are considering "rigid" (i.e. position independent) translations.

It follows that all fields, quite independently of their specific type of tensor (or spinor) representation, transform as scalar objects under global translations:

$$\psi'\big(x'\big) \equiv \psi'(x + \epsilon) = \psi(x). \tag{1.26}$$

Their infinitesimal variation $\delta\psi$ at a fixed space–time position (for instance, at the point x) can then be obtained by expanding in Taylor series the above transformation, evaluated at the translated point $x - \epsilon$, in the limit $\epsilon \to 0$. To the first order in ϵ, in particular, we have

$$\psi'(x) = \psi(x - \epsilon) \simeq \psi(x) - \epsilon^\mu \partial_\mu \psi(x) + \cdots, \tag{1.27}$$

so that

$$\delta\psi(x) \equiv \psi'(x) - \psi(x) = -\epsilon^\mu \partial_\mu \psi. \tag{1.28}$$

Let us now compute the conserved Nöther current for the case in which the global translations (1.23) represent an exact symmetry of the physical system we are considering.

We note, first of all, that the four-dimensional space–time measure is invariant under global translations, $d^4x' = d^4x$, according to the transformation rule (1.11). We can also note that the translational invariant systems correspond to the so-called "isolated" systems, for which the Lagrangian density also transforms as a scalar, $\mathcal{L}'(\psi'(x')) = \mathcal{L}(\psi(x))$, quite independently of the space–time integration measure. Hence, for a system invariant under global translations, we have

$$\int d^4x'\, \mathcal{L}'\big(\psi'(x')\big) = \int d^4x\, \mathcal{L}\big(\psi(x)\big), \tag{1.29}$$

which means that the action itself is also an invariant quantity.

In such a case we can put $K^\mu = 0$ in the general definition of symmetry, and consider the simpler condition (1.18). By imposing the validity of the equations of motion (1.8), using for δx^μ, $\delta\psi$ the explicit forms (1.25), (1.28), and taking into account that the parameters ϵ^μ are constant, we finally arrive at the conservation equations

$$\epsilon^\nu \partial_\mu \Theta_\nu{}^\mu = 0, \tag{1.30}$$

where we have defined

$$\Theta_\nu{}^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \partial_\nu \psi - \mathcal{L}\, \delta^\mu_\nu \tag{1.31}$$

(the sign of $\Theta_\nu{}^\mu$, in principle arbitrary, has been chosen in this way for later convenience, in agreement with our metric conventions).

Since the four parameters ϵ^ν are independent, it follows from Eq. (1.30) that there are four vector currents which are separately conserved, $\Theta_\nu{}^\mu$, $\nu = 1, \ldots, 4$—namely, a current for each of the four components of ϵ^ν. We thus recover a simple example of the class of transformations (1.21) considered at the end of the previous section. In this example we have $n = 4$, the index A of Eq. (1.21) corresponds to the space–time index ν, and the infinitesimal variations of Eq. (1.21) correspond to

$$\delta_\nu x^\mu = \partial_\nu x^\mu, \qquad \delta_\nu \psi = -\partial_\nu \psi. \tag{1.32}$$

We should note, in particular, that ν is a vector index in Minkowski space–time, so that the object $\Theta_\nu{}^\mu$ defined by Eq. (1.31) is a rank-two tensor, the so-called *canonical energy-momentum density* tensor.

In order to explain (and physically justify) the name given to this tensor we should recall, first of all, that any conserved current J^μ (satisfying the continuity equation $\partial_\mu J^\mu = 0$) can always be associated to a conserved "charge" (or constant of motion) defined by an appropriate space–time integration.

Let us consider, to this purpose, a portion Ω of Minkowski space–time which extends to infinity along the three space-like directions, and which is bounded along the time-like direction by two parallel Euclidean hyperplanes Σ_1 and Σ_2 (i.e. by two three-dimensional, space-like hypersurfaces with time-like normal n_μ, such that $n_\mu n^\mu = 1$, see Fig. 1.1). By integrating the continuity equation over Ω, by applying

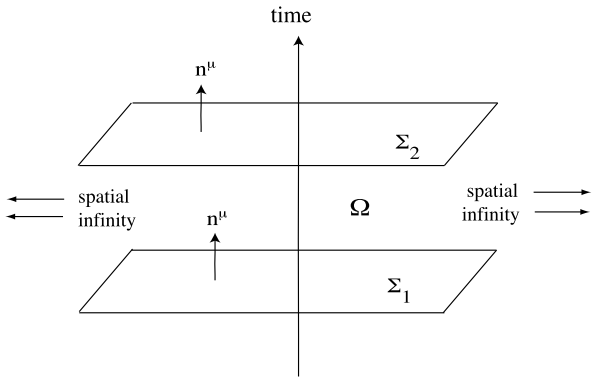

Fig. 1.1 The portion Ω of the Minkowski space–time, bounded by two three-dimensional hyperplanes Σ_1 and Σ_2 extending up to spatial infinity

the Gauss theorem, and assuming that the field contributions to J^μ are localized at a finite distance from the origin (i.e. that J^μ approaches zero quickly enough at $x \to \pm\infty$), we obtain

$$0 = \int_\Omega d^4x\, \partial_\mu J^\mu = \int_{\partial\Omega} J^\mu\, dS_\mu = \int_{\Sigma_2} J^\mu\, dS_\mu - \int_{\Sigma_1} J^\mu\, dS_\mu. \tag{1.33}$$

The opposite sign of the last two integrals is due to the fact that, according to the Gauss theorem, we have to consider the "outgoing" flow of J^μ on $\partial\Omega$, namely the flux oriented along the normal in the outward direction.

The above equation implies that the flux of J^μ does not depend on the given hypersurface,

$$\int_{\Sigma_2} J^\mu\, dS_\mu = \int_{\Sigma_1} J^\mu\, dS_\mu. \tag{1.34}$$

We can then evaluate the product $J^\mu\, dS_\mu$, in particular, in the rest frame of an inertial observer whose four-velocity is parallel to the normal n^μ. In that case $n^\mu = \delta^\mu_0$, $dS_0 = d^3x$, $dS_i = 0$, and the hyperplanes Σ_1, Σ_2 are represented by $t = \text{const}$ hypersurfaces, intersecting the time axis at the points t_1 and t_2, respectively. We then obtain from Eq. (1.34) the explicitly time-independent (i.e. conserved) quantity Q such that

$$\begin{aligned} Q(t_2) &= \frac{1}{c}\int_{\Sigma_2} J^\mu\, dS_\mu \equiv \frac{1}{c}\int_{t=t_2} J^0\, d^3x \\ &= Q(t_1) = \frac{1}{c}\int_{\Sigma_1} J^\mu\, dS_\mu \equiv \frac{1}{c}\int_{t=t_1} J^0\, d^3x = \text{const} \end{aligned} \tag{1.35}$$

(the convenient normalization factor $1/c$ has been inserted for dimensional reasons, as we shall see below).

The above result is valid for any divergenceless current J^μ, $\partial_\mu J^\mu = 0$. For a translationally invariant system, in particular, we have four divergenceless currents $\Theta_\nu{}^\mu$, $\nu = 1, \ldots, 4$, hence we can define four constants of motion (or conserved

charges) P_ν,

$$P_\nu = \frac{1}{c}\int_\Sigma \Theta_\nu{}^\mu \, dS_\mu = \frac{1}{c}\int_{t=\text{const}} \Theta_\nu{}^0 \, d^3x, \tag{1.36}$$

associated to the four parameters ϵ^ν specifying the translations along the four space–time directions. From the results of elementary analytical mechanics, on the other hand, we already know that the translational invariance along a spatial axis $\hat{x}^i$ is associated to the conservation of the momentum p_i along that axis, while the invariance under time translations is associated to energy conservation. We can thus interpret the four conserved quantities (1.36) as the four components of the canonical four-momentum vector, $P_\nu = (p_i, \mathcal{E}/c)$, and the components of the tensor $\Theta_\nu{}^\mu$—which must be integrated over a spatial volume to reproduce P_ν—as a *density* of energy and momentum.

We can also understand, at this point, the need for the dimensional factor $1/c$ introduced before. Consider in fact the time-like component of the momentum, $P_0 = \mathcal{E}/c$, where $\mathcal{E}$ is the total energy of the system. From Eq. (1.31) we have

$$\Theta_0{}^0 = \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right)\dot{\psi} - \mathcal{L}, \tag{1.37}$$

where $\dot{\psi} = \partial\psi/\partial t$, and where $\partial\mathcal{L}/\partial\dot{\psi}$ is the momentum canonically conjugate to the field ψ. It follows that $\Theta_0{}^0$ exactly represents the Hamiltonian density $\mathcal{H}$ which, for an isolated system, also coincide with its total energy density. The volume integral of $\Theta_0{}^0$ over all three-dimensional space, divided by c, thus correctly corresponds to the component P_0 of the four-momentum, in agreement with the definition (1.36).

Let us finally notice that the canonical tensor (1.31) *is not*, in general, a symmetric tensor (i.e. $\Theta_{\nu\mu} \neq \Theta_{\mu\nu}$). We know, on the other hand, that the definition of the conserved Nöther current is not unique: using this property, we can always introduce an equivalent form of the energy-momentum current which is not only conserved but also symmetric in the two tensor indices, as it will be explicitly shown in Sect. 1.3.

1.2.1 A Comment on the Non-uniqueness of the Definition

As already stressed in Sect. 1.1, it is always possible to add to the Lagrangian a total divergence without affecting the equations of motion, hence without breaking the symmetries of the given systems.

In particular, we can add to $\mathcal{L}$ the term $\overline{\mathcal{L}} = \partial_\alpha f^\alpha(\psi)$ without breaking the translational invariance of the system. The added term $\partial_\alpha f^\alpha$ provides a new contribution (let us call it $\overline{\Theta}_\nu{}^\mu$) to the canonical energy-momentum: such a contribution, however, turns out to be automatically divergence-free, $\partial_\mu \overline{\Theta}_\nu{}^\mu = 0$ (see Exercise 1.2). Hence, the total energy-momentum $\Theta + \overline{\Theta}$ associated to the new Lagrangian $\mathcal{L} + \overline{\mathcal{L}}$ is still conserved,

$$\partial_\nu\left(\Theta_\mu{}^\nu + \overline{\Theta}_\mu{}^\nu\right) = 0, \tag{1.38}$$

as Θ and $\overline{\Theta}$ are both separately conserved.

The new tensor $\Theta + \overline{\Theta}$ is different, in general, from the original canonical tensor Θ. However, the constants of motion associated to Θ and to $\Theta + \overline{\Theta}$ are exactly the same. In fact, by applying to $\overline{\mathcal{L}}$ the definition (1.31) we obtain

$$\overline{\Theta}_{\mu}{}^{\nu} = \partial_{\mu} f^{\nu} - \delta^{\nu}_{\mu} \partial_{\alpha} f^{\alpha} \tag{1.39}$$

(see Exercise 1.2, Eq. (1.109)). It follows that

$$\overline{P}_i = \frac{1}{c} \int \overline{\Theta}_i{}^0 \, d^3x = \frac{1}{c} \int d^3x \, \partial_i f^0, \tag{1.40}$$

$$\overline{P}_0 = \frac{1}{c} \int \overline{\Theta}_0{}^0 \, d^3x = -\frac{1}{c} \int d^3x \, \partial_i f^i. \tag{1.41}$$

The above integrals are all vanishing, since they reduce to integrals of the components of $f^{\mu}(\psi)$ evaluated on the border of the spatial hypersurface Σ, namely at spatial infinity where the fields are vanishing (if we are considering a physical system localized on a finite portion of space). The conserved four-momentum P_{ν} of Eq. (1.36) is thus the same for both Θ and $\Theta + \overline{\Theta}$, hence the two energy-momentum tensors are physically equivalent.

1.3 Lorentz Transformations and Canonical Angular Momentum

Another important symmetry of the Minkowski space–time is associated to the invariance under *global* rotations of the restricted (i.e. proper, orthochronous) Lorentz group, represented by the coordinate transformations

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{1.42}$$

where Λ is a constant matrix satisfying the conditions:

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}, \quad \det \Lambda = 1, \quad \Lambda^0{}_0 \geq 1 \tag{1.43}$$

($\eta_{\mu\nu}$ is the Minkowski metric). By expanding Λ around the identity we can set, to first order,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} + \cdots, \qquad x'^{\mu}(x) = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} + \cdots, \tag{1.44}$$

and by imposing the conditions (1.43) of the orthochronous $SO(1,3)$ group we find that ω has to be an antisymmetric matrix, $\omega_{\mu\nu} = \omega_{[\mu\nu]}$. We can then write

$$\delta x^{\mu} = x'^{\mu} - x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu} \equiv \frac{1}{2} \big(\omega^{\mu\nu} - \omega^{\nu\mu} \big) x_{\nu}, \tag{1.45}$$

where the six independent (and constant) components of $\omega_{\mu\nu}$ represent the six infinitesimal parameters of the considered global transformation.

The corresponding infinitesimal variation of the field can now be obtained by recalling that the restricted Lorentz group is a (non-compact) Lie group, and that a generic transformation can be represented in exponential form as follows:

$$\psi'(x') = U\psi(x), \quad U = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}. \tag{1.46}$$

The antisymmetric operator $S_{\mu\nu} = -S_{\nu\mu}$ contains the six generators of the group transformations— in our case three rotations and three boosts along the three spatial axes—and satisfies the so-called "Lie algebra" of the $SO(3,1)$ group, represented by the commutation rules:

$$\left[S^{\mu\nu}, S^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha}S^{\mu\beta} - \eta^{\nu\beta}S^{\mu\alpha} - \eta^{\mu\alpha}S^{\nu\beta} + \eta^{\mu\beta}S^{\nu\alpha}\right). \tag{1.47}$$

The explicit form of the generators obviously depends on the Lorentz representation of the field we are considering.

By expanding around the identity the transformation (1.46), and adopting a convenient notation by which we generically denote with $A, B, \ldots$ the full set of (tensor or spinor) Lorentz indices of the field ψ, we can then approximate the field transformation, to first order in ω, as

$$\psi'^{A}(x') = \left[\delta^{A}_{B} - \frac{i}{2}\omega_{\mu\nu}\left(S^{\mu\nu}\right)^{A}{}_{B} + \cdots\right]\psi^{B}(x). \tag{1.48}$$

For a scalar field, in particular, we have no explicit Lorentz indices and the corresponding generators are vanishing, $S^{\mu\nu} = 0$. For a vector field the indices $A, B, \ldots$ correspond to the space–time indices $\alpha, \beta, \ldots$ running from 0 to 3, and the corresponding six generators are represented by the 4×4 matrices $(S^{\mu\nu})^{\alpha}{}_{\beta}$: one matrix for each of the six independent combinations of the pair of antisymmetric indices μ, ν. The explicit form of these generators can be obtained by imposing on Eq. (1.48) to reduce to the infinitesimal vector transformation (1.44), namely by imposing

$$-\frac{i}{2}\omega_{\mu\nu}\left(S^{\mu\nu}\right)^{\alpha}{}_{\beta}\,\psi^{\beta} = \omega^{\alpha}{}_{\beta}\psi^{\beta}. \tag{1.49}$$

One then finds

$$\left(S^{\mu\nu}\right)^{\alpha}{}_{\beta} = i\left(\eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta}\right), \tag{1.50}$$

and it can be checked that, for these matrices, the Lie algebra (1.47) turns out to be automatically satisfied. An so on for higher-rank representations of the Lorentz group (see e.g. Chap. 13 for the generators of the spinor representation).

Here we are interested, in particular, in the infinitesimal variation of a generic field at a given (fixed) space–time position, say $\delta\psi(x) = \psi'(x) - \psi(x)$. To this purpose we start with the general transformation (1.46), evaluated not at x but at the translated point $x - \delta x$, we expand the Lorentz transformation to first order in ω as

in Eq. (1.48), and we also expand in Taylor series the translated field $\psi(x-\delta x)$, for $\delta x \to 0$. We then obtain (by omitting, for simplicity, the Lorentz indices of the field)

$$\psi'(x) = U\psi(x-\delta x) = \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} + \cdots\right)\left[\psi(x) - \delta x^{\mu}\partial_{\mu}\psi(x) + \cdots\right]$$

$$\simeq \psi(x) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\psi(x) - \delta x^{\mu}\partial_{\mu}\psi(x) + \cdots . \tag{1.51}$$

Using Eq. (1.45) for δx^{μ} we finally arrive at

$$\delta\psi \equiv \psi'(x) - \psi(x) = \frac{1}{2}\omega_{\mu\nu}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} - iS^{\mu\nu}\right)\psi. \tag{1.52}$$

We have now all the required ingredients for the definition of the conserved currents associated to the global transformations of the restricted Lorentz group. For such a group the four-volume element d^4x is automatically invariant—as in the case of the global translations—so that, assuming that the Lagrangian density is also separately Lorentz invariant, we can set $K^{\mu} = 0$, and apply the stronger symmetry condition (1.18). By imposing the equations of motion, and using the explicit form of the global infinitesimal variations (1.45), (1.52), we then obtain

$$\frac{1}{2}\omega_{\alpha\beta}\partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}\left(-iS^{\alpha\beta} + x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}\right)\psi + \mathcal{L}\left(\eta^{\alpha\mu}x^{\beta} - \eta^{\beta\mu}x^{\alpha}\right)\right]$$

$$= \frac{1}{2}\omega_{\alpha\beta}\partial_{\mu}\left[-i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}S^{\alpha\beta}\psi + \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}\partial^{\beta}\psi - \mathcal{L}\eta^{\mu\beta}\right)x^{\alpha}\right.$$

$$\left. - \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}\partial^{\alpha}\psi - \mathcal{L}\eta^{\mu\alpha}\right)x^{\beta}\right] = 0. \tag{1.53}$$

We should also note that the expression within the round brackets, on the right-hand side of the above equation, exactly corresponds the canonical energy-momentum tensor (1.31). The above equation is thus satisfied for arbitrary values of the (independent) parameters $\omega_{\alpha\beta}$ provided that

$$\partial_{\mu}J^{\mu\alpha\beta} = 0, \tag{1.54}$$

where

$$J^{\mu\alpha\beta} = S^{\mu\alpha\beta} + x^{\alpha}\Theta^{\mu\beta} - x^{\beta}\Theta^{\mu\alpha} = J^{\mu[\alpha\beta]}, \tag{1.55}$$

and where

$$S^{\mu\alpha\beta} = -i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}S^{\alpha\beta}\psi = S^{\mu[\alpha\beta]}. \tag{1.56}$$

Since $J^{\mu\alpha\beta} = J^{\mu[\alpha\beta]}$, this tensor has $4 \times 6 = 24$ independent components, corresponding to six conserved vector currents J^μ and thus to six constants of motion:

$$J^{\alpha\beta} = \frac{1}{c}\int_\Sigma J^{\mu\alpha\beta}\, dS_\mu = J^{[\alpha\beta]}, \tag{1.57}$$

associated to the invariance under boosts and rotations along the three spatial directions. By recalling that Θ represents the density of energy and momentum we can easily recognize, in the last part of the current (1.55),

$$x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha} \equiv L^{\mu\alpha\beta}, \tag{1.58}$$

the relativistic expression for the tensor density of *orbital angular momentum*. The additional contribution $S^{\mu\alpha\beta}$, which explicitly depends on the generators of the given Lorentz representation—and then on the intrinsic properties of the considered field—represents instead the tensor density of *intrinsic angular momentum* (intrinsic spin) of the given field (indeed, for a scalar field $S = 0$).

The tensor $J^{\mu\alpha\beta}$ thus represents the density of total angular momentum of the given system, and its volume integral (1.57) defines the total relativistic angular momentum tensor $J^{\alpha\beta}$, given by the sum of the orbital and intrinsic contributions.

1.3.1 Symmetrization of the Energy-Momentum Tensor

The conservation law of the angular momentum current helps us to understand the physical origin of the index asymmetry of the canonical energy-momentum tensor, $\Theta_{\mu\nu} \neq \Theta_{\nu\mu}$.

In fact, let us write explicitly Eq. (1.54), using the property $\partial_\mu \Theta^{\mu\nu} = 0$. We obtain

$$\partial_\mu S^{\mu\alpha\beta} + \Theta^{\mu\beta}\delta^\alpha_\mu - \Theta^{\mu\alpha}\delta^\beta_\mu = 0, \tag{1.59}$$

from which:

$$\Theta^{[\alpha\beta]} = -\frac{1}{2}\partial_\mu S^{\mu\alpha\beta}. \tag{1.60}$$

This clearly shows how the antisymmetric part of Θ is sourced by the spin density tensor, and explains why an antisymmetric part is unavoidably present in case of fields with nonzero intrinsic angular momentum. However, the above result also suggests a method to redefine of the energy-momentum tensor and obtain a new, physically equivalent expression, which is still conserved but symmetric.

Such a method, also known as the "Belinfante–Rosenfeld" symmetrization procedure, basically consists of subtracting from $\Theta^{\alpha\beta}$ the spin contributions, switching from Θ to a new tensor T such that

$$T^{\alpha\beta} = \Theta^{\alpha\beta} + \frac{1}{2}\partial_\mu\left(S^{\mu\alpha\beta} - S^{\alpha\beta\mu} + S^{\beta\mu\alpha}\right). \tag{1.61}$$

It can be easily checked that

$$T^{[\alpha\beta]} = -\frac{1}{2}\partial_\mu\left(S^{[\alpha\beta]\mu} + S^{[\beta\alpha]\mu}\right) \equiv 0, \tag{1.62}$$

and that

$$\partial_\beta T^{\alpha\beta} = \frac{1}{2}\partial_\beta\partial_\mu\left(S^{\mu\alpha\beta} + S^{\beta\mu\alpha}\right) = -\partial_\beta\partial_\mu S^{[\mu\beta]\alpha} \equiv 0. \tag{1.63}$$

The new tensor T is thus symmetric and automatically conserved. In addition, the difference between T and Θ is represented by a total divergence, which does not modify the constants of motion defined by the volume integration over an infinite space-like hypersurface, as discussed in the previous section.

The importance (and necessity) of the index symmetry for the energy-momentum tensor will become evident in the context of a relativistic theory of gravity, as we shall see later in Chap. 7. In such a context we will conveniently adopt an alternative definition of energy-momentum which always automatically provides the tensor in a symmetrized form (see in particular Sect. 7.2).

1.4 Examples of Energy-Momentum Tensor

In this last section we will introduce simple examples of energy-momentum tensors, for physical systems which will be frequently considered also in the subsequent chapters. Let us start with the case of a relativistic scalar field.

1.4.1 Scalar Field

Let us consider a classical scalar field ψ, which for simplicity we assume to be real, self-interacting through the potential $V(\phi)$. The total Lagrangian density $\mathcal{L}$ is given by the sum of the potential contribution and of the kinetic term, quadratic in the field gradients. Using natural units ($\hbar = c = 1$), and adopting for the kinetic part the usual canonical normalization, we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{1.64}$$

The momentum, canonically conjugate to the field variable ψ, is then given by

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi, \tag{1.65}$$

and the equation of motion (1.8) becomes

$$\partial_\mu\partial^\mu\phi \equiv \Box\phi = -\frac{\partial V}{\partial\phi}. \tag{1.66}$$

For a free massive field, in particular, $V = m^2\phi^2/2$, and the above equation reduces to the well-known Klein–Gordon equation,

$$\left(\Box + m^2\right)\phi = 0. \tag{1.67}$$

By applying the definition (1.31) to the scalar-field Lagrangian (1.64) we are led to the corresponding canonical energy-momentum tensor:

$$\Theta_\nu{}^\mu = \partial_\nu\phi\partial^\mu\phi - \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi\,\delta^\mu_\nu + V\delta^\mu_\nu. \tag{1.68}$$

We can easily check that this tensor is automatically symmetric, in agreement with the fact that the intrinsic angular momentum of a scalar field is zero (see Sect. 1.3.1). We can also check, by using the equation of motion (1.66), that this tensor is conserved (in the absence of external interactions). We have, in fact:

$$\partial_\mu\Theta_\nu{}^\mu = (\partial_\mu\partial_\nu\phi)\partial^\mu\phi + \partial_\nu\phi\Box\phi - (\partial_\nu\partial_\alpha\phi)\partial^\alpha\phi + \partial_\nu\phi\frac{\partial V}{\partial\phi} \equiv 0. \tag{1.69}$$

In the above equation the first and third term on the right-hand side automatically cancel each other, while the second and fourth term cancel by virtue of the equation of motion (1.66).

1.4.2 Electromagnetic Field

The electromagnetic forces are associated to a vector interaction, described by the vector potential A_ν. The kinetic term of the free field is quadratic in the first derivatives of A_ν, and it is represented by the so-called Maxwell Lagrangian density,

$$\mathcal{L} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right). \tag{1.70}$$

The general relations previously given for the equations of motion and for the energy-momentum tensor of a generic field ψ can now be applied to the vector case with the obvious replacement $\psi \to A_\nu$. The conjugate momentum is then given by

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.71}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. In the absence of sources the field equations (1.8) will then reproduce the usual Maxwell equations in vacuum, $\partial_\mu F^{\mu\nu} = 0$.

From the canonical definition (1.31) we also obtain the corresponding energy-momentum tensor,

$$\Theta_\alpha{}^\mu = -\frac{1}{4\pi}F^{\mu\nu}\partial_\alpha A_\nu + \frac{1}{16\pi}F^2\delta^\mu_\alpha, \tag{1.72}$$

which is not symmetric, in agreement with the general discussion of Sect. 1.3.1 (for a vector field, in fact, the intrinsic angular momentum is non-vanishing). However, the above tensor can be symmetrized by applying the Belinfante–Rosenfeld procedure, i.e. by adding a divergenceless term which cancels the intrinsic contributions of the spin density. In our case such an additional term has the following form:

$$\overline{\Theta}_\alpha{}^\mu = \frac{1}{4\pi} F^{\mu\nu}\partial_\nu A_\alpha, \tag{1.73}$$

and leads to the new tensor

$$T_\alpha{}^\mu = \Theta_\alpha{}^\mu + \overline{\Theta}_\alpha{}^\mu = -\frac{1}{4\pi}\left(F^{\mu\nu}F_{\alpha\nu} - \frac{1}{4}F^2\delta^\mu_\alpha\right). \tag{1.74}$$

It can be immediately checked that this tensor is symmetric, $T_{\alpha\mu} = T_{\mu\alpha}$, and traceless, $T_\mu{}^\mu = 0$. It is also instructive to compute its components in terms of the electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$), by using the explicit definition of $F_{\mu\nu}$:

$$\begin{aligned} &F^{i0} = E^i = -F_{i0}, \qquad F^{ij} = -\epsilon^{ijk}B_k = F_{ij}, \\ &F^2 \equiv F_{\mu\nu}F^{\mu\nu} = 2\big(B^2 - E^2\big). \end{aligned} \tag{1.75}$$

We then find that T_0^0 correctly provides the canonical energy density of the electromagnetic field,

$$T_0{}^0 = \frac{1}{8\pi}\big(E^2 + B^2\big), \tag{1.76}$$

and that the off-diagonal components T_0^i correctly reproduce the components of the well-known Poynting vector,

$$T_0{}^i = \frac{1}{4\pi}\epsilon^{ijk}E_j B_k = \frac{1}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})^i, \tag{1.77}$$

controlling the density of energy flux of the field.

We should notice, finally, that the energy-momentum (1.74) is referred to the Lagrangian of the free electromagnetic field, and is thus conserved only for free fields (in vacuum, without sources).

For a better clarification of this important point let us compute the divergence $\partial_\mu T^{\alpha\mu}$, and use the full Maxwell equations,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu, \qquad \partial_{[\mu}F_{\nu\alpha]} = 0, \tag{1.78}$$

taking into account the possible presence of charged sources described by the electromagnetic current J^μ. An explicit computation then leads to

$$\partial_\mu T_\alpha{}^\mu = -\frac{1}{4\pi}\left(\frac{4\pi}{c}J^\nu F_{\alpha\nu} + F^{\mu\nu}\partial_\mu F_{\alpha\nu} - \frac{1}{2}F^{\mu\nu}\partial_\alpha F_{\mu\nu}\right)$$

$$
\begin{aligned}
&= -\frac{1}{c} F_{\alpha\nu} J^{\nu} - \frac{1}{8\pi} F^{\mu\nu} (\partial_{\mu} F_{\alpha\nu} - \partial_{\nu} F_{\alpha\mu} - \partial_{\alpha} F_{\mu\nu}) \\
&= -\frac{1}{c} F_{\alpha\nu} J^{\nu} + \frac{1}{16\pi} F^{\mu\nu} \partial_{[\mu} F_{\nu\alpha]} \\
&= -\frac{1}{c} F_{\alpha\nu} J^{\nu}. \qquad (1.79)
\end{aligned}
$$

It follows that the divergence of $T_{\alpha\mu}$ is vanishing, and the energy-momentum of the electromagnetic fields is separately conserved, only in the absence of any coupling to the source term represented by the current J^{μ}. In the presence of sources the divergence is zero, as we shall see, only for the *total* energy-momentum tensor—i.e. for the tensor of the system "fields plus sources"—which corresponds in that case the true conserved Nöther current. This point will be explicitly illustrated by the simple example of the next section.

1.4.3 Point-Like Particle

Let us consider a free point-like particle of mass m and spin zero. The canonical energy-momentum is a symmetric tensor, and its explicit form can be deduced from the invariance of the action under global translations, following the same procedure as in the previous cases.

Such a canonical method will be explicitly presented in Exercise 1.4, starting from the action of a free relativistic particle. In this section we will obtain the energy-momentum tensor with a more direct approach, by noting that a point-like particle evolving along the trajectory $\boldsymbol{x} = \boldsymbol{x}(t)$, where t is the time parameter of a given inertial observer, is characterized by a mass density ρ_m given by

$$\rho_m = m\, \delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big). \qquad (1.80)$$

The delta-function distribution localizes, at any given time t, the point mass m at the spatial position of the particle. The four-momentum vector of the particle, as a function of time, can thus be written as

$$P^{\mu} = m u^{\mu} = \int d^3x\, \rho_m(\boldsymbol{x}) \frac{dx^{\mu}}{d\tau} = m \int d^3x\, \delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big) \frac{dx^{\mu}}{d\tau}, \qquad (1.81)$$

where $u^{\mu} = dx^{\mu}(t)/d\tau$ is the particle four-velocity along the given trajectory, and τ is the particle proper time.

A comparison with Eq. (1.36), which relates the momentum vector P and the canonical tensor Θ, then immediately leads to the relation:

$$\Theta^{\mu 0} = mc\, \delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big) \frac{dx^{\mu}}{d\tau}. \qquad (1.82)$$

We can also note, on the other hand, that $c = dx^0/dt$. By extending the previous relation to all coordinate components we thus arrive at the following expression for the energy-momentum tensor of a point-like particle:

$$\Theta^{\mu\nu} = m\,\delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt}. \tag{1.83}$$

The above expression is neither explicitly symmetric nor explicitly covariant. However, the symmetry of $\Theta^{\mu\nu}$ can be easily displayed if we recall that, for a free relativistic particle,

$$\frac{dt}{d\tau} = \gamma = \frac{\mathcal{E}}{mc^2}, \tag{1.84}$$

where γ is the Lorentz factor and $\mathcal{E}$ the relativistic kinetic energy. By multiplying and dividing by γ we can thus rewrite Θ in the form

$$\Theta^{\mu\nu} = m^2c^2\,\delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big)\frac{u^\mu u^\nu}{\mathcal{E}}, \tag{1.85}$$

which is equivalent to Eq. (1.83), but which is now explicitly symmetric in the two indices μ and ν.

In order to recast Eq. (1.83) in an explicitly covariant form we shall use the properties of the Dirac delta-function, which provides us with the identity:

$$\begin{aligned}\Theta^{\mu\nu}(\boldsymbol{x}, t) &= c\int dt'\,\delta\big(ct - ct'\big)\Theta^{\mu\nu}\big(\boldsymbol{x}, t'\big)\\ &= mc\int dt'\,\delta^4\big(x - x\big(t'\big)\big)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt'},\end{aligned} \tag{1.86}$$

where t is a generic integration variable. Switching to proper time, and parametrizing the particle world-line as $x = x(\tau)$, our integral becomes

$$\Theta^{\mu\nu} = mc\int d\tau\,\delta^4\big(x - x(\tau)\big)u^\mu u^\nu. \tag{1.87}$$

This form of the energy-momentum tensor is not only symmetric but also explicitly covariant, as $\delta^4(x)$ is a scalar for the global transformations of the $SO(3, 1)$ group, while the product of the two four-velocity vectors is clearly a tensor. We stress that the same result can be directly obtained also from the action of a free point-like particle, as shown in the Exercise 4.1.

Let us finally consider the covariant divergence of Θ, which we can separate into two parts as follows:

$$\partial_\nu\Theta^{\mu\nu} = \partial_i\Theta^{\mu i} + \partial_0\Theta^{\mu 0}. \tag{1.88}$$

By using for Θ the expression (1.83) we find that the only contribution to the spatial derivatives comes from the delta-function, so that

$$\partial_i\Theta^{\mu i} = mu^\mu\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}\delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big)$$

$$= -mu^{\mu}\frac{dx^{i}(t)}{dt}\frac{d}{dx^{i}(t)}\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big)$$

$$= -mu^{\mu}\frac{d}{dt}\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big). \tag{1.89}$$

Notice that in the second equality we have replaced the gradient along a generic direction, x^{i}, with the gradient along the particle trajectory, $x^{i}(t)$, thanks to the differentiation rule $\partial_{x} f(x-y) = -\partial_{y} f(x-y)$, valid for any function f depending on the difference of two variables.

For the time derivative of $\Theta^{\mu 0}$, instead, we have

$$\partial_{0}\Theta^{\mu 0} = \frac{d}{dt}\big(mu^{\mu}\big)\,\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big) + mu^{\mu}\frac{d}{dt}\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big). \tag{1.90}$$

By adding the two contributions (1.89), (1.90) we finally arrive at

$$\partial_{\nu}\Theta^{\mu\nu} = m\frac{du^{\mu}}{dt}\,\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big). \tag{1.91}$$

We can thus conclude that the above energy-momentum tensor has a vanishing divergence—and is thus separately conserved—only in the case of a free particle, satisfying the equation of motion $du^{\mu}/dt = 0$. In the presence of external forces there is instead a possible energy-momentum transfer between the external system and the particle, and the conserved quantity, in that case, is the *total* energy-momentum of the system "particle plus external forces".

An important example of this effect can be obtained by assuming that the point-like particle has an electrical charge e, and is interacting with a background electromagnetic field described by the tensor $F_{\mu\nu}$. The time evolution of the particle is then described by the well-known relativistic equation of the Lorentz force,

$$m\frac{du^{\mu}}{d\tau} = \frac{e}{c}F^{\mu}{}_{\nu}\frac{dx^{\nu}}{d\tau} \tag{1.92}$$

(see e.g. the book [32] of the bibliography). By parametrizing the trajectory with the time t of a generic inertial frame, multiplying by $d\tau/dt$ the above equation, and inserting the result into the conservation equation (1.91), we obtain

$$\partial_{\nu}\Theta^{\mu\nu} = \frac{e}{c}F^{\mu}{}_{\nu}\,\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big)\frac{dx^{\nu}}{dt}. \tag{1.93}$$

By using the definition of the electromagnetic current for a point-like particle,

$$J^{\nu} = \rho_{\mathrm{em}}\frac{dx^{\nu}}{dt} = e\delta^{3}\big(\boldsymbol{x}-\boldsymbol{x}(t)\big)\frac{dx^{\nu}}{dt}, \tag{1.94}$$

we can finally rewrite Eq. (1.93) as

$$\partial_{\nu}\Theta^{\mu\nu} = \frac{1}{c}F^{\mu}{}_{\nu}J^{\nu}. \tag{1.95}$$

A comparison with the divergence of the electromagnetic energy-momentum, Eq. (1.79), immediately shows that the sum of the two divergences is identically vanishing. This result is a particular example of the principle of conservation of the total energy-momentum tensor, $T^{\mu\nu} + \Theta^{\mu\nu}$, containing both the field and the particle contributions.

1.4.4 Perfect Fluid

Our last example is devoted to the case of a fluid whose elementary components have zero (or negligible) interactions—the so-called "perfect fluid" system. This type of fluid has no viscosity or internal friction, and its spatial distribution looks exactly isotropic to any observer locally at rest with a given fluid element. Also, if the fluid components have no intrinsic spin, the fluid energy-momentum is described by a symmetric canonical tensor.

In the rest system of a fluid element (the so-called "comoving" system), the energy-momentum tensor has then the following components:

$$T_0{}^0 = \rho, \qquad T_i{}^j = -p\delta_i^j, \qquad T_i{}^0 = 0. \tag{1.96}$$

We have denoted with ρ the proper energy density of the fluid, while p represents the internal pressure. In an generic frame, where the fluid is in motion with a velocity described by the four-vector u^μ, the components of $T_{\mu\nu}$ are given by

$$T_\mu{}^\nu = (\rho + p)\frac{u_\mu u^\nu}{c^2} - p\delta_\mu^\nu. \tag{1.97}$$

It can be easily checked that in the rest frame, where dove $u^i = 0$ and $u^0 = c$, the components of $T_{\mu\nu}$ reduce to those of Eq. (1.96).

The free evolution of a perfect fluid is characterized by the conservation of the energy-momentum, $\partial_\nu T_\mu{}^\nu = 0$, as well as by the conservation of the density of fluid particles. This second property is expressed by the continuity equation of the current N^μ,

$$N^\mu = nu^\mu, \quad \partial_\mu N^\mu = 0, \tag{1.98}$$

where n is the number of particles per unit of proper volume, i.e. the density of particles in the fluid rest frame. It may be useful to note that, as a consequence of these two conservation laws, the evolution of a free fluid is *adiabatic*.

In fact, by considering the conservation of the energy-momentum (1.97) (and setting for simplicity $c = 1$), we obtain

$$\begin{aligned} 0 &= u^\mu \partial_\nu T_\mu{}^\nu = u^\mu \partial_\nu \big[(\rho + p)u_\mu u^\nu\big] - u^\mu \partial_\mu p \\ &= \partial_\nu \big[(\rho + p)u^\nu\big] - u^\nu \partial_\nu p, \end{aligned} \tag{1.99}$$

where we have used

$$u^\mu \partial_\nu u_\mu = \frac{1}{2}\partial_\nu \left(u^\mu u_\mu\right) \equiv 0. \tag{1.100}$$

By multiplying and dividing by n, and using Eq. (1.98), we are led to

$$nu^\nu \partial_\nu \frac{(\rho + p)}{n} - u^\nu \partial_\nu p, \tag{1.101}$$

from which

$$nu^\nu \partial_\nu \left(\frac{\rho}{n}\right) + nu^\nu p\, \partial_\nu \left(\frac{1}{n}\right) = 0, \tag{1.102}$$

or, in differential form,

$$d\left(\frac{\rho}{n}\right) + p\, d\left(\frac{1}{n}\right) = 0. \tag{1.103}$$

Let us now recall that the rest energy density ρ is the energy per unit of proper volume, $\rho = E/V$, and n is the number of particles per unit of proper volume, $n = n_0/V$, where $n_0 =$ constant thanks to the conservation law (1.98). We can thus rewrite the above equation in the explicit thermodynamic form

$$dE + p\, dV = 0, \tag{1.104}$$

which clearly implies the conservation of the total entropy, $T\, dS = 0$.

Let us conclude by noticing that the free evolution of a perfect fluid keeps adiabatic even when the fluid is minimally coupled to an external gravitational field, and that this property has important applications in a cosmological context (see e.g. the book [51] of the bibliography).

1.5 Exercises Chap. 1

1.1 Equations of Motion and Total Divergence Show that the two Lagrangian densities, $\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{L}_2 = \mathcal{L} + \overline{\mathcal{L}}$, where

$$\mathcal{L} = \mathcal{L}(\psi, \partial\psi), \qquad \overline{\mathcal{L}} = \partial_\alpha f^\alpha, \quad f^\alpha = f^\alpha(\psi) \tag{1.105}$$

lead to the same equations of motion for the field ψ.

1.2 Energy-Momentum Tensor for a Total Divergence Show that the canonical energy-momentum tensor $\overline{\Theta}_\mu{}^\nu$ obtained from the Lagrangian density $\overline{\mathcal{L}}(\psi, \partial\psi) = \partial_\alpha f^\alpha(\psi)$ turns out to be automatically divergence-free, for any given $f^\alpha(\psi)$.

1.3 The Spin Four-Vector The center of mass of a physical system is at rest in the inertial frame $\mathcal{R}'$, and it is placed at the origin of that coordinate system. The orbital angular momentum is vanishing, but the system has an intrinsic angular momentum

lying in the (x', y') plane, with components J'_x and J'_y. Determine the intrinsic angular momentum in the inertial frame $\mathcal{R}$ where the physical system is moving with constant velocity v along the positive direction of the x axis.

1.4 Translational Symmetry of the Free Point-Particle Action Compute the canonical energy-momentum tensor starting from the action of a free (massive) point-like particle, and imposing the invariance under global infinitesimal translations.

1.6 Solutions

1.1 Solution By varying the action associated to $\mathcal{L}_1$ and to $\mathcal{L}_2$ we obtain, in both cases, the Euler–Lagrange equations (1.8). The difference between the two equations is represented by the term

$$\Delta = \left[\frac{\partial}{\partial\psi} - \partial_\mu \frac{\partial}{\partial(\partial_\mu\psi)}\right]\partial_\alpha f^\alpha. \tag{1.106}$$

This term, however, is identically vanishing, regardless of the specific choice of $f^\alpha(\psi)$. In fact, if we compute

$$\begin{aligned}
\Delta_1 &= \frac{\partial}{\partial\psi}\left(\partial_\alpha f^\alpha\right) = \frac{\partial}{\partial\psi}\left(\partial_\alpha\psi\frac{\partial f^\alpha}{\partial\psi}\right) = \partial_\alpha\psi\frac{\partial^2 f^\alpha}{\partial\psi^2},\\
\Delta_2 &= \frac{\partial}{\partial(\partial_\mu\psi)}\left(\partial_\alpha f^\alpha\right) = \delta^\mu_\alpha\frac{\partial f^\alpha}{\partial\psi} = \frac{\partial f^\mu}{\partial\psi},\\
\Delta_3 &= \partial_\mu\frac{\partial}{\partial(\partial_\mu\psi)}\left(\partial_\alpha f^\alpha\right) = \partial_\mu\left(\frac{\partial f^\mu}{\partial\psi}\right) = \frac{\partial^2 f^\mu}{\partial\psi^2}\,\partial_\mu\psi
\end{aligned} \tag{1.107}$$

we immediately obtain $\Delta \equiv \Delta_1 - \Delta_3 = 0$.

1.2 Solution For the Lagrangian $\overline{\mathcal{L}}$ we have

$$\begin{aligned}
\overline{\mathcal{L}} &= \partial_\alpha f^\alpha = \partial_\alpha\psi\frac{\partial f^\alpha}{\partial\psi};\\
\frac{\partial}{\partial(\partial_\nu\psi)}\overline{\mathcal{L}} &= \delta^\nu_\alpha\frac{\partial f^\alpha}{\partial\psi} = \frac{\partial f^\nu}{\partial\psi}.
\end{aligned} \tag{1.108}$$

By applying the canonical definition (1.31) we obtain the energy-momentum tensor:

$$\begin{aligned}
\overline{\Theta}_\mu{}^\nu &= \frac{\partial f^\nu}{\partial\psi}\,\partial_\mu\psi - \delta^\nu_\mu\partial_\alpha f^\alpha\\
&= \partial_\mu f^\nu - \delta^\nu_\mu\partial_\alpha f^\alpha,
\end{aligned} \tag{1.109}$$

which is automatically divergence-free,

$$\partial_\nu \overline{\Theta}_\mu{}^\nu = \partial_\nu \partial_\mu f^\nu - \partial_\mu \partial_\alpha f^\alpha \equiv 0, \tag{1.110}$$

since $\partial_\mu \partial_\alpha = \partial_\alpha \partial_\mu$.

1.3 Solution Let us first decompose the total angular momentum (1.57) into intrinsic and orbital part,

$$J^{\alpha\beta} = \Sigma^{\alpha\beta} + L^{\alpha\beta}, \quad L^{\alpha\beta} = x^\alpha P^\beta - x^\beta P^\alpha, \tag{1.111}$$

where Σ and L are obtained by the volume integration of the corresponding densities $S^{\mu\alpha\beta}$ and $L^{\mu\alpha\beta}$ (see Eqs. (1.55), (1.58)). Due to the presence of the orbital part, the angular momentum tensor J is not invariant under coordinate translations of the type $x^\mu \to x^\mu + a^\mu$. In that case, in fact:

$$J^{\alpha\beta} \to J^{\alpha\beta} + a^\alpha P^\beta - a^\beta P^\alpha. \tag{1.112}$$

In order to select the intrinsic component of the angular momentum—which should be unaffected by this type of coordinate transformations—we can conveniently introduce the so-called Pauli–Lyubanskii spin vector S^μ, defined by

$$S_\mu = \frac{1}{2c} \epsilon_{\mu\alpha\beta\nu} J^{\alpha\beta} u^\nu, \tag{1.113}$$

where u^ν is the four-velocity of the considered physical system. If $P^\mu = m u^\mu$ then the orbital part $L^{\alpha\beta}$ does not contributes to S_μ, because $P^{[\beta} u^{\nu]} = 0 = P^{[\alpha} u^{\nu]}$. Notice that S_μ has only three independent components, as it satisfies the constraint $u^\mu S_\mu \equiv 0$.

In the rest frame $\mathcal{R}'$ of our physical system we have $u^i = 0$, $u^0 = c$, and the intrinsic spin vector S'_μ has components

$$\begin{aligned} S'_1 &= J'^{23} = J'_x, \qquad S'_2 = J'^{31} = J'_y, \\ S'_3 &= J'^{12} = 0, \qquad S'_0 = 0. \end{aligned} \tag{1.114}$$

The components of S_μ in a different inertial frame $\mathcal{R}$ are related to the above components of S'_μ by the Lorentz transformation $S'^\mu = \Lambda^\mu{}_\nu S^\nu$. For the special case we are considering the frame $\mathcal{R}'$ is in motion with respect to $\mathcal{R}$ along the positive direction of the x axis. By applying the inverse Lorentz transformation we then obtain

$$\begin{aligned} S^1 &= \gamma\big(S'^1 + \beta S'^0\big), \qquad S^2 = S'^2, \\ S^3 &= S'^3, \qquad S^0 = \gamma\big(S'^0 + \beta S'^1\big), \end{aligned} \tag{1.115}$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Hence:

$$\begin{aligned} S_1 &= J_x = \gamma J'_x, \qquad S_2 = J_y = J'_y, \\ S_3 &= J_z = 0, \qquad S_0 = \beta\gamma S'^1 = -\beta\gamma J'_x. \end{aligned} \tag{1.116}$$

Note that the Lorentz transformation induces a deformation of the three-dimensional spin vector $\boldsymbol{S}$ in the (x, y) plane, but the modulus of the four-vector S_μ is left unchanged. In fact

$$\begin{aligned} S'_\mu S'^\mu &= -\left(J'^2_x + J'^2_y\right), \\ S_\mu S^\mu &= S_0^2 - S_1^2 - S_2^2 = -J'^2_x \gamma^2 \left(1 - \beta^2\right) - J'^2_y = -\left(J'^2_x + J'^2_y\right) \\ &= S'_\mu S'^\mu. \end{aligned} \tag{1.117}$$

1.4 Solution The time evolution of a point-like mass describes a one-dimensional trajectory $x^\mu = x^\mu(\tau)$ in Minkowski space–time (the so-called "world-line" of the particle), and the action associated to the free-particle evolution is proportional to the line integral along such a trajectory,

$$S = -mc \int \sqrt{dx_\mu \, dx^\mu} = -mc \int_{\tau_1}^{\tau_2} \sqrt{\dot{x}_\mu \dot{x}^\mu} \, d\tau \equiv \int_{\tau_1}^{\tau_2} L(x, \dot{x}) \, d\tau. \tag{1.118}$$

Here L is the effective Lagrangian, and $\dot{x} = dx/d\tau$. The trajectory is parametrized by a variable τ which we shall assume to be Lorentz invariant, and the action S has been normalized in such a way as to reproduce the non-relativistic, canonically normalized action in the limit $|dx^i/d\tau| \ll c$.

By varying the action with respect to x^μ with the boundary conditions $\delta x^\mu(\tau_1) = 0 = \delta x^\mu(\tau_2)$, and imposing $\delta S = 0$, we easily obtain the equation of motion in the general form:

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = \frac{d}{d\tau} \left(\frac{\dot{x}_\mu}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} \right) = 0. \tag{1.119}$$

By identifying τ with the particle proper time we are led to the constraint $\dot{x}_\alpha \dot{x}^\alpha = c^2 = \text{const}$, and the equation of motion for the free particle finally reduces to the well-known condition of vanishing four-acceleration, $\ddot{x}^\mu = 0$.

Since the position of the point-like particle is localized in space–time along the one-dimensional trajectory $x^\mu(\tau)$, the action (1.118) can be conveniently rewritten also as a four-volume integral with invariant measure d^4x and with an effective "delta-function" Lagrangian density $\mathcal{L}$, such that

$$\begin{aligned} S &= \int d^4x \, \mathcal{L}(x, \dot{x}), \\ \mathcal{L}(x, \dot{x}) &= -mc \int d\tau \, \sqrt{\dot{x}_\mu \dot{x}^\mu} \, \delta^4\big(x - x(\tau)\big). \end{aligned} \tag{1.120}$$

Considering this action, and performing an infinitesimal coordinate transformation $x^\mu \to x^\mu + \delta x^\mu$, we have to take into account that the Lagrangian density $\mathcal{L}$ depends not only on $\dot{x}$, but also on x (due to the presence of the delta function). The infinitesimal variation is thus given, in general, by

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \frac{d}{d\tau} \big(\delta x^\mu\big)$$

$$= \left[\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right] \delta x^\mu + \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta x^\mu \right). \tag{1.121}$$

By imposing that the Euler–Lagrange equations are satisfied, for any δx^μ, it follows that the Lagrangian is invariant under the given transformation provided that the last term of the above equation is vanishing.

For the Lagrangian (1.120), in particular, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta x^\mu = -mc \int d\tau \, \frac{\dot{x}_\mu}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} \delta^4\big(x - x(\tau)\big) \, \delta x^\mu. \tag{1.122}$$

By differentiating with respect to τ along the particle trajectory we then find a delta-function contribution,

$$\frac{d}{d\tau} \delta^4\big(x - x(\tau)\big) = \dot{x}^\nu \partial_\nu \delta^4\big(x - x(\tau)\big). \tag{1.123}$$

We find no contribution, instead, from the time derivative of $\dot{x}_\mu / \sqrt{\dot{x}_\alpha \dot{x}^\alpha}$, thanks to the equation of motion (1.119). If we consider, in particular, a global translation, $\delta x^\mu = \epsilon^\mu = \text{constant}$, and if we identify τ with the proper time parameter, as before, then the invariance condition $\delta \mathcal{L} = 0$ reduces to

$$-mc\,\epsilon^\mu \int d\tau \, \dot{x}_\mu \dot{x}^\nu \partial_\nu \delta^4\big(x - x(\tau)\big) \equiv -\epsilon^\mu \partial_\nu \Theta_\mu{}^\nu = 0, \tag{1.124}$$

where the conserved tensor

$$\Theta_\mu{}^\nu = mc \int d\tau \, \delta^4\big(x - x(\tau)\big) u_\mu u^\nu \tag{1.125}$$

exactly reproduces the energy-momentum tensor of a point particle, in agreement with the previous result presented in Eq. (1.87).

Let us finally check that the Euler–Lagrange equations for the Lagrangian density (1.120) are the same as the free-particle equations of motion presented in Eq. (1.119). We have, in fact,

$$\left(\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) \delta x^\mu = -mc \int d\tau \left[\frac{d}{d\tau} \left(\frac{\dot{x}_\mu}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} \right) \delta^4\big(x - x(\tau)\big) + \frac{\dot{x}_\mu}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} \dot{x}^\nu \partial_\nu \delta^4\big(x - x(\tau)\big) \right] \delta x^\mu, \tag{1.126}$$

and also

$$\frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu = -mc \int d\tau \, \sqrt{\dot{x}_\alpha \dot{x}^\alpha} \, \partial_\nu \delta^4\big(x - x(\tau)\big) \, \delta x^\nu. \tag{1.127}$$

Along the particle trajectory, on the other hand, we can set $\delta x^\mu = \dot{x}^\mu \, d\tau$. By taking the difference of the two terms (1.126), (1.127) we then find that the derivatives of the delta function disappear, and that the Euler–Lagrange equations exactly reproduce the equations of motion (1.119).

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Chapter 2
Towards a Relativistic Theory of Gravity

The equations of Newton's gravitational theory, which provide the theoretical foundations of Kepler's celestial mechanics and seem to describe so well the gravitational force at all macroscopical scales, are *not* compatible, unfortunately, with the principles of Einstein's special relativity.

The Newton equations, in fact, predict for the gravitational effects an infinite speed of propagation in any medium. Also, they do not take into account the possible transformation properties of the gravitational field of forces from one reference frame to another. The Newtonian theory defines indeed the forces generated by static matter sources, but gives us no hint about the forces produced by *moving* sources. Hence, the theory may describe the gravitational field of a mass M through the static potential $\phi(r) = -GM/r$ only in the non-relativistic approximation, i.e. in the regime where the modulus of the potential energy $m\phi$ of a test mass m is negligible with respect to its rest energy mc^2, namely:

$$\frac{GM}{rc^2} \ll 1. \tag{2.1}$$

A correct description of gravity in the relativistic regime thus require an appropriate generalization of Newton's theory. Which kind of generalization? A natural answer seems to be suggested by the close formal analogy existing between the Newton force among static masses and the Coulomb electrostatic force among electric charges. In the same way as the Coulomb potential corresponds to the fourth component of the electromagnetic vector potential, the Newton potential might correspond to the component of a four-vector, and the relativistic gravitational interaction might be represented by an appropriate vector field, in close analogy with the electromagnetic theory.

Such an attractive speculation, however, has to be immediately discarded for a very simple reason: vector-like interactions produce *repulsive* static interactions between sources of the same sign, while—as is well known—the static gravitational interaction between masses of the same sign is *attractive*.

Another simple (and formally consistent) possibility is based on the assumption that the Newton potential may be treated as an invariant under a general change

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of frame, i.e. that gravity may be correctly described by a relativistic *scalar* field. However, also this hypothesis has to be discarded on the ground of phenomenological results, even if the reasons, this time, are more subtle. In view of our subsequent applications it is worthwhile to recall here one of these reasons, concerning the precession of planetary orbits.

Let us consider the motion of a relativistic test body of mass m, interacting with a central (i.e. radially oriented) field of forces described by the scalar potential $U(r) = -GM/r$. The dynamics of the problem is controlled by the relativistic Lagrangian

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - mU, \tag{2.2}$$

where $v^2 = v_i v^i$ and $v^i = dx^i/dt$. The kinetic term of this Lagrangian can be directly obtained from the free-particle action (1.118), by parametrizing the particle trajectory with the time coordinate t of a generic inertial system, $x^\mu = x^\mu(t)$.

It can be easily shown that the angular momentum of this dynamical system is conserved, and that the motion is confined on a plane (as $\boldsymbol{x} \times \boldsymbol{\nabla} U = 0$). By introducing on that plane a system of polar coordinates $\{r, \varphi\}$,

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \tag{2.3}$$

and specializing U as the gravitational potential produced by a central body of mass M, we explicitly obtain the Lagrangian

$$L = -mc^2\left[1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\right]^{1/2} + \frac{GMm}{r}, \tag{2.4}$$

where the dot denotes differentiation with respect to t.

In the above Lagrangian there are two cyclic coordinates, φ and t, and the corresponding conjugate momenta are associated to constants of motion defining, respectively, the total angular momentum and the total energy (or Hamiltonian) of our system. We can set, in particular,

$$\frac{\partial L}{\partial \dot{\varphi}} = m\gamma r^2\dot{\varphi} = mh = \text{const}, \tag{2.5}$$

$$H = v^i\frac{\partial L}{\partial v^i} - L = m\gamma c^2 + mU = m\alpha = \text{const}, \tag{2.6}$$

where γ is the Lorentz factor,

$$\gamma = \left[1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\right]^{-1/2}, \tag{2.7}$$

and where h and α are constant parameters determined by the initial conditions.

Let us now combine the two relations (2.5) and (2.6) by eliminating $\dot{\varphi}^2$, differentiate the result with respect to φ, and define $u = 1/r$. By excluding the case of

circular orbits (corresponding to $dr/d\phi = 0$) we then arrive at the following equation of motion (see Exercise 2.1),

$$u'' + k^2 u = \frac{k^2}{p}, \tag{2.8}$$

where a prime denotes differentiation with respect to φ, and where the two constants k and p are defined by

$$k^2 = 1 - \frac{c^2 r_0^2}{4h^2}, \qquad \frac{k^2}{p} = \frac{\alpha r_0}{2h^2}, \qquad r_0 = \frac{2GM}{c^2}. \tag{2.9}$$

The general solution of Eq. (2.8) depends on two integration constants (which we will call e and φ_0), and can be obtained by adding to the particular solution $u = p^{-1}$ the general solution of the corresponding homogeneous equation. In particular, for a motion confined on a finite portion of space (like the planetary orbits of our interest), the general solution can be conveniently written as follows:

$$u = \frac{1}{p}\big[1 + e\cos k(\varphi - \varphi_0)\big], \tag{2.10}$$

where $0 < e < 1$. In the non-relativistic limit $c \to \infty$ we have $r_0 \to 0$, $\alpha \to \infty$, the product αr_0 stays finite and we find $k^2 \to 1$. We thus recover, in this limit, the well known equation describing (in polar coordinates) an ellipse of eccentricity e and perihelion position $\varphi = \varphi_0$.

If we take into account the relativistic corrections, using for k the exact value (2.9), we find that the motion is still confined in a portion of space extending between a minimum and a maximum distance from the origin, but such a motion does not correspond anymore to a closed trajectory: the orbit of the planet is not a stationary ellipse but rather an open curve with flower petal shape, i.e. a "rosette-like" path. In that case the position of closest approach to the source—the so-called perihelion—is periodically reached after the angle subtended by the motion is not $\varphi - \varphi_0 = 2\pi$, but rather $k(\varphi - \varphi_0) = 2\pi$ (see Eq. (2.10)). To each planetary revolution is thus associated an angular shift of the perihelion position given by

$$\Delta\varphi = \frac{2\pi}{k} - 2\pi = 2\pi\left(\frac{1}{k} - 1\right) \simeq 2\pi\left(\frac{c^2 r_0^2}{8h^2}\right) = \frac{\pi G^2 M^2}{c^2 h^2} \tag{2.11}$$

(we have used the approximation $c^2 r_0^2/h^2 \ll 1$, which is well satisfied by the parameters of a typical planetary orbit).

A theory of gravity based on a relativistic scalar potential thus predicts for the planetary orbits a small effect of perihelion precession, described by Eq. (2.11). An effect of this type actually exists in our solar system, and has been measured for various planets with a series of long (more than secular) and precise astronomical observations.

However, the prediction (2.11) obtained with a scalar model of gravity is in sharp quantitative contrast with all observational results: for the planet Mercury,

for instance, the observed precession amounts to 43 arc-seconds per century, while Eq. (2.11) only gives 7 arc-seconds per century. This difference largely exceeds all possible experimental and systematic errors.[1] A scalar model of the gravitational interaction cannot thus represent a satisfactory relativistic generalization of Newton's theory.

An alternative approach to a relativistic gravitational theory, which is (up to now) in good agreement with all available observations, and which admits, at a classical level, also an interesting geometric interpretation, is the model of *tensor* gravitational interactions adopted by the so-called theory of general relativity.

The starting point of this successful approach is a radical extension of the symmetry principles of special relativity, asserting the physical equivalence of all *inertial* frames. Such an equivalence is lifted to a more general level by the assumption that:

- *the physical laws are the same in all frames*,

without special restrictions to the class of the inertial frames. This assumption necessarily leads to the so-called "principle of general covariance", stating that:

- *the physical laws are covariant with respect to general coordinate transformations*,

and not only with respect to Lorentz transformations. The two above assumptions, which represent a rather natural—and even trivial, apparently—generalization of the special-relativistic postulates, are at the ground of the theory of general relativity, and have a revolutionary impact on the whole physics. In fact, they necessarily imply that the rigid, pseudo-Euclidean structure of the Minkowski space–time has to be abandoned in favor of a more general geometrical structure.

For an explicit illustration of this important point let us recall that, under a general coordinate transformation $x^\mu \to x'^\mu$, the coordinate differentials are related by

$$dx^\mu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right) dx'^\nu, \tag{2.12}$$

where the term in round brackets represents the inverse Jacobian matrix of the given transformation. Let us suppose that we start with the coordinates x^μ of an inertial frame, where the infinitesimal space–time interval assumes the Minkowski form:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{2.13}$$

The same interval, given as a function of the new coordinates x'^μ, will acquire in general a different form. From Eq. (2.12) we obtain, in fact,

$$ds^2 = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}\, dx'^\alpha\, dx'^\beta \equiv g_{\alpha\beta}\big(x'\big)\, dx'^\alpha\, dx'^\beta, \tag{2.14}$$

[1] As we shall see in Chap. 10, the perihelion precession predicted by general relativity is approximately six times larger that the special-relativistic result (2.11), and is thus in good agreement with the astronomical observations.

where we have defined

$$g_{\alpha\beta}(x') = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}. \tag{2.15}$$

This explicitly shows that a generic coordinate transformation—unlike the special case of the Lorentz transformations—does not preserve the infinitesimal interval of the Minkowski space–time.

If we plan to extend the class of physically equivalent systems so as to include also non-inertial frames, then we must be prepared to accept a space–time interval (also called "line-element") ds^2 which is not rigidly determined as a pseudo-Euclidean combination of the differential dx^2, but as a more general (possibly position-dependent) combination of such differentials.

2.1 The Postulates of the Riemannian Geometry

As pointed out in the previous section, a relativistic model based on the principle of general covariance induces on the space–time a geometrical structure which includes the Minkowski one, but which is compatible also with generalized geometrical schemes. In order to formulate a predictive physical model we need then to introduce some "working hypotheses", and provide a more precise characterization of the space–time geometry we are willing to assume.

Let us consider, for this purpose, the following basic postulates:

- the space–time interval ds^2 can be expressed as a homogeneous quadratic form (with position-dependent coefficients) of the coordinate differentials:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu. \tag{2.16}$$

- the space–time interval ds^2 is invariant under general coordinate transformations:

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)\, dx^\mu\, dx^\nu = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}\, dx'^\alpha\, dx'^\beta \\ &= g'_{\alpha\beta}(x')\, dx'^\alpha\, dx'^\beta = ds'^2. \end{aligned} \tag{2.17}$$

As we shall see below, this second assumption is exactly equivalent to the requirement that the coefficients $g_{\mu\nu}$ of the quadratic form ds^2—i.e. the components of the so-called "metric" of the space–time manifold—transform as the components of a second-rank (covariant) tensor, namely:

$$g'_{\alpha\beta}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \tag{2.18}$$

(see Chap. 3).

Starting with the above hypotheses we are led to a geometric model of Riemannian type, namely a model which extends to manifolds with four (or more) space–time dimensions the approach suggested by Gauss for the intrinsic description of two-dimensional surfaces.

It is appropriate to recall, for this purpose, that the geometric properties of a generic n-dimensional hypersurface Σ_n can be described in two ways. A first possibility is based on the *extrinsic* approach, based on the embedding of Σ_n into a higher-dimensional Euclidean (or pseudo-Euclidean) manifold $\mathcal{M}_D$, with $D > n$, parametrized by the coordinates X^A, with line-element:

$$ds^2 = \eta_{AB}\, dX^A\, dX^B, \quad A, B = 1, \ldots, D. \tag{2.19}$$

Let us consider the case $D = n + 1$, for simplicity. The hypersurface Σ_n can be represented as a submanifold of $\mathcal{M}_{n+1}$ defined by one condition relating the $n + 1$ coordinates X^A, namely $f(X^A) = 0$. We can take, as a simple example, a two-dimensional spherical surface S_2 of radius a, embedded in the external 3-dimensional Euclidean space $\mathcal{R}_3$ parametrized by the Cartesian coordinates X^i, $i = 1, 2, 3$. The given surface is then defined by the coordinate condition

$$f\big(X^i\big) \equiv X_1^2 + X_2^2 + X_3^2 - a^2 = 0, \tag{2.20}$$

representing indeed a sphere of radius a with center at the origin $X^i = 0$.

There is, however, a second possible approach, of *intrinsic* type, able to describe the geometry of Σ_n without referring to the coordinates X^A of the external manifold, and using instead a set of coordinates ξ^μ, $\mu = 1, \ldots, n$ defined on Σ_n itself. We need to this purpose a set of parametric equations,

$$X^A = X^A\big(\xi^\mu\big), \quad A, B = 1, \ldots, n+1, \ \mu = 1, \ldots, n, \tag{2.21}$$

describing the immersion of Σ_n into $\mathcal{M}_{n+1}$. The line-element (2.19), restricted to the hypersurface Σ_n, is then given by

$$ds^2 = \left[\eta_{AB}\frac{\partial X^A(\xi)}{\partial \xi^\mu}\frac{\partial X^B(\xi)}{\partial \xi^\nu}\right] d\xi^\mu\, d\xi^\nu = g_{\mu\nu}(\xi)\, d\xi^\mu\, d\xi^\nu. \tag{2.22}$$

The variable $g_{\mu\nu}(\xi)$, defined by the terms in square bracket of the above equation, is the so-called "induced metric" on the given hypersurface.

We may thus conclude that it is always possible to describe the geometry of Σ_n in terms of its intrinsic coordinates ξ^μ only, introducing, however, a line-element which is not, in general, Euclidean (or pseudo-Euclidean) like that of the embedding space $\mathcal{M}_{n+1}$. In fact, let us consider again the example of the spherical surface S_2 embedded in $\mathcal{R}_3$. We may choose as intrinsic coordinates on S_2 the angular variables $\xi^\mu = \{\theta, \varphi\}$ of the spherical coordinate system, related to the Cartesian coordinates of $\mathcal{R}_3$ by the following parametric equations:

$$X_1 = a\sin\theta\cos\varphi, \qquad X_2 = a\sin\theta\sin\varphi, \qquad X_3 = a\cos\theta. \tag{2.23}$$

By differentiating these equations, and inserting the result into the Euclidean line-element of $\mathcal{R}_3$, we obtain for the spherical surface S_2 the interval:

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.24}$$

Hence we find on the S_2 sphere an intrinsic non-Euclidean geometry, described by the Riemannian metric $g_{\mu\nu}(\theta, \varphi)$ with components:

$$g_{11} = a^2, \qquad g_{22} = a^2 \sin^2\theta, \qquad g_{12} = g_{21} = 0. \tag{2.25}$$

A four-dimensional Riemann space–time is then characterized by a geometric structure which generalizes the intrinsic Gaussian description of the two-dimensional surfaces. It should be clearly stressed, however, that the Riemann assumptions introduced at the beginning of this section are neither the only possible nor the more general ones.

In fact, we could even start with less restrictive assumptions, and impose (for instance) on the interval ds to be a homogeneous form of first degree in the coordinate differentials. In that case we can write $ds = F(x, dx)$, where the function F satisfies the condition

$$F(x, \lambda\, dx) = \lambda F(x, dx), \tag{2.26}$$

for any value of the parameter λ. As a simple example of this type we may consider the interval:

$$ds = \left(dx_1^4 + dx_2^4 + \cdots\right)^{1/4}. \tag{2.27}$$

The condition (2.26) defines the so-called *Finsler geometry*, and characterizes a space–time structure which is different from (and more general than) the Riemann one. In fact, the Riemann assumption (2.16) satisfies as a particular case the Finsler condition (2.26) (just like the Minkowski line-element being only a particular case of the more general Riemannian interval). Conversely, there are examples of space–time intervals—like that of Eq. (2.27)—which satisfy the Finsler hypothesis but not the Riemannian postulates introduced before.

In view of the various possible types of geometric structure, which include the Minkowski and the Riemann geometries within schemes of increasing level of generality and complexity, one may wonder what is the geometric model more appropriate to our physical space–time. The principle of general covariance tell us that the Minkowski geometry has to be generalized, but does not select a unique prescription. Can we obtain useful hints from other physical principles?

An answer to this question will be presented in the following section.

2.2 The Equivalence Principle

In order to formulate a theory of gravity based on a generalized relativity principle we need a generalized space–time geometry, "adapted" to the physical properties of the gravitational interaction.

One of the most important (and most typical) property of such interaction is expressed by the so-called "equivalence principle", stating that:

the gravitational interaction can always be locally eliminated,

where *locally* means at any given space–time point and in its infinitesimal neighborhood. Basically, this property arises from the fact that the effects of the gravitational interactions are *locally* indistinguishable from the effects of an accelerated frame, so that the gravitational effects can be eliminated by the simple application of an acceleration of appropriate sign and intensity.

It is important to stress that such a complete elimination of the interaction, for any given physical system, is achieved as a consequence of the *universality* of the gravitational coupling. In fact, as is well known since Galileo, all test bodies respond to a given external gravitational field with the same acceleration, which means that the ratio between the "gravitational charge" (i.e. the gravitational mass) and the inertial mass is the same for all bodies.

Gravity is the only fundamental interaction characterized by this type of universality. In the case of the electromagnetic interaction, for instance, there is no analogous of the equivalence principle because test bodies with different electric charges are differently affected by the same external field: by introducing an accelerated frame we can then locally eliminate at most the electromagnetic force acting on a particular type of charge, but not on *all* the charges of a given physical system. Hence, the electromagnetic interaction *is not* locally eliminable, unlike the gravitational interaction.

If we want to formulate a relativistic gravitational theory by introducing a new geometrical space–time structure, more general than the Minkowski one, we must require—in agreement with the principle of equivalence—that the effects of the new geometry can always be locally eliminated, i.e. that the new geometry can always be reduced, locally, to the Minkowski one.

This property is not satisfied, in general, in the Finsler case, but is always satisfied in the case of a Riemann space–time. In fact, if the space–time interval satisfies the properties (2.16), (2.17), then we can always introduce an appropriate system of coordinates—the so-called "locally inertial" system—where the Riemann metric $g_{\mu\nu}$ exactly reduces to $\eta_{\mu\nu}$ at a chosen space–time point. In the neighborhood of that point the geometry locally approaches the Minkowskian form.

For a simple visualization of this geometric property we may recall the example of the spherical surface S_2 already considered in the previous section. The intrinsic geometry of S_2 is not described by an Euclidean line-element; however, at any given point of S_2 we can always introduce an Euclidean plane tangent to the sphere, and approximate the local geometry of the sphere, in the neighborhood of that point, with the Euclidean geometry of the tangent plane. In the same way, in the case of a four-dimensional Riemann space–time we can always introduce at any given point a "flat" tangent manifold characterized by a Minkowskian metric structure, and locally approximate the Riemann geometry with the tangent Minkowski geometry.

For a more explicit illustration of the local reduction of the metric to the Minkowski form we shall now consider a metric g which satisfy the Riemann postulates (2.16), (2.17), and show that we can always find a coordinate transformation

$x \to x'$ such that the transformed metric exactly reduces to the Minkowski metric η at a given point x_0, namely that $g'(x_0) = \eta$. We can choose, for simplicity, a coordinate system x' which coincides with x at the reference position x_0.

Let us consider the inverse transformation $x = x(x')$, expanded in a Taylor series around $x' = x_0$:

$$x^\mu(x') \simeq x_0^\mu + \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{x'=x_0} (x'^\nu - x_0^\nu) + \frac{1}{2}\left(\frac{\partial x^\mu}{\partial x'^\alpha \partial x'^\beta}\right)_{x'=x_0} (x'^\alpha - x_0^\alpha)(x'^\beta - x_0^\beta) + \cdots . \tag{2.28}$$

Such a transformation is locally determined, to first order, once we are given the 16 (constant) matrix coefficients

$$I^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{x'=x_0} . \tag{2.29}$$

The metric transformation under a general change of coordinates, on the other hand, is fixed by Eq. (2.18). Such a transformation, evaluated at $x' = x = x_0$, and applied to the condition $g'(x_0) = \eta$, leads to

$$g'_{\alpha\beta}(x_0) = I^\mu{}_\alpha I^\nu{}_\beta \, g_{\mu\nu}(x_0) = \eta_{\alpha\beta}. \tag{2.30}$$

Since the initial metric $g_{\mu\nu}(x_0)$ is known, the above non-homogeneous system of equations provides 10 equations for the 16 unknown components of the matrix $I^\mu{}_\nu$. Such a system always admits non-trivial solutions for the coefficients $I^\mu{}_\nu$, so that we can always determine the infinitesimal coordinate transformation reducing the given metric to the Minkowski form at the chosen space–time point.

It should be noted that the system of (2.30) does not completely fix all the coefficients of $I^\mu{}_\nu$, but determines a class of solution depending on $16 - 10 = 6$ parameters. Otherwise stated, the coordinate transformation reducing the metric to the Minkowski form is only defined modulo six arbitrary degrees of freedom. This freedom physically corresponds to the residual possibility of performing local changes of frames, even after having imposed $g(x_0) = \eta$, by applying an arbitrary Lorentz transformation (which depends indeed on six parameters and which, as is well known, does not modify the local Minkowski metric).

More generally, if the two coordinate systems are not assumed to coincide at the point x_0, the sought transformation will be determined modulo the four additional constant parameters $x^\mu(x_0)$ which are associated to the zeroth-order term of the Taylor expansion (2.28), and which are to be added to the six parameters mentioned before. In fact, the most general transformations preserving the Minkowski metric are those of the Poincaré group, which include space–time translations plus Lorentz rotations and which depend indeed on $6 + 4 = 10$ parameters.

In conclusion we can say that the Riemann geometry, thanks to its local properties, provides the space–time manifold with a structure which is able to incorporate

and to generalize (consistently with the equivalence principle) the geometry of special relativity, and is then suitable (at least in principle) for a possible geometric description of the gravitational interaction. A few notions about the formalism and the computational techniques to be used in the context of a Riemann manifold will be introduced in the next chapter.

2.3 Exercises Chap. 2

2.1 Relativistic Motion in a Central Gravitational Field Deduce the equation of motion (2.8) by combining equations (2.5) and (2.6) which define, respectively, the constants of motion h and α.

2.2 Four-Dimensional Pseudo-Sphere Let us consider a four-dimensional hypersurface Σ_4 with pseudo-Euclidean signature, $g_{\mu\nu} = (+,-,-,-)$, parametrized by the four intrinsic coordinates $x^\mu = (ct, x^i)$, and embedded in a five-dimensional Minkowski space–time with coordinates z^A, $A = 0, 1, 2, 3, 4$. The hypersurface is described by the following parametric equations:

$$\begin{aligned} z^0 &= \frac{c}{H}\sinh(Ht) + \frac{H}{2c}e^{Ht}x_i x^i, \\ z^i &= e^{Ht}x^i, \\ z^4 &= \frac{c}{H}\cosh(Ht) - \frac{H}{2c}e^{Ht}x_i x^i, \end{aligned} \tag{2.31}$$

where H is a constant. Show that the given hypersurface Σ_4 represents a four-dimensional pseudo-sphere (or hyperboloid), and determine its intrinsic metric, i.e. the metric induced on Σ_4 by the embedding equations (2.31).

2.4 Solutions

2.1 Solution By expressing the derivatives with respect to t as

$$\dot{r} = r'\dot{\varphi}, \quad r' = \frac{dr}{d\varphi}, \tag{2.32}$$

we can rewrite Eq. (2.5) in the form

$$\dot{\varphi}^2 = \frac{h^2}{r^4}\left(1 - \frac{r'^2}{c^2}\dot{\varphi}^2 - \frac{r^2}{c^2}\dot{\varphi}^2\right), \tag{2.33}$$

and obtain

$$\dot{\varphi}^2 = \frac{h^2}{r^4}\left[1 + \frac{h^2}{r^4 c^2}\left(r'^2 + r^2\right)\right]^{-1}. \tag{2.34}$$

We can also conveniently obtain, from Eq. (2.6),

$$\frac{1}{\gamma^2} \equiv 1 - \frac{1}{c^2}\left(r'^2 + r^2\right)\dot{\varphi}^2 = \frac{c^4}{(\alpha + \frac{GM}{r})^2}. \tag{2.35}$$

By inserting the expression (2.34) for $\dot{\varphi}^2$, and inverting the previous equation, we are led to

$$\frac{1}{c^4}\left(\alpha + \frac{GM}{r}\right)^2 = 1 + \frac{h^2}{r^4 c^2}\left(r'^2 + r^2\right). \tag{2.36}$$

Let us now replace the radial variable r with the new variable $u = 1/r$, such that $r' = -u'/u^2$, and differentiate with respect to φ both terms of the above equation. We obtain

$$u'\left(u'' + u\right) = u'\left(\frac{r_0\alpha}{2h^2} + \frac{r_0^2 c^2}{4h^2}u\right), \tag{2.37}$$

where $r_0 = 2GM/c^2$. This equation always admits the trivial solution $u' = 0$, namely $r = \text{const}$, which describes a circular trajectory in the orbital plane. If we exclude the case of circular orbits, $u' \neq 0$, we can divide by u' and we arrive at the equation

$$u'' + u = \frac{r_0\alpha}{2h^2} + \frac{r_0^2 c^2}{4h^2}u, \tag{2.38}$$

which, by using the definitions (2.9), exactly reduces to the equation of motion (2.8).

2.2 Solution By squaring the coordinates z^A of Eq. (2.31), and computing the squared modulus $z^A z_A$ in the five-dimensional Minkowski space–time, we easily find that the hypersurface Σ_4 satisfies the equation:

$$\eta_{AB} z^A z^B = \left(z^0\right)^2 - \left(z^1\right)^2 - \left(z^2\right)^2 - \left(z^3\right)^2 - \left(z^4\right)^2 = -\frac{c^2}{H^2} = \text{const}. \tag{2.39}$$

This equation describes a four-dimensional pseudo-sphere of radius $R = c/H$ (compare for instance this result with Eq. (2.20), describing a two-dimensional spherical surface). Due to the pseudo-Euclidean signature of the external manifold, the space–time sections of Σ_4—e.g. the sections defined by $z^2 = z^3 = z^4 = 0$—are represented by hyperbolas instead of circles. Our hypersurface can thus be interpreted as a four-dimensional hyperboloid of revolution.

The intrinsic metric $g_{\mu\nu}$ induced by the parametric equations $z^A = z^A(x^\mu)$ is defined by

$$g_{\mu\nu} = \frac{\partial z^A}{\partial x^\mu}\frac{\partial z^B}{\partial x^\nu}\eta_{AB}, \tag{2.40}$$

according to Eq. (2.22). A simple differentiation of the equations (2.31) with respect to x gives us

$$g_{00} = \frac{1}{c^2}\left(\frac{\partial z^0}{\partial t}\right)^2 - \frac{1}{c^2}\left|\frac{\partial z^i}{\partial t}\right|^2 - \frac{1}{c^2}\left(\frac{\partial z^4}{\partial t}\right)^2 = 1,$$

$$g_{ij} = \frac{\partial z^0}{\partial x^i}\frac{\partial z^0}{\partial x^j} - \frac{\partial z^k}{\partial x^i}\frac{\partial z^l}{\partial x^j}\delta_{kl} - \frac{\partial z^4}{\partial x^i}\frac{\partial z^4}{\partial x^j} = -\delta_{ij}e^{2Ht}, \tag{2.41}$$
$$g_{0i} = 0.$$

The intrinsic geometry of our hyperboloid, in the given system of coordinates x^μ, is then described by the line-element:

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = c^2\,dt^2 - e^{2Ht}|d\boldsymbol{x}|^2. \tag{2.42}$$

This space–time interval represents a possible parametrization of the so-called de Sitter geometry (see for instance [45]), which has important applications in a cosmological context (see for instance [31]).

Chapter 3
Tensor Calculus in a Riemann Manifold

Motivated by the discussion of the previous chapter let us then consider a space–time with a Riemann geometrical structure and a pseudo-Euclidean signature. Namely, let us assume that the physical space–time is a differentiable manifold[1] equipped with a metric g which defines scalar products according to the postulates of Sect. 2.1, and which is represented by a real symmetric 4×4 matrix with time-like and space-like eigenvalues of opposite sign. Following our conventions, we will choose a positive time-like eigenvalue:

$$g_{\mu\nu} = \mathrm{diag}(+, -, -, -). \tag{3.1}$$

We will assume, also, that the so-called "affine connection" of our Riemannian manifold is symmetric and compatible with the metric (see Sect. 3.5 below).

We recall, first of all, that the metric eigenvalues—like any matrix eigenvalues—are left unchanged by the action of the so-called "similarity transformations", represented by the map $g \to g' = U^{-1} g U$, where U is an arbitrary 4×4 matrix. However, the metric eigenvalues *may change* under a general coordinate transformation. In that case, in fact, the metric transformation is determined by Eq. (2.18) which, by introducing the Jacobian matrix $J^{\mu}{}_{\nu}$ defined by

$$J^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}, \qquad \left(J^{-1}\right)^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}, \tag{3.2}$$

can be rewritten as

$$g'_{\alpha\beta} = \left(J^{-1}\right)^{\mu}{}_{\alpha}\, g_{\mu\nu} \left(J^{-1}\right)^{\nu}{}_{\beta} \equiv \left(J^{-1}\right)^{T}{}_{\alpha}{}^{\mu}\, g_{\mu\nu} \left(J^{-1}\right)^{\nu}{}_{\beta}, \tag{3.3}$$

or, in compact matrix form:

$$g' = \left(J^{-1}\right)^{T} g\, J^{-1}. \tag{3.4}$$

[1] Differentiable manifold: a topological Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

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This type of transformation is called a "congruence", and does not preserve in general the eigenvalues of the matrix g. However, it preserves the number of eigenvalues of a given sign, so that the $3+1$ signature of the metric is left unchanged. This result is also known as the "Sylvester theorem".

In the context of the Riemann geometry the basic notion of inertial observer (or inertial reference system), typical of special relativity, is replaced by the notion of *general coordinate system*, also called "chart" in the language of differential geometry. The functional relations among different charts are no longer linear, unlike the case of Lorentz transformations. Also, a single chart could be not sufficiently extended to map all points of the given Riemann manifold: in that case we must introduce a collection of charts, called "atlas".

In the region where two charts are intersecting each space–time point is mapped by two different coordinate systems, $\{x\}$ and $\{x'\}$: it is thus possible, in that region, to define the coordinate transformation $x \to x'$. According to our assumptions on the space–time geometry such a transformation must correspond to a *diffeomorphism*, i.e. to a bijective, invertible map which is differentiable and has a differentiable inverse. Such a transformation, in particular, has to be characterized by a non-vanishing Jacobian determinant.

Let us consider, as a simple example, the transformation from the polar coordinate system $\{r, \varphi\}$ to the Cartesian system $\{x, y\}$, defined by Eq. (2.3). The corresponding Jacobian determinant is given by $\det J \equiv |\partial x'/\partial x| = r$, so that the transformation is defined everywhere except that at the origin $r = 0$, where it cannot be inverted. Hence, the polar chart is not adequate to provide a full map of the Euclidean plane $\mathcal{R}_2$ (unlike the Cartesian chart which, instead, can parametrize the whole $\mathcal{R}_2$ plane).

The introduction of the Riemann geometric structure, and the implementation of a generalized principle of relativity according to which all charts must be treated on the same footing, requires a consistent classification of the geometric objects based on the general notion of diffeomorphic mapping (replacing the special relativistic classification based on the particular case of Lorentz transformations). The rest of this chapter will be devoted to a short presentation of the main aspects of this geometrical formalism.

3.1 Covariant and Contravariant Tensors

A geometric object y defined on a Riemann space–time manifold is represented by a collection of differentiable functions $y_A(x)$, called "components", which under the map $x \to x'$ are transformed as follows:

$$y_A(x) \to y'_A\big(x'\big) = Y_A\big[y_A(x), x'(x)\big]. \tag{3.5}$$

In general, the new components y'_A (referred to the new chart x') depend on the old components and on the new coordinates through a function Y_A, whose explicit form

is uniquely prescribed by the class of objects we are considering. If the transformation is homogeneous then the components provide us with a basis for the representation of the associated transformation group on the space–time manifold.

Consider, for instance, the map $x \to x'$ with Jacobian matrix J given by Eq. (3.2). An object ϕ is called a scalar if it is characterized by the identical transformation:

$$\phi'(x') = \phi(x). \tag{3.6}$$

An object A^μ is called a *contravariant vector* (or a tensor of type $(1,0)$) if it is transformed as the coordinate differentials,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\, dx^\nu, \tag{3.7}$$

namely as

$$A'^\mu(x') = J^\mu{}_\nu A^\nu(x). \tag{3.8}$$

An object B_μ is called a *covariant vector* (or a tensor of type $(0,1)$) if it is transformed as the gradient,

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}, \tag{3.9}$$

namely as

$$B'_\mu(x') = \left(J^{-1}\right)^\nu{}_\mu B_\nu(x). \tag{3.10}$$

It should be noted that, besides the direct transformations giving us the new components as a function of the old ones, we could also consider the associated inverse transformations, which for contravariant and covariant vectors are given, respectively, by

$$A^\mu(x) = \left(J^{-1}\right)^\mu{}_\nu A'^\nu(x'), \tag{3.11}$$

$$B_\mu(x) = J^\nu{}_\mu B'_\nu(x') \tag{3.12}$$

(they are obtained by inverting Eqs. (3.8), (3.10)).

The definition of vectors (i.e. of tensor objects of rank one) can be easily extended to objects of arbitrary rank by noting that a covariant (or contravariant) tensor of rank r is transformed as the direct product of r covariant (or contravariant) vectors. In particular, a "mixed" tensor of type (n,m), is an object which has rank n with respect to the contravariant representation and rank m with respect to the covariant representation of the given coordinate transformation. This object has a total of 4^{n+m} components, and is transformed according to the rule:

$$\begin{aligned} &T'^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x') \\ &\quad = J^{\mu_1}{}_{\alpha_1}\cdots J^{\mu_n}{}_{\alpha_n}\left(J^{-1}\right)^{\beta_1}{}_{\nu_1}\cdots\left(J^{-1}\right)^{\beta_m}{}_{\nu_m} T^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}(x). \end{aligned} \tag{3.13}$$

It should be noted that for $r = 2$ the above equation exactly reproduces a similarity transformation (with $U = J^{-1}$). In that case we have, in fact,

$$T'^{\mu}{}_{\nu} = J^{\mu}{}_{\alpha} T^{\alpha}{}_{\beta} \left(J^{-1}\right)^{\beta}{}_{\nu}, \tag{3.14}$$

or, in matrix form,

$$T' = J T J^{-1}. \tag{3.15}$$

In that special case the eigenvalues of the matrix $T^{\mu}{}_{\nu}$ are thus preserved, for any given coordinate transformation.

The geometric difference between covariant and contravariant components can be easily illustrated by introducing on the space–time manifold four basis vectors $\{\boldsymbol{e}_{\mu}\}$, $\mu = 1, 2, 3, 4$, defined so as to be "orthonormal" with respect to the given Riemann metric. Namely, their scalar product satisfies the condition:

$$\boldsymbol{e}_{\mu} \cdot \boldsymbol{e}_{\nu} = g_{\mu\nu} \tag{3.16}$$

(strictly speaking, we are considering a "pseudo-orthonormality" condition, which reduces to the standard orthonormality condition only in the context of Euclidean manifold with $g_{\mu\nu} = \delta_{\mu\nu}$). Any given vector $\boldsymbol{A}$ can then be represented as a linear combination of the four basis vectors,

$$\boldsymbol{A} = A^{\mu} \boldsymbol{e}_{\mu}, \tag{3.17}$$

and the coefficients A^{μ} of such a linear combination correspond to the contravariant components of the vector (namely to the components which can reproduce the vector, in the limit of an Euclidean space, if summed up according to the so-called "parallelogram law"). The covariant components A_{μ}, instead, are those obtained by considering the scalar projection of the vector $\boldsymbol{A}$ on the different basis vectors:

$$A_{\mu} = \boldsymbol{A} \cdot \boldsymbol{e}_{\mu}. \tag{3.18}$$

It is immediately clear that the two objects A^{μ} and A_{μ} may coincide only if the given basis identifies a reference system of Cartesian type, namely a set of "orthogonal" axes. In a general "curvilinear" system, instead, the combination of Eqs. (3.17), (3.18) leads to

$$A_{\mu} = A^{\nu} \boldsymbol{e}_{\nu} \cdot \boldsymbol{e}_{\mu} = g_{\mu\nu} A^{\nu}. \tag{3.19}$$

This relation generalizes to the Riemann case the well known property of the Minkowski metric of connecting covariant and contravariant components.

In particular, if the effect of $g_{\mu\nu}$ is that of "lowering indices" (as shown by the above equation), the effect of the contravariant components of the metric is the opposite one. This can be shown either by defining a "dual" basis $\widetilde{\boldsymbol{e}}^{\mu}$ such that $\widetilde{\boldsymbol{e}}^{\mu} \cdot \boldsymbol{e}_{\nu} = \delta^{\mu}_{\nu}$, and repeating the same procedure as before; or by noting that the contravariant components $g^{\mu\nu}$ are also the components of the inverse matrix-representation of the metric tensor. Let us illustrate this second point, as an instructive application of the properties of the Riemann geometrical structure.

We first notice, for this purpose, that the mixed components of the metric coincide with the components of the identity tensor,

$$g_\mu{}^\nu = \delta_\mu{}^\nu. \tag{3.20}$$

According to the basic postulates of the Riemann geometry, in fact, the metric is transformed as a second-rank tensor (see Eq. (2.18)); also, as discussed in Sect. 2.2, it is always possible to find a chart where $g_{\mu\nu}$ is locally reduced to the Minkowski form $\eta_{\mu\nu}$, so that the mixed components $g_\mu{}^\nu$ are reduced to $\eta_\mu^\nu = \delta_\mu^\nu$. The mixed components of the diagonal tensor δ_μ^ν, however, are transformed according to Eq. (3.14), and are invariant with respect to any coordinate transformation. Hence, if Eq. (3.20) holds in a special (locally inertial) chart, also holds in any chart.

On the other hand, according to Eq. (3.19), the components of $g_\mu{}^\nu$ can be obtained by lowering one of the two indices of the contravariant metric components. We have then the relation

$$g_{\mu\alpha} g^{\alpha\nu} = g_\mu{}^\nu = \delta_\mu{}^\nu, \tag{3.21}$$

which can be rewritten in compact matrix form as $gg^{-1} = I$, in full agreement with the role of inverse matrix for the contravariant representation of the metric tensor. By applying $g^{\rho\mu}$ to both sides of Eq. (3.19), and using Eq. (3.21), we finally obtain

$$g^{\rho\mu} A_\mu = A^\rho, \tag{3.22}$$

which represents the "dual" counterpart of the relation (3.19).

Using the above results, we may conclude this section by noticing that the scalar product between vector-like objects can be explicitly written in various forms, all equivalent to the contraction of covariant and contravariant indices:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A^\mu \boldsymbol{e}_\mu \cdot B^\nu \boldsymbol{e}_\nu = A^\mu B^\nu g_{\mu\nu} = A^\mu B_\mu = g^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu, \tag{3.23}$$

(an obvious extension of tensor properties from the Minkowski to the Riemann space–time). Further aspects of the Riemann geometry, with no analogous in the Minkowski space–time, will be illustrated in the following sections.

3.2 Tensor Densities

The tensor objects introduced in the previous section are particular cases of a more general class of geometric objects, called *tensor densities*, and characterized by two parameters: the rank r and the weight w.

A tensor density of rank r (for instance, of contravariant type) and weight w is a geometric object A with 4^r components which, under the coordinate transformation $x \to x'$, is transformed as follows:

$$A'^{\mu_1 \cdots \mu_r} = J^{\mu_1}{}_{\nu_1} \cdots J^{\mu_r}{}_{\nu_r} A^{\nu_1 \cdots \nu_r} (\det J)^w. \tag{3.24}$$

Namely, a density A is transformed as a tensor with respect to its r indices but, unlike the pure tensor case, the old components of A are multiplied by the power w of the Jacobian determinant. The weight w is thus a positive (or negative) dimensionless parameter counting the number of times $\det J$ (or its inverse) enters the transformation law.

Note that any tensor can be classified as a special type of tensor density of weight $w = 0$, so that, if we are limiting ourselves to a subclass of coordinate transformations characterized by the condition $\det J = 1$ (for instance, to the transformations of the proper Lorentz group, in the context of special relativity), the difference between tensor and tensor densities completely disappears. Note also that there are tensor densities of contravariant, covariant and mixed type. A general density T of mixed type (n, m) and weight w is transformed according to the tensor rule (3.13), with the right-hand side of the transformation multiplied by $(\det J)^w$.

As a simple example of density we may consider the four-volume element d^4x, which is transformed as a *scalar* density of weight $w = 1$. Under a general coordinate transformation we have, in fact,

$$d^4x \to d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x = \det J \, d^4x. \tag{3.25}$$

Another example of scalar density is given by the determinant of a tensor, and in particular by the determinant of the metric tensor $g_{\mu\nu}$, which is a scalar density of weight $w = -2$. In fact, by considering the determinant of the transformation rule (2.18), we obtain

$$\det g' = \left|\frac{\partial x}{\partial x'}\right|^2 \det g \equiv (\det J)^{-2} \det g. \tag{3.26}$$

It follows that the square root of $\det g_{\mu\nu}$ is a scalar density of weight $w = -1$, and that the object

$$d^4x \sqrt{-g} \tag{3.27}$$

is a pure scalar (because the associated weight is $w = 0$). Note that we have introduced the standard notation $g \equiv \det g_{\mu\nu}$ (which we shall adopt from now on), and that we have considered the square root of $-g$ since $g < 0$.

Let us finally consider the transformation property of a geometric object largely used in many tensor calculations: the so-called Levi-Civita (or alternating) symbol $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{[\mu\nu\rho\sigma]}$, fully antisymmetric with respect to all indices, normalized by the condition $\epsilon^{0123} = 1$. In the context of a general Riemannian structure this object is transformed as a tensor density of rank 4 and weight $w = -1$.

This conclusion can be reached by considering the standard expansion of the Jacobian determinant as a product of minors along a row (or a column) of the given Jacobian matrix. Such an expansion can be expressed in the following compact form:

$$\det J = J^0{}_\mu J^1{}_\nu J^2{}_\rho J^3{}_\sigma \epsilon^{\mu\nu\rho\sigma} = \epsilon^{0123} \det J, \tag{3.28}$$

which implies the tensor relation:

$$\epsilon^{\alpha\beta\gamma\delta} \det J = J^{\alpha}{}_{\mu} J^{\beta}{}_{\nu} J^{\gamma}{}_{\rho} J^{\delta}{}_{\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{3.29}$$

On the other hand, if we consider the coordinate mapping associated to the Jacobian matrix J, and require that the numerical values $\pm 1, 0$ of the components of the alternating symbol are the same in all charts, we must impose on the new chart that $\epsilon'^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}$. By inserting this condition into the above equation we are led to the transformation law

$$\epsilon'^{\alpha\beta\gamma\delta} = J^{\alpha}{}_{\mu} J^{\beta}{}_{\nu} J^{\gamma}{}_{\rho} J^{\delta}{}_{\sigma} \epsilon^{\mu\nu\rho\sigma} (\det J)^{-1}, \tag{3.30}$$

which defines indeed a tensor density of rank 4 and weight $w = -1$. By recalling that the scalar density $\sqrt{-g}$ has also weight $w = -1$, we can thus obtain a "true" (i.e. $w = 0$) totally antisymmetric tensor by adopting the definition

$$\eta^{\mu\nu\rho\sigma} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}, \tag{3.31}$$

valid in a general Riemann manifold.

To get the corresponding covariant version we first apply the standard lowering indices procedure,

$$\eta_{\alpha\beta\gamma\delta} = g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}. \tag{3.32}$$

Then we use the general expansion of the determinant of the matrix $g_{\mu\nu}$, which, from Eq. (3.29), is given by

$$-g\, \epsilon_{\alpha\beta\gamma\delta} = g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma} \epsilon^{\mu\nu\rho\sigma} \tag{3.33}$$

(the minus sign is due to our metric conventions, which imply $\epsilon_{0123} = -\epsilon^{0123} = -1$). We thus obtain the relation

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\, \epsilon_{\alpha\beta\gamma\delta}, \tag{3.34}$$

which defines the covariant form of the totally antisymmetric tensor (in terms of the corresponding Levi-Civita tensor density).

It is important to note that the symbol $\epsilon_{\alpha\beta\gamma\delta}$ is a covariant tensor density of weight $w = 1$ (namely, it has a weight opposite to that of the contravariant Levi-Civita symbol). We also note that in the contraction of $\eta_{\alpha\beta\gamma\delta}$ with $\eta^{\mu\nu\rho\sigma}$ the metric determinant disappears: the result is fully determined by the contraction of the corresponding Levi-Civita symbols, and is thus the same as that obtained in the Minkowski space–time manifold.

3.2.1 Contraction Rules for Totally Antisymmetric Tensors

We are reporting here, for later use, the explicit result for the product of two completely antisymmetric tensors. Let us first define the convenient symbol $\delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}$, denoting the determinant of the following $n\times$n matrix:

$$\delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n} = \det\begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_n} \\ \delta^{\mu_2}_{\nu_1} & \cdots & \delta^{\mu_2}_{\nu_n} \\ \cdots & \cdots & \cdots \\ \delta^{\mu_n}_{\nu_1} & \cdots & \delta^{\mu_n}_{\nu_n} \end{pmatrix}. \tag{3.35}$$

By applying the explicit definitions (3.31) and (3.34) we then obtain

$$\eta_{\mu\nu\rho\sigma}\eta^{\mu\nu\rho\sigma} = -4!, \tag{3.36}$$

$$\eta_{\mu\nu\rho\alpha}\eta^{\mu\nu\rho\sigma} = -3!\delta^{\sigma}_{\alpha}, \tag{3.37}$$

$$\eta_{\mu\nu\alpha\beta}\eta^{\mu\nu\rho\sigma} = -2!\delta^{\rho\sigma}_{\alpha\beta} \equiv -2!\left(\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\sigma}_{\alpha}\delta^{\rho}_{\beta}\right), \tag{3.38}$$

$$\eta_{\mu\alpha\beta\gamma}\eta^{\mu\nu\rho\sigma} = -\delta^{\nu\rho\sigma}_{\alpha\beta\gamma}, \tag{3.39}$$

$$\eta_{\alpha\beta\gamma\delta}\eta^{\mu\nu\rho\sigma} = -\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}. \tag{3.40}$$

3.3 Infinitesimal Transformations, Isometries and Killing Vectors

The transformation rules introduced in the previous sections do not describe the *local* transformation of a geometric object if the old and new components of the object are referred to the same system of coordinates.

In fact, for any given map $x \to x' = f(x)$, the old components of an object A, evaluated at a space–time point P specified by the coordinates x, are related to the new components A' evaluated at the point of coordinates $x' = f(x)$. This last point coincides with P if its coordinates are referred to the new chart, but corresponds to a different space–time point P', with coordinates $f(x) \neq x$, if its position is referred to the original system of coordinates. In summary, the considered transformations are of the type:

$$A(x) \to A'\big(f(x)\big). \tag{3.41}$$

The local variation of the geometric object—i.e. the variation of its components *at fixed* x—can be easily computed, however, for all transformations that can be expanded in power series around the identity. Such transformations are parametrized, to first order, by an infinitesimal vector ξ^{μ}(the so-called *generator*) as follows:

$$x'^{\mu} = f^{\mu}(x) \simeq x^{\mu} + \xi^{\mu}(x) + \mathcal{O}\big(\xi^2\big). \tag{3.42}$$

The inverse transformation, to first order in ξ, is given by

$$x^\mu = \left(f^{-1}\right)^\mu(x') \simeq x'^\mu - \xi^\mu(x') + \mathcal{O}(\xi^2). \tag{3.43}$$

By expanding in Taylor series the components $A'(x')$ around $x' = x$ (in the limit $\xi \to 0$) we obtain $A'(x)$, and we can then compute the corresponding local variation $\delta A(x) \equiv A'(x) - A(x)$. This is also called "functional variation" or *gauge transformation*, where the word "gauge" refers here to the symmetry properties associated to the covariance under general coordinate transformations. Such a computation can be extended in a straightforward way to any desired order of our series expansion.

Let us immediately present a simple example by considering the scalar transformation (3.6), for a general map $x' = f(x)$:

$$\phi'\left(f(x)\right) = \phi(x). \tag{3.44}$$

If we are interested in the local variation of the scalar ϕ at the space–time position x it is convenient to express the transformation not at x but rather at the translated point of coordinates $f^{-1}(x)$, where the above transformation takes the (exact) equivalent form:

$$\phi'(x) = \phi\left(f^{-1}(x)\right). \tag{3.45}$$

If we are considering, in particular, an infinitesimal transformation satisfying Eqs. (3.42), (3.43) we can expand in Taylor series the right-hand side of the above equation, and obtain

$$\phi'(x) = \phi\left(f^{-1}(x)\right) \simeq \phi(x - \xi) \simeq \phi(x) - \xi^\mu \partial_\mu \phi(x) + \cdots, \tag{3.46}$$

where we have neglected all terms of order ξ^2 and higher. The local (or functional) variation of the scalar field, under the infinitesimal coordinate transformation generated by ξ, is then given, to first order, by

$$\delta_\xi \phi \equiv \phi'(x) - \phi(x) = -\xi^\mu \partial_\mu \phi. \tag{3.47}$$

This result is in principle different from that of Eq. (1.28), referring to the special case of global infinitesimal translations, because we have here a position-dependent generator, $\xi^\mu = \xi^\mu(x)$.

The consequences of such a coordinate dependence, i.e. the effects of the *locality* of our transformations, are more evident if we consider the functional variation of a higher-rank object, for instance the variation of a contravariant vector $A^\mu(x)$.

By applying the general rule (3.8)—expressed not at x but at the coordinate position $f^{-1}(x)$—to the infinitesimal transformation (3.42), and expanding in power series up to the first order in ξ, we obtain

$$\begin{aligned} A'^\mu(x) &= \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x - \xi) = \left(\delta^\mu_\nu + \partial_\nu \xi^\mu + \cdots\right)\left(1 - \xi^\alpha \partial_\alpha + \cdots\right) A^\nu(x) \\ &= A^\mu(x) - \xi^\alpha \partial_\alpha A^\mu + A^\nu \partial_\nu \xi^\mu + \cdots \end{aligned} \tag{3.48}$$

from which:

$$\delta_\xi A^\mu \equiv A'^\mu(x) - A^\mu(x) = -\xi^\alpha \partial_\alpha A^\mu + A^\nu \partial_\nu \xi^\mu. \tag{3.49}$$

The second contribution to the variation, proportion to the gradients of ξ, is a consequence of the *local* nature of the considered transformation. Such a contribution indeed disappears in the limit of rigid (or global) translations with $\xi^\mu = \text{const}$.

Following the same procedure we can evaluate the local variation of a covariant vector, starting with the transformation rule (3.10) and taking into account that the components of the inverse Jacobian matrix are obtained by differentiating Eq. (3.43) with respect to x'. We then find

$$\begin{aligned} B'_\mu(x) &= \frac{\partial x^\nu}{\partial x'^\mu} B_\nu(x - \xi) = \left(\delta^\nu_\mu - \partial_\mu \xi^\nu + \cdots\right)\left(1 - \xi^\alpha \partial_\alpha + \cdots\right) B_\nu(x) \\ &= B_\mu(x) - \xi^\alpha \partial_\alpha B_\mu - B_\nu \partial_\mu \xi^\nu + \cdots, \end{aligned} \tag{3.50}$$

from which:

$$\delta_\xi B_\mu \equiv B'_\mu(x) - B_\mu(x) = -\xi^\alpha \partial_\alpha B_\mu - B_\nu \partial_\mu \xi^\nu. \tag{3.51}$$

Note that, working to the first order in ξ, we have identified $\partial \xi^\nu(x')/\partial x'^\mu$ with $\partial \xi^\nu(x)/\partial x^\mu$. We should note, also, that the last term of the above equation has a different sign with respect to the corresponding term of Eq. (3.49).

It is finally convenient, for later use, to compute the local variation of the metric tensor. By applying the general rule (3.3) to the infinitesimal transformation (3.43), and expanding in power series, we obtain

$$g'_{\mu\nu}(x) = \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha + \cdots\right)\left(\delta^\beta_\nu - \partial_\nu \xi^\beta + \cdots\right)\left(1 - \xi^\rho \partial_\rho + \cdots\right) g_{\alpha\beta}(x), \tag{3.52}$$

from which, to first order in ξ:

$$\delta_\xi g_{\mu\nu} \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\xi^\alpha \partial_\alpha g_{\mu\nu} - g_{\mu\alpha} \partial_\nu \xi^\alpha - g_{\alpha\nu} \partial_\mu \xi^\alpha. \tag{3.53}$$

By repeating this procedure for the contravariant components of the metric we are led to the analogous expression:

$$\delta_\xi g^{\mu\nu} = -\xi^\alpha \partial_\alpha g^{\mu\nu} + g^{\mu\alpha} \partial_\alpha \xi^\nu + g^{\alpha\nu} \partial_\alpha \xi^\mu. \tag{3.54}$$

A coordinate transformation which leaves the metric locally invariant—i.e. a transformation which satisfies the condition[2] $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$—is called an *isometry*, and the generator ξ^μ of the corresponding transformation is called a *Killing vector*. The Killing vectors of a metric are then determined by the condition $\delta_\xi g_{\mu\nu} = 0$

[2]When this condition is satisfied one also says that the metric has the property of "form-invariance".

(or, equivalently, $\delta_\xi g^{\mu\nu} = 0$) which, once the metric components are given, becomes a partial differential equation for the components of the vector ξ^μ:

$$\xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\alpha\nu}\partial_\mu \xi^\alpha = 0. \tag{3.55}$$

As we shall see below, such an equation can be rewritten in a more compact form by using the covariant derivation formalism that will be introduced in the next section. However, already from the above form we can easily verify, for instance, that the transformations of the Poincaré group are isometries of the Minkowski space–time manifold, i.e. that the six generators of the Lorentz rotations and the four generators of the global translations are Killing vectors for the Minkowski metric (see the Exercises 3.1 and 3.2).

The set of isometries associated to a given metric corresponds to an important symmetry group of the considered space–time manifold. Indeed, if such symmetries are known (namely, if the Killing vectors are known), they can be used to choose the most convenient (or "adapted") system of coordinates in which the geometric description of the manifold turns out to be greatly simplified.

Let us suppose, for instance, that our manifold admits a time-like Killing vector ξ^μ. If we choose a chart in which the time coordinate is aligned along the ξ^μ direction, such that $\xi^\mu = \delta^\mu_0$, we will find in this chart that the Killing condition $\delta g_{\mu\nu} = 0$ reduces to $\partial_0 g_{\mu\nu} = 0$ (see Eq. (3.55)), so that the corresponding metric is explicitly time-independent. In this chart, in particular, $\xi_\mu = g_{\mu 0}$ and $\xi^\mu \xi_\mu = g_{00}$. Similar simplifications are possible if the space–time admits space-like or null Killing vectors.

Let us finally mention that the local variation of a tensor object T along the direction defined by the flow of a vector ξ^μ is also called the *Lie derivative* of T with respect to ξ, and is denoted by $\mathcal{L}_\xi T$. More precisely, the action of the Lie derivative on T is the same (but with the opposite sign, for definition) as the action of the differential operator $\delta_\xi T$ introduced previously.[3] Hence, for any Killing vector ξ, the differential condition $\delta_\xi g_{\mu\nu} = 0$ can also be rewritten as a condition of vanishing Lie derivative,

$$\mathcal{L}_\xi \, g_{\mu\nu} = 0 = \mathcal{L}_\xi \, g^{\mu\nu}. \tag{3.56}$$

3.3.1 Second-Order Infinitesimal Transformations

To conclude this section we will briefly illustrate how the computation of the local variations can be extended to second order. Such an extension turns out to be of crucial importance in various modern applications of the cosmological perturbation theory (see e.g. the books [13, 33, 53] quoted in the bibliography), and is required,

[3] This means that the functional variation we have considered can also be interpreted, geometrically, as the effect of an infinitesimal translation along the curve with parametric equation $x^\mu = x^\mu(\lambda)$ and tangent vector $\xi^\mu = dx^\mu/d\lambda$.

in particular, for a correct interpretation of the large-scale observations which are nowadays releasing data with ever increasing precision.

To second order, the expansion of the coordinate transformation $x' = f(x)$ around the identity is characterized in general by two vector generators, ξ_1^μ (first-order generator) and ξ_2^μ (second-order generator), and can be parametrized as follows:

$$x'^\mu = f^\mu(x) \simeq x^\mu + \xi_1^\mu(x) + \frac{1}{2}\xi_2^\mu(x) + \frac{1}{2}\xi_1^\nu \partial_\nu \xi_1^\mu(x) + \cdots , \tag{3.57}$$

where ξ_2 and ξ_1^2 are of the same order. The inverse transformation, to second order, is given by

$$x^\mu = \big(f^{-1}\big)^\mu\big(x'\big) \simeq x'^\mu - \xi_1^\mu\big(x'\big) - \frac{1}{2}\xi_2^\mu\big(x'\big) + \frac{1}{2}\xi_1^\nu \partial_\nu \xi_1^\mu\big(x'\big) + \cdots \tag{3.58}$$

(see Exercise 3.3). By applying the above definitions to the transformation rules of tensor objects, and expanding in Taylor series, we can easily generalize all previous first-order computations.

Let us consider, for instance, the scalar case, and expand the right-hand side of Eq. (3.45) keeping all terms up to second order. By using Eq. (3.58) we obtain

$$\begin{aligned}\phi\big(f^{-1}(x)\big)\big|_{f^{-1}(x)=x} &\simeq \phi(x) + \left(-\xi_1^\mu - \frac{1}{2}\xi_2^\mu + \frac{1}{2}\xi_1^\nu \partial_\nu \xi_1^\mu\right)\partial_\mu \phi \\ &\quad + \frac{1}{2}\big(-\xi_1^\mu + \cdots\big)\big(-\xi_1^\nu + \cdots\big)\partial_\mu \partial_\nu \phi + \cdots \\ &= \phi(x) - \xi_1^\mu \partial_\mu \phi - \frac{1}{2}\xi_2^\mu \partial_\mu \phi + \frac{1}{2}\xi_1^\nu \partial_\nu\big(\xi_1^\mu \partial_\mu \phi\big) + \cdots . \end{aligned} \tag{3.59}$$

Comparing with $\phi'(x)$ we then find that the local variation, to second order, is given by

$$\delta_\xi^{(2)}\phi \equiv \phi'(x) - \phi(x) = -\left(\xi_1^\mu + \frac{1}{2}\xi_2^\mu\right)\partial_\mu \phi + \frac{1}{2}\xi_1^\nu \partial_\nu\big(\xi_1^\mu \partial_\mu \phi\big). \tag{3.60}$$

With the same procedure we can extend to second order the local variations of higher-rank tensors.

3.4 Covariant Derivative and Affine Connection

For a consistent embedding of our physical models into a generalized geometrical structure we need, besides the metric, also a geometric object called *affine connection* (or affinity). Such an object is required for the construction of differential operators characterized by the correct covariance properties with respect to general coordinate transformations.

In fact, unlike the coordinate differential dx^μ which is transformed like a vector (see Eq. (3.7)), the ordinary differential of a generic vector A^μ *does not* behaves, in general, like a vector. This can be easily checked by differentiating, for instance, the transformation (3.11): from the explicit definition (3.2) of the Jacobian matrix we obtain

$$dA^\mu = \left(J^{-1}\right)^\mu{}_\nu \, dA'^\nu + \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\nu} \, A'^\nu \, dx'^\alpha . \tag{3.61}$$

The last term, which is vanishing only for a constant Jacobian matrix—namely, for the special case of linear coordinate transformations—modifies the canonical transformation rule of a vector, and breaks the general covariance of the model we are considering.

In order to compensate such a correction (and restore the broken diffeomorphism symmetry) let us define a generalized differential operator by adding to dA^μ a new term δA^μ, which takes into account the possible variation of A associated to its parallel transport from x to $x + dx$, and which we assume to depend linearly on A and on dx. More precisely, let us introduce a covariant differential operator DA^μ such that

$$DA^\mu = dA^\mu + \delta A^\mu \equiv dA^\mu + \Gamma_{\alpha\beta}{}^\mu \, dx^\alpha \, A^\beta . \tag{3.62}$$

The coefficients $\Gamma_{\alpha\beta}{}^\mu$ of the new term represent the components of a suitable "compensating field" (or gauge field), which is transformed in such a way as to restore the appropriate transformation law for DA^μ. Given the fact that A and dx are vectors, while dA is not a vector, it follows that Γ cannot be a tensor: it corresponds indeed to a new type of geometric object called "affine connection".

The geometric properties of Γ are fixed by its transformation law, which is determined, in turn, by the requirement that DA^μ is transformed as a contravariant vector, namely by the condition:

$$DA^\mu = \left(J^{-1}\right)^\mu{}_\nu \left(DA^\nu\right)' . \tag{3.63}$$

Let us explicitly write down both sides of the above equation as a function of Γ and Γ', respectively. By using Eqs. (3.61), (3.62) the left-hand side can be rewritten as

$$\left(J^{-1}\right)^\mu{}_\nu \, dA'^\nu + \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} \, dx'^\alpha \, A'^\beta + \Gamma_{\lambda\sigma}{}^\mu \left(J^{-1}\right)^\lambda{}_\alpha \left(J^{-1}\right)^\sigma{}_\beta \, dx'^\alpha \, A'^\beta , \tag{3.64}$$

and the right-hand side as

$$\left(J^{-1}\right)^\mu{}_\nu \left(dA'^\nu + \Gamma'_{\alpha\beta}{}^\nu \, dx'^\alpha \, A'^\beta\right). \tag{3.65}$$

By equating the coefficients of all terms proportional to $dx'^\alpha \, A'^\beta$ present in Eq. (3.63), simplifying similar terms, and multiplying by $J^\rho{}_\mu$, we finally obtain the transformation rule:

$$\Gamma'_{\alpha\beta}{}^\rho = J^\rho{}_\mu \left(J^{-1}\right)^\lambda{}_\alpha \left(J^{-1}\right)^\sigma{}_\beta \Gamma_{\lambda\sigma}{}^\mu + \left(\frac{\partial x'^\rho}{\partial x^\mu}\right)\left(\frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta}\right). \tag{3.66}$$

For a linear coordinate transformation the last term is vanishing, and the connection Γ is transformed exactly as a third-rank tensor. For a general transformation, instead, we obtain a relation $\Gamma \to \Gamma'(\Gamma)$ which is not homogeneous, confirming the non-tensorial nature of this object. It should be noted, however, that the antisymmetric part of the connection,

$$Q_{\alpha\beta}{}^{\rho} = \Gamma_{[\alpha\beta]}{}^{\rho}, \tag{3.67}$$

called *torsion*, always transforms as a tensor: in fact, if we take the antisymmetric part of Eq. (3.66) with respect to α and β we find that the non-homogeneous term automatically disappears, being symmetric. The affine connection Γ thus contains, in general, $4^3 = 64$ components, among which only $6 \times 4 = 24$ (the components of the torsion Q) correspond to tensor representations of the diffeomorphism group.

The symmetric part of the connection, $\Gamma_{(\alpha\beta)}{}^{\mu}$, has 40 components (all of non-tensorial type) and satisfies a property which has an interesting physical meaning, as we shall see below: it can be locally set to zero in a given chart, without being necessarily zero in all charts. Namely, we can always find a "locally inertial" system of coordinates (see Sect. 2.2) where not only the metric is locally reduced to the Minkowski form, but also the symmetric part of Γ is locally zero, at any arbitrarily given point x_0.

In order to illustrate this important property of the connection let us consider the coordinate transformation (2.28) already introduced in Chap. 2, and let us impose that, in the new chart $\{x'\}$, the symmetric part of Γ' is vanishing at the point x_0. By applying the transformation rule of the affine connection, Eq. (3.66), and imposing $\Gamma'_{\alpha\beta}{}^{\rho}(x_0) = 0$, we then obtain the condition:

$$\left(\frac{\partial^2 x^{\mu}}{\partial x'^{\alpha}\partial x'^{\beta}}\right)_{x_0} = -I^{\lambda}{}_{(\alpha} I^{\sigma}{}_{\beta)} \Gamma_{\lambda\sigma}{}^{\mu}(x_0), \tag{3.68}$$

(we have also used the explicit definition (2.29) of the matrix $I^{\mu}{}_{\nu}$). We can now recall that the components of Γ in the old chart $\{x\}$ are known everywhere, and that the components of the matrix $I^{\mu}{}_{\nu}$ are determined by the condition $g(x_0) = \eta$ (see Eq. (2.30)). It follows that the previous equation completely fixes the 40 coefficients of the second-order term of the coordinate transformation (2.28), thus determining, to second order, the map to the locally inertial frame where the symmetric part of the connection is vanishing and the space–time geometry locally reduces to the Minkowski form.

Once the covariant differential is given, we can easily introduce the corresponding covariant derivative (which we shall denote with the symbol $\nabla_{\alpha} A^{\mu}$) as the limit for $dx^{\alpha} \to 0$ of the ratio between DA^{μ} and dx^{α}. We obtain

$$\nabla_{\alpha} A^{\mu} = \partial_{\alpha} A^{\mu} + \Gamma_{\alpha\beta}{}^{\mu} A^{\beta}, \tag{3.69}$$

(the first term, arising from the ordinary differential, reproduces the usual partial derivative). It should be noted that the two terms contributing to $\nabla_{\alpha} A^{\mu}$ do not behave, separately, as tensor objects, but their sum is a tensor, since both DA^{μ} and

dx^α satisfy the correct transformation properties. The covariant differential operator ∇_α (also called the covariant gradient) thus transforms according to the vector representation of the diffeomorphism group

Given the action of ∇_α on the contravariant object A^μ, the corresponding action on a covariant object like B_μ can be immediately derived by considering the scalar product $B_\mu A^\mu$, and by noting that the transformation of a scalar quantity does not involve the Jacobian matrix, so that the covariant differentiation simply reduces to the ordinary differential operation. A straightforward application of the Leibniz rule to the derivative of the scalar product then leads to

$$\begin{aligned}\nabla_\alpha\big(B_\mu A^\mu\big) &= (\nabla_\alpha B_\mu)A^\mu + B_\mu \nabla_\alpha A^\mu \\ &\equiv \partial_\alpha\big(B_\mu A^\mu\big) = (\partial_\alpha B_\mu)A^\mu + B_\mu \partial_\alpha A^\mu .\end{aligned} \tag{3.70}$$

Inserting for $\nabla_\alpha A^\mu$ the result (3.69) we obtain

$$A^\mu \nabla_\alpha B_\mu + \Gamma_{\alpha\beta}{}^\mu A^\beta B_\mu = A^\mu \partial_\alpha B_\mu , \tag{3.71}$$

from which, by equating the coefficients of A^μ,

$$\nabla_\alpha B_\mu = \partial_\alpha B_\mu - \Gamma_{\alpha\mu}{}^\beta B_\beta . \tag{3.72}$$

Note that the sign of the connection contribution to this covariant derivative is the opposite to the sign appearing in the derivative of a contravariant object, see Eq. (3.69).

In a similar way we can obtain the rule for the covariant derivative of a tensor of arbitrary rank and type, simply by noticing that a tensor of type (n, m) transforms as the product of n contravariant and m covariant vectors. By defining

$$T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \equiv A^{\mu_1}\cdots A^{\mu_n} A_{\nu_1}\cdots A_{\nu_m} , \tag{3.73}$$

and applying the Leibniz rule to the derivative of a product, we obtain

$$\begin{aligned}\nabla_\alpha T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = &\big(\nabla_\alpha A^{\mu_1}\big)A^{\mu_2}\cdots A^{\mu_n} A_{\nu_1}\cdots A_{\nu_m} \\ &+ A^{\mu_1}\big(\nabla_\alpha A^{\mu_2}\big)\cdots A^{\mu_n} A_{\nu_1}\cdots A_{\nu_m} + \cdots \\ &+ A^{\mu_1}\cdots A^{\mu_n}(\nabla_\alpha A_{\nu_1})A_{\nu_2}\cdots A_{\nu_m} \\ &+ \cdots .\end{aligned} \tag{3.74}$$

Finally, by using the known prescriptions (3.69), (3.72) we are led to

$$\begin{aligned}\nabla_\alpha T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = &\,\partial_\alpha T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \\ &+ \Gamma_{\alpha\beta}{}^{\mu_1} T^{\beta\mu_2\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \Gamma_{\alpha\beta}{}^{\mu_2} T^{\mu_1\beta\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \cdots \\ &- \Gamma_{\alpha\nu_1}{}^{\beta} T^{\mu_1\cdots\mu_n}{}_{\beta\nu_2\cdots\nu_m} - \Gamma_{\alpha\nu_2}{}^{\beta} T^{\mu_1\cdots\mu_n}{}_{\nu_1\beta\cdots\nu_m} - \cdots .\end{aligned} \tag{3.75}$$

We can summarize the above results by saying that the covariant derivative of a tensor is built from the corresponding partial derivative, with the addition of one

contracted connection contribution for each of the given tensor indices. According to our conventions, such additional contributions will enter with the positive sign and with the formal prescription of Eq. (3.69) if the index is of contravariant type, with the negative sign and with the prescription of Eq. (3.72) if the index is of covariant type. For a mixed second-rank tensor, for instance, the covariant derivative is given by

$$\nabla_\alpha T^\mu{}_\nu = \partial_\alpha T^\mu{}_\nu + \Gamma_{\alpha\beta}{}^\mu T^\beta{}_\nu - \Gamma_{\alpha\nu}{}^\beta T^\mu{}_\beta. \tag{3.76}$$

The introduction of few simple rules of covariant differential calculus is postponed to Sect. 3.6, to insert before a necessary (and more detailed) discussion of the properties of the affine connection, which we are going to present in Sect. 3.5.

3.4.1 Autoparallel Curves

The notion of covariant differentiation based on Eq. (3.62) can be applied, in particular, to the tangent vector of a given curve, and to its variation along the curve itself.

Let us consider a curve embedded in a Riemann manifold, described by the parametric equation $x^\mu = x^\mu(\tau)$, with tangent vector $u^\mu = dx^\mu/d\tau$ (note that u^μ is a vector, as τ is a scalar parameter). An infinitesimal displacement along the curve is expressed by $dx^\mu = u^\mu\, d\tau$, and the covariant differential (3.62) for a displacement along the curve then takes the form

$$DA^\mu = dA^\mu + \Gamma_{\alpha\beta}{}^\mu u^\alpha A^\beta\, d\tau. \tag{3.77}$$

The limit of the ratio $DA^\mu/d\tau$, for $d\tau \to 0$, defines the covariant derivative of A^μ along the curve:

$$\frac{DA^\mu}{d\tau} = \frac{dA^\mu}{d\tau} + \Gamma_{\alpha\beta}{}^\mu u^\alpha A^\beta. \tag{3.78}$$

Such a derivative can also be rewritten, in an equivalent form, as the partial covariant derivative of A^μ projected on the tangent u^μ, namely as

$$u^\alpha \nabla_\alpha A^\mu = u^\alpha \left(\partial_\alpha A^\mu + \Gamma_{\alpha\beta}{}^\mu A^\beta\right) \equiv \frac{DA^\mu}{d\tau}. \tag{3.79}$$

Let us now consider the covariant differential of the tangent vector, Du^μ. A curve is called *autoparallel*, or *affine geodesic*, if the covariant derivative of the tangent along the curve itself is vanishing, namely if:

$$\frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma_{\alpha\beta}{}^\mu u^\alpha u^\beta = 0. \tag{3.80}$$

This condition certifies that the tangent is "covariantly constant" along the curve, and thus generalizes the condition of constant tangent vector typical of the straight

trajectories in Euclidean space. Hence, the autoparallel curve generalizes the notion of straight line to the case of manifolds equipped with a non-vanishing affine connection.

It is important to stress that Eq. (3.80) only involves the symmetric part of the connection, since $\Gamma_{\alpha\beta}{}^{\mu}$ is contracted with the symmetric tensor $u^{\alpha}u^{\beta}$. Such a symmetric part is not a tensor and, as discussed previously, can always be locally eliminated. This means that equation (3.80) for the affine geodesic can always be reduced, locally, to the equation of a straight line ($d^2x^{\mu}/d\tau^2 = 0$).

3.5 Torsion, Non-metricity and Christoffel Symbols

Up to now the affine connection has been presented as an independent geometric object which is required, in addition to the metric, to provide the space–time manifold with a consistent geometric structure. The metric, needed to define scalar products, accounts for the local distortion of the modulus of a vector with respect to the Euclidean space; the connection, needed to define covariant differentiation, accounts for a possible distortion in both magnitude and direction of a vector when it is transported from one space–time point to another. In general, both the metric g and the connection Γ are to be specified for a full characterization of the space–time geometry.

At this point, however, we are in the position of distinguishing between two possible types of geometric structure. If g and Γ are completely independent objects then we have a manifold equipped with a geometric structure which is called *metric-affine*. If, instead, Γ can be expressed in terms of g, then the metric—by itself—is enough to describe the geometry of the given manifold, and we say that we are in the presence of a *metric* structure.

The second possibility is the one implemented in the context of the Riemann geometric model, used for the formulation of Einstein's theory of general relativity. In such a context one imposes a set of suitable conditions on the 64 independent components of Γ, so that the remaining components can be fully expressed in terms of the metric. Such conditions, as we shall see, are suggested by the phenomenological properties of the classical gravitational interactions experienced by macroscopic bodies.

In order to make explicit a possible relation between the metric and the connection let us start computing the covariant derivative of the metric tensor, $\nabla_{\alpha} g_{\mu\nu}$, and apply a cyclic permutation of the three indices α, μ, ν. According to the rules of covariant differentiation presented in Sect. 3.4 we obtain

$$\nabla_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma_{\alpha\mu}{}^{\beta} g_{\beta\nu} - \Gamma_{\alpha\nu}{}^{\beta} g_{\mu\beta} \equiv N_{\mu\nu\alpha}, \tag{3.81}$$

$$\nabla_{\mu} g_{\nu\alpha} = \partial_{\mu} g_{\nu\alpha} - \Gamma_{\mu\nu}{}^{\beta} g_{\beta\alpha} - \Gamma_{\mu\alpha}{}^{\beta} g_{\nu\beta} \equiv N_{\nu\alpha\mu}, \tag{3.82}$$

$$\nabla_{\nu} g_{\alpha\mu} = \partial_{\nu} g_{\alpha\mu} - \Gamma_{\nu\alpha}{}^{\beta} g_{\beta\mu} - \Gamma_{\nu\mu}{}^{\beta} g_{\alpha\beta} \equiv N_{\alpha\mu\nu}. \tag{3.83}$$

We have defined, for further convenience, the tensor $N_{\mu\nu\alpha} = \nabla_{\alpha} g_{\mu\nu}$ which is symmetric in the first two indices.

We multiply the first equation by $1/2$, the second and the third one by $-1/2$, and then we add the three equations by combining all terms so as to reconstruct the symmetric and antisymmetric part of the connection:

$$\frac{1}{2}(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}) + \Gamma_{(\mu\nu)\alpha} - \Gamma_{[\alpha\mu]\nu} - \Gamma_{[\alpha\nu]\mu}$$
$$= \frac{1}{2}(N_{\mu\nu\alpha} - N_{\nu\alpha\mu} - N_{\alpha\mu\nu}). \tag{3.84}$$

By recalling the definition of the torsion tensor, Eq. (3.67), we now add $Q_{\mu\nu\alpha} = \Gamma_{[\mu\nu]\alpha}$ to both terms of the above equation, and isolate on the left-hand side the full affine connection, moving all remaining terms on the right-hand side:

$$\begin{aligned}\Gamma_{(\mu\nu)\alpha} + Q_{\mu\nu\alpha} &\equiv \Gamma_{\mu\nu\alpha} \\ &= \frac{1}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \\ &\quad + Q_{\mu\nu\alpha} + Q_{\alpha\mu\nu} + Q_{\alpha\nu\mu} + \frac{1}{2}(N_{\mu\nu\alpha} - N_{\nu\alpha\mu} - N_{\alpha\mu\nu}).\end{aligned} \tag{3.85}$$

By multiplying by $g^{\rho\alpha}$, in order to recast the connection in its canonical form (namely, in order to raise the third index), we finally obtain

$$\Gamma_{\mu\nu}{}^{\rho} = \left\{{}_{\mu\nu}{}^{\rho}\right\} - K_{\mu\nu}{}^{\rho} + W_{\mu\nu}{}^{\rho}, \tag{3.86}$$

where

$$\left\{{}_{\mu\nu}{}^{\rho}\right\} = \frac{1}{2}g^{\rho\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \tag{3.87}$$

is the so-called *Christoffel symbols*,

$$K_{\mu\nu}{}^{\rho} = -\left(Q_{\mu\nu}{}^{\rho} - Q_{\nu}{}^{\rho}{}_{\mu} + Q^{\rho}{}_{\mu\nu}\right) \tag{3.88}$$

is the so-called *contortion* tensor, and

$$W_{\mu\nu}{}^{\rho} = \frac{1}{2}\left(N_{\mu\nu}{}^{\rho} - N_{\nu}{}^{\rho}{}_{\mu} - N^{\rho}{}_{\mu\nu}\right) \tag{3.89}$$

is the so-called *non-metricity* tensor.

The result of this simple computation is very important and instructive, as it clearly illustrates the possibility of three different and independent contributions to the affine connection: (i) from the partial derivatives of the metric, (ii) from the torsion, and (iii) from the covariant derivatives of the metric tensor. The first and third term, $\{\}$ and W, are symmetric in μ, ν, and contribute only to the symmetric part $\Gamma_{(\mu\nu)}{}^{\rho}$ of the connection. The second term, $-K$, has an antisymmetric part which coincides with the torsion tensor, $-K_{[\mu\nu]}{}^{\rho} = Q_{\mu\nu}{}^{\rho} = \Gamma_{[\mu\nu]}{}^{\rho}$, but its symmetric part is also non-vanishing, and is given by $-K_{(\mu\nu)}{}^{\rho} = Q^{\rho}{}_{\mu\nu} + Q^{\rho}{}_{\nu\mu}$.

It follows that there are various possible classes of connections, differing for the conditions we impose on their components. In particular, a connection is called symmetric if $Q_{\mu\nu}{}^{\rho} = \Gamma_{[\mu\nu]}{}^{\rho} = 0$, and is called *metric-compatible* if $N_{\mu\nu\rho} = \nabla_{\rho} g_{\mu\nu} = 0$. Different connections are associated to space–time manifolds with different geometric structures, and it seems appropriate to introduce here three examples, corresponding to an increasing level of generality.

- In the context of the Riemann geometry and of Einstein's theory of general relativity one assumes that the connection is symmetric ($Q = 0$) and metric-compatible ($\nabla g = 0$). In such a case $K = 0 = W$, and the full connection simply reduces to the Christoffel connection,

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\alpha} (\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu}) \equiv \left\{ {}_{\mu\nu}{}^{\rho} \right\}. \tag{3.90}$$

  Hence, the metric alone is enough to fully determine the space–time geometry. In addition, since the connection is symmetric, it does not contain any tensor piece and can always be eliminated in a suitable "locally inertial" chart, in agreement with the equivalence principle.
- If the connection is metric-compatible ($\nabla g = 0$) but not symmetric ($Q \neq 0$) one is considering a Riemann–Cartan geometric structure, which is at the ground of a generalized gravitational theory called the "Einstein–Cartan theory". In such a case the connection $\Gamma = \{\} - K$ contains a tensor contribution associated to the torsion Q, which cannot be locally eliminated. Such a geometric structure seems to be in disagreement with the typical properties of the gravitational interaction, hence not suitable for a geometric theory of the gravitational field, at least at a macroscopic level. However, as we shall see in Chap. 14, the presence of torsion seems to be unavoidable in the context of supersymmetric theories unifying gravity with the other fundamental interactions.
- Finally, if the connection is neither symmetric ($Q \neq 0$) nor metric-compatible ($\nabla g \neq 0$), one has a manifold with a *metric-affine* geometric structure, and the connection contains in general all possible contributions appearing in Eq. (3.86). A possible example of such a geometry is provided by the so-called "Weyl model" (characterized, however, by a vanishing torsion), suggested long ago to include the electromagnetic field into the geometric structure of the space–time, but later abandoned. We should say that, unlike torsion, adding to the connection the non-metricity tensor is not supported (at least at present) by compelling physical motivations.

In the rest of this book we will always assume—unless otherwise stated—that we are working with a symmetric and metric-compatible connection, which can be expressed in the Christoffel form of Eq. (3.90).

3.6 Useful Rules of Covariant Differentiation

Various important applications of the covariant differentiation formalism require the calculation of the trace of the Christoffel connection which, from Eq. (3.90), is given

by

$$\Gamma_{\mu\nu}{}^{\nu} = \frac{1}{2} g^{\nu\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \equiv \frac{1}{2} g^{\nu\alpha} \partial_\mu g_{\nu\alpha} \tag{3.91}$$

(the last two terms cancel each other as they are antisymmetric in ν, α, and are contracted with $g^{\nu\alpha}$ which is symmetric). In this section it will be shown that the trace $\Gamma_{\mu\nu}{}^{\nu}$ can be simply expressed in a form which only depends on the determinant of the metric tensor, and that such a form turns out to be particularly convenient when computing the covariant derivative of tensor densities, the covariant divergence, and the covariant d'Alembert operator.

3.6.1 Trace of the Christoffel Connection

Let us start with Eqs. (3.32)–(3.34), relating the determinant of the metric tensor to the completely antisymmetric tensor. By differentiating Eq. (3.33) we obtain

$$-dg\, \epsilon_{\alpha\beta\gamma\delta} = d(g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma})\, \epsilon^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \big[(dg_{\alpha\mu}) g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma} + \cdots \big]. \tag{3.92}$$

The square brackets contain three additional terms (which we have omitted, for simplicity) similar to the first one, and proportional to the differentials $dg_{\beta\nu}$, $dg_{\gamma\rho}$, $dg_{\delta\sigma}$. Dividing both members by $\sqrt{-g}$, and recalling the definitions (3.31), (3.34), we can then rewrite the above equation as

$$\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta} \left(\frac{dg}{g} \right) \equiv \eta_{\alpha\beta\gamma\delta} \left(\frac{dg}{g} \right) = \eta^{\mu}{}_{\beta\gamma\delta} (dg_{\alpha\mu}) + \cdots . \tag{3.93}$$

Multiplying by $\eta^{\alpha\beta\gamma\delta}$, and using the relations (3.36), (3.37), we finally obtain

$$\begin{aligned} 4! \left(\frac{dg}{g} \right) &= 3! \big[g^{\alpha\mu} dg_{\alpha\mu} + g^{\beta\nu} dg_{\beta\nu} + \cdots \big] \\ &= 3!\, 4\, g^{\alpha\mu} dg_{\alpha\mu}, \end{aligned} \tag{3.94}$$

from which

$$\frac{dg}{g} = \frac{2}{\sqrt{-g}} d(\sqrt{-g}) = g^{\alpha\beta}\, dg_{\alpha\beta} = -g_{\alpha\beta}\, dg^{\alpha\beta} \tag{3.95}$$

(for the last equality we have used the metric property $d(g^{\alpha\beta} g_{\alpha\beta}) \equiv 0$). Hence:

$$\frac{2}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) = g^{\alpha\beta}\, \partial_\mu g_{\alpha\beta}, \tag{3.96}$$

and the trace (3.91) of the Christoffel connection can be expressed as

$$\Gamma_{\mu\nu}{}^{\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) = \partial_\mu (\ln \sqrt{-g}). \tag{3.97}$$

3.6.2 Covariant Derivatives of Tensor Densities

Given a tensor density $V^{\mu\nu\cdots}$, of rank r and weight w, let us first note that the gradient ∇_α transforms as a covariant vector under a general diffeomorphism, so that the covariant derivative of V must correspond to an object which has the same weight w and a tensor rank $r+1$.

Let us then recall that $\sqrt{-g}$ is a scalar density of weight $w=-1$, so that the object $(-g)^{w/2}V^{\mu\nu\cdots}$ has weight $w=0$ (namely, it is a tensor). The covariant derivative of this last object can thus be computed by applying the standard rules presented in Sect. 3.4, and the result is an object of zero weight and rank $r+1$. If this result is then multiplied by $(-g)^{-w/2}$ we can finally obtain a density of rank $r+1$ and weight w, as desired.

By applying the above procedure we will then define the covariant derivative of a density of weight w as follows:

$$ {}^{(w)}\nabla_\alpha V^{\mu\nu\cdots} \equiv (-g)^{-w/2}\nabla_\alpha\big[(-g)^{w/2}V^{\mu\nu\cdots}\big], \tag{3.98} $$

where ∇_α is the usual gradient operator for tensor objects, while ${}^{(w)}\nabla_\alpha$ is the corresponding operator for densities of weight w. An explicit computation of the covariant derivative of the term in square brackets, performed according to the tensor rules of Sect. 3.4, leads to

$$ \begin{aligned} {}^{(w)}\nabla_\alpha V^{\mu\nu\cdots} &= \partial_\alpha V^{\mu\nu\cdots} + \Gamma_{\alpha\beta}{}^{\mu}V^{\beta\nu\cdots} + \Gamma_{\alpha\beta}{}^{\nu}V^{\mu\beta\cdots} + \cdots \\ &\quad + (-g)^{-w/2}\partial_\alpha(-g)^{w/2}V^{\mu\nu\cdots}. \end{aligned} \tag{3.99} $$

The equation has obviously to be completed by the addition of all connection contributions associated to all (covariant and contravariant) indices carried by V.

The above differential operation differs from the covariant derivative of a tensor for the presence of the last term, which contains the determinant of the metric tensor and which seems to be qualitatively different from the preceding contributions. It can be easily checked, however, that also the last term can be expressed in terms of the Christoffel connection. By exploiting the results (3.96), (3.97) we have, in fact,

$$ (-g)^{-w/2}\partial_\alpha(-g)^{w/2} = \frac{w}{2g}\partial_\alpha g = w\,\partial_\alpha(\ln\sqrt{-g}) = w\Gamma_{\alpha\beta}{}^{\beta}, \tag{3.100} $$

and the covariant derivative of the tensor density can be rewritten in an equivalent (more compact) form as

$$ {}^{(w)}\nabla_\alpha V^{\mu\nu\cdots} = \nabla_\alpha V^{\mu\nu\cdots} + w\Gamma_{\alpha\beta}{}^{\beta}\,V^{\mu\nu\cdots}. \tag{3.101} $$

If $w=0$ our object is a tensor and we recover the standard definitions of Sect. 3.4, i.e. ${}^{(w)}\nabla_\alpha \to \nabla_\alpha$.

3.6.3 Covariant Divergence and d'Alembert Operator

It is finally useful to derive a compact expression for the covariant divergence of a vector, $\nabla_\mu A^\mu$. By applying the rules of Sect. 3.4 we have

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma_{\mu\alpha}{}^\mu A^\alpha. \tag{3.102}$$

Hence, by using Eq. (3.97) for the trace of Γ:

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g})A^\alpha \equiv \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}A^\alpha\right). \tag{3.103}$$

Starting from this result we can simply arrive at a covariant formulation of the Gauss theorem in the context of a Riemann space–time manifold.

Indeed, if we integrate on a four-dimensional space–time region Ω the covariant divergence of a vector, expressed as in of Eq. (3.102), we cannot directly apply the usual Gauss theorem because of the presence of the second term with the connection. The integration measure d^4x, on the other hand, is not a scalar under general coordinate transformations: as already stressed in Eq. (3.27), the correct (i.e. scalar) integration measure on a four-dimensional Riemann manifold is given by $d^4x\sqrt{-g}$.

By adopting this last integration measure, and using for the divergence the result of Eq. (3.103), we can then express the integral in a form which is generally covariant, even if it contains a simple ordinary divergence. In that case we can directly apply the usual Gauss theorem (see e.g. Eq. (1.33)) and obtain

$$\int_\Omega d^4x\,\sqrt{-g}\,\nabla_\mu A^\mu = \int_\Omega d^4x\,\partial_\mu\left(\sqrt{-g}A^\mu\right) = \int_{\partial\Omega} dS_\mu\,\sqrt{-g}A^\mu, \tag{3.104}$$

where $dS_\mu\sqrt{-g}$ is the covariant measure for the integration of the outgoing flux of A^μ on the boundary $\partial\Omega$ of the given space–time region.

As a second application of Eq. (3.103) we may consider the computation of the covariant d'Alembert operator, $\nabla_\mu\nabla^\mu$, applied to a scalar function ψ. By definition, the d'Alembert operator corresponds to the divergence of the gradient operator: hence, by applying Eq. (3.103),

$$\nabla_\mu\nabla^\mu\psi = \nabla_\mu\partial^\mu\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right) = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right). \tag{3.105}$$

We can also write, in a more explicit form,

$$g^{\mu\nu}\partial_\mu\partial_\nu\psi + \partial_\nu\psi\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\right), \tag{3.106}$$

suggesting that this expression can be greatly simplified if the metric satisfies the so-called "harmonic gauge" condition, defined by $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$. The use of such a gauge is particularly convenient, as we shall see in Chap. 9, for the study of gravitational waves in the approximation of the linearized Einstein equations.

3.7 Exercises Chap. 3

3.1 Isometries of the Minkowski Space–Time Solve the Killing equation (3.55) for the Minkowski metric.

3.2 Boosts and Killing Vectors Compute the components of the Killing vector associated to a boost along the z axis in the Minkowski space–time manifold.

3.3 Inverse Infinitesimal Transformation Check that Eq. (3.58) exactly represents, to second order, the inverse of the infinitesimal transformation (3.57).

3.4 Killing Equation Show that the Killing equation (3.55) can be rewritten in explicit covariant form as

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{3.107}$$

3.5 Trace of the Christoffel Connection Deduce Eq. (3.95) from the following definition of the determinant of a generic matrix M:

$$\det M = e^{\mathrm{Tr}\ln M}. \tag{3.108}$$

3.6 Covariant Derivative of g Check that the covariant derivative of the metric determinant g, computed with the Christoffel connection, is identically vanishing.

3.7 Covariant Derivative of the Totally Antisymmetric Tensor Show that the Christoffel connection satisfies the condition

$$\nabla_\alpha \eta^{\mu\nu\rho\sigma} = 0, \tag{3.109}$$

where η is the totally antisymmetric tensor defined by Eq. (3.31).

3.8 Solutions

3.1 Solution By setting $g_{\mu\nu} = \eta_{\mu\nu}$ into the Killing equation (3.55) we obtain the condition

$$\partial_{(\mu}\xi_{\nu)} = 0. \tag{3.110}$$

By applying the partial derivative operator, and using the property $\partial_\alpha\partial_\mu = \partial_\mu\partial_\alpha$, we arrive at:

$$\partial_{(\alpha}\partial_\mu\xi_{\nu)} = 0. \tag{3.111}$$

On the other hand, by using the above property, we also have

$$\partial_{[\alpha}\partial_\mu\xi_{\nu]} = 0. \tag{3.112}$$

By adding the last two equations, and using Eq. (3.110), we obtain

$$\partial_\mu \partial_\nu \xi_\alpha = 0, \tag{3.113}$$

whose integration leads to

$$\partial_\nu \xi_\alpha = \omega_{\nu\alpha} = \text{const}. \tag{3.114}$$

A second integration then leads to the general solution

$$\xi_\alpha = c_\alpha + \omega_{\mu\alpha} x^\mu, \tag{3.115}$$

where $c_\alpha = \text{const}$, and where the matrix ω has to be antisymmetric in order to satisfy the Killing condition (3.110).

By varying the independent components of c_α and $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ one obtains, respectively, the four generators of the global translations (see Eq. (1.23)) and the six generators of the global Lorentz rotations (see Eq. (1.44)). One then recovers the Poincaré group as the maximal isometry group of the Minkowski space–time geometry.

3.2 Solution The Lorentz matrix Λ for a boost along z has the following nonzero components: $\Lambda^0{}_0 = \Lambda^3{}_3 = \gamma$, $\Lambda^0{}_3 = \Lambda^3{}_0 = -\beta\gamma$, $\Lambda^1{}_1 = \Lambda^2{}_3 = 1$. By expanding Λ around the identity (for a small enough velocity parameter), and setting

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \omega^\mu{}_\nu, \tag{3.116}$$

we find that the nonzero components of ω are given by $\omega^0{}_3 = \omega^3{}_0 = -\beta = -v/c$. By exploiting the result of the previous exercise, Eq. (3.115), one then finds that the Killing vector associated to the given boost has the following nonzero components:

$$\xi_0 = \omega^3{}_0 x_3 = \beta z, \qquad \xi_3 = \omega^0{}_3 x_0 = -\beta c t. \tag{3.117}$$

It can be easily checked that the Killing equation (3.110) turns out to be identically satisfied by the vector $\xi_\mu = (\beta z, 0, 0, -\beta c t)$.

3.3 Solution Let us insert into the various terms of Eq. (3.58) the expression of x'^μ given by Eq. (3.57), keeping all contributions up to second order, and dropping higher-order contributions. We obtain

$$\begin{aligned}\left(f^{-1}\right)^\mu(x') \simeq x^\mu &+ \xi_1^\mu(x) + \frac{1}{2}\xi_2^\mu(x) + \frac{1}{2}\xi_1^\nu \partial_\nu \xi_1^\mu(x) \\ &- \xi_1^\mu(x+\xi_1) - \frac{1}{2}\xi_2^\mu(x) + \frac{1}{2}\xi_1^\nu \partial_\nu \xi_1^\mu(x).\end{aligned} \tag{3.118}$$

Let us then expand in Taylor series around x the fifth term of the above equation:

$$\xi_1^\mu(x+\xi_1) \simeq \xi_1^\mu(x) + \xi_1^\nu \partial_\nu \xi_1^\mu(x) + \cdots. \tag{3.119}$$

Inserting this expansion into Eq. (3.118) we find that all terms cancel each other except the first one, and we are led to

$$\left(f^{-1}\right)^{\mu}\left(x'\right) = x^{\mu}, \tag{3.120}$$

in agreement with our assumption.

3.4 Solution By exploiting the properties of the Christoffel connection (in particular, the metric-compatible condition $\nabla g = 0$), we can write

$$\nabla_{\mu}\xi_{\nu} = \nabla_{\mu}\left(g_{\nu\alpha}\xi^{\alpha}\right) = g_{\nu\alpha}\nabla_{\mu}\xi^{\alpha} = g_{\nu\alpha}\left(\partial_{\mu}\xi^{\alpha} + \Gamma_{\mu\beta}{}^{\alpha}\xi^{\beta}\right). \tag{3.121}$$

Hence:

$$2\nabla_{(\mu}\xi_{\nu)} = g_{\nu\alpha}\partial_{\mu}\xi^{\alpha} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + \left(g_{\nu\alpha}\Gamma_{\mu\beta}{}^{\alpha} + g_{\mu\alpha}\Gamma_{\nu\beta}{}^{\alpha}\right)\xi^{\beta}. \tag{3.122}$$

On the other hand, by imposing $\nabla_{\beta} g_{\mu\nu} = 0$, we also have

$$\partial_{\beta} g_{\mu\nu} = \Gamma_{\beta\mu}{}^{\alpha} g_{\alpha\nu} + \Gamma_{\beta\nu}{}^{\alpha} g_{\mu\alpha}. \tag{3.123}$$

Inserting this last result into Eq. (3.122), and using the symmetry of g and of the first two indices of the connection Γ, we find

$$2\nabla_{(\mu}\xi_{\nu)} = g_{\nu\alpha}\partial_{\mu}\xi^{\alpha} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + \xi^{\alpha}\partial_{\alpha} g_{\mu\nu}, \tag{3.124}$$

so that

$$\nabla_{(\mu}\xi_{\nu)} = -\frac{1}{2}\delta_{\xi}\, g_{\mu\nu}, \tag{3.125}$$

where $\delta_{\xi} g_{\mu\nu}$ is defined by Eq. (3.53). With the same procedure we can also find

$$\nabla^{(\mu}\xi^{\nu)} = \frac{1}{2}\delta_{\xi}\, g^{\mu\nu}, \tag{3.126}$$

where $\delta_{\xi} g^{\mu\nu}$ is defined by Eq. (3.54). The condition $\nabla_{(\mu}\xi_{\nu)} = 0$ (or $\nabla^{(\mu}\xi^{\nu)} = 0$) is thus equivalent to the Killing equation $\delta_{\xi} g_{\mu\nu} = 0$ (or $\delta_{\xi} g^{\mu\nu} = 0$, respectively) which ensures the local invariance of the metric under the coordinate transformations generated by the vector ξ^{μ}.

3.5 Solution By differentiating Eq. (3.108) we get

$$d(\det M) = \det M \,\mathrm{Tr}\left(M^{-1}\, dM\right). \tag{3.127}$$

Let us then replace M by the matrix associated to the covariant representation of the metric tensor, $g_{\mu\nu}$, and recall that the inverse matrix, in this case, is associated to the contravariant representation $g^{\mu\nu}$ (see Eq. (3.21)). The above equations thus becomes

$$\frac{dg}{g} = \mathrm{Tr}\left(g^{\alpha\beta}\, dg_{\beta\nu}\right) = g^{\alpha\beta}\, dg_{\alpha\beta}, \tag{3.128}$$

in full agreement with Eq. (3.95).

3.6 Solution The determinant g of the matrix $g_{\mu\nu}$ is a scalar density of weight $w = 2$. By applying the appropriate rule of covariant differentiation, Eq. (3.101), and the result (3.97) for the trace of the Christoffel connection, we obtain

$$\begin{aligned} {}^{(w)}\nabla_\alpha g &= \partial_\alpha g - 2\Gamma_{\alpha\beta}{}^{\beta} g = \partial_\alpha g - 2g\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} \\ &= \partial_\alpha g - 2g\frac{1}{2g}\partial_\alpha g \equiv 0. \end{aligned} \tag{3.129}$$

This result is also an obvious consequence of the fact that we are using the Christoffel connection which is metric-compatible, i.e. which satisfies $\nabla_\alpha g_{\mu\nu} = 0$.

3.7 Solution By applying the definition (3.31) of the tensor $\eta^{\mu\nu\rho\sigma}$, and the definition (3.75) of covariant derivative, we have

$$\begin{aligned} \nabla_\alpha \eta^{\mu\nu\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma}\partial_\alpha(-g)^{-1/2} + \Gamma_{\alpha\beta}{}^{\mu}\eta^{\beta\nu\rho\sigma} \\ &\quad + \Gamma_{\alpha\beta}{}^{\nu}\eta^{\mu\beta\rho\sigma} + \Gamma_{\alpha\beta}{}^{\rho}\eta^{\mu\nu\beta\sigma} + \Gamma_{\alpha\beta}{}^{\sigma}\eta^{\mu\nu\rho\beta}. \end{aligned} \tag{3.130}$$

We should now recall that the components of the totally antisymmetric tensor η are non-vanishing only if the four indices are all different. In a four-dimensional space–time, on the other hand, there are only four possible values for each index. Hence, the four terms containing the connection in the previous equation are nonzero only if $\beta = \mu$ in the first term, $\beta = \nu$ in the second term, $\beta = \rho$ in the third term and $\beta = \sigma$ in the fourth term.

The sum of these four terms thus reproduces the trace of the connection. By using the result (3.97) we finally obtain

$$\begin{aligned} \nabla_\alpha \eta^{\mu\nu\rho\sigma} &= \eta^{\mu\nu\rho\sigma}(-g)^{1/2}\partial_\alpha(-g)^{-1/2} + \Gamma_{\alpha\beta}{}^{\beta}\eta^{\mu\nu\rho\sigma} \\ &= \eta^{\mu\nu\rho\sigma}\left[(-g)^{1/2}\partial_\alpha(-g)^{-1/2} + (-g)^{-1/2}\partial_\alpha(-g)^{1/2}\right] \\ &\equiv 0. \end{aligned} \tag{3.131}$$

Chapter 4
Maxwell Equations and Riemann Geometry

If we adopt a model of space–time based on the Riemann geometry, we have to face the problem of how to transfer to such a generalized context the old, standard results of relativistic physics obtained in the context of the Minkowski geometry. The equivalence principle tell us that the equations of special relativity still hold in a suitable inertial chart, but only *locally*, over a space–time region of small enough (infinitesimal) extension (see Sect. 2.2). In order to be globally extended on a general Riemann manifold, such equations are be suitably generalized.

The correct generalization procedure is provided by the so-called *minimal coupling principle*, which will be introduced in the next section and which will be applied, in this chapter, to the theory of the electromagnetic interactions. The validity of such a procedure is not restricted to the electromagnetic phenomena, however, and can be generally extended to all known physical systems and interactions. In the following chapters the minimal coupling principle will be indeed applied to many (and in largely different) physical situations.

4.1 The Minimal Coupling Principle

The generalized relativity principle introduced in Chap. 2 imposes on the physical laws to respect an exact covariance property under the action of general coordinate transformations.

If the physical system we are considering is described by equations which are already covariant in the context of a Minkowski geometric structure, then such a system can be easily embedded into a more general Riemann structure—namely, its equations can be lifted to a general-covariant form—by applying a standard procedure called "minimal coupling principle". Such a procedure, in practice, amounts to the following set of operations:

- replace the Minkowski metric with the Riemann metric, $\eta_{\mu\nu} \to g_{\mu\nu}$, for all scalar products and all raising (and lowering) indices operations;

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- replace all partial (and total) derivatives with the corresponding covariant derivatives, $\partial_\mu \to \nabla_\mu$;
- use the appropriate powers of $\sqrt{-g}$ to saturate to zero the weights of all tensor densities. In the action integral, in particular, the covariant measure is given by the prescription $d^4x \to d^4x\sqrt{-g}$.

By applying the above procedure to the equations of motion or—better—to the action of the physical system, the system turns out to be "coupled" to the geometry of the given Riemann manifold. The coupling is "minimal", in the sense that it depends only on the metric and on its first derivatives (the connection), and thus it can be locally neutralized in the limit in which $g \to \eta$ and $\Gamma \to 0$ (in agreement with the principle of equivalence). A coupling procedure introducing higher-order derivatives of the metric would involve the space–time curvature (see Chap. 6), and it would be impossible to eliminate the associated (non-local) "tidal" forces.

Also, such a coupling is *universal*, in the sense that it necessarily affects (in the same way) all physical systems, with no exceptions. Obviously, the coupling is implemented differently for different geometric objects (the explicit form of the covariant derivative, for instance, depends on the type of object we are considering). To the best of our present knowledge, however, there is no physical system which is "geometrically neutral", i.e. insensitive to the geometric properties (and in particular to the curvature) of the given Riemann manifold.

Let us finally notice that the minimal coupling principle is not a peculiar aspect of the Riemann model of space–time, but is also a typical ingredient of the so-called *gauge theories*, where such a principle is needed to restore the invariance of the theory under a group of local symmetry transformations. Indeed, even in the Riemann case the minimal coupling is introduced to make the model covariant with respect to the group of diffeomorphism transformations, thus lifting to a local level the symmetry associated to the "rigid" (i.e. global) Lorentz transformations of the Minkowski space–time. From this point of view, as already stressed in Sect. 3.4, the connection Γ represents the "gauge field" associated to a local symmetry.

This last point will be further discussed and better clarified in Chap. 12. Here we will limit ourselves to note that the "gauge paradigm" seems to be, at present, a crucial ingredient of all successful models of the fundamental interactions. The above link between gauge theories and Riemann space–time thus suggests that also the Riemann geometry could provide a suitable tool to describe a fundamental interaction, like (in particular) the gravitational one.

4.2 Coupling Geometry and Electromagnetic Fields

By applying the minimal coupling principle to the definition of the electromagnetic field tensor $F_{\mu\nu}$ we may note, first of all, that the usual relation between fields and

potentials is left unchanged:

$$\begin{aligned} F_{\mu\nu} \to & \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ & = \partial_\mu A_\nu - \partial_\nu A_\mu - \left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right) A_\alpha \\ & = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv F_{\mu\nu}. \end{aligned} \tag{4.1}$$

The connection contribution automatically disappears, because of the symmetry property $\Gamma_{[\mu\nu]}{}^\alpha = 0$. However, the geometric independence of the relation between fields and potentials is not an accidental consequence of using a model with a symmetric connection—as it might appear from the above equations—being a very general result, which holds even in the presence of torsion.

In fact, the electromagnetic potential (like all vector potentials associated to a gauge field, Abelian or not-Abelian) must be represented, geometrically, by the differential 1-form $A = A_\mu\, dx^\mu$ (see Appendix A). Such an object transforms as a scalar in the Minkowski space–time locally tangent to the given Riemann manifold (see Chap. 12): hence, the exterior covariant derivative of the 1-form A associated to the potential always reduces to an ordinary derivative, without any further explicit contribution induced by the connection. Of course, if the model is characterized by a symmetric connection (as in the case we are considering), the distinction between vector representation and 1-form representations is no longer relevant to the geometric coupling of the potential.

In any case, the fact that the relation between $F_{\mu\nu}$ and A_ν is left unchanged has two important consequences.

The first consequence is that the minimal coupling principle does not modify the Maxwell equations concerning the divergence of the magnetic field and the curl of the electric field, $\partial_{[\alpha} F_{\mu\nu]} = 0$. In fact, let us compute the covariant gradient of $F_{\mu\nu}$:

$$\nabla_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - \Gamma_{\alpha\mu}{}^\beta F_{\beta\nu} - \Gamma_{\alpha\nu}{}^\beta F_{\mu\beta}. \tag{4.2}$$

By taking the totally antisymmetric part of this equations we exactly recover the result

$$\nabla_{[\alpha} F_{\mu\nu]} = \partial_{[\alpha} F_{\mu\nu]} = 0, \tag{4.3}$$

where the connection contributions are absent, again, due to the index symmetry $\Gamma_{[\mu\nu]}{}^\alpha = 0$. Hence, the coupling to the geometry does not modify this sector of the Maxwell equations.

The second, important consequence concerns the invariance of $F_{\mu\nu}$ under the (electromagnetic) gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f, \tag{4.4}$$

generated by an arbitrary scalar function $f(x)$. Such an invariance still holds, and still implies the exact conservation of the electric charge, quite independently of the space–time geometry in which the fields (and the sources) are embedded.

In fact, let us consider the action describing the dynamics of the electromagnetic field and its interactions with the charged currents. By applying the minimal coupling principle to the corresponding Minkowskian action we have

$$S = -\int_{\Omega} d^4x\,\sqrt{-g}\left(\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}\widetilde{J}^{\mu}A_{\mu}\right). \tag{4.5}$$

Here all scalar products are performed with the Riemann metric g, and $\widetilde{J}$ is the electromagnetic current obtained by generalizing (according to the minimal coupling procedure) the corresponding current J of the Minkowski space–time.

The variation of this action induced by a gauge transformation of the potential, $\delta A_{\mu} = \partial_{\mu} f$, is given by

$$\begin{aligned}\delta S &= -\frac{1}{c}\int_{\Omega} d^4x\,\sqrt{-g}\,\widetilde{J}^{\mu}\partial_{\mu} f \\ &= -\frac{1}{c}\int_{\Omega} d^4x\,\partial_{\mu}\big(\sqrt{-g}\,\widetilde{J}^{\mu} f\big) + \frac{1}{c}\int_{\Omega} d^4x\, f\,\partial_{\mu}\big(\sqrt{-g}\,\widetilde{J}^{\mu}\big)\end{aligned} \tag{4.6}$$

(since $\delta F_{\mu\nu} = 0$). By applying the Gauss theorem we find that the first contribution to δS is vanishing, provided the current $\widetilde{J}$ approaches zero rapidly enough on the boundary $\partial\Omega$ of the four-dimensional integration volume (which is always true if our physical sources are localized on a finite portion of space). In that case, given the arbitrary value of the gauge function f, we can conclude that the action is gauge invariant provided that

$$\partial_{\mu}\big(\sqrt{-g}\,\widetilde{J}^{\mu}\big) = 0. \tag{4.7}$$

The above condition can also be rewritten as

$$\nabla_{\mu}\widetilde{J}^{\mu} = 0, \tag{4.8}$$

where we have used the result (3.103) for the covariant divergence of a vector.

We can say, therefore, that the gauge symmetry of the action is still associated to the presence of a conserved current (according to the Nöther theorem), but such a current ($\widetilde{J}$) seems to be different from its Minkowski analogue (J). Different currents, on the other hand, may correspond to different conserved charges (recall Eqs. (1.33)–(1.35), which show how to compute the conserved quantity associated to a given current). Hence, it would seem that the conserved charge depends not only on the intrinsic properties of the electromagnetic sources, but also on the metric, and thus on the geometric properties of the space–time in which the sources are embedded.

Actually, such a geometric influence on the electric charge is a fictitious effect, as can be easily checked by recalling the explicit relation existing between J and $\widetilde{J}$ according to the minimal coupling principle.

Let us first notice, in fact, that the current density of the Minkowski space–time is given by $J^{\mu} = \rho\, dx^{\mu}/dt$, where ρ is the electric charge density. Multiplying by d^4x

(which is a scalar measure for the proper Lorentz group) one obtains the four-vector

$$J^\mu \, d^4x = c \, dq \, dx^\mu , \tag{4.9}$$

where the infinitesimal scalar $dq = \rho \, d^3x$ gives the total charge in the volume element d^3x, while dx^μ is the infinitesimal displacement along the "world line" spanned by the evolution of the charged sources. By applying the minimal coupling principle to the previous equation we find that, in a general Riemann space–time, the corresponding four-vector is given by

$$\widetilde{J}^\mu \, d^4x \, \sqrt{-g} = c \, dq \, dx^\mu . \tag{4.10}$$

It follows that $\widetilde{J}^\mu = J^\mu / \sqrt{-g}$, and that Eqs. (4.7), (4.8) are nothing more than a transcription, in explicitly covariant form, of the conservation equation $\partial_\mu J^\mu = 0$ for the electromagnetic current J^μ of the Minkowski space–time.

The charge (of a given source) conserved in a Riemann space–time thus exactly coincides with the conserved charge (of the same source) in the Minkowski space–time.

4.3 The Generalized Maxwell Equations

As shown in the previous section, the generalization of the Minkowski space–time to the case of a Riemann space–time does not modify neither the relation between electromagnetic fields and potentials, nor the conservation law of the electric charge. Hence, one may wonder if there is or not some new physical effect associated with the presence of a generalized geometric structure. The answer is in the affirmative: there is indeed a modification of the dynamical equations describing the propagation of the electromagnetic fields in a general Riemann context. Such equations turn out to be crucially dependent on the geometric properties of the space–time itself.

In order to elucidate this effect let us recall the action (4.5), which we rewrite as

$$S = \int_\Omega d^4x \, \sqrt{-g} \, \mathcal{L}(A, \partial A), \tag{4.11}$$

where $\mathcal{L}$ is the term in round brackets of Eq. (4.5). By varying with respect to A_ν, and imposing the condition of stationary action, we get the corresponding Euler–Lagrange equations

$$\partial_\mu \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\mu A_\nu)} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial A_\nu}, \tag{4.12}$$

written for the "effective" Lagrangian $\sqrt{-g}\mathcal{L}$ (which is not a scalar, but a scalar density of weight $w = -1$). By computing the partial derivatives, and dividing by $\sqrt{-g}$, we then arrive at the equations of motion:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) = \frac{4\pi}{c} \widetilde{J}^\nu . \tag{4.13}$$

Let us now recall that

$$\begin{aligned}\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma_{\mu\alpha}{}^{\mu} F^{\alpha\nu} + \Gamma_{\mu\alpha}{}^{\nu} F^{\mu\alpha} \\ &= \partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g})F^{\alpha\nu} \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) \end{aligned} \tag{4.14}$$

(the last term of the first line is vanishing because $F^{\mu\alpha} = F^{[\mu\alpha]}$, and we have used Eq. (3.97) for the trace of the Christoffel connection). Equation (4.13) can thus be rewritten as

$$\nabla_\mu F^{\mu\nu} = \frac{4\pi}{c}\widetilde{J}^\nu. \tag{4.15}$$

This is exactly the equation we would obtain by applying the minimal coupling principle directly to the Maxwell equations in Minkowski space (see Eq. (1.78)), and is clearly different from the original electromagnetic equations.

To summarize our results, and emphasize the possible physical effects of the minimal electromagnetic coupling to the Riemann geometry, it is now convenient to write down the full set of generalized Maxwell equations in terms of the field variables that are *unchanged* with respect to the Minkowski space–time. Those variables are the covariant tensor $F_{\mu\nu}$ (see Eq. (4.1)) and the vector current $J^\mu = \sqrt{-g}\widetilde{J}^\mu$ (see Eqs. (4.9) and (4.10)). From Eqs. (4.3), (4.13) we then obtain

$$\begin{aligned}&\partial_\mu\left(\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\right) = \frac{4\pi}{c}J^\nu, \qquad \partial_{[\mu}F_{\alpha\beta]} = 0, \\ &F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.\end{aligned} \tag{4.16}$$

All geometric contributions, described by the Riemann metric g, here appear explicitly, and their form suggests the existence of a close analogy between the electromagnetic equations written in a Riemann space–time and the same equations written in the presence of a continuous optical medium.

4.3.1 Analogy with the Maxwell Equations in an Optical Medium

In the presence of a continuous dielectric medium, and in the context of the Minkowski space–time, it is well known that the Maxwell equations can be conveniently expressed in terms of two different field tensors. The usual tensor $F_{\mu\nu}$, whose components $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B^k$ describe the electromagnetic field in vacuum, related to the total charge density and current density; and another tensor $G^{\mu\nu}$, whose components $G^{i0} = D^i$ and $G^{ij} = -\epsilon^{ijk}H_k$ describe the electromagnetic field inside the medium (also called the "induction field"), related to the free

charge density and current density. The two fields F and G satisfy the following equations:

$$\begin{gathered} \partial_\mu G^{\mu\nu} = \frac{4\pi}{c} J^\nu, \qquad \partial_{[\mu} F_{\alpha\beta]} = 0, \\ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \end{gathered} \tag{4.17}$$

and are related by the so-called "constitutive equation",

$$G^{\mu\nu} = \chi^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{4.18}$$

which describes the electromagnetic properties of the considered medium.

The tensor χ satisfies the general properties

$$\chi^{\mu\nu\alpha\beta} = \chi^{[\mu\nu][\alpha\beta]} = \chi^{\alpha\beta\mu\nu}, \qquad \chi^{[\mu\nu\alpha\beta]} = 0. \tag{4.19}$$

In order to give a simple example we can take, in particular, an isotropic and non-conducting medium, characterized by a dielectric constant ϵ and a magnetic permeability μ. In that case, and in the reference system at rest with the medium, we have

$$\chi^{i0j0} = -\epsilon\delta^{ij}, \qquad \chi^{ijkl} = \frac{1}{2\mu}\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right), \tag{4.20}$$

so that Eq. (4.18) provides us with the well known constitutive relations:

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H}. \tag{4.21}$$

A comparison of Eqs. (4.16) and (4.17) clearly shows that a Riemann space–time manifold formally behaves, from an electrodynamic point of view, as a continuous optical medium whose dielectric properties are controlled by the metric, through an "effective" constitutive tensor given by

$$\chi^{\mu\nu\alpha\beta} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right). \tag{4.22}$$

This analogy is not only a formal result. Indeed, as we shall see later in Chap. 8, a space–time geometry described by a suitable Riemann metric is able to deflect and slow down light rays—and, more generally, any other electromagnetic signal—just like a transparent, inhomogeneous dielectric. Also: a geometry described by a non-static metric, with off-diagonal components $g^{0i} \neq 0$, my act as an optically active medium, able to rotate the polarization plane of a propagating electromagnetic wave. Further effects of the Riemann geometry on the propagation of light rays and electromagnetic signal will be illustrated in the following chapter.

Let us conclude this chapter by noticing, however, that it would be wrong to take too seriously such an analogy between optical media and geometry. There are indeed various crucial differences between the set of Eqs. (4.16)—describing electromagnetic fields in vacuum, embedded in a Riemann space–time—and the

set of Eqs. (4.17)—describing electromagnetic fields in a medium, embedded in the Minkowski space–time—which prevent a total physical equivalence of the two above configurations. Unlike a real dielectric medium, in fact, the "geometric medium" satisfies the equivalence principle, and thus affects in a universal way all physical systems.

In this respect, an important example concerns the Cherenkov effect. In a typical dielectric medium, where the photon speed is slowed down, it may become possible for a charged particle to propagate with a velocity faster than light in that medium. It that case, as is well known, there is emission of Cherenkov radiation.

In the geometric analogue of the dielectric medium, on the contrary, the Cherenkov effect cannot occur [18]. A curved geometry, indeed, may slow down the photon propagation but, at the same time, it also slows down—exactly at the same rate—the propagation of any other signal and test particle. If a particle is slower than light in the empty Minkowski space, then it will keep slower than light also in empty Riemann space, quite independently of the given geometry. Only a dielectric medium can act in a non-universal way, slowing down photons more than other types of particle, and triggering the emission of Cherenkov radiation.

4.4 Exercises Chap. 4

4.1 Electrostatic Field in a Spherically Symmetric Space–Time Compute the electrostatic field of a point-like charge e embedded in a space–time manifold described by the following Riemann metric:

$$g_{00} = f(r), \qquad g_{ij} = -\delta_{ij}, \qquad g_{i0} = 0, \tag{4.23}$$

where $r = (x_i x^i)^{1/2}$.

4.2 Conformal Invariance of the Maxwell Equations Derive the propagation equation for the vector potential $\boldsymbol{A}$ in the absence of charged sources, in the radiation gauge ($\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, $A_0 = 0$), and in a space–time geometry described by the Riemann metric

$$g_{00} = 1, \qquad g_{ij} = -a^2(t)\delta_{ij}, \qquad g_{i0} = 0 \tag{4.24}$$

(using for simplicity natural units in which $c = 1$). Show that such equation can always be recast in the standard form of the d'Alembert wave equation through a suitable redefinition of the time variable.

4.5 Solutions

4.1 Solution Let us consider the electromagnetic equations (4.16), and set

$$J^i = 0, \qquad J^0 = ec\delta^{(3)}(x), \qquad F_{ij} = 0, \qquad F_{0i} = E_i. \tag{4.25}$$

Noticing that $\sqrt{-g} = f^{1/2}$ and $g^{00} = f^{-1}$ we get

$$\partial_i\left(\sqrt{-g}\,g^{ij}\,g^{00}F_{j0}\right) = \partial_i\left(f^{-1/2}E^i\right) = 4\pi e\delta^{(3)}(x). \tag{4.26}$$

Let us now introduce a scalar function $\chi(r)$ such that

$$f^{-1/2}E^i = -\partial^i\chi. \tag{4.27}$$

Inserting this definition into Eq. (4.26), and solving the Poisson equation for the scalar potential χ, we easily obtain $\chi = e/r$, from which

$$E^i = -f^{1/2}\partial^i\chi = f^{1/2}\frac{ex^i}{r^3}. \tag{4.28}$$

4.2 Solution Let us insert the metric components (4.24) into Eqs. (4.16), and note that

$$g^{00} = 1, \qquad g^{ij} = -a^{-2}\delta^{ij}, \qquad \sqrt{-g} = a^3. \tag{4.29}$$

By using the radiation gauge, $A_0 = 0$, $\partial^i A_i = 0$, we obtain

$$-\partial_0\left(a\delta^{ij}\partial_0 A_j\right) + \frac{1}{a}\delta^{kj}\delta^{il}\partial_k\partial_j A_l = 0, \tag{4.30}$$

from which, dividing by a,

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\dot{a}}{a}\frac{\partial}{\partial t} - \frac{\nabla^2}{a^2}\right)\boldsymbol{A} = 0, \tag{4.31}$$

where $\dot{a} = da/dt$, and where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the usual Laplace operator of three-dimensional Euclidean space (we have set $c = 1$).

The above equation can be recast in standard d'Alembertian form by introducing a new time parameter τ, related to t by the differential condition $dt = a\,d\tau$. Using such a new coordinate we have, in fact,

$$\begin{aligned}
\frac{\partial A}{\partial \tau} &= a\frac{\partial A}{\partial t}, \\
\frac{\partial^2 A}{\partial \tau^2} &= a\frac{\partial}{\partial t}\left(a\frac{\partial A}{\partial t}\right) = a^2\frac{\partial^2 A}{\partial t^2} + a\dot{a}\frac{\partial A}{\partial t},
\end{aligned} \tag{4.32}$$

and Eq. (4.31) can be rewritten as

$$\left(\frac{\partial^2}{\partial \tau^2} - \nabla^2\right)\boldsymbol{A} = 0. \tag{4.33}$$

It is important to stress that this result is a consequence of the so-called *conformal invariance* of the Maxwell Lagrangian,

$$\sqrt{-g}\,g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{4.34}$$

which is invariant under the following class of transformations

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = f(x) g_{\mu\nu}, \qquad g^{\mu\nu} \to \widetilde{g}^{\mu\nu} = f^{-1}(x) g^{\mu\nu} \tag{4.35}$$

(called "local scale transformations", or also "Weyl transformations"). As a consequence of this invariance, the form of the Maxwell equations is the same in the two (different) geometries described by the metrics g and $\widetilde{g}$ related by the transformation (4.35).

With the time redefinition $t \to \tau$, on the other hand, the line-element of the space–time (4.24) takes the form

$$ds^2 = dt^2 - a^2\, dx_i\, dx^i = a^2\big(d\tau^2 - dx_i\, dx^i\big), \tag{4.36}$$

and the geometry is described by a new metric $\widetilde{g}_{\mu\nu}$,

$$\widetilde{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}. \tag{4.37}$$

This metric is called "conformally flat", as it is related to the Minkowski metric η by a transformation of type (4.35), with $f = a^2$. Since the Maxwell equations must be identical in the two metrics $\widetilde{g}$ and η, we can immediately conclude that the wave equation for the vector potential, if expressed in terms of the coordinate τ of the metric $\widetilde{g}$, must coincide with the equation one would obtain in the space–time described by the Minkowski metric η (namely, with the d'Alembert wave equation), as indeed obtained in Eq. (4.33).

Chapter 5
Test Bodies and Signals in a Riemann Space–Time

In the previous chapters we have introduced a possible generalization of the space–time geometry based on the model of Riemann manifold. We have illustrated the main properties and the new formal aspects of this geometrical structure, and we have discussed how to embed in a general Riemann context a physical model originally formulated in the Minkowski space–time. It is time, now, to make more explicit the close connection existing between space–time geometry and gravitational interactions.

In this chapter it will be shown that all dynamical effects of Newton's gravitational theory can be faithfully reproduced with the introduction of a suitable space–time metric. But it will be also shown that such a geometric description of the gravitational interaction is more than a simple rephrasing of known physics in a new language: the geometric representation predicts indeed new gravitational effects which are absent in the usual formulation of the Newton theory, and which have been confirmed by experimental observations of ever increasing precision.

5.1 Geodesic Motion of Free Particles

To introduce a possible geometric description of the gravitational effects we can conveniently start with the free motion of a point-like test body, embedded in an arbitrary metric background.

Let us consider the simple case of a particle of mass m, described in the Minkowski space–time by the free-particle action of Exercise 1.4, Eq. (1.118). By applying the minimal coupling principle of Sect. 4.1 we obtain the generalized action

$$\begin{aligned} S &= -mc \int ds = -mc \int \sqrt{dx^\mu \, dx_\mu} = -mc \int \sqrt{dx^\mu \, dx^\nu \, g_{\mu\nu}} \\ &= -mc \int d\tau \, \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}, \end{aligned} \tag{5.1}$$

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which is written in a form appropriate to any given Riemann manifold. In the last line we have denoted with a dot the derivative with respect to the parameter τ, which is a scalar under general coordinate transformations and which parametrizes the so-called "world line" $x^\mu = x^\mu(\tau)$, describing the space–time trajectory of the particle.

The free-particle action can be written also in a simpler form, classically equivalent to the previous one for all dynamical purposes, but containing no square-root term. This can be done by introducing the auxiliary scalar field $V(\tau)$ (acting as a Lagrange multiplier, with dimensions $[V] = M^{-1}$), and considering the action:

$$S = -\frac{1}{2}\int d\tau \left(V^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} + m^2 c^2 V\right) \equiv \int d\tau\, L(x, \dot{x}). \tag{5.2}$$

By varying with respect to V we obtain indeed the constraint:

$$\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = m^2 c^2 V^2. \tag{5.3}$$

Solving for V, and inserting into Eq. (5.2), we exactly recover the action (5.1).

The equations of motion for a free particle can thus be obtained by varying either the first type or the second type of action. The second one—also called Polyakov action—is well defined also in the limit of massless particles, unlike the action (5.1).

Let us then vary the action (5.2) with respect to the position x^μ of our test body, and choose the parameter τ in such a way that it turns out to be proportional to the proper-time coordinate along the world-line of the body. With this choice of the "time gauge" the auxiliary field V reduces to a constant (see Eq. (5.3)), and its factorized contribution does not affect the equations of motion. We have, in fact,

$$\frac{\partial L}{\partial x^\mu} = -\frac{1}{2V}\dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta}, \tag{5.4}$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{1}{V} g_{\mu\nu}\dot{x}^\nu, \tag{5.5}$$

and the Euler–Lagrange equations give

$$\begin{aligned} 0 &= -\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} + \frac{\partial L}{\partial x^\mu} \\ &= g_{\mu\nu}\ddot{x}^\nu + \dot{x}^\alpha \dot{x}^\nu \partial_\alpha g_{\mu\nu} - \frac{1}{2}\dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta} \\ &= g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\dot{x}^\alpha \dot{x}^\beta (\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}). \end{aligned} \tag{5.6}$$

Multiplying by $g^{\rho\mu}$ we are finally led to

$$\ddot{x}^\rho + \Gamma_{\alpha\beta}{}^\rho \dot{x}^\alpha \dot{x}^\beta = 0, \tag{5.7}$$

where Γ is the Christoffel connection defined in Eq. (3.90). The above equation exactly reproduces the equation of the so-called *geodesic* curve. Point-like test bodies,

freely propagating across a Riemann space–time described by the metric $g_{\mu\nu}$, thus move along the geodesics of the given metric.

Given its variational derivation, it is evident that a geodesic path extremizes the distance among two given points of the space–time manifold. It is also evident, by comparing Eqs. (5.6) to Eq. (3.80), that a geodesic coincides with an autoparallel curve only if the connection coincides with the Christoffel one (which is indeed the case for the geometric model we are considering). Within a more general geometric structure, including torsion and non-metricity contributions (see Eq. (3.86)), a point-like test body will still move along the geodesics defined by the Christoffel connection—in agreement with the result of a variational computation based on the minimal action principle. In that case, however, the trajectory of the motion will not correspond any longer to an autoparallel curve.

A physical trajectory, in particular, may correspond to a time-like or a null geodesic characterized, respectively, by $\dot{x}^\mu \dot{x}_\mu > 0$ or $\dot{x}^\mu \dot{x}_\mu = 0$. In the first case our test body is massive: multiplying by the mass, and setting $m\dot{x}^\mu = mu^\mu = p^\mu$, the equation of motion (5.7) can also be rewritten as

$$\frac{dp^\mu}{d\tau} + \Gamma_{\alpha\beta}{}^\mu u^\alpha p^\beta = 0, \tag{5.8}$$

or, in differential form, as

$$Dp^\mu \equiv dp^\mu + \Gamma_{\alpha\beta}{}^\mu \, dx^\alpha \, p^\beta = 0 \tag{5.9}$$

(see the definition (3.77) of covariant differentiation along a curve). This means that the four-momentum vector of the body is covariantly constant—namely, it is transported parallel to itself—along the trajectory of the motion (let us recall, to this purpose, the discussion of Sect. 3.4.1).

In the case of light-like trajectories of massless particles the differential condition (5.9) is still valid, but with the constraint $p^\mu p_\mu = 0$. If we have a wave signal—instead of a particle—and we may consider the approximation of geometric optics, we can describe its propagation in terms of the wave vector k^μ (instead of using the momentum vector p^μ). The corresponding trajectory, in that case, will thus be determined by the parallel transport of the vector k^μ:

$$Dk^\mu = dk^\mu + \Gamma_{\alpha\beta}{}^\mu \, dx^\alpha \, k^\beta = 0. \tag{5.10}$$

Let us conclude this section by noting that the above results for the free evolution of test particles and signals provide strong support to the idea of a geometric representation of the gravitational effects, for two important reasons.

The first reason is that the geodesic equation (5.7) is in agreement with the principle of equivalence. A geodesic motion, in fact, is of *locally inertial* type, since its equation can always be reduced to the free equation $\ddot{x} = 0$ when (locally) $\Gamma = 0$. Also, the geodesic trajectory is fully independent of the mass of the test body, for all bodies, and this property is automatically obtained without any assumption concerning the equality of inertial and gravitational masses (which is instead required in the context of Newton's theory).

The second reason is that the geodesic equation is also able to reproduce the Newtonian equation of motion, in the non-relativistic limit of small velocities and weak enough gravitational fields, through the introduction of a suitable metric background. This point will be illustrated in the next section.

5.2 The Newtonian Limit

Consider a test particle of mass m, interacting with a gravitational field described by the Newtonian potential $\phi(x)$ (see the Lagrangian (2.2)). Suppose that the field is *weak:*

$$|\phi| \ll c^2 \tag{5.11}$$

(i.e. that the gravitational potential energy is negligible with respect to the rest mass energy); that the field is *static:*

$$\dot{\phi} = 0 \tag{5.12}$$

(more generally, that the time derivative of the potential is negligible with respect to its spatial gradients, $|\partial_t \phi| \ll |\partial_i \phi|$); and, finally, that the velocities of the involved bodies are *non-relativistic:*

$$\left|v^i\right| = \left|\frac{dx^i}{dt}\right| \ll c. \tag{5.13}$$

In that regime, the action associated to the Lagrangian (2.2) takes the form:

$$\begin{aligned} S &= -mc^2 \int dt \left(\sqrt{1 - \frac{v^2}{c^2}} + \frac{\phi}{c^2}\right) \\ &\simeq \int dt \left(-mc^2 + \frac{1}{2}mv^2 - m\phi\right). \end{aligned} \tag{5.14}$$

On the other hand, the action of a massive particle embedded in a space–time geometry described by the metric $g_{\mu\nu}$ is given by Eq. (5.1), and can be explicitly written as

$$\begin{aligned} S &= -mc \int dt \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \\ &= -mc \int dt \left(g_{00}c^2 + g_{ij}v^i v^j + 2g_{0i}cv^i\right)^{1/2}. \end{aligned} \tag{5.15}$$

If we take the following metric:

$$g_{00} = \left(1 + 2\frac{\phi}{c^2}\right), \qquad g_{ij} = -\delta_{ij}, \qquad g_{0i} = 0, \tag{5.16}$$

the action becomes

$$S = -mc^2 \int dt \left(1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2}\right)^{1/2}. \tag{5.17}$$

By using the approximations (5.11), (5.13), and expanding the square root to lowest order in ϕ/c^2 and v^2/c^2, we finally obtain the expression

$$S \simeq -mc^2 \int dt \left(1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{\phi}{c^2}\right), \tag{5.18}$$

which exactly coincides with the action (5.14).

It follows that the space–time geometry described by the metric (5.16) can reproduce the dynamical effects of the gravitational interaction in the so-called *Newtonian limit*, in which the gravitational field is weak and static, and the test bodies' velocities are non-relativistic, as specified by Eqs. (5.11)–(5.13). We can check that, in this limit, the geodesic equation for the metric (5.16) is equivalent to the usual equation of motion of Newton's gravitational theory.

For this purpose it is convenient to separate Eq. (5.7) into its time and space components:

$$\ddot{x}^0 + \Gamma_{\alpha\beta}{}^0 \dot{x}^\alpha \dot{x}^\beta = 0, \tag{5.19}$$

$$\ddot{x}^i + \Gamma_{\alpha\beta}{}^i \dot{x}^\alpha \dot{x}^\beta = 0. \tag{5.20}$$

Using for the connection the definition (3.90) we can note, first of all, that the deviation of the metric from the Minkowski form is due to the presence of ϕ, and that the non-zero contributions to the metric gradients are proportional to $\partial g \sim \partial\phi$. Neglecting powers of ϕ of second (and higher) order we can thus use the Minkowski metric when applied to ∂g, and compute the connection (in the weak field limit) as follows:

$$\Gamma_{\alpha\beta}{}^\mu \simeq \frac{1}{2}\eta^{\mu\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta}). \tag{5.21}$$

By inserting the metric (5.16) we find

$$\Gamma_{00}{}^0 = 0, \qquad \Gamma_{ij}{}^0 = 0, \qquad \Gamma_{0i}{}^j = 0, \qquad \Gamma_{ij}{}^k = 0, \tag{5.22}$$

since the metric is static, diagonal, and only the spatial gradients of g_{00} may provide non-vanishing contributions. The non-zero components of Γ, in the Newtonian limit, are then given by

$$\Gamma_{0i}{}^0 = \frac{1}{c^2}\partial_i\phi, \qquad \Gamma_{00}{}^i = \frac{1}{c^2}\delta^{ij}\partial_j\phi. \tag{5.23}$$

The component $\Gamma_{0i}{}^0$, on the other hand, contributes to Eq. (5.19) with a term proportional to $v^i\partial_i\phi$, which we can neglect if we limit ourselves to the first order in

ϕ/c^2 and v/c. From the two geodesic equations (5.19), (5.20) we thus obtain the conditions

$$\ddot{x}^0 = 0, \tag{5.24}$$

$$\ddot{x}^i + \delta^{ij}\partial_j\phi\left(\frac{\dot{x}^0}{c}\right)^2 = 0, \tag{5.25}$$

valid in the Newtonian approximation.

Let us now recall that the dot denotes the derivative with respect to the scalar parameter τ (see Sect. 5.1), covariantly defined along the world-line. The integration of Eq. (5.24) gives then

$$\dot{x}^0 = c\frac{dt}{d\tau} = \alpha = \text{const}, \tag{5.26}$$

where α is an arbitrary integration constant. Inserting this result into the left-hand side of Eq. (5.25) we obtain

$$\dot{x}^i = \frac{dt}{d\tau}\frac{dx^i}{dt} = \frac{\alpha}{c}v^i, \qquad \ddot{x}^i = \frac{\alpha^2}{c^2}\frac{dv^i}{dt}, \tag{5.27}$$

and we can rewrite Eq. (5.25) in the following final (vector) form:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -\nabla\phi, \tag{5.28}$$

which exactly reproduces the well-known Newtonian equation of motion. .

The above computation clearly shows that the known, non-relativistic gravitational dynamics can be reproduced with the introduction of a geometric structure of Riemannian type and, in particular, with a space–time described (in Cartesian coordinates) by the line-element

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = \left(1 + \frac{2\phi}{c^2}\right)c^2\,dt^2 - |d\boldsymbol{x}|^2 \tag{5.29}$$

(see the metric (5.16)). However, it is important to stress that this geometric reformulation of the gravitational equations also predicts new gravitational effects which are absent in the usual Newtonian context, and which will be illustrated in the next section.

5.3 Time Dilatation and Frequency Shifts

If we accept a geometric description of the Newton gravitational field based on the generalized (non-Minkowskian) line-element (5.29), then we must accept, as a direct and unavoidable consequence, also a modification of the special-relativistic relation between the proper-time interval $d\tau$—typical of a given physical process—

and the corresponding interval of coordinate time dt, associated to the particular coordinate system we are using.

In the case of the Minkowski metric it is well known that such a relation only depends on the motion of the observer with respect to the considered physical process: on finds indeed $dt/d\tau = \gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor associated to the relative motion of the two inertial charts.

In the case of the metric (5.29), instead, one finds that the relation between the time intervals depends not only on the state of relative motion, but also on the *spatial position* of the observer with respect to the considered process: there is indeed a difference of time intervals even inside the same chart, in the absence of relative motion, for physical processes occurring at different spatial positions. This effect is common to all space–time geometries characterized by a metric component g_{00} which depends on the spatial coordinates, as is indeed the case of Eq. (5.29).

We should recall, in fact, that the proper-time interval between two events corresponds, by definition, to the scalar interval ds/c evaluated in the reference system in which the spatial separation between the given events is vanishing, $dx^i = 0$. If the metric component g_{00} is not a constant, such a quantity depends on the position even within the same coordinate system. For instance, given a physical process observed at the point x_1, the relation between proper and coordinate time intervals is given by

$$d\tau_1 = \sqrt{g_{00}(x_1)}\, dt. \tag{5.30}$$

Similarly, if the same process is observed at the point x_2, we have

$$d\tau_2 = \sqrt{g_{00}(x_2)}\, dt \tag{5.31}$$

(notice that dt is the time interval which would be locally measured in the absence of geometric distortion of the Minkowski space–time, and thus is the same at all points). It follows that

$$\frac{d\tau_1}{d\tau_2} = \left[\frac{g_{00}(x_1)}{g_{00}(x_2)}\right]^{1/2}, \tag{5.32}$$

which determines the relative difference of the measured time intervals as a function of the spatial localization of the observer.

It should be noted that in the metric (5.29) we have a negative gravitational potential, $\phi < 0$, so that $g_{00} < 1$. If we compare the proper-time interval between two events measured at the point x_1, where $\phi(x_1) \neq 0$, with the corresponding interval measured at infinity, where $\phi_\infty = 0$, $g_{00} = 1$ and $dt = d\tau$, we get, from Eq. (3.52):

$$d\tau_\infty = \frac{d\tau_1}{\sqrt{g_{00}(x_1)}} = \frac{d\tau_1}{[1 + \frac{2\phi(x_1)}{c^2}]^{1/2}} > d\tau_1. \tag{5.33}$$

It follows that the duration of a process, in the presence of a Newtonian gravitational field, appears to be increased with respect to the duration one would expect, for the same process, in the absence of gravity. This is the famous effect of *time dilatation*, certainly not predicted by Newton's gravitational theory.

The experimental measure of this effect can be conveniently based on the observation of a periodic process, and on the comparison of the period (or of the frequency) of the given process at different spatial positions.

We may consider, for instance, a monochromatic wave emitted at the point x_e and received at x_r. The ratio between the period of the signal at the two different positions is determined by Eq. (5.32), with $x_1 = x_e$ and $x_2 = x_r$. The corresponding frequencies, ω_e and ω_r, are related of course by the inverse ratio, and from Eq. (5.32) we immediately obtain

$$\frac{\omega_r}{\omega_e} = \left[\frac{g_{00}(x_e)}{g_{00}(x_r)}\right]^{1/2}. \tag{5.34}$$

It may be useful, also in view of many phenomenological applications, to show that the same relation can also be derived with a different (more general) procedure, by exploiting the notion of "static observer".

Let us consider, in fact, a class of observers characterized by a four-velocity u^μ with vanishing spatial components,

$$u^i = 0, \qquad u^0 = \frac{c}{\sqrt{g_{00}}} \tag{5.35}$$

(they are at rest, or "comoving", in the given geometric background, and the time component of their four-velocity is appropriately normalized in such a way that $g_{\mu\nu}u^\mu u^\nu = c^2$). Let us suppose that both the source and the receiver of the transmitted signal are at rest with respect to two local static observers, placed at the points x_e and x_r. Let us assume, finally, that the signal propagation can be appropriately described by the wave vector $k^\mu = (\boldsymbol{k}, \omega/c)$. The frequency locally observed at the positions x_e and x_r are then given, respectively, by the scalar projections $(k^\mu u_\mu)_{x_e}$ and $(k^\mu u_\mu)_{x_r}$.

In a locally inertial system, where $g_{00} = 1$, the results of the two scalar projections are obviously the same because both observers are static, and there is no Doppler effect due to a relative motion. The equality between the two projections, on the other hand, is a scalar relation valid in all coordinate systems, so that we can write, in general,

$$\left(g_{\mu\nu}k^\mu u^\nu\right)_{x_e} = \left(g_{\mu\nu}k^\mu u^\nu\right)_{x_r}. \tag{5.36}$$

For a diagonal metric (with $g_{0i} = 0$) we then obtain the relation

$$\omega_e\sqrt{g_{00}(x_e)} = \omega_r\sqrt{g_{00}(x_r)} \tag{5.37}$$

which exactly reproduces Eq. (5.34), as anticipated.

5.3.1 The Frequency Shift in a Newtonian Field

Let us now concentrate on the metric (5.29), which describes the effects of a Newtonian gravitational field, and let us apply to this metric the previous results concern-

ing the geometric shift of frequencies. By expanding the square roots appearing in Eq. (5.34) we immediately obtain, to first order in ϕ/c^2,

$$\frac{\omega_r}{\omega_e} \simeq \left(1+\frac{\phi_e}{c^2}\right)\left(1-\frac{\phi_r}{c^2}\right) \simeq 1-\frac{1}{c^2}(\phi_r-\phi_e), \tag{5.38}$$

from which

$$\frac{\Delta\omega}{\omega} \equiv \frac{\omega_r-\omega_e}{\omega_e} = -\frac{1}{c^2}(\phi_r-\phi_e) \equiv -\frac{\Delta\phi}{c^2}. \tag{5.39}$$

Taking into account that our potential is negative, it follows that if the gravitational field is stronger at the emission than at the receiving position (i.e., if $\phi_e < \phi_r$) then the difference $\Delta\omega$ is negative, and then $\omega_r < \omega_e$. This means that the received frequency is "red-shifted" with respect to the emitted one.

The frequency emitted by an atom at rest on the surface of a very compact star, for instance, appears to be smaller (when detected by an outer observer) than the frequency emitted by an identical atom living on the surface of the Sun or of the Earth (where the gravitational field is weaker). But the opposite effect is also possible, of course: if $\phi_e > \phi_r$ then Eq. (5.39) implies $\omega_r > \omega_e$, and the frequency—received in a region where the gravitational field is stronger than the field of the emission region—turns out to be "blue-shifted" (namely, higher than the one measured by the emitter).

Effects of this type are typically very small in the Newtonian limit. For instance, for a signal emitted by the surface of the Sun—with radius $R \sim 7 \times 10^{10}$ cm and mass $M \sim 10^{33}$ g—and received on the surface of the Earth, the red-shift is given by

$$\frac{\Delta\omega}{\omega} = -\left(\frac{GM}{Rc^2}\right)_{\text{Sun}} \sim -10^{-6}. \tag{5.40}$$

Nevertheless, the red-shift effect has been detected and experimentally confirmed even within the gravitational field of the Earth. The first experiment, performed by Pound and Rebka [42] in 1959, has observed the frequency shift of electromagnetic radiation propagating through a drop of about 23 meters on the Earth surface, and has confirmed the predictions of Eq. (5.39) to an accuracy of about 10 %. Further experiments have improved this accuracy, and also observed the gravitational red-shift of radiation emitted by the surface of compact stars like the "white dwarf" Sirius B.

There are also direct observations of the time dilatation effect (see Eq. (5.33)), performed by comparing the rate of atomic clocks carried by flying airplanes with the rate of identical clocks at rest on the Earth surface [23]. Such an effect turns out to be obviously amplified for the orbiting clocks placed on artificial satellites, at a great altitude: in that case the effect becomes strong enough to be taken into account (and automatically corrected) even by modern navigation systems like GPS (*Global Positioning System*). The orbiting clocks, affected by a weaker gravitational field, are faster than the terrestrial clocks by an amount of about 46 μs per day, and it may be interesting to note that this effect is dominant with respect to the kinematic

slowdown predicted by special relativity (which approximately amounts to 7.2 μs per day).

Let us conclude this section by noting that, for a signal described by the wave vector k^μ, the relation (5.39) between the spectral shift and the gravitational potential can also be obtained, more directly, from the condition (5.10) of geodesic propagation.

In fact, let us start with the metric (5.16), and use the associated connection (valid in the Newtonian approximation) already computed in Eqs. (5.22), (5.23). For the time component of k^μ we then obtain

$$\begin{aligned}\frac{d\omega}{c} &= -\Gamma_{\alpha\beta}{}^0\, dx^\alpha\, k^\beta \\ &= -\Gamma_{0i}{}^0\left(dx^0\, k^i + \frac{\omega}{c}\, dx^i\right) \\ &= -\frac{1}{c^2}\partial_i\phi\left(c\, dt\, k^i + \frac{\omega}{c}\, dx^i\right).\end{aligned} \tag{5.41}$$

The spatial components of wave vector are given by $k^i = (\omega/w)n^i$, where n^i is the unit vector pointing to the propagation direction, and w is the modulus of the phase velocity of the signal, related to the modulus v of the group velocity by $w = c^2/v$. It follows that $k^i\,\partial_i\phi$ is a mixed term of order $v^i\,\partial_i\phi$, which can be neglected in the Newtonian approximation. In that approximation we are left with

$$\frac{d\omega}{\omega} = -\frac{1}{c^2}\, dx^i\, \partial_i\phi = -\frac{1}{c^2}\, d\phi, \tag{5.42}$$

which indeed reproduces (in differential form) the previous result of Eq. (5.39).

5.4 Exercises Chap. 5

5.1 Time-Dependent Spectral Shift A photon propagates along the (light-like) geodesics associated to the following line-element:

$$ds^2 = c^2\, dt^2 - a^2(t)|d\boldsymbol{x}|^2, \tag{5.43}$$

which describes a space–time with a time-dependent geometry. The photon is emitted at a time t_e and received at a time t_r. Compute the frequency shift of the received photon with respect to the emitted one in terms of the geometric parameter $a(t)$.

5.2 Hyperbolic Geodesic Motion Determine the time-like geodesic trajectories associated to a motion along the x axes, in the space–time manifold parametrized by the coordinates $x^0 = ct$, $x^i = (x, y, z)$, and described by the following line-element:

$$ds^2 = \left(\frac{t_0}{t}\right)^2 \left(c^2\, dt^2 - dx^2\right) - dy^2 - dz^2, \tag{5.44}$$

where t_0 is a constant.

5.5 Solutions

5.1 Solution In order to evaluate the frequency shift experienced by the photon along its geodesic path we shall use the condition of parallel transport of the four-momentum vector, given by Eq. (5.9).

We recall, first of all, that in the Minkowski space–time a photon of frequency ω is characterized by an energy $\mathcal{E} = \hbar\omega$ and a momentum $p^i = (\hbar\omega/c)n^i$, where n^i is the unit vector pointing to the propagation direction. In the space–time described by the line-element (5.43) the four-momentum p^μ of the photon has then the following components:

$$p^0 = \frac{\hbar\omega}{c}, \qquad p^i = \frac{n^i}{a(t)}\frac{\hbar\omega}{c}. \tag{5.45}$$

The factor a^{-1} appearing in the spatial part of the vector is prescribed by the minimal coupling principle, in order to satisfy the covariant version of the null normalization condition:

$$g_{\mu\nu}\, p^\mu\, p^\nu = \left(p^0\right)^2 - a^2(t)|\boldsymbol{p}|^2 = 0. \tag{5.46}$$

The non-vanishing components of the connection associated to the metric (5.43) are given by

$$\Gamma_{0i}{}^j = \frac{1}{ac}\frac{da}{dt}\delta_i^j, \qquad \Gamma_{ij}{}^0 = \frac{a}{c}\frac{da}{dt}\delta_{ij} \tag{5.47}$$

(we have used the definition (3.90)). By applying the geodesic equation (5.9) we then obtain

$$\begin{aligned} dp^0 &= d\left(\frac{\hbar\omega}{c}\right) = -\Gamma_{ij}{}^0\, dx^i\, p^j \\ &= -\frac{\hbar\omega}{c^2}\frac{da}{dt}\delta_{ij}\, dx^i\, n^j. \end{aligned} \tag{5.48}$$

Let us now recall that a light-like geodesic is characterized by a null space–time interval, $dx_\mu\, dx^\mu = ds^2 = 0$. A photon propagating along the spatial direction n^i, across a space–time geometry specified by Eq. (5.43), must then follow a trajectory which satisfies the differential condition

$$c\, dt\, n^i = a\, dx^i. \tag{5.49}$$

Inserting this result into Eq. (5.48), using $\delta_{ij}n^i n^j = 1$, and dividing by $\hbar/c$, we obtain

$$\frac{d\omega}{\omega} = -\frac{da}{a}. \tag{5.50}$$

The integration of this equation immediately gives the time dependence of ω as a function of the time dependence of the geometric parameter $a(t)$:

$$\omega(t) = \frac{\omega_0}{a(t)}, \tag{5.51}$$

where ω_0 is an integration constant, representing the corresponding frequency in the Minkowski space–time (where $a = 1$). The spectral shift between the emitted frequency $\omega_e \equiv \omega(t_e)$ and the received frequency $\omega_r \equiv \omega(t_r)$ is then fixed by the ratio:

$$\frac{\omega_r}{\omega_e} = \frac{a(t_e)}{a(t_r)}. \tag{5.52}$$

It may be noted, finally, that if $a(t_r) > a(t_e)$ then we obtain $\omega_r < \omega_e$, namely the received frequency is red-shifted with respect to the emitted one. This is a typical effect of the cosmological gravitational field which permeates our Universe on very large scales of distance, and which can be indeed described (in first approximation) by a geometry of the type (5.43) (see e.g. the books [31, 45, 51] quoted in the bibliography).

5.2 Solution The geometry of the manifold (5.44) is described by the metric

$$\begin{aligned} g_{00} &= \left(\frac{t_0}{t}\right)^2 = -g_{11}, \qquad g_{22} = g_{33} = -1, \\ g^{00} &= \left(\frac{t}{t_0}\right)^2 = -g^{11}, \qquad g^{22} = g^{33} = -1, \end{aligned} \tag{5.53}$$

and the non-zero components of the associated connection (defined by Eq. (3.90)) are given by

$$\Gamma_{00}{}^0 = \Gamma_{11}{}^0 = \Gamma_{01}{}^1 = \Gamma_{10}{}^1 = -\frac{1}{ct}. \tag{5.54}$$

Let us explicitly write down the geodesic equation (5.7), by setting $x^0 = ct$ and recalling that the dot denotes differentiation with respect to the parameter τ, which we can identify with the proper time along the particle trajectory:

$$c\ddot{t} - \frac{1}{ct}\left(c^2\dot{t}^2 + \dot{x}^2\right) = 0, \tag{5.55}$$

$$\ddot{x} - \frac{2}{t}\dot{x}\dot{t} = 0, \tag{5.56}$$

$$\ddot{y} = 0, \tag{5.57}$$

$$\ddot{z} = 0. \tag{5.58}$$

We are considering, in particular, a one-dimensional motion along the x axes. Eq. (5.56) can thus be easily integrated, and gives

$$\dot{x} = \frac{\alpha}{t_0} t^2, \tag{5.59}$$

where α is an integration constant with the dimensions of an acceleration (the parameter t_0 has been inserted for further convenience). Instead of integrating also Eq. (5.55) we now observe that a time-like geodesic must satisfy the normalization condition

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \left(\frac{t_0}{t}\right)^2 \left(c^2 \dot{t}^2 - \dot{x}^2\right) = c^2, \tag{5.60}$$

which, combined with Eq. (5.59), gives

$$c^2 \dot{t}^2 = \frac{c^2}{t_0^2} t^2 + \frac{\alpha^2}{t_0^2} t^4. \tag{5.61}$$

By using this result to eliminate the proper-time variable τ (in terms of t) from Eq. (5.59), we obtain

$$\frac{dx}{dt} = \frac{\dot{x}}{\dot{t}} = \frac{\alpha t}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}}. \tag{5.62}$$

A second integration finally gives us the equation of the trajectory,

$$x(t) = \overline{x} + \int dt \, \frac{\alpha t}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}} = \overline{x} + \frac{c^2}{\alpha} \sqrt{1 + \frac{\alpha^2 t^2}{c^2}}, \tag{5.63}$$

where $\overline{x}$ is an integration constant depending on the initial conditions. Note that this solution also automatically satisfies Eq. (5.55), as can be explicitly checked by differentiating with respect to τ.

The above trajectory can be geometrically interpreted by squaring $x - \overline{x}$, and moving ct to the left-hand side:

$$(x - \overline{x})^2 - c^2 t^2 = \frac{c^4}{\alpha^2}. \tag{5.64}$$

In the plane (x, ct) this is an hyperbola with center at the point of coordinates $x = \overline{x}$ and $t = 0$, and asymptotes given by the light-cone $x = \overline{x} \pm ct$. The time-like geodesics of the given geometry thus exactly reproduce the trajectories of a uniformly accelerated motion in Minkowski space, characterized by a four-acceleration vector with constant modulus $a_\mu a^\mu = \alpha^2$.

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Chapter 6
Geodesic Deviation and Curvature Tensor

The fact that a particular Riemannian geometry may suitably represent the effects of a Newtonian gravitational field legitimates our expectations that the Riemann model of space–time may describe gravity in the relativistic regime as well.

For the full geometric formulation of a successful gravitational theory, however, we still lack a crucial ingredient: the notion of curvature of the space–time manifold, controlled in a covariant way by the associated curvature tensor (also called Riemann tensor). It is just this tensor which is able to distinguish, in a non-ambiguous way, a gravity-dominated space–time geometry from the standard, gravity-free geometry of the Minkowski space–time.

We have shown, in particular, that a suitable metric can reproduce the dynamical effects of a given model of gravity. The explicit form of the metric, however, depends not only on the intrinsic geometry, but also on the chart we are using to parametrize the space–time manifold: for instance, even in the Minkowski space–time we could globally introduce a metric $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$ through an appropriate coordinate transformation, thus simulating the effects of a possible gravitational field (see e.g. Exercise 6.1). We are thus unavoidably led to the question: how to identify, geometrically, the presence (or the absence) of a true gravitational field, without possible ambiguities due to the choice of the coordinates?

The answer to this question necessarily involves the space–time curvature, as we shall see in the next section.

6.1 The Equation of Geodesic Deviation

A successful geometric representation of the gravitational dynamics must comply with the basic properties of the gravitational interaction. In this respect we should recall, first of all, the equivalence principle (see Sect. 2.2), which asserts that the gravitational effects are locally undistinguishable from the physical effects experienced in an accelerated frame.

Such an equivalence enables us to eliminate gravity in a sufficiently small region of space–time, by introducing a suitable chart which supports a locally inertial

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frame. A typical example of such a possibility is provided by a free-falling elevator in the gravitational field of the Earth: a test body inside the falling elevator will fluctuate freely as if the elevator would be placed in empty space, in a region free from any gravitational field.

However, if we put two test bodies (instead of one) inside the elevator, then there is an important physical difference between the two configurations previously mentioned—i.e., free-fall in a given field and real absence of field—which soon clearly emerges. Suppose, for instance, that the two bodies are initially at rest at the initial time t_0. Then, for $t > t_0$, they will keep both at rest in the absence of a real external field; on the contrary, they will start approaching each other with a relative accelerated motion if the elevator is free falling.

The relative motion is unavoidable, due to the fact the test bodies are falling along geodesic trajectories which are not parallel, but converging toward the source of the physical field. So, even if the relative velocity of the two bodies is initially vanishing, $v(t_0) = 0$, their initial relative acceleration, $a(t_0)$, is always non-vanishing. Here we arrive at the point which is crucial for our discussion.

For any arbitrary gravitational field we can always (and entirely) eliminate the gravitational acceleration at any given point in space and at any given time, but there is no way to eliminate the acceleration between two different points—no matter how separated—at any time. If we take two points on two different geodesics they will be always characterized by a relative acceleration which cannot be eliminated (not even locally), and which is due to gravity, whose action tends to distort and focalize the trajectories. In the absence of gravity, on the contrary, the geodesics of all free bodies—quite independently of the chosen chart—are the straight lines of the Minkowski space–times, and their relative acceleration is vanishing.

We are thus led to the following conclusion: given a metric defined on the space–time manifold, and given a bundle of geodesic curves associated to that metric, the relative acceleration between points belonging to different geodesics only depends on the bending of the world-lines produced by the gravitational interaction, and can be used to denote, unambiguously, the presence (or the absence) of a physical gravitational field. In order to achieve a sound geometric representation of the gravitational interaction it thus important to determine in a precise quantitative way such acceleration, which is described by the so-called equation of "geodesic deviation" that we will now explicitly derive.

Let us consider two free test bodies embedded in a space–time geometry described by the Riemann metric $g_{\mu\nu}$, and evolving along two geodesic trajectories parametrized by the scalar variable τ (which we can identify with the proper-time parameter). We shall assume that the two bodies are very close in space, and that the infinitesimal separation between the two geodesics $x^\mu(\tau)$ and $y^\mu(\tau)$ is controlled by the (space-like) four-vector $\xi^\mu(\tau)$, such that

$$y^\mu(\tau) = x^\mu(\tau) + \xi^\mu(\tau). \tag{6.1}$$

We look for a differential equation determining the time evolution of the geodesic separation, restricting our computation to the first order in ξ^μ (also called the vector of "geodesic deviation").

For this purpose we start with the explicit expression of the two geodesic equations,

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}{}^\mu \dot{x}^\alpha \dot{x}^\beta = 0, \tag{6.2}$$

$$\ddot{x}^\mu + \ddot{\xi}^\mu + \Gamma_{\alpha\beta}{}^\mu (x+\xi)\big(\dot{x}^\alpha + \dot{\xi}^\alpha\big)\big(\dot{x}^\beta + \dot{\xi}^\beta\big) = 0 \tag{6.3}$$

(the dot denotes differentiation with respect to τ, see Eq. (5.7)). In the second geodesic we can expand the connection in Taylor series for $\xi \to 0$, and neglect all contributions of order ξ^2 and higher. We obtain

$$\ddot{x}^\mu + \ddot{\xi}^\mu + \big[\Gamma_{\alpha\beta}{}^\mu(x) + \xi^\nu \partial_\nu \Gamma_{\alpha\beta}{}^\mu(x) + \cdots\big]\big(\dot{x}^\alpha \dot{x}^\beta + 2\dot{x}^\alpha \dot{\xi}^\beta + \cdots\big) = 0. \tag{6.4}$$

By subtracting Eq. (6.2) we are finally led to

$$\ddot{\xi}^\mu + 2\Gamma_{\alpha\beta}{}^\mu \dot{x}^\alpha \dot{\xi}^\beta + \xi^\nu \big(\partial_\nu \Gamma_{\alpha\beta}{}^\mu\big)\dot{x}^\alpha \dot{x}^\beta = 0, \tag{6.5}$$

which gives the relative acceleration between two infinitely close geodesics, as a function of the connection and of its first derivatives, evaluated locally.

The above result is not suitable for a direct geometric interpretation, being written in a form which is not explicitly covariant. However, this difficulty can be overcome by using the definition (3.78) of the covariant derivative along a curve. By applying such a definition to the vector ξ^μ along the geodesic $x^\mu(\tau)$, we first obtain

$$\frac{D\xi^\mu}{d\tau} = \dot{\xi}^\mu + \Gamma_{\alpha\beta}{}^\mu \dot{x}^\alpha \xi^\beta. \tag{6.6}$$

The computation of the second covariant derivative,

$$\begin{aligned}\frac{D^2\xi^\mu}{d\tau^2} &= \frac{d}{d\tau}\frac{D\xi^\mu}{d\tau} + \Gamma_{\lambda\sigma}{}^\mu \dot{x}^\lambda \frac{D\xi^\sigma}{d\tau} \\ &= \ddot{\xi}^\mu + \Gamma_{\alpha\beta}{}^\mu\big(\ddot{x}^\alpha \xi^\beta + \dot{x}^\alpha \dot{\xi}^\beta\big) + \dot{x}^\nu \big(\partial_\nu \Gamma_{\alpha\beta}{}^\mu\big)\dot{x}^\alpha \xi^\beta \\ &\quad + \Gamma_{\lambda\sigma}{}^\mu \dot{x}^\lambda \big(\dot{\xi}^\sigma + \Gamma_{\alpha\beta}{}^\sigma \dot{x}^\alpha \xi^\beta\big),\end{aligned} \tag{6.7}$$

then leads to an explicit relation between the acceleration $\ddot{\xi}^\mu$ and its corresponding covariant expression $D^2\xi^\mu/d\tau^2$. By inserting into this relation the value of $\ddot{\xi}^\mu$ given by Eq. (6.5) and the value of $\ddot{x}^\mu$ given by Eq. (6.2), we find that all terms with $\dot{x}\dot{\xi}$ disappear, and we finally obtain

$$\frac{D^2\xi^\mu}{d\tau^2} = -\dot{x}^\beta \dot{x}^\alpha \xi^\nu \big(\partial_\nu \Gamma_{\beta\alpha}{}^\mu - \partial_\alpha \Gamma_{\beta\nu}{}^\mu + \Gamma_{\beta\alpha}{}^\rho \Gamma_{\rho\nu}{}^\mu - \Gamma_{\beta\nu}{}^\rho \Gamma_{\alpha\rho}{}^\mu\big). \tag{6.8}$$

This result can be rewritten in compact form as

$$\frac{D^2\xi^\mu}{d\tau^2} = -\xi^\nu R_{\nu\alpha\beta}{}^\mu \dot{x}^\beta \dot{x}^\alpha, \tag{6.9}$$

where

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - \Gamma_{\nu\rho}{}^{\beta}\Gamma_{\mu\alpha}{}^{\rho} \tag{6.10}$$

is a geometric object corresponding to a fourth-rank tensor, known as the *Riemann tensor*. The tensor nature of this object can be deduced from Eq. (6.9) and from the fact that ξ^α and $\dot{x}^\beta$ are both true vectors.

Equation (6.9) (called equation of geodesic deviation) determines the relative covariant acceleration of two geodesics with (infinitesimal) space-like separation parametrized by the vector ξ^μ. Since this acceleration is directly related to the presence of a gravitational field, and since the same acceleration is geometrically controlled by the Riemann tensor (6.10), it follows that this tensor is the object to be used in order to characterize the presence or the absence of gravity, and to provide a possible geometric representation of its physical effects.

The Riemann tensor, on the other hand, is also the geometric object which describes in a covariant way the curvature properties of a Riemann manifold (see Exercise 6.2 and the discussion of Sect. 6.3), and which is able to distinguish it unambiguously from the "flat" Minkowski space–time. In fact, there is a rigorous result of differential geometry stating that the vanishing of the Riemann tensor is a necessary and sufficient condition for the existence of a global coordinate transformation reducing the metric *everywhere* to the Minkowski form, $g_{\mu\nu} = \eta_{\mu\nu}$ (see for instance the book [38] of the bibliography).

In other words, a given metric $g_{\mu\nu}(x)$ describes a "curved" space–time if and only if $R_{\mu\nu\alpha\beta}(g) \neq 0$. Otherwise the given metric g corresponds to a particular "accelerated" parametrization of the Minkowski space–time, but the associated geodesic deviation of Eq. (6.9) is vanishing, and the geometry does not describe any true gravitational effect.

We are thus led to the important (and interesting) conclusion that the physical effects of the gravitational field can be geometrically associated to the presence of space–time curvature. In order to formulate a relativistic theory of gravity based on the geometric properties of the space–time manifold we must then specify how the gravitational sources may "induce" curvature, and how such a curvature propagates in space and evolves in time.

The discussion of these problems will be postponed to Chap. 7, after a necessary and more detailed study of the properties of the Riemann tensor, to be presented in the following sections.

6.2 The Riemann Curvature Tensor

A fourth-rank tensor, in a four-dimensional space–time, has in general $4^4 = 256$ components. The number of independent components of the Riemann tensor is much smaller, however, because of the symmetry properties of its indices and of the identities it satisfies.

A first property, already evident from the definition (6.10), is the asymmetry in the first two indices:

$$R_{\mu\nu\alpha}{}^{\beta} = R_{[\mu\nu]\alpha}{}^{\beta}. \tag{6.11}$$

A second (related) property of the Riemann tensor—written in fully covariant form as a tensor of type $(0, 4)$—is the invariance with respect to the exchange of the first pair of indices with the second one:

$$R_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha}{}^{\rho} g_{\rho\beta} = R_{\alpha\beta\mu\nu} \tag{6.12}$$

(see Exercise 6.3). As a consequence, this tensor must be antisymmetric also in the last two indices, so that

$$R_{\mu\nu\alpha\beta} = R_{[\mu\nu][\alpha\beta]}. \tag{6.13}$$

This means that $R_{\mu\nu\alpha\beta}$ can be written as the tensor product of two second-rank antisymmetric tensors, and that the total number of independent components is reduced from 256 to $6 \times 6 = 36$.

The list of the index symmetries is not yet complete, however. If we take the totally antisymmetric part with respect to the first three indices we obtain the condition

$$R_{[\mu\nu\alpha]}{}^{\beta} = 0, \tag{6.14}$$

also known as the "*I Bianchi identity*". As can be easily checked from the definition (6.10), this property is a direct consequence of the symmetry of the Christoffel connection ($\Gamma_{[\alpha\beta]}{}^{\mu} = 0$), hence it is not satisfied in the presence of torsion. In our case, however, it is valid, it imposes $4 \times 4 = 16$ conditions on the components of the Riemann tensor, and we are eventually left with $36 - 16 = 20$ independent components only.

There is also another property which concerns the covariant derivatives of the Riemann tensor (and which does not affect the total number of independent components). It is the so-called "*II Bianchi identity*":

$$\nabla_{[\lambda} R_{\mu\nu]\alpha}{}^{\beta} = 0. \tag{6.15}$$

This identity can be easily derived by working in the locally inertial chart where $\Gamma = 0$ (the partial derivatives of Γ, however, are non-vanishing). In that chart we have

$$\nabla_{\lambda} R_{\mu\nu\alpha}{}^{\beta}\big|_{\Gamma=0} = \partial_{\lambda}\partial_{\mu}\Gamma_{\nu\alpha}{}^{\beta} - \partial_{\lambda}\partial_{\nu}\Gamma_{\mu\alpha}{}^{\beta}. \tag{6.16}$$

By antisymmetrizing in λ, μ and ν we find that both terms on the right-hand side of the above equation are vanishing, so that also the left-hand side is vanishing. But if a tensor is zero in a given chart then it is zero in all charts, as expressed in fact by the identity (6.15).

Let us now recall, following the discussion of the previous section, that a non-zero Riemann tensor characterizes a geometry "physically" different from the

Minkowski geometry, because it describes a space–time manifold "bent" by the action of the gravitational interaction. In the Minkowski space–time, on the other hand, we have partial differential operators which commute among each other, while in a Riemann manifold the partial derivatives are to be replaced by covariant derivatives. If a given space–time is genuinely different from the Minkowski one, then its geometry should be characterized by covariant derivatives which cannot be globally reduced to partial derivatives and which cannot be represented by commuting operators, in general. We can thus expect that the Riemann tensor, which controls the deviations from the Minkowski geometry, also determines the commutator of two covariant derivatives.

This is indeed the case, as we can explicitly check by computing the second covariant derivative of a vector field A^α. By applying the general definitions of Sect. 3.4 we obtain

$$\begin{aligned}\nabla_\mu\nabla_\nu A^\alpha &= \nabla_\mu\big(\partial_\nu A^\alpha + \Gamma_{\nu\beta}{}^\alpha A^\beta\big)\\ &= \partial_\mu\partial_\nu A^\alpha + \big(\partial_\mu\Gamma_{\nu\beta}{}^\alpha\big)A^\beta + \Gamma_{\nu\beta}{}^\alpha\partial_\mu A^\beta\\ &\quad + \Gamma_{\mu\beta}{}^\alpha\big(\partial_\nu A^\beta + \Gamma_{\nu\rho}{}^\beta A^\rho\big) - \Gamma_{\mu\nu}{}^\rho\big(\partial_\rho A^\alpha + \Gamma_{\rho\beta}{}^\alpha A^\beta\big). \quad (6.17)\end{aligned}$$

Considering the commutation brackets, and using the symmetry property $\Gamma_{[\mu\nu]}{}^\rho = 0$, we find that all terms containing the partial derivatives of A cancel out each other, and we are left with

$$\begin{aligned}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A^\alpha &= \big(\partial_\mu\Gamma_{\nu\beta}{}^\alpha - \partial_\nu\Gamma_{\mu\beta}{}^\alpha\big)A^\beta\\ &\quad + \big(\Gamma_{\mu\rho}{}^\alpha\Gamma_{\nu\beta}{}^\rho - \Gamma_{\nu\rho}{}^\alpha\Gamma_{\mu\beta}{}^\rho\big)A^\beta, \quad (6.18)\end{aligned}$$

namely with

$$[\nabla_\mu, \nabla_\nu]A^\alpha = R_{\mu\nu\beta}{}^\alpha A^\beta. \quad (6.19)$$

We can say, therefore, that two covariant derivatives applied to a vector field commute if and only if the space–time geometry has a vanishing curvature.

We will conclude this section by introducing the possible contractions of the Riemann tensor. Contracting an index of the first pair with another index of the second pair we obtain the so-called *Ricci tensor*,

$$R_{\nu\alpha} \equiv R_{\mu\nu\alpha}{}^\mu = R_{(\nu\alpha)}, \quad (6.20)$$

which is symmetric, second-rank tensor, $R_{\nu\alpha} = R_{\alpha\nu}$. The symmetry property can be directly derived from its explicit definition,

$$R_{\nu\alpha} = \partial_\mu\Gamma_{\nu\alpha}{}^\mu - \partial_\nu\Gamma_{\mu\alpha}{}^\mu + \Gamma_{\mu\rho}{}^\mu\Gamma_{\nu\alpha}{}^\rho - \Gamma_{\nu\rho}{}^\mu\Gamma_{\mu\alpha}{}^\rho, \quad (6.21)$$

by recalling that $\Gamma_{\nu\alpha}{}^\mu = \Gamma_{(\nu\alpha)}{}^\mu$, by using the result (3.97),

$$\partial_\nu\Gamma_{\alpha\mu}{}^\mu = \partial_\nu\partial_\alpha(\ln\sqrt{-g}), \quad (6.22)$$

and by noticing that

$$\Gamma_{\nu\rho}{}^{\mu}\Gamma_{\mu\alpha}{}^{\rho} = \Gamma_{\alpha\mu}{}^{\rho}\Gamma_{\rho\nu}{}^{\mu} = \Gamma_{\alpha\rho}{}^{\mu}\Gamma_{\mu\nu}{}^{\rho}. \tag{6.23}$$

The trace of the Ricci tensor defines the so-called *scalar curvature*,

$$R = R_{\nu}{}^{\nu} = g^{\nu\alpha}R_{\nu\alpha}. \tag{6.24}$$

Finally, by combining the Ricci tensor with the scalar curvature we obtain the so-called *Einstein tensor*,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{6.25}$$

which plays an important role in the dynamical equations of the gravitational field, as we shall see in the following chapter.

It is important to notice that the Einstein tensor is symmetric, $G_{\mu\nu} = G_{\nu\mu}$, and that its covariant divergence is zero,

$$\nabla_{\nu}G_{\mu}{}^{\nu} = 0. \tag{6.26}$$

This relation, also called *contracted Bianchi identity*, is a direct consequence of the identity (6.15), which can we written explicitly as

$$\nabla_{\lambda}R_{\mu\nu\alpha}{}^{\beta} + \nabla_{\mu}R_{\nu\lambda\alpha}{}^{\beta} + \nabla_{\nu}R_{\lambda\mu\alpha}{}^{\beta} = 0. \tag{6.27}$$

In fact, if we compute the covariant divergence of the Ricci tensor, and use the above equation, we obtain

$$\nabla_{\nu}R_{\mu}{}^{\nu} = \nabla_{\nu}R_{\alpha\mu}{}^{\nu\alpha} = -\nabla_{\alpha}R_{\mu\nu}{}^{\nu\alpha} - \nabla_{\mu}R_{\nu\alpha}{}^{\nu\alpha}, \tag{6.28}$$

namely

$$2\nabla_{\nu}R_{\mu}{}^{\nu} = \nabla_{\mu}R, \tag{6.29}$$

from which

$$\nabla_{\nu}\left(R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R\right) = 0, \tag{6.30}$$

which exactly reproduces Eq. (6.26).

6.3 A Simple Example: Constant-Curvature Manifolds

The aim of this section is to compute the Riemann tensor for a manifold with constant curvature, with an arbitrary number of dimensions, and with a pseudo-Euclidean signature, $g_{\mu\nu} = (+, -, -, -, \dots)$. It will be shown, in particular, that the mixed representation of type $(2, 2)$ (where the Riemann tensor takes the form

$R_{\mu\nu}{}^{\alpha\beta}$) has constant components, directly related to the constant radius of curvature of the manifold we are considering.

Let us consider a D-dimensional hypersurface Σ_D (with one time-like and $D-1$ space-like dimensions), embedded in a $(D+1)$-dimensional Minkowski space–time parametrized by the coordinates X^A, with line-element:

$$ds^2 = \eta_{AB}\, dX^A\, dX^B, \quad A, B = 0, 1, \ldots, D. \tag{6.31}$$

The hypersurface Σ_D is described by the equation

$$\eta_{AB} X^A X^B = -\frac{1}{k}, \tag{6.32}$$

where k is a constant, with dimensions of the inverse of a length squared.

If $k > 0$ the above equation describes a "pseudo-hypersphere" of squared radius $a^2 = 1/k$ and two-dimensional space–time sections of hyperbolic type (see e.g. Eq. (2.39) in the solution of Exercise 2.2). If $k < 0$ the equation describes instead a higher-dimensional hyperboloid. In all cases we have a manifold with a constant radius of curvature.

For the computation of the Riemann tensor we can conveniently parametrize the hypersurface Σ_D by using the so-called "stereographic" coordinates $\{x^\mu\}$, $\mu = 0, 1, \ldots, D-1$, which simply coincide with the time coordinate and with the first $D-1$ spatial coordinates of the external Minkowski space–time. If we call y the additional spatial coordinate of the external space, we can then write

$$\begin{aligned} &X^A = \delta^A_\mu x^\mu, \quad A = 0, 1, \ldots, D-1, \\ &X^A = y, \quad A = D \end{aligned} \tag{6.33}$$

(see Exercises 2.2, 6.5, 6.6 for possible alternative parametrizations of the same hypersurface).

The intrinsic coordinates x^μ are restricted to the hypersurface Σ_D, hence they must satisfy the constraint (6.32), which takes the form

$$\eta_{\mu\nu} x^\mu x^\nu - y^2 = -\frac{1}{k}. \tag{6.34}$$

By differentiating we obtain

$$\eta_{\mu\nu} x^\mu\, dx^\nu = y\, dy, \tag{6.35}$$

from which

$$dy^2 = \frac{1}{y^2}\left(x_\mu\, dx^\mu\right)^2 = \frac{x_\mu x_\nu\, dx^\mu\, dx^\nu}{\frac{1}{k} + x_\alpha x^\alpha}. \tag{6.36}$$

By using the last equation to eliminate dy^2 in Eq. (6.31) we finally arrive at the line-element

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu - dy^2$$

$$= \eta_{\mu\nu}\, dx^\mu\, dx^\nu - k\frac{x_\mu x_\nu}{1 + k x_\alpha x^\alpha}\, dx^\mu\, dx^\nu. \tag{6.37}$$

The intrinsic metric of Σ_D, namely the tensor $g_{\mu\nu}$ such that $ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$, then takes the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - k\frac{x_\mu x_\nu}{1 + k x_\alpha x^\alpha}, \tag{6.38}$$

where x^μ are the coordinates of the stereographic chart we are considering. This is the metric of a constant-curvature manifold, with a curvature controlled by the parameter k which can be positive, negative, or vanishing. For $k = 0$ we obviously recover the flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, which describes the Minkowski hyperplane with a constant but vanishing curvature.

Let us now compute the Riemann tensor for this metric. The first step is to obtain the connection, which in the stereographic chart takes the simple form

$$\Gamma_{\nu\alpha}{}^\beta = -k g_{\nu\alpha} x^\beta \tag{6.39}$$

(see the computation of Exercise 6.4). By applying the definition (6.10) we have then

$$R_{\mu\nu\alpha}{}^\beta = -k\big(\partial_\mu g_{\nu\alpha} x^\beta + g_{\nu\alpha}\delta^\beta_\mu\big) + \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\alpha}{}^\rho - \{\mu \leftrightarrow \nu\}, \tag{6.40}$$

where the symbol $\{\mu \leftrightarrow \nu\}$ denotes that all the preceding terms are to be inserted with the index μ replaced by ν and vice versa. Thanks to the metric properties of the Christoffel connection ($\nabla_\mu g_{\nu\alpha} = 0$, see Sect. 3.5) we can also set

$$\partial_\mu g_{\nu\alpha} = \Gamma_{\mu\nu}{}^\rho g_{\rho\alpha} + \Gamma_{\mu\alpha}{}^\rho g_{\nu\rho}. \tag{6.41}$$

Inserting this result into Eq. (6.40), and using the explicit form (6.39) of the connection, we find that all quadratic terms ($\sim \Gamma^2$) cancel each other, so that we are left with

$$\begin{aligned} R_{\mu\nu\alpha}{}^\beta &= \Gamma_{\mu\nu}{}^\rho \Gamma_{\rho\alpha}{}^\beta + \Gamma_{\mu\alpha}{}^\rho \Gamma_{\nu\rho}{}^\beta - k g_{\nu\alpha}\delta^\beta_\mu + \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\alpha}{}^\rho - \{\mu \leftrightarrow \nu\} \\ &\equiv k\big(g_{\mu\alpha}\delta^\beta_\nu - g_{\nu\alpha}\delta^\beta_\mu\big). \end{aligned} \tag{6.42}$$

Multiplying by $g^{\rho\alpha}$, in order to switch to a mixed tensor of type $(2, 2)$, we finally obtain

$$R_{\mu\nu}{}^{\rho\beta} = k\big(\delta^\rho_\mu \delta^\beta_\nu - \delta^\rho_\nu \delta^\beta_\mu\big). \tag{6.43}$$

All components are constant, as anticipated at the beginning of this section, and proportional to the inverse of the squared curvature radius.

The Ricci contraction and the scalar curvature can now be calculated, respectively, from their definitions (6.20) and (6.24). By recalling that we are considering a D-dimensional manifold we obtain

$$R_\mu{}^\beta \equiv R_{\mu\nu}{}^{\nu\beta} = -k(D-1)\delta^\beta_\mu, \tag{6.44}$$

and

$$R \equiv R_\mu{}^\mu = -kD(D-1). \tag{6.45}$$

If $D = 2$ and $k = 1/a^2$ we recover, in particular, the results of Exercise 6.2 concerning a two-dimensional spherical surface (modulo a sign difference, due to the negative signature we are using here for the space-like dimensions).

Let us conclude by noticing that the constant-curvature manifolds considered in this section are also called "maximally symmetric" manifolds. In fact, they always admit $D(D+1)/2$ isometries, which is the maximum allowed number of isometries in D space–time dimensions, as can be checked by solving the Killing equation (3.55) and finding all possible allowed Killing vectors (see also Sect. 7.4). A trivial example is provided by the Minkowski space–time in $D = 4$, which has vanishing constant curvature, and which admits the 10-parameter Poincaré group as the maximal group of isometries.

A less trivial example corresponds to the so-called "*de Sitter* space–time" describing a four-dimensional pseudosphere with constant positive curvature, which also admits a 10-parameter group of isometries (different from the Poincaré group), called "de Sitter group". This type of manifold, which can be obtained as an exact solution of the gravitational Einstein equations (see Chap. 7), seems to play a fundamental role in the description of the large-scale geometry during the primordial stages of the cosmological evolution (see e.g. the books [31, 33] of the bibliography). Other possible parametrizations of the de Sitter manifold, different from the stereographic one, are introduced and discussed in Exercises 2.2, 5.2 and 6.6).

6.4 Exercises Chap. 6

6.1 Rindler Metric Consider a Minkowski space–time geometry, and a transformation from the coordinates $x^\mu = (ct, x, y, z)$ of a given inertial system to the new coordinates $x'^\mu = (ct', x', y, z)$, where x' and t' are given by

$$x = x' \cosh\big(ct'\big), \qquad ct = x' \sinh\big(ct'\big). \tag{6.46}$$

Compute the space–time metric $g'_{\mu\nu}(x')$ in the new chart $\{x'^\mu\}$, and check that the curvature tensor associated to this metric is vanishing. Finally, compare the space–time region spanned by the new chart $\{x'^\mu\}$ to the region spanned by the chart $\{x^\mu\}$.

6.2 Gauss Curvature of a Spherical Surface Compute the components of the curvature tensor for a two-dimensional spherical surface of radius a described by the line-element (2.24), and check that the scalar curvature R corresponds to the Gauss curvature $2/a^2$.

6.3 A Property of the Riemann Tensor Show that

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}, \tag{6.47}$$

using the explicit definition of $R_{\mu\nu\alpha\beta}$ and the symmetry properties of the metric tensor.

6.4 Stereographic Coordinates Check that the Christoffel connection associated to the metric (6.38), written in stereographic coordinates, takes the form (6.39).

6.5 Euclidean Hypersphere Compute the hypersurface "area" and the hypervolume of an n-dimensional sphere Σ_n, with radius a and Euclidean signature. The manifold Σ_n is embedded in a $(n+1)$-dimensional Euclidean space parametrized by the coordinates X^A, and is represented by the equation

$$X_1^2 + X_2^2 + \cdots + X_{n+1}^2 = a^2, \qquad A = 1, 2, \ldots, n+1. \tag{6.48}$$

Suppose that the intrinsic geometry of the hypersphere is parametrized by n angular coordinates of spherical-polar type,

$$\xi^\mu = (a\theta_1, a\theta_2, \ldots, a\theta_{n-1}, a\varphi), \tag{6.49}$$

where

$$0 \le \theta_i \le \pi, \quad i = 1, \ldots, n-1, \quad 0 \le \varphi \le 2\pi. \tag{6.50}$$

6.6 Static Parametrization of the de Sitter Manifold Show that the line-element

$$ds^2 = \left(1 - \frac{r^2}{a^2}\right)c^2\,dt^2 - \left(1 - \frac{r^2}{a^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{6.51}$$

where a is a constant, describes in polar coordinates a four-dimensional space–time with constant positive curvature. Check that the above metric and the metric (2.42) of the Exercises 2.2 correspond to different parametrizations (both incomplete) of the same space–time manifold.

6.5 Solutions

6.1 Solution By differentiating Eq. (6.46) we obtain

$$\begin{aligned} dx &= dx' \cosh\left(ct'\right) + x' c\, dt' \sinh\left(ct'\right), \\ c\, dt &= dx' \sinh\left(ct'\right) + x' c\, dt' \cosh\left(ct'\right). \end{aligned} \tag{6.52}$$

Let us insert dx and dt, expressed in terms of dx' and dt', into the Minkowski line-element:

$$\begin{aligned} ds^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 \\ &= x'^2 c^2\,dt'^2 - dx'^2 - dy^2 - dz^2. \end{aligned} \tag{6.53}$$

By introducing a new metric $g'_{\mu\nu}(x')$ for the chart x'^{μ} the above line-element can be rewritten as

$$ds^2 = g'_{\mu\nu}(x')\, dx'^{\mu}\, dx'^{\nu}, \tag{6.54}$$

where

$$g'_{00} = x'^2, \qquad g'_{11} = g'_{22} = g'_{33} = -1. \tag{6.55}$$

The non-vanishing components of the connection associated to this metric are given by

$$\Gamma'^{\,0}_{01} = \Gamma'^{\,0}_{10} = \frac{1}{x'}, \qquad \Gamma'^{\,1}_{01} = x'. \tag{6.56}$$

By applying the definition (6.10) we then find that all the components of the Riemann tensor are vanishing. In fact, using the result (6.56), and applying the antisymmetry properties of the Riemann tensor (see Sect. 6.2), it follows that the only terms possibly different from zero are $R'_{101}{}^{0}$ and $R'_{100}{}^{1}$. But even in those cases we find

$$\begin{aligned} R'_{101}{}^{0} &= \partial_1 \Gamma'^{\,0}_{01} + \Gamma'^{\,0}_{10}\Gamma'^{\,0}_{01} = -\frac{1}{x'^2} + \frac{1}{x'^2} \equiv 0, \\ R'_{100}{}^{1} &= \partial_1 \Gamma'^{\,1}_{00} - \Gamma'^{\,1}_{00}\Gamma'^{\,0}_{10} = 1 - 1 \equiv 0. \end{aligned} \tag{6.57}$$

It should be noted that the result $R'_{\mu\nu\alpha\beta} = 0$ is an obvious consequence of the fact that the metric $g'_{\mu\nu}(x')$ has been obtained through a global coordinate transformation from the metric $\eta_{\mu\nu}$. Hence, by applying the inverse transformation, we can *always and everywhere* reduce $g'_{\mu\nu}$ to the Minkowski metric η, which has $\Gamma(\eta) = 0$ and then $R(\Gamma) = 0$.

The metric (6.55), however, cannot be extended to the whole Minkowski space–time but only to a portion of it, i.e. to the so-called "Rindler manifold". In fact, the coordinates x' and ct' do not cover all the Minkowski plane (x, ct), but only the region "external" to the light-cone bounded by the bisecting lines $x = \pm ct$.

This point can be checked by noting that the transformations (6.46) imply

$$\frac{ct}{x} = \tanh(ct'), \qquad x^2 - c^2t^2 = x'^2. \tag{6.58}$$

The first equation describes, at fixed t', a straight line crossing the origin of the plane (x, ct), and forming with the x axes an angle ranging from $-\pi/4$ to $\pi/4$. The second equations describes, at fixed x', a hyperbola centered on the origin of the plane (x, ct), which has asymptotes given by the light-cone $x = \pm ct$, and which intersects the x axes at the points $x = \pm x'$. By varying x' and t' from $-\infty$ to $+\infty$, and taking into account that the point $x' = 0$ has to be excluded (since the transformation (6.46) is singular at $x' = 0$), we find that the two curves of Eq. (6.58) span the portion of Minkowski plane defined by the conditions:

$$x > |ct|, \qquad x < -|ct|. \tag{6.59}$$

This region defines the so-called "Rindler manifold".

6.2 Solution It is convenient to multiply the angular coordinates by the radius a of the sphere, so as to use normalized variables with the correct dimensions of length: $x^1 = a\theta$, $x^2 = a\varphi$. With such coordinates the line-element (2.24) defines the dimensionless metric:

$$g_{11} = 1 = g^{11}, \qquad g_{22} = \sin^2\theta = \left(g^{22}\right)^{-1}, \tag{6.60}$$

and the non-vanishing components of the connection are given by

$$\Gamma_{22}{}^1 = -\frac{1}{a}\sin\theta\cos\theta, \qquad \Gamma_{12}{}^2 = \Gamma_{21}{}^2 = \frac{1}{a}\frac{\cos\theta}{\sin\theta}. \tag{6.61}$$

The non-vanishing components of the Riemann tensor, in two dimensions, are only $R_{121}{}^2$ and $R_{122}{}^1$. By applying the definition (6.10) we find

$$R_{121}{}^2 = -\frac{1}{a^2}, \qquad R_{122}{}^1 = \frac{1}{a^2}\sin^2\theta, \tag{6.62}$$

so that

$$R_{12}{}^{12} = -R_{12}{}^{21} = -\frac{1}{a^2}. \tag{6.63}$$

The associated scalar curvature,

$$R = R_{\mu\nu}{}^{\nu\mu} = R_{12}{}^{21} + R_{21}{}^{12} = \frac{2}{a^2}, \tag{6.64}$$

exactly corresponds to the Gauss curvature for a spherical surface of constant radius a (see also Eq. (6.45)).

6.3 Solution Let us check the tensor relation (6.47) in the locally inertial chart, where $g = \text{cost}$ and $\Gamma = 0$, but $\partial\Gamma \neq 0$ and $\partial^2 g \neq 0$. We set

$$R_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha}{}^{\rho}\, g_{\rho\beta}, \qquad R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu}{}^{\rho}\, g_{\rho\nu}, \tag{6.65}$$

and use the definition (6.10). From the first equation we have

$$\begin{aligned} R_{\mu\nu\alpha\beta}|_{\Gamma=0} &= g_{\beta\rho}\left(\partial_\mu \Gamma_{\nu\alpha}{}^{\rho} - \partial_\nu \Gamma_{\mu\alpha}{}^{\rho}\right) \\ &= \frac{1}{2} g_{\beta\rho}\partial_\mu\left[g^{\rho\sigma}\left(\partial_\nu g_{\alpha\sigma} + \partial_\alpha g_{\nu\sigma} - \partial_\sigma g_{\nu\alpha}\right)\right] - \{\mu \leftrightarrow \nu\}. \end{aligned} \tag{6.66}$$

Since $g_{\beta\rho}g^{\rho\sigma} = \delta^\sigma_\beta$ we find that the first contribution in square brackets disappears, and the above expression reduces to

$$R_{\mu\nu\alpha\beta}|_{\Gamma=0} = \frac{1}{2}(\partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\mu\partial_\beta g_{\nu\alpha}) - \frac{1}{2}(\partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\nu\partial_\beta g_{\mu\alpha}). \tag{6.67}$$

Similarly, we obtain

$$R_{\alpha\beta\mu\nu}|_{\Gamma=0} = \frac{1}{2}(\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\alpha\partial_\nu g_{\beta\mu}) - \frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\beta\partial_\nu g_{\alpha\mu}). \tag{6.68}$$

The two results (6.67) and (6.68) coincide so that, in the locally inertial chart, the relation (6.47) is satisfied. Being a tensor relation its validity obviously extends to any other system of coordinates.

6.4 Solution The partial derivative of the metric (6.38) is given by

$$\partial_\alpha g_{\mu\nu} = -\frac{k}{1+kx^2}(\eta_{\mu\alpha}x_\nu + \eta_{\nu\alpha}x_\mu) + \frac{2k^2}{(1+kx^2)^2}x_\mu x_\nu x_\alpha, \tag{6.69}$$

where $x^2 \equiv \eta_{\alpha\beta}x^a x^\beta$, and where the indices of the stereographic coordinates x^μ are raised and lowered with the Minkowski metric. From the metricity condition (6.41) we also obtain

$$\partial_\alpha g_{\mu\nu} = \Gamma_{\alpha\mu\nu} + \Gamma_{\alpha\nu\mu}. \tag{6.70}$$

With a cyclic permutation (see also Eq. (3.85) with $Q = 0$, $N = 0$) we are then led to

$$\begin{aligned}\Gamma_{\alpha\mu\nu} &= \frac{1}{2}(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu}) \\ &= -\frac{k}{1+kx^2}\eta_{\mu\alpha}x_\nu + \frac{k^2}{(1+kx^2)^2}x_\mu x_\nu x_\alpha \\ &\equiv -\frac{k}{1+kx^2}g_{\mu\alpha}x_\nu.\end{aligned} \tag{6.71}$$

In the second line we have used the result (6.69), in the third line the definition (6.38).

By inverting the matrix (6.38) we also find that the contravariant components of the metric are given by

$$g^{\mu\nu} = \eta^{\mu\nu} + kx^\mu x^\nu \tag{6.72}$$

(we can easily check that the relation $g^{\mu\alpha}g_{\mu\beta} = \delta^\alpha_\beta$ is identically satisfied). We then finally obtain

$$\begin{aligned}\Gamma_{\alpha\mu}{}^\beta \equiv g^{\beta\nu}\Gamma_{\alpha\mu\nu} &= -\frac{k}{1+kx^2}g_{\mu\alpha}x_\nu\left(\eta^{\beta\nu} + kx^\beta x^\nu\right) \\ &= -kg_{\alpha\mu}x^\beta,\end{aligned} \tag{6.73}$$

which exactly reproduces the result presented in Eq. (6.39).

6.5 Solution Let us adopt an induction procedure, starting with the two-dimensional sphere Σ_2.

For $n = 2$ we have $\xi^\mu = (a\theta_1, a\varphi)$, and the parametric equations describing the embedding of the two-sphere Σ_2 in the external Euclidean space are

$$\begin{aligned} X_1 &= a \sin\theta_1 \cos\phi, \\ X_2 &= a \sin\theta_1 \sin\phi, \\ X_3 &= a \cos\theta_1. \end{aligned} \tag{6.74}$$

Differentiating, and inserting the result into the Euclidean line-element, we obtain

$$ds^2 = \delta_{AB}\, dX^A\, dX^B = a^2\big(d\theta_1^2 + \sin^2\theta_1\, d\phi^2\big) \tag{6.75}$$

(see also Eq. (2.24)), which corresponds to the diagonal metric

$$g_{\mu\nu} = \mathrm{diag}\big(1, \sin^2\theta_1\big). \tag{6.76}$$

The covariant infinitesimal measure on the spherical surface is thus the following:

$$\sqrt{\det g_{\mu\nu}}\, d^2\xi = a^2 \sin\theta_1\, d\theta_1\, d\phi \tag{6.77}$$

and its integration gives the total surface

$$S_2(a) = a^2 \int_0^\pi d\theta_1 \sin\theta_1 \int_0^{2\pi} d\varphi = 4\pi a^2. \tag{6.78}$$

Finally, by integrating in dr, from 0 to a, a generic spherical surface $S_2(r)$ of radius r, we obtain the volume of three-dimensional Euclidean space enclosed within the given surface Σ_2:

$$V_3(a) = \int_0^a dr\, S_2(r) = \int_0^a dr\, 4\pi r^2 = \frac{4}{3}\pi a^3. \tag{6.79}$$

Let us repeat now the same procedure for the spherical manifold Σ_3 with $n = 3$ dimensions, parametrized by the intrinsic angular coordinates $\xi^\mu = (a\theta_2, a\theta_1, a\varphi)$. This manifold is described by the four parametric equations

$$\begin{aligned} X_1 &= a \sin\theta_2 \sin\theta_1 \cos\phi, \\ X_2 &= a \sin\theta_2 \sin\theta_1 \sin\phi, \\ X_3 &= a \sin\theta_2 \cos\theta_1, \\ X_4 &= a \cos\theta_2. \end{aligned} \tag{6.80}$$

Differentiating we get the line-element

$$ds^2 = a^2\big(d\theta_2^2 + \sin^2\, d\theta_2\, d\theta_1^2 + \sin^2\, d\theta_2 \sin^2\theta_1\, d\phi^2\big), \tag{6.81}$$

from which

$$\sqrt{\det g_{\mu\nu}}\, d^3\xi = a^3 \sin^2\theta_2 \sin\theta_1\, d\theta_2\, d\theta_1\, d\phi. \tag{6.82}$$

By integrating we finally obtain the "area" of the 3-dimensional spherical hypersurface,

$$S_3(a) = a^3 \int_0^\pi d\theta_2 \sin^2\theta_2 \int_0^\pi d\theta_1 \sin\theta_1 \int_0^{2\pi} d\varphi = 2\pi^2 a^3, \tag{6.83}$$

and the corresponding four-dimensional "hypervolume" of the enclosed Euclidean space:

$$V_4(a) = \int_0^a dr\, S_3(r) = \int_0^a dr\, 2\pi^2 r^3 = \frac{\pi^2}{2} a^4. \tag{6.84}$$

By extending the above procedure to the general case of a n-dimensional spherical hypersurface, parametrized by the n angular coordinates $\xi^\mu = (a\theta_1, \ldots, a\theta_{n-1}, a\varphi)$, we easily arrive at the following line-element of the hypersphere:

$$\begin{aligned} ds^2 = a^2\big(&d\theta_{n-1}^2 + \sin^2\theta_{n-1}\, d\theta_{n-2}^2 + \sin^2\theta_{n-1}\sin^2\theta_{n-2}\, d\theta_{n-3}^2 + \cdots \\ &+ \sin^2\theta_{n-1}\sin^2\theta_{n-2}\sin^2\theta_{n-3}\cdots\sin^2\theta_1\, d\varphi^2\big), \end{aligned} \tag{6.85}$$

which leads to the covariant measure:

$$\sqrt{\det g_{\mu\nu}}\, d^n\xi = a^n \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-1}\theta_{n-1}\, d\theta_1\, d\theta_2 \cdots d\theta_{n-1}\, d\varphi. \tag{6.86}$$

Hence:

$$S_n(a) = 2\pi a^n \int_0^\pi d\theta_1 \sin\theta_1 \int_0^\pi d\theta_2 \sin^2\theta_2 \cdots \int_0^\pi d\theta_{n-1} \sin^{n-1}\theta_{n-1}. \tag{6.87}$$

Using the result

$$\int_0^\pi \sin^p x\, dx = \frac{\sqrt{\pi}\,\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}, \tag{6.88}$$

where Γ is the Euler Gamma function,[1] we obtain

$$S_n(a) = 2\pi a^n \pi^{\frac{n-1}{2}} \left[\frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} \frac{\Gamma(2)}{\Gamma(\frac{5}{2})} \cdots \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\right]. \tag{6.89}$$

Inside the square brackets, all Gamma functions in the numerator cancel with those in the denominator, except the first one in the numerator and the last one in the denominator. The surface of the n-dimensional hypersphere is then the following:

$$S_n(a) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} a^n. \tag{6.90}$$

[1] See for instance [14].

The integral in dr finally gives the hypervolume of the Euclidean space enclosed within Σ_n:

$$V_{n+1}(a) = \int_0^a dr\, S_n(r) = \frac{2\pi^{\frac{n+1}{2}}}{(n+1)\Gamma(\frac{n+1}{2})} a^{n+1}. \tag{6.91}$$

6.6 Solution Let us work with the coordinates $x^\mu = (ct, r, \theta, \phi)$, and consider a spherically symmetric geometry of the type (6.51), described by the general metric

$$\begin{aligned} g_{00} = \quad & f(r) = \frac{1}{g^{00}}, \qquad g_{11} = -\frac{1}{f(r)} = \frac{1}{g^{11}}, \\ g_{22} = \quad & -r^2 = \frac{1}{g^{22}}, \qquad g_{33} = -r^2 \sin^2\theta = \frac{1}{g^{33}}, \end{aligned} \tag{6.92}$$

where f is a function only of r. The non-zero components of the Christoffel connection (denoting with a prime the derivative with respect to r) are given by

$$\begin{aligned} &\Gamma_{01}{}^0 = \frac{1}{2}\frac{f'}{f}, \qquad \Gamma_{00}{}^1 = \frac{1}{2} f f', \qquad \Gamma_{33}{}^1 = -r\, f \sin^2\theta, \\ &\Gamma_{11}{}^1 = -\frac{1}{2}\frac{f'}{f}, \qquad \Gamma_{22}{}^1 = -r\, f, \qquad \Gamma_{12}{}^2 = \frac{1}{r}, \\ &\Gamma_{33}{}^2 = -\sin\theta\cos\theta, \qquad \Gamma_{13}{}^3 = \frac{1}{r}, \qquad \Gamma_{23}{}^3 = \frac{\cos\theta}{\sin\theta}. \end{aligned} \tag{6.93}$$

An explicit computation of the Riemann tensor for this metric shows that $R_{\mu\nu\alpha}{}^\beta$ is non-zero only if $\mu = \alpha$ and $\nu = \beta$, or $\mu = \beta$ and $\nu = \alpha$. We thus obtain the following non-vanishing components:

$$\begin{aligned} &R_{01}{}^{01} = -\frac{1}{2} f'', \qquad R_{02}{}^{02} = R_{03}{}^{03} = R_{12}{}^{12} = R_{13}{}^{13} = -\frac{1}{2r} f', \\ &R_{23}{}^{23} = -\frac{1}{r^2}(f-1). \end{aligned} \tag{6.94}$$

The corresponding Ricci tensor is diagonal, with components:

$$\begin{aligned} R_0{}^0 &= R_1{}^1 = \frac{1}{2} f'' + \frac{1}{r} f', \\ R_2{}^2 &= R_3{}^3 = \frac{1}{r} f' + \frac{1}{r^2}(f-1). \end{aligned} \tag{6.95}$$

The scalar curvature is finally given by

$$R = \frac{4}{r} f' + f'' + \frac{2}{r^2}(f-1). \tag{6.96}$$

Let us now consider the particular case of the metric (6.51). In that case we have

$$f = 1 - \frac{r^2}{a^2}, \qquad f' = -2\frac{r}{a^2}, \qquad f'' = -\frac{2}{a^2}, \tag{6.97}$$

and from Eqs. (6.94)–(6.96) we directly obtain, respectively, the following non-zero components of the Riemann tensor:

$$R_{01}{}^{01} = R_{02}{}^{02} = R_{03}{}^{03} = R_{12}{}^{12} = R_{13}{}^{13} = R_{23}{}^{23} = \frac{1}{a^2}, \tag{6.98}$$

of the Ricci tensor:

$$R_0{}^0 = R_1{}^1 = R_2{}^2 = R_3{}^3 = -\frac{3}{a^2}, \tag{6.99}$$

and the scalar curvature:

$$R = -\frac{12}{a^2}. \tag{6.100}$$

A comparison with Eqs. (6.43)–(6.45), for $D = 4$, immediately leads us to conclude that the metric (6.51) describes a manifold with constant positive curvature $k = 1/a^2$. Such a metric thus corresponds to a static parametrization of the de Sitter space–time manifold.

It is instructive to compare the above parametrization with the different parametrization of the de Sitter space–time used in Exercise 2.2. The two different charts correspond indeed to a metric which in one case is static, while in the other is time-dependent. The two line-elements (6.51) and (2.42) are so different that we could even think they refer to physically different manifolds.

However, we can easily convince ourself that the two metrics represent the same manifold if we consider the hypersurface embedded in a 5-dimensional Minkowski space–time (parametrized by the coordinates z^A, $A = 1, \ldots, 4$) and described by the following parametric equations:

$$\begin{aligned} z^0 &= \sqrt{a^2 - r^2}\,\sinh\left(\frac{ct}{a}\right), \\ z^1 &= r\sin\theta\cos\varphi, \\ z^2 &= r\sin\theta\sin\varphi, \\ z^3 &= r\cos\theta, \\ z^4 &= \sqrt{a^2 - r^2}\,\cosh\left(\frac{ct}{a}\right). \end{aligned} \tag{6.101}$$

Such an hypersurface satisfies the equation

$$\eta_{AB} z^a z^B = -a^2, \tag{6.102}$$

and thus exactly reproduces the pseudosphere of Eq. (2.39), with radius $a^2 = c^2/H^2$. On the other hand, by differentiating equations (6.101) with respect to ct, r,

θ, φ, and inserting the result into the line-element of the 5-dimensional Minkowski space–time, we get

$$\begin{aligned} ds^2 &= \eta_{AB}\, dz^A\, dz^B \\ &= \left(1 - \frac{r^2}{a^2}\right) c^2\, dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big), \end{aligned} \tag{6.103}$$

namely, we exactly recover the line-element (6.51). This confirms that in both cases we are considering the same space–time manifold, parametrized by different systems of coordinates.

Let us conclude by noticing that neither the coordinates of Eq. (6.101), nor the coordinates of Exercise 2.2, Eq. (2.31), are able to provide a full covering of the de Sitter manifold (i.e. of the four-dimensional pseudosphere of Eq. (6.102)).

If we take the coordinates (2.31), for instance, then we can see that the condition $z^0 \geq -z^4$ is always satisfied as x^i and t are varied from $-\infty$ to $+\infty$ (the border condition $z^0 = -z^4$ is reached only in the limit $t \to -\infty$). If we take the sections $x^i = 0$ of the de Sitter manifold we then find, with the given coordinates, that we can only span the branch $z^4 > 0$ of the hyperbola $z_4^2 - z_0^2 = c^2/H^2$, but *not* the associated branch with $z^4 < 0$. The same is true for the coordinates defined by the parametrization of Eq. (6.101), which implies $z^0 \geq -z^4$ and $z^0 \leq z^4$.

The two charts we have considered are thus incomplete. A full covering of the de Sitter manifold (6.102) is provided, instead, by the chart $x^\mu = (ct, \chi, \theta, \varphi)$ defined by the following parametric equations:

$$\begin{aligned} z^0 &= cH^{-1} \sinh(Ht), \\ z^1 &= cH^{-1} \cosh(Ht) \sin\chi \sin\theta \cos\varphi, \\ z^2 &= cH^{-1} \cosh(Ht) \sin\chi \sin\theta \sin\varphi, \\ z^3 &= cH^{-1} \cosh(Ht) \sin\chi \cos\theta, \\ z^4 &= cH^{-1} \cosh(Ht) \cos\chi. \end{aligned} \tag{6.104}$$

Here $c/H = a$, and t ranges from $-\infty$ to $+\infty$, χ and θ range from 0 to π, while φ ranges from 0 to 2π (see e.g. the book [45] of the bibliography). We leave to the reader the check that, for this chart, the line-element of the de Sitter manifold takes the form

$$ds^2 = c^2\, dt^2 - \frac{c^2}{H^2} \cosh^2(Ht)\big[d\chi^2 + \sin^2\chi\big(d\theta^2 + \sin^2 d\varphi^2\big)\big]. \tag{6.105}$$

By setting $cH^{-1} \sin\chi = r$ this line-element can also be rewritten as

$$ds^2 = c^2\, dt^2 - \cosh^2(Ht)\left[\frac{dr^2}{1 - \frac{H^2}{c^2} r^2} + r^2\big(d\theta^2 + \sin^2 d\varphi^2\big)\right], \tag{6.106}$$

a form more frequently used in many cosmological applications.

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Chapter 7
The Einstein Equations for the Gravitational Field

With the inclusion of the Riemann tensor, introduced in the previous chapter, we have completed the list of the geometric ingredients needed to formulate a relativistic theory of gravity: the metric, the affine connection, and the curvature.

From the study of the geodesic equation we have learned that the connection—determined by the first derivatives of the metric—describes the effects of the gravitational forces, thus assigning to the metric the role of effective gravitational "potential". We have also learned, from the equation of geodesic deviation, that any physical gravitational field is unavoidably and unambiguously associated to a curvature tensor—which contains the square of the connection, hence the square of the first derivatives of the metric. A relativistic theory of the gravitational field, similar to the classical field theories based on second-order differential equations for the potentials, can thus be formulated in a straightforward way by including the metric into the matter part of the action through the minimal coupling procedure, and by using the curvature tensor as an effective "kinetic term" for the metric itself.

In this chapter we will present an action of this type, whose variation leads to the famous Einstein equations for the gravitational field. We shall develop all details of the required variational computation, which has some interesting unconventional side not always properly explained in many textbooks on general relativity. We shall then illustrate various physical aspects of those equations, with emphasis on the properties of the energy-momentum tensor, such as its role as source of the space–time curvature and the important implications of its covariant conservation equation.

7.1 Gravitational Action and Field Equations

Let us start with the matter action S_m for a general physical system ψ, described by the Lagrangian density $\mathcal{L}_m(\psi, \partial\psi)$. We can easily lift this action to a general-covariant (more precisely, diffeomorphism invariant) form, by coupling ψ to the

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geometry according to the "minimal coupling principle" (see Chap. 4):

$$S_m = \int_\Omega d^4x\,\sqrt{-g}\,\mathcal{L}_m(\psi, \nabla\psi, g). \tag{7.1}$$

Note that this generalized action contains both the connection Γ (implicitly present inside the covariant derivatives $\nabla\psi$) and the metric g. The metric, in particular, appears in both the Lagrangian (where it is needed for the definition of scalar products) and the covariant integration measure (see in particular Sect. 3.2).

Let us add to the above action a kinetic term for the metric, selected in such a way that it is related to the geometric curvature and is a scalar under general coordinate transformations. The simplest choice—corresponding to the so-called "Einstein–Hilbert action"—is the following:

$$S_{\mathrm{EH}} = -\frac{1}{2\chi}\int_\Omega d^4x\,\sqrt{-g}\,R, \tag{7.2}$$

where R is the scalar curvature (see Eq. (6.24)), and χ is a dimensional constant which controls the strength of the coupling between matter and geometry (and which we may regard, for the moment, as an arbitrary parameter). The precise value of χ will be determined in the next chapter; we can already notice, however, that within our conventions the action has dimensions of energy times length, $[S] = EL$, the scalar curvature has dimensions $[R] = L^{-2}$, hence the dimensions of χ are $[\chi] = E^{-1}L$.

It should be mentioned that a curvature-dependent, scalar action can be obtained also by self-contracting the components of the Riemann and Ricci tensors. We could consider, for instance, the following gravitational action:

$$S \propto \int_\Omega d^4x\,\sqrt{-g}\big(\alpha_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + R_{\mu\nu}R^{\mu\nu} + \alpha_3 R^2\big), \tag{7.3}$$

where α_1, α_2, α_3 are arbitrary coefficients. More generally, we could regard the Einstein–Hilbert term R/χ as the lowest-order contribution of a series of arbitrarily high powers of the curvature tensor and of its contractions. In such a case we should replace R/χ in Eq. (7.2) with an expression of the following type:

$$\frac{1}{\chi}\big(R + \lambda^2 R^2 + \lambda^4 R^3 + \lambda^6 R^4 + \cdots\big), \tag{7.4}$$

where R^n denotes the nth power of the curvature tensor, and where λ is a constant with dimensions of length, required for dimensionality reasons (indeed, all terms inside the round brackets must have dimensions L^{-2}).

Actually, it is well known that higher-order terms like those appearing in Eq. (7.4) can be induced by quantum-loop corrections to the Einstein action (7.2): one finds, in that case, that λ is related to the coupling constant χ by the relation $\lambda^2 \sim \hbar c\chi$, clearly showing that all such corrections disappear in the classical limit $\hbar \to 0$. In addition, corrections to the Einstein action in the form of an infinite power series

of curvature terms are also predicted by string theory (see e.g. the books [19, 22, 41, 58] of the bibliography): in that case λ coincides with the string length λ_s, the fundamental parameter of string theory.

Since the curvature contains the square of the metric derivatives, $R \sim (\partial g)^2$, it follows that higher powers of the curvature contain powers of ∂g higher than second, and lead to higher-order (and highly complicated) differential equations of motion. However, as clearly shown by the expansion (7.4), the higher-power terms may become important with respect to the linear term only for $\lambda^2|R| \gtrsim 1$, namely for values of the space–time curvature sufficiently larger than the scale λ^{-2} (i.e. for small enough values of the geometric curvature radius, $|R|^{-1/2} < \lambda$).

On the other hand, a curvature scale of the order of λ^{-2} is always much higher—in quantum theory as well as in the string-theory case—than the curvature scales typically associated to the macroscopic gravitational fields that will be considered in the present textbook. Hence, for our purpose, we can safely limit our attention to the Einstein action (7.2) (bearing in mind that its validity regime is bounded by the condition $\lambda^2|R| \ll 1$).

It is important to stress, at this point, that the connection contributes to the curvature with two different types of term, a linear one and a quadratic one, $R \sim \partial\Gamma + \Gamma^2$. The Einstein action thus contains, besides the square of the first derivatives of the metric, also terms which are linear in the second derivatives of the metric, $\partial\Gamma \sim \partial^2 g$. These last contributions, as we shall see, appear under the action integral in the form of a total divergence: hence, by applying the Gauss theorem, they lead to a "flux integral" (over the boundary of the considered space–time region Ω) of terms that are linear in the first derivatives of the metric. We can then write, in a compact symbolic way:

$$\int_\Omega \partial^2 g \sim \int_{\partial\Omega} \partial g. \tag{7.5}$$

By varying the Einstein–Hilbert action with respect to the metric we thus find a boundary contribution which is proportional to *the variation of the first derivatives of the metric*, $\delta\partial g$, and which is non-zero, in general, even if we impose, as usual, that the metric variation is vanishing ($\delta g = 0$) on the boundary $\partial\Omega$. In such a case, in fact, one gets zero variational contributions from the metric gradients along all directions lying on the hypersurface $\partial\Omega$, but a non-zero contribution from the gradient along the direction normal to $\partial\Omega$.

In order to completely eliminate the variational contribution of $\delta\partial g$ (and thus obtain the standard Euler–Lagrange equations) we need an additional piece of the action, represented by an integral over the boundary hypersurface $\partial\Omega$ and defined in such a way that its variation exactly cancels the contributions of all $\delta\partial g$ terms arising from the Einstein action. The full gravitational action we shall consider—in order to derive second-order equations for the metric through a correct application of the usual variational formalism—is thus the following:

$$S = S_{\mathrm{EH}} + S_{\mathrm{YGH}} + S_m. \tag{7.6}$$

The additional term S_{YGH} is called York–Gibbons–Hawking action (it is named from those who first clarified this important point of variational calculus [21, 56]), and its explicit form will be specified below.

We will now impose that the full action (7.6) is stationary with respect to local variations of the metric tensor, $\delta_g S = 0$, assuming the validity of the boundary condition $(\delta g)_{\partial\Omega} = 0$.

Let us start with the Einstein action S_{EH}. Considering all possible variational contributions, using the result

$$\delta_g \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{7.7}$$

(see Chap. 3, Eq. (3.95)), and recalling the definition (6.25) of the Einstein tensor $G_{\mu\nu}$, we obtain

$$\begin{aligned}
\delta_g S_{\mathrm{EH}} &= -\frac{1}{2\chi}\int_\Omega d^4x\, \delta\big(\sqrt{-g}R\big) = -\frac{1}{2\chi}\int_\Omega d^4x\, \delta\big(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}\big) \\
&= -\frac{1}{2\chi}\int_\Omega d^4x\, \big(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}R_{\mu\nu}\delta\sqrt{-g} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\big) \\
&= -\frac{1}{2\chi}\int_\Omega d^4x\, \sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right] \\
&= -\frac{1}{2\chi}\int_\Omega d^4x\, \sqrt{-g}\big[G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\big].
\end{aligned} \tag{7.8}$$

7.1.1 Boundary Contributions

The second term of the last line of the above equation corresponds to the previously mentioned boundary contribution. In fact, let us compute the variation of the Ricci tensor using its explicit definition (6.21):

$$\delta R_{\nu\alpha} = \partial_\mu\big(\delta\Gamma_{\nu\alpha}{}^{\mu}\big) + \delta\Gamma_{\mu\rho}{}^{\mu}\Gamma_{\nu\alpha}{}^{\rho} + \Gamma_{\mu\rho}{}^{\mu}\delta\Gamma_{\nu\alpha}{}^{\rho} - \{\mu \leftrightarrow \nu\}. \tag{7.9}$$

By applying the definition of covariant derivative we have

$$\nabla_\mu\big(\delta\Gamma_{\nu\alpha}{}^{\mu}\big) = \partial_\mu\big(\delta\Gamma_{\nu\alpha}{}^{\mu}\big) + \Gamma_{\mu\rho}{}^{\mu}\delta\Gamma_{\nu\alpha}{}^{\rho} - \Gamma_{\mu\nu}{}^{\rho}\delta\Gamma_{\rho\alpha}{}^{\mu} - \Gamma_{\mu\alpha}{}^{\rho}\delta\Gamma_{\nu\rho}{}^{\mu}, \tag{7.10}$$

and we are led to the result

$$\delta R_{\nu\alpha} = \nabla_\mu\big(\delta\Gamma_{\nu\alpha}{}^{\mu}\big) - \nabla_\nu\big(\delta\Gamma_{\mu\alpha}{}^{\mu}\big), \tag{7.11}$$

also known as the *contracted Palatini identity*. It follows that the contribution of $\delta R_{\mu\nu}$ to Eq. (7.8) can be written in the form of a total divergence as

$$
\begin{aligned}
&-\frac{1}{2\chi}\int_{\Omega} d^4x\,\sqrt{-g}\,g^{\nu\alpha}\delta R_{\nu\alpha} \\
&\quad=-\frac{1}{2\chi}\int_{\Omega} d^4x\,\sqrt{-g}\,\nabla_\mu\big(g^{\nu\alpha}\delta\Gamma_{\nu\alpha}{}^{\mu}-g^{\mu\alpha}\delta\Gamma_{\alpha\nu}{}^{\nu}\big)
\end{aligned}
\tag{7.12}
$$

(we have used the metric property $\nabla g=0$). It is important to stress that the argument of the covariant divergence (namely, the quantity inside the round brackets of the above equation) exactly behaves as a rank-one contravariant tensor, even if expressed in terms of the Christoffel connection (see Exercise 7.1 for an equivalent, but explicitly covariant, version of the same expression).

We can now apply the Gauss theorem (see Sect. 3.6) to represent the above variational contribution in the form of a flux integral over the boundary $\partial\Omega$ of the considered space–time domain Ω:

$$
\begin{aligned}
&-\frac{1}{2\chi}\int_{\partial\Omega} dS_\mu\,\sqrt{-g}\big(g^{\nu\alpha}\delta\Gamma_{\nu\alpha}{}^{\mu}-g^{\mu\alpha}\delta\Gamma_{\alpha\nu}{}^{\nu}\big) \\
&\quad=-\frac{1}{2\chi}\int_{\partial\Omega} d^3\xi\,\sqrt{|h|}n_\mu\big(g^{\nu\alpha}\delta\Gamma_{\nu\alpha}{}^{\mu}-g^{\mu\alpha}\delta\Gamma_{\alpha\nu}{}^{\nu}\big).
\end{aligned}
\tag{7.13}
$$

In the second line we have explicitly introduced the covariant volume-element $d^3\xi\sqrt{|h|}$ on the boundary hypersurface, oriented along the normal n^μ such that

$$
g_{\mu\nu}n^\mu n^\nu=\epsilon,\quad \epsilon=\pm 1 \tag{7.14}
$$

(the plus or minus sign corresponds, respectively, to the case of time-like or space–like normal). Also, h is the determinant of the so-called "induced metric" $h_{\mu\nu}$ on the hypersurface $\partial\Omega$, defined in such a way as to be tangent to the given hypersurface:

$$
h_{\mu\nu}=g_{\mu\nu}-\epsilon n_\mu n_\nu,\qquad h_{\mu\nu}n^\nu\equiv 0. \tag{7.15}
$$

Let us now explicitly evaluate the variational contribution (7.13) by imposing on the metric to be fixed on the boundary, $(\delta g)_{\partial\Omega}=0$. By using the definition (3.90) of the Christoffel connection, and omitting all terms associated to a vanishing contribution, we find

$$
\begin{aligned}
&\big[n_\mu\big(g^{\nu\alpha}\delta\Gamma_{\nu\alpha}{}^{\mu}-g^{\mu\alpha}\delta\Gamma_{\alpha\nu}{}^{\nu}\big)\big]_{\partial\Omega} \\
&\quad=n^\mu g^{\nu\alpha}\frac{1}{2}(\partial_\nu\delta g_{\alpha\mu}+\partial_\alpha\delta g_{\nu\mu}-\partial_\mu\delta g_{\nu\alpha}) \\
&\qquad-n^\alpha g^{\nu\mu}\frac{1}{2}(\partial_\alpha\delta g_{\nu\mu}+\partial_\nu\delta g_{\alpha\mu}-\partial_\mu\delta g_{\alpha\nu}) \\
&\quad=-g^{\nu\alpha}n^\mu\partial_\mu\delta g_{\nu\alpha}+n^\mu g^{\nu\alpha}\partial_\nu\delta g_{\alpha\mu}.
\end{aligned}
\tag{7.16}
$$

In order to separate metric gradients along normal and tangential directions (with respect to the boundary $\partial\Omega$) we can now conveniently apply the definition (7.15) of the induced metric $h_{\mu\nu}$. In terms of $h_{\mu\nu}$, in fact, the previous equation can be rewritten as

$$\begin{aligned}&\left(-g^{\nu\alpha}n^{\mu}+n^{\nu}g^{\mu\alpha}\right)\partial_{\mu}\delta g_{\nu\alpha}\\&\quad=\left[-n^{\mu}\left(h^{\nu\alpha}-\epsilon n^{\nu}n^{\alpha}\right)+n^{\nu}\left(h^{\mu\alpha}-\epsilon n^{\mu}n^{\alpha}\right)\right]\partial_{\mu}\delta g_{\nu\alpha}\\&\quad=-h^{\nu\alpha}n^{\mu}\partial_{\mu}\delta g_{\nu\alpha}+n^{\nu}h^{\mu\alpha}\partial_{\mu}\delta g_{\nu\alpha}.\end{aligned}\tag{7.17}$$

In the second term of the last line the gradient of δg is project—by the induced metric—tangentially to the hypersurface $\partial\Omega$. The boundary condition we are assuming implies that such a tangential contribution is vanishing, $(h^{\mu\alpha}\partial_{\mu}\delta g)_{\partial\Omega}=0$, so that we are left with the first term only, where the gradient of δg is project along the normal direction. The variational contribution of the Ricci tensor thus reduces to the final simplified form given by:

$$-\frac{1}{2\chi}\int_{\Omega}d^{4}x\,\sqrt{-g}g^{\nu\alpha}\delta R_{\nu\alpha}=\frac{1}{2\chi}\int_{\partial\Omega}d^{3}\xi\,\sqrt{|h|}h^{\nu\alpha}n^{\mu}\partial_{\mu}\delta g_{\nu\alpha}.\tag{7.18}$$

This term is in general non-vanishing, and has to be canceled by the variation of an (appropriate) additional piece of the action.

Let us consider, to this purpose, the boundary action S_{YGH} defined on $\partial\Omega$ and written in a general as

$$S_{\mathrm{YGH}}=-\frac{1}{2\chi}\int_{\partial\Omega}dS_{\mu}\,\sqrt{-g}V^{\mu}=-\frac{1}{2\chi}\int_{\partial\Omega}d^{3}\xi\,\sqrt{|h|}n_{\mu}V^{\mu}.\tag{7.19}$$

The vector V^{μ} is expected to contain the first derivatives of the metric, since the variation $\delta_g S_{\mathrm{YGH}}$ must cancel the contribution of Eq. (7.18). Apart from that, however, V^{μ} is not uniquely defined if we use a variational formalism where the metric and its tangential derivatives are kept fixed on the space–time boundary.[1] In that case, in fact, boundary terms differing for arbitrary functions of the metric $g_{\mu\nu}$, of the normal vector n_{μ}, and of their tangential derivatives $h^{\alpha\beta}\partial_{\beta}g_{\mu\nu}$, $h^{\alpha\beta}\partial_{\beta}n_{\mu}$, provide exactly the same contribution to $\delta_g S_{\mathrm{YGH}}$ (note that the variation of n^{μ} is obtained by differentiating Eq. (7.14), and is thus proportional to the variation of $g_{\mu\nu}$).

A possible example of boundary action, with a simple covariant formulation and a clear geometric interpretation, is obtained by considering the so-called "extrinsic curvature" $K_{\mu\nu}$ of the boundary, defined by:

$$K_{\mu\nu}=h_{\mu}^{\alpha}h_{\nu}^{\beta}\nabla_{\alpha}n_{\beta}=K_{\nu\mu},\qquad K_{\mu\nu}n^{\nu}=0,\tag{7.20}$$

[1] In particular, with an appropriate choice of the boundary action, it is always possible to obtain a total action which is simply quadratic in the connection (and thus contains the square of the first derivatives of the metric, without second derivatives), and which reproduces the same equations of motion as any other choice of $S_{\mathrm{EH}}+S_{\mathrm{YGH}}$ (see e.g. [32]).

and using as Lagrangian density the scalar projection

$$n_\mu V^\mu = 2K \equiv 2h^{\mu\nu}K_{\mu\nu} = 2h^{\mu\nu}\big(\partial_\mu n_\nu - \Gamma_{\mu\nu}{}^\alpha n_\alpha\big). \tag{7.21}$$

By performing the corresponding variation (omitting all terms whose variational contribution is vanishing because of our boundary conditions) we easily get

$$\begin{aligned}\delta\big(\sqrt{|h|}2K\big)_{\partial\Omega} &= 2\sqrt{|h|}h^{\mu\nu}\big(\partial_\mu \delta n_\nu - n_\alpha \delta\Gamma_{\mu\nu}{}^\alpha\big)\\ &= -2\sqrt{|h|}h^{\mu\nu}n_\alpha\delta\Gamma_{\mu\nu}{}^\alpha\\ &= -2\sqrt{|h|}h^{\mu\nu}n^\alpha\frac{1}{2}(\partial_\mu\delta g_{\nu\alpha} + \partial_\nu\delta g_{\mu\alpha} - \partial_\alpha\delta g_{\mu\nu})\\ &= \sqrt{|h|}h^{\mu\nu}n^\alpha\partial_\alpha\delta g_{\mu\nu}.\end{aligned} \tag{7.22}$$

Inserting this result into Eq. (7.19) we finally obtain

$$\delta_g S_{\rm YGH} = -\frac{1}{2\chi}\int_{\partial\Omega} d^3\xi\sqrt{|h|}\,h^{\mu\nu}n^\alpha\partial_\alpha\delta g_{\mu\nu}, \tag{7.23}$$

which exactly cancel the Ricci contribution (7.18). By adding Eqs. (7.8), (7.18) and (7.23) we are thus led to

$$\delta_g(S_{\rm EH} + S_{\rm YGH}) = -\frac{1}{2\chi}\int_\Omega d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}. \tag{7.24}$$

7.1.2 Contribution of the Matter Sources

In order to complete our variational procedure we need to vary the matter action (7.1). Taking into account that $\mathcal{L}_m$ may depend on $g_{\mu\nu}$ and on its derivatives we can write, in general,

$$\begin{aligned}\delta_g S_m &= \int_\Omega d^4x\left[\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(\partial_\alpha g^{\mu\nu})}\partial_\alpha\delta g^{\mu\nu} + \cdots\right]\\ &= \int_\Omega d^4x\left[\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(\partial_\alpha g^{\mu\nu})} + \cdots\right]\delta g^{\mu\nu}\end{aligned} \tag{7.25}$$

(in the second line we have applied the Gauss theorem and used the boundary condition $(\delta g)_{\partial\Omega} = 0$). We have omitted, for simplicity, all contributions with metric derivatives of second (and higher) order, as those terms are absent in the typical actions of most conventional physical systems. In any case, the result (7.25) can be expressed in a general compact form by introducing the symmetric tensor $T_{\mu\nu}$ such that

$$\delta_g S_m = \int_\Omega d^4x\,\delta_g(\sqrt{-g}\mathcal{L}_m) \equiv \frac{1}{2}\int_\Omega d^4x\,\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}. \tag{7.26}$$

Equivalently, using the language of the functional derivative, we can set

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{7.27}$$

where the symbol $\delta/\delta g^{\mu\nu}$ denotes the sequence of differential operations appearing inside the square brackets on the second line of Eq. (7.25)

7.1.3 Einstein Equations

By adding the variational contributions (7.24), (7.26), and imposing on the action to be stationary with respect to arbitrary variations $\delta g^{\mu\nu}$, we finally arrive at the Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \chi T_{\mu\nu}. \tag{7.28}$$

Tracing we have $G_\mu{}^\mu = -R = \chi T$, where $T = T_\mu{}^\mu$. By eliminating R in terms of T, according to the above trace relation, we can also rewrite the Einstein equations as

$$R_{\mu\nu} = \chi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \tag{7.29}$$

In the rest of this chapter we will discuss some important aspect of the above equations, starting with the physical interpretation of the tensor $T_{\mu\nu}$ that will be illustrated in the following section.

7.2 The Dynamical Energy-Momentum Tensor

The tensor object $T_{\mu\nu}$ defined by Eqs. (7.26), (7.27) is the so-called *dynamical* energy-momentum tensor (also called *metric* energy-momentum tensor).

The attribute "dynamical" can be easily explained by noting that this tensor plays the role of source for the space–time curvature, described by the left-hand side of the Einstein equations. The attribute "metric" clearly refers to its origin, namely to the fact that $T_{\mu\nu}$ has to be computed by varying the matter action with respect to the metric $g_{\mu\nu}$. Such a definition, by the way, automatically guarantees the symmetry property $T_{\mu\nu} = T_{\nu\mu}$. Much less clear, instead, is probably the reason why such a tensor should represent the energy and momentum density of the physical system described by the action S_m.

Let us recall, to this purpose, that in the first chapter of this book we have defined the canonical energy-momentum tensor as the "conserved current" associated to the translational invariance of the action (see in particular Sect. 1.2). Working in the context of a (special-relativistic) Minkowski space–time we have considered, in

particular, *global* space–time translations, depending on four constant parameters. In the context of a general Riemann geometry, however, the space–time manifold is not necessarily compatible with such a type of "rigid" symmetry based on global coordinate transformation. We have to consider instead the invariance of our physical system under *local* space–time translations, represented by transformations of the form

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{7.30}$$

where the parameters ξ^μ (which we shall be treat as infinitesimal) may depend on the space–time position.

In such a context, let us ask ourselves under which conditions a physical system described by the field ψ, embedded in a curved space–time and described by the generic matter action (7.1), turns out to be invariant under local infinitesimal translations. To get the answer let us apply the standard approach based on the Nöther theorem, already used in Sect. 1.2 for global translations in flat space–time. In the present case we have to compute the local variation of the action under the infinitesimal transformation (7.30) and to impose, as usual, the constraint that the (Euler–Lagrange) equations of motion for the field ψ are satisfied. Starting from the action (7.1) this leads to the condition

$$\delta_\xi S_m = \int_\Omega d^4x \left[\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi}\delta_\xi\psi + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\delta_\xi g^{\mu\nu} \right] = 0, \tag{7.31}$$

where $\delta_\xi\psi$ and $\delta_\xi g^{\mu\nu}$ denote, respectively, the local and independent variations of the field and of the metric, induced by the infinitesimal transformation (7.30), and computed to first order in the local parameter ξ^μ. They multiply the functional derivatives of the action density $\sqrt{-g}\mathcal{L}_m$, computed (at fixed position) with respect to ψ and $g^{\mu\nu}$. It should be stressed that there is no contribution to $\delta_\xi S_m$ directly arising from the variation of the coordinates themselves, $\delta x^\mu = \xi^\mu$, because both $d^4x\sqrt{-g}$ and $\mathcal{L}_m$ are scalar, invariant under general diffeomorphisms.

We can then notice that the first term of the above integrand exactly reproduces the Euler–Lagrange equations for the field ψ, and thus is vanishing if—according to the Nöther theorem—we are considering "on shell" configurations of the physical system, satisfying the classical equations of motion.

Considering the second term, we may recall that the local metric variation induced by an infinitesimal coordinate transformation of the type (7.30) has been already discussed in Sect. 3.3, and can be expressed in compact form as

$$\delta_\xi g^{\mu\nu} = \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu \tag{7.32}$$

(see in particular the solution to Exercise 3.4). We should also recall that the functional derivative of the action density defines the tensor $T_{\mu\nu}$, according to Eq. (7.27).

We thus obtain

$$\delta_\xi S_m = \frac{1}{2}\int_\Omega d^4x\,\sqrt{-g}\,T_{\mu\nu}\left(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu\right)$$
$$= \int_\Omega d^4x\,\sqrt{-g}\,T_{\mu\nu}\nabla^\mu\xi^\nu, \tag{7.33}$$

where we have exploited the symmetry of the energy-momentum tensor.

At this point, it is convenient to extract from the integrand a total divergence, and to rewrite the above equation as

$$\delta_\xi S_m = \int_\Omega d^4x\,\sqrt{-g}\left[\nabla_\mu\left(T_\nu{}^\mu\xi^\nu\right) - \xi^\nu\nabla_\mu T_\nu{}^\mu\right]. \tag{7.34}$$

The total-divergence term is equivalent—through the Gauss theorem—to the flux integral of a quantity proportional to $T_\mu{}^\nu$ on the boundary $\partial\Omega$ of the integration region. This integral is vanishing if the considered system is localized in a finite portion of space, and $T_{\mu\nu}$ approaches zero at a sufficiently fast rate at spatial infinity. In any case, a total divergence can also be absorbed into that part of the action that leads to the equations of motion, and gives no contribution to the variation $\delta_\xi S_m$.

We can thus conclude that the matter action is invariant under local infinitesimal translations, $\delta_\xi S_m = 0$, for arbitrary values of the parameter $\xi^\mu(x)$, if and only if $T_{\mu\nu}$ satisfies the covariant conservation law:

$$\nabla_\nu T_\mu{}^\nu = 0. \tag{7.35}$$

This entitles us to regard $T_{\mu\nu}$ as the correct generalized version of the energy-momentum tensor in the context of a (curved) space–time manifold with a Riemann geometric structure.

It is important to stress that the result (7.35) is also crucial for the formal consistency of the Einstein equations. In fact, the contracted Bianchi identity (6.26) prescribes a vanishing covariant divergence for the Einstein tensor, i.e. for the left-hand side of Eq. (7.28). Hence, the right-hand side too (i.e. $T_{\mu\nu}$) has to be divergenceless. The covariant divergence of $T_{\mu\nu}$, on the other hand, controls the local variation of the matter action under translations, according to Eq. (7.34). It follows that, for the consistency of the Einstein equations, the matter action has to be invariant under local infinitesimal translations—namely, under general diffeomorphisms of the type (7.30)—which also means that the sources must be coupled to the geometry in a general-covariant way.

The general covariance of the geometric theory we are considering—i.e. the intrinsic diffeomorphism invariance of the total action (7.6)—also emerges from the following remark.

The differential constraint

$$\nabla_\nu G_\mu{}^\nu = \chi\,\nabla_\nu T_\mu{}^\nu = 0 \tag{7.36}$$

imposes four conditions on the 10 components of the Einstein equations (7.28), and leaves us with six independent components. By solving such equations we can thus fix at most only six of the ten components of the metric tensor $g_{\mu\nu}$. A detailed study of the so-called "Cauchy problem" for the Einstein equations shows indeed that there are only six equations of true "dynamical" type, containing the second time-derivatives of the metric. The remaining four equations contain only first-order time derivatives, hence they represent "constraints" on the set of initial data, but do not contribute to determine the time evolution of the unknown metric components.

On the other hand, the presence in the metric of four arbitrary degrees of freedom is in full agreement with the covariance of the theory, by which it is always possible to change the coordinate system, $x^\mu \to x'^\mu$, and impose on the metric four arbitrary "gauge" conditions, thus fixing the residual degrees of freedom. Such conditions turn out to be very useful to simplify the Einstein equations, as we shall see explicitly in the next chapter.

7.2.1 Examples: Scalar and Vector Fields, Point-Like Sources

The dynamical tensor (7.27) generalizes the canonical energy-momentum to a version which is adapted to a covariant Riemannian context, and already automatically symmetrized. We will illustrate such properties, in this section, through the particular examples of a self-interacting scalar field, a massless vector field (the electromagnetic field) and a point-like massive particle.

Let us start with the scalar case, and consider a field ϕ which in Minkowski space is described by the Lagrangian density (1.64) (in units in which $\hbar = c = 1$). The corresponding covariant action in a generic Riemann manifold is obtained through the minimal coupling procedure (see Chap. 4), and is given by

$$S = \int_\Omega d^4x \, \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{7.37}$$

A comparison with Eq. (7.1) gives the following effective Lagrangian (or action density):

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\right). \tag{7.38}$$

We have explicitly written the scalar products in terms of the metric, since the above expression has be varied just with respect to $g^{\mu\nu}$, according to the definition (7.27).

For this particular case the effective Lagrangian depends on the metric but not on its derivatives, so that the functional derivative of Eq. (7.27) reduces to a simple partial derivative,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}}. \tag{7.39}$$

By using the result (7.7) we thus obtain

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\left[\frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V\right)\right]$$
$$= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + g_{\mu\nu}V(\phi), \tag{7.40}$$

which represents the covariant version of the canonical tensor (1.68) (already symmetric even in the canonical case, due to the absence of intrinsic angular momentum). It can be easily checked that the covariant divergence of this tensor is vanishing, provided the equations of motion of the scalar field are satisfied (see Exercise 7.2).

Let us repeat the same procedure for the electromagnetic field, which in a Riemann space–time is described by the covariant action (4.5). Let us consider, for simplicity, field propagation in the absence of sources, and set $J^\mu = 0$. The effective Lagrangian associated to the action (4.5) is

$$\sqrt{-g}\mathcal{L}_m = -\frac{\sqrt{-g}}{16\pi}\left(g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right), \tag{7.41}$$

and again we have no derivatives of the metric. By applying Eq. (7.39) we find

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\left[-\frac{\sqrt{-g}}{16\pi}2g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{2}\frac{\sqrt{-g}}{16\pi}g_{\mu\nu}F^2\right]$$
$$= -\frac{1}{4\pi}\left(F_\mu{}^\beta F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F^2\right), \tag{7.42}$$

where $F^2 \equiv F_{\alpha\beta}F^{\alpha\beta}$. We have thus obtained the covariant version of the canonical tensor already in its symmetrized form (see Eq. (1.74)).

It may be noted, at this point, that the definition of dynamical energy-momentum tensor can be used also in the flat space–time context, as a convenient technique to operate the symmetrization of the canonical tensor: we couple our system to a "fictitious" curved geometry, we vary with respect to the metric by applying the definition (7.27), and then we impose the limit $g_{\mu\nu} \to \eta_{\mu\nu}$.

Let us then consider, as our third example, a point-like particle, described in Minkowski space by the action (1.120) (see Exercise 1.4). In the presence of a curved geometry parametrized by the metric $g_{\mu\nu}$ the action becomes

$$S = mc\int_\Omega d^4x \int d\tau\,\sqrt{\dot{x}_\mu\dot{x}_\nu g^{\mu\nu}}\,\delta^4\big(x - x(\tau)\big) \tag{7.43}$$

(the sign has been chosen in order to match the conventions used for the Einstein action in Sect. 7.1). An important point to be stressed, in the above definition, is the absence of the factor $\sqrt{-g}$ in the integration measure over Ω: such an absence is due to the fact the distribution $\delta^4(x)$ transforms as a scalar density of weight $w = -1$

(see Sect. 3.2), so that $d^4x\,\delta^4(x)$ is already a true scalar under general coordinate transformations.

The corresponding effective Lagrangian density,

$$\sqrt{-g}\mathcal{L}_m = mc^2 \int d\tau\, \sqrt{\dot{x}_\mu \dot{x}_\nu g^{\mu\nu}}\, \delta^4\big(x - x(\tau)\big), \tag{7.44}$$

is localized with a delta-function distribution along the world-line of the particle, $x^\mu = x^\mu(\tau)$. This Lagrangian, again, depends on the metric but not on its derivatives, and by applying Eq. (7.39) we find

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\left[\frac{mc^2}{2}\int d\tau\, \frac{\dot{x}_\mu \dot{x}_\nu}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}}\, \delta^4\big(x - x(\tau)\big)\right]. \tag{7.45}$$

By identifying the parameter τ with the proper time of the particle we have $\dot{x}_\alpha \dot{x}^\alpha = c^2$ and we can write, finally,

$$T_{\mu\nu}(x) = \frac{mc}{\sqrt{-g}}\int d\tau\, \delta^4\big(x - x(\tau)\big) u_\mu u_\nu, \tag{7.46}$$

where $u_\mu = \dot{x}_\mu$ is the particle four-velocity along the particle world-line $x_\mu(\tau)$.

The above covariant expression generalizes the special-relativistic result (1.87) to a general-covariant context (we can check, in particular, that $\delta^4(x)/\sqrt{-g}$ exactly transforms as a scalar under general diffeomorphisms). The above expression can also be obtained from the alternative form of the free-particle action introduced in Eq. (5.2).

Let us finally notice that the energy-momentum tensor (7.46) can be written in an equivalent form which is *not* explicitly covariant, but is nevertheless convenient for our later applications. By parametrizing the particle trajectory with a new (generic) time coordinate t', and explicitly extracting the time component of the delta-function, we have

$$\begin{aligned} T_{\mu\nu}(\boldsymbol{x}, t) &= \frac{mc}{\sqrt{-g}}\int dt'\, \delta^4\big(x - x(t')\big) u_\mu \frac{dx_\nu}{dt'} \\ &\equiv c\int dt'\, \delta\big(x^0 - ct'\big) T_{\mu\nu}\big(\boldsymbol{x}, t'\big), \end{aligned} \tag{7.47}$$

from which

$$T_{\mu\nu}(\boldsymbol{x}, t) = \frac{m}{\sqrt{-g}}\delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big) u_\mu \frac{dx_\nu}{dt}, \tag{7.48}$$

or

$$T_{\mu\nu}(\boldsymbol{x}, t) = \frac{c}{\sqrt{-g}}\delta^3\big(\boldsymbol{x} - \boldsymbol{x}(t)\big)\frac{p_\mu p_\nu}{p^0}, \tag{7.49}$$

where $p^\mu = m\, dx^\mu/d\tau$. These two last expressions generalize, respectively, our previous results (1.83) and (1.85) to a space–time manifold with a general Riemannian structure.

7.3 The Einstein Equations with a Cosmological Constant

The Einstein action of Sect. 7.1 can be generalized by adding not only terms with higher powers of the curvature, but also terms with lower powers: we can add, in particular, a power-zero term, namely a constant. In fact, thanks to the determinant of the metric appearing in the covariant integration measure, even a constant Lagrangian may contribute to the dynamic of the gravitational equations.

Let us thus consider the following simple generalization of the Einstein action (7.2):

$$S = -\int_{\Omega} d^4x \sqrt{-g}\left(\frac{R}{2\chi} + \Lambda\right), \tag{7.50}$$

where Λ is a constant with dimensions of energy density. The variation of the new term with respect to the metric gives

$$\delta_g(-\sqrt{-g}\,\Lambda) = \frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\Lambda\, \delta g^{\mu\nu} \tag{7.51}$$

(see Eq. (7.7)). By adding the other variational contributions (see Eqs. (7.24), (7.26)) we are led to the following generalized equations:

$$G_{\mu\nu} = \chi(T_{\mu\nu} + g_{\mu\nu}\Lambda). \tag{7.52}$$

They are still compatible with the constraint (7.36), since $\nabla^{\nu} g_{\mu\nu} = 0$.

The new constant parameter Λ is usually called "cosmological constant" because, originally, it was introduced (by Einstein) just to obtain a set of equations for the cosmic gravitational field allowing solutions characterized by a time-independent geometry (thus implementing a model of "static" Universe). By solving (7.52) with the appropriate (positive) sign and the appropriate numerical value of Λ one finds, in fact, that the effective repulsive forces induced by Λ are able to counteract the gravitational attraction sourced by the matter described by $T_{\mu\nu}$, and to keep the Universe in a configuration of static equilibrium.

The presence (or at least the physical relevance) of the cosmological term $\Lambda g_{\mu\nu}$ in the gravitational equations was seriously challenged, in the past, by the large-scale astronomical observations, all of them clearly confirming—already since the discovery of the Hubble–Humason law—the "non-static" behavior of our Universe, and the continuous expansion of the cosmic geometry.

More recently, however, the importance (and even the necessity) of such a term has been reassessed, not only in the context of the "inflationary" models for the primordial Universe, but also for the present Universe (in view of the Supernovae observations attesting a state of accelerated expansion). In those case the role of the repulsive forces generated by Λ is no longer that of ensuring a static geometry, but of accelerating the time evolution of the cosmic geometry by overtaking the braking forces due to the other gravitational sources. The reader interested in these cosmological aspects is referred, for instance, to the books [11, 53] quoted in the bibliography.

Apart from the possible interpretations and applications of Λ in a cosmological context, it is clear, from Eq. (7.52), that the dynamical effect of introducing a constant term in the action is that of adding to the gravitational sources an effective energy-momentum tensor $\tau_{\mu\nu}$ proportional to the metric,

$$\tau_{\mu\nu} \equiv g_{\mu\nu}\Lambda. \tag{7.53}$$

This tensor can be formally interpreted as the energy-momentum of a perfect fluid with energy density $\rho = \Lambda$ and barotropic equation of state $p = -\rho$.

In fact, let us consider the fluido-dynamic tensor $T_{\mu\nu}$ introduced in Eq. (1.97), and let us apply the minimal coupling principle to switch from the Minkowski space to a more general geometric context. We obtain (in units $c = 1$)

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{7.54}$$

and we can immediately see that the result (7.53) for $\tau_{\mu\nu}$ is exactly reproduced if $\rho + p = 0$ and $-p = \rho = \Lambda$. But which kind of fluid, or which type of physical system, is characterized by such a form of the energy-momentum tensor?

According to Eq. (7.52) the contribution of $\tau_{\mu\nu}$ to the Einstein equations is independent on the matter Lagrangian, and is non-zero even in the absence of matter sources. This suggests the possibility of identifying $\tau_{\mu\nu}$ as an effective energy-momentum tensor associated *not* to a particular physical system, but to the space–time itself, *even if empty*. Indeed, if we include the zero-point energies of the quantum fluctuations of the vacuum—which are always present even if the corresponding classical fields or sources are vanishing—we find that the vacuum state of quantum field theories is characterized by a constant non-vanishing value of the averaged energy density, $\langle\rho\rangle \neq 0$, and that the expectation value of the corresponding energy-momentum tensor just takes the form:[2]

$$\langle T_{\mu\nu}\rangle = \langle\rho\rangle g_{\mu\nu}. \tag{7.55}$$

This suggests that the constant Λ could be interpreted as an averaged vacuum energy density. Like any other form of energy it contributes to the space–time curvature, acting as a gravitational source described by the effective energy-momentum tensor (7.53).

According to such an interpretation we can (and should) include into Λ all possible contributions to the vacuum energy density, of both classical and quantum type, taking into account all known interactions and their sources. For instance, a typical standard-model contribution is that provided by a scalar (Higgs-type) field ϕ, localized at an extremum of its potential $V(\phi)$. In that case, in fact, the equation of motion (7.94) (see Exercise 7.2) can be solved by setting $\phi = \phi_0$, where ϕ_0 is the value of the field at the extremum, such that $(\partial V/\partial\phi)_{\phi_0} = 0$. By inserting this

[2] See for instance [52]. The vacuum energy density is exactly vanishing only in the case of supersymmetric field-theory models. At low energy, however, supersymmetry is expected to be broken, and the vacuum of the broken phase has a non-zero energy density.

solution into the energy-momentum tensor (7.40) we obtain, for this scalar-field configuration,

$$T_{\mu\nu} = g_{\mu\nu} V(\phi_0), \tag{7.56}$$

which exactly reproduces Eq. (7.53) with $\Lambda = V(\phi_0)$.

However, to avoid conflicts with present observations of the cosmic large-scale geometry, the total value of Λ has to be extremely small:[3] in particular, it must satisfy the constraint $\Lambda \lesssim 6 \times 10^{-9}$ erg/cm^3, or, in units $\hbar = c = 1$, $\Lambda \lesssim 3 \times 10^{-47}$ GeV^4. The correct explanation and understanding of such a small scale is undoubtedly one of the main challenges of present theoretical physics.

Given the allowed values of Λ, its contribution to the field equations (7.52) can be safely neglected for all macroscopic and astronomic gravitational fields that will be studied in the subsequent chapters. From now on we shall thus use the Einstein equations *without* the cosmological constant. However, before forgetting its contribution, it should be mentioned that the presence of Λ leads to interesting solutions of the Einstein equations even in the absence of other sources.

By setting $T_{\mu\nu} = 0$, and tracing Eq. (7.52), we obtain, in fact,

$$R_{\mu\nu} = -\chi \Lambda g_{\mu\nu}, \qquad R = -4\chi\Lambda. \tag{7.57}$$

A comparison with Eqs. (6.44), (6.45) immediately shows that the cosmological constant induces on the space–time manifold a maximally symmetric geometry, with a constant curvature and a curvature parameter k which, in $D = 4$, is related to Λ by:

$$k = \frac{1}{3}\chi\Lambda. \tag{7.58}$$

In the presence of a positive cosmological constant—or, equivalently, of a perfect fluid with $\rho = -p = \text{cost}$, $\rho > 0$—the Einstein equations are then exactly solved by the de Sitter metric (see Sect. 6.3 and Exercise 6.6), describing a four-dimensional pseudo-hypersphere with constant curvature radius a, such that

$$a^2 = \frac{1}{k} = \frac{3}{\chi\Lambda}. \tag{7.59}$$

If $\Lambda < 0$ the solution describes instead a space–time with negative constant curvature, called *anti-de Sitter* manifold. This type of geometry does not seem to have cosmological or phenomenological applications, at present; however, it is a relevant formal ingredient of many supersymmetric models of gravity (see Chap. 14).

7.4 Energy-Momentum Conservation and Motion of Test Bodies

The object of this section is to illustrate how the covariant conservation equation (7.35) can be used to derive the equation of motion of a generic test body in a given

[3] See for instance *Particle Data Group*, at http://pdg.lbl.gov, for the last updated results.

(curved) space–time manifold. It will be shown, in particular, that the motion is geodesic only in the approximation in which the test body can be treated as a point-like particle (namely, as a body with negligible spatial extension and no internal structure).

In fact, in the presence of a composite internal structure, the gravitational field induces "tidal forces" among the various parts of the body: this, as we shall see, leads to an effective coupling between the internal momenta of the body (such as its intrinsic angular momentum) and the external space–time curvature. As a consequence, the trajectory of the motion deviates from a geodesic curve.

Let us start our discussion from Eq. (7.35), that we explicitly rewrite as

$$\begin{aligned}&\partial_\nu T^{\mu\nu} + \Gamma_{\nu\alpha}{}^{\mu} T^{\alpha\nu} + \Gamma_{\nu\alpha}{}^{\nu} T^{\mu\alpha} \\ &\quad = \partial_\nu T^{\mu\nu} + \Gamma_{\nu\alpha}{}^{\mu} T^{\alpha\nu} + \frac{1}{\sqrt{-g}}(\partial_\alpha \sqrt{-g}) T^{\mu\alpha} = 0 \end{aligned} \tag{7.60}$$

(we have used Eq. (3.97) for the trace of the Christoffel connection). By multiplying by $\sqrt{-g}$ we then obtain the condition

$$\partial_\nu \left(\sqrt{-g} T^{\mu\nu}\right) + \sqrt{-g}\, \Gamma_{\nu\alpha}{}^{\mu} T^{\alpha\nu} = 0, \tag{7.61}$$

which is equivalent to Eq. (7.35).

We will assume that the tensor $T_{\mu\nu}$ describes the energy-momentum distribution of a test body, i.e. of a physical system which does not affect in any significant way the geometry of the external space–time manifold, and which is localized on a finite portion of space. As a consequence, $T_{\mu\nu}$ will be non-zero only within a narrow (four-dimensional) "world-tube" centered around the (one-dimensional) "world-line" $z^\mu(t)$ describing the space–time evolution of the center of mass of the body.

In order to provide a direct explanation of how the resulting motion depends on the internal structure of the body, let us now integrate Eq. (7.61) over a suitable spatial hypersurface Σ which extends up to spatial infinity, and which intersects the world-tube of the body at a given time t. By separating the divergence into its space and times parts we have

$$\int_\Sigma d^3x\, \partial_i \left(\sqrt{-g} T^{\mu i}\right) + \frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, \sqrt{-g}\, T^{\mu 0} + \int_\Sigma d^3x\, \sqrt{-g}\, \Gamma_{\nu\alpha}{}^{\mu} T^{\alpha\nu} = 0. \tag{7.62}$$

By applying the Gauss theorem we find that the first term gives no contribution to the above equation (because $T_{\mu\nu}$, being localized in space, is vanishing at infinity), and the previous condition reduces to

$$\frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, \sqrt{-g}\, T^{\mu 0} + \int_\Sigma d^3x\, \sqrt{-g}\, \Gamma_{\nu\alpha}{}^{\mu} T^{\alpha\nu} = 0. \tag{7.63}$$

Let us first consider a point-like body, evolving along the space–time trajectory $x^\mu = z^\mu(t)$, and characterized by the energy-momentum distribution (7.48) (with

$x(t)$ obviously replaced by $z(t)$). The spatial integration can be easily performed thanks to the presence of $\delta^3(\boldsymbol{x} - \boldsymbol{z}(t))$, and we obtain

$$\frac{dp^\mu}{dt} + \Gamma_{\nu\alpha}{}^\mu p^\alpha \frac{dz^\nu}{dt} = 0, \tag{7.64}$$

where we have set $p^\mu = m\, dz^\mu/d\tau$. The multiplication by $dt/d\tau$ finally leads to the geodesic equation which—as already stressed in Sect. 5.1—describes the motion of a point-like particle in a generic Riemann manifold. This equation is also valid for an extended but structureless body, namely for a body with zero (or negligible) internal momenta.

In fact, let us consider the second term of Eq. (7.63) and expand the connection $\Gamma(x)$, inside the world-tube, in Taylor series around the center-of-mass position $z^\mu(t)$:

$$\Gamma_{\nu\alpha}{}^\mu(x) = \Gamma_{\nu\alpha}{}^\mu(z) + \left(\partial_\rho \Gamma_{\nu\alpha}{}^\mu\right)_z \left(x^\rho - z^\rho\right) + \cdots . \tag{7.65}$$

Let us suppose that the portion of tube intersecting the hypersurface Σ has a typical size $|\delta x| = |x - z|$ much smaller than the local space–time curvature radius: in that case $|\partial\Gamma||\delta x| \ll |\Gamma|$, and we can adopt a perturbative approach for all higher-order terms of the above equation. We can then write a sort of "multipole" expansion for the exact equation of motion (7.63):

$$\begin{aligned} &\frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, \sqrt{-g}\, T^{\mu 0} + \Gamma_{\nu\alpha}{}^\mu(z) \int_\Sigma d^3x\, \sqrt{-g}\, T^{\alpha\nu} \\ &\quad + \left(\partial_\rho \Gamma_{\nu\alpha}{}^\mu\right)_z \int_\Sigma d^3x\, \sqrt{-g}\, T^{\alpha\nu}\left(x^\rho - z^\rho\right) + \cdots = 0. \end{aligned} \tag{7.66}$$

Also, let us consider the divergence of $x^\alpha \sqrt{-g} T^{\mu\nu}$, which, by using Eq. (7.61), can be expressed as

$$\begin{aligned} \partial_\nu\left(x^\alpha \sqrt{-g} T^{\mu\nu}\right) &= \sqrt{-g} T^{\mu\alpha} + x^\alpha \partial_\nu\left(\sqrt{-g} T^{\mu\nu}\right) \\ &= \sqrt{-g} T^{\mu\alpha} - x^\alpha \sqrt{-g}\, \Gamma_{\nu\beta}{}^\mu T^{\beta\nu}. \end{aligned} \tag{7.67}$$

By integrating this equation over Σ and using the Gauss theorem we obtain

$$\begin{aligned} &\frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, \sqrt{-g}\, x^\alpha T^{\mu 0} - \int_\Sigma d^3x\, \sqrt{-g}\, T^{\mu\alpha} \\ &\quad + \int_\Sigma d^3x\, \sqrt{-g}\, \Gamma_{\nu\beta}{}^\mu T^{\beta\nu} x^\alpha = 0. \end{aligned} \tag{7.68}$$

The Taylor expansion of the connection finally gives

$$\frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\,\sqrt{-g}\,x^\alpha T^{\mu 0} - \int_\Sigma d^3x\,\sqrt{-g}\,T^{\mu\alpha} + \Gamma_{\nu\beta}{}^{\mu}(z)\int_\Sigma d^3x\,\sqrt{-g}\,T^{\beta\nu}x^\alpha + \left(\partial_\rho \Gamma_{\nu\beta}{}^{\mu}\right)_z \int_\Sigma d^3x\,\sqrt{-g}\,T^{\beta\nu}x^\alpha\left(x^\rho - z^\rho\right) + \cdots = 0. \tag{7.69}$$

Let us now consider the case of a test body for which all integrals of the type

$$\int_\Sigma d^3x\,\sqrt{-g}\,T^{\mu\nu}\delta x^\alpha, \quad \delta x^\alpha = x^\alpha - z^\alpha \tag{7.70}$$

(related to internal "dipolar" momenta) are vanishing, as well as all higher-order integrals of the type $\int T\delta x\delta x$, $\int T\delta x\delta x\delta x$, an so on (we are thus assuming the validity of a "monopole" approximation). In that case, by setting $x^\alpha = z^\alpha + \delta x^\alpha$ into Eq. (7.69), solving for the second integral in terms of the others, and inserting the result into Eq. (7.66) we obtain (after neglecting all multipolar and higher-order terms):

$$\frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\,\sqrt{-g}\,T^{\mu 0} + \Gamma_{\nu\alpha}{}^{\mu}(z)\frac{dz^\nu}{dt}\frac{1}{c}\int_\Sigma d^3x\,\sqrt{-g}\,T^{\alpha 0} = 0. \tag{7.71}$$

The definition

$$\frac{1}{c}\int_\Sigma d^3x\,\sqrt{-g}\,T^{\mu 0} = p^\mu, \tag{7.72}$$

generalizing the canonical result of Sect. 1.2, finally leads (in this approximation) to the geodesic equation (7.64).

However, if the internal momenta like those of Eq. (7.70) are not negligible, there are corrections to the equation of motion controlled by terms depending on the derivatives of the connection (see Eq. (7.66)), and thus related to the space–time curvature and its higher derivatives. As a useful (and interesting) example of such corrections we shall consider here one of the most typical cases: the motion of a body with a dipolar internal structure, i.e. the motion of a spinning test body.

We first note, for this purpose, that the equation of motion (7.66), as well as the momentum definition (7.72), are *not* expressed in an explicitly covariant form. Also, the quantity defined by Eq. (7.72) is not globally conserved (i.e. $\partial_\nu(\sqrt{-g}T^{\mu\nu}) \neq 0$, see Eq. (7.61)): hence, its value depends on the chosen integration hypersurface. The physical explanation of this second remark is that $T_{\mu\nu}$ correctly describes the energy-momentum of the test body, without including, however, the full contribution of the external gravitational field. It is well known, on the other hand, that for a composed system—like the interacting matter-geometry system—what is globally conserved is always the *total* energy-momentum distribution.

In order to derive an explicitly covariant version of the equations of motion we shall consider here the special (but physically realistic) case of a space–time geometry characterized by the presence of isometries, i.e. by the existence of one (or more) Killing vector ξ_μ (see Sect. 3.3). In that case we can easily define a globally conserved quantity (exactly like in Minkowski space–time) by projecting the energy-momentum tensor along the space–time direction selected by the given isometry.

Consider in fact the vector $J^\mu = T^{\mu\nu}\xi_\nu$ which, by construction, has a vanishing covariant divergence:

$$\nabla_\mu\left(T^{\mu\nu}\xi_\nu\right) = \xi_\nu \nabla_\mu T^{\mu\nu} + T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} \equiv 0 \tag{7.73}$$

(we have used the equations (7.35) and (3.107)). By integrating over the four-dimensional domain Ω, and applying the Gauss theorem, we find that the flux of J^μ on the boundary $\partial\Omega$ is vanishing,

$$\int_\Omega d^4x\,\sqrt{-g}\,\nabla_\mu\left(T^{\mu\nu}\xi_\nu\right) = \int_{\partial\Omega} dS_\mu\,\sqrt{-g}\,T^{\mu\nu}\xi_\nu = 0 \tag{7.74}$$

(for a localized $T_{\mu\nu}$ distribution). Let us now take for Ω the portion of space–time enclosed between two space-like hypersurfaces Σ_1 and Σ_2, intersecting the world-tube of the test body at two different times t_1, t_2 (see e.g. Fig. 1.1). By applying the same arguments presented in Sect. 1.2 (see in particular Eq. (1.33)) we can then conclude that the integral

$$\int_\Sigma dS_\mu\,\sqrt{-g}\,T^{\mu\nu}\xi_\nu = \text{const} \tag{7.75}$$

defines a conserved quantity, namely a quantity whose value does *not* depend on the particular choice of the hypersurface Σ.

The above conserved quantity depends on both $T_{\mu\nu}$ and the gravitational field (polarized along ξ_ν) present inside the world-tube of the test body. Since the integral only concerns the small section of the tube intersecting Σ, it can be evaluated by expanding ξ_ν around an arbitrary point of this section—in particular, around the position of the center of mass $z^\mu(\tau)$ (note that we are now using the proper time τ as evolution parameter).

To this aim let us first recall that, according to an important property of the Killing vectors, we can always express their second covariant derivatives in terms of the curvature tensor as follows:

$$\nabla_\alpha\nabla_\nu\xi_\mu = -R_{\mu\nu\alpha}{}^\beta\xi_\beta \tag{7.76}$$

(see Exercise 7.3). Thanks to this property, once ξ and its covariant derivative $\nabla\xi$ is given at a space–time point z, all higher derivatives of ξ in z are determined by Eq. (7.76) and by its derivatives, and can thus be expressed as a linear combinations of $\xi(z)$ and $\nabla\xi(z)$.

On the other hand, the value of the Killing vector at a generic point x around z can always be constructed as a Taylor series in the expansion parameter $\delta x = x - z$: hence, according to the above property, it turns out that $\xi_\mu(x)$ will be completely determined as a linear combination of $\xi_\mu(z)$ and $\nabla_{[\mu}\xi_{\nu]}(z)$ (we have selected the antisymmetric part since $\nabla_{(\mu}\xi_{\nu)} = 0$). The coefficients of the linear combination will depend on x, z and on the given geometry, and will be the same for all Killing vectors of that metric. Incidentally this also explains why a Killing vector, in a D-dimensional manifold, depends linearly on $D + D(D-1)/2 = D(D+1)/2$ parameters, and why there are at most $D(D+1)/2$ linearly independent Killing vectors.

What we need, in particular, is the expansion of $\xi(x)$ in power series inside the world-tube, around the trajectory $z(\tau)$ of the center of mass of the body. We can thus write, to first order,

$$\xi_\nu(x) = \xi_\nu(z) + A_\nu{}^\beta(x,z)\delta x^\alpha \nabla_{[\alpha}\xi_{\beta]}(z) + \cdots, \tag{7.77}$$

where $\delta x^\alpha = x^a - z^a$, and where $A_\nu{}^\beta$ is a function of x, z which depends on the given geometric background. Let us insert this expansion into Eq. (7.75) and divide by c, assuming that all internal momenta of order higher than the dipole are negligible. We then obtain

$$\frac{1}{c}\int_\Sigma dS_\mu \sqrt{-g}\, T^{\mu\nu}\xi_\nu = \xi_\nu(z)p^\nu + \frac{1}{2}\nabla_{[\alpha}\xi_{\beta]}(z)S^{\alpha\beta} = \text{const}, \tag{7.78}$$

where we have defined

$$\begin{aligned} p^\nu &= \frac{1}{c}\int_\Sigma dS_\mu \sqrt{-g}\, T^{\mu\nu}, \\ S^{\alpha\beta} &= \frac{1}{c}\int_\Sigma dS_\mu \sqrt{-g}\big(T^{\mu\nu}A_\nu{}^\beta \delta x^\alpha - T^{\mu\nu}A_\nu{}^\alpha \delta x^\beta\big). \end{aligned} \tag{7.79}$$

By recalling the results already obtained in the Minkowski space–time we can easily identify—according to the minimal coupling procedure—the first integral of the above equations with the conserved four-momentum p^ν of the test body, and the second integral with the angular momentum tensor $S^{\alpha\beta}$ (of intrinsic type, since it involves internal momenta). In the flat space–time limit, indeed, we have $\sqrt{-g} \to 1$, $A_\nu{}^\alpha \to \delta^\alpha_\nu$, and the definitions of Eq. (7.79) exactly reduce to those of Eqs. (1.36) and (1.57) introduced in Chap. 1.

It is important to stress that the conserved quantity (7.78) is independent of the proper-time parameter τ, namely it is constant along the trajectory $z^\mu(\tau)$. Its covariant derivative along the curve $z^\mu(\tau)$ (see Sect. 3.4.1) gives then

$$\xi_\nu \frac{Dp^\nu}{d\tau} + p^\nu \frac{dz^\mu}{d\tau}\nabla_{[\mu}\xi_{\nu]} + \frac{1}{2}\nabla_{[\alpha}\xi_{\beta]}\frac{DS^{\alpha\beta}}{d\tau} + \frac{1}{2}S^{\alpha\beta}\frac{dz^\mu}{d\tau}\nabla_\mu\nabla_\alpha\xi_\beta = 0. \tag{7.80}$$

By using the property (7.76), and setting $v^\mu = dz^\mu/d\tau$, we finally arrive at the condition

$$\xi_\nu\left(\frac{Dp^\nu}{d\tau} + \frac{1}{2}R_{\alpha\beta\mu}{}^\nu S^{\alpha\beta}v^\mu\right) + \frac{1}{2}\nabla_{[\alpha}\xi_{\beta]}\left(\frac{DS^{\alpha\beta}}{d\tau} + v^\alpha p^\beta - v^\beta p^\alpha\right) = 0, \tag{7.81}$$

where we have explicitly grouped the coefficients of ξ and $\nabla\xi$. This condition should be valid for any given choice of the Killing vector, hence the coefficients of both terms in ξ and $\nabla\xi$ must be vanishing. This separately implies the two equations of motion governing the evolution of p and S, along the world-line $z(\tau)$, for a spinning test body:

$$\frac{Dp^\mu}{d\tau} + \frac{1}{2}R_{\alpha\beta\nu}{}^\mu S^{\alpha\beta}v^\nu = 0, \tag{7.82}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = p^\alpha v^\beta - p^\beta v^\alpha. \tag{7.83}$$

In the absence of intrinsic angular momentum ($S^{\alpha\beta} \to 0$) we immediately recover the usual equation of geodesic motion, $Dp^\mu/d\tau = 0$. We also find $p^{[a}v^{\beta]} = 0$, which is indeed the case if the two vectors p^μ and v^μ are proportional.

In the presence of spin there is instead a "tidal" coupling of the test body to the space–time curvature, and the trajectory deviates from a geodesic according to the generalized equation (7.82) (also called Dixon–Mathisson–Papapetrou [10, 37, 40] equation). In addition, the "kinematic" velocity vector $v^\mu = dz^\mu/d\tau$ is no longer parallel to the direction of the energy-momentum flux, identified by p^μ. In order to close the dynamical system of equations for the 14 variables p_μ, v^μ, $S_{\mu\nu}$, the 10 independent components of Eqs. (7.82), (7.83) have to be supplemented by a further vector condition, for instance by the condition $p_\nu S^{\mu\nu} = 0$ which better specifies the world-line of the center of mass of the body.

7.5 Exercises Chap. 7

7.1 Variational Contribution of the Ricci Tensor Show that the variational contribution of the Ricci tensor, appearing in Eq. (7.8), can be written in explicitly covariant form as

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu\left(g_{\alpha\beta}\nabla^\mu\delta g^{\alpha\beta} - \nabla_\nu\delta g^{\mu\nu}\right). \tag{7.84}$$

Check that this expression immediately leads to the boundary contribution (7.16).

7.2 Energy-Momentum Conservation for a Scalar Field Show that the covariant divergence of the dynamical energy-momentum tensor (7.40) is vanishing, provided the equations of motion of the scalar field are satisfied.

7.3 Second Covariant Derivative of a Killing Vector Derive the result of Eq. (7.76) by exploiting the properties of the Killing vectors and of the Riemann curvature tensor.

7.6 Solutions

7.1 Solution In order to obtain Eq. (7.84) we may conveniently adopt a local inertial system (see Sect. 2.2), where $g = \text{cost}$, $\Gamma = 0$, $\partial\Gamma \neq 0$, and where we can set $\delta g = 0$, keeping, however, $\partial\delta g \neq 0$. Starting with the definition (6.21) of the Ricci tensor we have, in this system,

$$\begin{aligned}\delta R_{\mu\nu}|_{\Gamma=0} &= \partial_\alpha\big(\delta\Gamma_{\mu\nu}{}^{\alpha}\big) - \partial_\mu\big(\delta\Gamma_{\alpha\nu}{}^{\alpha}\big) \\ &= \frac{1}{2}g^{\alpha\beta}\partial_\alpha(\partial_\mu\delta g_{\nu\beta} + \partial_\nu\delta g_{\mu\beta} - \partial_\beta\delta g_{\mu\nu}) - \frac{1}{2}g^{\alpha\beta}\partial_\mu\partial_\nu\delta g_{\alpha\beta}\,. \end{aligned} \tag{7.85}$$

By tracing we obtain

$$\big(g^{\mu\nu}\delta R_{\mu\nu}\big)_{\Gamma=0} = \partial^\beta\partial^\nu\delta g_{\nu\beta} - g^{\alpha\beta}\partial_\mu\partial^\mu\delta g_{\alpha\beta}\,. \tag{7.86}$$

By switching from the inertial frame to a more general coordinate frame our partial derivatives become covariant derivatives, and we are led to

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\nabla^\mu\delta g_{\alpha\beta}\,. \tag{7.87}$$

By recalling that $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$, and using the metric property $\nabla g = 0$, we finally arrive at the result (7.84):

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g_{\alpha\beta}\nabla_\mu\nabla^\mu\delta g^{\alpha\beta} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} \\ &\equiv \nabla_\mu\big(g_{\alpha\beta}\nabla^\mu\delta g^{\alpha\beta} - \nabla_\nu\delta g^{\mu\nu}\big). \end{aligned} \tag{7.88}$$

The boundary contribution (7.16) can be easily obtained starting from the covariant result (7.87). The integration of Eq. (7.87) over the space–time domain Ω, and the application of the Gauss theorem, gives, in fact,

$$-\frac{1}{2\chi}\int_{\partial\Omega} d^3\xi\,\sqrt{|h|}\,n^\mu\big(g^{\nu\alpha}\nabla_\alpha\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\delta g_{\alpha\beta}\big). \tag{7.89}$$

On the boundary, however, $\delta g = 0$, so that the only non-zero contributions to the above integral are provided by the partial-derivative terms inside the round brackets. We are thus led to the boundary contribution

$$n^\mu g^{\nu\alpha}\partial_\alpha\delta g_{\mu\nu} - g^{\alpha\beta}n^\mu\partial_\mu\delta g_{\alpha\beta}, \tag{7.90}$$

which is exactly the same as that given in Eq. (7.16).

7.2 Solution Let first derive the covariant equation of motion for the scalar field ϕ, coupled to the space–time geometry according to the action (7.37).

The variation of the action with respect to ϕ leads to the following Euler–Lagrange equations:

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi} \equiv \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial\phi} - \partial_\mu \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(\partial_\mu\phi)} = 0, \tag{7.91}$$

where, for the action density (7.38),

$$\begin{aligned} \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial\phi} &= -\sqrt{-g}\,\frac{\partial V}{\partial\phi}, \\ \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(\partial_\mu\phi)} &= \sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi. \end{aligned} \tag{7.92}$$

The equation of motion thus takes the form

$$\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\big)\phi + \frac{\partial V}{\partial\phi} = 0, \tag{7.93}$$

or, by recalling the definition (3.105) of the covariant D'Alembert operator:

$$\nabla_\mu\nabla^\mu\phi + \frac{\partial V}{\partial\phi} = 0. \tag{7.94}$$

Let us now compute the covariant divergence of the tensor (7.40):

$$\begin{aligned} \nabla_\nu T_\mu{}^\nu &= \nabla_\nu\big(\partial_\mu\phi\partial^\nu\phi\big) - \frac{1}{2}\nabla_\mu\big(\partial_\alpha\phi\partial^\alpha\phi\big) + \nabla_\mu V \\ &= (\nabla_\nu\partial_\mu\phi)\partial^\nu\phi + \partial_\mu\phi\nabla^2\phi - (\nabla_\mu\partial_\alpha\phi)\partial^\alpha\phi + \frac{\partial V}{\partial\phi}\partial_\mu\phi, \end{aligned} \tag{7.95}$$

where $\nabla^2 \equiv \nabla_\nu\nabla^\nu$. In the second line, the second and fourth terms cancel out each other thanks to the equation of motion (7.94), while the first and third term cancel thanks to the index symmetry:

$$\nabla_\nu\partial_\mu\phi = \partial_\nu\partial_\mu\phi - \Gamma_{\nu\mu}{}^\alpha\partial_\alpha\phi = \nabla_\mu\partial_\nu\phi. \tag{7.96}$$

Hence

$$\nabla_\nu T_\mu{}^\nu = 0. \tag{7.97}$$

7.3 Solution Let us apply the result (6.19) for the commutator of two covariant derivatives acting on a vector,

$$\nabla_\mu\nabla_\nu\xi_\alpha - \nabla_\nu\nabla_\mu\xi_\alpha = -R_{\mu\nu\alpha}{}^\beta\xi_\beta, \tag{7.98}$$

and consider the totally antisymmetrized part of this equation. From the Bianchi identity (6.14) we have $R_{[\mu\nu\alpha]}{}^{\beta} = 0$; hence we obtain

$$\nabla_\mu \nabla_\nu \xi_\alpha + \nabla_\nu \nabla_\alpha \xi_\mu + \nabla_\alpha \nabla_\mu \xi_\nu - \nabla_\nu \nabla_\mu \xi_\alpha - \nabla_\mu \nabla_\alpha \xi_\nu - \nabla_\alpha \nabla_\nu \xi_\mu = 0. \tag{7.99}$$

On the other hand, by using the property (3.107) of the Killing vectors,

$$\nabla_\nu \xi_\alpha = -\nabla_\alpha \xi_\nu, \tag{7.100}$$

we can rewrite Eq. (7.99) as

$$\nabla_\mu \nabla_\nu \xi_\alpha - \nabla_\nu \nabla_\mu \xi_\alpha = \nabla_\alpha \nabla_\nu \xi_\mu. \tag{7.101}$$

Combining the above equation with Eq. (7.98) we are finally led to

$$\nabla_\alpha \nabla_\nu \xi_\mu = -R_{\mu\nu\alpha}{}^{\beta} \xi_\beta, \tag{7.102}$$

which reproduces the result of Eq. (7.76).

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Chapter 8
The Weak-Field Approximation

The Einstein equations introduced in the previous chapter connect the space–time curvature to the energy-momentum density of the matter sources. Such equations will receive here a definitive gravitational interpretation, obtained by comparing their linearized version with the equations of Newton's gravitational theory. In this way we will be able to fix the value of the constant χ which controls the coupling of matter and geometry, and which we have previously introduced as a free dimensional parameter.

By solving the linearized Einstein equations we will obtain the space–time geometry associated to a weak and static gravitational field: we will find, in this way, interesting dynamic effects and new types of interaction between sources and geometry which were absent in the Newtonian limit discussed in Chap. 5. We will discuss, in particular, two effects: the bending of light rays and the radar-echo delay. Both effects concern the propagation of electromagnetic signals in the gravitational field of our solar system; their experimental verification, in both cases, has provided important support to the idea of a geometric description of gravity, in general, and to the Einstein gravitational theory, in particular.

8.1 Linearized Einstein Equations

Consider a space–time geometry which is only slightly different from that of the Minkowski space–time so that, in a Cartesian chart, the metric $g_{\mu\nu}$ can be expanded around the Minkowski metric. We will set, to zeroth order, $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and, to first order, $g^{(1)}_{\mu\nu} = h_{\mu\nu}$.

Neglecting all terms of order higher than the first we have then the expansion

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \tag{8.1}$$

where the symmetric tensor $h_{\mu\nu}$ describes small geometric fluctuations which can be treated perturbatively. By inserting this expansion into the Einstein equations,

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and neglecting all terms of order h^2 and higher, we can obtain a system of linear differential equations for $h_{\mu\nu}$ determining, in this approximation, the dynamical evolution of the deviations from the Minkowski geometry.

Let us first notice, to this aim, that the covariant and contravariant components of h are connected (to first order) by the Minkowski metric:

$$\begin{aligned} h_\mu{}^\nu &= g^{\nu\alpha}h_{\mu\alpha} = \eta^{\nu\alpha}h_{\mu\alpha} + \mathcal{O}\big(h^2\big), \\ h \equiv h_\mu{}^\mu &= g^{\mu\nu}h_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} + \mathcal{O}\big(h^2\big). \end{aligned} \tag{8.2}$$

The inverse metric is then given by

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}, \tag{8.3}$$

and satisfies the condition

$$g^{\mu\alpha}g_{\nu\alpha} = \delta^\mu_\nu + h_\nu{}^\mu - h^\mu{}_\nu + \mathcal{O}\big(h^2\big) = \delta^\mu_\nu + \mathcal{O}\big(h^2\big). \tag{8.4}$$

Let us now compute the connection. To zeroth order we have the Minkowski metric, with a vanishing connection $\Gamma^{(0)\alpha}_{\mu\nu} = 0$. To first order in h, using Eqs. (8.1) and (8.3), we have

$$\Gamma^{(1)\beta}_{\nu\alpha} = \frac{1}{2}\eta^{\beta\rho}(\partial_\nu h_{\alpha\rho} + \partial_\alpha h_{\nu\rho} - \partial_\rho h_{\nu\alpha}). \tag{8.5}$$

Since the non-zero component of Γ are proportional to the gradients of h, we can then neglect all contributions of type Γ^2 when computing the first-order expression of the curvature tensor, and we obtain

$$\begin{aligned} R^{(1)}_{\mu\nu\alpha}{}^\beta &= \partial_\mu \Gamma^{(0)\beta}_{\nu\alpha} - \partial_\nu \Gamma^{(0)\beta}_{\mu\alpha} \\ &= \frac{1}{2}\eta^{\beta\rho}(\partial_\mu\partial_\alpha h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\alpha} - \partial_\nu\partial_\alpha h_{\mu\rho} + \partial_\nu\partial_\rho h_{\mu\alpha}). \end{aligned} \tag{8.6}$$

In order to write down the Einstein equations we need, in particular, the Ricci contraction, which in this approximation reduces to

$$R^{(1)}_{\nu\alpha} = R^{(1)}_{\mu\nu\alpha}{}^\mu = \frac{1}{2}\big(\partial_\mu\partial_\alpha h_\nu{}^\mu - \Box h_{\nu\alpha} - \partial_\nu\partial_\alpha h + \partial_\nu\partial_\rho h^\rho{}_\alpha\big), \tag{8.7}$$

where we have defined $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Inserting this result into the gravitational equations (7.29) we finally obtain

$$\frac{1}{2}\big(\partial_\mu\partial_\alpha h_\nu{}^\mu - \Box h_{\nu\alpha} - \partial_\nu\partial_\alpha h + \partial_\nu\partial_\rho h^\rho{}_\alpha\big) = \chi\left(T_{\nu\alpha} - \frac{1}{2}\eta_{\nu\alpha}T\right). \tag{8.8}$$

The above system of linear differential equations for the variable $h_{\mu\nu}$ provides a first-order approximation to the Einstein equations, valid for small deviations from the Minkowski geometry. In this approximation, and in order to be consistent with the contracted Bianchi identity, the tensor $T_{\nu\alpha}$ appearing on the right-hand side has

to be computed to the zeroth order in h (namely, it corresponds to the unperturbed energy-momentum tensor of the Minkowski space–time), and satisfies the standard conservation law $\partial^\nu T_{\mu\nu} = 0$ (see Exercise 8.1).

8.1.1 The Harmonic Gauge

The left-hand side of the previous equations can be further simplified, thanks to the covariance of our geometric model, by choosing a system of coordinates where the components of the metric tensor satisfy four appropriate "gauge conditions" (see the discussion of Sect. 7.2). In our context, in particular, it is convenient to consider the following conditions:

$$\partial_\nu \left(h_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu h \right) = 0, \tag{8.9}$$

which specify the so-called "harmonic gauge", or de Donder gauge (see also Exercise 8.2). By imposing such conditions we find that the first, third and fourth term of the Ricci tensor (8.7) exactly cancel out each other, and the linearized Einstein equations (8.8) are reduced to

$$\Box h_\nu{}^\alpha = -2\chi \left(T_\nu{}^\alpha - \frac{1}{2}\delta_\nu^\alpha T \right). \tag{8.10}$$

It is important to stress that we can always find a system of coordinates where the condition (8.9) is satisfied. In fact, let us consider the infinitesimal diffeomorphism connecting the original chart x^μ to a new chart $x'^\mu = x^\mu + \xi^\mu(x)$, where ξ satisfies the condition $|\partial_\alpha \xi^\mu| \ll 1$ (required for the validity of the expansion (8.1) and the consequent use of the linear approximation). The local variation of the metric under this "gauge" transformation has been computed in Sect. 3.3, and is given in general by Eq. (3.53). By inserting in that equation the expansion (8.1), i.e. by setting $g = \eta + h$, $g' = \eta + h'$, and neglecting terms of order h^2, ξ^2 and $h\xi$, we find

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{8.11}$$

Let us compute, in this new chart, the left-hand side of Eq. (8.9):

$$\partial_\nu \left(h'_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu h' \right) = \partial_\nu \left(h_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu h \right) - \Box \xi_\mu. \tag{8.12}$$

Considering the coordinate transformation generated by a vector ξ^μ which satisfies the condition

$$\Box \xi_\mu = \partial_\nu \left(h_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu h \right), \tag{8.13}$$

we can thus obtain a chart where the gauge condition (8.9) is satisfied. Note that, even if such a gauge condition already holds in our chart, we can still perform a

coordinate transformation and preserve the harmonic gauge, provided the generator of the transformation satisfies $\Box\xi_\mu = 0$. This procedure is very similar, formally, to the case of the Lorenz gauge of the electromagnetic theory (but with important physical differences due to the tensor nature of the field variable $h_{\mu\nu}$).

8.2 Space-Time Metric for a Weak and Static Field

We look now for solutions of the linearized equations (8.10) by assuming that the space–time geometry is time-independent ($\partial_0 h_{\mu\nu} = 0$) and that the sources are static (or, in any case, that they move with a negligible speed). The source stress tensor can then be approximated by setting $T_0^0 \simeq \rho c^2$, where ρ is the rest mass density, and $T_{ij} \simeq 0 \simeq T_{0j}$. In this limit $T \simeq T_0^0$, and Eq. (8.10) for h_{00} reduces to

$$\nabla^2 h_{00} = \chi \rho c^2. \tag{8.14}$$

Here $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the usual Laplace operator of three-dimensional Euclidean space.

It should be recalled, at this point, that we have already discussed in Sect. 5.2 the metric distortion predicted by the Newton theory of gravity in the limit of weak and static fields. According to Eq. (5.16) we have obtained, in particular, the result

$$h_{00} = g_{00} - \eta_{00} = \frac{2\phi}{c^2}, \tag{8.15}$$

where ϕ is the Newton gravitational potential. This value of h_{00} must now be recovered also in the context of the linearized Einstein equations, in order that such equations may correctly reproduce the known results of the Newtonian limit. Assuming that this is indeed the case, Eq. (8.14) becomes

$$\nabla^2\phi = \frac{1}{2}\chi\rho c^4. \tag{8.16}$$

The gravitational potential of the Newton theory, on the other hand, satisfies the well known Poisson equation,

$$\nabla^2\phi = 4\pi G\rho, \tag{8.17}$$

where G is the Newton constant. It follows that the Einstein equations provide a consistent generalization of Newton's gravitational theory—in the sense that they are able to reproduce the Newtonian results in the limit of weak and static fields—provided the parameter controlling the coupling of matter and geometry is fixed as follows:

$$\chi = \frac{8\pi G}{c^4}. \tag{8.18}$$

Note that this constant has dimensions $[\chi] = E^{-1}L$, as anticipated in Sect. 7.1.

Given the above identification, the linearized Einstein equation not only reproduce the old result for g_{00} in the Newtonian limit, but also provide new interesting results for the spatial part of the metric tensor.

In fact, let us consider Eq. (8.10) for the spatial components h_{ij}. With $T_{ij} = 0$ we get

$$\nabla^2 h_{ij} = \chi \delta_{ij} \rho c^2. \tag{8.19}$$

This equation is to be compared with Eq. (8.14) for h_{00} and its solution (8.15). Using the same coupling constant and the same boundary conditions the above equations imply $h_{ij} = \delta_{ij} h_{00}$, namely:

$$h_{ij} = \delta_{ij} \frac{2\phi}{c^2}, \tag{8.20}$$

from which

$$g_{ij} = \eta_{ij} + h_{ij} = -\delta_{ij} \left(1 - \frac{2\phi}{c^2} \right). \tag{8.21}$$

The full line-element solving the linearized Einstein equations, and describing the geometry generated by a weak and static field, is thus the following:

$$ds^2 = \left(1 + \frac{2\phi}{c^2} \right) c^2 \, dt^2 - \left(1 - \frac{2\phi}{c^2} \right) |d\boldsymbol{x}|^2, \tag{8.22}$$

where ϕ is a solution of Eq. (8.17).

Comparing this result with the line-element (5.29) (suggested by the pure Newtonian theory) we see that this approximate solution of the Einstein equations can reproduce all gravitational effects associated with the metric component g_{00} in the presence of weak and static sources (namely, the time-delay effects already discussed in Chap. 5). In addition, however, the solution (8.22) predicts (with the same sources) also a warping of the three-dimensional Euclidean space, hence new forms of gravitational interactions and further dynamic effects on the motion of test bodies and on the propagation of signals. Such effects will be discussed in the following sections.

8.3 The Bending of Light Rays

Let us consider an electromagnetic wave which propagates along a null geodesic of the metric (8.22) and which can be described, in the approximation of geometric optics, by the associated wave vector $k^\mu = (\boldsymbol{k}, \omega/c)$. Its geodesic path is determined (as discussed in Sect. 5.1) by the parallel transport of the vector k^μ, i.e. by the differential condition

$$dk^\mu + \Gamma_{\alpha\beta}{}^\mu \, dx^\alpha \, k^\beta = 0 \tag{8.23}$$

(see Eq. (5.10)).

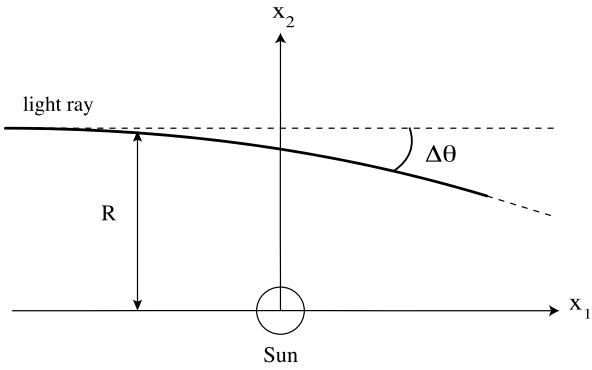

Fig. 8.1 Schematic view of the deflection process in the plane (x_1, x_2)

Let us suppose that the solution (8.22) describes a central gravitational field, generated by a source of mass M placed at the origin, so that $\phi = -GM/r$. Suppose, also, that the initial direction of the incident wave is asymptotically parallel to the x_1 axes, with impact parameter R (see Fig. 8.1). Considering the geodesic evolution of the wave in the plane (x_1, x_2) we want to compute the deflection angle $\Delta\theta$, to first order in ϕ/c^2.

We may assume, in particular, that we are working in the Sun's gravitational field, with a central mass $M \simeq 2 \times 10^{33}$ g, that the impact parameter is slightly larger than the Sun radius, i.e. $R \gtrsim 7 \times 10^{10}$ cm, and that the frequency of the incident wave is inside the visible band of the electromagnetic spectrum. In that case the wavelength $\lambda = 2\pi c/\omega$ of the light beam is much smaller than both the impact parameter and the curvature radius of the local space–time geometry, so that the approximation of geometric optics is well satisfied. In addition, the gravitational potential satisfies the condition $GM/Rc^2 \ll 1$, and the deviations from the Minkowski metric along the trajectory of the light ray are small enough to justify the use of the weak-field approximation.

In such a context we are entitled to assume that the deflection angle is small, $|\Delta\theta| \ll 1$, and that it can be safely identified with its tangent. We can then use the relation

$$\Delta\theta \simeq \frac{\Delta k^2}{k^1}, \tag{8.24}$$

where Δk^2 is the (small) component of the wave vector in the direction of the x_2 axes, arising under the action of the Sun gravitational field. The value of Δk^2 can be obtained from the infinitesimal variation predicted by the geodesic condition (8.23),

$$dk^2 = -\Gamma_{\alpha\beta}{}^2\, dx^\alpha\, k^\beta, \tag{8.25}$$

and integrating such variation along the whole trajectory.

In the weak-field approximation the connection is given by Eq. (8.5), and it is already a first-order quantity in ϕ/c^2. Working to first order we must then insert into Eq. (8.25) the zeroth-order expression of the vectors dx^α and k^β, namely the

vectors computed along the unperturbed trajectory of the light ray,

$$\begin{aligned} dx^\alpha &= \big(c\,dt, dx^1, 0, 0\big), \quad c\,dt = dx^1, \\ k^\beta &= \left(\frac{\omega}{c}, k^1, 0, 0\right), \quad \frac{\omega}{c} = k^1. \end{aligned} \tag{8.26}$$

Equation (8.25) thus reduces to

$$dk^2 = -\big(\Gamma_{00}{}^2 + 2\Gamma_{01}{}^2 + \Gamma_{11}{}^2\big)\frac{\omega}{c}\,dx^1. \tag{8.27}$$

With the metric (8.22), in particular, we have

$$\begin{aligned} \Gamma_{00}{}^2 &= \frac{1}{2}\partial_2 h_{00} = \partial_2\frac{\phi}{c^2}, \qquad \Gamma_{01}{}^2 = 0, \\ \Gamma_{11}{}^2 &= \frac{1}{2}\partial_2 h_{11} = \partial_2\frac{\phi}{c^2}. \end{aligned} \tag{8.28}$$

Hence

$$\begin{aligned} dk^2 &= -2\frac{\omega}{c^3}\partial_2\phi\,dx^1 = \frac{2\omega}{c^3}\partial_2\left(\frac{GM}{\sqrt{x_1^2 + x_2^2}}\right)dx^1 \\ &= -\frac{2\omega}{c^3}\frac{GMx_2}{(x_1^2 + x_2^2)^{3/2}}\,dx^1. \end{aligned} \tag{8.29}$$

This infinitesimal variation must to be integrated in dx^1, from $-\infty$ to $+\infty$, along the unperturbed trajectory (characterized by the condition $x_2 = R$). From Eq. (8.24) we obtain

$$\begin{aligned} \Delta\theta \simeq \frac{\Delta k^2}{k^1} &= \frac{c}{\omega}\int_{-\infty}^{+\infty}\big(dk^2\big)_{x_2=R} \\ &= -\frac{2GMR}{c^2}\int_{-\infty}^{+\infty}\frac{dx_1}{(x_1^2 + R^2)^{3/2}}. \end{aligned} \tag{8.30}$$

The integral in dx_1 can be easily solved by putting $x_1 = R\sinh z$, and gives

$$\int_{-\infty}^{+\infty}\frac{dx_1}{(x_1^2 + R^2)^{3/2}} = \frac{1}{R^2}\int_{-\infty}^{+\infty}\frac{dz}{\cosh^2 z} = \frac{1}{R^2}[\tanh z]_{-\infty}^{+\infty} = \frac{2}{R^2}. \tag{8.31}$$

In first approximation we thus arrive at the following deflection angle:

$$\Delta\theta \simeq -\frac{4GM}{Rc^2} \tag{8.32}$$

(also called the "Einstein angle").

For the Sun, and for a light ray (emitted from a distant star) which reaches our telescopes "grazing" the Sun—i.e. with an impact parameter of the order of the solar radius—the predicted angle of deflection is about 1.75 arc-seconds. This effect has been observed (for the first time in 1919) during the solar eclipses, and the theoretical prediction (8.32) has been repeatedly confirmed with a present level of accuracy of about one percent.

A better precision can be reached by measuring the deflection of radio waves (instead of light waves), i.e. of the signals we receive from astrophysical radio sources (like *quasars*), grazing the Sun. In this band of frequency there is no need to wait for an eclipse, and by using interferometric techniques—in particular VLBI, or *Very Long Baseline Interferometry*—the predictions of general relativity have been tested up to a precision of one part in 10^{-4}.

It should be stressed that the deflection mechanism illustrated in this section is at the ground of the so-called "gravitational lensing" effect, by which the gravitational field of cosmic bodies (like stars, galaxies, ...) is able to bend and to focus light rays, just like a transparent dielectric medium. This lensing effect can produce multiple images of the same object and, in particular, can make the image of a pointlike source into a series of arcs or circles of light (the so-called "Einstein rings"). This effect has also been observed,[1] and found to be in agreement with the predictions of Einstein's theory. The detailed study of the gravitational lensing effects represents today a powerful tool in many fields of astrophysics.

We note, finally, that the deflection angle of Eq. (8.32) is independent of the frequency (i.e. of the energy) of the incident wave. This is a consequence of the fact that the given wave (or signal, or massless test body) propagates along a null geodesic and satisfies the unperturbed dispersion relation $\omega(k) = ck$ (see Eq. (8.26)). If one considers instead the deflection of a massive body, propagating along time-like geodesics with an energy $E(p) = \hbar\omega = (c^2p^2 + m^2c^4)^{1/2}$, then, by repeating the previous calculations, one finds an energy-dependent deflection angle (see Exercise 8.3, Eq. (8.64)).

This means that, in the presence of a (small) photon mass m_γ, the gravitational field would behave like an optical prism, by deflecting different light frequencies at different angles and separating the colors present in a given light beam. The absence of any "prismatic effect" in all the observed phenomena of gravitational lensing thus gives an upper limit on the possible value of a photon mass m_γ. Such a limit, however, turns out to be much less stringent than other limits already existing on m_γ, due to experiments of more specific electromagnetic type.

8.4 The Radar-Echo Delay

Another interesting gravitational effect, also predicted by the solution (8.22) of the linearized Einstein equations, concerns a possible variation of the "time of flight"

[1] See for instance [35].

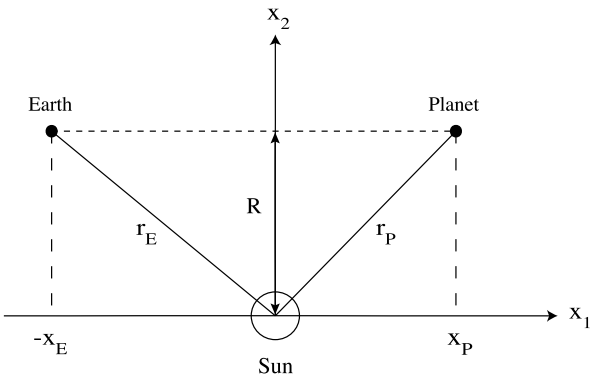

Fig. 8.2 Schematic view of the path of the radar signal in the plane (x_1, x_2)

of signals (and, in general, of test bodies) propagating in a given gravitational field, with respect to their propagation time in flat Minkowski space–time.

This effect can be illustrated by considering an electromagnetic wave (in particular, a radar signal) propagating in the solar gravitational field. The signal is emitted from the Earth, reflected by a planet, and received back on the Earth after passing at a minimum distance from the Sun equal to R (see Fig. 8.2). During the trip of the signal the relative motion of the planets is negligible, and we can assume that they are both fixed at a radial distance r_E and r_P from the origin. We will also assume, to get a first-order estimate of the time of flight, that the spatial path of the signal follows the unperturbed (straight line) trajectory, neglecting bending effects which would induce additional corrections, of higher order in the Newtonian potential.

In the absence of gravity the unperturbed trajectory, parallel to the x_1 axes, is covered with a velocity c, and the total time of round trip is obviously given by $2(x_P + x_E)/c$ (i.e., equal to the unperturbed distance divided by the unperturbed velocity). Let us ask how this result is changed, to first order, if we take into account that the space–time geometry is not the Minkowski one, but the one described by the line-element (8.22).

We note, to this aim, that the radar signal propagates along the null geodesics of the given geometry, so that its trajectory is characterized by the differential condition

$$\left(1 + \frac{2\phi}{c^2}\right)^{1/2} c\, dt = \left(1 - \frac{2\phi}{c^2}\right)^{1/2} dx_1, \tag{8.33}$$

namely (to first order in ϕ/c^2) by

$$dt = \frac{dx_1}{c}\left(1 - \frac{2\phi}{c^2}\right) = \frac{dx_1}{c}\left(1 + \frac{2GM}{rc^2}\right). \tag{8.34}$$

The time of the trip is thus affected by the gravitational potential, which warps both the time and spatial part of the geometry.

To compute the total time T of the round trip, to the first order in the gravitational corrections, we must integrate Eq. (8.34) along the unperturbed trajectory at fixed

$x_2 = R$. We then obtain

$$T = 2\int dt = \frac{2}{c}\int_{-x_E}^{x_P} dx_1\left(1 + \frac{2GM}{c^2\sqrt{x_1^2 + R^2}}\right)$$

$$= \frac{2}{c}(x_E + x_P) + \Delta t, \tag{8.35}$$

where Δt is the extra contribution due to the deviations from the Minkowski space–time:

$$\Delta t = \frac{4GM}{c^3}\int_{-x_E}^{x_P} \frac{dx_1}{\sqrt{x_1^2 + R^2}} = \frac{4GM}{c^3}\ln\left(\frac{\sqrt{x_P^2 + R^2} + x_P}{(\sqrt{x_E^2 + R^2} - x_E}\right)$$

$$= \frac{4GM}{c^3}\ln\left(\frac{r_P + x_P}{r_E - x_E}\right). \tag{8.36}$$

Since the argument of the logarithm is larger than one then the interval Δt is positive, and the effect of the gravitational field, in this case, is that of increasing the round-trip time (hence the name of radar-echo "delay").

It is evident from Eq. (8.36) that the smaller is the value of R, the larger is the influence of the solar gravitational field on the signal, so that the maximum value of the delay is obtained when R is just above the solar radius, namely when the Earth and the target planet are in the astronomical configuration called "conjunction". In that configuration we have $R \ll x_E, x_P$, and the argument of the logarithm can be approximated as follows:

$$\frac{r_P + x_P}{r_E - x_E} \simeq \frac{x_P(1 + \frac{R^2}{2x_P^2} + \cdots) + x_P}{x_E(1 + \frac{R^2}{2x_E^2} + \cdots) - x_E} \simeq \frac{2x_P 2x_E}{R^2}. \tag{8.37}$$

The time delay (8.36) then reduces to

$$\Delta t \simeq \frac{4GM}{c^3}\ln\left(\frac{4x_P x_E}{R^2}\right), \tag{8.38}$$

which represents the standard expression of the so-called "Shapiro effect" [48].

The above time delay (properly corrected by the additional time dilatation effects, locally due to the gravitational field present at the observer position) has been repeatedly measured by using, as target planets, both Mars and Venus. In the case of Mars, in particular, the radar signal has been reflected by the spacecraft *Viking*, after its landing on Mars in 1976. In that case the theoretical prediction for the time delay, Eq. (8.38), has been tested to an accuracy of one part in 10^3 thanks to an accurate experiment due to Reasenberg and Shapiro [44].

8.5 Velocity Measurements in the Presence of Gravity

The effect discussed in the previous section describes a delay, hence an effective "slowdown" of signals with respect to their trip time measured in the vacuum Minkowski space–time. It is as if the electromagnetic signals would propagate with an effective overall velocity which is smaller than c, because of the presence of the external gravitational "medium".

There is no doubt that the gravitational field can be *locally* eliminated, and that the *instantaneous* velocity of the signal (as well as the velocity of any massless particle) locally reduces to c in agreement with the laws of special relativity. The *mean* velocity between any two points in space is determined, however, by *non-local* measurements, and the effects due to the possible presence of a gravitational field, in that case, cannot be eliminated.

By taking into account proper-time distortion, one then finds that gravity can produce not only an effective "slowdown" of the propagation velocities, but also, in some cases, an effective "speedup", depending on the observer position and on the given kinematic configuration.[2] Let us illustrate this point with a simple computation performed in the weak-field geometry of Eq. (8.22), with a central gravitational potential $\phi = -GM/r$.

We will consider, for simplicity, the radial propagation of a light signal between two points of coordinates r_1 and r_2, with $r_1 < r_2$. The proper length of the trip, in a static geometry described by the line-element

$$ds^2 = g_{00}c^2\,dt^2 + g_{ij}\,dx^i\,dx^j, \tag{8.39}$$

is a constant, and is given in general by

$$\Delta\ell = \int_{r_1}^{r_2} \sqrt{|g_{ij}\,dx^i\,dx^j|}. \tag{8.40}$$

In our case, to first order in ϕ/c^2, we obtain from Eq. (8.22):

$$\Delta\ell = \int_{r_1}^{r_2} \left(1 - \frac{\phi}{c^2}\right) dr = r_2 - r_1 + \frac{GM}{c^2} \ln \frac{r_2}{r_1}, \quad r_2 > r_1. \tag{8.41}$$

The time interval along a radial null geodesic of the metric (8.22), on the other hand, is defined by the differential condition

$$dt = \frac{dr}{c}\left(1 - \frac{2\phi}{c^2}\right) \tag{8.42}$$

(see also Eq. (8.34)). The duration of the given trip, referred to the *proper* time of a static observer located at one end of the trajectory (say r_1), is thus given by (see

[2] This last effect has been emphasized, in particular, in connection with recent discussions of the possible existence (and detection) of superluminal particles, see e.g. [3, 34].

Eq. (5.30)):

$$\Delta\tau(r_1) = \sqrt{g_{00}(r_1)}\,\Delta t = \frac{1}{c}\left(1+\frac{\phi_1}{c^2}\right)\int_{r_1}^{r_2}\left(1-\frac{2\phi}{c^2}\right)dr$$
$$= \frac{1}{c}(r_2-r_1)\left[1-\frac{GM}{c^2 r_1}+\frac{2GM}{c^2(r_2-r_1)}\ln\frac{r_2}{r_1}\right], \quad r_2 > r_1. \tag{8.43}$$

The effective velocity $v(r_1)$, referred to an observer at the position r_1, is finally defined as follows:

$$\frac{v(r_1)}{c} = \frac{\Delta\ell}{\Delta\tau(r_1)} = 1+\frac{GM}{c^2 r_1}-\frac{GM}{c^2(r_2-r_1)}\ln\frac{r_2}{r_1}, \quad r_2 > r_1. \tag{8.44}$$

It can be easily checked that for $r_2 > r_1$ the above equation always gives $v(r_1) > c$, thus defining an effective "superluminal" propagation velocity. The opposite is true if the velocity is referred to an observer located at the opposite end of the trajectory ($r = r_2$), where the effective velocity is locally defined by

$$\frac{v(r_2)}{c} = \frac{\Delta\ell}{\Delta\tau(r_2)} = 1+\frac{GM}{c^2 r_2}-\frac{GM}{c^2(r_2-r_1)}\ln\frac{r_2}{r_1}, \quad r_2 > r_1. \tag{8.45}$$

In that case one always obtains $v(r_2) < c$, namely a "subluminal" velocity.

It should be stressed, however, that the terms subluminal and superluminal are somewhat misleading here, since the velocity $v(r_i)$, $i = 1, 2$, *is* indeed the effective mean velocity locally ascribed to the trip along the light-cone, in the warped geometry we have considered. And, being referred to light-cone propagation, it is $v(r_i)$ (and *not* c) that defines the maximum allowed (boundary) velocity for the propagation of physical signals exchanged between r_1 and r_2.

We note, finally, that $v(r_i) \to c$ for $r_2 \to r_1$, so that we always recover c as the instantaneous velocity of light-cone propagation for a *local* physical process (the same value as in flat Minkowski space–time, as expected).

8.6 Exercises Chap. 8

8.1 Bianchi Identity in the Linear Approximation Show that the linearized equation (8.8) is compatible with the contracted Bianchi identity provided the energy-momentum tensor satisfies the unperturbed conservation equation

$$\partial^\alpha T_{\mu\alpha} = 0. \tag{8.46}$$

8.2 Harmonic Gauge Show that the harmonic gauge condition $g^{\alpha\beta}\Gamma_{\alpha\beta}{}^\mu = 0$ can also be expressed in the equivalent form $\partial_\nu(\sqrt{-g}\,g^{\mu\nu}) = 0$, used in Sect. 3.6. Check that in the linear approximation such a condition reduces to Eq. (8.9).

8.3 Gravitational Deflection of a Massive Particle Working in the weak-field approximation, compute the angle of deflection of a particle of mass m impacting (with energy E and impact parameter R) on a central gravitational field described by the metric (8.22), with effective potential given by $\phi = -GM/r$.

8.4 Central Forces Linearly Depending on the Velocity Consider the deflection of a massive particle impacting on a field of central forces linearly depending on the particle velocity,

$$F^\mu = \frac{dp^\mu}{d\tau} = G^\mu{}_\nu u^\nu, \tag{8.47}$$

assuming that the field is static and that, in this limit, the forces are reduced to

$$\frac{dp^i}{d\tau} = G^i{}_0 u^0 = -m\frac{u^0}{c}\partial_i\phi = -\frac{p^0}{c}\partial_i\phi, \tag{8.48}$$

where $\phi = -GM/r$. Show that the deflection angle $\Delta\theta$, computed to first order in ϕ, approaches zero when the particle velocity approaches the light velocity.

8.7 Solutions

8.1 Solution The contracted Bianchi identity (see Eq. (6.30)), as well as the consistency of the Einstein equations written in the linear approximation, imply that the ordinary divergence of the left-hand side of Eq. (8.8) must be equal to the divergence of the right-hand side of the same equation.

The divergence of the left-hand side gives

$$\partial^\alpha R^{(1)}_{\nu\alpha} = \frac{1}{2}\partial_\nu\left(-\Box h + \partial^\alpha\partial_\rho h^\rho{}_\alpha\right). \tag{8.49}$$

On the other hand, by computing the trace of Eq. (8.8), we have

$$\partial_\mu\partial^\nu h_\nu{}^\mu - \Box h = -\chi T. \tag{8.50}$$

Hence, Eq. (8.49) can be rewritten as

$$\partial^\alpha R^{(1)}_{\nu\alpha} = -\frac{1}{2}\chi\,\partial_\nu T. \tag{8.51}$$

The divergence of the right-hand side of Eq. (8.8) gives

$$\chi\,\partial^\alpha T_{\nu\alpha} - \frac{1}{2}\chi\,\partial_\nu T. \tag{8.52}$$

The two equations (8.51), (8.52) are then compatible if and only if the energy-momentum tensor satisfies the usual conservation equation

$$\partial^\alpha T_{\nu\alpha} = 0. \tag{8.53}$$

8.2 Solution By using the definition of the Christoffel connection we have

$$g^{\alpha\beta}\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})$$
$$= g^{\alpha\beta}g^{\mu\nu}\partial_\alpha g_{\beta\nu} - \frac{1}{2}g^{\alpha\beta}\partial^\mu g_{\alpha\beta}. \tag{8.54}$$

Let us note that $\partial_\alpha(g^{\mu\nu}g_{\beta\nu}) = \partial_\alpha\delta^\mu_\beta = 0$, and use Eq. (3.96). We obtain

$$g^{\alpha\beta}\Gamma_{\alpha\beta}{}^{\mu} = g^{\alpha\beta}g_{\beta\nu}\partial_\alpha g^{\mu\nu} - \frac{1}{\sqrt{-g}}\partial^\mu\sqrt{-g}$$
$$= -\partial_\nu g^{\mu\nu} - \frac{1}{\sqrt{-g}}g^{\mu\nu}\partial_\nu\sqrt{-g}$$
$$= -\frac{1}{\sqrt{-g}}\partial_\nu\big(\sqrt{-g}g^{\mu\nu}\big). \tag{8.55}$$

The condition of harmonic gauge can thus be equivalently expressed in the two following ways:

$$g^{\alpha\beta}\Gamma_{\alpha\beta}{}^{\mu} = 0, \qquad \partial_\nu\big(\sqrt{-g}g^{\mu\nu}\big) = 0. \tag{8.56}$$

In the linear approximation we can use the expansion (8.1), (8.3) for the metric and Eq. (8.5) for the connection. Within such an approximation we find that the gauge condition $g^{\alpha\beta}\Gamma_{\alpha\beta}{}^{\mu} = 0$ reduces—modulo corrections of order h^2 and higher—to

$$g^{\alpha\beta}\Gamma_{\alpha\beta}{}^{\mu} = \eta^{\alpha\beta}\Gamma^{(1)\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta})$$
$$= \partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h = \partial_\alpha\left(h^{\mu\alpha} - \frac{1}{2}\eta^{\mu\alpha}h\right) = 0. \tag{8.57}$$

This equation exactly coincides with the condition (8.9) used in Sect. 8.1.1.

8.3 Solution We may consider the same configuration described in Sect. 8.3 for the deflection of a light ray, with the only difference that the wave vector k^μ is replaced by the four-momentum vector $p^\mu = (\boldsymbol{p}, E/c)$ of the massive test particle. Eq. (8.25) is then replaced by

$$dp^2 = -\Gamma_{\alpha\beta}{}^{2}\,dx^\alpha p^\beta, \tag{8.58}$$

where

$$dx^\alpha = \big(c\,dt, dx^1, 0, 0\big), \quad c\,dt = \frac{c}{v}dx^1 = \frac{E}{pc}dx^1,$$
$$k^\beta = \left(\frac{E}{c}, p, 0, 0\right), \quad E = \big(p^2c^2 + m^2c^4\big)^{1/2}. \tag{8.59}$$

Note that we have called p the initial momentum along the x_1 axes, and we have used the standard relativistic relation $p = Ev/c^2$ for a particle of velocity v along the unperturbed trajectory. Taking into account that ${\Gamma_{01}}^2 = 0$, Eq. (8.58) gives

$$dp^2 = -{\Gamma_{00}}^2 \left(\frac{E^2}{pc^2}\right) dx^1 - {\Gamma_{11}}^2 p\, dx^1, \tag{8.60}$$

so that, using for the connection the result (8.28), we obtain

$$dp^2 = -\frac{2p^2c^2 + m^2c^4}{pc^4} \frac{GMx_2}{(x_1^2 + x_2^2)^{3/2}} dx^1. \tag{8.61}$$

Let us now proceed exactly as in Sect. 8.3, dividing by the incident momentum and integrating the transverse momentum dp^2 along the whole unperturbed trajectory:

$$\begin{aligned} \Delta\theta \simeq \frac{\Delta p^2}{p} &= \frac{1}{p} \int \left(dp^2\right)_{x_2=R} \\ &= -\frac{2p^2c^2 + m^2c^4}{pc^2} \frac{GMR}{c^2} \int_{-\infty}^{+\infty} \frac{dx_1}{(x_1^2 + R^2)^{3/2}}. \end{aligned} \tag{8.62}$$

We should recall that this relation is valid for $|\Delta\theta| \ll 1$, so that the momentum p of the test body cannot be arbitrarily small (to avoid breaking, for instance, the approximation of geometric optics). By exploiting the solution of the integral (8.31) we finally obtain the result

$$\Delta\theta(p) = -\frac{2GM}{Rc^2} \left(2 + \frac{m^2c^2}{p^2}\right), \tag{8.63}$$

which can also expressed as a function of the total energy as

$$\Delta\theta(E) = -\frac{2GM}{Rc^2} \left(1 + \frac{E^2}{E^2 - m^2c^4}\right). \tag{8.64}$$

In the limit $m \to 0$ we exactly recover the Einstein angle (3.82), describing an energy-independent deflection.

8.4 Solution We will follow the procedure used also in the previous Exercise, by assuming that the particle has an initial momentum p asymptotically aligned along the x_1 axes, and an impact parameter R. The deflection angle, to first order in ϕ, is then given by

$$\Delta\theta \simeq \frac{\Delta p^2}{p} = \frac{1}{p} \int \left(dp^2\right)_{x_2=R}, \tag{8.65}$$

where

$$\left(dp^2\right)_{x_2=R} = \left(\frac{dp^2}{d\tau}\, d\tau\right)_{x_2=R} = \left(\frac{dp^2}{d\tau}\right)_{x_2=R} \left(\frac{m}{p}\right) dx^1. \tag{8.66}$$

Inserting dp^2 into Eq. (8.65), and using the static limit described by Eq. (8.48), we obtain the angle

$$\Delta\theta = -\frac{2GM}{Rc^2}\frac{mcp^0}{(p)^2}, \tag{8.67}$$

which can be written in terms of the velocity $v = p/(m\gamma)$ as follows:

$$\Delta\theta = -\frac{2GM}{Rc^2}\frac{c^2}{v^2\gamma} = -\frac{2GM}{Rc^2}\frac{c^2}{v^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2}. \tag{8.68}$$

Note that if $v \to c$ then $\Delta\theta \to 0$, so that a massless particle (like the photon) is unaffected by the field of forces we have considered.

It is instructive to compare this result with the one obtained in the previous exercise. The geodesic force predicted by general relativity is *quadratic* (instead of being linear, like in the case considered here) in the velocity of the test bodies. This implies, in particular, a different velocity dependence of the deflection angle. The general-relativistic angle (8.64), written as a function of the velocity $v = pc^2/E$, takes indeed the form

$$\Delta\theta(v) = -\frac{2GM}{Rc^2}\frac{c^2}{v^2}\left(1 + \frac{v^2}{c^2}\right) \tag{8.69}$$

(to be compared with Eq. (8.68)). When $v \to c$ the deflection in this case is nonvanishing, and one recovers the Einstein angle (8.32).

Chapter 9
Gravitational Waves

The linearized Einstein equations (8.10) describe the gravitational dynamics in the approximation in which the deviations from the Minkowski geometry, represented by $h_{\mu\nu}$, are small enough to be treated as first-order perturbations. Such an approximation can be successfully applied to the static gravitational field of many astrophysical sources, as we have seen in the previous chapter.

The linearized approximation holds, however, even if the perturbations $h_{\mu\nu}$ are time-dependent. In that case they describe geometric fluctuations which in vacuum propagate at the speed of light, and interact with a coupling strength determined by the Newton constant: they are the so-called gravitational waves, another new and important prediction of Einstein's theory of general relativity.

In this chapter we will introduce the main properties of these waves, focusing on the aspects that are at the grounds of the present techniques of detection. Since the coupling strength is extremely weak, a direct experimental detection of the gravitational waves is still lacking. However, thanks to the instruments allowed by present technology—some of which are already operating, other in the design stage—we can reasonably expect that such a detection will not be long awaited for (see for instance the books [8, 36] of the bibliography).

In any case, we should not forget that the gravitational waves have been already detected—although indirectly—through the observation of the orbital period of binary astrophysical systems. The emission of gravitational radiation from those systems, in fact, produces a decrease of the period which has been observed, experimentally measured, and found to be in agreement with the predictions of general relativity (see Sect. 9.2.4).

9.1 Propagation of Metric Fluctuations in Vacuum

In the absence of sources, $T_{\mu\nu} = 0$, the linearized equation (8.10) reduces to a wave equation for the propagation in the Minkowski vacuum of the symmetric tensor field $h_{\mu\nu}$,

$$\Box h_{\mu\nu} = 0, \quad h_{\mu\nu} = h_{\nu\mu}, \tag{9.1}$$

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satisfying the harmonic gauge condition (8.9):

$$\partial^{\nu} h_{\mu\nu} = \frac{1}{2}\partial_{\mu} h. \tag{9.2}$$

The above set of equations is formally very similar to the set of equations for the vacuum propagation of electromagnetic waves, $\Box A_{\mu} = 0$, where A_{μ} is the vector potential in the Lorenz gauge $\partial^{\mu} A_{\mu} = 0$. Since the d'Alembert operator is the same, in both cases the solutions describe signals propagating at the speed of light. There are important dynamical differences, however, due to the fact that $h_{\mu\nu}$ is a tensor while A^{μ} is a vector.

In fact, as already stressed in Chap. 2, the forces generated by identical static sources are attractive if they are transmitted by a tensor, repulsive if they are transmitted by a vector. The basic reason for this difference traces back to the fact that a tensor field, when quantized, corresponds to massless spin-2 particles (the gravitons), while the vector field corresponds to massless particles of spin 1 (the photons). From a classical point of view this aspect is reflected by the properties of the polarization states (in particular, by the helicity configurations) of the wave solutions, which we shall discuss now in detail for the tensor case.

9.1.1 Polarization and Helicity States

The symmetric tensor field $h_{\mu\nu}$ has in general 10 independent components, which are reduced to six after imposing the four conditions of the Lorenz gauge. Let us show that we can always apply four additional conditions on the solutions of Eqs. (9.1), (9.2), so as to end up with two independent components only. We will show, also, that such independent components can be chosen in such a way that $h_{\mu\nu} \neq 0$ only if the indices μ, ν refer to the spatial plane orthogonal to the direction of wave propagation.

Let us start with a general (retarded) solution of Eq. (9.1) describing (for instance) propagation along the x_1 axis, namely:

$$h_{\mu\nu}(\boldsymbol{x}, t) = h_{\mu\nu}\big(x^1 - ct\big). \tag{9.3}$$

The gauge condition (9.2) then implies

$$\partial^0 h_{\mu 0} + \partial^1 h_{\mu 1} = \frac{1}{2}\partial_{\mu} h. \tag{9.4}$$

On the other hand, given a function f which depends on $x^1 - ct$, we have, in general,

$$\partial_0 f\big(x^1 - ct\big) = -\partial_1 f\big(x^1 - ct\big) = \partial^1 f\big(x^1 - ct\big). \tag{9.5}$$

The gauge condition thus reduces to

$$\partial^1 (h_{\mu 0} + h_{\mu 1}) = \frac{1}{2}\partial_{\mu} h. \tag{9.6}$$

Let us now consider an infinitesimal diffeomorphism $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, generated by the vector ξ_μ such that, in the new chart $\{x'\}$, the metric fluctuations satisfy the condition:

$$h'_{\mu 0} = 0. \tag{9.7}$$

According to Eq. (8.11) such a transformation is defined by

$$\begin{aligned} h'_{\mu 0} &= h_{\mu 0} - \partial_\mu \xi_0 - \partial_0 \xi_\mu = 0, \\ \Box \xi_\mu &= 0, \end{aligned} \tag{9.8}$$

where the second condition on ξ_μ has been imposed in order to preserve the validity of the harmonic gauge (see Eq. (8.13)). The non-homogeneous system of equations (9.8) always admits nontrivial solutions for the unknown variables ξ_μ, so that we can always find a system of coordinates where Eq. (9.7) is valid.

In that system (dropping for simplicity the prime) we have $h_{\mu 0} = 0$, and the gauge condition (9.6) reduces to

$$\partial^1 h_{\mu 1} = \frac{1}{2} \partial_\mu h. \tag{9.9}$$

Consider, in particular, the component $\mu = 0$ of the above equation. Since $h_{01} = 0$ we get $\partial_0 h = 0$, from which $h =$ constant. This means that there are no dynamical degrees of freedom associated to the trace of the tensor field, so that we can always impose on our solution the condition

$$h = 0, \tag{9.10}$$

through an appropriate choice of boundary conditions. But if we set $h = 0$ in Eq. (9.9) we immediately obtain $h_{\mu 1} =$ constant. Hence we can also impose

$$h_{\mu 1} = 0, \tag{9.11}$$

modulo non-dynamical contributions which can be absorbed by the integration constants.

By combining the conditions (9.7), (9.10), (9.11) we find that the only nonzero components of the wave correspond to h_{22}, h_{23}, h_{32}, h_{33}, and satisfy the constraints $h_{23} = h_{32}$ (symmetry condition) and $h_{22} = -h_{33}$ (traceless condition). This means that, in the system of coordinates we have selected, the tensor field of the gravity wave has only two independent components, and it is nonzero only along directions lying on the plane orthogonal to the propagation direction. This choice of coordinates is also called "TT gauge"—i.e. transverse-traceless gauge—and in this particular case the harmonic gauge of Eq. (9.2) can be expressed by the two separate conditions

$$\partial^\nu h_{\mu\nu} = 0, \quad h = 0. \tag{9.12}$$

In this gauge, the diagonal and off-diagonal tensor components are usually denoted, respectively, by h_+ and $h_\times$. In our case, in particular, we have

$$h_+ = h_{22} = -h_{33}, \qquad h_\times = h_{23} = h_{32}, \tag{9.13}$$

and the solution of the gravity-wave equation, in the TT gauge, takes the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_+ & h_\times \\ 0 & 0 & h_\times & -h_+ \end{pmatrix}. \tag{9.14}$$

More generally, any solution of the linearized Einstein equations in vacuum—i.e. any gravitational wave freely propagating in the Minkowski space–time—can be represented (in the TT gauge) as a linear combination of its components h_+ and $h_\times$ by introducing two polarization tensors, $\epsilon^{(1)}_{\mu\nu}$, $\epsilon^{(2)}_{\mu\nu}$, such that

$$h_{\mu\nu} = \epsilon^{(1)}_{\mu\nu} h_+ + \epsilon^{(2)}_{\mu\nu} h_\times. \tag{9.15}$$

The two tensors $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are constant, traceless, and different from zero only in the plane orthogonal to the propagation direction. For a motion along the x_1 axis, in particular, we have

$$\epsilon^{(1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \epsilon^{(2)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{9.16}$$

These two tensors satisfy the "orthonormality" relation

$$\mathrm{Tr}\{\epsilon^{(i)}\epsilon^{(j)}\} \equiv \epsilon^{(i)}_{\mu\nu}\epsilon^{(j)\mu\nu} = 2\delta^{ij}, \quad i, j = 1, 2, \tag{9.17}$$

thus defining two linearly independent polarization states.

As in the case of electromagnetic waves, also in the gravitational case we can introduce states of circular polarization through a suitable combination (with complex coefficients) of the linear polarization states. The corresponding tensors of circular polarization, in particular, are given by

$$\epsilon^{(\pm)}_{\mu\nu} = \frac{1}{2}\left(\epsilon^{(1)}_{\mu\nu} \pm i\epsilon^{(2)}_{\mu\nu}\right). \tag{9.18}$$

They satisfy the orthonormality conditions

$$\begin{aligned} \mathrm{Tr}\{\epsilon^{(+)}\epsilon^{*(-)}\} &\equiv \epsilon^{(+)}_{\mu\nu}\epsilon^{*(-)\mu\nu} = 0, \\ \mathrm{Tr}\{\epsilon^{(+)}\epsilon^{*(+)}\} &= \mathrm{Tr}\{\epsilon^{(-)}\epsilon^{*(-)}\} = 1, \end{aligned} \tag{9.19}$$

following from Eq. (9.17). It is important to recall that the transformation properties of $\epsilon^{(\pm)}$ under rotations around the propagation axis are directly related to the so-called *helicity* of the wave, namely to the intrinsic angular momentum carried by the wave and projected along the propagation direction.

We can say, more precisely, that a wave ψ propagating along the $\hat{\boldsymbol{x}}$ axis has helicity h if, under a rotation of an angle θ around $\hat{\boldsymbol{x}}$, ψ is transformed as

$$\psi \to \psi' = e^{ih\theta}\psi. \tag{9.20}$$

In our case, for a plane wave propagating in the x_1 direction, we must consider the transformation

$$\epsilon'^{(\pm)}_{\mu\nu} = U_\mu{}^\alpha U_\nu{}^\beta \epsilon^{(\pm)}_{\alpha\beta}, \tag{9.21}$$

where

$$U_\mu{}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{9.22}$$

is the matrix representing a rotation around x_1. By inserting the explicit representation of the polarization tensors (9.16), (9.18) we easily find

$$\epsilon'^{(\pm)}_{\mu\nu} = e^{\pm 2i\theta}\epsilon^{(\pm)}_{\mu\nu} \tag{9.23}$$

(see Exercise 9.2). The gravitational waves are thus characterized by two states of circular polarization with helicity ± 2.

Summarizing the discussions of this section, we can say that (i) the solutions of the d'Alembert equation for the tensor perturbations of the four-dimensional Minkowski geometry contain two independent polarization states; (ii) the associated circular polarization states have helicity ± 2. These two results tell us that the gravitational wave solutions, if quantized according to the standard procedure of quantum field theory, describe particles with (i) zero mass and intrinsic angular momentum (i.e. spin) parallel or antiparallel to their propagation direction; also, (ii) their spin is 2 (in units $\hbar$). These particles are the *gravitons*, which represent the quanta of the gravitational field, just like the photons are the quanta of the electromagnetic field.

9.2 Radiation Emission in the Quadrupole Approximation

Let us now discuss the process of radiation emission, starting from the linearized equations (8.10), and including the matter sources. By computing the trace,

$$\Box h = \frac{16\pi G}{c^4} T, \tag{9.24}$$

and eliminating T in terms of h, Eq. (8.10) can be rewritten as

$$\Box\psi_\mu{}^\nu = -\frac{16\pi G}{c^4} T_\mu{}^\nu, \tag{9.25}$$

where

$$\psi_\mu{}^\nu = h_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu h, \qquad \partial_\nu \psi_\mu{}^\nu = 0, \qquad \partial_\nu T_\mu{}^\nu = 0. \tag{9.26}$$

Note that the harmonic gauge condition—i.e. the zero divergence of $\psi_{\mu\nu}$—is perfectly consistent with the conservation equation of the unperturbed energy-momentum tensor (in agreement with the contracted Bianchi identity, as discussed in Exercise 8.1).

The solutions of the above linear (but inhomogeneous) set of equations can now be obtained with the standard techniques of the retarded Green functions, and can be written in general as

$$\psi_{\mu\nu}(\boldsymbol{x}, t) = -\frac{4G}{c^4} \int d^3x' \, \frac{T_{\mu\nu}(\boldsymbol{x}', t')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{9.27}$$

Here $t' = t - |\boldsymbol{x} - \boldsymbol{x}'|/c$ is the so-called retarded time, and $T_{\mu\nu}$ is the energy-momentum of the sources evaluated in the Minkowski space–time, to zeroth order in the metric fluctuations. For the particular case of a static, point-like source of mass M, with $T_{00}(x') = Mc^2\delta^3(\boldsymbol{x}')$, Eq. (9.27) gives immediately:

$$\psi_{00} = -\frac{4GM}{c^2|\boldsymbol{x}|}, \tag{9.28}$$

in agreement with the solutions previously given in Eqs. (8.15), (8.20), and corresponding to $\psi_{00} = 4\phi/c^2$.

For a generic energy-momentum distribution the solution (9.27) can be approximated, at large enough distance from the sources, by expanding the denominator $|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$ in close analogy with the well-known case of the multipole expansion of the retarded electromagnetic potentials. If we look at the flux of emitted radiation, at large distances, we find, however, an important difference from the electromagnetic case.

To lowest order, in fact, the radiated electromagnetic power turns out to be controlled by the second time-derivative of the *dipole moment* of the sources ($dE/dt \propto |\ddot{\boldsymbol{d}}|^2$). The lowest-order emission of gravitational radiation is controlled, instead, by the third derivative of the *quadrupole moment* of the sources ($dE/dt \propto |\dddot{Q}|^2$). This is due to the fact that, for an isolated system of massive sources, the total momentum $p_T = \sum_i m_i \dot{\boldsymbol{x}}_i$ must be conserved, so that

$$\ddot{\boldsymbol{d}} \sim \frac{d^2}{dt^2} \sum_i m_i \boldsymbol{x}_i = \frac{d}{dt} \boldsymbol{p}_T = 0. \tag{9.29}$$

A similar argument applies to the emission of dipolar magnetic radiation, which is forbidden in the electromagnetic theory for the conservation of the total angular momentum.

Since a gravitational system cannot emit dipole radiation it follows that, to lowest-order, we can expect an outgoing flux of gravitational waves only from those systems characterized by a nonvanishing (and non-constant) quadrupole moment. In order to illustrate this point let us first show how, sufficiently far from the sources (in the so-called "radiation zone"), the solution of Eq. (9.27) for the field ψ is directly related to the quadrupole moment of the matter distribution.

9.2.1 Gravitational Field in the Radiation Zone

We start by considering the conservation equation $\partial_\nu T^{\mu\nu} = 0$, for the spatial components $\mu = i$:

$$\partial^k T_{ik} + \partial^0 T_{i0} = 0, \tag{9.30}$$

and for the time component $\mu = 0$:

$$\partial^k T_{0k} + \partial^0 T_{00} = 0. \tag{9.31}$$

Let us multiply Eq. (9.30) by x_j and integrate over a space-like hypersurface Σ, extending to infinity, defined by $t =$ constant:

$$\int_\Sigma d^3x\, \partial^k (x_j\, T_{ik}) - \int_\Sigma d^3x\, T_{ij} + \frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, T_{i0} x_j = 0. \tag{9.32}$$

By applying the Gauss theorem one finds that the first integral does not contribute (because $T_{ik} = 0$ at spatial infinity for a localized system), so that

$$\int_\Sigma d^3x\, T_{ij} = \frac{1}{2c}\frac{d}{dt}\int_\Sigma d^3x\, (T_{i0} x_j + T_{j0} x_i) \tag{9.33}$$

(we have symmetrized the right-hand side because T_{ij} is symmetric). Let us then multiply Eq. (9.31) by $x_i x_j$ and integrate, using again the Gauss theorem. We obtain

$$-\int_\Sigma d^3x\, T_{0k}\partial^k (x_i x_j) + \frac{1}{c}\frac{d}{dt}\int_\Sigma d^3x\, T_{00} x_i x_j = 0, \tag{9.34}$$

and then, inserting this result in the right-hand side of Eq. (9.33):

$$\int_\Sigma d^3x\, T_{ij} = \frac{1}{2c^2}\frac{d^2}{dt^2}\int_\Sigma d^3x\, T_{00} x_i x_j. \tag{9.35}$$

We will now assume that the sources of Eq. (9.27) are localized in a region centered around the origin of our coordinate system, with a typical extension characterized by the distance scale $\overline{x}$ (so that $T_{\mu\nu}(x') = 0$ for $|\boldsymbol{x}'| \gg \overline{x}$). If we are interested

in the emission of radiation of wavelength $\lambda \gg \overline{x}$, we may consider the solution of Eq. (9.27) at a large distance from the sources, in particular at a point P of coordinates $\boldsymbol{x}$ such that $|\boldsymbol{x}| \equiv R \gg \lambda$ (from which we have also $|\boldsymbol{x}| \gg \overline{x}$). In that limit (i.e., in the so-called radiation zone) we can expand the denominator of the integrand (9.27) by keeping only the zeroth-order term, namely by setting $|\boldsymbol{x} - \boldsymbol{x}'| \simeq |\boldsymbol{x}| = R$, and the solution becomes

$$\psi_{\mu\nu}(\boldsymbol{x}, t) = -\frac{4G}{Rc^4} \int d^3x' \, T_{\mu\nu}\big(\boldsymbol{x}', t'\big). \tag{9.36}$$

For the spatial components ψ_{ij}, in particular, we can use Eq. (9.35), where we put $T_{00} = \rho c^2$. We then obtain

$$\psi_{ij}(\boldsymbol{x}, t) = -\frac{2G}{Rc^4} \frac{d^2}{dt^2} \int d^3x' \, \rho\big(x', t'\big) x'_i x'_j. \tag{9.37}$$

We only need the spatial components of the tensor fluctuations because, in the radiation zone ($R \gg \lambda \gg \overline{x}$), the solution (9.27) can be approximated by a plane wave which, as discussed in Sect. 9.1.1, has nonvanishing components along spatial directions only—in particular, the directions orthogonal to the (radial, outgoing) propagation direction. In that regime we can also use the TT gauge, namely adopt a system of coordinates where the trace of the tensor field $h = -\psi$ is vanishing. In such a gauge we have $\psi_{ij} = h_{ij}$, and the solution (9.37) takes the form

$$h_{ij} = -\frac{2G}{3Rc^4} \ddot{Q}_{ij}, \tag{9.38}$$

where the dots denote time derivatives, and where we have introduced the (traceless) quadrupole moment of the sources,

$$Q_{ij} = \int d^3x' \, \rho\big(x', t'\big)\big(3x'_i x'_j - |\boldsymbol{x}'|^2 \delta_{ij}\big) \tag{9.39}$$

(to be evaluated at the retarded time t').

9.2.2 Energy-Momentum Tensor of a Gravitational Wave

For the computation of the energy flux radiated to infinity we need the energy-momentum $\tau_{\mu\nu}$ of the gravitational wave, in the TT gauge. In order to obtain $\tau_{\mu\nu}$ in the correctly normalized form it is convenient to start from the effective action of the metric fluctuations, i.e. from the action that varied with respect to $h_{\mu\nu}$ leads to the wave equation (9.1), after imposing the gauge conditions (9.12).

Let us consider, to this purpose, the Einstein action (7.2), and use for the metric the weak field approximation (8.1). By expanding the action S up to terms quadratic

in metric fluctuations, $S = S^{(0)} + S^{(1)} + S^{(2)}$, we can write the general form of the quadratic contribution $S^{(2)}$ as follows:

$$S^{(2)} = -\frac{1}{2\chi}\int d^4x \left[\left(\sqrt{-g}g^{\nu\alpha}\right)^{(2)} R^{(0)}_{\nu\alpha} + \left(\sqrt{-g}g^{\nu\alpha}\right)^{(0)} R^{(2)}_{\nu\alpha} + \left(\sqrt{-g}g^{\nu\alpha}\right)^{(1)} R^{(1)}_{\nu\alpha}\right]. \tag{9.40}$$

The first term of the above integral is vanishing because to zeroth order the metric coincides with the Minkowski one, hence $R^{(0)}_{\nu\alpha} = 0$. For the second term we have

$$\begin{aligned} \left(\sqrt{-g}g^{\nu\alpha}\right)^{(0)} &= \eta^{\nu\alpha}, \\ R^{(2)}_{\nu\alpha} &= \partial_\mu \Gamma^{(2)\mu}_{\nu\alpha} + \Gamma^{(1)\mu}_{\mu\rho}\Gamma^{(1)\rho}_{\nu\alpha} - \{\mu \leftrightarrow \nu\}, \end{aligned} \tag{9.41}$$

and we can separately consider various contributions.

The contributions linear in $\Gamma^{(2)}$,

$$\eta^{\nu\alpha}\partial_\mu \Gamma^{(2)\mu}_{\nu\alpha}, \qquad -\eta^{\nu\alpha}\partial_\nu \Gamma^{(2)\mu}_{\mu\alpha}, \tag{9.42}$$

can be written as total divergences, which do not affect the equations of motion of $h_{\mu\nu}$. The first contribution quadratic in $\Gamma^{(1)}$ can be explicitly written as

$$\eta^{\nu\alpha}\Gamma^{(1)\mu}_{\mu\rho}\Gamma^{(1)\rho}_{\nu\alpha} = \Gamma^{(1)\mu}_{\mu\rho}\frac{1}{2}\eta^{\beta\rho}\left(\partial^\alpha h_{\alpha\beta} + \partial^\nu h_{\nu\beta} - \partial_\beta h\right) \equiv 0, \tag{9.43}$$

and it is identically vanishing because of the gauge condition (9.12). The second contribution quadratic in $\Gamma^{(1)}$ can be written as

$$\begin{aligned} &-\eta^{\nu\alpha}\Gamma^{(1)\mu}_{\nu\rho}\Gamma^{(1)\rho}_{\mu\alpha} \\ &\quad = -\frac{1}{4}\eta^{\mu\beta}(\partial_\nu h_{\rho\beta} + \partial_\rho h_{\nu\beta} - \partial_\beta h_{\rho\nu})\left(\partial_\mu h^{\nu\rho} + \partial^\nu h_\mu{}^\rho - \partial^\rho h_\mu{}^\nu\right), \end{aligned} \tag{9.44}$$

and, neglecting total divergences, it reduces to

$$-\frac{1}{4}h_\mu{}^\nu \Box h_\nu{}^\mu. \tag{9.45}$$

We are left now with the last term of the action (9.40). Let us first notice that

$$\left(\sqrt{-g}g^{\nu\alpha}\right)^{(1)} = (\sqrt{-g})^{(0)}\left(g^{\nu\alpha}\right)^{(1)} = -h^{\nu\alpha}, \tag{9.46}$$

since $(\sqrt{-g})^{(1)}$ is proportional to h, hence it is vanishing in the TT gauge. By using the result (8.7) and the gauge conditions (9.12) we obtain

$$\left(\sqrt{-g}g^{\nu\alpha}\right)^{(1)} R^{(1)}_{\nu\alpha} = \frac{1}{2}h^{\nu\alpha}\Box h_{\nu\alpha}. \tag{9.47}$$

We finally sum up the two contributions (9.45), (9.47), integrate by part, and neglect a total divergence. The effective action (9.40) reduces to

$$S^{(2)} = \frac{c^4}{32\pi G}\int d^4x\,\frac{1}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta}. \tag{9.48}$$

This is the action controlling the evolution of a gravitational wave freely propagating in the Minkowski space–time, and satisfying the gauge conditions (9.12).

The corresponding dynamical energy-momentum tensor $\tau_{\mu\nu}$ can now be obtained by applying the standard definition of Sect. 7.2. We have to rewrite the action in terms of arbitrary curvilinear coordinates, to vary with respect to the effective metric $g_{\mu\nu}$, and to impose that the equations of motion for $h_{\mu\nu}$ are satisfied. We obtain

$$\begin{aligned}\delta S^{(2)} &\equiv \frac{1}{2}\int d^4x\,\sqrt{-g}\,\tau_{\mu\nu}\delta g^{\mu\nu}\\ &= \frac{c^4}{32\pi G}\int d^4x\,\sqrt{-g}\,\frac{1}{2}\big(\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\delta g^{\mu\nu}+\cdots\big),\end{aligned} \tag{9.49}$$

modulo terms that are vanishing for a wave satisfying the equation $\Box h_{\mu\nu}=0$ and the TT gauge conditions. It follows that

$$\tau_{\mu\nu} = \frac{c^4}{32\pi G}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}. \tag{9.50}$$

We can easily check that this tensor, for a wave satisfying Eqs. (9.1), (9.12), is traceless and conserved,

$$\partial^\nu \tau_{\mu\nu} = 0 \tag{9.51}$$

(see Exercise 9.3).

9.2.3 Radiated Power

Starting from the conservation equation (9.51) we can now compute the power (i.e. the energy per unit time) carried away from the sources by the emitted gravitational radiation. By integrating Eq. (9.51) over a finite volume V centered on the sources, and using the Gauss theorem, we have

$$\frac{1}{c}\frac{d}{dt}\int_V d^3x\,\tau_\mu{}^0 = -\int_V d^3x\,\partial_i\tau_\mu{}^i = -\int_S \tau_\mu{}^i\,d\sigma_i, \tag{9.52}$$

where $d\sigma_i$ is the area-element on the closed two-dimensional surface S bounding the volume V. Hence, for $\mu=0$,

$$\frac{dE}{dt} = -c\int_S \tau_0{}^i\,d\sigma_i \equiv -\int_S dI. \tag{9.53}$$

The left-hand side of the above equation gives the power radiated away from the volume V. On the right-hand side, $c\tau_0{}^i$ is the flux of gravitational energy in the $\hat{x}_i$ direction, while $dI = c\tau_0{}^i\, d\sigma_i$ represents the intensity of radiated energy, per unit time, across an infinitesimal surface element.

In order to compute the total emitted power let us then consider a sphere of radius R centered on the sources, and compute the intensity dI of radiated energy in the element of solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$, along a generic radial direction identified by the unit vector n^i:

$$dI = c\tau_0{}^i n_i R^2\, d\Omega. \tag{9.54}$$

We can choose, for instance, a wave propagating along the x_1 axis. By computing the energy-momentum tensor (9.50) for the wave solution (9.13), and using the property $\partial^1 h_{ij} = \partial_0 h_{ij}$, we obtain

$$dI = \frac{c^3}{16\pi G}\left(\dot{h}_{22}^2 + \dot{h}_{23}^2\right) R^2\, d\Omega, \tag{9.55}$$

where the dot denotes differentiation with respect to $t = x^0/c$. More generally, for a propagation direction specified by the polar angles $\{\theta, \phi\}$, and corresponding to

$$n_1 = \sin\theta\cos\varphi, \qquad n_2 = \sin\theta\sin\varphi, \qquad n_3 = \cos\theta, \tag{9.56}$$

the radiated intensity can be written in the form

$$dI = \frac{c^3}{16\pi G}\left[\frac{1}{4}\left(\dot{h}_{ij}n^i n^j\right)^2 + \frac{1}{2}\dot{h}_{ij}\dot{h}^{ij} - \dot{h}_{ik}\dot{h}^k{}_j n^i n^j\right] R^2\, d\Omega \tag{9.57}$$

(see Exercise 9.4). Assuming that the radius R is large enough, and that we are evaluating dI in the radiation zone, we can also apply Eq. (9.38) to eliminate $h_{\mu\nu}$ in terms of the quadrupole moment of the sources. The previous equation then becomes

$$dI = \frac{G}{36\pi c^5}\left[\frac{1}{4}\left(\dddot{Q}_{ij}n^i n^j\right)^2 + \frac{1}{2}\dddot{Q}_{ij}\dddot{Q}^{ij} - \dddot{Q}_{ik}\dddot{Q}^k{}_j n^i n^j\right] d\Omega, \tag{9.58}$$

and shows that, at large enough distances, the radiated power becomes independent on R and fully controlled by $\dddot{Q}$.

To perform the angular integration in $d\Omega$ over the unit two-sphere we recall that $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. Using for n_i the polar representation (9.56) we easily obtain

$$\int_\Omega d\Omega = 4\pi, \qquad \int_\Omega d\Omega\, n_i n_j = \frac{4\pi}{3}\delta_{ij}, \tag{9.59}$$

$$\int_\Omega d\Omega\, n_i n_j n_k n_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{9.60}$$

(see Exercise 9.5). Integrating, and using the symmetry property ($Q_{ij} = Q_{ji}$) and the traceless property ($Q_{ij}\delta^{ij} = 0$) of the quadrupole moment, we finally arrive at

the following expression for the radiated gravitational power:

$$\frac{dE}{dt} = -\int_{\Omega} dI = -\frac{G}{36\pi c^5} 4\pi \dddot{Q}_{ij} \dddot{Q}^{ij} \left(\frac{1}{30} + \frac{1}{2} - \frac{1}{3}\right) = -\frac{G}{45c^5} \dddot{Q}_{ij} \dddot{Q}^{ij}. \quad (9.61)$$

In the case of sources undergoing periodic motions it is convenient to consider the time average (over a period T) of the radiated power. By defining

$$\langle \cdots \rangle = \frac{1}{T} \int_0^T dt (\cdots) \quad (9.62)$$

we obtain

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{G}{45c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle. \quad (9.63)$$

A direct application of this result to the simple case of a harmonic oscillator will be presented in Exercise 9.6. In the next section we will consider instead the case of a binary star system, and show that the energy loss in the form of emitted gravitational radiation is responsible for a decrease of the rotational period of such a system. This effect has been observed, and found to be in agreement with the predictions of general relativity.

9.2.4 Example: A Binary Star System

The power emitted by a system of accelerated masses in the form of quadrupole gravitational radiation, according to Eq. (9.63), is very small. We can easily realize that this is indeed the case by considering, as a typical example of gravity-wave source available in the laboratory, a linear oscillator of mass m, frequency ω and size L. In such a case Eq. (9.63) gives

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{48G}{45c^5} m^2 L^4 \omega^6 \quad (9.64)$$

(see Exercise 9.6). If we use $m = 1$ kg, $L = 1$ m and $\omega = 10$ Hz we obtain a radiated power of about 10^{-40} erg/s, or 10^{-47} Watt (which is well below the detection sensitivity allowed by present technology).

Much more powerful sources, however, are available in an astrophysical context. Simple and well-known examples are provided by the binary systems, in particular by those in which the two components are very close to each other, orbiting at very high velocity around their center of mass. The mechanism of gravity-wave emission is in principle the same as the one of the laboratory oscillator, but the effect turns out to be enormously enhanced because of the masses coming into play.

Consider indeed two celestial bodies (for instance two stars) of masses m_1 and m_2, orbiting in the (x_1, x_2) plane around their common center of mass, with high

but non-relativistic velocities. Let us assume that the system can be described, in first approximation, as a point-like source of reduced mass

$$M = \frac{m_1 m_2}{m_1 + m_2}, \tag{9.65}$$

rotating with angular velocity ω on a circular orbit of radius a, described by the equations

$$x_1 = a\cos\omega t, \qquad x_2 = a\sin\omega t, \qquad x_3 = 0. \tag{9.66}$$

In that case

$$\rho = M\delta(x_1 - a\cos\omega t)\delta(x_2 - a\sin\omega t)\delta(x_3), \tag{9.67}$$

and the components of the quadrupole moment (9.39) are given by

$$\begin{aligned} &Q_{11} = Ma^2\big(3\cos^2\omega t - 1\big), \qquad Q_{22} = Ma^2\big(3\sin^2\omega t - 1\big), \\ &Q_{33} = -Ma^2, \qquad Q_{12} = Q_{21} = 3Ma^2\cos\omega t\sin\omega t. \end{aligned} \tag{9.68}$$

The computation of the time derivatives then gives

$$\begin{aligned} &\dddot{Q}_{11} = 24Ma^2\omega^3\sin\omega t\cos\omega t = -\dddot{Q}_{22}, \\ &\dddot{Q}_{12} = -12Ma^2\omega^3\big(\cos^2\omega t - \sin^2\omega t\big). \end{aligned} \tag{9.69}$$

The time average over a period $T = 2\pi/\omega$, according to the definition (9.62), leads to

$$\begin{aligned} &\big\langle \sin^2\omega t\cos^2\omega t\big\rangle = \frac{1}{8}, \\ &\big\langle\big(\cos^2\omega t - \sin^2\omega t\big)^2\big\rangle = \big\langle\cos^2 2\omega t\big\rangle = \frac{1}{2}. \end{aligned} \tag{9.70}$$

Inserting the above results into Eq. (9.63) we finally see that the given binary system radiated gravitational waves with averaged power:

$$\left\langle \frac{dE}{dt}\right\rangle = -\frac{32G}{5c^5}\,M^2a^4\omega^6. \tag{9.71}$$

In order to estimate the emission power of a typical system we can take M of the order of the solar mass, $M \sim 10^{33}$ g, an orbital radius of about ten times the solar radius, $a \sim 10^{11}$ cm, and a period of few hours, e.g. $\omega \sim 10^{-4}$ Hz. The corresponding power turns out to be $\sim 10^{27}$ erg/s, or 10^{20} Watt. If such a system is inside our Galaxy, we can assume that its mean distance from the Earth is about $R \sim 10^{20}$ cm. The energy flux we can receive is thus typically the following:

$$\Phi = \frac{1}{4\pi R^2}\left|\left\langle \frac{dE}{dt}\right\rangle\right| \sim 10^{-14}\ \frac{\text{erg}}{\text{cm}^2\,\text{sec}} = 10^{-21}\ \frac{\text{Watt}}{\text{cm}^2}. \tag{9.72}$$

This is certainly much bigger than the flux we could receive from the laboratory oscillator considered before (even placing our detector at the distance of one cm); however, it is still too small for a direct detection, given the present instrumental sensitivities. Nonetheless, the gravitational radiation emitted by such a binary system can be indirectly detected by observing the variation it induces on the orbital period of the system.

In order to illustrate this possibility we need to connect the total energy of the system to its rotational period. It will be enough, to this purpose, to describe the unperturbed binary system in the Newtonian approximation, assuming for simplicity that the two components have the same masses, $m_1 = m_2 = m$, and are rotating around their center of mass with frequency ω, on a circular orbits of radius r. The emitted power is then given by Eq. (9.71) with $M = m/2$ and $a = 2r$.

The total energy of the system, in the Newtonian approximation, is given by

$$E = m\omega^2 r^2 - \frac{Gm^2}{2r}. \tag{9.73}$$

In addition, the equilibrium between gravitational and centrifugal forces imposes the condition (the well-known Kepler "third law")

$$m\omega^2 r = \frac{Gm^2}{4r^2}. \tag{9.74}$$

By eliminating r in terms of ω in Eq. (9.73) we can then express the energy in terms of the frequency only,

$$E(\omega) = -\left(\frac{G}{4}\right)^{2/3} m^{5/3}\omega^{2/3}. \tag{9.75}$$

Differentiating, and introducing the period $T = 2\pi/\omega$, we obtain

$$\frac{dE}{E} = \frac{2}{3}\frac{d\omega}{\omega} = -\frac{2}{3}\frac{dT}{T}, \tag{9.76}$$

so that the time variation of the period and of the energy can be finally related by

$$\frac{dT}{dt} = -\frac{3}{2}\frac{T}{E}\frac{dE}{dt}. \tag{9.77}$$

For the binary system, on the other hand, the total energy E is negative, as can be checked by eliminating $m\omega^2$ in Eq. (9.73) through Eq. (9.74):

$$E = -\frac{Gm^2}{4r} < 0. \tag{9.78}$$

It follows that $\dot{T}$ and $\dot{E}$ have the same sign, and that the loss of energy due to the emission of gravitational radiation produces a decrease of the rotational period; such a decrease can be directly computed by inserting in the right-hand side of Eq. (9.77) the radiate power given (in the quadrupole approximation) by Eq. (9.71).

This effect has been observed and experimentally measured in the binary system discovered by Hulse and Taylor [26] (Nobel Prizes for Physics in 1993), a system where one of the two components is the pulsar PSR B1913+16 (a compact, highly magnetized neutron star). Highly accurate observations, continuously carried out for many years, have shown that the rotational period of that system (about 7 hours and 45 minutes) decreases at a rate dT/dt of about 76.5 μs per year. This result agrees with the predictions of general relativity, concerning the emission of quadrupole radiation, to an accuracy of about 0.2%.

So, there is no doubt that gravitational waves exist, and that they may be correctly described by the Einstein gravitational equations. However, the challenge of a direct experimental detection is still open. Some aspects of the phenomenology of gravitational waves relevant for their interactions with detectors will be briefly introduced in the following sections.

9.3 The Interaction of Monochromatic Waves with Massive Bodies

To understand the mechanism of gravity-wave detection we should discuss, first of all, the response of a system of test masses to the local transit of a gravitational wave. The detector operation principles are grounded indeed on the relative motion of masses induced by the wave—just like the electromagnetic detectors are based on the motion of test charges. We should then start with the equation of geodesic deviation (see Sect. 6.1),

$$\frac{D^2\eta^\mu}{d\tau^2} + \eta^\nu R_{\nu\alpha\beta}{}^\mu u^\alpha u^\beta = 0, \tag{9.79}$$

which provides the local gravitational acceleration between two test masses with spatial separation η^μ. This is the basic equation for the detection mechanism, for all possible types of gravitational "antenna".

Let us consider two test masses, initially at rest, and characterized by a (small enough) initial separation $\eta^\mu = L^\mu = (0, L^i) = \text{const}$. When subjected to the action of a wave, represented by the Riemann tensor $R_{\mu\nu\alpha\beta}$, they tend to move from the equilibrium position, following the evolution predicted by Eq. (9.79). Assuming that the displacements are small, that the motion is non-relativistic, and that the gravitational field of the wave is weak enough, we can put

$$\eta^\mu = L^\mu + \xi^\mu, \quad |\xi| \ll |L|, \tag{9.80}$$

we can approximate the four-velocity as $u^\mu = (c, \mathbf{0})$, and we can limit our computation to first order in ξ and in the weak field $h_{\mu\nu}$ of the wave. In this limit the equation of geodesic deviation reduces to

$$\ddot{\xi}^i = -L^j R^{(1)}_{j00}{}^i c^2, \tag{9.81}$$

where the dot denotes differentiation with respect to t, and $R^{(1)}_{\mu\nu\alpha}{}^{\beta}$ is the Riemann tensor computed to the first order in h (see Eq. (8.6)).

The gravitational field of the wave can be conveniently expressed in the TT gauge, where $h_{\mu 0} = 0$ (see Sect. 9.1.1). In that case only the third term of Eq. (8.6) gives a nonvanishing contribution to $R^{(1)}_{j00}{}^{i}$,

$$R^{(1)}_{j00}{}^{i} = \frac{1}{2c^2}\delta^{ik}\ddot{h}_{jk} = \frac{1}{2c^2}\ddot{h}_j{}^{i}, \tag{9.82}$$

and we obtain

$$\ddot{\xi}^i = -\frac{1}{2}L^j\ddot{h}_j{}^{i}. \tag{9.83}$$

Let us consider, in particular, a monochromatic plane wave propagating in the x_3 direction, with frequency $\omega = ck$ and nonzero components only in the transverse (x_1, x_2) plane:

$$h_{ij} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos\big[k(z-ct)+\phi\big]. \tag{9.84}$$

We have introduced an arbitrary phase ϕ and a 2×2 matrix representing the components $h_{11} = -h_{22}$ e $h_{12} = h_{21}$. For this wave

$$\ddot{h}_{ij} = -k^2c^2h_{ij} = -\omega^2 h_{ij}, \tag{9.85}$$

and the equations of motion (9.83) become

$$\begin{aligned} \ddot{\xi}^1 &= -\frac{\omega^2}{2}\big(L^1h_+ + L^2h_\times\big)\cos(kz-\omega t+\phi), \\ \ddot{\xi}^2 &= -\frac{\omega^2}{2}\big(L^1h_\times - L^2h_+\big)\cos(kz-\omega t+\phi). \end{aligned} \tag{9.86}$$

For a better illustration of the relative motion of the test masses we can imagine a set of massive particles localized on the (x_1, x_2) plane, and arranged so as to form a circle of radius $L/2$. Consider an incident wave with polarization of type h_+, amplitude $h_+ = f$ (and polarization component $h_\times = 0$). The force acting on the particles varies periodically with time, changing from a state characterized by $\cos(kz-\omega t+\phi) = 1$ and by

$$\ddot{\xi}^1 = -\frac{\omega^2}{2}Lf, \qquad \ddot{\xi}^2 = \frac{\omega^2}{2}Lf \tag{9.87}$$

(maximum attractive force along x_1 and maximum repulsion along x_2), to a state characterized by $\cos(kz-\omega t+\phi) = -1$ and by

$$\ddot{\xi}^1 = \frac{\omega^2}{2}Lf, \qquad \ddot{\xi}^2 = -\frac{\omega^2}{2}Lf, \tag{9.88}$$

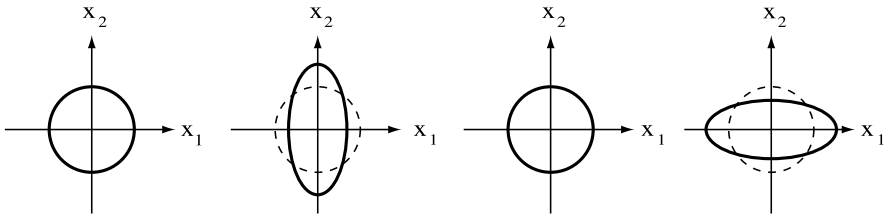

Fig. 9.1 Response to the polarization mode h_+ of massive free particles, arranged in a circle in the plane transverse to the wave propagation

(maximum repulsive force along x_1 and maximum attraction along x_2). Following such a periodic variation of $h_+(t)$, the circle of particles thus undergoes a series of subsequent alternate compressions and expansions along the orthogonal axes x_1, x_2, as illustrated in Fig. 9.1.

Consider now an incident wave of polarization type $h_\times$ (with component $h_+ = 0$) and amplitude $h_\times = f$, the same as before. The equations of motion (9.86) for the $h_\times$ mode,

$$\begin{aligned} \ddot{\xi}^1 &= -\frac{\omega^2}{2} L^2 f \cos(kz - \omega t + \phi), \\ \ddot{\xi}^2 &= -\frac{\omega^2}{2} L^1 f \cos(kz - \omega t + \phi), \end{aligned} \tag{9.89}$$

can be exactly reduced to the equations for the h_+ mode after a rotation of $\pi/4$ in the (x_1, x_2) plane. In fact, by defining

$$\begin{pmatrix} \widetilde{\xi}^1 \\ \widetilde{\xi}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{L}^1 \\ \widetilde{L}^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} L^1 \\ L^2 \end{pmatrix}, \tag{9.90}$$

we obtain from Eq. (9.89)

$$\begin{aligned} \ddot{\widetilde{\xi}}^1 &= -\frac{\omega^2}{2} \widetilde{L}^1 f \cos(kz - \omega t + \phi), \\ \ddot{\widetilde{\xi}}^2 &= \frac{\omega^2}{2} \widetilde{L}^2 f \cos(kz - \omega t + \phi), \end{aligned} \tag{9.91}$$

reproducing the system (9.86) for $h_\times = 0$ and $h_+ = f$. The action of the $h_\times$ mode on the circle of particles is thus the same as that of the h_+ mode, but is referred to a "tilted" system of orthogonal axes, at an angle of 45 degrees with respect to the previous configuration (see Fig. 9.2).

These types of distortion (or "stress") induced on a given distribution of test masses are typical of the two polarization states of a tensor (i.e. spin-2) wave. All gravitational detectors aim at amplifying, as much as possible, such distortions produced by the wave on the system of masses playing the role of gravitational "an-

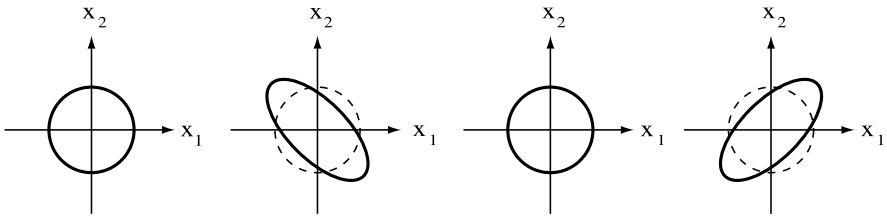

Fig. 9.2 Response to the polarization mode $h_\times$ of massive free particles, arranged in a circle in the plane transverse to the wave propagation

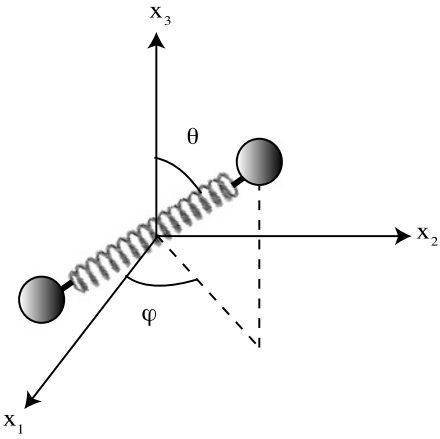

Fig. 9.3 Orientation of the oscillator with respect to a set of Cartesian axes. The incident gravitational wave propagates along the x_3 axis

tenna", and at subtracting all "noise" effects, i.e. all possible types of mass vibration (thermal, seismic, etc.) not directly due to the wave.

9.4 The Damped Oscillator as a Gravitational Detector

The simplest example of gravitational detector is provided by the damped mechanical oscillator, which we can interpret as an (ideal) macroscopic model for a system of vibrating masses.

Suppose we have two masses M connected by a spring of length L (at rest), and orientation described by the polar angles θ, φ with respect to Cartesian coordinate system (see Fig. 9.3). Let us study the response of this oscillator to a plane wave propagating along the positive direction of the x_3 axis, of polarization type h_+, which can be parametrized as follows:

$$h_{ij} = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} e^{i(kz-\omega t)}, \tag{9.92}$$

where $\omega = ck$.

Let us first observe, to this aim, that the mass separation in the (x_1, x_2) is described by the two-dimensional vector $\boldsymbol{L}$ with components:

$$L_1 = L\sin\theta\cos\varphi, \qquad L_2 = L\sin\theta\sin\varphi. \tag{9.93}$$

Assuming that the wavelength of the incident radiation is much larger than the size of the oscillator ($kL \ll 1$), we can rewrite Eq. (9.83) for the small oscillations in the (x_1, x_2) plane as follows:

$$\begin{aligned} \ddot{\xi}^1 &= -\frac{\omega^2}{2} hL\sin\theta\cos\varphi e^{-i\omega t}, \\ \ddot{\xi}^2 &= \frac{\omega^2}{2} hL\sin\theta\sin\varphi e^{-i\omega t}. \end{aligned} \tag{9.94}$$

By projecting this acceleration along the oscillator axis we obtain the relative acceleration of the two masses produced by the gravitational wave:

$$\begin{aligned} \ddot{\xi} &\equiv \ddot{\xi}^1\cos\varphi\sin\theta + \ddot{\xi}^2\sin\varphi\sin\theta \\ &= -\frac{\omega^2}{2} hLe^{-i\omega t}\sin^2\theta\cos 2\varphi. \end{aligned} \tag{9.95}$$

Let us finally add to this acceleration the one due to the elastic force of the spring, together with a possible damping term proportional to $\dot{\xi}$, with characteristic time τ_0. We then arrive at the equation

$$\ddot{\xi} + \frac{\dot{\xi}}{\tau_0} + \omega_0^2\xi = -\frac{\omega^2}{2} hLe^{-i\omega t}\sin^2\theta\cos 2\varphi, \tag{9.96}$$

which describes the response of the oscillator to radiation of frequency $\omega \ll c/L$, propagating along the direction specified by the angles θ and φ with respect to its axis. The damping time τ_0 and the proper frequency ω_0 are typical of the given oscillator, and correspond to intrinsic parameters of the detector depending on its geometric structure and internal composition.

Equation (9.96) is the basic equation describing the response of the so-called "resonant-mass" detectors. We may consider, for instance, the ideal case in which the oscillator axis is orthogonal to the direction of the incident wave, so that $\theta = \pi/2$ and $\varphi = 0$ (or $\varphi = \pi/2$). By solving Eq. (9.96) we then find, in the stationary regime, the following particular exact solution:

$$\xi(t) = \frac{\omega^2}{2}\frac{hLe^{-i\omega t}}{\omega^2 - \omega_0^2 + \frac{i\omega}{\tau_0}}. \tag{9.97}$$

The maximum response is obtained, in this case, when the wave frequency coincides with the proper frequency of the oscillating masses, $\omega \simeq \omega_0$. In this "resonant" regime the solution becomes

$$\xi(t) = -\frac{i}{2}\omega_0\tau_0 hLe^{-i\omega t}. \tag{9.98}$$

In the resonant regime, the appropriate variable for a precise characterization of the detector efficiency is the so-called "cross section" σ, defined as the power dissipated by the system with respect to the energy flux of the incident radiation. For the oscillator we are considering the power loss is given by $P_d = E_v/\tau_0$, where $E_v = M|\dot{\xi}|^2$ is the kinetic energy of the two vibrating masses. The energy flux of the polarized wave (9.92), propagating along the x_3 axis, is obtained by the energy-momentum tensor (9.50), and found to be:

$$c{\tau_0}^3 = \frac{c^3}{16\pi G}|\dot{h}_{11}|^2 = \frac{\omega^2 c^3}{16\pi G}|h|^2. \tag{9.99}$$

The cross section, in our case, is then defined as

$$\sigma = \frac{P_d}{c{\tau_0}^3} = \frac{16\pi GM|\dot{\xi}|^2}{\tau_0 \omega^2 c^3 |h|^2}. \tag{9.100}$$

Since in the resonant regime the oscillation amplitude is given by Eq. (9.98), we finally obtain

$$\sigma = \frac{4\pi GM}{c^3}\omega_0^2 L^2 \tau_0 = \frac{4\pi GM}{c^3}\frac{Q^2 L^2}{\tau_0}, \tag{9.101}$$

where $Q = \omega_0 \tau_0$ is the so-called "quality factor" of the detector. It should be noted that, at the resonance, $|\xi| = QLh/2$. Hence the detector efficiency increases, in that case, with both the Q factor and the size of the detector.

The resonant antennas presently in operation have typical sizes $L \sim 10^2$ cm, typical quality factors $Q \sim 10^5$, and—through sophisticated systems of electronic amplification—are sensitive to oscillation amplitudes $|\xi| \sim 10^{-15}$ cm. At the resonance frequency they can thus detect gravitational fluctuations of local amplitude $|h| \sim 10^{-22}$ cm. Nevertheless, the intensity of the emitted radiation is so small, and the astrophysical sources so far away, that no detectable signal has reached so far the presently operating antennas.

9.4.1 The Presently Operating Detectors

It seems appropriate to conclude this chapter with a concise list of the gravitational detectors which are presently in the operational phase (or in a design phase, planned to be operative in a near future).

There are today two types of detector which have been developed at a high technological level, and can be most efficiently applied to the search of gravitational waves of astrophysical origin: resonant bars and interferometric antennas.

The resonant bars are big aluminum cylinders, which start vibrating when they are crossed by a gravitational wave, thus behaving (in principle) like the elementary oscillator discussed before. Their typical resonant frequency is $\omega_0 \sim 1$ kHz. The

thermal noise is eliminated by cooling down these bars to temperatures smaller than one Kelvin degree.

Among the most powerful and sensitive bars we should mention NAUTILUS (at the Frascati national INFN laboratory, Italy) AURIGA (at the Legnaro national INFN laboratory, Italy), EXPLORER (at CERN, Geneva), ALLEGRO (in Louisiana, USA), NIOBE (in Australia).

We should say that these bars, of cylindrical shape, will probably evolve towards new types of resonant detector of polyhedrical (or even spherical) shape, of solid or hollow type. Among such possible types of (future) detector we should mention the TIGA project (Louisiana, USA), the GRAL project (Leiden, Germany), and the DUAL project (INFN, Italy) of a hollow sphere. These new detectors should improve, in various ways, the performance of present bar detectors because—unlike bars—they can identify the direction of the incident waves, are sensitive to radiation of scalar type (like the dilaton radiation of generalized gravitational theories), and, finally, may have a good response also at frequencies higher than the kHz band.

The second class of operating detectors is based on the technology of laser interferometers, with very long arms, of kilometric size. The mirrors placed at the ends of the two arms start vibrating under the gravity-wave action, and produce a perturbation in the interference fringes, with maximal sensitivity around the frequency band of about 100 Hz. The laser beams travel inside long vacuum tubes, but there is no need of cooling down the system (as is instead the case for the bar detectors).

Among the most sensitive interferometers we should mention LIGO, with arms of 4 km, which exists in two twin versions (in the Washington state and in Louisiana, USA); VIRGO, with arms of 3 km (at Cascina, Italy); GEO, with arms of 600 m (at Hannover, Germany); TAMA, with arms of 300 m (in Japan).

All gravitational antennas mentioned so far are designed to work on the Earth surface, and are thus unavoidably subject to environment noises of geophysical type (seismic and micro-seismic). This necessarily limits the detector sensitivity at low frequency: actually, it excludes from the sensitivity band all waves with frequencies $\omega \lesssim 1$ Hz because, at those frequencies, the detector vibrations induced by micro-seismic effects are always much larger than those produced by any realistic flux of gravitational radiation.

In order to evade this annoying limitation there is (still in a design phase) a class of "space interferometers": unmanned spacecrafts which, orbiting around the Sun, can exchange among themselves collimated laser beams, thus effectively operating as interferometers with arms of enormous length. Being in space, they are free from seismic noise and are thus sensitive to gravitational fluctuations of frequency much smaller than those typical of terrestrial interferometers.

We should mention, in particular, the LISA project (a collaboration of the space agencies ESA and NASA), concerning three spacecraft at a relative distance of about five million kilometers one from the other, and expected to reach the maximum sensitivity at a frequency around $\omega = 10^{-3}$ Hz; the BBO project (NASA), concerning four spacecraft, with maximal sensitivity around $\omega = 10^{-1}$ Hz; and the DECIGO project (Japan), also similar to BBO.

The planned sensitivities of all such detectors (and of their advanced versions), either resonant of interferometric, on the Earth or in space, should enable us to detect

in a not so far future the gravitational radiation of the most powerful astrophysical sources located inside our Galaxy. Even more: thanks to such detectors we could be able to measure a possible relic background of cosmic gravitational radiation, produced during the very early phases of our Universe, and still present today on a extremely large spectrum of frequencies (in principle, from 10^{-18} Hz to the GHz range). For a detailed discussion of this last point the interested reader is referred to the books [8, 19] quoted in the bibliography.

9.5 Exercises Chap. 9

9.1 Polarization States in D Dimensions Determine the number of independent polarization states for the tensor fluctuation of the metric, h_{AB}, in a D-dimensional Minkowski space–time.

9.2 Helicity of the Gravitational Waves Derive Eq. (9.23) for a plane wave propagating along the x_1 axis.

9.3 Energy-Momentum of a Gravitational Wave Consider a gravitational wave propagating along the x_1 axis, in the TT gauge and in the Minkowski space–time. Show that the energy-momentum tensor (9.50) associated to this wave satisfies the following properties:

$$\tau_\nu{}^\nu = 0, \qquad \partial^\nu \tau_{\mu\nu} = 0. \tag{9.102}$$

9.4 Radiated Flux Along an Arbitrary Direction Deduce Eq. (9.57) for the intensity of the gravitational radiation emitted along the direction identified by an arbitrary unit vector n_i, using Eq. (9.55) which gives the radiation intensity in the x_1 direction.

9.5 Angular Integration of the Radiation Flux Compute the angular integrals (9.59), (9.60) for the unit vector defined in polar coordinates by Eq. (9.56).

9.6 Quadrupole Radiation from a Harmonic Oscillator Apply Eq. (9.63) to compute the (time-averaged) radiation power of a point particle of mass m harmonically oscillating along the x_3 axis, with frequency ω and amplitude L.

9.6 Solutions

9.1 Solution Let us apply the same procedure of Sect. 9.1.1, with the only difference that the tensor indices A, B range from 0 to $D-1$. In that case a symmetric, second-rank tensor like h_{AB} has a total number of

$$\frac{D^2 - D}{2} + D = \frac{1}{2} D(D+1) \tag{9.103}$$

independent components (we have considered the off-diagonal elements, divided by two, and added the number of diagonal elements). We can then impose on these independent components D gauge conditions (using for instance the harmonic gauge), and other D conditions through a coordinate transformation which preserves the chosen gauge. We end up, in this way, with a total number

$$n = \frac{1}{2}D(D+1) - 2D = \frac{1}{2}D(D-3) \tag{9.104}$$

of independent degrees of freedom (and then of independent polarization states) for the dynamical evolution of tensor metric fluctuations in vacuum.

For $D = 4$ we obtain $n = 2$, in agreement with the conclusion of Sect. 9.1.1. In a five-dimensional space–time, instead, a gravitational wave has a number $n = 5$ of independent polarization states.

9.2 Solution Adopting a compact block notation for the 2×2 matrices, and using the explicit definitions (9.16), (9.18), (9.22), we can set

$$\epsilon^{(\pm)}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\pm} \end{pmatrix}, \qquad U_{\mu}{}^{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \tag{9.105}$$

where

$$\epsilon^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} = \frac{1}{2}(\sigma_3 \pm i\sigma_1), \qquad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{9.106}$$

and σ_1, σ_3 are the Pauli matrices in the representation where σ_3 is diagonal. We then obtain

$$\epsilon'^{(\pm)}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon'^{\pm} \end{pmatrix} \tag{9.107}$$

where

$$\epsilon'^{\pm} = R\epsilon^{\pm}R^T. \tag{9.108}$$

A simple matrix computation finally leads to

$$\epsilon'^{\pm} = e^{\pm 2i\theta}\epsilon^{\pm}. \tag{9.109}$$

9.3 Solution Using the definition (9.50) we can explicitly write the conditions (9.102) as follows:

$$\partial^{\nu} h^{\alpha\beta}\partial_{\nu} h_{\alpha\beta} = 0, \tag{9.110}$$

$$\partial^{\nu}\left(\partial_{\mu} h^{\alpha\beta}\partial_{\nu} h_{\alpha\beta}\right) = 0. \tag{9.111}$$

Since $\Box h_{\alpha\beta} = 0$, they are both satisfied provided the trace condition (9.110) is satisfied.

For a wave propagating along the x_1 axis we have $h_{\alpha\beta} = h_{\alpha\beta}(x^1 - ct)$, and the trace τ_ν^ν becomes

$$\begin{aligned}\partial^\nu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} &= \partial^0 h^{\alpha\beta}\partial_0 h_{\alpha\beta} + \partial^1 h^{\alpha\beta}\partial_1 h_{\alpha\beta} \\ &= 2\partial^0 h_{22}\partial_0 h_{22} + 2\partial^0 h_{23}\partial_0 h_{23} \\ &\quad + 2\partial^1 h_{22}\partial_1 h_{22} + 2\partial^1 h_{23}\partial_1 h_{23} \end{aligned} \tag{9.112}$$

(we have used Eq. (9.13)). For each component of h_{ij}, on the other hand,

$$\partial^0 h_{ij} = \partial_0 h_{ij} = -\partial_1 h_{ij} = \partial^1 h_{ij}. \tag{9.113}$$

All terms of Eq. (9.112) thus identically sum up to zero.

9.4 Solution The intensity of the radiated energy along the direction identified by the unit vector $\boldsymbol{n}$ must correspond to a scalar expression in 3-dimensional Euclidean space, which depends on $\dot{h}_{ij}$ and n_i, which is quadratic in $\dot{h}_{ij}$, and which reduces to Eq. (9.55) for a wave propagating along x_1. We can then consider the more general quadratic scalar form in $\dot{h}$,

$$A(\boldsymbol{n}) = \alpha_1\big(\dot{h}_{ij}n^i n^j\big)^2 + \alpha_2\,\dot{h}_{ij}\dot{h}^{ij} + \alpha_3\,\dot{h}_{ik}\dot{h}^k{}_j n^i n^j, \tag{9.114}$$

and determine the arbitrary coefficients α_1, α_2, α_3 by imposing that, for $\boldsymbol{n} = \boldsymbol{n}_1 = (1, 0, 0)$, the above equation reduces to

$$A(\boldsymbol{n}_1) = \dot{h}_{22}^2 + \dot{h}_{23}^2 \equiv \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2 + \dot{h}_{23}^2. \tag{9.115}$$

We have used the condition $h_{22} = -h_{33}$, which holds for a wave propagating along x_1, in order to express $A(\boldsymbol{n}_1)$ in terms of all nonvanishing components of h.

From Eq. (9.114) we have

$$\begin{aligned} A(\boldsymbol{n}_1) = {} & \alpha_1\dot{h}_{11}^2 + \alpha_2\big(\dot{h}_{11}^2 + \dot{h}_{22}^2 + \dot{h}_{33}^2 + 2\dot{h}_{12}^2 + 2\dot{h}_{13}^2 + 2\dot{h}_{23}^2\big) \\ & + \alpha_3\big(\dot{h}_{11}^2 + \dot{h}_{12}^2 + \dot{h}_{13}^2\big). \end{aligned} \tag{9.116}$$

By eliminating $\dot{h}_{11}$ with the generic traceless condition

$$\dot{h}_{11} = -(\dot{h}_{22} + \dot{h}_{33}), \tag{9.117}$$

and imposing the equality with Eq. (9.115), we arrive at the following system of equations:

$$\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= \frac{1}{4}, \qquad & 2\alpha_1 + 2\alpha_2 + 2\alpha_3 &= -\frac{1}{2}, \\ 2\alpha_2 &= 1, \qquad & 2\alpha_2 + \alpha_3 &= 0. \end{aligned} \tag{9.118}$$

The first equation is obtained by equating the coefficients of $\dot{h}_{22}^2$ and $\dot{h}_{33}^2$, the second by equating the coefficients of $\dot{h}_{22}\dot{h}_{33}$, the third by equating the coefficients of $\dot{h}_{23}^2$, the fourth by equating the coefficients of $\dot{h}_{12}^2$ and $\dot{h}_{13}^2$. The solution is

$$\alpha_1 = \frac{1}{4}, \qquad \alpha_2 = \frac{1}{2}, \qquad \alpha_3 = -1. \tag{9.119}$$

By inserting these values in Eq. (9.114) we then arrive at the quadratic form of Eq. (9.57).

9.5 Solution Let us first notice that

$$\int_\Omega d\Omega = \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta \, d\theta = 4\pi. \tag{9.120}$$

From the definition (9.56) we have

$$\begin{aligned} \int_\Omega d\Omega \, n_1^2 &= \int_0^{2\pi} d\varphi \, \cos^2\varphi \int_0^\pi \sin\theta \, d\theta \, \sin^2\theta \\ &= \int_0^{2\pi} d\varphi \, \frac{1}{2}(1+\cos 2\varphi) \int_{-1}^1 d(\cos\theta) \left(1-\cos^2\theta\right) \\ &= \frac{4\pi}{3}. \end{aligned} \tag{9.121}$$

Similarly,

$$\int_\Omega d\Omega \, n_2^2 = \int_\Omega d\Omega \, n_3^2 = \frac{4\pi}{3}, \tag{9.122}$$

while the result is vanishing if we integrate $n_1 n_2$, $n_1 n_3$, and $n_2 n_3$. Hence:

$$\int_\Omega d\Omega \, n_i n_j = \frac{4\pi}{3}\delta_{ij}, \tag{9.123}$$

in agreement with Eq. (9.59).

Consider now the integral

$$\int_\Omega d\Omega \, n_i n_j n_k n_l. \tag{9.124}$$

By using the definition (9.56) we find that the integral is vanishing if three or more indices are different. Otherwise we obtain

$$\int_\Omega d\Omega \, n_1^2 n_2^2 = \int_\Omega d\Omega \, n_1^2 n_3^2 = \int_\Omega d\Omega \, n_2^2 n_3^2 = \frac{4\pi}{15}, \tag{9.125}$$

and

$$\int_\Omega d\Omega \, n_1^4 = \int_\Omega d\Omega \, n_2^4 = \int_\Omega d\Omega \, n_3^4 = \frac{4\pi}{5}. \tag{9.126}$$

We can thus express the result in compact form as follows:

$$\int_{\Omega} d\Omega\, n_i n_j n_k n_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{9.127}$$

in agreement with Eq. (9.60)

9.6 Solution The motion of the given oscillator is described by the equations

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3(t) = L\cos\omega t, \tag{9.128}$$

and its quadrupole moment is given by

$$Q_{ij} = \int d^3x\, \rho\big(3x_i x_j - r^3\delta_{ij}\big), \tag{9.129}$$

where (for a point-like mass):

$$\rho = m\delta(x_1)\delta(x_2)\delta(x_3 - L\cos\omega t), \qquad r^2 = L^2\cos^2\omega t. \tag{9.130}$$

By performing the volume integration we then obtain

$$Q_{11} = Q_{22} = -mL^2\cos^2\omega t, \qquad Q_{33} = 2mL^2\cos^2\omega t. \tag{9.131}$$

It should be noted that Q is diagonal, and that it satisfies the traceless condition $\delta^{ij}Q_{ij} = 0$.

By computing the time derivatives of Q we find

$$\dddot{Q}_{11} = \dddot{Q}_{22} = -8mL^2\omega^3\cos\omega t\sin\omega t, \tag{9.132}$$

$$\dddot{Q}_{33} = 16mL^2\omega^3\cos\omega t\sin\omega t, \tag{9.133}$$

so that

$$\dddot{Q}_{ij}\dddot{Q}^{ij} = \dddot{Q}_{11}^2 + \dddot{Q}_{22}^2 + \dddot{Q}_{33}^2 = 384\,m^2L^4\omega^6\cos^2\omega t\sin^2\omega t. \tag{9.134}$$

The average over a period $T = 2\pi/\omega$ gives

$$\frac{1}{T}\int_0^T dt\, \cos^2\omega t\sin^2\omega t = \frac{1}{8}. \tag{9.135}$$

By applying Eq. (9.63) we finally find that the averaged power emitted by the oscillator in the form of quadrupole gravitational radiation is given by

$$\left\langle\frac{dE}{dt}\right\rangle = -\frac{G}{45c^5}\langle\dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = -\frac{48G}{45c^5}\,m^2L^4\omega^6. \tag{9.136}$$

Chapter 10
The Schwarzschild Solution

So far we have only used the linearized Einstein equations, and considered geometric configurations typical of the weak-field approximation. In this chapter we will apply for the first time the full Einstein equations without approximations, and we will obtain a particular exact solution for a static, spherically symmetric gravitational field.

Such a solution—the well known Schwarzschild solution—will be used to illustrate one of the most celebrated predictions of general relativity: the perihelion precession of the planetary orbits. Such an effect, experimentally known since the nineteenth century for the planets of our solar system, has provided one of the most convincing observational test of the Einstein theory.

It should be immediately stressed, however, that the Schwarzschild solution plays a very important role in general relativity not only for its phenomenological applications, but also for its formal properties. It gives indeed a simple and basic example of how gravity can modify the (classical) causal structure of space–time, by introducing "event horizons" which limit our possibility to get information from a given portion of space (the interior of the so-called "black hole"). Also, extrapolated to the limit $r \to 0$, it represents one of the simplest models of geometric singularity, i.e. of "geodesically incomplete" space–time manifold.

10.1 Spherically Symmetric Einstein Equations in Vacuum

Let us look for solutions of the Einstein equations (7.29) describing the geometry associated to a spherically symmetric gravitational field, generated by a central source. We are interested, in particular, in the vacuum geometry ("externally" to the matter sources): we can thus set $T_{\mu\nu} = 0$, and in that case the equations simply reduce to $R_{\mu\nu} = 0$. All we need, therefore, are the components of the Ricci tensor for a metric $g_{\mu\nu}$ which describes a spherically symmetric three-dimensional space.

This means, more precisely, that the spatial part (g_{ij}) of our metric has to be rotationally invariant, i.e. it must admit the rotation group $SO(3)$ as its isometry

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group. We can also say, using the terminology of Sect. 6.3, that it must be possible to choose an "adapted" foliation of the space–time manifold based on three-dimensional space-like sections containing maximally symmetric subspaces with $n = 2$ dimensions, and then characterized by $n(n+1)/2 = 3$ Killing vectors (corresponding, in this case, to the three generators of the spatial rotations).

Using polar coordinates, $x^\mu = (ct, r, \theta, \varphi)$, the above condition can be easily satisfied by imposing that the space–time sections specified by fixed values of t and r correspond to two-dimensional spherical surfaces. The most general line-element satisfying this property is then the following:

$$ds^2 = A_1(r,t)c^2\,dt^2 - A_2(r,t)\,dr^2 - A_3(r,t)\,dr\,dt - A_4(r,t)\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big), \tag{10.1}$$

where A_i, $i = 1, \ldots, 4$, are arbitrary real functions of r and t. At fixed values of t and r we have, in fact, $dr = 0 = dt$, and we recover the line-element of a two-dimensional sphere of constant radius $a = A_4^{1/2}$ (see Eq. (2.24)).

Before starting the computation of the Ricci tensor it is convenient to note that the above metric can be further simplified, by imposing appropriate gauge conditions which preserve the required spherical symmetry. In particular, we can introduce two new coordinates $\tilde{t}$ and $\tilde{r}$ defined by the transformation

$$t = f_1(\tilde{t}, \tilde{r}), \qquad r = f_2(\tilde{t}, \tilde{r}) \tag{10.2}$$

(which does not involve the angular variables), and choose the two arbitrary functions f_1, f_2 in such a way that, in the new chart, $\tilde{A}_3 = 0$ and $\tilde{A}_4 = \tilde{r}^2$.

In such a new chart, dropping for simplicity the tilde symbol, and adopting the following (largely used) notation:

$$g_{00} = A_1 = e^{\nu(r,t)}, \qquad g_{11} = -A_2 = -e^{\lambda(r,t)}, \tag{10.3}$$

the line-element takes the form

$$ds^2 = e^{\nu}c^2\,dt^2 - e^{\lambda}\,dr^2 - r^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big). \tag{10.4}$$

The two functions ν and λ only depend on r and t, and will now be determined by imposing that the metric satisfies the Einstein equations in vacuum.

Let us first notice, to this aim, that the chosen metric $g_{\mu\nu}$ is represented by a diagonal matrix,

$$g_{\mu\nu} = \mathrm{diag}\big(e^{\nu}, -e^{\lambda}, -r^2, -r^2\sin^2\theta\big), \tag{10.5}$$

so that the (contravariant) components of the inverse metric are simply given by

$$g^{\mu\nu} = \mathrm{diag}\big(e^{-\nu}, -e^{-\lambda}, -r^{-2}, -r^{-2}\sin^{-2}\theta\big). \tag{10.6}$$

Let us also recall that, in our case, $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and apply the standard definition (3.90) of the Christoffel connection. Denoting with a dot and a

prime, respectively, the partial derivatives with respect to t and r, we find that the non-zero components of the connection are the following:

$$\begin{aligned}
&\Gamma_{00}{}^{0} = \frac{\dot{\nu}}{2c}, \qquad \Gamma_{00}{}^{1} = \frac{\nu'}{2} e^{\nu-\lambda}, \qquad \Gamma_{01}{}^{0} = \frac{\nu'}{2}, \\
&\Gamma_{01}{}^{1} = \frac{\dot{\lambda}}{2c}, \qquad \Gamma_{11}{}^{0} = \frac{\dot{\lambda}}{2c} e^{\lambda-\nu}, \qquad \Gamma_{11}{}^{1} = \frac{\lambda'}{2}, \\
&\Gamma_{12}{}^{2} = \frac{1}{r}, \qquad \Gamma_{13}{}^{3} = \frac{1}{r}, \qquad \Gamma_{22}{}^{1} = -r e^{-\lambda}, \\
&\Gamma_{23}{}^{3} = \frac{\cos\theta}{\sin\theta}, \qquad \Gamma_{33}{}^{1} = -r \sin^2\theta e^{-\lambda}, \qquad \Gamma_{33}{}^{2} = -\sin\theta\cos\theta.
\end{aligned} \tag{10.7}$$

We are now in the position of computing the components of the Ricci tensor, and imposing the vacuum Einstein equations $R_{\mu\nu} = 0$. Such a system of equations takes a simpler form if we use, in particular, the mixed components of the Ricci tensor, $R_\nu{}^\mu \equiv g^{\mu\alpha} R_{\nu\alpha}$. By using the definition (6.21), and equating to zero all nonvanishing components, we obtain

$$R_1{}^1 = e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} - \frac{\lambda'}{r}\right) - \frac{e^{-\nu}}{c^2}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) = 0, \tag{10.8}$$

$$R_2{}^2 = R_3{}^3 = \frac{1}{r^2}\left[e^{-\lambda}\left(1 + \frac{r\nu'}{2} - \frac{r\lambda'}{2}\right) - 1\right] = 0, \tag{10.9}$$

$$R_0{}^0 = e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'}{r}\right) - \frac{e^{-\nu}}{c^2}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) = 0, \tag{10.10}$$

$$R_1{}^0 = \frac{e^{-\nu}}{cr}\dot{\lambda} = 0, \qquad R_0{}^1 = -\frac{e^{-\lambda}}{cr}\dot{\lambda} = 0. \tag{10.11}$$

In the next section we will show that this set of equations admits a simple exact solution for the two unknown functions ν and λ.

10.2 The Birkhoff Theorem and the Schwarzschild Solution

We start with the two equations (10.11), which imply $\dot{\lambda} = 0$ and then $\lambda = \lambda(r)$. With such a condition all terms with the time-derivatives of λ drop from the above equations. We are left with three equations for the two unknowns λ and ν but, as we shall see, only two equations are independent.

By subtracting Eq. (10.8) from Eq. (10.10) we obtain the condition

$$\nu' + \lambda' = 0, \tag{10.12}$$

whose integration gives

$$\nu + \lambda = f(t), \tag{10.13}$$

where f is an arbitrary function of the time coordinate only. Since $\lambda = \lambda(r)$ it follows that the time and the radial dependence, in the g_{00} component of the metric, can be factorized as

$$g_{00}c^2\,dt^2 = e^{\nu}c^2\,dt^2 \equiv e^{-\lambda(r)}e^{f(t)}c^2\,dt^2. \tag{10.14}$$

Performing the coordinate transformation $t \to \tilde{t}$, where $\tilde{t}$ is defined by

$$e^{f(t)/2}\,dt = d\tilde{t}, \tag{10.15}$$

the spherical symmetry is preserved, and we can always eliminate the time dependence of g_{00} (i.e. of ν) by absorbing such a dependence into the new time coordinate $\tilde{t}$. Hence, we can look for a solution depending only on the radial coordinate, and satisfying the condition

$$\nu(r) = -\lambda(r). \tag{10.16}$$

It is appropriate, at this point, to introduce the definition of *static* metric: a metric is said to be "static" if there is a chart where $g_{i0} = 0$ and where all nonvanishing metric components are time independent, $\partial_0 g_{\mu\nu} = 0$. We can then summarize the previous result by saying that a spherically symmetric metric, which satisfies the Einstein equations in vacuum, is necessarily static. This proposition coincides with the statement of the so-called *Birkhoff theorem*.

It should be stressed, for the sake of clarity, that a static metric is characterized by the presence of a time-like Killing vector ξ^{μ}, which, as discussed in Sect. 3.3, guarantees the existence of a chart in which $\partial_0 g_{\mu\nu} = 0$. This condition, satisfied by all metrics of *stationary* type, does not ensure, however, the validity of the additional condition $g_{i0} = 0$, required for the metric to be static. This additional condition is satisfied, and the metric is static (and not only stationary), if and only if the Killing vector satisfies $\xi_{[\mu}\nabla_{\nu}\xi_{\alpha]} = 0$ (see Exercise 10.1).

Using the result (10.16) we can now easily solve Eq. (10.9), which becomes

$$e^{\nu}\left(1 + r\nu'\right) \equiv \left(e^{\nu}r\right)' = 1. \tag{10.17}$$

Integrating, and dividing by r, we obtain

$$e^{\nu} = 1 - \frac{2m}{r} = e^{-\lambda}, \tag{10.18}$$

where we have called $-2m$ the integration constant, which has dimension of length (the presence of the negative sign has an important phenomenological motivation, as will be explained in the next subsection). We thus arrive at the well known Schwarzschild solution, which corresponds to the line-element

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2\,dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{10.19}$$

and which describes the geometry of the empty space–time bent by the spherically symmetric gravitational field surrounding a central source.

Let us immediately notice that this metric has a singularity at $r = 2m$, where $g_{00} \to 0$ and $g_{11} \to \infty$. For $r < 2m$ the sign of the components g_{00} and g_{11} is reversed, and the given coordinates are no longer appropriate to parametrize the space–time geometry. This point will be discussed in detail in Sect. 10.4.

Let us notice, also, that the solution (10.19) not only satisfies Eq. (10.9) and a linear combination of Eqs. (10.8), (10.10), but also separately satisfies Eq. (10.8) and Eq. (10.10) (which are exactly equivalent in the case of the Schwarzschild solution). In fact,

$$e^{\nu}\nu' = \frac{2m}{r^2}, \qquad e^{\nu}\left(\nu'' + \nu'^2\right) = -\frac{4m}{r^3}, \tag{10.20}$$

so that

$$R_1{}^1 = e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{r}\right) = -\frac{2m}{r^3} + \frac{2m}{r^3} = 0. \tag{10.21}$$

10.2.1 The Weak-Field Limit

In order to physically interpret the integration constant $-2m$, and understand the reason of the negative sign we have chosen, let us rewrite the Schwarzschild solution in the so-called "isotropic" chart, characterized by a radial coordinate $\tilde{r}$ such that

$$r = \tilde{r}\left(1 + \frac{m}{2\tilde{r}}\right)^2. \tag{10.22}$$

In this chart

$$dr = d\tilde{r}\left(1 - \frac{m^2}{4\tilde{r}^2}\right), \tag{10.23}$$

and the line-element (10.19) becomes

$$ds^2 = \left(\frac{1 - \frac{m}{2\tilde{r}}}{1 + \frac{m}{2\tilde{r}}}\right)^2 c^2\, dt^2 - \left(1 + \frac{m}{2\tilde{r}}\right)^4 \left[d\tilde{r}^2 + \tilde{r}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]. \tag{10.24}$$

Switching from polar to Cartesian coordinates through the standard transformation

$$\begin{aligned} &x_1 = \tilde{r}\sin\theta\cos\varphi, \qquad x_2 = \tilde{r}\sin\theta\sin\varphi, \qquad x_3 = \tilde{r}\cos\theta, \\ &\tilde{r} = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2} = |\boldsymbol{x}|, \end{aligned} \tag{10.25}$$

we then obtain

$$ds^2 = \left(\frac{1 - \frac{m}{2|\boldsymbol{x}|}}{1 + \frac{m}{2|\boldsymbol{x}|}}\right)^2 c^2\, dt^2 - \left(1 + \frac{m}{2|\boldsymbol{x}|}\right)^4 |d\boldsymbol{x}|^2. \tag{10.26}$$

These new coordinates are called isotropic because the spatial part of the metric is exactly the same along any particular spatial directions, as clearly shown by the above equation.

Let us now consider the limit of large distances from the central source, $|\boldsymbol{x}| \to \infty$. In this limit we can expand the line-element for $m/|\boldsymbol{x}| \ll 1$, and obtain

$$ds^2 = \left(1 - \frac{2m}{|\boldsymbol{x}|}\right)c^2\,dt^2 - \left(1 + \frac{2m}{|\boldsymbol{x}|}\right)|d\boldsymbol{x}|^2. \tag{10.27}$$

But at arbitrarily large distances from the sources the gravitational field becomes arbitrarily weak, and our exact solution must reproduce the metric obtained by solving the linearized Einstein equations in the weak-field approximation (see Eq. (8.22)).

Comparing our limit (10.27) with the approximated solution (8.22), and identifying $-2m/|\boldsymbol{x}|$ with $2\phi/c^2$, we find that the Schwarzschild solution may describe the realistic gravitational field of a central source provided the integration constant of the Einstein equations is related to the total mass M of the central body by

$$2m = \frac{2GM}{c^2}. \tag{10.28}$$

The constant parameter $2m$ is dimensionally a length, and is called the "Schwarzschild radius". The negative sign present in the solution is needed to obtain a field of forces of attractive type, together with a positive value for mass of the source, $M > 0$.

10.3 Perihelion Precession

The Schwarzschild solution provides a good approximation to the gravitational field of the Sun in interplanetary space. The planets move, in first approximation, as point-like test bodies along the geodesics of the Schwarzschild metric. Since the radial coordinates of the planets are much bigger than the Schwarzschild radius of the Sun (which is of the order of one kilometer), the planetary motion takes place in the weak-field regime $r \gg 2m$, and can be safely described in the context of the Schwarzschild metric (10.19) without ambiguity problems due to a possible exchange of roles of the time and radial coordinates.

In order to determine the general relativistic orbits let us start with the geodesic equation, written in the convenient (not explicitly covariant) form of Eq. (5.6):

$$\frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^\nu\right) = \frac{1}{2}\dot{x}^\alpha\dot{x}^\beta\partial_\mu g_{\alpha\beta} \tag{10.29}$$

(the dot denotes differentiation with respect to the proper time τ). We will use for $g_{\mu\nu}$ the general representation (10.5) (with $\lambda = -\nu$), and we will consider the separate integration of the various components of the geodesic equation.

The component $\mu = 0$,

$$\frac{d}{d\tau}\left(e^\nu\dot{x}^0\right) = 0, \tag{10.30}$$

can be immediately integrated, and gives

$$\dot{x}^0 = e^{-\nu} k, \tag{10.31}$$

where k is a constant of motion associated to the invariance under time translation (i.e. to the conservation of the total energy of the system).

The component $\mu = 2$ gives

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = \frac{1}{2}\dot{\varphi}^2 \frac{\partial}{\partial\theta}\left(r^2 \sin^2\theta\right), \tag{10.32}$$

namely

$$r^2\ddot{\theta} + 2\dot{r}\dot{\theta} - r^2\dot{\varphi}^2 \sin\theta\cos\theta = 0. \tag{10.33}$$

Choosing as initial conditions $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$, the above equation implies $\ddot{\theta} = 0$, and is identically satisfied by $\theta = \pi/2 =$ constant. This means that the motion is confined to a plane (like in the non-relativistic case), and that we can always chose our reference system in such a way that the plane of the motion coincides with the equatorial plane $\theta = \pi/2$. In all subsequent computation we will adopt this choice, which greatly simplifies all equations.

The component $\mu = 3$,

$$\frac{d}{d\tau}\left(r^2\dot{\varphi}\right) = 0, \tag{10.34}$$

can also immediately integrated, and gives

$$\dot{\varphi} = \frac{h}{r^2}, \tag{10.35}$$

where h is a constant of motion associated to the rotational invariance (i.e. to the conservation of the angular momentum) in the equatorial plane $\theta = \pi/2$.

Finally, we need the equation for the radial motion, corresponding to the $\mu = 1$ component of the geodesic equation. Instead of directly considering such an equation, however, we can conveniently exploit the normalization of the four-velocity vector, which imposes the condition $\dot{x}^\mu \dot{x}_\mu = c^2$. By expressing $\dot{x}^0$ and $\dot{\varphi}$ in terms of the integration constants (10.31), (10.35), and using $\dot{\theta} = 0$, $\theta = \pi/2$, we obtain

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \equiv e^{-\nu}k^2 - e^{-\nu}\dot{r}^2 - \frac{h^2}{r^2} = c^2. \tag{10.36}$$

Solving with respect to $\dot{r}$ we arrive at an equation $\dot{r} = \dot{r}(r)$ which describes the radial motion of the given test body.

In the case of planetary motions of orbital type, confined within a finite portion of the equatorial plane, it is convenient to use $r = r(\varphi)$ instead of $r = r(t)$ as the parametric equation. To this aim, let us denote with a prime the derivative with respect to φ, and express $\dot{r}$ as $\dot{r} = r'\dot{\varphi}$. It is also a common practice, in the context

of celestial mechanics, to introduce the variable $u = r^{-1}$, such that $r' = -u'u^{-2}$. Using Eq. (10.35) we have then

$$\dot{r} = -u'u^{-2}\dot{\varphi} = -hu', \tag{10.37}$$

and the condition (10.36) can be rewritten as

$$e^{-\nu}k^2 - e^{-\nu}h^2u'^2 - h^2u^2 = c^2. \tag{10.38}$$

By multiplying by $e^{\nu}h^{-2}$, and differentiating with respect to φ, we finally obtain the following equation for the geodesic motion in the equatorial plane $\theta = \pi/2$:

$$2u'u'' + 2uu' - 6mu^2u' - \frac{2mc^2}{h^2}u' = 0. \tag{10.39}$$

The above equations can be satisfied in two ways. A first possibility is $u' = 0$, namely $r =$ constant. This corresponds to a circular orbit of constant radius, but this is not the case we are interested in, because a circular motion cannot be associated, of course, to any type of precession. If $u' \neq 0$, instead, we can divide by u', and Eq. (10.39) reduces to

$$u'' + u = \frac{mc^2}{h^2} + 3mu^2, \tag{10.40}$$

which is the *exact* equation for the (non-circular) orbit of a planet in the Schwarzschild gravitational field. All differences from the corresponding Newtonian equation are contained in the last term $3mu^2$, which represents the relativistic corrections due to the space–time curvature.

Since such corrections are small with respect to the other terms ($mu = m/r \ll 1$, hence $mu^2 \ll u$) it will be enough, to reach our purpose, to solve the geodesic equation with a perturbative expansion, by defining:

$$u = u_{(0)} + u_{(1)} + \cdots, \tag{10.41}$$

where the zeroth-order term $u_{(0)}$ satisfies the unperturbed Newtonian equation

$$u''_{(0)} + u_{(0)} = \frac{mc^2}{h^2}. \tag{10.42}$$

The general exact solution for $u_{(0)}$ is then given by

$$u_{(0)} = \frac{mc^2}{h^2}\big[1 + e\cos(\varphi - \varphi_0)\big], \tag{10.43}$$

where φ_0 and e are integration constants (see also Eq. (2.10) in the non-relativistic limit $k \to 1$). For $0 \leq e \leq 1$ this solution describes, in polar coordinates, an ellipse of eccentricity e and major semi-axis

$$a = \frac{h^2}{mc^2(1 - e^2)}. \tag{10.44}$$

In order to compute the "post-Newtonian" corrections let us insert the expansion (10.41) into the exact equation (10.40). To first order we obtain for $u_{(1)}$ the following equation:

$$u''_{(1)} + u_{(1)} = 3mu^2_{(0)} = \frac{3m^3c^4}{h^4}\left[1 + 2e\cos(\varphi - \varphi_0) + e^2\cos^2(\varphi - \varphi_0)\right], \tag{10.45}$$

where the relativistic term, evaluated for the unperturbed solution, plays the role of source for the first-order corrections (notice that, working in the weak-field approximation, we have neglected the contribution $6mu_{(0)}u_{(1)} \ll u_{(1)}$). Considering orbits of small eccentricity ($e \ll 1$) we can neglect the term $e^2\cos^2\varphi$ with respect to $e\cos\varphi$. Also, the constant term on the right-hand side of the above equation can be absorbed into the Newtonian part of the solution, simply by rescaling the constant h that determines the orbital parameters. In this way we end up with the following equation for the $u_{(1)}$ correction:

$$u''_{(1)} + u_{(1)} = \frac{6m^3c^4}{h^4}e\cos(\varphi - \varphi_0), \tag{10.46}$$

with particular exact solution:

$$u_{(1)} = \frac{3m^3c^4}{h^4}e\varphi\sin(\varphi - \varphi_0). \tag{10.47}$$

By including the first-order corrections due to the Schwarzschild geometry we thus arrive at the following (approximated) solution of the orbital equation:

$$u \simeq u_{(0)} + u_{(1)} = \frac{mc^2}{h^2}\left[1 + e\cos(\varphi - \varphi_0) + \frac{3m^2c^2}{h^2}e\varphi\sin(\varphi - \varphi_0)\right]. \tag{10.48}$$

Let us now define

$$\Delta\varphi = \frac{3m^2c^2}{h^2}\varphi, \tag{10.49}$$

and observe that $|\Delta\varphi| \sim 3mu_{(0)} \sim 3m/r \ll 1$. By applying the subtraction formula $\cos(\alpha - \beta)$ for a small angle, $|\beta| \ll 1$, i.e. by using the relation

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \simeq \cos\alpha + \beta\sin\alpha, \tag{10.50}$$

we can finally rewrite the solution (10.48) as

$$u = \frac{mc^2}{h^2}\left[1 + e\cos(\varphi - \varphi_0 - \Delta\varphi)\right]. \tag{10.51}$$

This solution is now to be compared with the Newtonian solution (10.43). The orbit described by the general relativistic solution is still confined between a minimal

and a maximal distance from the origin,

$$\frac{h^2}{mc^2(1+e)} \leq r \leq \frac{h^2}{mc^2(1-e)}, \tag{10.52}$$

but—unlike the Newtonian ellipse (10.43)—it is no longer a closed orbit: it is an open curve describing a "rosetta-like" path (see also the introduction to Chap. 2). If we consider, in particular, the position of closest approach to the central source (the so-called perihelion), we find that, after the angle subtended by the motion is $\varphi - \varphi_0 = 2\pi$, the perihelion position is shifted with respect to the starting position at $\varphi = \varphi_0$. To each planetary revolution is associated an angular shift of the perihelion given by

$$\Delta\varphi(2\pi) = \frac{6\pi m^2 c^2}{h^2} = \frac{6\pi G^2 M^2}{h^2 c^2} \tag{10.53}$$

(where we have used the definition of Schwarzschild radius (10.28)). We can immediately notice that this effect, induced by the curved space–time geometry, is about six times bigger than the one obtained by including the kinematic corrections due to special relativistic effects (see Eq. (2.11)).

Using the definition (10.44) of major semi-axis, Eq. (10.53) can also be rewritten as

$$\Delta\varphi(2\pi) = \frac{6\pi GM}{a(1-e^2)c^2}. \tag{10.54}$$

Expressed in this form, it is evident that the smaller is a (at fixed e), the larger is the shift effect. In fact, it is just in the case of Mercury that we have observed the largest value of perihelion shift: after a long series of highly accurate astronomical measurements, dating back to the second half of the 18th century, and after subtracting all known precession contributions due to the interference of other planets, we are left with a residual effect (anomalous with respect to the Newtonian theory) of about 43.11 arc-seconds per century.

The result (10.54), applied to Mercury, predicts a perihelion shift $\Delta\varphi = 0.1038$ arc-seconds per revolution. Given that in a century there is time for Mercury to perform 415 solar revolution, the theoretical prediction reproduces the observed effect up to an accuracy of about one percent. This agreement is very good, taking into account that there are many possible sources of systematic errors (such as deviations from exact spherical symmetry of the solar field, and so on), which can independently contribute to a small shift, to be added to the Schwarzschild contribution of Eq. (10.54).

10.4 Event Horizon and Kruskal Coordinates

Let us now suppose that the source of the metric (10.19) has a very compact extension, and is fully contained within a region centered on the origin, of radius $r < 2m$.

In that case it makes sense to consider the Schwarzschild solution also in the strong field regime $r \sim 2m$. We should recall, in fact, that such solution only holds in vacuum, and then it can be applied, at most, up to the external surface of the central matter distribution acting as gravitational source. Inside the source we must solve the Einstein equations with $T_{\mu\nu} \neq 0$.

It is still unclear, at present, whether or not such very compact bodies may really exist in Nature. There are indirect indications, at the level of astrophysical observations, that seems to confirm their existence: we can say, however, that a definitive experimental confirmation is still lacking. Nevertheless, the study of the Schwarzschild solution in the regime $r \leq 2m$ is of great theoretical interest, as an example of space–time manifold with a causal structure qualitatively different from that of the Minkowski space–time. The Schwarzschild manifold is characterized, in particular, by a horizon at $r = 2m$, and by a singularity at $r = 0$.

In order to illustrate the first property we may consider a central body of size $r > 2m$, which is uniformly collapsing in the radial direction without breaking the spherical symmetry of the original configuration.The surface of the body, for an external observer placed at a distance $r_1 > r$ from the center, will always keep *outside* the Schwarzschild radius, as if such radius would represent an insurmountable barrier.

More precisely, the proper-time interval $\Delta\tau$ needed to reach the radial coordinate $r = 2m$ starting from $r > 2m$ (an interval which is finite for an observer at rest on the surface of the collapsing body, as can be easily checked by integrating the radial geodesic equation), becomes an *infinite* time interval for any external observer at rest at r_1 (for any $r_1 > 2m$), because of the time-dilatation effect produced by the gravitational field. In fact, by applying the results of Sect. 5.3 to the Schwarzschild metric, we have

$$\Delta\tau(r_1) = \left[\frac{g_{00}(r_1)}{g_{00}(r)}\right]^{1/2} \Delta\tau = \left(1 - \frac{2m}{r_1}\right)^{1/2} \frac{\Delta\tau}{(1 - \frac{2m}{r})^{1/2}} \underset{r\to 2m}{\longrightarrow} \infty \qquad (10.55)$$

(see Eq. (5.32)).

This means that the surface $r = 2m$ provides a simple example of what is called an "event horizon", i.e. a surface of infinite redshift. Suppose, in fact, that the surface of the collapsing body is continuously emitting signals (for instance, electromagnetic radiation) with proper frequency ω, in the outward direction. The signals are received by the static observer at $r_1 > r$ with a frequency which is redshifted by the action of the gravitational field (see Eq. (5.34)), and which is given by

$$\omega(r_1) = \left[\frac{g_{00}(r)}{g_{00}(r_1)}\right]^{1/2} \omega = \left(1 - \frac{2m}{r}\right)^{1/2} \frac{\omega}{(1 - \frac{2m}{r_1})^{1/2}} \underset{r\to 2m}{\longrightarrow} 0. \qquad (10.56)$$

As the collapsing surface approaches the Schwarzschild radius the emitted signal is received with smaller and smaller frequencies (i.e. smaller and smaller energies), until it completely disappears when emitted from the position $r = 2m$. No signal can reach an external observer starting from the spherical surface of radius $2m$, which

thus looks black, dark, as if no radiation could (classically) leave that surface.[1] It is just because of this effect that the region enclosed inside that surface is called "black hole".[2]

It must be stressed, at this point, that the presence of a horizon at $r = 2m$, characterized by the singularity of the metric (10.19), by the divergence of the collapsing time (10.55), and by the infinite redshift (10.56), does not necessarily implies that the surface $r = 2m$ has to be interpreted as a "physically" singular region of space–time (namely, as an inaccessible region, excluded from the physical space–time). That things are not this way is suggested, first of all, by the study of the curvature tensor, since the scalar objects formed with this tensor tend to diverge in the presence of a space–time singularity.

More precisely, it can be shown that the regularity of all curvature scalars is a *necessary* (but not sufficient) condition for the absence of space–time singularities (see e.g. the book [25] of the bibliography). For a geometry satisfying the vacuum Einstein equations, in particular, there are four nonvanishing scalar object which can be formed with the metric and its first and second derivatives, without introducing covariant derivatives of the curvature tensor:[3]

$$\begin{aligned} &R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \qquad R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma}\eta^{\mu\nu\alpha\beta}, \\ &R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma}R^{\mu\nu\alpha\beta}, \qquad R_{\mu\nu\rho\sigma}R_{\alpha\beta\lambda\delta}R^{\mu\nu\alpha\beta}\eta^{\rho\sigma\lambda\delta}. \end{aligned} \tag{10.57}$$

For the Schwarzschild solution all these scalars are regular at $r = 2m$. Considering, for instance, the square of the Riemann tensor we have

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48m^2}{r^6} \tag{10.58}$$

(see Exercise 10.2). All scalars, however, unambiguously denote the presence of a singularity at $r = 0$.

The fact that the curvature is regular at a given space–time position, while the metric is not, is a situation—often encountered in the context of differential geometry—typically due to a "bad" choice of the coordinate system. In particular, the chart we have used to give the Schwarzschild solution in the form of Eq. (10.19) is perfectly appropriate to describe the space–time region characterized by $r > 2m$, but—because of the horizon singularity—is not appropriate to cover the whole space–time manifold describing the gravitational field produced in vacuum by a central source. If so, there must exist a chart (let us call it $\{\bar{x}^\mu\}$) completing the previous one, namely a chart in which the Schwarzschild solution can be extended even

[1] Actually, radiation can be emitted thanks to quantum effects, as first shown by [24].

[2] There is a curious coincidence concerning the name of the physicist who discovered this metric: Schwarzschild, in German language, means indeed "black shield".

[3] If the metric is not Ricci-flat, i.e. if $R_{\mu\nu} \neq 0$ and $R \neq 0$, the number of such scalar objects raises from 4 to 14.

below $r = 2m$ without encountering metric singularities, down to the (unavoidable) physical singularity present at $r = 0$.

Such a chart $\{\bar{x}^\mu\}$ represents, in the language of differential geometry, the *maximal analytical extension* of the coordinate system for the given space–time manifold, and is characterized by the following properties. For a regular (or, more precisely, *geodesically complete*) manifold, all the geodesics of $\{\bar{x}^\mu\}$ can be extended for arbitrary values of their affine parameter without encountering singularities, starting from any point of the given manifold. If the manifold is not regular (i.e., if it is geodesically incomplete), then some of the geodesic of $\{\bar{x}^\mu\}$ can get to an abrupt end by hitting a physical space–time singularity (like, for instance, the one at $r = 0$ in the Schwarzschild solution). However, all geodesics that are not hitting singularities (if any) must be arbitrarily extensible, like in the previous case.

For a simple illustration of the above notions we can take, for instance, a two-dimensional (pseudo-Euclidean) section M_2 of the Minkowski space–time M_4. This section is, of course, a regular manifold: the Cartesian chart $\{\bar{x}^\mu\} = (x, ct)$ provides an example of maximal analytical extension for the M_2 coordinates, because its geodesics—the straight lines of the pseudo-Euclidean plane—can be extended without any geometrical obstruction from $-\infty$ to $+\infty$, starting from any given point of M_2.

If we consider instead the so-called Rindler coordinates (ξ, η), defined by

$$x = \xi \cosh \eta, \qquad ct = \xi \sinh \eta \tag{10.59}$$

(we are assuming that η is dimensionless), then we have a chart that—as shown in Exercise 6.1—only cover a portion of the M_2 manifold, defined by the conditions $x > |ct|$ and $x < -|ct|$ (the so-called Rindler manifold, i.e. the portion of M_2 "external" to the light cone $x = \pm ct$). The Rindler geodesics do not hit any singular point of space–time (because M_2 has no singularities); however, they cannot be arbitrarily extended (unlike the Cartesian straight lines), as there are indeed geodesics reaching the border of the Rindler manifold in a finite proper-time interval (see Exercise 10.3), and on the border they must necessarily come to an end. It follows that the coordinates (ξ, η) do not represent a maximal analytical extension for the M_2 manifold, but a chart defined on M_2 which can be possibly extended to the whole manifold (as done indeed by the transformation (10.59)).

In the case of the Schwarzschild geometry the situation is very similar to the one described above, with the important difference that the Schwarzschild manifold is not a regular one, because of the presence of a singularity at $r = 0$. Hence, whatever the used chart, there will always be geodesics ending at the origin in a finite interval of their time parameter. In the chart of Eq. (10.19), however, there is a horizon at $r = 2m$ where the metric becomes singular (but not the space–time manifold). Since the geodesics can reach the horizon in a finite proper-time interval, we can expect that those coordinates can be extended, just like the coordinates of the Rindler chart defined on M_2.

The maximal analytical extension for the space–time manifold associated to the Schwarzschild solution is provided by the chart parametrized by the so-called

Kruskal coordinates (u, v), related to the coordinates (r, ct) by a transformation not involving angular variables. Outside the horizon ($r > 2m$) the (dimensionless) Kruskal coordinates are defined by

$$\begin{aligned} u &= \pm\left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \cosh\left(\frac{ct}{4m}\right), \\ v &= \pm\left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \sinh\left(\frac{ct}{4m}\right). \end{aligned} \tag{10.60}$$

Inside the horizon ($r < 2m$) they are defined by

$$\begin{aligned} u &= \pm\left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \sinh\left(\frac{ct}{4m}\right), \\ v &= \pm\left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \cosh\left(\frac{ct}{4m}\right). \end{aligned} \tag{10.61}$$

It is understood that, in both cases, we take u and v with the same sign (see for instance the book [39] of the bibliography for a detailed derivation of the above transformations).

It can be easily checked that the Kruskal coordinates always satisfy the condition

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) e^{r/2m}, \tag{10.62}$$

both outside and inside the horizon. Their ratio, instead, gives

$$\frac{v}{u} = \tanh\left(\frac{ct}{4m}\right), \qquad r > 2m, \tag{10.63}$$

outside the horizon, and

$$\frac{u}{v} = \tanh\left(\frac{ct}{4m}\right), \qquad r < 2m, \tag{10.64}$$

inside the horizon. These last three relations will be useful when discussing the causal structure of the geometry associated to the Schwarzschild solution, as we will see in the next section.

Finally, it is instructive to rewrite the line-element (10.19) in terms of the Kruskal coordinates. Let us start with the region $r > 2m$. By differentiating Eq. (10.62) we have

$$dr = \frac{8m^2}{r} e^{-r/2m} (u\, du - v\, dv). \tag{10.65}$$

By differentiating Eq. (10.63), and using Eq. (10.60) for u^2, we obtain

$$c\, dt = \frac{8m^2}{r - 2m} e^{-r/2m} (u\, dv - v\, du). \tag{10.66}$$

Inserting this result into Eq. (10.19), and simplifying, we arrive at:

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} \left(dv^2 - du^2\right) - r^2 \left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{10.67}$$

Exactly the same result is obtained if the above procedure is applied to the case $r < 2m$. This provides explicit evidence that the Schwarzschild solution, written in the Kruskal chart, is regular at $r = 2m$ and keeps regular, as expected, for any r (excluding the central singularity at $r = 0$).

10.4.1 Causal Structure of the "Black Hole" Geometry

The line-element (10.67) represents the exact Schwarzschild solution (10.19), expressed in terms of Kruskal coordinates. It describes the vacuum geometry associated to the spherically symmetric gravitational field produced by a matter source centered around the origin. Unlike the line-element (10.19), the Kruskal parametrization can also be applied to the region $r < 2m$, and in principle down to $r = 0$ for a point-like source.

The metric (10.67) thus provides an appropriate model for the so-called *eternal black hole*, namely for an ideal physical system that, after completing the phase of gravitational collapse, has reached a final stable configuration, static and infinitely concentrated. Such a final state is probably not very realistic, from a phenomenological point of view; however, the study of this metric is especially instructive for what concerns the possible geometric properties of the space–time in the limit of very strong gravitational fields.

In order to discuss the properties of the space–time described by Eq. (10.67) we can conveniently focus our attention on its two-dimensional sections, parametrized by the coordinates u and v (the so-called "Kruskal plane"). Using Eq. (10.62) we can note, first of all, that the Schwarzschild horizon $r = 2m$ corresponds to the bisecting lines of the Kruskal plane, $u = \pm v$. Also, using Eqs. (10.63), (10.64), we can see that the line $u = v$ corresponds to $t = +\infty$, the line $u = -v$ to $t = -\infty$ (see Fig. 10.1, left panel). This in agreement with the already noticed fact that, for a static observer positioned at $r > 2m$, the time needed by a free falling test body to reach the Schwarzschild radius is infinite.

We also obtain, from Eqs. (10.63), (10.64), that the space–time sections at constant t are represented in the Kruskal plane by straight lines crossing the origin, $u/v = \text{constant}$. From Eq. (10.62) we obtain, instead, that the sections at constant r are hyperbolas, of two possible types:

$$\begin{aligned} u^2 - v^2 &= \text{const} > 0, \qquad r > 2m, \\ u^2 - v^2 &= \text{const} < 0, \qquad r < 2m. \end{aligned} \tag{10.68}$$

Depending on the sign of $u^2 - v^2$ those curves lie outside the horizon (sectors I and III of the Kruskal plane), or inside the horizon (sectors II and IV of the Kruskal plane) as illustrated in Fig. 10.1, right panel.

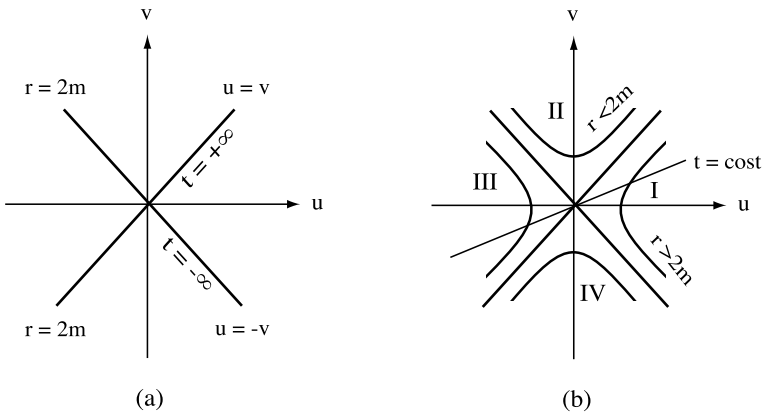

Fig. 10.1 *Left panel* (**a**): the Schwarzschild horizon in the Kruskal plane. *Right panel* (**b**): the sections $t = \text{const}$ are straight lines crossing the origin; the sections $r = \text{const}$ are hyperbolas, outside the horizon (sectors I and III) and inside the horizon (sectors II and IV)

We may recall, at this point, the previously mentioned analogy between the Kruskal plane (u, v) and the Minkowski plane (x, ct). There is, in particular, a close similarity between the curves at constant r outside the horizon, and the hyperbolic trajectories of uniformly accelerated observers in Minkowski space. This is more than a formal analogy, as a static observer located at constant r in the Schwarzschild field is indeed experiencing a constant acceleration due to the attraction of the central gravitational field.

In addition, the uniformly accelerated observers of Minkowski space have a horizon (corresponding to the asymptote of their hyperbolic trajectory) represented by the light cone $x = \pm ct$; in the Kruskal plane, the asymptotes of the hyperbolas correspond to the Schwarzschild horizon $u = \pm v$. We can say, in this context, that the role of the coordinates (r, ct) for the Schwarzschild space–time is just the analogue of that of the Rindler coordinates (10.59) for the Minkowski space–time. Indeed, just like the Rindler chart (ξ, η) only covers the portion of Minkowski plane exterior to the light cone, the chart (r, ct) only covers the portion of Kruskal plane exterior to the Schwarzschild horizon ($r > 2m$, or $u^2 > v^2$, i.e. the space–time sectors I and III of Fig. 10.1).

If we look at the sectors II and IV, inside the horizon, we find, however, an important difference between the Kruskal and the Minkowski plane. In the Minkowski plane the physically accessible space–time extends to infinity, while in the Kruskal plane the allowed portion of space–time is limited by the hyperbola $u^2 - v^2 = -1$, corresponding to the singularity at $r = 0$ (see Eq. (10.62)).

This is because the Kruskal chart is the maximal analytical extension of a space–time manifold which *is not* geodesically complete (due to the curvature singularity at $r = 0$). If we draw, in the Kruskal plane, a radial geodesic which evolves along the positive direction of the time-like axis v, starting from the sector I, we find that this trajectory can reach the sector II by crossing without problems the Schwarzschild

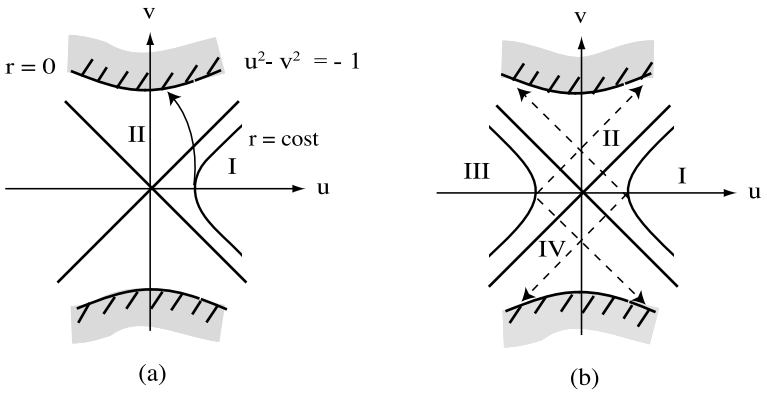

Fig. 10.2 *Left panel* (**a**): the physically accessible portion of Kruskal plane is limited by the hyperbola $r = 0$, which is reached in a finite proper time by a free falling observer. *Right panel* (**b**): the dashed lines denote possible light-like trajectories (of advanced and retarded type) in the Kruskal plane. The space–time sectors II, III and IV cannot communicate with sector I. The sectors I and III are causally disconnected

radius, but it has to stop, suddenly, as soon as it reaches (in a finite proper time) the hyperbola corresponding to the $r = 0$ singularity (see Fig. 10.2, left panel).

It is important to stress that the horizon $r = 2m$ can be crossed by physical (time-like or null) trajectories only from the outside ($r > 2m$) towards the inside ($r < 2m$), but not the opposite: in order to "exit" from sector I, in fact, the trajectory should be tilted at an angle larger than 45 degrees with respect to the vertical (time) axis, thus becoming space-like (and corresponding to a superluminal velocity). Once inside sector I, it becomes impossible for an observer to exit, or to send signals outside sector I. The Schwarzschild horizon thus (classically) behaves as a semipermeable membrane, which can be crossed in one direction only.

Let us notice, also, that the sectors I and II of the Kruskal plane have a symmetric counterpart (spatially reflected and time reversed) in the sectors III and IV, respectively—namely, in the sectors where we apply the coordinate transformations (10.60), (10.61) with the minus sign for both coordinates. Such twin sectors disappear if we impose that the points (u, v) and $(-u, -v)$ of the Kruskal plane are to be topologically identified, as is probably natural to assume (let us recall, in this respect, that the Einstein equations control the geometry of the space–time manifold, but leave its topology completely undetermined).

In the absence of any topological identification, and in the hypothesis that the sectors III and IV are real and physically distinct from their "copies" I and II, it should be stressed that they cannot send, in any way, signals towards the sector I (where, presumably, are localized the observers with whom we can properly identify ourselves).

The sector IV is also called "white hole", being isometric to the interior of the Schwarzschild horizon but with a sign of v which is the opposite of that of sector II: hence, it corresponds to a black hole where the time coordinate runs in the reversed

direction. It follows that the portion of horizon surrounding such a white hole, defined by the equations $r = 2m$, $v < 0$, can be crossed (in principle) only by time-like and light-like trajectories which, for an observer of sector I, are directed "towards the past" (again, from the outside towards to inside, but not the opposite). See for instance the book [25] of the bibliography.

Finally, if we consider an hypothetical exchange of signals between sector I and sector III, we can see from Fig. 10.2 (right panel) that their trajectory in the Kruskal plane should be characterized by a slope larger than 45 degrees with respect to the time axis, corresponding to superluminal velocities. Hence, the sectors I and III turns out to be causally disconnected.

The property of the Schwarzschild horizon to behave as a semipermeable surface, and its ability of producing a (classically) impenetrable shielding of some space–time regions with respect to others, has repeatedly suggested the possibility of applying to the black hole solutions a macroscopic "thermodynamical" description, associating to the horizon a well defined entropy proportional to its area [6]. A discussion of these aspects of black hole physics is, however, outside the scope of this short introduction, and the interested reader is referred, for instance, to the book [50] of the bibliography.

10.5 Exercises Chap. 10

10.1 Killing Vectors and Static Gravitational Fields A space–time manifold is characterized by a time-like killing vector ξ^μ. Show that the geometry is static (i.e. that there is a chart where the metric tensor satisfies the conditions $\partial_0 g_{\mu\nu} = 0$ and $g_{i0} = 0$) if and only if

$$\xi_{[\alpha} \nabla_\mu \xi_{\nu]} = 0. \tag{10.69}$$

10.2 Squared Riemann Tensor for the Schwarzschild Metric Compute the curvature invariant $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ for the Schwarzschild metric (10.19).

10.3 Geodesic Motion in the Rindler Manifold Consider the two-dimensional Rindler manifold described by the metric

$$ds^2 = \xi^2 \, d\eta^2 - d\xi^2, \tag{10.70}$$

and show that a test particle moving geodesically from ξ_0 towards the origin reaches the point $\xi = 0$ (on the border of the Rindler manifold) in a finite proper-time interval. This shows that the Rindler chart does not represent a maximally analytically extended system of coordinates for the Minkowski space–time.

10.6 Solutions

10.1 Solution Let us choose a chart where the time axis is oriented along the ξ^μ direction, namely a chart where $\xi^\mu = \delta^\mu_0$. In this case $\xi_\mu = g_{\mu 0}$ and $\xi^\mu \xi_\mu = g_{00} > 0$. The Killing condition $\delta_\xi g_{\mu\nu} = 0$, written explicitly according to Eq. (3.53), reduces to

$$\partial_0 g_{\mu\nu} = 0, \tag{10.71}$$

so that the metric is not time dependent. We also obtain, in this chart,

$$\nabla_\mu \xi_\nu = \nabla_\mu g_{\nu\alpha} \xi^\alpha = g_{\nu\alpha} \nabla_\mu \xi^\alpha = g_{\nu\alpha} \Gamma_{\mu 0}{}^\alpha = \frac{1}{2}(\partial_\mu g_{0\nu} - \partial_\nu g_{\mu 0}) = \partial_{[\mu} g_{\nu]0}. \tag{10.72}$$

If the metric is static it satisfies the condition $g_{i0} = 0$, so that

$$\xi_{[\alpha} \nabla_\mu \xi_{\nu]} = g_{0[\alpha} \partial_\mu g_{\nu]0} \equiv 0 \tag{10.73}$$

(because, in the above expression, the metric is non vanishing only for $\alpha = 0$ and $\nu = 0$).

Conversely, let us suppose that Eq. (10.69) is satisfied, and show that we can always find a chart where $g_{i0} = 0$.

Let us keep for the moment the coordinates where $\xi^\mu = \delta^\mu_0$ and the metric is time independent. Writing explicitly Eq. (10.69), and contracting with ξ^α, we obtain

$$\begin{aligned} &\xi^\alpha (\xi_\alpha \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\alpha + \xi_\nu \nabla_\alpha \xi_\mu - \xi_\alpha \nabla_\nu \xi_\mu - \xi_\mu \nabla_\alpha \xi_\nu - \xi_\nu \nabla_\mu \xi_\alpha) \\ &= \xi^2 \nabla_\mu \xi_\nu + \frac{1}{2} \xi_\mu \nabla_\nu \xi^2 + \xi_\nu \xi^\alpha \nabla_\alpha \xi_\mu - \{\mu \leftrightarrow \nu\} = 0, \end{aligned} \tag{10.74}$$

where $\xi^2 = \xi^\mu \xi_\mu$. From Eq. (10.72) we have

$$\xi^\alpha \nabla_\alpha \xi^\mu = \nabla_0 \xi_\mu = \partial_{[0} g_{\mu]0} = -\frac{1}{2} \partial_\mu g_{00} = -\frac{1}{2} \nabla_\mu \xi^2. \tag{10.75}$$

Inserting this result into Eq. (10.74), and dividing by ξ^4, we are led to the condition

$$\begin{aligned} &\xi^{-2} (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu) - \xi_\mu \nabla_\nu \xi^{-2} + \xi_\nu \nabla_\mu \xi^{-2} \\ &\equiv \nabla_\mu \left(\xi^{-2} \xi_\nu\right) - \nabla_\nu \left(\xi^{-2} \xi_\mu\right) = 0, \end{aligned} \tag{10.76}$$

which is solved by

$$\xi_\nu = \xi^2 \partial_\nu \phi, \tag{10.77}$$

where ϕ is an arbitrary scalar function. In the chart where we are working, on the other hand, we have $\xi_0 = g_{00} = \xi^2$, which implies $\partial_0 \phi = 1$, namely

$$\phi = x^0 + f\left(x^i\right), \tag{10.78}$$

where f is an arbitrary function of the spatial coordinates.

Let us now consider the coordinate transformation

$$x^0 \to x'^0 = \phi = x^0 + f\left(x^i\right), \quad x^i \to x'^i = x^i. \tag{10.79}$$

The components of our Killing vector are left unchanged,

$$\xi'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \xi^{\nu} = \frac{\partial x'^{\mu}}{\partial x^0} = \delta_0^{\mu}, \tag{10.80}$$

and so is also the metric component g_{00}:

$$g'_{00} = \frac{\partial x^{\alpha}}{\partial x'^0} \frac{\partial x^{\beta}}{\partial x'^0} g_{\alpha\beta} = \delta_0^{\alpha} \delta_0^{\beta} g_{\alpha\beta} = g_{00}. \tag{10.81}$$

For the mixed components, instead, we find

$$\begin{aligned} g'_{i0} &= \frac{\partial x^{\alpha}}{\partial x'^i} \frac{\partial x^{\beta}}{\partial x'^0} g_{\alpha\beta} = g_{\alpha\beta} \delta_0^{\beta} \left(\delta_j^{\alpha} \delta_i^j - \delta_0^{\alpha} \partial_i f\right) \\ &= g_{i0} - g_{00} \partial_i f \equiv 0. \end{aligned} \tag{10.82}$$

The result is zero because, in the old chart,

$$g_{i0} = \xi_i = \xi^2 \partial_i \phi = g_{00} \partial_i f. \tag{10.83}$$

This shows that if the Killing vector satisfies the condition (10.69) it is always possible to find a chart where the metric components g_{i0} are all vanishing, as appropriate to a geometry of static type.

10.2 Solution The Schwarzschild metric (10.19) has the same structure as that of the metric (6.92) introduced in Exercise 6.6, with

$$f(r) = g_{00} = -\frac{1}{g_{11}} = 1 - \frac{2m}{r}. \tag{10.84}$$

By using the result (6.94) we then immediately obtain the following non-zero components of the Riemann tensor:

$$\begin{aligned} &R_{01}{}^{01} = -\frac{1}{2} f'' = \frac{2m}{r^3}, \qquad R_{23}{}^{23} = -\frac{1}{r^2}(f-1) = \frac{2m}{r^3}, \\ &R_{02}{}^{02} = R_{03}{}^{03} = R_{12}{}^{12} = R_{13}{}^{13} = -\frac{1}{2r} f' = -\frac{m}{r^3}. \end{aligned} \tag{10.85}$$

Hence:

$$\begin{aligned} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} &= R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} \\ &= 4R_{01}^2{}^{01} + 4R_{02}^2{}^{02} + 4R_{03}^2{}^{03} + 4R_{12}^2{}^{12} + 4R_{13}^2{}^{13} + 4R_{23}^2{}^{23} \\ &= \frac{48m^2}{r^6}. \end{aligned} \tag{10.86}$$

10.3 Solution The connection for the metric (10.70) has been already computed in Exercise 6.1. The time-component of the geodesic equation,

$$\ddot{\eta} + \frac{2}{\xi}\dot{\eta}\dot{\xi} = 0, \tag{10.87}$$

gives

$$\dot{\eta} = k\xi^{-2}, \tag{10.88}$$

where k is an integration constant, and the dot denotes differentiation with respect to the proper time τ. From the normalization of the four-velocity vector we have

$$\dot{x}^{\mu}\dot{x}_{\mu} = \xi^2\dot{\eta}^2 - \dot{\xi}^2 = \frac{k^2}{\xi^2} - \dot{\xi}^2 = c^2. \tag{10.89}$$

Separating the variables, and integrating, we then obtain the proper-time interval $\Delta\tau$ needed to reach the origin (where the Rindler parametrization of the Minkowski space–time is no longer defined):

$$\Delta\tau = -\int_{\xi_0}^{0} \frac{\xi\, d\xi}{\sqrt{k^2 - c^2\xi^2}} = \frac{1}{c^2}\left(k - \sqrt{k^2 - c^2\xi_0^2}\right). \tag{10.90}$$

This integral is convergent, and the time interval is finite.

Note that the integration constant k can be expressed in terms of the velocity $\dot{\xi}_0$ at the initial time $\tau = 0$. From Eq. (10.89) we have, in fact,

$$k^2 = \xi_0^2\left(c^2 + \dot{\xi}_0^2\right), \tag{10.91}$$

and we can also rewrite the final result (10.90) in the form

$$\Delta\tau = \frac{\xi_0}{c^2}\left(\sqrt{c^2 + \dot{\xi}_0^2} - \dot{\xi}_0\right). \tag{10.92}$$

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Chapter 11
The Kasner Solution

The Schwarzschild solution describes a space–time geometry whose three-dimensional spatial sections are rotational invariant, and thus isotropic, without preferred directions. In this chapter we will present another exact solution of the Einstein equations with a spatial geometry which is *homogeneous*, i.e. position-independent, but *anisotropic*, and thus with a different curvature along different spatial directions.

Anisotropic models of space–time are frequently used in a cosmological context, where they are applied to phenomenological studies of the symmetry properties of our Universe and to theoretical studies of the primordial epochs, close to the initial singularity regime.

Also, anisotropic manifolds play an important role in the context of unified models of all interactions formulated in a higher-dimensional space–time (like, for instance, superstring models). In fact, if our space–time has more than four dimensions, then the spatial geometry must be necessarily anisotropic in order to be compatible with the large-scale expansion of three spatial dimensions, while all the remaining dimensions simultaneously contract—or perhaps keep frozen—down to scales of distance so small as to be inaccessible (so far) to a direct detection.

The metric that we will consider in this chapter is translational invariant along all spatial directions, and the geometry admits the spatial translations as its typical isometry group. In three dimensions, the Abelian translation group is only one of the *nine* different groups, in general non-Abelian, which represent all possible (three-parameter) groups of isometries of a homogeneous space.

The different geometries associated to the different isometry groups are usually classified with Roman numbers running from I to IX, and form the so-called class of "Bianchi models" (see e.g. the books [47, 57] of the bibliography). The model discussed here corresponds to the simplest case (the only one with an Abelian isometry group), and is known in the literature as a model of Bianchi-I type.

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11.1 Einstein Equations for Homogeneous and Anisotropic Metrics

The simplest generalization of the Minkowski metric which preserves the homogeneity of the spatial geometry, while introducing an arbitrary spatial anisotropy, corresponds to a metric whose spatial components may depend on the time coordinate through dimensionless functions $a_i(t)$, $i = 1, 2, 3, \ldots$, which have, in general, a different form along different spatial directions.

Let us then consider an anisotropic space–time whose line-element, in the chart where the metric is diagonal, can be written as follows:

$$ds^2 = c^2\, dt^2 - \sum_{i=1}^{d} a_i^2(t)\, dx_i^2. \tag{11.1}$$

We have assumed that our manifold contains d spatial dimensions, with $d \geq 3$. The generators of the translations along the $\hat{x}_i$ axes are Killing vectors for this geometry, which admits spatial translations as an Abelian isometry group with d parameters. The metric corresponding to the above line-element is a Bianchi-I type metric, written in the so-called "synchronous" gauge in which $g_{00} = 1$ and $g_{0i} = 0$. We will present now the Einstein equations for this metric, assuming that the sources can be described as a perfect fluid satisfying the same symmetry properties (namely, invariance under spatial translations) as the geometry.

Let us start with the metric

$$g_{00} = 1, \qquad g_{ij} = -a_i^2 \delta_{ij}, \tag{11.2}$$

whose contravariant components are given by

$$g^{00} = 1, \qquad g^{ij} = -\frac{\delta^{ij}}{a_i^2}, \tag{11.3}$$

where $a_i = a_i(t)$. *An important warning*: in the above equations, and in all the subsequent equations of this chapter, there is *not* a sum over repeated indices. Throughout this chapter the sum, whenever needed, will be always explicitly expressed with the summation symbol (as in Eq. (11.1)).

By applying the definition (3.90) we can easily obtain the connection components. Denoting with a dot the derivative with respect to $x^0 = ct$, and defining $H = \dot{a}/a$, we find the following nonvanishing components:

$$\Gamma_{0i}{}^{j} = \frac{\dot{a}_i}{a_i}\delta_i^j \equiv H_i \delta_i^j, \qquad \Gamma_{ij}{}^{0} = a_i \dot{a}_i \delta_{ij}. \tag{11.4}$$

The corresponding Ricci tensor turns out to be diagonal,

$$\begin{aligned} R_0{}^0 &= -\sum_i \frac{\ddot{a}_i}{a_i} = -\sum_i \left(\dot{H}_i + H_i^2\right), \\ R_i{}^j &= -\delta_i^j \left(\dot{H}_i + H_i \sum_k H_k\right), \end{aligned} \tag{11.5}$$

where the sum over Latin indices goes from 1 to d. The scalar curvature is then given by

$$R = R_0{}^0 + \sum_i R_i{}^i = -\sum_i \left(2\dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2. \tag{11.6}$$

Let us now suppose that the gravitational sources can be represented, at least in first approximation, as a perfect fluid spatially distributed in a homogeneous but anisotropic way. Namely, as a fluid without any viscosity or internal friction, with an energy density ρ and a pressure which do not depend on the position but only on time, and with a pressure p_i which can be different along the different spatial directions. We shall assume that the fluid is "comoving" with the geometry, namely that it is at rest in the reference frame in which the metric has the form (11.1). Recalling the general definition (1.96) we can thus write its energy-momentum tensor in diagonal form as follows:

$$T_0{}^0 = \rho(t), \qquad T_i{}^j = -p_i(t)\delta_i^j. \tag{11.7}$$

We have now in the position of writing down explicitly the Einstein equations (7.28). The $(0, 0)$ component of the Einstein tensor gives

$$\frac{1}{2}\left(\sum_i H_i\right)^2 - \frac{1}{2}\sum_i H_i^2 = \chi\rho, \tag{11.8}$$

while the spatial components lead to

$$\delta_i^j \left(\dot{H}_i + H_i \sum_k H_k\right) - \frac{1}{2}\delta_i^j \sum_k \left(2\dot{H}_k + H_k^2\right) - \frac{1}{2}\delta_i^j \left(\sum_k H_k\right)^2 = \chi p_i \delta_i^j. \tag{11.9}$$

This is a system of $d + 1$ differential equations for the $2d + 1$ unknowns $\{a_i, \rho, p_i\}$. To obtain a solution, we need further information.

In our case the additional information is provided by the d equations of state, $p_i = p_i(\rho)$, which relate the pressure components to the energy density of the fluid. We can take, for instance, a perfect fluid of "barotropic" type, which satisfies the conditions

$$\frac{p_i}{\rho} = w_i = \text{const}, \tag{11.10}$$

and assume that the constant coefficients w_i (depending on the intrinsic properties of the fluid components) are known. By eliminating everywhere p_i in terms of ρ we are then left with $d+1$ equations and $d+1$ unknowns.

For a barotropic fluid, in particular, we can easily obtain a relation connecting the energy density $\rho(t)$ to the geometric parameters $a_i(t)$. Consider in fact the covariant conservation of the energy-momentum tensor (11.7), following from the Einstein equations and from the contracted Bianchi identity (7.36):

$$\partial_\nu T_\mu{}^\nu + \Gamma_{\nu\alpha}{}^\nu T_\mu{}^\alpha - \Gamma_{\nu\mu}{}^\alpha T_\alpha{}^\nu = 0. \tag{11.11}$$

By using Eqs. (11.4), (11.7) we find that the component $\mu = i$ of the conservation equation is identically satisfied, while the component $\mu = 0$ gives

$$\dot{\rho} + \sum_i H_i(\rho + p_i) = 0. \tag{11.12}$$

We note that the same result can also be obtained directly from the Einstein equations, by differentiating Eq. (11.8) and using Eq. (11.9). If the fluid is barotropic, we can exploit the equations of state (11.10), and the conservation equation becomes

$$\frac{\dot{\rho}}{\rho} = -\sum_i (1+w_i)\frac{\dot{a}_i}{a_i}. \tag{11.13}$$

Separating the variables, and integrating, we finally arrive at

$$\rho = \rho_0 \prod_{i=1}^{d} a_i^{-(1+w_i)}, \tag{11.14}$$

where ρ_0 is an integration constant. Using this result into the Einstein equations we can eliminate ρ, and solve the equations for the geometric unknowns $a_i(t)$ (also called "scale factors").

11.2 Higher-Dimensional Solutions in Vacuum

An anisotropic geometry, like the one of the previous section, is compatible with nontrivial solutions of the Einstein equations even in the absence of sources.

Consider in fact the case $\rho = 0$, $p_i = 0$, and look for solutions of Eqs. (11.8), (11.9) using for the scale factor a power-law behavior,

$$a_i = \left(\frac{t}{t_0}\right)^{\beta_i}, \qquad H_i = \frac{\dot{a}_i}{a_i} = \frac{\beta_i}{ct}, \qquad \dot{H}_i = -\frac{\beta_i}{c^2 t^2}, \tag{11.15}$$

where β_i and t_0 are constant parameters. In that case the equations can be solved exactly, and the solution we obtain, in the limit $t \to 0$, is valid even in the presence

of sources because, in that limit, the contributions of the matter sources tend to become subleading with respect to the contributions of the geometry to the Einstein equations.

By inserting the particular expressions (11.15) of H and $\dot{H}$ into Eqs. (11.8), (11.9) (with $\rho = p_i = 0$) the time-dependence factorize out, and we are left with two algebraic equations for the unknown powers β_i. From Eq. (11.8) we obtain the condition

$$\left(\sum_i \beta_i\right)^2 = \sum_i \beta_i^2, \tag{11.16}$$

while from Eq. (11.9), summing up all diagonal components, we obtain

$$-\sum_i \beta_i + \left(\sum_i \beta_i\right)^2 + d\sum_i \beta_i - \frac{d}{2}\sum_i \beta_i^2 - \frac{d}{2}\left(\sum_i \beta_i\right)^2 = 0. \tag{11.17}$$

By eliminating $\sum_i \beta_i^2$ through Eq. (1.16) we can also rewrite the above condition as

$$(d-1)\sum_i \beta_i + (1-d)\left(\sum_i \beta_i\right)^2 = 0. \tag{11.18}$$

The system of equations (11.16), (11.18) can now be satisfied in two ways.

A first possibility corresponds to the conditions

$$\sum_i \beta_i = 0 = \sum_i \beta_i^2, \tag{11.19}$$

which, however, is associated to the trivial solutions $\beta_i = 0$, $a_i =$ constant, corresponding to the Minkowski space–time.

If, on the contrary, $\sum_i \beta_i \neq 0$, we can divide Eq. (11.18) by $\sum_i \beta_i$, and we obtain the conditions

$$\sum_i \beta_i = 1 = \sum_i \beta_i^2, \tag{11.20}$$

which characterizes the so-called Kasner solution. Any metric of the type (11.1), with $a_i \sim t_i^{\beta}$, and with the constant parameters β_i satisfying the conditions (11.20), represents an exact solution of the Einstein equations in vacuum. Note that such solution is *necessarily anisotropic*, since there are no real solutions to the Kasner conditions (11.20) with the same value for all β_i, for any given number $d \geq 2$ of spatial dimensions.

Let us to stress some important properties of this exact solution. We note, first of all, that the Kasner solution is singular for $t \to 0$. A computation of the squared Riemann tensor gives, in fact,

$$R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \sim \frac{1}{t^4}. \tag{11.21}$$

In addition, near the singularity the solutions is insensitive to the possible presence of gravitational sources (provided they are not too "exotic").

Let us suppose, in fact, that the sources can be described as a barotropic fluid, and insert the Kasner solution into their energy density (11.14). Comparing the evolution of $\rho(t)$ and of the geometric contributions to the Einstein equations we then obtain the ratio

$$\frac{\rho}{H_i^2} \sim \frac{\rho}{\dot{H}_i} \sim t^{1-\sum_i \beta_i w_i} \tag{11.22}$$

(we have used the Kasner condition $\sum_i \beta_i = 1$). In the case of "conventional" matter sources, characterized by equations of state such that $|w_i| < 1$ (more precisely, such that $\sum_i \beta_i w_i < 1$), the exponent of t is always positive: in the limit $t \to 0$ the matter contributions thus become negligible with respect to the other terms of the Einstein equations, and the Kasner solution keeps valid, in this limit, even in the presence of sources.

It is interesting to note, finally, that the sign of the parameters β_i, in order to satisfy the Kasner conditions (11.20), cannot be the same for all parameters. This means (if we recall the definitions (11.15) of the scale factors a_i) that, for $t \to \infty$, the geometry must be *expanding* along some directions (the ones with $\beta_i > 0$), and *contracting* along others (the ones with $\beta_i < 0$). In other words, there *must* be contracting dimensions, together with the expanding ones, for the Kasner solution to be valid.

As anticipated at the beginning of this chapter, the Kasner solution is thus a natural candidate to describe a phase of "spontaneous" dimensional reduction, during which the action of the gravitational interaction can effectively disentangle the time evolution of the different spatial dimensions (by inflating some dimensions and shrinking others). Let us consider, for instance, a 5-dimensional space–time, and a Kasner solution with parameters $\beta_i = (1/2, 1/2, 1/2, -1/2)$. The conditions (11.20) are satisfied, and the corresponding line-element is given by

$$ds^2 = c^2\, dt^2 - \left(\frac{t}{t_0}\right)^{1/2} \left(dx_1^2 + dx_2^2 + dx_3^2\right) - \left(\frac{t}{t_0}\right)^{-1/2} dy^2 \tag{11.23}$$

(we have called y the coordinate along the fifth dimension). The three-dimensional spatial geometry expands, for $t \to \infty$, while the additional spatial dimension shrinks to smaller and smaller scales of proper distances.

The only way to evade the constraint of a Kasner solution with positive and negative parameters β_i is provided by the "almost trivial" case, characterized by only one nonvanishing coefficient,

$$\beta_i = (1, 0, 0, 0, \dots), \tag{11.24}$$

corresponding to the line-element

$$ds^2 = c^2\, dt^2 - \left(\frac{t}{t_0}\right)^2 dx_1^2 - dx_2^2 - dx_3^2 - \cdots . \tag{11.25}$$

This solution describes the so-called "Milne space–time", which is, however, a globally flat manifold. It can be easily checked, in fact, that the Riemann tensor for the above metric is identically vanishing, and that the line-element (11.25) can be globally reduced to the Minkowski line-element through an appropriate coordinate transformation (see Exercise 11.1).

11.3 Exercises Chap. 11

11.1 The Milne Space–Time Show that the Milne line-element (11.25) can be obtained from the Minkowski line-element through the global transformation

$$ct = ct' \cosh\left(\frac{x'}{\lambda}\right), \qquad x = ct' \sinh\left(\frac{x'}{\lambda}\right), \tag{11.26}$$

where λ is a constant parameter, and (ct, x) are the coordinates of the Minkowski plane. Compute the Riemann tensor for the Milne metric, and show that the Milne coordinates (ct', x') only cover the region *inside* the light cone of the Minkowski plane.

11.2 Anisotropic Einstein Equations from a Variational Principle Deduce Eqs. (11.8), (11.9), in the absence of sources, starting from the effective Einstein action for a Bianchi-I type metric, and using a variational procedure.

11.4 Solutions

11.1 Solution By differentiating Eq. (11.26) we obtain

$$\begin{aligned} c\,dt &= c\,dt' \cosh\left(\frac{x'}{\lambda}\right) + dx'\,\frac{ct'}{\lambda} \sinh\left(\frac{x'}{\lambda}\right), \\ dx &= c\,dt' \sinh\left(\frac{x'}{\lambda}\right) + dx'\,\frac{ct'}{\lambda} \cosh\left(\frac{x'}{\lambda}\right). \end{aligned} \tag{11.27}$$

Inserting this result into the Minkowski line-element we obtain the Milne line-element,

$$ds^2 = c^2\,dt^2 - dx^2 = c^2\,dt'^2 - \left(\frac{ct'}{\lambda}\right)^2 dx'^2, \tag{11.28}$$

with the same Milne metric as that of Eq. (11.25),

$$g_{00} = 1, \qquad g_{11} = -\left(\frac{t'}{t_0}\right)^2, \tag{11.29}$$

where $t_0 = \lambda/c$.

The Riemann tensor for this metric is identically vanishing. Using for the connection components the results (11.4) we have, in fact,

$$\Gamma_{01}{}^{1} = \frac{1}{ct'}, \qquad \Gamma_{11}{}^{0} = \frac{t'}{ct_0^2}, \tag{11.30}$$

so that

$$\begin{aligned} R_{101}{}^{0} &= -\frac{1}{c^2 t_0^2} + \frac{1}{c^2 t_0^2} \equiv 0, \\ R_{100}{}^{1} &= \frac{1}{c^2 t'^2} - \frac{1}{c^2 t'^2} \equiv 0. \end{aligned} \tag{11.31}$$

Let us finally notice that, from the transformation (11.26), we have

$$\frac{x}{ct} = \tanh\left(\frac{x'}{\lambda}\right), \qquad c^2 t^2 - x^2 = c^2 t'^2. \tag{11.32}$$

The first equation, at fixed x', represents a straight line crossing the origin of the Minkowski plane, and forming with the ct axis an angle between $-\pi/4$ and $\pi/4$. The second equation, at fixed t', represents a hyperbola centered on the origin, with asymptotes given by the lines $x = \pm ct$, which intersects the ct axis for $t = \pm t'$. By varying x' and t' the two curves span the portion of Minkowski plane internal to the light cone, defined by

$$ct > |x|, \qquad ct < -|x|, \tag{11.33}$$

namely the region also called "Milne space–time". This region is just the complement of the so-called Rindler space–time, covering the region exterior to the light cone (see Exercise 6.1).

11.2 Solution In order to obtain all equations of motion, and in particular the $(0, 0)$ component of the Einstein equations, the effective action must contain all relevant metric components. So, let us start with anisotropic metric (11.2) without imposing the synchronous gauge $g_{00} = 1$, and define

$$g_{00} = N^2(t), \qquad g_{ij} = -a_i^2(t)\delta_{ij}. \tag{11.34}$$

The nonvanishing components of the connection are now given by

$$\Gamma_{0i}{}^{j} = H_i \delta_i^j, \qquad \Gamma_{ij}{}^{0} = \frac{a_i \dot{a}_i}{N^2}\delta_{ij}, \qquad \Gamma_{00}{}^{0} = F, \tag{11.35}$$

where $F = \dot{N}/N$, and the scalar curvature becomes

$$R = \frac{1}{N^2}\left[2F\sum_i H_i - \sum_i \left(2\dot{H}_i + H_i^2\right) - \left(\sum_i H_i\right)^2\right]. \tag{11.36}$$

This is more general than Eq. (11.6), because of the contributions of $g_{00} = N^2$. We have, also,

$$\sqrt{-g} = N \prod_i a_i , \tag{11.37}$$

and the effective Einstein action takes the form

$$\begin{aligned} S &= -\frac{1}{2\chi} \int d^{d+1}x \, \sqrt{-g}\, R \\ &= -\frac{1}{2\chi} \int d^d x \, \frac{dt}{N} \prod_i a_i \left[2F \sum_i H_i - \sum_i (2\dot{H}_i + H_i^2) - \left(\sum_i H_i \right)^2 \right]. \end{aligned} \tag{11.38}$$

We can note, at this point, that

$$\begin{aligned} &\frac{d}{dt} \left[\frac{2}{N} \prod_i a_i \sum_i H_i \right] \\ &\quad = \frac{1}{N} \prod_i a_i \left[2 \sum_i \dot{H}_i - 2F \sum_i H_i + 2 \left(\sum_i H_i \right)^2 \right]. \end{aligned} \tag{11.39}$$

By eliminating through the above relation the terms linear in F and $\dot{H}$ of Eq. (11.38), we can rewrite the action (modulo a total time derivative) in the following standard quadratic form:

$$S = -\frac{1}{2\chi} \int \frac{dt}{N} \prod_i a_i \left[\left(\sum_i H_i \right)^2 - \sum_i H_i^2 \right]. \tag{11.40}$$

Notice that the variable N has no kinetic term, hence it plays the role of auxiliary field (or Lagrange multiplier), and can be set to a constant—after the variation—with an appropriate gauge choice.

We are now in the position of deriving the field equations by varying with respect to the variables N, a_i, and imposing the condition of stationary action, $\delta S = 0$. The variation with respect to N gives the constraint

$$\left(\sum_i H_i \right)^2 - \sum_i H_i^2 = 0, \tag{11.41}$$

corresponding to Eq. (11.8) with $\rho = 0$.

In order to vary with respect to the spatial metric components we can conveniently set $a_i = \exp \alpha_i$, so that $H_i = \dot{\alpha}_i$, and the effective action becomes

$$S = -\frac{1}{2\chi} \int dt \, L(\alpha_i, \dot{\alpha}_i), \tag{11.42}$$

where

$$L = \frac{\exp(\sum_i \alpha_i)}{N}\left[\left(\sum_i \dot{\alpha}_i\right)^2 - \sum_i \dot{\alpha}_i^2\right]. \tag{11.43}$$

The variation with respect to α_i leads to the Lagrange equations of motion for this new variable. Computing the derivatives, and then imposing the synchronous gauge $N = 1$, we obtain

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_i} &= \exp\left(\sum_k \alpha_k\right)\left[\left(\sum_k \dot{\alpha}_k\right)^2 - \sum_k \dot{\alpha}_k^2\right], \\
\frac{\partial L}{\partial \dot{\alpha}_i} &= \exp\left(\sum_k \alpha_k\right)\left[2\sum_k \dot{\alpha}_k - 2\dot{\alpha}_i\right], \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} &= \exp\left(\sum_k \alpha_k\right)\sum_k \dot{\alpha}_k\left[2\sum_k \dot{\alpha}_k - 2\dot{\alpha}_i\right] \\
&\quad + \exp\left(\sum_k \alpha_k\right)\left[2\sum_k \ddot{\alpha}_k - 2\ddot{\alpha}_i\right].
\end{aligned} \tag{11.44}$$

The Lagrange equations for α_i thus imply:

$$\left(\sum_k \dot{\alpha}_k\right)^2 - 2\dot{\alpha}_i \sum_k \dot{\alpha}_k + 2\sum_k \ddot{\alpha}_k - 2\ddot{\alpha}_i + \sum_k \dot{\alpha}_k^2 = 0. \tag{11.45}$$

Multiplying by $-1/2$, and replacing $\dot{\alpha}_i$ with H_i, we can rewrite the above equation in the form

$$\dot{H}_i + H_i \sum_k H_k - \sum_k \dot{H}_k - \frac{1}{2}\sum_k H_k^2 - \frac{1}{2}\left(\sum_k H_k\right)^2 = 0, \tag{11.46}$$

which exactly coincides with the component $i = j$ of Eq. (11.9), with $p_i = 0$.

Chapter 12
Vierbeins and Lorentz Connection

The geometric description of gravity so far presented has been developed using the language of classical differential geometry, based, in particular, on the notions of Riemannian metric g and Christoffel connection Γ. The curvature of the space–time manifold, its dynamical evolution, and its interaction with the matter sources has been described in terms of differential equations for the variables g and Γ.

The aim of this chapter is to introduce an alternative (but fully equivalent) approach to the description of a Riemannian manifold based on the notions of vierbein V and Lorentz connection ω. This alternative language is particularly appropriate to embed spinor fields in a curved space–time—hence to describe the gravitational interactions of fermions—as we shall see in the next chapter.

Also, and most important, this alternative geometric formalism naturally leads to the formulation of general relativity as a gauge theory for a local symmetry group, thus putting gravity on the same footing of the other fundamental (strong, weak and electromagnetic) interactions. It will be shown, in particular, that the gauge symmetry of the gravitational interactions is the local Lorentz symmetry, and that the space–time curvature can be interpreted, in such a context, as the Yang–Mills field for the Lorentz connection, which plays the role of the (non-Abelian) gauge potential.

These important aspects of the gravitational interactions, as well as the possibility of extending the local symmetry from the Lorentz to the Poincaré group, will be further illustrated in Appendix A.

12.1 Projection on the Flat Tangent Space

We have already stressed, in Sect. 2.2, that the geometry of a Riemann space–time can always be *locally* approximated by the Minkowski geometry, i.e. that we can always introduce, at any given point of a Riemann manifold, a "flat" tangent manifold described by the Minkowski metric.

We can thus locally characterize the geometry of a four-dimensional Riemann space–time $\mathcal{R}_4$ by introducing, at any given point x, a "quadruplet" of covariant

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vectors V_μ,

$$V_\mu^a(x), \quad a = 0, 1, 2, 3, \tag{12.1}$$

forming an orthonormal basis in the local Minkowski space $\mathcal{M}_4$ tangent to $\mathcal{R}_4$ at the given point x. They are orthonormal with respect to the Minkowski metric η^{ab} of the tangent manifold, i.e. they satisfy the condition:

$$g^{\mu\nu} V_\mu^a V_\nu^b = \eta^{ab}. \tag{12.2}$$

These vectors are called "tetrads" or, using the German language, "vierbeins" (four legs), which becomes "vielbeins" (many legs) if the manifold is higher-dimensional.

It is important, at this point, to clarify the notations we are going to use throughout this chapter. Differently from the previous chapters, small Latin indices $a, b, \ldots$ will range from 0 to 3, and will be used to denote tensor indices in the flat tangent space (namely, they are indices labeling tensor representations of the local Lorentz group, raised and lowered by the Minkowski metric η). The Greek indices $\mu, \nu, \ldots$ also will range from 0 a 3, but they will refer to tensor objects defined on the Riemann manifold (hence they transform covariantly under diffeomorphisms, and are raised and lowered by the Riemann metric g).

In the technical language of differential geometry the general-covariant, Greek indices are also called *holonomic* indices, while the Latin indices, referred to the tangent-space transformations, are called *anholonomic* indices. In this book we will use a simpler and more direct terminology, using the following definitions:

$$\begin{aligned} a, b, c, \ldots \quad &\Longrightarrow \quad \text{flat } \textit{indices} \quad \text{(or Lorentz indices)}, \\ \mu, \nu, \alpha \ldots \quad &\Longrightarrow \quad \text{curved } \textit{indices} \quad \text{(or Riemann indices)}. \end{aligned}$$

These conventions will be used also in the following chapters, unless otherwise specified.

Let us now notice that Eq. (12.2), rewritten in a mixed tensor form as

$$V_\mu^a V_b^\mu = \delta_b^a, \tag{12.3}$$

defines the inverse (or *dual*) basis of contravariant vectors V_a^μ, which are also orthonormal with respect to the Minkowski metric:

$$g_{\mu\nu} V_a^\mu V_b^\nu = \eta_{ab}. \tag{12.4}$$

By inverting the relations (12.2), (12.4) we obtain

$$g_{\mu\nu} = V_\mu^a V_\nu^b \eta_{ab}, \qquad g^{\mu\nu} = V_a^\mu V_b^\nu \eta^{ab}. \tag{12.5}$$

These two equations, together with the previous ones, enable us to express the metric tensor in terms of the four vierbeins V_μ^a and of their inverse V_a^μ.

To avoid a possible ambiguity of sign, and also in view of forthcoming applications, it is finally convenient to normalize the four vectors V_μ^a in such a way that

$$\sqrt{-g} = \sqrt{|\det g_{\mu\nu}|} = \left|\det V_\mu^a\right| \equiv V. \tag{12.6}$$

After this fixing, the local knowledge of the vector field $V_\mu^a(x)$ fully determines the Riemann metric $g_{\mu\nu}(x)$ at any given point of the space–time manifold, modulo a residual freedom of the choice of the vierbeins due to the action of the Lorentz transformations $\Lambda^a{}_b$ in the local tangents space. We can immediately check, in fact, that the vector V_μ^a and the transformed vector $V_\mu'^a = \Lambda^a{}_b V_\mu^b$ determine to the same Riemann metric:

$$\begin{aligned} g'_{\mu\nu} &= V_\mu'^a V_\nu'^b \eta_{ab} = \Lambda^a{}_i \Lambda^b{}_j V_\mu^i V_\nu^j \eta_{ab} \\ &= \eta_{ij} V_\mu^i V_\nu^j \equiv g_{\mu\nu} \end{aligned} \tag{12.7}$$

(we have used the condition $\Lambda^T \eta \Lambda = \eta$).

By exploiting the properties of the vierbein field and of their inverse, any geometrical object defined on the Riemann space–time manifold can be locally projected on the tangent Minkowski space, simply by contracting its curved indices with those of V_μ^a or V_a^μ. Given a second-rank tensor B, for instance, we can consider the projections

$$\begin{aligned} B^{\mu\nu} &\longrightarrow B^{ab} = V_\mu^a V_\nu^b B^{\mu\nu}, \\ B_{\mu\nu} &\longrightarrow B_{ab} = V_a^\mu V_b^\nu B_{\mu\nu}. \end{aligned} \tag{12.8}$$

Conversely, we can switch from the tangent space down to the Riemann manifold through the inverse projection. The Minkowski metric η_{ab}, just to give an another example, is the projection of the Riemann metric on the local tangent space (see Eq. (12.4)).

It is important to stress, in this context, that if we start with an object (say $B_{\mu\nu}$) which is a tensor under general coordinate transformations, after the projection we end up with a new object B_{ab} which is a tensor under local Lorentz (tangent-space) transformations, but is a *scalar* under general coordinate transformations (since it has no curved indices). In this sense, the vierbeins are objects of "mixed" type, transforming as general-covariant vectors with respect to their curved index, and as Lorentz vectors with respect to the flat index:

$$V_\mu^a \to \widetilde{V}_\mu^a = \frac{\partial x^\nu}{\partial x'^\mu} \Lambda^a{}_b V_\nu^b. \tag{12.9}$$

And here we arrive at a crucial point of our discussion.

Thanks to the projection transforming curved into flat indices, we can move from the diffeomorphisms of the Riemann manifold to the Lorentz transformations on the tangent manifold. The tangent manifold, however, varies from point to point, so that the corresponding Lorentz transformations are *local* transformations, represented by

matrices $\Lambda = \Lambda(x)$. The requirement of general covariance, which is at the ground of a geometric model of gravity formulated in a curved space–time manifold, thus translates—within the vierbein formalism—into the requirement of *local Lorentz invariance* (obviously, if the space–time manifold is globally flat then the Lorentz invariance becomes global, and we recover the usual special-relativistic structure).

The presence of an intrinsic local symmetry in the context of a geometric theory of gravity allows now an interesting comparison with the gauge theories of fundamental interactions. In order to make such a comparison the more direct and explicit as possible, the next section will be devoted to an essential summary of the formalism of gauge theories.

12.1.1 Local Symmetries and Gauge Fields

Suppose we have a field ψ whose action is invariant under the global symmetry transformation $\psi \to \psi' = U\psi$, where U represents the transformation of an n-parameters Lie group and can then be parametrized, in general, as follows:

$$U = e^{-i\epsilon^A X_A}, \tag{12.10}$$

with $A = 1, \ldots, n$. The parameters ϵ^A are real constant coefficients, while the operators X_A—which are Hermitian if the representation is unitary—are the generators of the transformation, and satisfy the so-called Lie algebra of the group:

$$[X_A, X_B] = i f_{AB}{}^C X_C. \tag{12.11}$$

The structure constants $f_{AB}{}^C = -f_{BA}{}^C$ are all vanishing only if the group is Abelian.

If we have a global transformation (namely, if all parameters ϵ^A are constant) then the field and the field gradients transform in the same way,

$$\partial_\mu \psi' = \partial_\mu (U\psi) = U \partial_\mu \psi, \tag{12.12}$$

and the action, assumed to be built on a Lagrangian quadratic in the fields and its gradients, $\mathcal{L} \sim \psi^\dagger \psi + (\partial\psi)^\dagger \partial\psi$, turns out to be automatically invariant. If we have instead a local transformation, with $\epsilon^A = \epsilon^A(x)$, then the field gradients are transformed differently from the field itself,

$$\partial_\mu \psi \to \partial_\mu \psi' = \partial_\mu (U\psi) = U \partial_\mu \psi + (\partial_\mu U)\psi, \tag{12.13}$$

and the kinetic term of the above action is no longer invariant.

The invariance under local transformations (also called *gauge invariance*) can be restored by replacing the ordinary gradient operator ∂_μ with a generalized differential operator, the so-called *gauge covariant derivative*, that we shall denotes with D_μ (to avoid confusing it with the covariant derivative ∇_μ defined in Riemann

spaces). The operator D_μ is defined in such a way that the covariant derivative of the field ψ is transformed just like the field, i.e.

$$D_\mu \psi \to (D_\mu \psi)' = U D_\mu \psi, \tag{12.14}$$

even in the case of local transformations. The replacement $\partial_\mu \to D_\mu$ in the field Lagrangian leads to a kinetic term of the type $\mathcal{L} \sim (D\psi)^\dagger D\psi$, and makes the action automatically invariant according to the procedure suggested by the principle of minimal coupling (already discussed for the geometric case in Sect. 4.1).

In order to define the gauge covariant derivative we must introduce a set of n vector fields (the so-called "gauge potentials") A_μ, one for each generator of the symmetry group,

$$X_A \longrightarrow A_\mu^A, \quad A = 1, \ldots, n. \tag{12.15}$$

We then construct the differential operator D_μ as follows:

$$D_\mu = \partial_\mu - i g A_\mu^A X_A, \tag{12.16}$$

where g is a coupling constant, depending on the model of interaction we are considering. The transformation properties of the fields A_μ^A are then determined by imposing the transformation law (12.14).

In this context it is convenient to adopt a compact formalism, and introduce the gauge potential (or *connection*) $A_\mu \equiv A_\mu^A X_A$, defined by saturating the group indices with the corresponding generators. The condition (12.14) then implies

$$\begin{aligned} D'_\mu \psi' &= \left(\partial_\mu - i g A'_\mu\right) U \psi = U \partial_\mu \psi - i g A'_\mu U \psi + (\partial_\mu U)\psi \\ &= U D_\mu \psi = U(\partial_\mu - i g A_\mu)\psi = U \partial_\mu \psi - i g U A_\mu \psi. \end{aligned} \tag{12.17}$$

Multiplying from the right by U^{-1} we then find that the transformation law for A_μ, required for the validity of Eq. (12.14), is given by

$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U) U^{-1}. \tag{12.18}$$

We can conclude by saying that if a model, globally symmetric with respect to a Lie group of transformations represented by U, is minimally coupled—through the covariant derivative (12.16)—to a gauge potential satisfying the transformation law (12.18), then the model also becomes locally symmetric under the corresponding group of transformations represented by $U(x)$.

12.2 Local Lorentz Invariance and Covariant Derivative

In Sect. 12.1 we have seen that a *general-covariant* geometric model, adapted to a curved space–time manifold, must be *locally Lorentz invariant* if referred to the tangent-space manifold described by the vierbein formalism.

We have seen also that, in order to lift a global symmetry group to the local level, we must introduce "covariant" differential operators, constructed with the gauge fields associated to that symmetry. The suitable formalism, to this purpose, is provided by the so-called gauge theories, and the required procedure has been recalled in Sect. 12.1.1. In this section we will directly apply such a procedure to the group of local Lorentz symmetry in the tangent Minkowski space–time.

Let us first note that the restricted (i.e. proper, orthochronous) Lorentz group is a 6-parameter Lie group, and that a generic transformation can be represented in exponential form as follows:

$$U = e^{-\frac{i}{2}\omega_{ab}J^{ab}}. \tag{12.19}$$

The matrix $\omega_{ab} = -\omega_{ba}$ is antisymmetric and contains six real parameters, while the six generators $J_{ab} = -J_{ab}$ satisfy the Lie algebra of $SO(3,1)$:

$$\left[J^{ab}, J^{cd}\right] = i\left(\eta^{ad}J^{bc} - \eta^{ac}J^{bd} - \eta^{bd}J^{ac} + \eta^{bc}J^{ad}\right). \tag{12.20}$$

In order to restore the symmetry for local transformations with $\omega_{ab} = \omega_{ab}(x)$, we must associate to the six generators J^{ab} six independent gauge vectors ω_μ,

$$J^{ab} \longrightarrow \omega_\mu{}^{ab} = -\omega_\mu{}^{ba} \tag{12.21}$$

(representing the components of the so-called "Lorentz connection", or "spin connection"), and introduce a Lorentz covariant derivative, defined by:

$$D_\mu = \partial_\mu - \frac{i}{2}\omega_\mu{}^{ab}J_{ab}. \tag{12.22}$$

The factor $1/2$ has been adopted for future convenience (and also to follow standard conventions). By applying the procedure of Sect. 12.1.1 we then find that the Lorentz covariant derivative of a field is transformed like the field itself provided the Lorentz connection $\omega_\mu \equiv \omega_\mu{}^{ab}J_{ab}$ obeys the following transformation law:

$$\omega'_\mu = U\omega_\mu U^{-1} - 2i(\partial_\mu U)U^{-1}, \tag{12.23}$$

which exactly reproduces Eq. (12.18) with $g = 1/2$.

Let us immediately consider an explicit example, concerning the covariant derivative of a vector-valued (tangent-space) field A^a. This field is a scalar under general coordinate transformations (as no curved index is present), and a (contravariant) Lorentz vector under the local tangent-space transformation

$$A'^a = \Lambda^a{}_b(x)A^b, \tag{12.24}$$

represented by the position-dependent Lorentz matrix $\Lambda^a{}_b(x)$. We can easily check that the ordinary gradient of A is not transformed in the required covariant way, i.e. that

$$\left(\partial_\mu A^a\right)' = \Lambda^a{}_b(x)\partial_\mu A^b + \left(\partial_\mu \Lambda^a{}_b\right)A^b \neq \Lambda^a{}_b(x)\partial_\mu A^b \tag{12.25}$$

because of the locality of $\Lambda(x)$.

In order to restore the local symmetry, and apply to A the definition of covariant derivative (12.22), we need the explicit form of the generators J for the vector representation of the Lorentz group. This form can be obtained by starting from the infinitesimal form of the transformation (12.24). By recalling Eq. (1.44), and expanding $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b + \cdots$, we obtain, to first order,

$$\delta A^a = \omega^a{}_b A^b. \tag{12.26}$$

On the other hand, using for Λ the exponential representation (12.19), and expanding around the identity,

$$\Lambda^a{}_b = \delta^a_b - \frac{i}{2}\omega^{ij}(J_{ij})^a{}_b + \cdots, \tag{12.27}$$

we also obtain

$$\delta A^a = -\frac{i}{2}\omega^{ij}(J_{ij})^a{}_b A^b. \tag{12.28}$$

By equating the two infinitesimal variations (12.26), (12.28), and solving for J, we find that the six vector generators J_{ij} are represented by 4×4 matrices defined as follows:

$$(J_{ij})^a{}_b = i\left(\eta_{jb}\delta^a_i - \eta_{ib}\delta^a_j\right). \tag{12.29}$$

Note that for these matrices the Lie algebra (12.20) turns out to be automatically satisfied.

By using the above generators, we can now explicitly write down the Lorentz covariant derivative for the contravariant vector A^a in the local tangent space:

$$\begin{aligned} D_\mu A^a &= \partial_\mu A^a - \frac{i}{2}\omega^{ij}_\mu (J_{ij})^a{}_b A^b \\ &\equiv \partial_\mu A^a + \omega_\mu{}^a{}_b A^b. \end{aligned} \tag{12.30}$$

It can be immediately checked that the correct transformation property is satisfied,

$$\left(D_\mu A^a\right)' = \Lambda^a{}_b D_\mu A^b, \tag{12.31}$$

provided the connection ω obeys the transformation law (12.23) (see Exercise 12.1).

In Chap. 13 we will discuss in detail the explicit form of the covariant derivative for the spinor representations of the Lorentz group. In this chapter we concentrate our attention on the tensor representations and we note that—as in the case of the Riemann covariant derivative ∇_μ—the Lorentz derivative of a vector can be easily extended to tensor objects with an arbitrary number of covariant and contravariant indices. It will be enough, for this purpose, to use the Leibnitz rule for the derivative of a product and note that, for a tangent-space scalar, the covariant operator D_μ simply reduces to ∂_μ.

To obtain the derivative of a covariant vector B_a, for instance, we can consider the scalar product $A^a B_a$, and impose

$$\partial_\mu \left(A^a B_a\right) = D_\mu \left(A^a B_a\right) = A^a D_\mu B_a + B_a \left(\partial_\mu A^a + \omega_\mu{}^a{}_b A^b\right). \tag{12.32}$$

Solving for $D_\mu B_a$ we get

$$D_\mu B_a = \partial_\mu B_a - \omega_\mu{}^b{}_a B_b. \tag{12.33}$$

An so on for tensor objects of arbitrary rank.

According to our conventions, specified by the explicit differentiation rules (12.30), (12.33), it follows that the Lorentz connection must operate on all Lorentz indices of the differentiated field, and must be applied with the positive sign for contravariant indices (as in Eq. (12.30)), and with the negative sign for covariant indices (as in Eq. (12.33)). For a mixed Lorentz tensor of rank $r = 2$ we have, for instance,

$$D_\mu A^a{}_b = \partial_\mu A^a{}_b + \omega_\mu{}^a{}_c A^c{}_b - \omega_\mu{}^c{}_b A^a{}_c. \tag{12.34}$$

Note that the index position is important, since $\omega_\mu{}^{ab} \neq \omega_\mu{}^{ba}$.

12.2.1 The Metricity Condition for the Vierbeins

Summarizing the discussion of the previous sections we can say that, using the vierbein formalism, all geometric objects of the Riemann space–time can be projected on the local tangent space. For the projected fields we can introduce a covariant derivative based on the Lorentz connection which preserves the local Lorentz invariance of the Minkowski tangent space, consistently with the general covariance of the underlying Riemann manifold.

It is now time to ask whether a relation can be established between the Lorentz and the Riemann covariant derivatives and, in particular, between the Christoffel connection Γ and the Lorentz connection ω. If so, since Γ can be expressed in terms of the metric g, and g can be expressed in terms of the vierbeins V, we may expect the existence of a precise functional relationship $\omega = \omega(V)$ giving the Lorentz connection in terms of the vierbein field. This would make the two systems of geometrical variables, $\{g, \Gamma\}$ and $\{V, \omega\}$, completely equivalent, in all respects, for the consistent formulation of a geometric model of gravity.

The answer to the above question can be obtained by considering the covariant derivative of the vierbein field. As already stressed, the vierbein is a mixed object, with a curved index in the Riemann manifold and a Lorentz index in the flat tangent manifold. Hence, the total covariant derivative must be constructed with both the connection Γ, to make the differential operator covariant with respect to general diffeomorphisms acting on the curved index, and the connection ω, to restore covariance with respect to local Lorentz transformations acting on the flat index. More

precisely, we must set:

$$\begin{aligned}\nabla_\mu V^a_\nu &= \partial_\mu V^a_\nu + \omega_\mu{}^a{}_b V^b_\nu - \Gamma_{\mu\nu}{}^\alpha V^a_\alpha \\ &\equiv D_\mu V^a_\nu - \Gamma_{\mu\nu}{}^\alpha V^a_\alpha \end{aligned} \tag{12.35}$$

where, in the second line, we have explicitly used the definition of the Lorentz covariant derivative (12.30).

We can now apply our general hypotheses on the geometric structure of the space–time manifold. We recall, in particular, the assumption of a "metric-compatible" geometry, which imposes on the metric the condition of vanishing covariant derivative (see the discussion of Sect. 3.5). Using Eq. (12.5), such a condition can be expressed as

$$\nabla_\alpha g_{\mu\nu} = \nabla_\alpha \left(V^a_\mu V^b_\nu \eta_{ab} \right) = 2\eta_{ab} V^a_\mu \nabla_\alpha V^b_\nu + V^a_\mu V^b_\nu \nabla_\alpha \eta_{ab} = 0. \tag{12.36}$$

The covariant derivative of the Minkowski metric, however, is identically vanishing. In fact, by using the general rule (12.33) for the derivative of the covariant Lorentz indices, we have:

$$\nabla_\alpha \eta_{ab} = -\omega_\alpha{}^c{}_a \eta_{cb} - \omega_\alpha{}^c{}_b \eta_{ac} = -(\omega_{\alpha ba} + \omega_{\alpha ab}) \equiv 0, \tag{12.37}$$

because of the antisymmetry property of the Lorentz connection (see Eq. (12.21)). The metricity condition (12.36) thus imposes on the vierbein the differential condition:

$$\nabla_\mu V^a_\nu = 0 \tag{12.38}$$

(also known in the literature as the "vierbein postulate").

Using the expression (12.35) for the covariant derivative, we can explicitly rewrite the metricity condition in the following form:

$$\partial_\mu V^a_\nu + \omega_\mu{}^a{}_b V^b_\nu = \Gamma_{\mu\nu}{}^\alpha V^a_\alpha. \tag{12.39}$$

This equation provides an answer to the question raised at the beginning of this section: the two connections ω and Γ are not independent. By expressing Γ in terms of g, and g in terms of V, we can solve the above equation for ω and determine everywhere the Lorentz connection as a function of the vierbeins and its first partial derivatives.

The same result, however, can also be obtained through a faster procedure that will be illustrated in the following section.

12.3 The Levi-Civita Connection and the Ricci Rotation Coefficients

To obtain a compact expression for the Lorentz connection as a function of the vierbein field we start with the metricity condition (12.39), which we rewrite as

follows:

$$\partial_\mu V^c_\nu + \omega_\mu{}^c{}_\nu - \Gamma_{\mu\nu}{}^c = 0 \tag{12.40}$$

(we have explicitly used the vierbein field to operate projections between flat and curved indices). By taking the antisymmetric part, and recalling the definition of the torsion tensor (3.67), we have:

$$\partial_{[\mu} V^c_{\nu]} + \omega_{[\mu}{}^c{}_{\nu]} - Q_{\mu\nu}{}^c = 0. \tag{12.41}$$

We stress that the possible presence of an antisymmetric part in the connection Γ is not at all excluded by the metricity condition (see Sect. 3.5); we can thus compute ω by taking into account a nonzero torsion contribution, and obtain for the Lorentz connection the most general result compatible with a vierbein field satisfying the metricity postulate.

Let us now project the above relation on the flat tangent space (by contracting curved indices with $V^\mu_a V^\nu_b$): we obtain

$$C_{ab}{}^c + \frac{1}{2}\big(\omega_a{}^c{}_b - \omega_b{}^c{}_a\big) - Q_{ab}{}^c = 0, \tag{12.42}$$

where

$$C_{ab}{}^c = V^\mu_a V^\nu_b \partial_{[\mu} V^c_{\nu]} = C_{[ab]}{}^c \tag{12.43}$$

are the so-called *Ricci rotation coefficients*. Let us rewrite three times the above relation by applying a cyclic permutation of the indices a, b, c, and multiplying by -1 the second and the third equation:

$$\begin{aligned} C_{abc} + \frac{1}{2}(\omega_{acb} - \omega_{bca}) - Q_{abc} &= 0, \\ -C_{bca} - \frac{1}{2}(\omega_{cba} - \omega_{cab}) + Q_{bca} &= 0, \\ -C_{cab} - \frac{1}{2}(\omega_{bac} - \omega_{abc}) + Q_{cab} &= 0. \end{aligned} \tag{12.44}$$

Summing up these three equations, and using the symmetry property $\omega_{abc} = \omega_{a[bc]}$, we find that the ω terms of the first and third equation exactly cancel among each other, while those of the second equations combine together to give

$$\omega_{cab} = C_{cab} - C_{abc} + C_{bca} - (Q_{cab} - Q_{abc} + Q_{bca}). \tag{12.45}$$

Let us finally present the result in canonical form, by rising the indices a and b, and projecting the index c on the curved Riemann manifold: we thus arrive at the expression

$$\omega_\mu{}^{ab} = \gamma_\mu{}^{ab} + K_\mu{}^{ab}, \tag{12.46}$$

where

$$\gamma_\mu{}^{ab} = V_\mu^c \left(C_c{}^{ab} - C^{ab}{}_c + C^b{}_c{}^a \right) \tag{12.47}$$

is the so-called *Levi-Civita connection*, and where

$$K_\mu{}^{ab} = -V_\mu^c \left(Q_c{}^{ab} - Q^{ab}{}_c + Q^b{}_c{}^a \right) \tag{12.48}$$

is the contortion tensor (which obviously coincides with the one defined in Eq. (3.88), except for the tangent-space projection).

If we restrict ourselves to a geometric model with vanishing torsion (as in the case of Einstein's theory of general relativity) then the Lorentz connection coincides with the Levi-Civita connection, and is fully determined by the Ricci rotation coefficients (i.e. by the vierbeins and their first partial derivatives) according to Eq. (12.47). In the following sections, and in the rest of this book, we shall always assume (unless otherwise stated) that $Q = 0$ and that $\omega_\mu{}^{ab} = \gamma_\mu{}^{ab}$.

It may be useful, also in view of our subsequent computations, to summarize the symmetry properties of the indices of the various geometric objects contributing to the Lorentz connection. From Eqs. (12.43), (12.45), (12.47), and (12.48) we have:

$$\begin{aligned} &C_{abc} = C_{[ab]c}, \qquad Q_{abc} = Q_{[ab]c}, \qquad \omega_{abc} = \omega_{a[bc]}, \\ &\gamma_\mu{}^{ab} = \gamma_\mu{}^{[ab]}, \qquad K_\mu{}^{ab} = K_\mu{}^{[ab]}. \end{aligned} \tag{12.49}$$

12.3.1 The Curvature Tensor and the Gravitational Action

To complete our short presentation of the geometric formalism based on the vierbein field and on the Lorentz connection we still need the expression of the curvature—in particular, of the Riemann tensor—in terms of these new variables. After such a last step we will be able to express the gravitational Einstein equations using a new approach which is equivalent to the standard one, but which has the virtue of disclosing the local symmetries encoded into the geometric model and—as we shall see in the following chapter—of allowing a direct coupling of the fermion fields to the geometry.

In order to express the curvature as a function of V and ω we start considering the second covariant derivative of the tangent-space vector field A^a. The first covariant derivative obviously coincides with the Lorentz derivative $D_\mu A^a$, given by Eq. (12.30) (since A^a has no curved indices). The second derivative, instead, acts on both the flat index a and the curved index ν, so that

$$\begin{aligned} \nabla_\mu \nabla_\nu A^a = {} & \partial_\mu \left(\partial_\nu A^a + \omega_\nu{}^a{}_b A^b \right) \\ & + \omega_\mu{}^a{}_c \left(\partial_\nu A^c + \omega_\nu{}^c{}_b A^b \right) - \Gamma_{\mu\nu}{}^\alpha D_\alpha A^a . \end{aligned} \tag{12.50}$$

Considering the commutator of the two covariant derivatives we find that the terms which are symmetric in μ and ν drop out, and we are left with

$$[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu] A^a = \left[\partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b \right] A^b - \{ \mu \leftrightarrow \nu \} \tag{12.51}$$

(as already stressed, we are considering a torsionless geometry with $\Gamma_{[\mu\nu]}{}^{\alpha} = 0$).

On the other hand, as discussed in Sect. 6.2, the commutator of two covariant derivatives acting on a "curved-index" vector A^{α} is controlled by the Riemann tensor, and is given by Eq. (6.19). By expressing A^{α} as a projection from the tangent space, $A^{\alpha} = V_a^{\alpha} A^a$, and using the metricity properties of the vierbeins ($\nabla V = 0$), Eq. (6.19) can then be rewritten as

$$V_a^{\alpha}[\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}]A^a = R_{\mu\nu\beta}{}^{\alpha}(\Gamma) V_b^{\beta} A^b, \tag{12.52}$$

where $R_{\mu\nu\beta}{}^{\alpha}(\Gamma)$ is the Riemann tensor (6.10), computed as usual from the Christoffel connection. Comparing this commutator with that of Eq. (12.51), and inverting the projections, we easily get the sought expression connecting the Riemann tensor to the Lorentz connection and its first derivatives. In compact form:

$$R_{\mu\nu\beta}{}^{\alpha}(\Gamma) = V_a^{\alpha} V_{\beta}^b R_{\mu\nu}{}^a{}_b(\omega), \tag{12.53}$$

where we have defined what we may call "the Lorentz curvature" tensor as

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_{\mu}\omega_{\nu}{}^{ab} - \partial_{\nu}\omega_{\mu}{}^{ab} + \omega_{\mu}{}^a{}_c\, \omega_{\nu}{}^{cb} - \omega_{\nu}{}^a{}_c\, \omega_{\mu}{}^{cb}. \tag{12.54}$$

It is worth stressing that the right-hand side of Eq. (12.53) exactly represent the projection (on the curved indices α and β) of the so-called "Yang–Mills field" for the Lorentz connection. This connection, on the other hand, is the gauge potential associated to the local Lorentz symmetry of tangent space. The terms quadratic in ω appearing in the curvature tensor are thus a consequence of the non-Abelian character of the considered local symmetry group. A geometric model of the gravitational interaction based on the dynamics of the space–time curvature—like general relativity—thus acquires, in this context, a natural interpretation as a gauge theory for the local Lorentz group.

There is, however, an important difference between general relativity and a gauge theory of conventional type, due to the fact that the Einstein action is linear, instead of quadratic, in the Yang–Mills field (i.e. in the curvature). The physical reason of such a difference traces back to the fact that the gauge potential (i.e. the connection), in the case of gravity, is a so-called "composed" field, namely it is a function of another variable—the metric or the vierbein—which turns out to be the fundamental dynamical variable. This does not excludes, however, the possibility of considering interaction models based on gravitational actions containing quadratic or higher powers of the curvature.

Let us confine ourselves, at least for the moment, to the standard context of the Einstein theory, and let us conclude this chapter by showing how we can equivalently formulate the Einstein equations in terms of the "gauge" variables $\{V, \omega\}$, instead of using the more conventional "geometric" variables $\{g, \Gamma\}$. We shall consider, for brevity, the gravitational part of the action, assuming that all matter sources are absent.

We can use Eq. (12.6) for the metric determinant, and Eq. (12.53) for the curvature. The scalar curvature (6.24) is thus given by

$$R = R_{\mu\nu}{}^{\nu\mu} = V_a^{\mu} V_b^{\nu} R_{\mu\nu}{}^{ab}(\omega), \tag{12.55}$$

and the Einstein action becomes

$$S = -\frac{1}{2\chi}\int d^4x\sqrt{-g}\,R(\Gamma) = -\frac{1}{2\chi}\int d^4x\,V\,V_a^\mu V_b^\nu R_{\mu\nu}{}^{ab}(\omega), \tag{12.56}$$

where $R_{\mu\nu}{}^{ab}$ is given by Eq. (12.54). The field equations can then be obtained in two ways.

A first possibility is to eliminate everywhere the Lorentz connection in terms of the vierbeins through Eq. (12.46), thus obtaining an action depending only on the vierbeins and their first and second derivatives. The variation with respect to the vierbeins is then performed as in the case of the metric-dependent action.

A second possibility is to treat vierbeins and Lorentz connection as independent variables, and compute the separate variation of the action with respect to V and to ω. This second procedure, called first-order variational formalism, or also "Palatini formalism", is particularly convenient when the action is written in the language of differential forms (see Appendix A). Here it will be adopted as an instructive exercise, in order to illustrate other properties of the vierbeins and of the Lorentz connection.

Let us notice, first of all, that by applying the product rules of the completely antisymmetric tensors (see Sect. 3.2), the Einstein action (12.56) can be written in the following, more convenient form as

$$S = \frac{1}{8\chi}\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V_\alpha^c V_\beta^d R_{\mu\nu}{}^{ab}(\omega) \tag{12.57}$$

(see Exercise 12.2). Let us then varying with respect to the Lorentz connection ω, contained inside R, at fixed V. From the definition (12.54) we have

$$\delta_\omega R_{\mu\nu}{}^{ab} = D_\mu\delta\omega_\nu{}^{ab} - D_\nu\delta\omega_\mu{}^{ab}, \tag{12.58}$$

where

$$D_\mu\delta\omega_\nu{}^{ab} = \partial_\mu\delta\omega_\nu{}^{ab} + \omega_\mu{}^a{}_c\delta\omega_\nu{}^{cb} + \omega_\mu{}^b{}_c\delta\omega_\nu{}^{ac}. \tag{12.59}$$

Inserting this result into the action (12.57), and integrating by parts, we obtain (modulo a total derivative):

$$\delta_\omega S = -\frac{1}{2\chi}\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V_\alpha^c\big(D_\mu V_\beta^d\big)\delta\omega_\nu{}^{ab}. \tag{12.60}$$

We have used the fact that the Lorentz covariant derivative of $\epsilon^{\mu\nu\alpha\beta}$ is vanishing because the object has no flat indices, while the analogous derivative of ϵ^{abcd} is vanishing because the Lorentz connection is antisymmetric (see Exercise 12.3).

By imposing on the action to be stationary, $\delta_\omega S = 0$, we obtain the condition

$$D_{[\mu}V_{\beta]}^d = 0, \tag{12.61}$$

which exactly reproduces Eq. (12.41) (obtained from the metricity postulate) for the case of the torsionless geometry we are considering. Solving for ω we recover

the expression of the Levi-Civita connection, which is now obtained not as a consequence of our geometric assumptions, but as a dynamical "field equation" for the gauge connection.

Let us finally vary the action (12.57) with respect to the vierbein V, at fixed ω. According to the Palatini formalism there are no contributions from R, which only contains the independent variable ω. We thus obtain

$$\delta_V S = \frac{1}{4\chi} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} V_\beta^d R_{\mu\nu}{}^{ab} \delta V_\alpha^c, \tag{12.62}$$

and the condition of stationary action gives

$$\epsilon^{\mu\nu\alpha d} \epsilon_{abcd} R_{\mu\nu}{}^{ab} = 0. \tag{12.63}$$

By using again the product rules of completely antisymmetric tensors, and the relation (12.53) between Lorentz and Riemann curvature (which we can apply thanks to the variational result (12.61)), we find that the above equations can be explicitly rewritten in tensor form as follows:

$$R^\alpha{}_c - \frac{1}{2} V_c^\alpha R \equiv V_c^\beta \left(R^\alpha{}_\beta - \frac{1}{2} \delta_\beta^\alpha R \right) = 0 \tag{12.64}$$

(see Exercise 12.4). We thus exactly recover the standard form of the Einstein equations in vacuum.

12.4 Exercises Chap. 12

12.1 Local Transformation of the Lorentz Covariant Derivative Check that Eq. (12.31) holds provided the Lorentz connection satisfies the transformation law (12.23).

12.2 Einstein Action in the Vierbein Formalism Show that the Einstein action (12.56) can be equivalently rewritten in the form (12.57).

12.3 Lorentz Derivative of the Totally Antisymmetric Tensor Show that $D_\mu \epsilon^{abcd} = 0$.

12.4 Einstein Equations in the Vierbein Formalism Check that Eq. (12.63) is equivalent to the vacuum Einstein equations.

12.5 Solutions

12.1 Solution Let us explicitly write down the left-hand side of Eq. (12.31):

$$\left(D_\mu A^a\right)' = \Lambda^a{}_b \partial_\mu A^b + \left(\partial_\mu \Lambda^a{}_b\right) A^b + \omega'^a_\mu{}_b \Lambda^b{}_c A^c. \tag{12.65}$$

Note also that, for a vector field,

$$\omega_\mu \equiv \omega_\mu{}^{ij}(J_{ij})^a{}_b = 2i\,\omega_\mu{}^a{}_b \tag{12.66}$$

(see Eq. (12.29)). For the vector representation, the transformation (12.23) we can thus be rewritten as follows:

$$\omega'^{\,a}_{\mu\ b} = \left[\Lambda\omega_\mu\Lambda^{-1}\right]^a{}_b - \left[(\partial_\mu\Lambda)\Lambda^{-1}\right]^a{}_b. \tag{12.67}$$

Inserting this result into Eq. (12.65), and simplifying, we obtain

$$\begin{aligned}\left(D_\mu A^a\right)' &= \Lambda^a{}_b\partial_\mu A^b + \Lambda^a{}_b\,\omega_\mu{}^b{}_c A^c \\ &\equiv \Lambda^a{}_b D_\mu A^b,\end{aligned} \tag{12.68}$$

which reproduces indeed the transformation of Eq. (12.31).

12.2 Solution Let us consider the result (3.38), and express one of the two antisymmetric tensors as a projection from its tangent-space counterpart, namely as

$$\eta^{\mu\nu\alpha\beta}V^a_\rho V^b_\sigma V^c_\alpha V^d_\beta\epsilon_{abcd} = -2\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma\right). \tag{12.69}$$

By inverting the projection on the curved indices ρ and σ we obtain

$$\eta^{\mu\nu\alpha\beta}\epsilon_{abcd}V^c_\alpha V^d_\beta = -2\left(V^\mu_a V^\nu_b - V^\nu_a V^\mu_b\right). \tag{12.70}$$

Finally, by using the definitions (3.31) and (12.6), we can rewrite the above equation as follows:

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V^c_\alpha V^d_\beta = -4V\,V^{[\mu}_a V^{\nu]}_b. \tag{12.71}$$

Let us now observe that $R_{\mu\nu}{}^{ab}$ is antisymmetric in the first two indices, so that

$$\begin{aligned}-V\,V^\mu_a V^\nu_b R_{\mu\nu}{}^{ab} &= -V\,V^{[\mu}_a V^{\nu]}_b R_{\mu\nu}{}^{ab} \\ &= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V^c_\alpha V^d_\beta R_{\mu\nu}{}^{ab}.\end{aligned} \tag{12.72}$$

Dividing by 2χ, and integrating over d^4x, we thus arrive at the Einstein action in the form of Eq. (12.57).

12.3 Solution By applying the definition of Lorentz covariant derivative to the contravariant tensor representation in tangent space we have

$$\begin{aligned}D_\mu\epsilon^{abcd} &= \omega_\mu{}^a{}_i\epsilon^{ibcd} + \omega_\mu{}^b{}_i\epsilon^{aicd} \\ &\quad + \omega_\mu{}^c{}_i\epsilon^{abid} + \omega_\mu{}^d{}_i\epsilon^{abci}.\end{aligned} \tag{12.73}$$

Since ϵ is a totally antisymmetric tensor, the four terms on the right-hand side of the above equation can be nonzero only if, in each of these terms, the two flat indices

of the Lorentz connection are equal. But the connection is antisymmetric, so that $\omega_\mu{}^i{}_i = 0$. The covariant derivative of ϵ is thus identically vanishing.

12.4 Solution For the antisymmetric tensors of Eq. (12.63) we can use the product rule (3.39) with three curved indices locally projected on the tangent space, i.e.

$$\epsilon^{\mu\nu\alpha d}\epsilon_{abcd} = -V^{\mu\nu\alpha}_{abc} \equiv -\det\begin{pmatrix} V_a^\mu & V_a^\nu & V_a^\alpha \\ V_b^\mu & V_b^\nu & V_b^\alpha \\ V_c^\mu & V_c^\nu & V_c^\alpha \end{pmatrix}. \tag{12.74}$$

Equation (12.63) then provides (recall the definition of R of Eq. (12.55))

$$\begin{aligned}\epsilon^{\mu\nu\alpha d}\epsilon_{abcd}R_{\mu\nu}{}^{ab} &= -\big(R_{\mu\nu}{}^{\mu\nu}V_c^\alpha + R_{\mu\nu}{}^{\nu\alpha}V_c^\mu + R_{\mu\nu}{}^{\alpha\mu}V_c^\nu \\ &\quad - R_{\mu\nu}{}^{\mu\alpha}V_c^\nu - R_{\mu\nu}{}^{\nu\mu}V_c^\alpha - R_{\mu\nu}{}^{\alpha\nu}V_c^\mu\big) \\ &= -2RV_c^\alpha + 4R_c{}^\alpha = 4V_c^\beta\left(R_\beta{}^\alpha - \frac{1}{2}\delta^\alpha_\beta R\right) = 0,\end{aligned} \tag{12.75}$$

and is thus perfectly equivalent to the Einstein equations in vacuum.

Chapter 13
The Dirac Equation in a Gravitational Field

The aim of this chapter is to introduce a topic which is often omitted in books adopting a traditional approach to general relativity (with due exceptions, see e.g. [51]): the gravitational interactions of spinors.

Such an omission can be easily justified, from a phenomenological point of view, if we consider the weakness of gravity compared with the strength of all the other interactions acting on spinors at a microscopic level. There is no doubt, indeed, that electroweak and strong interactions are dominant with respect to gravity in the density, energy and temperature regimes typical of ordinary matter configurations.

This is no longer true, however, in more "exotic" and extreme regimes, like (for instance) those characterizing the primordial state of our Universe. In fact, as shown by studies of the so-called "renormalization group" equations, the effective coupling constants of all fundamental interactions are "running" as a function of the energy scale, and tend to converge to the same unified value at very high energies.

Also, and most important, the gravitational interactions of spinors cannot be neglected in the context of those theoretical models aiming at a unified descriptions of all fields and interactions (such as modern superstring models). Spinor fields are indeed required to represent the fundamental components of matter like quarks and leptons, and the graviton cannot be excluded from the multiplet of fields interacting with spinors, otherwise the theory would be formally (and physically) inconsistent.

Finally, it should be mentioned that—according to recent "brane models" of the Universe—the gravitational coupling strength could become much higher than Newtonian (and even comparable to the strength of the other interactions) along the extra spatial dimensions, possibly existing "externally" to our four-dimensional space–time. In particular, if the TeV energy scale is the threshold energy needed to probe this effect (as suggested by various theoretical arguments and models), then the gravitational interactions of spinors could directly show up even in present (or near-future) accelerator experiments.

The discussion of this chapter will be focused on the particular case of Dirac spinors, to utilize a model which is supposed to be well known to all students. We will introduce the gravitational interactions by projecting the Dirac action on the Minkowski space–time locally tangent to the (curved) Riemann manifold, and im-

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plementing the local Lorentz invariance through the minimal coupling principle. This procedure, based on the formalism of vierbein fields and Lorentz connection (see Chap. 12), can also be applied to Weyl or Majorana spinors without changes. The results we will obtain may provide the classical starting point for a subsequent quantization, to be performed according to the standard rules of quantum field theory.

13.1 A Concise Summary of the Spinor Formalism

We start by recalling the basic equations and properties of the Dirac spinor model in the Minkowski space–time. This will be useful to fix our notations and show our conventions for the geometrical objects that will be projected on the flat space locally tangent to the Riemann manifold.

We stress, in particular, that the vector/tensor indices referring to the Minkowski space–time will be denoted by small Latin letters, according to the conventions already introduced in the previous chapter; spinor indices, instead, will always be implicit, according to the usual conventions. Also, the system of natural units with $\hbar = 1 = c$ will be used everywhere throughout this chapter.

In the absence of gravity (namely, in a globally flat space–time manifold) the Dirac equation for a spinor field ψ of mass m,

$$i\gamma^a \partial_a \psi - m\psi = 0, \tag{13.1}$$

can be derived from the following action:

$$S = \int d^4x \left(i\overline{\psi}\gamma^a \partial_a \psi - m\overline{\psi}\psi \right). \tag{13.2}$$

Here ψ is a field with four complex components, transforming as a spinor representation of the restricted Lorentz group and of the spatial reflections (or parity) transformations. We have introduced the standard notation $\overline{\psi} = \psi^\dagger \gamma^0$, where the "dagger" symbol denotes transposition and complex conjugation. We have also denoted with γ^a, $a = 0, 1, 2, 3$, the 4×4 Dirac matrices satisfying the so-called Clifford algebra,

$$2\gamma^{(a}\gamma^{b)} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}. \tag{13.3}$$

With our convention for the metric signature we have, in particular,

$$\begin{aligned} &\left(\gamma^0\right)^2 = 1, \qquad \left(\gamma^0\right)^\dagger = \gamma^0, \\ &\left(\gamma^i\right)^2 = -1, \qquad \left(\gamma^i\right)^\dagger = -\gamma^i, \quad i = 1, 2, 3. \end{aligned} \tag{13.4}$$

We can finally introduce the matrix γ^5 such that

$$\begin{aligned} &\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad \left(\gamma^5\right)^2 = 1, \qquad \left(\gamma^5\right)^\dagger = \gamma^5, \\ &\left\{\gamma^5, \gamma^a\right\} \equiv \gamma^5\gamma^a + \gamma^a\gamma^5 = 0 \end{aligned} \tag{13.5}$$

(from now on the anticommutation operator will also be denoted by curly brackets).

The explicit form of the Dirac matrices depends on the chosen representation. For the purpose of this chapter it will be enough to recall here the so-called Weyl (or "chiral") representation, such that the Dirac field takes the form

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{13.6}$$

where ψ_L, ψ_R are two-component Weyl spinors, i.e. spinor representations of the Lorentz group with helicity $-1/2$ (for ψ_L) and $+1/2$ (for ψ_R). Within such a chiral representation we have (using a convenient "block" notation for 2×2 matrices)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{13.7}$$

where σ^i are the usual Pauli matrices satisfying:

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \quad i, j = 1, 2, 3. \tag{13.8}$$

Quite independently of the chosen representation, the Dirac action (13.2) is invariant under *global* Lorentz transformations

$$\psi \longrightarrow \psi' = U\psi, \quad U = e^{-\frac{i}{4}\omega^{ab}\sigma_{ab}}, \tag{13.9}$$

where we have included into the antisymmetric tensor $\omega_{ab} = -\omega_{ba}$ the six independent (and real) constant parameters, while

$$\sigma_{ab} = \frac{i}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a) = i\,\gamma_{[a}\gamma_{b]} \tag{13.10}$$

are the six corresponding generators. Note that the factor $1/4$ appearing at the exponent of the transformation is in agreement with the general definition (12.19), because the intrinsic angular momentum of the Dirac field is associated to the operator

$$J_{ab} = \frac{\sigma_{ab}}{2}. \tag{13.11}$$

Indeed, it is this operator that satisfies the commutation relations of the Lie algebra of the $SO(3,1)$ group (see Sect. 1.3):

$$\left[\frac{1}{2}\sigma_{ab}, \frac{1}{2}\sigma_{cd}\right] = \frac{i}{2}(\eta_{ad}\sigma_{bc} - \eta_{ac}\sigma_{bd} - \eta_{bd}\sigma_{ac} + \eta_{bc}\sigma_{ad}), \tag{13.12}$$

as can be checked by using the properties of the Dirac matrices.

It should be noted, also, that $\sigma_{ab}^\dagger \neq \sigma_{ab}$, so that the representation (13.9) is not a unitary representation: $U^\dagger \neq U^{-1}$. In fact, an explicit computation shows that

$$U^{-1} = \gamma^0 U^\dagger \gamma^0. \tag{13.13}$$

This transformation property, on the other hand, guarantees the Lorentz invariance of the bilinear term $\overline{\psi}\psi$:

$$\overline{\psi}'\psi = \psi^{\dagger\prime}\gamma^0\psi' = \psi^\dagger U^\dagger\gamma^0 U\psi = \psi^\dagger\gamma^0\big(\gamma^0 U^\dagger\gamma^0\big)U\psi = \overline{\psi}\psi, \tag{13.14}$$

and thus the global Lorentz invariance of the Dirac action.

Let us conclude this section by recalling (for later applications) how the explicit form of the spinor generators (13.10) can be obtained from the Lorentz invariance of the total action.

For the global Lorentz transformations we shall use the parametrization (13.9), with σ_{ab} unknown. The transformation of the Dirac equation (13.1) gives

$$i\gamma^a\partial'_a\psi' - m\psi' = i\gamma^a\big(\Lambda^{-1}\big)^b{}_a U\partial_b\psi - mU\psi = 0. \tag{13.15}$$

By applying U^{-1} from the left we obtain the condition:

$$U^{-1}\gamma^a U\big(\Lambda^{-1}\big)^b{}_a = \gamma^b. \tag{13.16}$$

The multiplication by $\Lambda^c{}_b$ gives then the Lorentz transformation of the Dirac matrices,

$$U^{-1}\gamma^c U = \Lambda^c{}_b\gamma^b, \tag{13.17}$$

also useful for our subsequent computations. Let us expand the transformation around the identity, by setting, to first order,

$$\Lambda^a{}_b = \delta^a_b + \omega^a{}_b + \cdots, \qquad U = 1 - \frac{i}{4}\omega^{ab}\sigma_{ab} + \cdots. \tag{13.18}$$

By inserting this expansion into Eq. (13.17), and solving for σ_{ab}, we finally arrive at the result (13.10).

13.2 A Covariant and Locally Lorentz-Invariant Dirac Equation

The coupling of a Dirac spinor to a background gravitational field can be introduced by applying the same procedure already used in all previous cases, namely by embedding the action into the curved space–time manifold as prescribed by the minimal coupling principle.

Such a coupling procedure basically requires three steps (see Sect. 4.1). First of all the integration measure d^4x has to be replaced by its covariant generalization:

$$d^4x \to d^4x\,\sqrt{-g} \equiv d^4x\,V \tag{13.19}$$

(recall that $V \equiv |\det V^a_\mu|$ is the determinant of the vierbein field, see Eq. (12.6)). Second, the scalar products defined with the Minkowski metric η must be rewritten

in terms of the Riemann metric g. In the particular case of the Dirac action this implies, following the index conventions of Chap. 12,

$$\gamma^a \partial_\alpha \to \gamma^\mu \partial_\mu, \tag{13.20}$$

where γ^μ are the Dirac matrices of the flat tangent space locally projected by the vierbeins down to the Riemann manifold (see Sect. 12.1):

$$\gamma^\mu = V_a^\mu \gamma^a. \tag{13.21}$$

The algebraic relation satisfied by γ^μ is similar to that of Eq. (13.3), but with η replaced by g:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = V_a^\mu V_b^\nu \left(\gamma^a \gamma^b + \gamma^b \gamma^a\right) = 2 V_a^\mu V_b^\nu \eta^{ab} = 2 g^{\mu\nu} \tag{13.22}$$

(we have used the vierbein property (12.5)).

Third, we have to replace partial derivatives with covariant derivatives. In our case the field ψ can be regarded as a Lorentz spinor locally defined on the flat tangent space:[1] then the field has no curved Riemann indices, and its total covariant derivative reduces to a Lorentz covariant derivative (see Sect. 12.2). By using the definition (12.22) and the explicit form of the Lorentz generators (13.11) we thus obtain

$$\begin{aligned} \partial_\mu \psi \to \nabla_\mu \psi \equiv D_\mu \psi &= \left(\partial_\mu - \frac{i}{2} \omega_\mu{}^{ab} \frac{\sigma_{ab}}{2}\right) \psi \\ &= \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{[a} \gamma_{b]}\right) \psi, \end{aligned} \tag{13.23}$$

where $\omega_\mu{}^{ab}$ is the Lorentz connection (or gauge potential) introduced in Chap. 12 to restore local Lorentz symmetry.

With the above prescriptions the Dirac action in a curved Riemann manifold takes the following form:

$$S = \int d^4x \sqrt{-g} \left(i \overline{\psi} \gamma^\mu \nabla_\mu \psi - m \overline{\psi} \psi\right). \tag{13.24}$$

This action is clearly a scalar with respect to general coordinate transformations, but is also invariant under local Lorentz transformations ($\psi' = U(x)\psi$) defined in the flat tangent space.

In fact, by adopting the convenient notation

$$\omega_\mu \equiv \omega_\mu{}^{ab} \frac{\sigma_{ab}}{2}, \tag{13.25}$$

[1] An alternative (but scarcely used) method of coupling spinors to a curved geometry is based on the possible representation of spinors in terms of totally antisymmetric tensor fields. This representation, know as the Dirac–Kähler spinor formalism [28], actually dates back to much earlier work by Landau and Ivanenko [27].

and using Eq. (12.23) for the transformations of the Lorentz connection, we easily find that

$$\begin{aligned}(D_\mu\psi)' &= \left(\partial_\mu - \frac{1}{2}\omega'_\mu\right)U\psi \\ &= (\partial_\mu U)\psi + U\partial_\mu\psi - \frac{i}{2}U\omega_\mu\psi - (\partial_\mu U)\psi \\ &= U D_\mu\psi. \end{aligned} \tag{13.26}$$

This means that the covariant derivative of the Dirac field transforms exactly as the field itself. It follows that the kinetic term of the action (13.24)—besides the mass term—is also locally Lorentz invariant. This can be checked by considering its tangent space projection, using the relation $\overline{\psi}' = \overline{\psi}U^{-1}$ following from Eq. (13.13), and using the transformation law (13.17) for the Dirac matrices. In fact:

$$\begin{aligned}\left(\overline{\psi}\gamma^\mu\nabla_\mu\psi\right)' &= \left(\overline{\psi}\gamma^a D_a\psi\right)' = \overline{\psi}U^{-1}\gamma^a\left(\Lambda^{-1}\right)^b{}_a(D_b\psi)' \\ &= \overline{\psi}U^{-1}\gamma^a U\left(\Lambda^{-1}\right)^b{}_a D_b\psi = \overline{\psi}\gamma^b D_b\psi \\ &= \overline{\psi}\gamma^\mu\nabla_\mu\psi. \end{aligned} \tag{13.27}$$

Once the action is known, we can treat ψ and $\overline{\psi}$ as independent Lagrangian variables, and the Dirac equation for a spinor ψ, minimally coupled to the given space–time geometry, is immediately obtained by varying the action with respect to $\overline{\psi}$. This gives

$$i\gamma^\mu D_\mu\psi - m\psi = 0. \tag{13.28}$$

More explicitly, according to the definitions (13.21), (13.23), we can write the above equation in the form

$$i\gamma^a V_a^\mu\partial_\mu\psi - m\psi + \frac{i}{4}\omega_{\mu ab}V_c^\mu\gamma^c\gamma^{[a}\gamma^{b]}\psi = 0, \tag{13.29}$$

where (from Eq. (12.45)):

$$V_c^\mu\omega_{\mu ab} = \omega_{cab} = C_{cab} - C_{abc} - C_{bca}, \tag{13.30}$$

and where C_{abc} are the Ricci rotation coefficients defined by Eq. (12.43).

13.3 Geometry Couplings to the Axial and Vector Currents

Let us discuss in more detail the gravitational interaction described by the last term of Eq. (13.29), which we shall call for simplicity $M(\omega)$:

$$M(\omega)\psi = \frac{i}{4}\omega_{cab}\gamma^c\gamma^{[a}\gamma^{b]}\psi. \tag{13.31}$$

This term, as we shall see, can be explicitly separated as a coupling of the background geometry to the vector and to the axial-vector part of the spinor current.

In order to implement such a splitting let us consider the totally antisymmetric part of the product of three Dirac matrices,

$$6\gamma^{[a}\gamma^{b}\gamma^{c]} = \big(\gamma^{a}\gamma^{b}\gamma^{c} + \gamma^{b}\gamma^{c}\gamma^{a} + \gamma^{c}\gamma^{a}\gamma^{b} - \gamma^{a}\gamma^{c}\gamma^{b} - \gamma^{b}\gamma^{a}\gamma^{c} - \gamma^{c}\gamma^{b}\gamma^{a}\big), \tag{13.32}$$

which—by applying the anticommutation rules (13.3)—can also be rewritten as follows:

$$\gamma^{[a}\gamma^{b}\gamma^{c]} = \gamma^{a}\gamma^{b}\gamma^{c} - \gamma^{a}\eta^{bc} + \gamma^{b}\eta^{ca} - \gamma^{c}\eta^{ab}. \tag{13.33}$$

We need, in particular, the part antisymmetric with respect to b and c, which can be extracted from the above equation as follows:

$$\gamma^{a}\gamma^{[b}\gamma^{c]} = \gamma^{[a}\gamma^{b}\gamma^{c]} + 2\eta^{a[b}\gamma^{c]}. \tag{13.34}$$

We should also notice that the matrix γ^5, defined by Eq. (13.5), can be conveniently (and covariantly) expressed, with our conventions, as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{abcd}\gamma^{a}\gamma^{b}\gamma^{c}\gamma^{d} \tag{13.35}$$

(recall that $\epsilon_{0123} = -\epsilon^{0123} = -1$). By exploiting the product rules satisfied by the totally antisymmetric tensors (given in Sect. 3.2) we thus arrive at the relation:

$$\gamma^{[a}\gamma^{b}\gamma^{c]} = -i\epsilon^{abcd}\gamma^5\gamma_d. \tag{13.36}$$

Using this result into Eq. (13.34) we find that $M(\omega)$ can be finally expressed in the form:

$$M(\omega) = \frac{1}{4}\omega_{abc}\epsilon^{abcd}\gamma^5\gamma_d + \frac{i}{2}\omega_{a}{}^{a}{}_{c}\gamma^{c}. \tag{13.37}$$

This expression tell us that the trace of the Lorentz connection, $\omega_{a}{}^{a}{}_{c}$, is coupled to the vector current of the Dirac field, while the totally antisymmetric part, $\omega_{[abc]}$, is coupled to the axial-vector current.

On the other hand, according to the explicit definition (13.30) of the Lorentz connection, we have

$$\omega_{[abc]} = C_{[abc]}, \qquad \omega_{a}{}^{a}{}_{c} = C_{ca}{}^{a}. \tag{13.38}$$

The Dirac equation (13.29) can thus be rewritten in the equivalent (but more convenient) form

$$i\gamma^{a}V_{a}^{\mu}\partial_{\mu}\psi - m\psi + \frac{1}{4}C_{[abc]}\epsilon^{abcd}\gamma^5\gamma_d\psi + \frac{i}{2}C_{ca}{}^{a}\gamma^{c}\psi = 0, \tag{13.39}$$

where the background geometry is directly described by the Ricci rotation coefficients

$$C_{ab}{}^{c} = V_a^{\mu} V_b^{\nu} \partial_{[\mu} V_{\nu]}^{c} \tag{13.40}$$

(see Sect. 12.3). Equation (13.39) clearly shows how the gravitational interaction of the spinor is completely determined by the set of vierbein fields V_{μ}^{a} (associated to the given metric) and by the corresponding first-order derivatives of the vierbeins (see also Exercise 13.1).

13.4 Symmetrized Form of the Covariant Dirac Action

It may instructive to derive Eq. (13.39) starting with the Dirac action written in a form which is symmetric with respect to ψ and $\overline{\psi}$ (more appropriate to a quantized version of the theory). Such a symmetric form can be easily obtained by adding to the covariant action (13.24) the corresponding hermitian conjugate (h.c.) expression:

$$S = \int d^4x \sqrt{-g}\, \frac{1}{2}\big(i\overline{\psi}\gamma^a D_a \psi - m\overline{\psi}\psi + \text{h.c.}\big). \tag{13.41}$$

The associated Lagrangian density, in explicit form, is then given by

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{2}\sqrt{-g}\big[\overline{\psi}\gamma^a \partial_a \psi - \big(\overline{\psi}\gamma^a \partial_a \psi\big)^{\dagger}\big] \\ & + \frac{i}{8}\sqrt{-g}\,\omega_{abc}\big[\overline{\psi}\gamma^a \gamma^{[b}\gamma^{c]}\psi - \big(\overline{\psi}\gamma^a \gamma^{[b}\gamma^{c]}\psi\big)^{\dagger}\big] - \sqrt{-g}\, m\overline{\psi}\psi.\end{aligned} \tag{13.42}$$

Let us separately consider the various contributions.

By using the relation

$$\gamma^0\big(\gamma^a\big)^{\dagger}\gamma^0 = \gamma^a, \tag{13.43}$$

we first obtain, for the hermitian conjugation of the kinetic term (second term in the first square brackets):

$$-\frac{i}{2}\sqrt{-g}\,\partial_a \psi^{\dagger}\big(\gamma^a\big)^{\dagger}\gamma^0\psi = -\frac{i}{2}\sqrt{-g}\,\partial_a \overline{\psi}\gamma^a\psi. \tag{13.44}$$

Consider then the conjugate of the connection contribution (second term in the second square brackets),

$$-\frac{i}{8}\sqrt{-g}\,\omega_{abc}\psi^{\dagger}\frac{1}{2}\big[\big(\gamma^b\gamma^c\big)^{\dagger} - \big(\gamma^c\gamma^b\big)^{\dagger}\big]\big(\gamma^a\big)^{\dagger}\gamma^0\psi. \tag{13.45}$$

By recalling the properties (13.3), (13.4) of the Dirac matrices we have

$$\begin{aligned}\big(\gamma^a\big)^{\dagger}\gamma^0 &= \gamma^0\gamma^a, \\ \big(\gamma^b\gamma^c\big)^{\dagger}\gamma^0 &= -\gamma^0\big(\gamma^b\gamma^c\big), \quad b \neq c,\end{aligned} \tag{13.46}$$

so that the contribution of Eq. (13.45) becomes

$$\frac{i}{8}\sqrt{-g}\,\omega_{abc}\,\overline{\psi}\gamma^{[b}\gamma^{c]}\gamma^{a}\psi. \tag{13.47}$$

Summing up all contributions we are thus led to the following effective Lagrangian:

$$\mathcal{L} = \sqrt{-g}\Big[\frac{i}{2}\big(\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\overline{\psi}\gamma^{\mu}\psi\big) - m\overline{\psi}\psi \\ + \frac{i}{8}\sqrt{-g}\,\omega_{abc}\,\overline{\psi}\big(\gamma^{a}\gamma^{[b}\gamma^{c]} + \gamma^{[b}\gamma^{c]}\gamma^{a}\big)\psi\Big]. \tag{13.48}$$

We should recall, at this point, Eq. (13.33). Its antisymmetrization with respect to b and c gives Eq. (13.34). After the cyclic permutation $\{abc\} \to \{bca\}$, a new antisymmetrization with respect to b and c gives

$$\gamma^{[b}\gamma^{c]}\gamma^{a} = \gamma^{[a}\gamma^{b}\gamma^{c]} - 2\eta^{a[b}\gamma^{c]}. \tag{13.49}$$

By adding Eqs. (13.34), (13.49), and inserting the result into Eq. (13.48), we obtain for the symmetrized Lagrangian the following final expression:

$$\mathcal{L} = \frac{i}{2}\sqrt{-g}\big(\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\overline{\psi}\gamma^{\mu}\psi\big) - \sqrt{-g}\,m\overline{\psi}\psi + \frac{i}{4}\sqrt{-g}\,\omega_{[abc]}\,\overline{\psi}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi. \tag{13.50}$$

Here—perhaps surprisingly—the Lorentz connection is directly coupled only to the axial-vector part of the spinor current. The coupling to the vector current, present in the Dirac equation of the previous section, would seem to be absent.

Actually this conclusion would be wrong, because the above symmetrized Lagrangian also contains a new term—with the derivative of $\overline{\psi}$—which couples ψ to $\sqrt{-g}$ and to γ^{μ}:

$$-\frac{i}{2}\sqrt{-g}\,\partial_{\mu}\overline{\psi}\gamma^{\mu}\psi. \tag{13.51}$$

Such a term gives an additional contribution to the equation of motion which, as we shall see, exactly reproduces the trace of the Lorentz connection and the gravitational coupling to the vector spinor current.

In fact, let us write down the Euler–Lagrange equations following from the variation of the action density (13.50) with respect to $\overline{\psi}$. The partial derivative with respect to the field is

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = \sqrt{-g}\left(\frac{i}{2}\gamma^{\mu}\partial_{\mu}\psi - m\psi + \frac{i}{4}\omega_{[abc]}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi\right). \tag{13.52}$$

The corresponding canonical momentum is

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\overline{\psi})} = -\frac{i}{2}\sqrt{-g}\,\gamma^{\mu}\psi, \tag{13.53}$$

and its derivative gives

$$\begin{aligned}\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} &= -\frac{i}{2}\sqrt{-g}\left[\gamma^\mu \partial_\mu \psi + (\partial_\mu \gamma^\mu)\psi + \frac{1}{\sqrt{-g}}(\partial_\mu \sqrt{-g})\gamma^\mu \psi\right] \\ &= -\frac{i}{2}\sqrt{-g}\left[\gamma^\mu \partial_\mu \psi + \frac{1}{\sqrt{-g}}(\partial_\mu \sqrt{-g}\gamma^\mu)\psi\right] \\ &= -\frac{i}{2}\sqrt{-g}\left[\gamma^\mu \partial_\mu \psi + \frac{1}{\sqrt{-g}}(\partial_\mu \sqrt{-g}\, V_a^\mu)\gamma^a \psi\right]. \end{aligned} \tag{13.54}$$

The last term of this equation, on the other hand, can be expressed through the trace of the Lorentz connection,

$$\omega_a{}^a{}_b = \frac{1}{\sqrt{-g}}(\partial_\mu \sqrt{-g}\, V_b^\mu) \tag{13.55}$$

(see Exercise 13.2). Hence:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = -\frac{i}{2}\sqrt{-g}\big(\gamma^\mu \partial_\mu \psi + \omega_a{}^a{}_b \gamma^b \psi\big). \tag{13.56}$$

By subtracting Eqs. (13.52) and (13.56) we finally obtain the spinor equation of motion:

$$i\gamma^a V_a^\mu \partial_\mu \psi - m\psi + \frac{i}{4}\omega_{[abc]}\gamma^{[a}\gamma^b\gamma^{c]}\psi + \frac{i}{2}\omega_a{}^a{}_b \gamma^b \psi = 0. \tag{13.57}$$

If we introduce γ^5 through Eq. (13.36), and express the connection in terms of the Ricci coefficients (using Eq. (13.38)), we exactly recover the covariant Dirac equation (13.39) presented in the previous section.

13.5 Exercises Chap. 13

13.1 Dirac Equation in a Conformally Flat Space–Time Write down the explicit form of the Dirac equation for a massive spinor in a conformally flat space–time geometry, described by the metric

$$g_{\mu\nu}(x) = f^2(x)\eta_{\mu\nu}. \tag{13.58}$$

13.2 Trace of the Lorentz Connection Derive Eq. (13.55) for the trace of the Lorentz connection.

13.3 Energy-Momentum Tensor of a Dirac Spinor Compute the dynamical energy-momentum tensor (7.27) for a free Dirac field embedded in a curved Riemannian geometry.

13.6 Solutions

13.1 Solution The vierbein field associated to the metric (13.58), and defined in such a way as to satisfy Eq. (12.5), is given by

$$V_\mu^a = f\delta_\mu^a, \qquad V_a^\mu = f^{-1}\delta_a^\mu. \tag{13.59}$$

The computation of the corresponding Ricci rotation coefficients, from Eq. (12.43), leads to

$$C_{ab}{}^c = \frac{1}{2f^2}\left(\delta_b^c\delta_a^\mu - \delta_a^c\delta_b^\mu\right)\partial_\mu f. \tag{13.60}$$

The trace of the Lorentz connection, according to Eq. (13.38), is then

$$\omega_b{}^b{}_a = C_{ab}{}^b = \frac{3}{2f^2}\delta_a^\mu\partial_\mu f. \tag{13.61}$$

Let us now compute the antisymmetric part of the connection. From Eq. (13.60) we have

$$C_{abc} = \frac{1}{2f^2}\left(\eta_{cb}\delta_a^\mu - \eta_{ca}\delta_b^\mu\right)\partial_\mu f. \tag{13.62}$$

Hence, according to Eq. (13.38):

$$\omega_{[abc]} = C_{[abc]} \equiv 0. \tag{13.63}$$

It follows that the Dirac equation (13.39) (or (13.57)) reduces to

$$\left(if^{-1}\gamma^a\delta_a^\mu\partial_\mu - m + \frac{3i}{4f^2}\gamma^a\delta_a^\mu\partial_\mu f\right)\psi = 0. \tag{13.64}$$

Multiplying by f we finally obtain

$$\left(i\gamma^a\delta_a^\mu\partial_\mu - mf + i\frac{3}{4}\gamma^a\delta_a^\mu\partial_\mu \ln f\right)\psi = 0. \tag{13.65}$$

We may thus conclude that the coupling to a conformally flat geometry generates an effective, position-dependent mass term $\widetilde{m} = mf$, and an "effective potential" described by the last term of the above equation.

13.2 Solution Let us start with the metric condition for the vierbein field, Eq. (12.39), which we can rewrite as

$$\omega_\mu{}^a{}_\nu = \Gamma_{\mu\nu}{}^a - \partial_\mu V_\nu^a. \tag{13.66}$$

Let us compute its trace, by applying V_a^μ:

$$\begin{aligned}\omega_a{}^a{}_\nu &= \Gamma_{\mu\nu}{}^\mu - V_a^\mu\partial_\mu V_\nu^a \\ &= \frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g} + V_\nu^a\partial_\mu V_a^\mu.\end{aligned} \tag{13.67}$$

Note that in the second line we have used the trace of the Christoffel connection and the relation

$$\partial_\mu \left(V_\nu^a V_a^\mu\right) = \partial_\mu \left(\delta_\nu^\mu\right) = 0. \tag{13.68}$$

By multiplying Eq. (13.67) by V_b^ν we finally obtain

$$\omega_a{}^a{}_b = \frac{1}{\sqrt{-g}} \partial_b \sqrt{-g} + \partial_\mu V_b^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V_b^\mu\right), \tag{13.69}$$

which exactly reproduces Eq. (13.55) used in Sect. 13.4.

13.3 Solution Let us consider the covariant action (13.41), symmetrized with respect to ψ and $\overline{\psi}$. Using our previous results for the Lagrangian (13.48) we can rewrite the symmetrized action in compact form as follows:

$$S = \int d^4x \sqrt{-g} \left[\frac{i}{2} g^{\mu\nu} (\overline{\psi} \gamma_\mu D_\nu \psi - D_\nu \overline{\psi} \gamma_\mu \psi) - m \overline{\psi} \psi \right], \tag{13.70}$$

where

$$\begin{aligned} D_\nu \psi &= \partial_\nu \psi + \frac{1}{4} \omega_{\nu ab} \gamma^{[a} \gamma^{b]} \psi, \\ D_\nu \overline{\psi} &= \partial_\nu \overline{\psi} - \frac{1}{4} \omega_{\nu ab} \overline{\psi} \gamma^{[a} \gamma^{b]}. \end{aligned} \tag{13.71}$$

In order to obtain the dynamical energy-momentum tensor we have now to vary the action with respect to the metric, by imposing that the equations of motion of the Dirac field are satisfied (see e.g. Sect. 7.2). By applying the standard definition (7.27), in particular, we obtain

$$\delta S = \frac{1}{2} \int d^4x \, \sqrt{-g} \, T_{\mu\nu} \delta g^{\mu\nu}, \tag{13.72}$$

where

$$T_{\mu\nu} = i \overline{\psi} \gamma_{(\mu} D_{\nu)} \psi - i D_{(\nu} \overline{\psi} \gamma_{\mu)} \psi \tag{13.73}$$

is the sought energy-momentum tensor.

It should be noted that the variation of $\sqrt{-g}$ (which is present in the action) does not contribute to $T_{\mu\nu}$ as a consequence of the equations of motion, which impose on the Dirac field the conditions:

$$i \gamma^\mu D_\mu \psi = m \psi, \qquad i D_\mu \overline{\psi} \gamma^\mu = -m \overline{\psi}. \tag{13.74}$$

Chapter 14
Supersymmetry and Supergravity

In this chapter we will discuss the gravitational interactions of simple physical systems containing bosonic and fermionic degrees of freedom, taking into account the possibility that these different components are related among themselves by infinitesimal field transformations.

In the case in which the equations of motion of the system are left invariant by such transformations we will say that the transformations represent a *supersymmetry* (SUSY) operation for the given system. The supersymmetry is global if the transformations depend on a set of constant parameters, local if such parameters are position dependent.

Local supersymmetry, as we shall see, can only be implemented within a generally covariant context, namely in the context of physical models which are formulated in a curved space–time and which thus automatically include the gravitational interactions. Gravitational models containing bosonic and fermionic sources, and invariant under local supersymmetry transformations, are called *supergravity* (SUGRA) models.

In this chapter, after presenting a few explicit examples of global supersymmetry for elementary physical systems in Minkowski space–time, we will introduce the simplest supergravity model with two fundamental components only (the graviton and the gravitino). To this purpose, we shall start by listing here the basic properties characterizing the parameters of a generic SUSY transformation (for an exhaustive introduction to supersymmetry and supergravity the interested reader is referred, for instance, to the book [55] of the final Bibliography).

Let us suppose that the infinitesimal transformation connecting a boson field $B(x)$ to a fermion field $F(x)$ is of the type

$$B \to B' = B + \delta B, \quad \delta B = \epsilon F, \tag{14.1}$$

where ϵ symbolically represents a set of constant, infinitesimal parameters. Given that B is a field of integer spin, F of half-integer spin, we can immediately conclude that the parameter ϵ must be represented by a spinorial object, in order to restore the correct statistical properties of the above equation. In a quantum version

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of the model the components of ϵ (and of $\overline{\epsilon} = \epsilon^{\dagger}\gamma^0$) must commute with B and anticommute with F (and among themselves).

In addition, if the boson is represented by a real field, $B = B^*$, it may be convenient to formulate the supersymmetric model by associating to F a Majorana fermion field (since, in that case, we can always choose for F a representation with real components). In that case the parameter ϵ also must correspond to a Majorana spinor, i.e. it must satisfy the condition

$$\epsilon = \epsilon^c, \quad \epsilon^c = C\,\overline{\epsilon}^T, \tag{14.2}$$

where C is the charge conjugation operator, defined by

$$C^T = -C, \qquad C^{-1}\gamma^{\mu}C = -\left(\gamma^{\mu}\right)^T \tag{14.3}$$

(the superscript T denotes, as usual, the "transpose" operation). For the Majorana spinors we can assume that the anticommutation properties also hold at a classical level, namely that the variables ϵ^A satisfy a Grassmann algebra of the type

$$\left\{\epsilon^A, \epsilon^B\right\} = 0 = \left\{\epsilon^A, \overline{\epsilon}^B\right\}. \tag{14.4}$$

Finally, let us consider the physical dimensions of the parameter ϵ. In a four-dimensional space–time, and in units $\hbar = c = 1$, we have $[B] = M$, $[F] = M^{3/2}$, so that ϵ must have dimensions $[\epsilon] = M^{-1/2}$. It follows that the transformation of the fermion field F completing the set of SUSY transformations must be of the type

$$F \to F' = F + \delta F, \quad \delta F = \epsilon\,\partial B. \tag{14.5}$$

This means that we may expect, for dimensional reasons, the presence of a gradient operator in the transformation of the fermion field. It is just that presence, as we shall see, that triggers the connection between SUSY transformations and space–time translations, and leads from local SUSY to supergravity models.

Notation Throughout this chapter the spinor indices will be denoted by capital Latin letters. Also, when discussing models of global supersymmetry, we will always use Greek letters for the Lorentz vector indices, as they will be referred to a global Minkowski space–time, without any possible confusion with the flat tangent space. Finally, we will always adopt natural units with $\hbar = c = 1$.

14.1 Global Supersymmetry in Flat Space–Time

A very simple example of globally supersymmetric system can be obtained by considering two particles– with spin 0 and spin 1/2—represented, respectively, by a real scalar field ϕ and a Majorana spinor ψ in flat Minkowski space–time.

Consider the following infinitesimal transformation connecting ϕ and ψ:

$$\phi \to \phi + \delta\phi, \qquad \psi \to \psi + \delta\psi, \tag{14.6}$$

where

$$\delta\phi = \overline{\epsilon}\psi, \qquad \delta\psi = -\frac{i}{2}\gamma^{\mu}\epsilon\partial_{\mu}\phi, \tag{14.7}$$

and where

$$\epsilon = \epsilon^{c} = C\,\overline{\epsilon}^{T} = \text{const}, \qquad \psi = \psi^{c} = C\,\overline{\psi}^{T}. \tag{14.8}$$

We can easily check that this transformation preserves the Lagrangian density of our scalar-spinor system,

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + i\,\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi, \tag{14.9}$$

modulo a total divergence which does not contributes to the equations of motion.

Let us compute, first of all, the infinitesimal variation of the conjugate field $\overline{\psi}$. From Eq. (14.7) we obtain

$$\delta\overline{\psi} = \frac{i}{2}\big(\gamma^{\mu}\epsilon\partial_{\mu}\phi\big)^{\dagger}\gamma^{0} = \frac{i}{2}\epsilon^{\dagger}\gamma^{\mu\dagger}\gamma^{0}\partial_{\mu}\phi = \frac{i}{2}\overline{\epsilon}\gamma^{\mu}\partial_{\mu}\phi, \tag{14.10}$$

where we have used Eq. (13.46). The total variation of the Lagrangian under the transformation (14.7) is then given by

$$\delta\mathcal{L} = \partial^{\mu}\phi\overline{\epsilon}\partial_{\mu}\psi + \frac{1}{2}\overline{\psi}\gamma^{\mu}\gamma^{\nu}\epsilon\partial_{\mu}\partial_{\nu}\phi - \frac{1}{2}\overline{\epsilon}\gamma^{\mu}\gamma^{\nu}\partial_{\nu}\psi\partial_{\mu}\phi. \tag{14.11}$$

Let us now apply the properties of the Dirac matrices in the Minkowski space–time

$$\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu} = \gamma^{(\mu}\gamma^{\nu)}\partial_{\mu}\partial_{\nu} = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu} \equiv \Box, \tag{14.12}$$

and extract a total divergence from the first and third contribution to $\delta\mathcal{L}$. Eq. (14.11) can then be rewritten as

$$\delta\mathcal{L} = \partial_{\mu}\big(\overline{\epsilon}\psi\,\partial^{\mu}\phi\big) - \overline{\epsilon}\psi\Box\phi + \frac{1}{2}\overline{\psi}\epsilon\Box\phi - \frac{1}{2}\partial_{\nu}\big(\overline{\epsilon}\gamma^{\mu}\gamma^{\nu}\psi\,\partial_{\mu}\phi\big) + \frac{1}{2}\overline{\epsilon}\psi\Box\phi. \tag{14.13}$$

All terms containing $\Box\phi$ cancel each other, since $\overline{\epsilon}\psi = \overline{\psi}\epsilon$ (see Exercise 14.1). We are thus left with a total divergence term, which can be rewritten as

$$\delta\mathcal{L} = \partial_{\mu}K^{\mu}, \tag{14.14}$$

where

$$\begin{aligned} K^{\mu} &= \overline{\epsilon}\psi\,\partial^{\mu}\phi - \frac{1}{2}\overline{\epsilon}\gamma^{\nu}\gamma^{\mu}\psi\,\partial_{\nu}\phi \\ &= \overline{\epsilon}\psi\,\partial^{\mu}\phi - \frac{1}{2}\overline{\epsilon}\big(-\gamma^{\mu}\gamma^{\nu} + 2\eta^{\mu\nu}\big)\psi\,\partial_{\nu}\phi \\ &= \frac{1}{2}\overline{\epsilon}\gamma^{\mu}\gamma^{\nu}\psi\,\partial_{\nu}\phi. \end{aligned} \tag{14.15}$$

The equations of motion are left unchanged under the transformation $\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$ (see Sect. 1.1), hence the considered transformation (14.7) represents a symmetry for the system described by the Lagrangian (14.9). We stress (in view of our subsequent discussion) that the above result has been obtained without using the equations of motion of ϕ and ψ.

Let us now compute the commutator of two infinitesimal transformations, with parameters ϵ_1 and ϵ_2, applied to the scalar field ϕ. We have

$$\begin{aligned} \delta_1\phi &= \overline{\epsilon}_1\psi, \\ \delta_2\delta_1\phi &= \overline{\epsilon}_1\delta_2\psi = -\frac{i}{2}\overline{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\phi, \end{aligned} \tag{14.16}$$

and then

$$\begin{aligned} (\delta_2\delta_2 - \delta_1\delta_2)\phi &= -\frac{i}{2}\big(\overline{\epsilon}_1\gamma^\mu\epsilon_2 - \overline{\epsilon}_2\gamma^\mu\epsilon_1\big)\partial_\mu\phi \\ &= -i\big(\overline{\epsilon}_1\gamma^\mu\epsilon_2\big)\partial_\mu\phi. \end{aligned} \tag{14.17}$$

In the second line we have used the relation $\overline{\epsilon}_2 = -\epsilon_2^T C^{-1}$ (see Exercise 14.1); we have also applied Eq. (14.3) and the anticommuting properties of the Majorana spinors, which imply:

$$\overline{\epsilon}_2\gamma^\mu\epsilon_1 = -\epsilon_2^T C^{-1}\gamma^\mu C\overline{\epsilon}_1^T = \epsilon_2^T\gamma^{\mu T}\overline{\epsilon}_1^T = -\big(\overline{\epsilon}_1\gamma^\mu\epsilon_2\big)^T = -\overline{\epsilon}_1\gamma^\mu\epsilon_2. \tag{14.18}$$

As explicitly shown by the result (14.17), the commutator of two infinitesimal supersymmetry transformations is proportional to an infinitesimal translation generated by the gradient operator, with parameter ξ^μ proportional to $\overline{\epsilon}_1\gamma^\mu\epsilon_2$. When $\epsilon = \epsilon(x)$, in particular, we obtain a local translation with parameter $\xi^\mu = \xi^\mu(x)$, which is equivalent to a general (infinitesimal) coordinate transformation, $x^\mu \to x^\mu + \xi^\mu(x)$. It follows that the invariance under local supersymmetry can be implemented only in a model which is also invariant under general coordinate transformations, i.e. in model formulated in the context of a curved space–time, which automatically includes the gravitational interaction. We obtain, in this way, gravitational models which are locally supersymmetric, and which are called *supergravity models*.

A comparison between translations and SUSY transformations also suggests to introduce, for each spinor parameter ϵ^A of the infinitesimal transformation (14.7), a spinor (Majorana) generator Q_A, such that

$$\delta\phi = \overline{\epsilon}\psi \equiv \big(\overline{\epsilon}^A Q_A\big)\phi \tag{14.19}$$

(let us recall here that capital Latin letters are used to label the spinor components). In this case the commutator of two SUSY transformations becomes

$$\begin{aligned}[\delta_2, \delta_1]\phi &= \left(\overline{\epsilon}_2^A Q_A \overline{\epsilon}_1^B Q_B - \overline{\epsilon}_1^B Q_B \overline{\epsilon}_2^A Q_A\right)\phi \\ &= \left(\overline{\epsilon}_2^A Q_A \overline{Q}_B \epsilon_1^B + \overline{Q}_B \overline{\epsilon}_2^A \epsilon_1^B Q_A\right)\phi \\ &= \overline{\epsilon}_2^A \{Q_A, \overline{Q}_B\} \epsilon_1^B \phi. \end{aligned} \tag{14.20}$$

In the second line we have used the relations $\overline{\epsilon}_1 Q = \overline{Q}\epsilon_1$ and $\epsilon_1^B \overline{\epsilon}_2^A = -\overline{\epsilon}_2^A \epsilon_1^B$, while in the third line the relations $\overline{Q}_B \overline{\epsilon}_2^A = -\overline{\epsilon}_2^A \overline{Q}_B$ and $\epsilon_1^B Q_A = -Q_A \epsilon_1^B$, following from the anticommuting properties of the Majorana spinors. Comparing this result with Eq. (14.17), and using Eq. (14.18), we can immediately conclude that the SUSY generators obey the following anticommutation relations:

$$\{Q_A, \overline{Q}_B\} = i\gamma^\mu_{AB}\partial_\mu = \left(\gamma^\mu P_\mu\right)_{AB}. \tag{14.21}$$

Here we have explicitly displayed the spinor indices A, B of the Dirac matrices, and we have denoted with $P_\mu = i\partial_\mu$ the standard generator of the space–time translations.

Since the translations are elements of the Poincaré group, the above relation suggests a possible supersymmetric extension of such a group obtained by adding the spinor generators Q_A to the Poincaré generators P_μ, $J_{\mu\nu}$, and by including into the Lie algebra both commutation and anticommutation relations.

Such a generalization indeed exists, has a consistent group structure, and corresponds to the so-called "super-Poincaré" group based on the set of generators $\{P_\mu, J_{\mu\nu}, Q_A\}$, which satisfy a Lie algebra called "graded" algebra (or super-algebra). The study of super-groups and of the associated super-manifolds (parametrized by an equal number of bosonic and fermionic coordinates) provides a powerful tool for the formulation of supersymmetry and supergravity models (see e.g. the book [7] of the final Bibliography).

14.1.1 Example: The Wess–Zumino Model

The supersymmetric model introduced in the previous section is not algebraically consistent, because the algebra of its generators is not a closed algebra. One finds, in particular, that the result (14.17), connecting the SUSY transformations to an infinitesimal translation, cannot be obtained if the transformations are applied to ψ instead of ϕ (as done previously).

This is due to the fact that the model (14.9) has a different number of bosonic and fermionic degrees of freedom. In fact, the real scalar field has only one component, while a Majorana spinor has four real components. Working "on-shell", namely imposing that the equations of motion are satisfied, $\Box\phi = 0 = i\gamma^\mu\partial_\mu\psi$, reduces the number of independent spinor components from four to two, but the number of degrees of freedom still does not coincide.

This problem can be easily solved by increasing the number of bosonic components, as done in the so-called Wess–Zumino model [54] which contains three real fields (a scalar A, a pseudo-scalar B and a Majorana spinor $\psi = \psi^c$), and is described by the free Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu A \partial^\mu A + \frac{1}{2}\partial_\mu B \partial^\mu B + i\overline{\psi}\gamma^\mu \partial_\mu \psi \tag{14.22}$$

(we have omitted, for simplicity, terms describing possible mutual interactions). By imposing the equations of motion,

$$\Box A = 0, \qquad \Box B = 0, \qquad i\gamma^\mu \partial_\mu \psi = 0, \tag{14.23}$$

we are left with two bosonic and two fermionic degrees of freedom, because the Dirac equation imposes two (Weyl) conditions on the four real components of the spinor, thus halving the number of independent components. The on-shell version of this model is thus appropriate to implement a structure which is both supersymmetric and algebraically consistent.

In fact, the Wess–Zumino model is globally supersymmetric with respect to the following transformations:

$$\begin{aligned} \delta A &= \overline{\epsilon}\psi, \\ \delta B &= i\overline{\epsilon}\gamma^5 \psi, \\ \delta \psi &= -\frac{i}{2}\gamma^\mu \partial_\mu \big(A + i\gamma^5 B\big)\epsilon, \end{aligned} \tag{14.24}$$

where $\epsilon = \epsilon^c$ is a constant (Majorana) spinor parameter. The variation of the Lagrangian (14.22) induced by the above transformations can be put in the form of a total divergence, $\delta\mathcal{L} = \partial_\mu K^\mu$, even without using the equations of motion, just like in the model of the previous section.

Unlike the previous model, however, the commutator of two SUSY transformations now gives the same result whatever the field (A, B, ψ) on which it is applied, *provided* one applies the spinor field equations of motion. One obtains, in particular,

$$[\delta_2, \delta_1] \begin{pmatrix} A \\ B \\ \psi \end{pmatrix} = -i\overline{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu \begin{pmatrix} A \\ B \\ \psi \end{pmatrix}, \tag{14.25}$$

in agreement with the result of Eq. (14.17) (see Exercise 14.2). Without using the equations of motion, on the contrary, the algebra does not close because the model only contains two bosonic degrees of freedom, to be compared with the four fermionic degrees of freedom.

However, it is possible to obtain an algebraically consistent model even "off-shell" (i.e. without imposing the equations of motion) by adding to the Lagrangian (14.22) two bosonic fields of "auxiliary" type (namely, without kinetic terms):

a scalar F and a pseudo-scalar G. The new Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B - F^2 - G^2\right) + i\overline{\psi}\gamma^\mu \partial_\mu \psi, \tag{14.26}$$

is invariant (again, modulo a total divergence) under the following global supersymmetry transformations:

$$\begin{aligned}
\delta A &= \overline{\epsilon}\psi, \\
\delta B &= i\overline{\epsilon}\gamma^5\psi, \\
\delta\psi &= -\frac{i}{2}\gamma^\mu\partial_\mu\left(A + i\gamma^5 B\right)\epsilon + \frac{1}{2}\left(F - i\gamma^5 G\right)\epsilon, \\
\delta F &= -i\overline{\epsilon}\gamma^\mu\partial_\mu\psi, \\
\delta G &= \overline{\epsilon}\gamma^\mu\gamma^5\partial_\mu\psi.
\end{aligned} \tag{14.27}$$

This model has the same number (four) of bosonic and fermionic off-shell degrees of freedom, and the commutator of two transformations leads to a consistent result—proportional to an effective translation—irrespectively of the field to which it is applied, and even without using the equations of motion.

14.2 The Rarita–Schwinger Field

Another simple (but important) example of global supersymmetry is obtained by considering a system of two particles—with spin 2 and spin 3/2—in the flat Minkowski space–time. This example is particularly relevant for a subsequent extension to the case of local SUSY transformations and for the formulation of the simplest model of supergravity.

We should recall, first of all, that a spin-3/2 particle (called "gravitino", in this context) is represented by the vector-spinor Rarita–Schwinger field, ψ_μ^A. This field simultaneously provides a vector representation of the Lorentz group with index μ and a spinor representation with index A. It contains in general $4 \times 4 = 16$ complex components, which become real if the spinor is of Majorana type.

The action for a Rarita–Schwinger field in Minkowski space can be written as

$$S = \int d^4x\, \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\alpha\psi_\beta, \tag{14.28}$$

where the sum over spinor indices is implicit. This action is invariant under the "gauge" transformation

$$\psi_\mu \to \psi_\mu + \partial_\mu\lambda, \tag{14.29}$$

where λ is a spinor field. The variation with respect to $\overline{\psi}_\mu$ gives the equation of motion

$$R^\mu \equiv i\epsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\nu\partial_\alpha\psi_\beta = 0. \tag{14.30}$$

By using the properties of the Dirac matrices, and the invariance under the transformation (14.29), the above equation can be reduced to a set of conditions which have a simpler interpretation, and are of more convenient use for the subsequent supersymmetric applications.

In order to obtain the equivalent form of the equations of motion we first contract R^μ with γ_μ, and use the results of Eqs. (13.33), (13.34). We are led to

$$\begin{aligned}\frac{1}{2}\gamma_\mu R^\mu &= -\frac{1}{2}\gamma_\mu\gamma^{[\mu}\gamma^\alpha\gamma^{\beta]}\partial_\alpha\psi_\beta \\ &= -\frac{1}{2}\gamma_\mu\big(\gamma^\mu\gamma^\alpha\gamma^\beta - 2\eta^{\mu\alpha}\gamma^\beta\big)\partial_{[\alpha}\psi_{\beta]} \\ &= -\gamma^{[\alpha}\gamma^{\beta]}\partial_\alpha\psi_\beta \\ &= -\frac{1}{2}\gamma^\alpha\gamma^\beta\partial_\alpha\psi_\beta + \frac{1}{2}\big(2\eta^{\beta\alpha} - \gamma^\alpha\gamma^\beta\big)\partial_\alpha\psi_\beta \\ &= -\gamma^\alpha\partial_\alpha\big(\gamma^\beta\psi_\beta\big) + \partial^\alpha\psi_\alpha. \end{aligned} \tag{14.31}$$

Let us then consider the expression

$$A^\nu = \frac{1}{2}\gamma^\nu\big(\gamma_\mu R^\mu\big) - R^\nu, \tag{14.32}$$

and note that (using again the properties of the Dirac matrices):

$$\begin{aligned} R^\nu &= -\gamma^\nu\gamma^{[\alpha}\gamma^{\beta]}\partial_\alpha\psi_\beta + 2\eta^{\nu\alpha}\gamma^\beta\partial_{[\alpha}\psi_{\beta]} \\ &= -\gamma^\nu\gamma^a\partial_\alpha\big(\gamma^\beta\psi_\beta\big) + \gamma^\nu\partial^\alpha\psi_\alpha + \partial^\nu\big(\gamma^\beta\psi_\beta\big) - \gamma^\beta\partial_\beta\psi^\nu. \end{aligned} \tag{14.33}$$

By inserting this form of R^ν into the last term of Eq. (14.32) we find that the first two terms exactly cancel with the result of Eq. (14.31), and we are left with

$$A^\nu = \gamma^\beta\big(\partial_\beta\psi^\nu - \partial^\nu\psi_\beta\big). \tag{14.34}$$

The Rarita–Schwinger equation, $R^\mu = 0$, implies the vanishing of the two expressions (14.31) and (14.34), and thus provides the differential conditions

$$\begin{aligned}\partial^\alpha\psi_\alpha - \gamma^a\partial_\alpha\big(\gamma^\beta\psi_\beta\big) &= 0, \\ \gamma^\mu(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu) &= 0. \end{aligned} \tag{14.35}$$

By exploiting the invariance under the transformation (14.29) we can now impose the gauge condition

$$\gamma^\mu\psi_\mu = 0. \tag{14.36}$$

Using this condition into Eqs. (14.35) we find that the gravitino equation is equivalent, in this gauge, to a Dirac equation for each of its vector components ψ_ν,

$$i\gamma^\mu\partial_\mu\psi_\nu = 0, \tag{14.37}$$

plus a transversality condition in the (Lorentz) vector index,

$$\partial^{\mu}\psi_{\mu} = 0. \tag{14.38}$$

The counting of the residual degrees of freedom now tell us that the (bosonic) vector components of the gravitino are reduced to two, as appropriate to a massless, transverse, gauge vector field (like the photon). In addition, assuming we are dealing with a Majorana spinor, the four real fermionic components (for each value of the vector index) have been halved because of the Dirac equation (14.37), and additionally reduced by half because of the gauge condition (14.36).

The set of (gauge-fixed) equations (14.36)–(14.38) thus describes a Majorana fermion field with a total number of $2 \times 1 = 2$ dynamical degrees of freedom, suitable to form a consistent (on-shell) supersymmetric system in combination with some boson field also possessing two dynamical degrees of freedom. A possible appropriate partner of this type is the graviton, as we shall see in the next section.

14.2.1 Global Supersymmetry in the Graviton–Gravitino System

In Chap. 9 have seen that the fluctuations of the Minkowski geometry can be described, in the linear approximation and in the TT gauge, by a symmetric tensor field $h_{\mu\nu}$ which satisfies the transversality and traceless conditions,

$$\partial^{\nu}h_{\mu\nu} = 0, \qquad \eta^{\mu\nu}h_{\mu\nu} = 0, \tag{14.39}$$

and has a free action given by Eq. (9.48):

$$S = \frac{1}{4}\int d^4x\, \partial_{\alpha}h^{\mu\nu}\partial^{\alpha}h_{\mu\nu} \tag{14.40}$$

(throughout this section we will use units in which $2\chi = 16\pi G/c^4 = 1$).

As already stressed in Sect. 9.1.1, this field has only two independent components, corresponding to the two polarization states of a massless, spin-2 particle (the graviton). The graviton–gravitino system, represented by the fields $h_{\mu\nu}$ and $\psi_{\mu} = \psi^{c}_{\mu}$ decoupled with each other and embedded in the Minkowski vacuum geometry, has thus the same number of bosonic and fermionic (on-shell) degrees of freedom, and may provide, in principle, another possible example of globally supersymmetric system.

That this is indeed the case can be easily seen by considering the following infinitesimal transformations between bosonic and fermionic components:

$$\begin{aligned} \delta h_{\mu\nu} &= \overline{\epsilon}(\gamma_{\mu}\psi_{\nu} + \gamma_{\nu}\psi_{\mu}), \\ \delta\psi_{\mu} &= \gamma^{[\alpha}\gamma^{\beta]}\epsilon\, \partial_{\alpha}h_{\mu\beta}, \end{aligned} \tag{14.41}$$

where $\epsilon = \epsilon^c$ is a constant (Majorana) spinor parameter. The Lagrangian density for the graviton–gravitino system can be obtained from the actions (14.28), (14.40),

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{3/2} = \frac{1}{4}\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\alpha\psi_\beta, \tag{14.42}$$

and its infinitesimal variation can be expressed in the form $\delta\mathcal{L} = \partial_\mu K^\mu$ without using the equations of motion (an explicit computation, proceeding along the lines of the example of Sect. 14.1, is reported in Exercise 14.3). The equations of motion for $h_{\mu\nu}$ and ψ_μ are thus invariant, and the system is globally supersymmetric.

14.3 $N = 1$ Supergravity in $D = 4$ Dimensions

Lifting a global supersymmetry to its local version is possible, as already stressed in Sect 14.1, provided the model is formulated in a general-covariant framework, taking into account the gravitational interactions. This suggests that the model described by the Lagrangian (14.42), which already includes gravity at the linearized level, could represent a natural starting point for the formulation of a locally supersymmetric model and for the study of its geometrical properties.

Let us then consider a tensor-spinor system of the Einstein–Rarita–Schwinger type, obtained by coupling the field ψ_μ to the geometry of a curved space–time manifold, and by using for the tensor field the exact, non-linear (Einstein) description based on the curvature of the space–time geometry. We start then with the action

$$S = \int d^4x \left(-\frac{1}{2\chi}\sqrt{-g}R + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu\gamma_5\gamma_\nu\nabla_\alpha\psi_\beta\right), \tag{14.43}$$

and ask whether it is appropriate to represent a simple (locally supersymmetric) model of supergravity. The answer is not necessarily affirmative, otherwise any model with an equal number of bosonic and fermionic components, written in a general-covariant context, would be automatically supersymmetric (which is not true).

Let us notice, first of all, that the covariant gravitino action has been obtained from Eq. (14.28) by applying the minimal coupling principle, and using the vierbein field to project the Dirac matrices from the Minkowski tangent space down to the curved space–time, as appropriate to a spinor field (see Sect. 13.2). We have used, in particular, the following prescriptions:

$$d^4x \to d^4x\sqrt{-g}, \qquad \gamma_a \to \gamma_\mu = V^a_\mu\gamma_a, \qquad \partial_\mu \to \nabla_\mu. \tag{14.44}$$

The absence of $\sqrt{-g}$ in the gravitino action is due to the replacement—appropriate to a curved space–time—of the antisymmetric density ϵ with the antisymmetric tensor η (see Sect. 3.2),

$$\epsilon^{\mu\nu\alpha\beta} \to \eta^{\mu\nu\alpha\beta} = \frac{\epsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}}, \tag{14.45}$$

which leads to

$$d^4x\,\sqrt{-g}\,\eta^{\mu\nu\alpha\beta} \equiv d^4x\,\epsilon^{\mu\nu\alpha\beta}. \tag{14.46}$$

For a complete specification of the action (14.43) we still need, however, an explicit definition of the covariant derivative $\nabla_\alpha \psi_\beta$, which in principle depends on the chosen model of space–time geometry. The gravitino field ψ_μ^A has a curved index μ which transforms as a vector under general coordinate transformations, and a flat index A transforming as a spinor under local Lorentz transformations in the Minkowski tangent space. Hence, its covariant derivative has to be an operator both general-covariant in the vector index and locally Lorentz invariant in the spinor index. By recalling the results of Chaps. 12 and 13 (in particular, Eq. (13.23)), we can then write the covariant derivative as follows:

$$\begin{aligned}\nabla_\mu \psi_\nu &= \partial_\mu \psi_\nu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{[a}\gamma_{b]}\psi_\nu - \Gamma_{\mu\nu}{}^\alpha \psi_\alpha \\ &\equiv D_\mu \psi_\nu - \Gamma_{\mu\nu}{}^\alpha \psi_\alpha.\end{aligned} \tag{14.47}$$

In the second line we have explicitly separated the Lorentz covariant derivative $D_\mu \psi_\nu$, acting on the spinor index, from the connection Γ acting on the vector index.

It should be stressed, at this point, that we have resumed the conventions of the two previous chapters: spinor indices are implicit, Latin letters $a, b, c, \ldots$ denote Lorentz indices in the flat tangent space, Greek letters denote tensor indices in the curved space–time manifold. Finally, ω is the Lorentz connection (see Sect. 12.3) while Γ is the space–time connection. We will leave undefined for the moment their specific expressions, because there are various possibilities to be considered, at least in principle.

(I) A first possibility, which would seem the most natural one in the context of the Einstein gravitational model discussed so far, is to adopt for the space–time manifold the Riemann geometric model. In this case the torsion is vanishing, $Q_{\mu\nu}{}^\alpha = \Gamma_{[\mu\nu]}{}^\alpha = 0$; the Lorentz connection is completely determined by the vierbeins, $\omega = \omega(V)$, and coincides with the Levi-Civita connection (12.47); finally, Γ coincides with the Christoffel connection Γ_g of Eq. (3.90), and drops out from the gravitino action because, in the absence of torsion, $\nabla_{[\alpha}\psi_{\beta]} = D_{[\alpha}\psi_{\beta]}$.

We arrive in this way to the model described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2\chi}\sqrt{-g}R(g, \Gamma_g) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu \gamma_5 \gamma_\nu D_\alpha(V)\psi_\beta, \tag{14.48}$$

where $D_\alpha(V) \equiv D_\alpha(\omega(V))$. Such a model, however, *is not* locally supersymmetric. To implement local supersymmetry one must add to this action non-minimal interaction terms which are quadratic in the spinor current of the gravitino, $J_{\mu\nu}{}^\alpha = \overline{\psi}_\mu \gamma^\alpha \psi_\nu$. However, since this current is antisymmetric in μ and ν, it could play the role of source for the antisymmetric part of the connection $Q_{\mu\nu}{}^\alpha$ (as we shall see in the next section). This fact, in particular, leads us to consider another possibility.

(II) A second possibility is to adopt for the space–time manifold the so-called Riemann–Cartan geometric structure, characterized by the presence of torsion,

$Q_{\mu\nu}{}^{\alpha} \neq 0$. In that case both connections include the contortion contributions,

$$\begin{aligned} \omega &= \omega(V, Q) = \omega(V) + K(Q), \\ \Gamma &= \Gamma(g, Q) = \Gamma_g - K(Q), \end{aligned} \tag{14.49}$$

as prescribed, respectively, by Eqs. (12.45) and (3.86), and we are led to the model described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2\chi}\sqrt{-g}R(g, \Gamma_g, Q) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_{\mu}\gamma_5\gamma_{\nu}\big[D_{\alpha}(V, Q)\psi_{\beta} - Q_{\alpha\beta}{}^{\rho}\psi_{\rho}\big], \tag{14.50}$$

where $D_{\alpha}(V, Q) \equiv D_{\alpha}(\omega(V, Q))$.

In this context the metric (or the vierbeins) and the connection become independent variables, and we have "one more" field equation (with respect to general relativity): the equation for the connection which, once solved, gives the torsion in terms of the spinor current of the gravitino:

$$Q_{\mu\nu}{}^{\alpha} \sim J_{\mu\nu}{}^{\alpha} = \overline{\psi}_{\mu}\gamma^{\alpha}\psi_{\nu} \tag{14.51}$$

(see Sect. 14.3.1).

Inserting this result into the action, and eliminating everywhere the torsion with $J_{\mu\nu}{}^{\alpha}$, we obtain quadratic interaction terms $\sim J^2$ just of the type needed to implement supersymmetry. Even in that case, however, the model we obtain *is not* locally supersymmetric (other corrections of order J^2 are needed).

(III) The correct supergravity model [9, 15], which is general-covariant and locally supersymmetric, and which includes all the quadratic terms of type $(\overline{\psi}\gamma\psi)^2$ required by supersymmetry, can be consistently formulated using the Einstein–Cartan geometric structure as in case II) above. However, we must drop the last term contributing to the gravitino Lagrangian (14.50).

Stated in more explicit form, a consistent supersymmetric model for the graviton–gravitino system must be constructed according to the following prescriptions.

- Use the Einstein–Cartan model for the space–time geometry, with a non-symmetric connection of type (14.49), and with the torsion Q determined by the gravitino as prescribed by the Palatini variational procedure.
- Include the torsion in the gravitational action, by using a scalar curvature computed with the connection (14.49).
- Couple the gravitino *only* to the Lorentz connection through the covariant prescription

$$\partial_{[\mu}\psi_{\nu]} \to D_{[\mu}\psi_{\nu]}. \tag{14.52}$$

This, in particular, removes the last torsion term of the Lagrangian (14.50).

A few comments are in order on this last, important point. Given that $D_{[\mu}\psi_{\nu]}$ is different from $\nabla_{[\mu}\psi_{\nu]}$ when the torsion is nonvanishing (see Eq. (14.47)), the prescription (14.52) would seem to correspond to a sort of "non-minimal" coupling.

However, we must note that the gravitino, in spite of being a fermion, is also (and in all respects) a gauge field in the vector index μ: actually, it is the "compensating field" needed to restore the invariance of the Einstein action under local supersymmetry transformations.

This clarification is crucial because, as already stressed in Sect. 4.2, a gauge vector field A_μ is represented, geometrically, by an object called a differential "1-form" (see Appendix A), and can be written as $A \equiv A_\mu\, dx^\mu$. As such, it has no explicit index in the space–time manifold, and its exterior covariant derivative always coincides with the gauge derivative, $\nabla A \equiv DA$.

It is true that the exterior derivative only contains the antisymmetric part of the derivative, as $\nabla A \equiv \nabla_{[\mu}\psi_{\nu]}\, dx^\mu \wedge dx^\nu$ (see Appendix A), but it is also true that the gauge-field action only contain just the antisymmetric part of ∇A. By referring to the property $\nabla A = DA$ we can say that, in a sense, the vector index of a gauge field couples to the covariant derivative just as if it were gravitationally neutral.[1]

This fundamental property applies to all gauge fields, as already stressed when discussing the gravitational coupling of the electromagnetic field in Sect. 4.2 (even if, in the context of a torsionless geometry, such a property becomes trivial). Applied to the gravitino field it explains the coupling prescription (14.52) and allows its interpretation as a "minimal" prescription, in spite of its apparent non-minimal aspect.

If there is no need to introduce the space–time connection Γ in the gravitino action, it becomes convenient to formulate the model in terms of the vierbeins V_μ^a and of the Lorentz connection $\omega_\mu{}^{ab}$ (besides the field ψ_μ, of course). By applying the previous prescriptions and the vierbein language of Chap. 12 we thus arrive at the supergravity Lagrangian

$$\mathcal{L} = -\frac{1}{2\chi} V R(V,\omega) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu \gamma_5 \gamma_\nu D_\alpha(\omega)\psi_\beta, \tag{14.53}$$

where $R(V,\omega)$ is the scalar curvature (12.55), where

$$D_\alpha(\omega) = \partial_\alpha + \frac{1}{4}\omega_\alpha{}^{ab}\gamma_{[a}\gamma_{b]}, \tag{14.54}$$

and where the connection $\omega = \omega(V,\psi)$ is to be computed in terms of the vierbeins and of the gravitino by imposing on the action to be stationary with respect to the variation of ω.

The above Lagrangian describes the so-called $N = 1$ supergravity (or "simple supergravity") model in $D = 4$ space–time dimensions. The nickname $N = 1$ denotes the presence of only one gravitino, required to restore the local supersymmetry of the Einstein action. If we want to include in the SUGRA model additional field

[1] A gauge field is not immune, obviously, to the gravitational interaction: it feels gravity through all the other forms of minimal coupling present in the action (see the discussion of the electromagnetic potential in Chap. 4).

components without breaking SUSY we must generalize the above structure by including additional gravitinos, which play the role of gauge fields for the new local supersymmetries. One thus obtains the so-called "extended supergravity" models with $N = 2, 3, \ldots, 8$ gravitinos (the case $N > 8$ would require the presence of fields with spin $5/2$ and higher, which do not seem to allow a consistent coupling to gravity in a standard field-theory context). The simplest extended model, the case $N = 2$, includes a new gauge vector field, and describes a consistent supersymmetric coupling between the field-doublet of spin $\{2, 3/2\}$ discussed here and the additional field-doublet $\{3/2, 1\}$.

Coming back to the "simple" case $N = 1$, it is time to note that the Lagrangian (14.53) is left invariant (modulo a total derivative) by the following local supersymmetry transformation (written in units $\chi = 8\pi G/c^4 = 1$):

$$\begin{aligned} \delta V_\mu^a &= \overline{\epsilon}(x)\gamma^a \psi_\mu, \\ \delta\psi_\mu &= -2D_\mu \epsilon(x) \equiv -2\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{[a}\gamma_{b]}\right)\epsilon(x), \end{aligned} \tag{14.55}$$

where $\epsilon = \epsilon^c$ is a position-dependent Majorana spinor parameter. The infinitesimal transformation of the connection follows from the previous ones after expressing ω in terms of V and ψ (see the next section):

$$\delta\omega = \frac{\delta\omega(V,\psi)}{\delta V}\delta V + \frac{\delta\omega(V,\psi)}{\delta\psi}\delta\psi. \tag{14.56}$$

However, there is no need to consider explicitly such a transformation because, in the variation of the Lagrangian $\delta\mathcal{L}$, the variation $\delta\omega$ is multiplied by $\delta\mathcal{L}/\delta\omega$, which is identically vanishing if we take into account the relation $\omega = \omega(V, \psi)$. Hence, in order to check the local supersymmetry of the model we only need the transformation rules of V and ψ (the explicit computation is presented in Exercise 14.4).

The explicit computation shows, in particular, that the variation $\delta\mathcal{L} = \partial_\mu K^\mu$ is vanishing if we use the gravitino equation of motion. Such equations of motion (to be derived in the next section) are also needed to close the algebra of the SUSY generators, since in this model the number of boson and fermion degrees if freedom is the same only on-shell.

The fundamental field components, in fact, are the vierbeins V_μ^a and the gravitino ψ_μ^A, which is a Majorana fermion. All indices ranges from 1 to 4, and thus each of the two fields is in general characterized by $4 \times 4 = 16$ real parameters. The symmetries of the model are diffeomorphism invariance, local Lorentz invariance and local SUSY. We can thus impose on the vierbeins six conditions by exploiting the Lorentz transformations, and other four conditions through a general coordinate transformation. We are left with six bosonic components (which are indeed the total degrees of freedom of a generic gravitational field in four dimensions, as already stressed in Sect. 7.2).

By exploiting a local SUSY transformation we can also impose four conditions on the gravitino, and leave 12 independent fermionic components. Their number is

halved (hence we obtain six components like in the bosonic case) only if we impose the equations of motion. Of course we can make consistent the generator algebra also off-shell, but at the price of adding six more bosonic degrees of freedom. The conventional choice, for this purpose, is that of adding three auxiliary fields: a scalar S, a pseudo-scalar P and an axial-vector A_μ (but there are also more complicated options, introducing $6 + n$ bosonic components and n fermionic components).

14.3.1 Field Equations for the Metric and the Gravitino

In order to obtain the field equations for the simple SUGRA model of the previous section we will adopt the Palatini formalism (see Sect. 12.3.1), and vary the Lagrangian (14.53) by treating V, ω and ψ as independent variables. Let us start with the variation with respect to ω, which explicitly determines the torsion produced by the gravitino and enable us to express the connection in terms of V and ψ.

Variation with Respect to the Connection

The variation of the gravitational part of the action is the same already performed in Sect. 12.3.1, and the result is that given by Eq. (12.60). The variation of the new (gravitino) part gives

$$\delta_\omega \mathcal{L}_{3/2} = \frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu \gamma_5 \gamma_\nu \gamma_{[a}\gamma_{b]}\psi_\beta \delta\omega_\alpha{}^{ab}. \tag{14.57}$$

By using the properties (13.34), (13.36) of the Dirac matrices we obtain

$$\begin{aligned}\gamma_5\gamma_\nu\gamma_{[a}\gamma_{b]} &= \gamma_5 V^c_\nu \gamma_c \gamma_{[a}\gamma_{b]} = \gamma_5 V^c_\nu(\gamma_{[a}\gamma_b\gamma_{c]} + 2\eta_{c[a}\gamma_{b]}) \\ &= -i\epsilon_{abcd}V^c_\nu \gamma^d + \gamma_5 V_{\nu a}\gamma_b - \gamma_5 V_{\nu b}\gamma_a.\end{aligned} \tag{14.58}$$

Inserting this result into $\delta_\omega \mathcal{L}_{3/2}$ we are led to

$$\begin{aligned}\delta_\omega \mathcal{L}_{3/2} = \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V^c_\nu \overline{\psi}_\mu \gamma^d \psi_\beta\, \delta\omega_\alpha{}^{ab} \\ + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}V_{\nu a}\overline{\psi}_\mu \gamma_5\gamma_b\psi_\beta\, \delta\omega_\alpha{}^{ab}.\end{aligned} \tag{14.59}$$

The second contribution to the above variation is identically vanishing because the current $\overline{\psi}_\mu\gamma_5\gamma_b\psi_\beta = \overline{\psi}_\beta\gamma_5\gamma_b\psi_\mu$ is symmetric in μ and β (see Eq. (14.90) of Exercise 14.2). By adding the first contribution to the variation of the gravitational action, Eq. (12.60), we obtain

$$D_{[\mu}V^a_{\nu]} = -\frac{\chi}{4}\overline{\psi}_\mu \gamma^a \psi_\nu, \tag{14.60}$$

which represent the field equation for the connection.

By using the metricity condition of the vierbein, Eqs. (12.40), (12.41), we can now observe that the left-hand side of the above equation exactly defines the torsion tensor $Q_{\mu\nu}{}^{\alpha}$. It follows that such a tensor is fully determined by the (Dirac) vector currents associated to the gravitino,

$$Q_{\mu\nu}{}^{a} = -\frac{\chi}{4}\overline{\psi}_{\mu}\gamma^{a}\psi_{\nu}. \tag{14.61}$$

Also, by exploiting the general result (12.45), we can immediately express the Lorentz connection as follows:

$$\begin{aligned}\omega_{\mu ab} &= V^{c}_{\mu}\omega_{cab} \\ &= V^{c}_{\mu}(C_{cab} - C_{abc} + C_{bca}) \\ &\quad + \frac{\chi}{4}V^{c}_{\mu}(\overline{\psi}_{c}\gamma_{b}\psi_{a} - \overline{\psi}_{a}\gamma_{c}\psi_{b} + \overline{\psi}_{b}\gamma_{a}\psi_{c}).\end{aligned} \tag{14.62}$$

We recall that C_{abc} denotes the Ricci rotation coefficients, defined by Eq. (12.43).

Variation with Respect to the Vierbeins

Let us now vary with respect to V. The contribution of the gravitational action has been already computed in Sect. 12.3.1, and the result is given by Eq. (12.62). In the gravitino part of the action, on the other hand, the vierbeins are explicitly contained only in the projected Dirac matrices $\gamma_{\nu} = V^{c}_{\nu}\gamma_{c}$ (as ψ and ω are independent variables). Hence:

$$\delta_{V}\mathcal{L}_{3/2} = -\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_{\mu}\gamma_{5}\gamma_{c}D_{\nu}\psi_{\beta}\delta V^{c}_{\alpha} \tag{14.63}$$

(we have chosen the appropriate names of the indices in order to directly compare this variation with the gravitational one of Eq. (12.62)). By adding the two contributions we obtain

$$G^{\alpha}{}_{c} = \chi\,\theta^{\alpha}{}_{c}, \tag{14.64}$$

where G is the Einstein tensor of Eq. (12.64), and

$$\theta^{\alpha}{}_{c} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_{\mu}\gamma_{5}\gamma_{c}D_{\nu}\psi_{\beta} \tag{14.65}$$

is the canonical energy-momentum tensor of the gravitino.

It should be noticed that such a tensor is not symmetric, but so is also the Einstein tensor present on the left-hand side of Eq. (14.64), as it is computed from a connection which includes torsion. However, it is always possible to rewrite Eq. (14.64) in a symmetric "Einsteinian" form, by making explicit the torsion contributions to the action (14.53) through the result (14.61), and separating them from the Riemannian part of the curvature and of the covariant derivative of the gravitino. By varying the

action (in this new form) with respect to the metric (or the vierbeins) we are then led to field equations containing the usual symmetric Einstein tensor and a symmetrized version of the dynamical energy-momentum tensor of the gravitino.

Variation with Respect to the Gravitino

Finally, by varying the action with respect to $\overline{\psi}_\mu$, we obtain the gravitino equation of motion,

$$R^\mu \equiv \epsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\nu D_\alpha \psi_\beta = 0. \tag{14.66}$$

This equation must satisfy the consistency condition $D_\mu R^\mu = 0$ (otherwise additional constraints would appear, and the coupling to the graviton would be inconsistent). It is instructive to show that the consistency condition is satisfied provided the equations of motion of the vierbeins and of the connection are also satisfied.

We should note, first of all, that by applying to R^μ the Lorentz covariant derivative we find two contributions, sourced, respectively, by ψ_β and by the vierbein field used to project the Dirac matrices $\gamma_\nu = V_\nu^a \gamma_a$:

$$D_\mu R^\mu = \epsilon^{\mu\nu\alpha\beta}\gamma_5\big[\gamma_\nu D_{[\mu} D_{\alpha]}\psi_\beta + \gamma_a D_\alpha \psi_\beta D_{[\mu} V^a_{\nu]}\big] \tag{14.67}$$

(the antisymmetrization is due to the contraction with $\epsilon^{\mu\nu\alpha\beta}$). The last contribution to DR (that we shall call A) is proportional to the torsion, and using the field equation (14.60) we immediately obtain (in units $\chi = 1$):

$$A = -\frac{1}{4}\big(\overline{\psi}_\mu \gamma^a \psi_\nu\big)\gamma_5\gamma_a D_\alpha \psi_\beta \epsilon^{\mu\nu\alpha\beta}. \tag{14.68}$$

Let us now compute the other contribution to DR, and show that the two contributions exactly cancel out.

We need the commutator of two Lorentz covariant derivatives applied to a spinor, and we recall, for this purpose, the general definition of D_μ in terms of the generators J_{ab} (see Eq. (12.22)). For a generic field ψ, using the generator algebra (12.20), we find

$$\begin{aligned}
[D_\mu, D_\nu]\psi &= -\frac{i}{2}\big(\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab}\big) J_{ab}\psi \\
&\quad - \frac{1}{4}\omega_\mu{}^{ab}\omega_\nu{}^{cd}[J_{ab}, J_{cd}]\psi \\
&= -\frac{i}{2} R_{\mu\nu}{}^{ab}(\omega) J_{ab}\psi,
\end{aligned} \tag{14.69}$$

where $R_{\mu\nu}{}^{ab}$ is the Lorentz curvature (12.54). In the case of a vector-valued field, using the generators (12.29), we then recover the result (12.51). For a spinor field we must use the generators (13.11) and we are led to

$$D_{[\mu} D_{\nu]}\psi = \frac{1}{2}[D_\mu, D_\nu]\psi = \frac{1}{8} R_{\mu\nu}{}^{ab}\gamma_{[a}\gamma_{b]}\psi. \tag{14.70}$$

The first contribution to the covariant divergence (14.67) (that we will call B) can then be written in form

$$B = \frac{1}{8}\gamma_5\gamma_\nu\gamma_{[a}\gamma_{b]}\psi_\beta R_{\mu\alpha}{}^{ab}\epsilon^{\mu\nu\alpha\beta}. \tag{14.71}$$

The combination of Dirac matrices appearing here has been already computed in Eq. (14.58). Using that result we have

$$\begin{aligned} B = &-\frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R_{\mu\alpha}{}^{ab}V_\nu^c\gamma^d\psi_\beta \\ &+\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}R_{\mu\alpha\nu}{}^b\gamma_5\gamma_b\psi_\beta. \end{aligned} \tag{14.72}$$

The first term (that we will call B_1) is proportional to the Einstein tensor. By exploiting Eq. (12.75) of Exercise 12.4 we have, in fact,

$$B_1 = -\frac{i}{8}R_{\mu\alpha}{}^{ab}V_{abd}^{\mu\alpha\beta}\gamma^d\psi_\beta = \frac{i}{2}G^\beta{}_d\gamma^d\psi_\beta. \tag{14.73}$$

Hence, according to the field equation (14.64):

$$B_1 = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}(\overline{\psi}_\mu\gamma_5\gamma_a D_\nu\psi_\beta)\gamma^a\psi_\alpha. \tag{14.74}$$

The second term of Eq. (14.72) (that we will call B_2) is proportional to $R_{[\mu\alpha\nu]}{}^b$, which is determined by the Bianchi identity for the curvature tensor. Ina Riemann geometric context this term is vanishing (see Sect. 6.2), and the gravitino equation would be inconsistent. In a Riemann–Cartan context, on the contrary, the Bianchi identity is modified. In fact, let us consider the commutator of two covariant derivatives, and apply Eq. (12.51):

$$2D_{[\mu}D_{\alpha]}V_\nu^b = R_{\mu\alpha}{}^b{}_cV_\nu^c = -R_{\mu\alpha\nu}{}^b. \tag{14.75}$$

Considering the totally antisymmetric part in μ, α, ν, and using again the torsion equation (14.60), we obtain

$$R_{[\mu\alpha\nu]}{}^b = -2D_{[\mu}D_\alpha V_{\nu]}^b = \frac{1}{2}D_{[\mu}\big(\overline{\psi}_\alpha\gamma^b\psi_{\nu]}\big) = \overline{\psi}_{[\alpha}\gamma^b D_\mu\psi_{\nu]}, \tag{14.76}$$

so that

$$B_2 = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\big(\overline{\psi}_\alpha\gamma^b D_\mu\psi_\nu\big)\gamma_5\gamma_b\psi_\beta. \tag{14.77}$$

To check that the sum of the three contributions $A + B_1 + B_2$ is vanishing we can now use the so-called Fierz identity. Given three 4-component spinors ξ, ψ, χ, this identity reads

$$(\overline{\xi}\psi)\chi_A = -\frac{1}{4}\sum_i\big(\overline{\xi}\Gamma^i\chi\big)(\Gamma_i\psi)_A, \tag{14.78}$$

where the symbol Γ^i denotes the 16 matrix operators representing a basis in the space of the 4×4 matrices, i.e.:

$$\Gamma^i = \left(1, \gamma^a, \sigma^{ab}, \gamma^a\gamma^5, \gamma^5\right), \quad a < b. \tag{14.79}$$

By applying the Fierz identity we can then rewrite B_1 as follows:

$$B_1 = \frac{1}{16}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu \Gamma^i \psi_\alpha\right)\gamma^a \Gamma_i \gamma_5 \gamma_a D_\nu \psi_\beta. \tag{14.80}$$

The nonvanishing contributions to this equation come from those terms with a current $\overline{\psi}_\mu \Gamma^i \psi_\alpha$ which is antisymmetric in μ and α: hence (for the anticommutation properties of the Majorana spinors) the only possible contributions may come from γ^μ and $\sigma^{\mu\nu}$. However, $\gamma^a \sigma_{\mu\nu}\gamma_a \equiv 0$, and we are left with

$$\begin{aligned} B_1 &= \frac{1}{16}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu \gamma^b \psi_\alpha\right)\gamma_5 \gamma^a \gamma_b \gamma_a D_\nu \psi_\beta \\ &= \frac{1}{16}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu \gamma^b \psi_\alpha\right)\gamma_5 \gamma^a (-\gamma_a\gamma_b + 2\eta_{ab}) D_\nu \psi_\beta \\ &= -\frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu \gamma^b \psi_\alpha\right)\gamma_5 \gamma_b D_\nu \psi_\beta \\ &= \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu \gamma^b \psi_\nu\right)\gamma_5 \gamma_b D_\alpha \psi_\beta \end{aligned} \tag{14.81}$$

(in the last step we have used the antisymmetry in the indices α and ν). By repeating the same procedure for B_2 we obtain

$$B_2 = \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\alpha \gamma^a \psi_\beta\right)\gamma_5 \gamma_a D_\mu \psi_\nu. \tag{14.82}$$

In conclusion we find $B_2 = B_1 = -A/2$, so that the three contributions (14.68), (14.81), (14.82) exactly sum up to zero, ensuring the validity of the consistency condition $D_\mu R^\mu = 0$ for the model of supergravity we have considered.

14.4 Exercises Chap. 14

14.1 Commutation Properties of the Majorana Spinors Show that, given two Majorana spinors $\epsilon = \epsilon^c$ and $\psi = \psi^c$, they satisfy the property

$$\overline{\epsilon}\psi = \overline{\psi}\epsilon. \tag{14.83}$$

14.2 On-Shell SUSY Transformations Derive the result (14.25) for the fields B and ψ of the Wess–Zumino model, using the properties of the Majorana spinors and imposing that the equations of motion are satisfied.

14.3 Global SUSY for the Graviton–Gravitino System Show that the effective Lagrangian (14.42) is left invariant (modulo a total divergence) by the global infinitesimal transformations (14.41).

14.4 Local SUSY for $N = 1$ Supergravity Compute the infinitesimal variation $\delta\mathcal{L}$ of the effective Lagrangian (14.53) under the local transformation (14.55), and show that the result is a total divergence, $\delta\mathcal{L} = \partial_\mu K^\mu$, which is vanishing if the equations of motion of the gravitino field are satisfied.

14.5 Solutions

14.1 Solution From the Majorana condition (14.2) we have

$$C^{-1}\epsilon = \overline{\epsilon}^T, \tag{14.84}$$

so that, using the properties (14.3) of the charge conjugation operator,

$$\overline{\epsilon} = \left(C^{-1}\epsilon\right)^T = \epsilon^T\left(C^{-1}\right)^T = -\epsilon^T C^{-1}. \tag{14.85}$$

Hence:

$$\overline{\epsilon}\psi = -\epsilon^T C^{-1} C\overline{\psi}^T = -\epsilon^T\overline{\psi}^T = (\overline{\psi}\epsilon)^T = \overline{\psi}\epsilon. \tag{14.86}$$

The last two steps are due to the fact that the spinors ϵ^A and ψ^A anticommute, and then

$$-\epsilon^T\overline{\psi}^T = -\epsilon^A\psi^*_A = \psi^{*A}\epsilon_A = \overline{\psi}\epsilon. \tag{14.87}$$

14.2 Solution By applying to B the infinitesimal transformations (14.24), and computing the commutator, we obtain

$$\begin{aligned}\delta_2\delta_1 B &= \delta_2\left(i\overline{\epsilon}_1\gamma^5\psi\right) = \frac{1}{2}\overline{\epsilon}_1\gamma^5\gamma^\mu\partial_\mu\left(A + i\gamma^5 B\right)\epsilon_2,\\ [\delta_2, \delta_1]B &= \frac{1}{2}\left(\overline{\epsilon}_1\gamma^5\gamma^\mu\epsilon_2\right)\partial_\mu A + \frac{i}{2}\left(\overline{\epsilon}_1\gamma^5\gamma^\mu\gamma^5\epsilon_2\right)\partial_\mu B - \{1 \leftrightarrow 2\}.\end{aligned} \tag{14.88}$$

The first term, proportional to $\partial_\mu A$, is symmetric in the exchange of the indices 1 and 2 and then it does not contributes. In fact, by recalling Eqs. (14.3) and (14.85), and using the property

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 = \left[\gamma^5, C\right], \tag{14.89}$$

we have

$$\begin{aligned}\overline{\epsilon}_1\gamma^5\gamma^\mu\epsilon_2 &= -\epsilon_1^T C^{-1}\gamma^5\gamma^\mu C\,\overline{\epsilon}_2^T = \epsilon_1^T\gamma^5\gamma^{\mu T}\overline{\epsilon}_2^T\\ &= -\left(\overline{\epsilon}_2\gamma^\mu\gamma^5\epsilon_1\right)^T = \overline{\epsilon}_2\gamma^5\gamma^\mu\epsilon_1.\end{aligned} \tag{14.90}$$

For the second term, proportional to $\partial_\mu B$, we note that

$$\gamma^5 \gamma^\mu \gamma^5 = -\gamma^\mu, \tag{14.91}$$

so that, by using the result (14.18), we finally obtain

$$[\delta_2, \delta_1]B = -i\big(\overline{\epsilon}_1 \gamma^\mu \epsilon_2\big)\partial_\mu B, \tag{14.92}$$

in agreement with Eq. (14.25).

Let us now consider the commutator of two infinitesimal transformations applied to ψ, starting from Eq. (14.24) and writing explicitly the spinor indices:

$$\delta_1 \psi_A = -\frac{i}{2}\partial_\mu A\big(\gamma^\mu \epsilon_1\big)_A + \frac{1}{2}\partial_\mu B\big(\gamma^\mu \gamma^5 \epsilon_1\big)_A. \tag{14.93}$$

Therefore:

$$[\delta_2, \delta_1]\psi_A = -\frac{i}{2}(\overline{\epsilon}_2 \partial_\mu \psi)\gamma^\mu \epsilon_1 + \frac{i}{2}\big(\overline{\epsilon}_2 \gamma^5 \partial_\mu \psi\big)\gamma^\mu \gamma^5 \epsilon_1 - \{1 \leftrightarrow 2\}. \tag{14.94}$$

It is convenient, at this point, to use the Fierz identity (14.78) to re-arrange the right-hand side of Eq. (14.94), so as to move to the right $\partial_\mu \psi$. We obtain, in this way,

$$\begin{aligned}[\delta_2, \delta_1]\psi_A = {} & \frac{i}{8}\sum_i \big(\overline{\epsilon}_2 \Gamma^i \epsilon_1\big)\gamma^\mu \Gamma_i \partial_\mu \psi \\ & - \frac{i}{8}\sum_i \big(\overline{\epsilon}_2 \Gamma^i \epsilon_1\big)\gamma^\mu \gamma^5 \Gamma_i \gamma^5 \partial_\mu \psi - \{1 \leftrightarrow 2\}\end{aligned} \tag{14.95}$$

and we note that the nonzero contributions to this expression come from the operators Γ^i for which $\overline{\epsilon}_2 \Gamma^i \epsilon_1$ is antisymmetric in the indices 1 and 2. For the properties of the Majorana spinor this is only possible for γ^μ and $\sigma^{\mu\nu}$ (defined by Eq. (13.10)).

For $\sigma^{\mu\nu}$, however, we have $\gamma^5 \sigma^{\mu\nu} \gamma^5 = \sigma^{\mu\nu}$, and the four terms of Eq. (14.95) identically cancel among each other. We are thus left with the contribution of γ^μ, which gives

$$\begin{aligned}[\delta_2, \delta_1]\psi &= \frac{i}{2}\big(\overline{\epsilon}_2 \gamma^\nu \epsilon_1\big)\gamma_\mu \gamma_\nu \partial^\mu \psi \\ &= \frac{i}{2}\big(\overline{\epsilon}_2 \gamma^\nu \epsilon_1\big)(-\gamma_\nu \gamma_\mu + 2\eta_{\mu\nu})\partial^\mu \psi.\end{aligned} \tag{14.96}$$

The first term on the right-hand side is zero for the equations of motion, which impose $\gamma_\mu \partial^\mu \psi = 0$. The final result, using Eq. (14.18), can then be rewritten as follows:

$$[\delta_2, \delta_1]\psi = -i\big(\overline{\epsilon}_1 \gamma^\mu \epsilon_2\big)\partial_\mu \psi, \tag{14.97}$$

in agreement with Eq. (14.25).

14.3 Solution Let us compute, first of all, the infinitesimal variation of the conjugate field $\overline{\psi}_\mu$. From the definition (14.41) of $\delta\psi_\mu$, using the result of Eq. (13.46), we obtain

$$\delta\overline{\psi}_\mu = \left(\gamma^{[\alpha}\gamma^{\beta]}\epsilon\right)^\dagger \gamma^0 \partial_\alpha h_{\mu\beta} = -\overline{\epsilon}\gamma^{[\alpha}\gamma^{\beta]}\partial_\alpha h_{\mu\beta}. \tag{14.98}$$

By varying the Lagrangian (14.42) we are then led to

$$\begin{aligned}\delta\mathcal{L} = {} & \partial^\alpha h^{\mu\nu}(\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\nu) \\ & + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\big[-\big(\overline{\epsilon}\gamma^{[\rho}\gamma^{\sigma]}\gamma^5\gamma^\nu\partial^\alpha\psi^\beta\big)\partial_\rho h^\mu{}_\sigma \\ & + \big(\overline{\psi}^\mu\gamma^5\gamma^\nu\gamma^{[\rho}\gamma^{\sigma]}\epsilon\big)\partial^\alpha\partial_\rho h^\beta{}_\sigma\big]\end{aligned} \tag{14.99}$$

(where all spinor objects have been enclosed in round brackets).

Consider the third (and last) contribution to the above equation. By extracting a total divergence,

$$\partial^\alpha V_\alpha \equiv \partial^\alpha\left(\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\overline{\psi}^\mu\gamma^5\gamma^\nu\gamma^{[\rho}\gamma^{\sigma]}\epsilon\,\partial_\rho h^\beta{}_\sigma\right), \tag{14.100}$$

by exploiting the anticommuting properties of the Majorana spinor $\overline{\psi}^\mu$ and ϵ, and by renaming the summation indices μ and β, such a last contribution can be rewritten as:

$$\partial^\alpha V_\alpha - \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\big(\overline{\epsilon}\gamma^5\gamma^\nu\gamma^{[\rho}\gamma^{\sigma]}\partial^\alpha\psi^\beta\big)\partial_\rho h^\mu{}_\sigma. \tag{14.101}$$

The variation (14.99) then reduces to

$$\begin{aligned}\delta\mathcal{L} = {} & \partial^\alpha h^{\mu\nu}(\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\nu) + \partial^\alpha V_\alpha \\ & - \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\,\overline{\epsilon}\gamma^5\big[\gamma^{[\rho}\gamma^{\sigma]}\gamma^\nu + \gamma^\nu\gamma^{[\rho}\gamma^{\sigma]}\big]\partial^\alpha\psi^\beta\partial_\rho h^\mu{}_\sigma.\end{aligned} \tag{14.102}$$

For the product of Dirac matrices we can use Eqs. (13.34), (13.36), (13.49), implying

$$\gamma^{[\rho}\gamma^{\sigma]}\gamma^\nu + \gamma^\nu\gamma^{[\rho}\gamma^{\sigma]} = 2\gamma^{[\nu}\gamma^\rho\gamma^{\sigma]} = -2i\epsilon^{\nu\rho\sigma\lambda}\gamma^5\gamma_\lambda. \tag{14.103}$$

Inserting this result into Eq. (14.102), and using the product rule (3.39) for completely antisymmetric tensors, we arrive at:

$$\delta\mathcal{L} = \partial^\alpha h^{\mu\nu}(\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\nu) + \partial^\alpha V_\alpha - \delta^{\rho\sigma\lambda}_{\mu\alpha\beta}\big(\overline{\epsilon}\gamma_\lambda\partial^\alpha\psi^\beta\big)\partial_\rho h^\mu{}_\sigma. \tag{14.104}$$

We now notice that the last term of the above expression in nonvanishing only for $\mu \neq \rho$ and $\mu \neq \sigma$, due to the gauge conditions (14.39). The only contribution of the symbol $\delta^{\rho\sigma\lambda}_{\mu\alpha\beta}$ thus comes from the term with $\mu = \lambda$, corresponding to

$$\delta^\lambda_\mu\big(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\sigma_\alpha\delta^\rho_\beta\big). \tag{14.105}$$

Inserting into Eq. (14.104) we obtain

$$\delta\mathcal{L} = \partial^\alpha h^{\mu\nu}(\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\nu) + \partial^\alpha V_\alpha - (\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\beta)\partial^\alpha h^{\mu\beta} + (\overline{\epsilon}\gamma_\mu\partial_\alpha\psi_\beta)\partial^\beta h^{\mu\alpha}. \tag{14.106}$$

The first and third contributions to $\delta\mathcal{L}$ cancel among each other. The last contribution can be rewritten as a total divergence,

$$\partial_a W^\alpha \equiv \partial_\alpha\big(\overline{\epsilon}\gamma_\mu\psi_\beta\partial^\beta h^{\mu\alpha}\big), \tag{14.107}$$

since $\partial_\alpha h^{\mu\alpha} = 0$ in the gauge (14.39). The total infinitesimal variation of the Lagrangian can be then rewritten in the final form

$$\delta\mathcal{L} = \partial_\alpha\big(V^\alpha + W^\alpha\big) \equiv \partial_\alpha K^\alpha, \tag{14.108}$$

where, according to the definitions (14.100) and (14.107),

$$K^\alpha = (\overline{\epsilon}\gamma_\mu\psi_\beta)\partial^\beta h^{\mu\alpha} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}(\overline{\psi}_\mu\gamma_5\gamma_\nu\gamma_{[\rho}\gamma_{\sigma]}\epsilon)\partial^\rho h_\beta{}^\sigma. \tag{14.109}$$

Using Eq. (14.58) and the properties of the Majorana spinors the current K can also be recast in the form

$$K^\alpha = \frac{1}{2}(\overline{\epsilon}\gamma_\mu\psi_\nu)\big(\partial^\nu h^{\mu\alpha} + \partial^\alpha h^{\mu\nu}\big) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\big(\overline{\psi}_\mu\gamma_5\gamma^\rho\epsilon\big)\partial_\nu h_{\beta\rho}. \tag{14.110}$$

14.4 Solution As discussed in Sect. 14.3, it will be enough to vary the action with respect to V and ψ. Hence, we must compute

$$\delta\mathcal{L} = \delta_V\mathcal{L}_2 + \delta_V\mathcal{L}_{3/2} + \delta_\psi\mathcal{L}_{3/2}, \tag{14.111}$$

where $\mathcal{L}_2$ and $\mathcal{L}_{3/2}$ denote, respectively, the gravitational and spinorial part of the Lagrangian (14.53). For simplicity, and for consistency with the transformations defined in (14.55), we will use everywhere the convention $\chi = 8\pi G/c^4 = 1$.

For the gravitational action there is only the variational contribution of the vierbeins, and using the results (12.62), (12.75) we can immediately conclude that

$$\delta_V\mathcal{L}_2 = \delta_V\left(-\frac{V}{2}R\right) = G^\mu{}_a\delta V^a_\mu = \big(\overline{\epsilon}\gamma^a\psi_\mu\big)G^\mu{}_a, \tag{14.112}$$

where G is the Einstein tensor (12.64).

Let us now vary the Rarita–Schwinger Lagrangian with respect to the gravitino, using the infinitesimal transformations $\delta\psi_\mu = -2D_\mu\epsilon$, $\delta\overline{\psi}_\mu = -2D_\mu\overline{\epsilon}$. We obtain

$$\begin{aligned}\delta_\psi\mathcal{L}_{3/2} &= -i\epsilon^{\mu\nu\alpha\beta}(D_\mu\overline{\epsilon}\gamma_5\gamma_\nu D_\alpha\psi_\beta + \overline{\psi}_\mu\gamma_5\gamma_\nu D_\alpha D_\beta\epsilon)\\ &= -i\epsilon^{\mu\nu\alpha\beta}\big[\overline{\psi}_\mu\gamma_5\gamma_\nu D_{[\alpha}D_{\beta]}\epsilon - \overline{\epsilon}\gamma_5\gamma_\nu D_{[\mu}D_{\alpha]}\psi_\beta\\ &\quad - (\overline{\epsilon}\gamma_5\gamma_a D_\alpha\psi_\beta)D_\mu V^a_\nu\big] + \text{total divergence}.\end{aligned} \tag{14.113}$$

We will first consider the joint contribution of the first two terms (which we will denote by C).

By exploiting the results for the commutator (14.70) we obtain

$$C = -\frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\psi}_\mu\gamma_5\gamma_\nu\gamma_{[a}\gamma_{b]}\epsilon R_{\alpha\beta}{}^{ab} - \overline{\epsilon}\gamma_5\gamma_\nu\gamma_{[a}\gamma_{b]}\psi_\beta R_{\mu\alpha}{}^{ab}\right). \tag{14.114}$$

The combination of Dirac matrices appearing here has been already evaluated in Eq. (14.58). Inserting that result we get

$$\begin{aligned} C = &-\frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V_\nu^c\left(\overline{\psi}_\mu\gamma^d\epsilon R_{\alpha\beta}{}^{ab} - \overline{\epsilon}\gamma^d\psi_\beta R_{\mu\alpha}{}^{ab}\right) \\ &-\frac{i}{4}\epsilon^{\mu\nu\alpha\beta}V_{\nu a}\left(\overline{\psi}_\mu\gamma_5\gamma_b\epsilon R_{\alpha\beta}{}^{ab} - \overline{\epsilon}\gamma_5\gamma_b\psi_\beta R_{\mu\alpha}{}^{ab}\right). \end{aligned} \tag{14.115}$$

By recalling the commutation property of the Majorana spinors we can also write

$$\overline{\psi}_\mu\gamma^d\epsilon = -\overline{\epsilon}\gamma^d\psi_\mu, \qquad \overline{\psi}_\mu\gamma_5\gamma_b\epsilon = \overline{\epsilon}\gamma_5\gamma_b\psi_\mu \tag{14.116}$$

(see Exercises 14.1 and 14.2). The last two terms of Eq. (14.115) thus cancel among each other, while the first two terms give

$$C = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}V_\nu^c R_{\mu\alpha}{}^{ab}\left(\overline{\epsilon}\gamma^d\psi_\beta\right) = -G^\beta{}_d\left(\overline{\epsilon}\gamma^d\psi_\beta\right) \tag{14.117}$$

(we have used Eq. (12.75)).

Let us consider now the last term of Eq. (14.113), and eliminate $D_{[\mu}V_{\nu]}^a$ through the torsion equation (14.60). By adding the C contribution we find that the variation of the gravitino Lagrangian (14.113) reduces to

$$\delta_\psi\mathcal{L}_{3/2} = -G^\mu{}_a\left(\overline{\epsilon}\gamma^a\psi_\mu\right) - \frac{i}{4}(\overline{\epsilon}\gamma_5\gamma_a D_\alpha\psi_\beta)\left(\overline{\psi}_\mu\gamma^a\psi_\nu\right)\epsilon^{\mu\nu\alpha\beta}. \tag{14.118}$$

We still need the variation of $\mathcal{L}_{3/2}$ with respect to V, which gives

$$\begin{aligned} \delta_V\mathcal{L}_{3/2} &= \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{\psi}_\mu\gamma_5\gamma_a D_\alpha\psi_b\delta V_\nu^a \\ &= \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}(\overline{\psi}_\mu\gamma_5\gamma_a D_\alpha\psi_b)\left(\overline{\epsilon}\gamma^a\psi_\nu\right). \end{aligned} \tag{14.119}$$

Operating on the spinors a Fierz rearrangement of type (14.78),

$$\delta_V\mathcal{L}_{3/2} = -\frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\left(\overline{\epsilon}\gamma^a\Gamma^i\gamma_5\gamma_a D_\alpha\psi_\beta\right)(\overline{\psi}_\mu\Gamma_i\psi_\nu), \tag{14.120}$$

we can note that the nonzero contribution, corresponding to a spinor current $\overline{\psi}_\mu\Gamma_i\psi_\nu$ antisymmetric in μ and ν, comes from the matrix $\Gamma^i = \gamma^a$. The possible term $\overline{\psi}_\mu\sigma_{\alpha\beta}\psi_\nu$, which is also antisymmetric, must be excluded because

$\gamma^a \sigma_{\alpha\beta} \gamma_a \equiv 0$. Therefore:

$$\begin{aligned}\delta_V \mathcal{L}_{3/2} &= -\frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\big(\overline{\epsilon}\gamma_5\gamma^a\gamma^b\gamma_a D_\alpha\psi_\beta\big)(\overline{\psi}_\mu\gamma_b\psi_\nu)\\ &= -\frac{i}{8}\epsilon^{\mu\nu\alpha\beta}\overline{\epsilon}\gamma_5\big(-\gamma^b\gamma^a+2\eta^{ab}\big)\gamma_a D_\alpha\psi_\beta(\overline{\psi}_\mu\gamma_b\psi_\nu)\\ &= \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}\big(\overline{\epsilon}\gamma_5\gamma^b D_\alpha\psi_\beta\big)(\overline{\psi}_\mu\gamma_b\psi_\nu). \end{aligned} \tag{14.121}$$

Adding all contributions,—given by Eqs. (14.112), (14.118) and (14.121)—we obtain for $\delta\mathcal{L}$ a vanishing result, modulo the total divergence that we have neglected in Eq. (14.113), and which is given by

$$\partial_\mu K^\mu = -i\, D_\mu\big(\epsilon^{\mu\nu\alpha\beta}\overline{\epsilon}\gamma_5\gamma_\nu D_\alpha\psi_\beta\big). \tag{14.122}$$

We can immediately check that $\partial_\mu K^\mu = 0$ if we impose the gravitino equations of motion $R^\mu = 0$, together with the consistency condition $D_\mu R^\mu = 0$, discussed in Sect. 14.3.1.

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Appendix A
The Language of Differential Forms

This appendix—with the only exception of Sect. A.4.2—does not contain any new physical notions with respect to the previous chapters, but has the purpose of deriving and rewriting some of the previous results using a different language: the language of the so-called differential (or *exterior*) forms. Thanks to this language we can rewrite all equations in a more compact form, where the tensor indices of the curved space–time are "hidden" inside the variables, with great formal simplifications and benefits (especially in the context of the variational computations).

The matter of this appendix is not intended to provide a complete nor a rigorous introduction to this formalism: it should be regarded only as a first, intuitive and operational approach to the calculus of differential forms (also called exterior calculus, or "Cartan calculus"). The main purpose is to quickly put the reader in the position of understanding, and also independently performing, various computations typical of a geometric model of gravity. The readers interested in a more rigorous discussion of differential forms are referred, for instance, to the book [49] of the bibliography.

Let us finally notice that in this appendix we will follow the conventions introduced in Chap. 12, Sect. 12.1: Latin letters $a, b, c, \ldots$ will denote Lorentz indices in the flat tangent space, Greek letters $\mu, \nu, \alpha, \ldots$ tensor indices in the curved manifold. For the matter fields we will always use natural units $\hbar = c = 1$. Also, unless otherwise stated, in the first three sections (A.1, A.2, A.3) we will assume that the space–time manifold has an arbitrary number D of dimensions, with signature $(+, -, -, -, \ldots)$.

A.1 Elements of Exterior Calculus

Let us start with the observation that the infinitesimal (oriented) surface-element $dx_1\, dx_2$ of a differentiable manifold is antisymmetric with respect to the exchange of the coordinates, $x_1 \to x_1' = x_2$ and $x_2 \to x_2' = x_1$, since the corresponding Jacobian

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determinant of the transformation is $|\partial x'/\partial x| = -1$. Hence:

$$\int dx_1\, dx_2 = -\int dx_2\, dx_1. \tag{A.1}$$

With reference to a generic volume element $dx_1\, dx_2 \cdots dx_D$ let us then introduce the composition of differentials called *exterior product* and denoted by the wedge symbol, $dx^\mu \wedge dx^\nu$, which is associative and antisymmetric, $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Let us define, in this context, an "exterior" differential form of degree p—or, more synthetically, a p-form—as an element of the linear vector space Λ_p spanned by the external composition of p differentials.

Any p-form can thus be represented as a homogeneous polynomial with a degree of p in the exterior product of differentials,

$$A \in \Lambda^p \quad \Longrightarrow \quad A = A_{[\mu_1 \cdots \mu_p]}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{A.2}$$

where $dx^{\mu_i} \wedge dx^{\mu_j} = -dx^{\mu_j} \wedge dx^{\mu_i}$ for any pair of indices, and where $A_{[\mu_1 \cdots \mu_p]}$ (the so-called "components" of the p form) correspond to the components of a totally antisymmetric tensor of rank p. A scalar ϕ, for instance, can be represented as a 0-form, a covariant vector A_μ as a 1-form A, with $A = A_\mu\, dx^\mu$, an antisymmetric tensor $F_{\mu\nu}$ as a 2-form F, with $F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, and so on.

In a D-dimensional manifold, the direct sum of the vector spaces Λ_p from 0 to D defines the so-called Cartan algebra Λ,

$$\Lambda = \bigoplus_{p=0}^{D} \Lambda_p. \tag{A.3}$$

In the linear vector space Λ the exterior product is a map $\Lambda \times \Lambda \to \Lambda$ which, in the coordinate differential base $dx^{\mu_1} \wedge dx^{\mu_2} \cdots$, is represented by a composition law which satisfies the properties of

(1) bilinearity:

$$\begin{aligned} &\left(\alpha\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} + \beta\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) \wedge d\, x^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}} \\ &\quad = (\alpha + \beta)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}} \end{aligned} \tag{A.4}$$

(α and β are arbitrary numerical coefficients);

(2) associativity:

$$\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) \wedge \left(dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}}\right) = dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+q}}; \tag{A.5}$$

(3) skewness:

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = dx^{[\mu_1} \wedge \cdots \wedge dx^{\mu_p]}. \tag{A.6}$$

This last property implies that the exterior product of a number of differentials $\mu_p > D$ is identically vanishing.

Starting with the above definitions, we can now introduce some important operations concerning the exterior forms.

A.1.1 Exterior Product

The exterior product between a p-form $A \in \Lambda_p$ and a q-form $B \in \Lambda_q$ is a bilinear and associative mapping $\wedge : \Lambda_p \times \Lambda_q \to \Lambda_{p+q}$, which defines the $(p+q)$-form C such that

$$C = A \wedge B = A_{\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+q}}. \tag{A.7}$$

The commutation properties of this product depend on the degrees of the forms we are considering (i.e. on the number of the components we have to switch), and in general we have the rule:

$$A \wedge B = (-1)^{pq} B \wedge A. \tag{A.8}$$

A.1.2 Exterior Derivative

The exterior derivative of a p form $A \in \Lambda_p$ can be interpreted (for what concerns the product rules) as the exterior product between the gradient 1-form $dx^\mu \partial_\mu$ and the p-form A. It is thus represented by the mapping $d : \Lambda_p \to \Lambda_{p+1}$, which defined the $(p+1)$-form dA such that

$$dA = \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}. \tag{A.9}$$

For a scalar ϕ, for instance, the exterior derivative is represented by the 1-form

$$d\phi = \partial_\mu \phi \, dx^\mu. \tag{A.10}$$

The exterior derivative of the 1-form A is represented by the 2-form

$$dA = \partial_{[\mu} A_{\nu]} \, dx^\mu \wedge dx^\nu, \tag{A.11}$$

and so on for higher degrees.

An immediate consequence of the definition (A.9) is that the second exterior derivative is always vanishing,

$$d^2 A = d \wedge dA \equiv 0, \tag{A.12}$$

regardless of the degree of the form A. We can also recall that a p-form A is called *closed* if $dA = 0$, and *exact* if it satisfies the property $A = d\phi$, where ϕ is a $(p-1)$-form. If a form is exact then it is (obviously) closed. However, if a form is closed then it is not necessarily exact (it depends on the topological properties of the manifold where the form is defined).

Another consequence of the definition (A.9) is that, in a space–time with a symmetric connection ($\Gamma_{\mu\nu}{}^\alpha = \Gamma_{\nu\mu}{}^\alpha$), the gradient ∂_μ appearing in the exterior-derivative operator can be always replaced by the covariant gradient ∇_μ. In fact,

$$\nabla_{\mu_1} A_{\mu_2 \mu_3 \ldots} = \partial_{\mu_1} A_{\mu_2 \mu_3 \ldots} - \Gamma_{\mu_1 \mu_2}{}^\alpha A_{\alpha \mu_3 \ldots} - \Gamma_{\mu_1 \mu_3}{}^\alpha A_{\mu_2 \alpha \ldots} - \cdots, \tag{A.13}$$

so that all connection terms disappear after antisymmetrization, and

$$dA = \nabla A \equiv \nabla_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}. \tag{A.14}$$

Finally, again from the definition (A.9) and from the commutation rule (A.8), we can obtain a generalized Leibnitz rule for the exterior derivative of a product. Consider, for instance, the exterior product of a p-form A and a q-form B. By recalling that d is a 1-form operator we have

$$\begin{aligned} d(A \wedge B) &= dA \wedge B + (-1)^p A \wedge dB, \\ d(B \wedge A) &= dB \wedge A + (-1)^q B \wedge dA. \end{aligned} \tag{A.15}$$

And so on for multiple products.

A.1.3 Duality Conjugation and Co-differential Operator

Another crucial ingredient for the application of this formalism to physical models is the so-called *Hodge-duality* operation, which associates to each p-form its $(D-p)$-dimensional "complement". The dual of a p-form $A \in \Lambda_p$ is a mapping $^\star : \Lambda_p \to \Lambda_{D-p}$, defining the $(D-p)$-form $^\star A$ such that

$$^\star A = \frac{1}{(D-p)!} A^{\mu_1 \cdots \mu_p} \eta_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D}\, dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}. \tag{A.16}$$

We should recall that the fully antisymmetric tensor η is related to the Levi-Civita antisymmetric density ϵ by the relation

$$\eta_{\mu_1 \cdots \mu_D} = \sqrt{|g|}\, \epsilon_{\mu_1 \cdots \mu_D} \tag{A.17}$$

(see Sect. 3.2, Eq. (3.34)). We should also note that the use of $\sqrt{|g|}$ instead of $\sqrt{-g}$ is due to the fact that the sign of $\det g_{\mu\nu}$, in an arbitrary number of D space–time dimensions, depends on the number (even or odd) of the $D-1$ spacelike components.

It may be useful to point out that the square of the duality operator does not coincides with the identity, in general. By applying the definition (A.16), in fact, we obtain

$$\begin{aligned} {}^\star\big({}^\star A\big) &= \frac{1}{p!(D-p)!} A_{\mu_1 \cdots \mu_p} \eta^{\mu_1 \cdots \mu_D} \eta_{\mu_{p+1} \cdots \mu_D \nu_1 \cdots \nu_p}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} \\ &= (-1)^{p(D-p)} (-1)^{D-1} \frac{1}{p!}\, \delta^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_p} A_{\mu_1 \cdots \mu_p}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p} \\ &= (-1)^{p(D-p)+D-1} A. \end{aligned} \tag{A.18}$$

The factor $(-1)^{D-1}$ comes from the product rules of the totally antisymmetric tensors since, in $D-1$ spatial dimensions (and with our conventions),

$$\epsilon_{012\ldots D-1} = (-1)^{D-1}\,\epsilon^{012\ldots D-1} = (-1)^{D-1}. \tag{A.19}$$

The product rules thus become, in general,

$$\eta_{\nu_1\cdots\nu_p\mu_{p+1}\cdots\mu_D}\eta^{\mu_1\cdots\mu_D} = (-1)^{D-1}(D-p)!\,\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}, \tag{A.20}$$

where $\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}$ is the determinant defined in Eq. (3.35). The factor $(-1)^{p(D-p)}$, instead, comes from the switching of the p indices of A with the $D-p$ indices of its dual (such a switching is needed to arrange the indices of η in a way to match the sequence of the product rule (A.20)).

We also note, for later applications, that the dual of the identity operator is directly related to the scalar integration measure representing the hypervolume element of the given space–time manifold. From the definition (A.16) we have, in fact,

$$\begin{aligned} {}^\star 1 &= \frac{1}{D!}\,\eta_{\mu_1\cdots\mu_D}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} \\ &= \sqrt{|g|}\,\epsilon_{012\ldots D-1}\,dx^0\wedge dx^1\cdots dx^{D-1} \\ &= (-1)^{D-1}\sqrt{|g|}\,d^Dx. \end{aligned} \tag{A.21}$$

Combining this result with the product rule

$$\eta_{\mu_1\cdots\mu_D}\eta^{\mu_1\cdots\mu_D} = (-1)^{D-1}D!, \tag{A.22}$$

we are led to the useful relation

$$dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} = \sqrt{|g|}\,d^Dx\,\eta^{\mu_1\cdots\mu_D} = d^Dx\,\epsilon^{\mu_1\cdots\mu_D}, \tag{A.23}$$

which will be frequently applied in our subsequent computations.

The duality operation is necessarily required in order to define the scalar products appearing, for instance, in all action integrals. Consider in fact the exterior product between a p-form A and the dual of another p-forma B. By using the definition (A.16) and the relation (A.23) we obtain

$$\begin{aligned} \int A\wedge{}^\star B &= \frac{1}{(D-p)!}\int A_{\mu_1\cdots\mu_p}B^{\nu_1\cdots\nu_p}\eta_{\nu_1\cdots\nu_p\mu_{p+1}\cdots\mu_D}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} \\ &= (-1)^{D-1}\int d^Dx\,\sqrt{|g|}\,A_{\mu_1\cdots\mu_p}B^{\nu_1\cdots\nu_p}\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} \\ &= (-1)^{D-1}p!\int d^Dx\,\sqrt{|g|}A_{\mu_1\cdots\mu_p}B^{\mu_1\cdots\mu_p} \end{aligned} \tag{A.24}$$

(in the second step we have applied the product rule (A.20)). The above result holds for forms of the same degree p (but p is arbitrary), and using Eq. (A.21) it can be

rewritten as

$$A \wedge {}^\star B = B \wedge {}^\star A = p!^\star 1\, A_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p}. \tag{A.25}$$

Let us finally observe that—through the application of the duality operation—we can express the divergence of a p form A by computing the exterior derivative of its dual, and by subsequently "dualizing" the obtained result. We obtain, in this way, the $(p-1)$-form ${}^\star(d{}^\star A)$ whose components exactly correspond to the divergence of the antisymmetric tensor $A_{[\mu_1 \cdots \mu_P]}$.

Consider, in fact, the exterior derivative of the dual form (A.16):

$$d{}^\star A = \frac{1}{(D-p)!} \partial_\alpha \left(\sqrt{|g|} A^{\mu_1 \cdots \mu_p} \right) \epsilon_{\mu_1 \cdots \mu_D}\, dx^\alpha \wedge dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}. \tag{A.26}$$

Computing the dual we obtain

$$\begin{aligned} {}^\star\left(d{}^\star A\right) &= \frac{1}{(p-1)!(D-p)!}\, \partial^\alpha \left(\sqrt{|g|} A_{\mu_1 \cdots \mu_p} \right) \epsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_D} \\ &\quad \times \frac{1}{\sqrt{|g|}} \epsilon_\alpha{}^{\mu_{p+1} \cdots \mu_D}{}_{\nu_1 \cdots \nu_{p-1}}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p-1}} \\ &= p(-1)^{D-1+(p-1)(D-p)} \nabla^\alpha A_{\alpha \nu_1 \cdots \nu_{p-1}}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p-1}}, \end{aligned} \tag{A.27}$$

where

$$\nabla_\alpha A^{[\alpha \nu_1 \cdots \nu_{p-1}]} = \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} A^{[\alpha \nu_1 \cdots \nu_{p-1}]} \right) \tag{A.28}$$

is the covariant divergence of a completely antisymmetric tensor, computed with a symmetric connection.

By exploiting the above result we can also define a further differential operation acting on the exterior forms, represented by the so-called "co-differential" operator (or exterior co-derivative). The co-differential of a p-form is a mapping $\delta : \Lambda_p \to \Lambda_{p-1}$, defining the $(p-1)$-forma δA such that

$$\delta A = p\, \nabla^\alpha A_{\alpha \mu_1 \cdots \mu_{p-1}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}. \tag{A.29}$$

A comparison with Eq. (A.27) shows that exterior derivative d and co-derivative δ are related by

$$\delta = (-1)^{D-1+(p-1)(D-p)} {}^\star d {}^\star. \tag{A.30}$$

The notions of duality, exterior derivative and exterior product introduced above will be enough for the pedagogical purpose of this appendix, and will be applied to the geometric description of gravity illustrated in the following sections.

A.2 Basis and Connection One-Forms: Exterior Covariant Derivative

The language of exterior forms is particularly appropriate, in the context of differential geometry, to represent equations projected on the flat tangent manifold. By using the vierbeins V_μ^a (see Chap. 12), in fact, we can introduce in the tangent Minkowski space–time a set of *basis* 1-*forms*

$$V^a = V_\mu^a \, dx^\mu, \tag{A.31}$$

and represent any given p-form $A \in \Lambda_p$ on this basis as

$$A = A_{[a_1 \cdots a_p]} V^{a_1} \wedge \cdots \wedge V^{a_p}, \tag{A.32}$$

where $A_{a_1 \cdots a_p} = A_{\mu_1 \cdots \mu_p} V_{a_1}^{\mu_1} \cdots V_{a_p}^{\mu_p}$ are the components of the form projected on the local tangent space. In this representation the formalism becomes completely independent of the particular coordinates chosen to parametrize the curved space–time manifold, at least until the equations are explicitly rewritten in tensor components.

In the absence of explicit curved indices (namely, of explicit representations of the diffeomorphism group), the full covariant derivative is reduced to a Lorentz-covariant derivative (see Sect. 12.2). By introducing the *connection* 1-*form*,

$$\omega^{ab} = \omega_\mu{}^{ab} \, dx^\mu, \tag{A.33}$$

where $\omega_\mu{}^{ab}$ is the Lorentz connection, we can then define the *exterior, Lorentz-covariant* derivative. Given a p-form $\psi \in \Lambda_p$, transforming as a representation of the Lorentz group with generators J_{ab} in the local tangent space, the exterior covariant derivative is a mapping $D : \Lambda_p \to \Lambda_{p+1}$, defining the $(p+1)$-form $D\psi$ such that

$$D\psi = d\psi - \frac{i}{2} \omega^{ab} J_{ab} \psi \tag{A.34}$$

(see Eq. (12.22)).

Consider, for instance, a p-form $A^a \in \Lambda_p$ vector-valued in the tangent space. The vector generators of the Lorentz group lead to the covariant derivative (12.30). The corresponding exterior covariant derivative is given by

$$DA^a = D_{\mu_1} A^a_{\mu_2 \cdots \mu_{p+1}} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} = dA^a + \omega^a{}_b \wedge A^b, \tag{A.35}$$

where dA^a is the ordinary exterior derivative of Sect. A.1.2. Since the operator D is a 1-form and A^a is a p-form, the derivative DA^a is a $(p+1)$-form. We should note that DA^a is transformed correctly as a vector under local Lorentz transformations,

$$DA^a \to \Lambda^a{}_b \big(DA^b \big), \tag{A.36}$$

since the connection 1-form is transformed as

$$\omega^a{}_b \to \Lambda^a{}_c \, \omega^c{}_k \big(\Lambda^{-1} \big)^k{}_b - (d\Lambda)^a{}_c \big(\Lambda^{-1} \big)^c{}_b. \tag{A.37}$$

This last condition is nothing more than the transformation law deduced in Exercise 12.1, Eq. (12.67), written, however, in the language of differential forms.

The above definition can be easily applied to other representations of the local Lorentz group. If we have, for instance, a tensor-valued p-form of mixed type, $A^a{}_b \in \Lambda_p$, and we recall the definition (12.34) of the covariant derivative of a tensor object, we can immediately write down the exterior covariant derivative as

$$DA^a{}_b = dA^a{}_b + \omega^a{}_c \wedge A^c{}_b - \omega^c{}_b \wedge A^a{}_c. \tag{A.38}$$

An so on for other representations of the local Lorentz group.

It is important to stress that the differential symbol D operates on the p-form in a way which is independent on p. Hence, the previous rules apply with no changes also to tensor-valued 0-forms. As an typical example we may quote here the metric η^{ab} of the tangent Minkowski space–time: computing its exterior covariant derivative we find

$$D\eta^{ab} = d\eta^{ab} + \omega^a{}_c \eta^{cb} + \omega^b{}_c \eta^{ac} = \omega^{ab} + \omega^{ba} \equiv 0 \tag{A.39}$$

(the result is vanishing thanks to the antisymmetry property of the Lorentz connection, $\omega^{ab} = \omega^{[ab]}$). Another important tensor-valued 0-form in the tangent space is the fully antisymmetric symbol ϵ^{abcd}. By applying the result of Exercise 12.3 we can easily compute the exterior covariant derivative $D\epsilon^{abcd}$ and check that, even in this case, this derivative is a vanishing 1-form.

The properties of the 1-form D, regarded as a mapping $D : \Lambda_p \to \Lambda_{p+1}$, are the same as those of the exterior derivative d. Given, for instance, a p-form A and a q-form B, the covariant derivative of their exterior product obeys the rules

$$\begin{aligned} D(A \wedge B) &= DA \wedge B + (-1)^p A \wedge DB, \\ D(B \wedge A) &= DB \wedge A + (-1)^q B \wedge DA \end{aligned} \tag{A.40}$$

(see Eq. (A.15)). The second covariant derivative, however, is in general non-vanishing, being controlled by the space–time curvature.

In fact, by applying the D operator to the generic $(p+1)$-form $D\psi$ of Eq. (A.34), we obtain

$$\begin{aligned} D^2\psi = D \wedge D\psi &= D_\alpha D_\beta \psi_{\mu_1 \cdots \mu_p}\, dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &= -\frac{i}{4} R_{\alpha\beta}{}^{ab}(\omega) J_{ab} \psi_{\mu_1 \cdots \mu_p}\, dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &= -\frac{i}{2} R^{ab} J_{ab} \wedge \psi, \end{aligned} \tag{A.41}$$

where $R_{\alpha\beta}{}^{ab}$ is the Lorentz connection (12.54), and where we have defined the curvature 2-form R^{ab} as

$$R^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab}\, dx^\mu \wedge dx^\nu$$

$$= \left(\partial_{[\mu}\omega_{\nu]} + \omega_{[\mu|}{}^{a}{}_{c}\omega_{|\nu]}{}^{cb}\right) dx^{\mu} \wedge dx^{\nu}$$
$$= d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}. \tag{A.42}$$

If (in particular) ψ is a vector field, $\psi \to A^{a}$, and J_{ab} correspond to the vector generators (12.29), then Eq. (A.41) becomes

$$D^{2}A^{a} = R^{a}{}_{b} \wedge A^{b}. \tag{A.43}$$

This equation exactly reproduces, in the language of exterior forms, the result (12.51) concerning the commutator of two covariant derivatives applied to a Lorentz vector.

We can finally check, as a simple exercise, that Eq. (A.43) can be directly obtained also by computing the exterior covariant derivative of Eq. (A.35). By applying the definition of D, and using the properties of the differential forms, we obtain, in fact:

$$\begin{aligned} D^{2}A^{a} &= D \wedge DA^{a} = d\left(DA^{a}\right) + \omega^{a}{}_{c} \wedge DA^{c} \\ &= d^{2}A^{a} + d\omega^{a}{}_{b} \wedge A^{b} - \omega^{a}{}_{b} \wedge dA^{b} + \omega^{a}{}_{c} \wedge \left(dA^{c} + \omega^{c}{}_{b} \wedge A^{b}\right) \\ &= \left(d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}\right) \wedge A^{b} \\ &\equiv R^{a}{}_{b} \wedge A^{b}, \end{aligned} \tag{A.44}$$

where R^{ab} is given by Eq. (A.42).

A.3 Torsion and Curvature Two-Forms: Structure Equations

We have seen in Chap. 12 that the Lorentz connection ω represents the non-Abelian "gauge potential" associated to the local Lorentz symmetry, and that the curvature $R(\omega)$ represents the corresponding "gauge field" (or Yang–Mills field). In the language of exterior forms the potential is represented by the connection 1-form, ω^{ab}, and the gauge field by the curvature 2-form, R^{ab}, both defined in the previous section.

In the previous section we have also introduced, besides the connection, another variable which is of fundamental importance for the formulation of a geometric model of the gravitational interactions: the 1-form V^{a}, acting as a basis in the Minkowski tangent space. By recalling the vierbein metricity condition, Eq. (12.40), and considering its antisymmetric part

$$D_{[\mu}V^{a}_{\nu]} \equiv \partial_{[\mu}V^{a}_{\nu]} + \omega_{[\mu}{}^{a}{}_{\nu]} = \Gamma_{[\mu\nu]}{}^{a} \equiv Q_{\mu\nu}{}^{a}, \tag{A.45}$$

we can then associate to the 1-form V^{a} the torsion 2-form R^{a} such that

$$R^{a} = Q_{\mu\nu}{}^{a}\, dx^{\mu} \wedge dx^{\nu} = D_{[\mu}V^{a}_{\nu]}\, dx^{\mu} \wedge dx^{\nu} = DV^{a}. \tag{A.46}$$

The equations which define the curvature and torsion 2-forms in terms of the connection and basis 1-forms,

$$R^a = DV^a = dV^a + \omega^a{}_b \wedge V^b, \tag{A.47}$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{A.48}$$

are called *structure equations*, as they control the geometric structure of the given manifold. The curvature, being the Yang–Mills field of the Lorentz group, satisfies a structure equation which is a direct consequence of the Lie algebra for that group, and which reflects the interpretation of the connection ω as the associated gauge potential. If also the torsion equation would be determined by the algebraic structure of some symmetry group, then also the 1-form V^a could be interpreted as a gauge potential, and the torsion 2-form as the corresponding gauge field.

In the following section it will be shown that the geometric structure described by Eqs. (A.47), (A.48) is a direct consequence of the algebraic structure of the Poincaré group. More precisely, it will be shown that the torsion and the curvature defined by the above equations exactly represent the components of the Yang–Mills field for a non-Abelian gauge theory based on the local Poincaré symmetry.

A.3.1 Gauge Theory for the Poincaré Group

Consider a local symmetry group G, characterized by n generators X_A, $A = 1, 2, \ldots, n$, which satisfy the Lie algebra

$$[X_A, X_b] = i f_{AB}{}^C X_C, \tag{A.49}$$

where $f_{AB}{}^C = -f_{BA}{}^C$ are the structure constant of the given Lie group.

In order to formulate the corresponding gauge theory (see Sect. 12.1.1), let us associate to each generator X_A the potential 1-form $h^A = h^A_\mu \, dx^\mu$, with values in the Lie algebra of the group, and define

$$h \equiv h^A_\mu X_A \, dx^\mu. \tag{A.50}$$

Let us then introduce the corresponding exterior covariant derivative,

$$D = d - \frac{i}{2} h, \tag{A.51}$$

which we have written in units in which $g = 1$, where g is the dimensionless coupling constant.

The exterior product of two covariant derivatives defines the 2-form $R = R^A X_A$, representing the gauge field (or curvature):

$$D^2\psi = D \wedge D\psi = \left(d - \frac{i}{2}h\right) \wedge \left(d - \frac{i}{2}h\right)\psi$$

$$= -\frac{i}{2}\, dh\, \psi + \frac{i}{2} h \wedge d\psi - \frac{i}{2} h \wedge d\psi - \frac{1}{4} h \wedge h\psi$$
$$= -\frac{i}{2} R\psi, \tag{A.52}$$

where

$$R = R^A X_A = dh - \frac{i}{2} h \wedge h. \tag{A.53}$$

Using the definition $h = h^A X_A$, and the Lie algebra (A.49), we then obtain

$$R^A X_A = \left(dh^A\right) X_A - \frac{i}{4} h^B \wedge h^C [X_B, X_c]$$
$$= \left(dh^A + \frac{1}{4} f_{BC}{}^A h^B \wedge h^C \right) X_A. \tag{A.54}$$

This clearly shows that the components of the gauge field,

$$R^A = dh^A + \frac{1}{4} f_{BC}{}^A h^B \wedge h^C, \tag{A.55}$$

are directly determined by the algebraic structure of the gauge group.

Let us now consider the Poincaré group, namely the group with the maximum number of isometries in the flat tangent space. It is characterized by ten generators,

$$X_A = \{P_a, J_{ab}\}, \tag{A.56}$$

where $J_{ab} = -J_{ba}$ (in this case the group index A ranges over the 4 components of the translation generators, P_a, and the six components of the generators of Lorentz rotations, J_{ab}). Let us associate to these generators an equal number of gauge potentials, represented by the 1-forms

$$h^A = \left\{V^a, \omega^{ab}\right\}, \tag{A.57}$$

where $\omega^{ab} = -\omega^{ba}$. The corresponding gauge (or Yang–Mills) field $R = R^A X_A$ can then be decomposed into translation and Lorentz-rotation components,

$$R = R^A X_A = R^a P_a + R^{ab} J_{ab}, \tag{A.58}$$

and the explicit form of the curvatures R^a and R^{ab} in terms of the potential V^a and ω^{ab} is fixed by the Lie algebra of the group, according to Eq. (A.55).

The Lie algebra of the Poincaré group is explicitly realized by the following commutation relations of generators:

$$\begin{aligned} [P_a, P_b] &= 0, \\ [P_a, J_{bc}] &= i(\eta_{ab} P_c - \eta_{ac} P_b), \\ [J_{ab}, J_{cd}] &= i(\eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{bc} J_{ad}). \end{aligned} \tag{A.59}$$

A comparison with the general relation (A.49) then tell us that the nonvanishing structure constant are

$$\begin{aligned} f_{a,bc}{}^{d} &= 2\eta_{a[b}\delta^{d}_{c]} = -f_{bc,a}{}^{d}, \\ f_{ab,cd}{}^{ij} &= 2\eta_{d[a}\delta^{i}_{b]}\delta^{j}_{c} - 2\eta_{c[a}\delta^{i}_{b]}\delta^{j}_{d}, \end{aligned} \tag{A.60}$$

where the indices (or pairs of indices) corresponding to the generators P_a and J_{ab}, respectively, have been separated by a comma. Inserting this result into the curvature (A.55) we then obtain the result that the gauge field associated to the translations,

$$\begin{aligned} R^{a} &= dV^{a} + \frac{1}{4} f_{b,cd}{}^{a} V^{b} \wedge \omega^{cd} + \frac{1}{4} f_{cd,b}{}^{a} \omega^{cd} \wedge V^{b} \\ &= dV^{a} + \frac{1}{2} f_{cd,b}{}^{a} \omega^{cd} \wedge V^{b} \\ &= dV^{a} + \eta_{bd}\delta^{a}_{c}\omega^{cd} \wedge V^{b} \\ &= dV^{a} + \omega^{a}{}_{b} \wedge V^{b} \equiv DV^{a}, \end{aligned} \tag{A.61}$$

exactly coincides with the torsion 2-form (A.47). Also, the gauge field associated to the Lorentz rotations,

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \frac{1}{4} f_{ij,cd}{}^{ab} \omega^{ij} \wedge \omega^{cd} \\ &= d\omega^{ab} + \frac{1}{2}\big(\eta_{di}\delta^{a}_{j}\delta^{b}_{c} - \eta_{ci}\delta^{a}_{j}\delta^{b}_{d}\big)\omega^{ij} \wedge \omega^{cd} \\ &= d\omega^{ab} + \frac{1}{2}\big(\omega_{d}{}^{a} \wedge \omega^{bd} - \omega_{c}{}^{a} \wedge \omega^{cb}\big) \\ &= d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}, \end{aligned} \tag{A.62}$$

exactly coincides with the Lorentz curvature (A.48).

A gravitational theory based on a Riemann–Cartan geometric structure, characterized by curvature and torsion, can thus be interpreted as a gauge theory for the Poincaré group. The Einstein theory of general relativity corresponds to the limiting case $R^a = DV^a = 0$ in which the torsion gauge field is vanishing, i.e. the potential associated to the translations is "pure gauge".

It is always possible, in principle, to formulate a model of space–time based on an arbitrary geometrical structure. In practice, however, the type of geometric structure which is more appropriate—and, sometimes, also necessarily required for the physical consistency of the model—turns out to be determined by the given gravitational sources.

We have seen, for instance, that a symmetric (and metric compatible) connection may provide a satisfactory description of the gravitational interactions of macroscopic bodies; in the case of the gravitino field, instead, the presence of torsion

is needed to guarantee a minimal and consistent gravitational coupling to the geometry. In Sects. A.4.1 and A.4.2 it will be shown that, in the context of the so-called Einstein–Cartan theory of gravity, the torsion tensor is determined by the sources themselves—just like the curvature tensor—through the field equations of the adopted model of gravity. Hence, in that case, torsion cannot be arbitrarily prescribed any longer.

A.3.2 Bianchi Identities

Let us conclude Sect. A.3 by showing how the Bianchi identities, expressed in the language of exterior forms, can be easily deduced by computing the exterior covariant derivative of the two structure equations (A.47), (A.48).

The covariant derivative of the torsion gives the first Bianchi identity, which reads

$$\begin{aligned} DR^a &= dR^a + \omega^a{}_b \wedge R^b \\ &= d\omega^a{}_b \wedge V^b - \omega^a{}_b \wedge dV^b + \omega^a{}_b \wedge dV^b + \omega^a{}_c \wedge \omega^c{}_b \wedge V^b \\ &= R^a{}_b \wedge V^b . \end{aligned} \tag{A.63}$$

The covariant derivative of the Lorentz curvature gives the second Bianchi identity, which reads

$$\begin{aligned} DR^{ab} &= dR^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} \\ &= d\omega^a{}_c \wedge \omega^{cb} - \omega^a{}_c \wedge d\omega^{cb} + \omega^a{}_c \wedge \left(d\omega^{cb} + \omega^c{}_i \wedge \omega^{ib}\right) \\ &\quad + \omega^b{}_c \wedge \left(d\omega^{ac} + \omega^a{}_i \wedge \omega^{ic}\right) \\ &\equiv 0. \end{aligned} \tag{A.64}$$

Note that the right-hand side of this equation is identically vanishing because, using the properties of the exterior forms introduced in Sects. A.1.1 and A.1.2, we have

$$\omega^b{}_c \wedge d\omega^{ac} = d\omega^a{}_c \wedge \omega^{bc} = -d\omega^a{}_c \wedge \omega^{cb}, \tag{A.65}$$

so that the first and the second-last term on the right-hand side exactly cancel each other. In addition,

$$\omega^b{}_c \wedge \omega^a{}_i \wedge \omega^{ic} = \omega^a{}_i \wedge \omega^i{}_c \wedge \omega^{bc} = -\omega^a{}_i \wedge \omega^i{}_c \wedge \omega^{cb}, \tag{A.66}$$

so that also the last and third to last term cancel each other.

The Bianchi identities (A.63), (A.64) hold, in general, in a geometric structure satisfying the metricity condition $\nabla g = 0$ (see Sect. 3.5), even in the case of nonvanishing torsion. In the absence of torsion we can easily check that the above identities are reduced to the known identities of the Riemann geometry, already presented in tensor form in Sect. 6.2.

In fact, by setting $R^a = 0$, we find that Eq. (A.63) becomes

$$R^A{}_b \wedge V^b = 0, \tag{A.67}$$

and thus implies

$$\frac{1}{2} R_{[\mu\nu|}{}^a{}_b V^b_{|\alpha]}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha = 0, \tag{A.68}$$

from which

$$R_{[\mu\nu}{}^a{}_{\alpha]} = -R_{[\mu\nu\alpha]}{}^a = 0, \tag{A.69}$$

which coincides with the first Bianchi identity (6.14).

From Eq. (A.64), on the other hand,

$$\frac{1}{2} D_{[\mu} R_{\alpha\beta]}{}^{ab} dx^\mu \wedge dx^\alpha \wedge dx^\beta = 0, \tag{A.70}$$

from which

$$D_{[\mu} R_{\alpha\beta]}{}^{ab} = 0. \tag{A.71}$$

In addition (see Chap. 12),

$$\nabla_\mu R_{\alpha\beta}{}^{ab} = D_\mu R_{\alpha\beta}{}^{ab} - \Gamma_{\mu\alpha}{}^\rho R_{\rho\beta}{}^{ab} - \Gamma_{\mu\beta}{}^\rho R_{\alpha\rho}{}^{ab}. \tag{A.72}$$

By computing the totally antisymmetric part in μ, α, β, we find that the Γ contributions disappear if the torsion is vanishing ($\Gamma_{[\mu\alpha]}{}^\rho = 0$). In that case Eq. (A.71) can be rewritten in the form

$$\nabla_{[\mu} R_{\alpha\beta]}{}^{ab} = 0, \tag{A.73}$$

which coincides with the first Bianchi identity (6.15).

A.4 The Palatini Variational Formalism

According to the variational method of Palatini, already introduced in Sect. 12.3.1, the connection and the vierbeins (or the metric) are to be treated as independent variables. In this section this method will be applied to the variation of the action written in the language of exterior forms: we will use, as fundamental independent variables, the basis 1-forms V^a and the connection 1-form ω^{ab}. We will also restrict, for simplicity, to a space–time manifold with $D = 4$ dimensions (our computations, however, can be extended without difficulty to the generic D-dimensional case).

Let us notice, first of all, that the gravitational action (12.56)—which corresponds to the integral of the scalar curvature density over a four-dimensional space–time region—can be written as the integral of a 4-form as follows:

$$S_g = \frac{1}{2\chi} \int R^{ab} \wedge {}^\star(V_a \wedge V_b). \tag{A.74}$$

Using the definition of Lorentz curvature, Eq. (A.42), the definition of dual, Eq. (A.16), and the relation (A.23) we have, in fact:

$$\begin{aligned} R^{ab} \wedge {}^\star(V_a \wedge V_b) &= \frac{1}{2} R_{\mu\nu}{}^{ab} \frac{1}{2} V_a^\alpha V_b^\beta \eta_{\alpha\beta\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \frac{1}{4} R_{\mu\nu}{}^{ab} V_a^\alpha V_b^\beta \eta_{\alpha\beta\rho\sigma} \eta^{\mu\nu\rho\sigma}\, d^4x \sqrt{-g} \\ &= -\frac{1}{2} R_{\mu\nu}{}^{ab} V_a^\alpha V_b^\beta \left(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu\right) d^4x \sqrt{-g} \\ &= -R\, d^4x \sqrt{-g} \end{aligned} \tag{A.75}$$

(in the second-last step we have used the product rule (A.20) in $D = 4$). The scalar curvature appearing here is defined as the following contraction of the Lorentz connection:

$$R = R_{\mu\nu}{}^{ab}(\omega) V_a^\mu V_b^\nu, \tag{A.76}$$

in agreement with Eq. (12.55).

The total action (for gravity plus matter sources) can then be written in the form

$$S_g = \frac{1}{2\chi} \int R^{ab} \wedge {}^\star(V_a \wedge V_b) + S_m(\psi, V, \omega), \tag{A.77}$$

where $\chi = 8\pi G/c^4$, ψ is the field representing the sources, and a possible appropriate boundary term is to be understood. In the following section this action will be varied with respect to V^a and ω^{ab}, in order to obtain the corresponding field equations.

A.4.1 General Relativity and Einstein–Cartan Equations

In order to vary the action (A.77) with respect to V let us explicitly rewrite the dual operation referred to the basis 1-form of the local tangent space, according to Eq. (A.32). We obtain

$${}^\star(V_a \wedge V_b) = \frac{1}{2} \epsilon_{abcd} V^c \wedge V^d. \tag{A.78}$$

The variation of the gravitational part of the action then gives

$$\begin{aligned} \delta_V S_g &= \frac{1}{4\chi} \int R^{ab} \wedge \left(\delta V^c \wedge V^d + V^c \wedge \delta V^d\right) \epsilon_{abcd} \\ &= \frac{1}{2\chi} \int \left(R^{ab} \wedge V^c \epsilon_{abcd}\right) \wedge \delta V^d, \end{aligned} \tag{A.79}$$

where we have used the anticommutation property of the exterior product of two 1-forms, $\delta V^c \wedge V^d = -V^d \wedge \delta V^c$, and the antisymmetry of the tensor ϵ in c and d.

We should now consider the additional contribution arising from the variation of the matter action, which we can write, in general, as

$$\delta_V S_m = \int \theta_d \wedge \delta V^d. \tag{A.80}$$

Here θ_d is a 3-form associated to the canonical energy-momentum density,

$$\theta_d = \frac{1}{3!}\theta_d{}^i \epsilon_{iabc} V^a \wedge V^b \wedge V^c, \tag{A.81}$$

whose explicit expression depends on the type of source we are including into our model (a few examples will be given below). By adding the two contributions (A.79), (A.80) we then obtain the field equations

$$\frac{1}{2} R^{ab} \wedge V^c \epsilon_{abcd} = -\chi \theta_d, \tag{A.82}$$

reproducing the Einstein gravitational equations as an equality between 3-forms, vector-valued in the tangent Minkowski manifold.

In order to switch to the standard tensor language let us extract the components of the forms using the definitions (A.42), (A.81), and multiply by the totally anti-symmetric tensor $\epsilon^{\mu\nu\alpha\beta}$. The left-hand side of Eq. (A.82) then gives

$$\frac{1}{4} R_{\mu\nu}{}^{ab} V^a_\alpha \epsilon_{abcd} \epsilon^{\mu\nu\alpha\beta} = R_d{}^\beta - \frac{1}{2} V_d^\beta R, \tag{A.83}$$

where we have used the result of Exercise 12.4 (Eq. (12.75)). The right-hand side gives

$$-\frac{\chi}{3!}\theta_d{}^i \epsilon_{iabc} \epsilon^{abc\beta} = \chi \theta_d{}^\beta. \tag{A.84}$$

The field equation (A.82) thus provides the tensor equality

$$G_d{}^\beta = \chi \theta_d{}^\beta, \tag{A.85}$$

where $G_d{}^\beta$ is the Einstein tensor (A.83).

The above equations are not completely determined, however, until we have not specified the connection to be used for the computation of the curvature, of the Einstein tensor, and of the energy-momentum tensor of the sources. To this aim we must consider the second field equation, obtained by varying the action (A.77) with respect to ω.

We start with the variation of the curvature $R^{ab}(\omega)$. From the definition (A.42) we have

$$\delta_\omega R^{ab} = d\delta\omega^{ab} + \delta\omega^a{}_c \wedge \omega^{cb} + \omega^a{}_c \wedge \delta\omega^{cb}$$

$$= d\delta\omega^{ab} + \omega^{a}{}_{c} \wedge \delta\omega^{cb} + \omega^{b}{}_{c} \wedge \delta\omega^{ac}$$
$$\equiv D\delta\omega^{ab}. \tag{A.86}$$

Let us now consider the gravitational action. Using the result (A.86), the definition of torsion (A.47), and the property $D\epsilon_{abcd} = 0$ (see Sect. A.2), we obtain

$$\delta_\omega S_g = \frac{1}{4\chi} \int D\delta\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}$$
$$= \frac{1}{4\chi} \int \left[D\left(\delta\omega^{ab} \wedge V^c \wedge V^d\right) + 2\delta\omega^{ab} \wedge R^c \wedge V^d \right] \epsilon_{abcd} \tag{A.87}$$

(for the sign of the last term we have used Eq. (A.40)).

The first term of the above integral corresponds to a total divergence and can be expressed, thanks to the Gauss theorem, in the form of a boundary contribution. In fact, it is the four-volume integral of the exterior covariant derivative of a scalar-valued 3-form, i.e. it is an integral of the type

$$\int_\Omega DA = \int_\Omega dA = \int_\Omega \partial_{[\mu} A_{\nu\alpha\beta]}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$
$$= \int_\Omega \partial_\mu \left(A_{\nu\alpha\beta} \eta^{\mu\nu\alpha\beta} \sqrt{-g} \right) d^4x = \int_{\partial\Omega} dS_\mu \sqrt{-g}\, \eta^{\mu\nu\alpha\beta} A_{\nu\alpha\beta}, \tag{A.88}$$

where, in our case,

$$A = \delta\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} \tag{A.89}$$

(we have used Eq. (A.23) and the Gauss theorem). Since A is proportional to $\delta\omega$ its contribution is vanishing, because the variational principle requires $\delta\omega = 0$ on the boundary $\partial\Omega$. We are thus left only with the second term of Eq. (A.87), which gives

$$\delta_\omega S_g = \frac{1}{2\chi} \int \delta\omega^{ab} \wedge R^c \wedge V^d \epsilon_{abcd}. \tag{A.90}$$

There is, however, a further possible contribution from the matter action S_m, whose variation with respect to ω can be expressed, in general, as

$$\delta_\omega S_m = \int \delta\omega^{ab} \wedge S_{ab}, \tag{A.91}$$

where $S_{ab} = -S_{ba}$ is an antisymmetric, tensor-valued 3-form related to the canonical density of intrinsic angular momentum. Its explicit from depends on the considered model of source (see the examples given below). Adding the two contributions (A.90) and (A.91) we finally obtain the relation

$$\frac{1}{2} R^c \wedge V^d \epsilon_{abcd} = -\chi S_{ab}, \tag{A.92}$$

which represents the field equation for the connection. Solving for ω, and inserting the result into Eq. (A.82), we have fully specified the geometry of the given model of gravity, and we can solve the equations to determine the corresponding dynamics. The two equations (A.82), (A.92) are also called *Einstein–Cartan equations*.

In the particular case in which there are no contributions to Eq. (A.92) from the matter sources—or the contributions S_{ab} are present, but are physically negligible—one obtains that the torsion is zero, and recovers the Einstein field equations of general relativity. In fact, if we rewrite Eq. (A.92) in tensor components, antisymmetrize, and recall the rule (12.74), we arrive at the condition

$$\frac{1}{2} Q_{[\mu\nu}{}^{c} V^{d}_{\alpha]} \epsilon_{abcd} \epsilon^{\mu\nu\alpha\beta} = \frac{1}{2} Q_{\mu\nu}{}^{c} V^{\mu\nu\beta}_{abc} = 0, \tag{A.93}$$

namely at

$$\begin{aligned} &\frac{1}{2}\left(Q_{ab}{}^{c} V^{\beta}_{c} + Q_{bc}{}^{c} V^{\beta}_{a} + Q_{ca}{}^{c} V^{\beta}_{b} - Q_{ac}{}^{c} V^{\beta}_{b} - Q_{ba}{}^{c} V^{\beta}_{c} - Q_{cb}{}^{c} V^{\beta}_{a}\right) \\ &\quad = Q_{ab}{}^{\beta} + Q_{b} V^{\beta}_{a} - Q_{a} V^{\beta}_{b} = 0, \end{aligned} \tag{A.94}$$

where $Q_b \equiv Q_{bc}{}^{c}$. Multiplying by V^{b}_{β} we find that the trace must be vanishing, $Q_a = 0$, and Eq. (A.94) reduces to:

$$Q_{ab}{}^{c} \equiv 0. \tag{A.95}$$

The condition of vanishing torsion, on the other hand, can also be written as $R^a = DV^a = 0$, namely as

$$D_{[\mu} V^{a}_{\nu]} \equiv \partial_{[\mu} V^{a}_{\nu]} + \omega_{[\mu}{}^{a}{}_{\nu]} = 0. \tag{A.96}$$

This equation, solved for ω, leads to the Levi-Civita connection of general relativity (see Eqs. (12.41)–(12.48) with $Q = 0$). With such a connection Eq. (A.85) exactly reduces to the Einstein field equations: to the left we recover the symmetric Einstein tensor, obtained from the usual Riemann tensor, and to the right we recover the symmetric (dynamical) energy-momentum tensor.

For a torsionless geometry, and in the language of the exterior forms, the covariant conservation law of the energy-momentum tensor can be obtained by computing the exterior covariant derivative of Eq. (A.82). In fact, the derivative of the left-hand side is identically vanishing,

$$\frac{1}{2} DR^{ab} \wedge V^{c} \epsilon_{abcd} = 0, \tag{A.97}$$

thanks to the second Bianchi identity (A.64). This immediately implies

$$D\theta_a = 0, \tag{A.98}$$

which reproduces to the conservation equation (7.35), when translated into the tensor language.

Let us notice, first of all, that Eq. (A.97) corresponds to the so-called "contracted Bianchi identity", written in the language of exterior forms. Switching to the tensor formalism—i.e. considering the components of the forms, and antisymmetrizing—we obtain, in fact:

$$\frac{1}{4}\nabla_\mu R_{\alpha\beta}{}^{ab} V_\nu^c \epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta} = 0. \tag{A.99}$$

We have replaced D_μ with ∇_μ because the difference between the two objects is represented by the contribution of the Christoffel symbols, which disappears after antisymmetrization in μ, α, β (see Eq. (A.72)). By using the result (12.75) for the product of the antisymmetric tensors the above equation then reduces to:

$$\nabla_\mu \left(R_c{}^\mu - \frac{1}{2} V_c^\mu R \right) = 0. \tag{A.100}$$

By exploiting the metricity condition $\nabla V = 0$ we can finally multiply by V_ν^c, and rewrite our result as

$$\nabla_\mu G_\nu{}^\mu = 0, \tag{A.101}$$

which coincides indeed with the contracted Bianchi identity (6.26).

Let us now consider the components of Eq. (A.98), use the definition (A.81), and antisymmetrize. By repeating the above procedure, and recalling that $\nabla_\mu \eta_{\rho\nu\alpha\beta} = 0$ (see Exercise 3.7), we get

$$\frac{1}{6}\nabla_\mu \theta_a{}^\rho \eta_{\rho\nu\alpha\beta}\eta^{\mu\nu\alpha\beta} = -\frac{1}{6}\nabla_\mu \theta_a{}^\mu = 0. \tag{A.102}$$

Multiplying by V_ν^a, and using $\nabla V = 0$, we finally arrive at the condition

$$\nabla_\mu \theta_\nu{}^\mu = 0, \tag{A.103}$$

which reproduces the covariant conservation of the energy-momentum tensor, in agreement with previous results (see Eq. (7.35)).

Example: Free Scalar Field

It is probably instructive to conclude our discussion of this generalized gravitational formalism with a simple example of matter field which is not source of torsion: a massless scalar field ϕ. Its action can be written (in units $\hbar = c = 1$):

$$S_m = -\frac{1}{2}\int d\phi \wedge {}^\star d\phi. \tag{A.104}$$

In fact, by applying the result (A.24) to the 1-form $d\phi$, we obtain

$$d\phi \wedge {}^\star d\phi = -d^4x \sqrt{-g}\, \partial_\mu \phi \partial^\mu \phi, \tag{A.105}$$

so that the above action exactly coincides with the canonical action (7.37) of a free scalar field (with $V(\phi) = 0$).

The variation with respect to ω—which does not appear in S_m—is trivially zero: we thus recover the torsionless condition (A.95), and the connection reduces to the standard form used in the context of general relativity.

The variation of the action (A.104) with respect to V represents a useful exercise for the calculus of exterior forms. Let us first notice that $\delta_V\, d\phi = 0$, and that a nonzero variational contribution is provided by the dual term only. By referring the dual to the tangent space basis we have, in particular:

$$^\star d\phi = \frac{1}{3!} V_i^\mu \partial_\mu \phi \, \epsilon^i{}_{abc} V^a \wedge V^b \wedge V^c. \tag{A.106}$$

Therefore:

$$\begin{aligned} \delta_V \left(^\star d\phi \right) = & \frac{1}{2} \partial^i \phi \, \epsilon_{iabc} \delta V^a \wedge V^b \wedge V^c \\ & - \frac{1}{3!} \delta V_\mu^j \partial_j \phi V_i^\mu \epsilon^i{}_{abc} V^a \wedge V^b \wedge V^c, \end{aligned} \tag{A.107}$$

where we have used the identity

$$\left(\delta V_i^\mu\right) V_\mu^j = -\left(\delta V_\mu^j\right) V_i^\mu, \tag{A.108}$$

following from the relation $V_\mu^j V_i^\mu = \delta_i^j$. Using again the definition of dual, we can rewrite Eq. (A.107) in compact form as follows:

$$\delta_V \left(^\star d\phi \right) = \partial^i \phi \delta V^a \wedge {}^\star (V_i \wedge V_a) - \partial_j \phi \, {}^\star \delta V^j. \tag{A.109}$$

The variation of the scalar-field action thus takes the form

$$\begin{aligned} \delta_V S_m &= -\frac{1}{2} \int \left[\partial^a \phi \, d\phi \wedge \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_a \phi \, d\phi \wedge {}^\star \delta V^a \right] \\ &= -\frac{1}{2} \int \left[\partial^a \phi \, d\phi \wedge {}^\star (V_a \wedge V_b) \wedge \delta V^b + \partial_a \phi \, {}^\star d\phi \wedge \delta V^a \right] \end{aligned} \tag{A.110}$$

(in the second step we have used, for the second term, the property $A \wedge {}^\star B = B \wedge {}^\star A$ which holds if the forms A and B are of the same degree). The field equation (A.82), in our case, becomes

$$\frac{1}{2} R^{ab} \wedge V^c \epsilon_{abcd} = \frac{\chi}{2} \left[\partial^a \phi \, d\phi \wedge {}^\star (V_a \wedge V_d) + \partial_d \phi \, {}^\star d\phi \right]. \tag{A.111}$$

The left-hand side, computed with a vanishing torsion, coincides with the usual symmetric Einstein tensor. Let us check that the right-hand side corresponds to the usual (symmetric) energy-momentum tensor of a massless scalar field.

By considering the components of the 3-form present on the right-hand side, and antisymmetrizing, we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{2}\partial^a\phi\partial_\mu\phi\,\epsilon_{adij}V^i_\nu V^j_\alpha\epsilon^{\mu\nu\alpha\beta}+\frac{1}{6}\partial_d\phi\partial^\rho\phi\,\eta_{\rho\mu\nu\alpha}\eta^{\mu\nu\alpha\beta}\right]\\
&\quad=-\frac{1}{2}\partial^a\phi\partial_\mu\phi\big(V^\mu_a V^\beta_d-V^\beta_a V^\mu_d\big)+\frac{1}{2}\partial_d\phi\partial^\beta\phi\\
&\quad=\partial_d\phi\partial^\beta\phi-\frac{1}{2}V^\beta_d\big(\partial_\mu\phi\partial^\mu\phi\big)=\theta_d{}^\beta,
\end{aligned}\tag{A.112}$$

which coincides indeed with the canonical tensor of Eq. (7.40) (for $V=0$).

A.4.2 Spinning Sources and Riemann–Cartan Geometry

As a simple example of space–time geometry with nonvanishing torsion we will consider here a model in which the gravitational source is a massless Dirac field, represented as a 0-form ψ, spinor-valued in the Minkowski tangent space. The matter action can then be written (in units $\hbar=c=1$) as

$$S_m=-i\int\overline{\psi}\gamma\wedge{}^\star D\psi,\tag{A.113}$$

where $\gamma=\gamma_a V^a$ is a 1-form, and ${}^\star D\psi$ is the 3-form obtained by dualizing the exterior covariant derivative of a spinor, defined according to Eq. (13.23). Using the result (A.24) we have, in fact,

$$-i\overline{\psi}\gamma\wedge{}^\star D\psi=i\overline{\psi}\gamma^\mu D_\mu\psi\,d^4x\,\sqrt{-g},\tag{A.114}$$

which leads to the covariant Dirac action (13.24).

By varying the spinor action with respect to V, and applying the definition (A.80), we obtain the 3-form

$$\theta_a=i\overline{\psi}\gamma_a{}^\star D\psi,\tag{A.115}$$

representing the gravitational source of the Einstein–Cartan gravitational equation (A.82). Note that this object is different from the dynamical energy-momentum tensor of the Dirac field computed in Exercise 13.3 (which is symmetric and acts as a source of the gravitational Einstein equations). In fact, by inserting θ_a in Eq. (A.82), extracting the components, antisymmetrizing, and finally projecting back to the curved space–time, we arrive at the following tensor equation:

$$G_{\alpha\beta}=i\chi\overline{\psi}\gamma_\alpha D_\beta\psi,\tag{A.116}$$

with a right-hand side which is explicitly *not* symmetric in α and β.

Such an asymmetry, inconsistent for the Riemann geometry, is appropriate instead to a Riemann–Cartan geometry with torsion. In that case, in fact, the left-hand side of Eq. (A.116) is to be computed with a non-symmetric affine connection (see Sect. 3.5), and turns out to be non-symmetric, unlike the usual Einstein tensor.

In order to explicitly compute the torsion produced by the Dirac source, the action (A.113) has to be varied with respect to the connection ω. We recall, to this aim, that

$$D\psi = d\psi + \frac{1}{4}\omega^{ab}\gamma_{[a}\gamma_{b]}\psi \tag{A.117}$$

(see Eq. (13.23)). We thus obtain

$$\begin{aligned}\delta_\omega S_m &= -\frac{i}{4}\int \overline{\psi}\gamma \wedge {}^\star\big(\delta\omega^{ab}\gamma_{[a}\gamma_{b]}\big)\psi \\ &= -\frac{i}{4}\int \delta\omega^{ab} \wedge \overline{\psi}\,{}^\star\gamma\gamma_{[a}\gamma_{b]}\psi,\end{aligned} \tag{A.118}$$

where ${}^\star\gamma = \gamma_c\,{}^\star V^c$, and where we have used the property $\gamma \wedge {}^\star\delta\omega = \delta\omega \wedge {}^\star\gamma$. By applying the definition (A.91) we find that the Einstein–Cartan equation (A.92) for the connection becomes

$$\frac{1}{2}R^c \wedge V^d\epsilon_{abcd} = \frac{i}{4}\chi\overline{\psi}\,{}^\star\gamma\gamma_{[a}\gamma_{b]}\psi. \tag{A.119}$$

The spinor current plays the role of source, and the torsion is no longer vanishing.

In order to obtain the explicit expression of the torsion tensor we must rewrite the above equation in components, and antisymmetrize. For the left-hand side we already know the result, reported in Eq. (A.94). By repeating the same procedure for the right-hand side we obtain

$$\frac{i}{4}\overline{\psi}\gamma^c\gamma_{[a}\gamma_{b]}\psi\frac{1}{6}V_c^\rho\eta_{\rho\mu\nu\alpha}\epsilon^{\mu\nu\alpha\beta} = \frac{i}{4}\overline{\psi}\gamma^\beta\gamma_{[a}\gamma_{b]}\psi, \tag{A.120}$$

and Eq. (A.119) becomes

$$Q_{ab}{}^\beta + Q_bV_a^\beta - Q_aV_b^\beta = \frac{i}{4}\chi\overline{\psi}\gamma^\beta\gamma_{[a}\gamma_{b]}\psi. \tag{A.121}$$

The multiplication by V_β^b now gives the torsion trace as

$$Q_a = i\frac{3}{8}\chi\overline{\psi}\gamma_a\psi, \tag{A.122}$$

so that, moving all trace terms to the right-hand side:

$$Q_{abc} = \frac{i}{4}\chi\overline{\psi}(\gamma_c\gamma_{[a}\gamma_{b]} - 3\eta_{c[a}\gamma_{b]})\psi. \tag{A.123}$$

By recalling the relations (13.34), (13.36) among the γ matrices we can finally rewrite the torsion tensor by explicitly separating the vector and axial-vector contributions of the Dirac current:

$$Q_{abc} = \frac{\chi}{4}\left(\epsilon_{abcd}\overline{\psi}\gamma^5\gamma^d\psi + i\overline{\psi}\gamma_{[a}\eta_{b]c}\psi\right). \tag{A.124}$$

Once the torsion is determined, the corresponding Lorentz connection is obtained by solving the metricity conditions for the vierbeins, and is given (according to Eqs. (12.46)–(12.48)) by

$$\omega_{cab} = \gamma_{cab} + K_{cab} \equiv \gamma_{cab} - (Q_{cab} - Q_{abc} + Q_{bca}), \tag{A.125}$$

where γ is the Levi-Civita connection. With $Q \neq 0$, the Lorentz curvature determined by ω contain the contributions of the contortion K and defines a non-symmetric Einstein tensor, thus modifying the field equations with respect to the equations of general relativity.

Another interesting consequence of the presence of torsion is the modification of the covariant form of the Dirac equation. The equation of motion following from the action (A.113), $i\gamma \wedge {}^\star D\psi = 0$, is still expressed in the standard form $i\gamma^\mu D_\mu\psi = 0$, but the covariant derivative (A.117) is referred to the connection (A.125). The presence of torsion then introduces into the spinor equation non-linear "contact" corrections, also called "Heisenberg terms".

They can be easily determined by inserting into the Lorentz connection the explicit torsion tensor (A.124), and separating the torsion contributions by defining

$$\begin{aligned} D &= d + \frac{1}{4}\gamma^{ab}\gamma_{[a}\gamma_{b]} + \frac{1}{4}K^{ab}\gamma_{[a}\gamma_{b]} \\ &= \overline{D} + \frac{1}{4}K^{ab}\gamma_{[a}\gamma_{b]}, \end{aligned} \tag{A.126}$$

where $\overline{D}$ is the spinor covariant derivative of general relativity (see Chap. 13), computed without torsion. We then obtain

$$\begin{aligned} i\gamma^\mu D_\mu\psi &= i\gamma^\mu\overline{D}_\mu\psi + \frac{i}{4}\gamma^\mu K_{\mu ab}\gamma^{[a}\gamma^{b]}\psi \\ &= i\gamma^\mu\overline{D}_\mu\psi + \frac{\chi}{16}\gamma^c\gamma^{[a}\gamma^{b]}\psi\left[\overline{\psi}(\gamma_b\eta_{ca} - \gamma_a\eta_{cb})\psi - i\epsilon_{abcd}\overline{\psi}\gamma^5\gamma^d\psi\right]. \end{aligned} \tag{A.127}$$

Non-linear terms of this type are required, for instance, in the covariant equation of the Rarita–Schwinger field to restore local supersymmetry, as already discussed in Sect. 14.3.

A.4.3 Example: A Simple Model of Supergravity

As a last application of the exterior calculus we will present here the action, and derive the corresponding field equations, for the $N = 1$ supergravity model of Sect. 14.3.

Representing the gravitino field as the 1-form $\psi = \psi_\mu dx^\mu$, spinor-valued in the tangent space, we can express the action for the Lagrangian (14.53) as follows,

$$S = \frac{1}{4\chi}\int R^{ab}\wedge V^c\wedge V^d\epsilon_{abcd} + \frac{i}{2}\int\overline{\psi}\wedge\gamma_5\gamma\wedge D\psi, \tag{A.128}$$

where $\gamma = \gamma_a V^a$, and where the operator D denotes the exterior, Lorentz-covariant derivative of Eq. (A.117).

The reformulation of the gravitational part of the action into the usual tensor language has already be presented in Eq. (A.75). For the spinor part of the action we can use Eq. (A.23), which leads to the more explicit form

$$\frac{i}{2}\overline{\psi}_\mu\gamma_5\gamma_\nu D_\alpha\psi_\beta\, dx^\mu\wedge dx^\nu\wedge dx^\alpha\wedge dx^\beta = \frac{i}{2}\overline{\psi}_\mu\gamma_5\gamma_\nu D_\alpha\psi_\beta\epsilon^{\mu\nu\alpha\beta}\, d^4x, \tag{A.129}$$

in full agreement with the Lagrangian (14.53).

The field equations are obtained by varying with respect to V, ω and $\overline{\psi}$. Starting with V we have

$$\begin{aligned}\delta_V S_{3/2} &= \frac{i}{2}\int\overline{\psi}\wedge\gamma_5\gamma_a\delta V^a\wedge D\psi\\ &= \frac{i}{2}\int\overline{\psi}\wedge\gamma_5\gamma_a D\psi\wedge\delta V^a.\end{aligned} \tag{A.130}$$

By adding the variation of the gravitational part of the action, Eq. (A.79), we immediately obtain

$$\frac{1}{2}R^{ab}\wedge V^c\epsilon_{abcd} = -\frac{i}{2}\chi\overline{\psi}\wedge\gamma_5\gamma_d D\psi. \tag{A.131}$$

Let us now translate this equation in the more convenient tensor language.

The tensor version of the left-hand side has been reported in Eq. (A.83). By extracting the tensor components of the right-hand side we are led to the equation

$$\begin{aligned}G_d{}^\beta &= -\frac{i}{2}\chi\overline{\psi}_\mu\gamma_5\gamma_d D_\nu\psi_\alpha\epsilon^{\mu\nu\alpha\beta}\\ &= \frac{i}{2}\chi\overline{\psi}_\mu\gamma_5\gamma_d D_\nu\psi_\alpha\epsilon^{\mu\nu\beta\alpha} \equiv \chi\theta_d{}^\beta,\end{aligned} \tag{A.132}$$

where $\theta_d{}^\beta$ is the canonical tensor (14.65). Hence, we exactly recover the result previously given in Eq. (14.64).

Let us now vary with respect to ω. By recalling the definition (A.117) of the spinor covariant derivative, and varying the gravitino action, we have

$$\delta_\omega S_{3/2} = \frac{i}{8}\int \delta\omega^{ab} \wedge \overline{\psi} \wedge \gamma_5 \gamma \gamma_{[a}\gamma_{b]} \wedge \psi. \tag{A.133}$$

By adding the variation of the gravitational action, Eq. (A.90), we arrive at the following field equation for the connection:

$$\frac{1}{2} R^c \wedge V^d \epsilon_{abcd} = -\frac{i}{8} \chi \overline{\psi} \wedge \gamma_5 \gamma \gamma_{[a}\gamma_{b]} \wedge \psi. \tag{A.134}$$

Let us notice that $\gamma = \gamma_c V^c_\nu \, dx^\nu = \gamma_\nu \, dx^\nu$, so that we can exploit the relation (14.58) to express the product of Dirac matrices $\gamma_5 \gamma_\nu \gamma_{[a}\gamma_{b]}$. By inserting the result into the above equation, and dropping terms which are vanishing for the anticommutation properties of the Majorana spinors (see Sect. 14.3.1), we are led to:

$$\begin{aligned} \frac{1}{2} R^c \wedge V^d \epsilon_{abcd} &= -\frac{1}{8} \chi \overline{\psi} \wedge V^c \gamma^d \wedge \psi \epsilon_{abcd} \\ &= -\frac{1}{8} \chi \overline{\psi} \gamma^c \wedge \psi \wedge V^d \epsilon_{abcd}. \end{aligned} \tag{A.135}$$

Note that in the second line we have used the property $V^c \wedge \psi = -\psi \wedge V^c$, and we have exchanged the names of the indices c and d. From the above equation, factorizing $V^d \epsilon_{abcd}$, we can immediately deduce that the torsion 2-form is given by

$$R^c = -\frac{1}{4} \chi \overline{\psi} \gamma^c \wedge \psi, \tag{A.136}$$

in agreement with the tensor result (14.60).

Let us finally vary the action with respect to $\overline{\psi}$. The result is the gravitino equation,

$$\frac{i}{2} \gamma_5 \gamma \wedge D\psi = 0. \tag{A.137}$$

By extracting the components, and antisymmetrizing, we arrive at the result

$$\frac{i}{2} \gamma_5 \gamma_\nu D_\alpha \psi_\beta \epsilon^{\mu\nu\alpha\beta} = 0, \tag{A.138}$$

which exactly reproduces the tensor equation (14.66).

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Appendix B
Higher-Dimensional Gravity

As already shown in various parts of this book (Chap. 11, Appendix A), there are no difficulties in writing the gravitational equations in space–time manifolds with a total number of dimensions $D > 4$. The problem, if any, is to understand the possible relevance/pertinence of such models for a geometric description of gravity at the macroscopic level, and find the possible corrections to the four-dimensional gravitational interactions induced by the presence of the extra dimensions.

Let us ask ourselves, first of all, why we should consider higher-dimensional models of gravity. The answer is simple: a higher-dimensional space–time is required by unified models of all fundamental interactions, such as supergravity and superstring models (see e.g. the books [7, 22, 41] of the bibliography). Ten-dimensional superstring theory, in particular, is at present the only theory able to unify gravity with all the other gauge interactions, as well as to provide a model of quantum gravity valid at all energy scales.

Given that a complete and theoretically consistent model of gravity needs to be formulated in a higher-dimensional space–time manifold, the question then becomes: how can we deduce, from such a model, the equations governing the gravitational interactions in $D = 4$?

The answer is provided by the so-called mechanism of "dimensional reduction", which basically tells us how our four-dimensional Universe is embedded into the higher-dimensional space. In this appendix we will briefly discuss two possibilities: the "old" Kaluza–Klein scenario, where the extra dimensions are compactified on a very small length scale, and the new "brane-world" scenario, where all fundamental interactions (but gravity) are confined on a four-dimensional "slice" of a higher-dimensional "bulk" manifold.

As in the case of Appendix A, it should be clearly stressed that the aim of this appendix is that of providing only a first, pedagogical introduction to the above-mentioned problems. The interested reader is referred to other books for an exhaustive presentation of this subject and for the discussion of its many aspects and problems (see e.g. the book [5] of the bibliography for the Kaluza–Klein scenario).

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B.1 Kaluza–Klein Gravity

The simplest example of higher-dimensional model gravity was provided almost one century ago by Kaluza and Klein [29, 30], and was inspired by the wish of providing a geometric description not only of gravity but also of the other fundamental interaction known at that time, namely the electromagnetic interaction. The basic idea was that of interpreting the electromagnetic potential A_μ as a component of the metric of a five-dimensional space–time $\mathcal{M}_5$, and the $U(1)$ gauge symmetry as an isometry of the five-dimensional geometry. This idea, as we shall see, can be extended (in principle) also to non-Abelian gauge fields, in the context of higher-dimensional manifolds with the appropriate geometric (and isometric) structure.

But let us start with the simple case of pure $D = 5$ gravity, described by the action

$$S = -\frac{M_5^3}{2} \int dx^5 \sqrt{|\gamma_5|}\, R_5. \tag{B.1}$$

Here γ_5 is the determinant of the five-dimensional metric γ_{AB}, R_5 is the Riemann scalar curvature computed from γ_{AB}, and $M_5^3 \equiv (8\pi G_5)^{-1}$ is the mass scale determining the effective gravitational coupling constant G_5 of the five-dimensional space–time $\mathcal{M}_5$. Note that we are working in units $\hbar = c = 1$ and that, in these units, the D-dimensional coupling constant has dimensions $[G_D] = L^{D-2} = M^{2-D}$. In $D = 4$ the coupling is controlled by the usual Newton constant G, related to the Planck-length (or mass) scale by $8\pi G = \lambda_{\rm P}^2 = M_{\rm P}^{-2}$.

A D-dimensional (symmetric) metric tensor has in general $D(D+1)/2$ independent components, which become 15 in $D = 5$. It is thus always possible to parametrize γ_{AB} in terms of a 4-dimensional metric tensor $g_{\mu\nu}$ (with 10 independent components), a 4-dimensional vector A_μ (with four independent components) and a scalar ϕ (with one independent component). Including (for later convenience) a possible conformal rescaling of γ_{AB} we can thus set:

$$\gamma_{AB} = w(\phi)\, \overline{\gamma}_{AB}, \tag{B.2}$$

where $w(\phi)$ is a positive (but arbitrary) scalar function of ϕ, and where

$$\overline{\gamma}_{\mu\nu} = g_{\mu\nu} - \phi A_\mu A_\nu, \qquad \overline{\gamma}_{\mu 4} = \overline{\gamma}_{4\mu} = \phi A_\mu, \qquad \overline{\gamma}_{44} = -\phi. \tag{B.3}$$

Conventions: Greek indices run from 0 to 3, capital Latin indices from 0 to 4, and we are assuming that ϕ is positive. The fifth dimension corresponds to the index 4. The inverse metric is given by $\gamma^{AB} = w^{-1}\overline{\gamma}_{AB}$, where

$$\overline{\gamma}^{\mu\nu} = g^{\mu\nu}, \qquad \overline{\gamma}^{\mu 4} = \overline{\gamma}^{4\mu} = A^\mu = g^{\mu\alpha} A_\alpha, \qquad \overline{\gamma}^{44} = -\phi^{-1} + g^{\alpha\beta} A_\alpha A_\beta, \tag{B.4}$$

and where $g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu$. It can be easily checked that $\gamma_{AC}\gamma^{CB} = \delta^B_A$.

The parametrization of γ_{AB} in terms of the multiplet of dimensionless fields $\{g_{\mu\nu}, A_\mu, \phi\}$ is fully general, up to now, but useful in our context to discuss the transformation properties of the metric under particular coordinate transformations.

In fact, let us consider the chart $z^A = \{x^\mu, y\}$ (we have called y the fifth coordinate z^4), and the coordinate transformation

$$x'^\mu = x^\mu, \qquad y'^\mu = y + f(x). \tag{B.5}$$

By applying the standard transformations rule of the metric tensor, Eq. (2.18), we readily obtain

$$\begin{aligned} g'_{\mu\nu}(x, y') &= g_{\mu\nu}(x, y), \qquad A'_\mu(x, y') = A_\mu(x, y) + \partial_\mu f(x), \\ \phi'(x, y') &= \phi(x, y). \end{aligned} \tag{B.6}$$

The result for A_μ suggests that a geometric model which is isometric with respect to the transformation (B.5) should include an Abelian gauge symmetry, associated to the vector component A_μ of the metric tensor.

That this is indeed the case is confirmed by the so-called "dimensional reduction" of the model from $\mathcal{M}_5$ down to our 4-dimensional space–time $\mathcal{M}_4$. The Kaluza–Klein approach to this process is based on the topological assumption that $\mathcal{M}_5$ has the product structure $\mathcal{M}_5 = \mathcal{M}_4 \otimes S_1$, where S_1 is a compact one-dimensional space, topologically equivalent to a circle of radius L_c, and then parametrized by a coordinate y such that $0 \le y \le 2\pi L_c$. In that case any field defined on $\mathcal{M}_5$ (including $g_{\mu\nu}$, A_μ and ϕ) is periodic in y, and can be expanded in Fourier series as

$$\begin{aligned} g_{\mu\nu}(z) &= \sum_{n=-\infty}^{\infty} g^{(n)}_{\mu\nu}(x) e^{iny/L_c}, \\ A_\mu(z) &= \sum_{n=-\infty}^{\infty} A^{(n)}_\mu(x) e^{iny/L_c}, \\ \phi(z) &= \sum_{n=-\infty}^{\infty} \phi^{(n)}(x) e^{iny/L_c}, \end{aligned} \tag{B.7}$$

where all Fourier components satisfy the reality condition, i.e. $(g^{(n)}_{\mu\nu})^* = g^{(-n)}_{\mu\nu}$, and so on.

Once the y-dependence is known, dimensional reduction is achieved by inserting these field components into the action (B.1) and integrating over the fifth coordinate. The result will be an effective four-dimensional action involving the (complicated) mutual interactions of the infinite "towers" of four-dimensional fields[1] (the Fourier modes $g^{(n)}_{\mu\nu}$, $A^{(n)}_\mu$, $\phi^{(n)}$) which, at least in the flat-space and perturbative regime, are characterized by a mass which is growing with n, i.e. $m_n = n/L_c$.

[1] Such an action is also characterized by an infinite number of four-dimensional symmetries, as we may discover by Fourier expanding like in Eq. (B.7) the parameters ξ^A of the infinitesimal coordinate transformation $z^A \to z^A + \xi^A(x^\mu, y)$. In fact, the assumed topology of $\mathcal{M}_5$ restrict us to coordinate transformations periodic in y, i.e. $\xi^A = \sum_n \xi^A_{(n)}(x) e^{iny/L_c}$ (see [12]).

This (low-energy) value of the mass can be easily obtained by expanding the full action around the trivial Minkowski background, $\gamma_{AB} = \eta_{AB} + h_{AB} + \cdots$. One then finds that the fluctuations h_{AB} satisfy in vacuum the five-dimensional d'Alembert equation,

$$\left(\partial_0^2 - \nabla^2 - \partial_y^2\right) h_{AB} = 0, \tag{B.8}$$

and that their Fourier components, taking into account the periodicity condition (B.7), are of the form $h \sim \exp(-ik_\mu x^\mu + iny/L_c)$. Hence they satisfy the dispersion relation

$$-\omega^2 + k^2 + \frac{n^2}{L_c^2} = 0, \tag{B.9}$$

typical of massive modes with $m^2 = n^2/L_c^2$.

If we assume that L_c is very small (after all, as we shall see in a moment, the size of the fifth dimension has to be small enough to explain why it cannot be experimentally resolved at the present available energies), it follows that the massive modes with $n \neq 0$ must be very heavy. In the low-energy limit we can thus limit ourselves (at least in first approximation) to the zero modes only, assuming that all fields appearing in the Kaluza–Klein model are independent of the fifth coordinate y. In such a simplified case we can check explicitly that the model describes a four-dimensional gravitational field $g_{\mu\nu}^{(0)}$, a massless scalar $\phi^{(0)}$ and an Abelian gauge vector $A_\mu^{(0)}$.

In fact, let us compute explicitly the action (B.1) with the metric (B.2), (B.3), assuming that g, A, ϕ depend only on x (and omitting the zero-mode index (0), for simplicity). The metric determinant is given by

$$\sqrt{|\gamma_5|} = \sqrt{-g}\, \phi^{1/2} w^{5/2}(\phi), \tag{B.10}$$

where $g = \det g_{\mu\nu}$. For a better illustration of the role played by the conformal factor $w(\phi)$ it is convenient to express the scalar curvature $R_5(\gamma)$, appearing in the action, in terms of the scalar curvature $\overline{R}_5(\overline{\gamma})$ computed for the conformally related metric $\overline{\gamma}_{AB}$. By recalling the general result for the conformal rescaling of the scalar curvature (see e.g. the book [19] of the bibliography) we obtain, for $\gamma_{AB} = w\,\overline{\gamma}_{AB}$ in $D = 5$,

$$R_5(\gamma) = w^{-1}\left[\overline{R}_5(\overline{\gamma}) - 4\overline{\nabla}_A \overline{\nabla}^A \ln w - 3(\overline{\nabla}_A \ln w)\left(\overline{\nabla}^A \ln w\right)\right] \tag{B.11}$$

(the symbol $\overline{\nabla}_A$ denotes the covariant derivative computed with the metric $\overline{\gamma}$). The five-dimensional action (B.1) then becomes

$$S = -\frac{M_5^3}{2} \int_0^{2\pi L_c} dy \int d^4x \sqrt{|\gamma_5|} R_5(\gamma)$$

$$= -\frac{M_5^3}{2}\int_0^{2\pi L_c} dy \int d^4x\,\sqrt{-g}\,\phi^{1/2} w^{3/2}(\phi)\big[\overline{R}_5(\overline{\gamma}) - 4\overline{\nabla}_A\big(\partial^A \ln w\big) - 3(\partial_A \ln w)\big(\partial^A \ln w\big)\big], \tag{B.12}$$

where we have replaced $\overline{\nabla}_A \ln w$ with $\partial_A \ln w$, since w is a scalar. By recalling that $\sqrt{|\overline{\gamma}_5|} = \sqrt{-g}\,\phi^{1/2}$ we have, also,

$$\overline{\nabla}_A\big(\partial^A \ln w\big) = \frac{1}{\sqrt{-g}\sqrt{\phi}}\,\partial_A\big(\sqrt{-g}\sqrt{\phi}\,\partial^A \ln w\big) = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\,\partial^\mu \ln w\big) + \frac{1}{2}\big(\partial^\mu \ln w\big)(\partial_\mu \ln \phi), \tag{B.13}$$

where we have replaced the index A with the index μ everywhere, since we are considering the limit in which all fields are independent of the fifth coordinate.

It is now evident, from the action (B.12), that by choosing $w(\phi) = \phi^{-1/3}$, i.e. $\ln w = -(1/3)\ln\phi$, we can eliminate the non-minimal coupling to ϕ present in the four-dimensional part of the integration measure. With such a choice the measure reduces to the canonical form $d^4x\sqrt{-g}$, hence the first term in the second line of Eq. (B.13) contributes to the action as a total divergence (and can be dropped), while the second term becomes quadratic in the first derivatives of $\ln\phi$, and contributes to the kinetic part of the scalar action (together with the last term of Eq. (B.12)). The action then reduces to:

$$S = -\frac{M_5^3}{2}\int_0^{2\pi L_c} dy \int d^4x\,\sqrt{-g}\left[\overline{R}_5(\overline{\gamma}) + \frac{1}{3}(\partial_\mu \ln\phi)\big(\partial^\mu \ln\phi\big)\right]. \tag{B.14}$$

Let us now evaluate the contribution directly arising from the scalar curvature of the five-dimensional metric $\overline{\gamma}_{AB}$. An explicit computation leads to

$$\sqrt{-g}\,\overline{R}_5(\overline{\gamma}) = \sqrt{-g}\left[R(g) + \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu \ln\phi)\big(\partial^\mu \ln\phi\big)\right], \tag{B.15}$$

modulo a total divergence. Here $R(g)$ is the scalar curvature associated to the four-dimensional metric $g_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By inserting this result into Eq. (B.14), integrating over y, and defining $\sigma = -(1/\sqrt{3})\ln\phi$, we finally end up with the action

$$S = -\frac{M_{\rm P}^2}{2}\int d^4x\,\sqrt{-g}\left(R + \frac{e^{-\sqrt{3}\sigma}}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma\right), \tag{B.16}$$

where we have identified the effective four-dimensional gravitational coupling with the usual Newton constant by setting:

$$M_{\rm P}^2 \equiv (8\pi G)^{-1} = 2\pi L_c M_5^3. \tag{B.17}$$

Note that the ratio between the four- and five-dimensional coupling constants turns out to be controlled by the compactification scale L_c. In particular, if the coupling

strength of $D = 5$ gravity is the same as in $D = 4$, i.e. $M_5 \sim M_{\rm P}$, then the size of the compact five dimension must be in the Planck-length range, $L_c \sim M_{\rm P}^{-1} \sim \lambda_{\rm P}$.

The above dimensionally reduced action, Eq. (B.16), shows that the zero-mode content of a five-dimensional theory of pure gravity with one spatial dimension compactified on a circle can reproduce a canonical model of four-dimensional gravity, coupled to an Abelian gauge vector A_μ and to a scalar "dilaton" field σ. It should be noted, in this context, that we have the interesting appearance of a non-minimal scalar-vector coupling in front of the standard Maxwell Lagrangian. The vector field, however, has to be appropriately rescaled ($A_\mu \to M_{\rm P} A_\mu/\sqrt{2}$) in order to match the usual canonical normalization.

B.1.1 Dimensional Reduction from $D = 4 + n$ Dimensions

The geometric description of gauge fields based on the Kaluza–Klein model of dimensional reduction can be extended to the case of non-Abelian symmetries, provided we consider space–time manifolds with a higher number of compact dimensions. The gauge group of the dimensionally reduced model corresponds, in that case, to the non-Abelian isometry group of the compact spatial dimensions.

Let us consider a space–time manifold $\mathcal{M}_D$ with $D = 4 + n$ dimensions and with a topological structure $\mathcal{M}_d = \mathcal{M}_4 \otimes \mathcal{K}_{D-4}$, where $\mathcal{K}_{D-4}$ is a compact n-dimensional space characterized by an isometry group G generated by a set of N Killing vectors $\{K^m_{(i)}\}$, where $i, j = 1, 2, \ldots, N$. Conventions: here and in the following subsections we will split the D-dimensional coordinates as $z^A = (x^\mu, y^m)$, where x^μ, with $\mu, \nu = 0, 1, 2, 3$, will denote coordinates on $\mathcal{M}_4$, while y^m, with $m, n = 4, 5, \ldots, D-1$, will denote coordinates on $\mathcal{K}_{D-4}$. The indices i, j, instead, are running over the N generators of the isometry group.

Suppose that the group is non-Abelian, namely that the Killing vectors $K^m_{(i)}$ satisfy a closed (non-trivial) algebra of commutation relations. Considering the differential operator $K_i \equiv K^m_i \partial_m$ (from now on we will omit, for simplicity, the round brackets on the group indices), and computing the commutation brackets

$$[K_i, K_j] = \left(K^m_i \partial_m K^n_j - K^m_j \partial_m K^n_i\right)\partial_n, \tag{B.18}$$

it can be easily shown that, if K_i and K_j are Killing vectors, then the right-hand side of the above equation is a Killing vector, too (recall the Killing properties illustrated in Sect. 3.3 and Exercise 3.4). We can thus write the general commutation rule

$$[K_i, K_j] = f_{ij}{}^k K_k, \quad i, j, k = 1, 2, \ldots, N, \tag{B.19}$$

where $f_{ij}{}^k = -f_{ji}{}^k$ are the structure constant of the given isometry group G.

Let us now generalize the previous parametrization of the higher-dimensional metric tensor γ_{AB} by introducing, in D dimensions, a symmetric 4×4 tensor $g_{\mu\nu}$, a symmetric $(D-4) \times (D-4)$ tensor ϕ_{mn}, and $D-4$ four-dimensional vectors

B_μ^m (the total number of components is again $D(D+1)/2$, as appropriate to γ_{AB}). More precisely, we shall use the following general ansatz:

$$\gamma_{AB} = w \begin{pmatrix} g_{\mu\nu} - \phi_{mn} B_\mu^m B_\nu^n & \phi_{mp} B_\mu^p \\ \phi_{np} B_\nu^p & -\phi_{mn} \end{pmatrix}, \tag{B.20}$$

where we have inserted also the so-called "warp" factor $w(\phi)$ (a function of $\phi \equiv \det \phi_{mn}$), possibly useful to restore the canonical normalization of the kinetic terms in the dimensionally reduced action. By computing $\gamma = \det \gamma_{AB}$ we obtain

$$\sqrt{|\gamma|} = w^{D/2} |\phi|^{1/2} \sqrt{|g|}, \tag{B.21}$$

and the inverse metric is given by

$$\gamma^{AB} = w^{-1} \begin{pmatrix} g^{\mu\nu} & B_\alpha^m g^{\mu\alpha} \\ B_\alpha^n g^{\nu\alpha} & -\phi^{mn} + g^{\alpha\beta} B_\alpha^m B_\beta^n \end{pmatrix}, \tag{B.22}$$

where $g^{\mu\alpha} g_{\nu\alpha} = \delta_\nu^\mu$ and $\phi^{mp} \phi_{pn} = \delta_n^m$.

We are now in the position of exploiting the isometries of the factorized geometry and showing that, after an appropriate dimensional reduction, to each one of the N isometries of the compact manifold $\mathcal{K}_{D-4}$ we can associate a vector transforming as a non-Abelian gauge potential of the reduced four-dimensional theory.

Following (and extending to higher D) the Kaluza–Klein mechanism of the previous section, we shall implement the dimensional reduction by considering an effective low-energy limit (a sort of "ground state" configuration) in which $g_{\mu\nu}$ depends only on x, ϕ_{mn} is constant in four-dimensional space–time (but may depend on y), and the vectors B_μ depend on x and may also depend on y, but only through the y-dependence of the Killing vectors. We thus set:

$$g_{\mu\nu} = g_{\mu\nu}(x), \qquad \phi_{\mu\nu} = \phi_{mn}(y), \qquad B_\mu^m(x, y) = A_\mu^i(x) K_i^m(y). \tag{B.23}$$

The metric $g_{\mu\nu}(x)$ and the N vector fields $A_\mu^i(x)$ (associated to the Killing generators K_i) play the role of the "zero-mode" fields $g_{\mu\nu}$, A_μ of the $D = 5$ model of the previous section. Let us now check that A_μ^i transforms as a non-Abelian gauge vector under the action of the isometry group G.

Consider an infinitesimal coordinate transformation $z'^A = z^A + \xi^A$, with generator

$$\xi^A = (\xi^\mu, \xi^m), \qquad \xi^\mu = 0, \qquad \xi^m(x, y) = \epsilon^i(x) K_i^m(y). \tag{B.24}$$

We know that the local infinitesimal variation of the D-dimensional metric can be written, in general, as

$$\delta\gamma_{AB} = -\xi^M \partial_M \gamma_{AB} - \gamma_{AM} \partial_B \xi^M - \gamma_{BM} \partial_A \xi^M \tag{B.25}$$

(see Eq. (3.53)). Let us concentrate on the variation of the mixed components $\gamma_{\mu m}$ which are given, according to Eqs. (B.20) and (B.23), by

$$\gamma_{\mu m} = B_\mu^n \phi_{mn} = A_\mu^i(x) K_{im}(y). \tag{B.26}$$

For the infinitesimal transformation with generator (B.24) we obtain

$$\delta\gamma_{\mu m} = -\gamma_{mn}\partial_\mu \xi^n - \gamma_{\mu n}\partial_m \xi^n - \xi^n \partial_n \gamma_{\mu m}, \tag{B.27}$$

from which, by taking into account the x and y dependence of γ, A, ϵ and K (see Eqs. (B.23), (B.24), (B.26)), we have

$$\delta\big(A^i_\mu K_{im}\big) = K_{im}\partial_\mu \epsilon^i - A^i_\mu K_{in}\big(\partial_m K^n_j\big)\epsilon^j - \epsilon^j K^n_j(\partial_n K_{im})A^i_\mu. \tag{B.28}$$

For the last term we can now use the algebra of the isometry group given in Eqs. (B.18), (B.19), which implies

$$K^n_j \partial_n K_{im} = K^n_i \partial_n K_{jm} + f_{ji}{}^k K_{km}. \tag{B.29}$$

Inserting this result into the last term of Eq. (B.28) we find, after renaming indices:

$$\begin{aligned}\delta\big(A^i_\mu K_{im}\big) = {} & K_{im}\big(\partial_\mu \epsilon^i - f_{kl}{}^i \epsilon^k A^l_\mu\big) \\ & - A^i_\mu \epsilon^j \big(K^n_i \partial_n K_{jm} + K_{in}\partial_m K^n_j\big).\end{aligned} \tag{B.30}$$

The contribution of the second line is identically vanishing thanks to the basic property of the Killing vectors $\widetilde{\nabla}_n K_m + \widetilde{\nabla}_m K_n = 0$ (see Exercise 3.4), where $\widetilde{\nabla}$ denotes the covariant derivative computed with the metric ϕ_{mn} of the compact space $\mathcal{K}_{D-4}$. In fact, for any given (fixed) pair of Killing vectors, of indices i and j, we have

$$\begin{aligned}& K^n_i \partial_n K_{jm} + K_{in}\partial_m K^n_j \\ & \quad = K^n_i \big(\partial_n K_{jm} + \partial_m K_{jn} - \widetilde{\Gamma}_{nm}{}^p K_{jp} - \widetilde{\Gamma}_{mn}{}^p K_{jp}\big) \\ & \quad = K^n_i \big(\widetilde{\nabla}_n K_{jm} + \widetilde{\nabla}_m K_{jn}\big) \equiv 0,\end{aligned} \tag{B.31}$$

where $\widetilde{\Gamma} = \Gamma(\phi)$, and where we have eliminated the partial derivatives of ϕ^{mn} by using the metricity condition $\widetilde{\nabla}_m \phi^{np} = 0$.

Finally, by considering a local variation of the vector A^i_μ at fixed K_i (namely, the field A_μ and the transformed field $A_\mu + \delta A_\mu$ are projected on the same Killing vectors), we have $\delta(A^i_\mu K_{im}) = K_{im}\delta A^i_\mu$, and we can rewrite the result (B.30) as

$$\delta A^i_\mu(x) = \partial_\mu \epsilon^i(x) - f_{kl}{}^i \epsilon^k(x) A^l_\mu(x). \tag{B.32}$$

This is clearly the infinitesimal transformation of the gauge potential of a non-Abelian symmetry group, with local parameter ϵ^i and structure constants $f_{ij}{}^k$.

In fact, let us consider the gauge transformation for the non-Abelian vector potential A_μ already derived (in finite form) in Eq. (12.18), and expand the group representation (12.10) as

$$U = 1 + i\epsilon^i X_i + \cdots, \tag{B.33}$$

where the generators X_i satisfy the Lie algebra:

$$[X_i, X_j] = i f_{ij}{}^k K_k. \tag{B.34}$$

In order to match the notation of this section we are denoting with $i, j = 1, 2, \ldots, N$ the indices with values in the group algebra. Also, we will use units in which the gauge coupling constant of Chap. 12 is fixed to $g = 1$. By expanding Eq. (12.18) to first order in ϵ we thus obtain

$$A'^i_\mu X_i = A^i_\mu X_i + i\epsilon^i A^j_\mu (X_i X_j - X_j X_i) + X_i \partial_\mu \epsilon^i. \tag{B.35}$$

Hence, by using Eq. (B.34),

$$\delta A^i_\mu \equiv A'^i_\mu - A^i_\mu = \partial_\mu \epsilon^i - f_{kl}{}^i \epsilon^k A^l_\mu, \tag{B.36}$$

which exactly coincides with the isometry transformation (B.32).

It can be added that, by inserting the metric ansatz (B.20), (B.23) into the higher-dimensional Einstein action (and choosing an appropriate warp factor), we end up with the canonical form of the four-dimensional Einstein–Yang–Mills action with metric $g_{\mu\nu}(x)$ and gauge potential A^i_μ. In this context we also obtain an interesting generalization of Eq. (B.17), namely a relation between the size of the compact dimensions and the scale of the higher-dimensional gravitational coupling G_D, defined by $8\pi G_D = M_D^{2-D}$.

The expansion of the D-dimensional Einstein action around the ground state configuration (B.20), (B.23) gives, in fact:

$$\begin{aligned} &-\frac{M_D^{D-2}}{2} \int d^D z \sqrt{|\gamma|}\, R_D \\ &= -\frac{M_D^{D-2}}{2} \int_{\mathcal{K}_{D-4}} d^{D-4} y\, w^{D/2} |\det \phi_{mn}|^{1/2} \int_{\mathcal{M}_4} d^4 x \sqrt{|g|} \big[R(g) + \cdots\big]. \end{aligned} \tag{B.37}$$

Consider the pure gravity sector, and call V_{D-4} the proper (finite) hypervolume of the compact Kaluza–Klein extra-dimensional space (including a possible warp-factor contribution),

$$V_{D-4} = \int_{\mathcal{K}_{D-4}} d^{D-4} y\, w^{D/2}(y) \big|\det \phi_{mn}(y)\big|^{1/2}. \tag{B.38}$$

Comparing Eq. (B.37) with the four-dimensional Einstein action,

$$-\frac{M_{\rm P}^2}{2} \int d^4 x \sqrt{|g|}\, R(g), \tag{B.39}$$

we immediately obtain

$$M_D^{D-2} V_{D-4} = M_{\rm P}^2. \tag{B.40}$$

Since $M_{\rm P}$ is known ($M_{\rm P} = (8\pi G)^{-1} \simeq 2.4 \times 10^{18}$ GeV), this is a constraint connecting the strength of the higher-dimensional gravitational coupling to the size and the number of the compact extra dimensions.

Let us consider, for instance, the simple isotropic case with a compactification scale of size L_c, the same for all $D-4$ extra dimensions. Then $V_{D-4} \sim L_c^{D-4}$ and Eq. (B.40) reduces to

$$M_D^{D-2} L_c^{D-4} \sim M_{\rm P}^2. \tag{B.41}$$

Again (as in $D = 5$) we obtain that a D-dimensional coupling of Newtonian strength, $M_D \sim M_{\rm P}$, implies a Planckian compactification scale, $L_c \sim M_{\rm P}^{-1} \sim 10^{-33}$ cm. However, larger compactification scales are in principle allowed for smaller values of the mass M_D. Solving Eq. (B.41) for L_c we obtain, in general,

$$L_c \sim 10^{-17}\ {\rm cm} \left(\frac{1\ {\rm TeV}}{M_D}\right)^{(D-2)/(D-4)} 10^{30/(D-4)}. \tag{B.42}$$

We have referred M_D to the TeV scale since this scale is, in a sense, preferred because of theoretical "prejudices" related to the solution of the so-called "hierarchy" problem and of the cosmological constant problem.

Concerning the present observational results, we should mention the existence of gravitational experiments[2] excluding the presence of extra dimensions down to length scales $L_c \lesssim 10^{-2}$ cm. According to Eq. (B.42) this is compatible with $M_D \sim 1$ TeV for a number $D \geq 6$ of extra compact dimensions. However, high-energy experiments probing the standard model of strong and electroweak interactions have excluded (up to now) the presence of extra dimensions down to scales $L_c \lesssim 10^{-15}$ cm. This seems to suggest $M_D \gg 1$ TeV, or $M_D \sim 1$ TeV but with an unexpectedly large number of extra dimensions, unless—as we shall see in Sect. B.2—there is some mechanism able to confine gauge interactions inside three-dimensional space, making them insensitive to the extra dimensions.

Before discussing this interesting possibility let us come back to the Kaluza–Klein scenario, with a compact extra-dimensional space and a topological structure $\mathcal{M}_d = \mathcal{M}_4 \otimes \mathcal{K}_{D-4}$. There is a problem, in $D > 5$, due to the fact that if we impose on the higher-dimensional metric γ_{AB} to satisfy the vacuum Einstein equations, and we look for low-energy solutions in which $\mathcal{M}_4$ coincides with the flat Minkowski space–time ($g_{\mu\nu} = \eta_{\mu\nu}$), then we find, for consistency, that the manifold $\mathcal{K}_{D-4}$ has to be "Ricci flat". This means, more precisely, that the Ricci tensor of the metric ϕ_{mn} must satisfy the condition $\widetilde{R}_{mn}(\phi) = 0$.

This is possible, of course: the compact manifold, for instance, could be a torus, or a Calabi–Yau manifold used in the compactification of superstring models. A Ricci-flat manifold, however, only admits *Abelian* isometries (see e.g. the book [5] of the bibliography), hence all Killing vectors are commuting ($f_{ij}{}^k = 0$) and the previous example reduces to a model with N Abelian gauge fields (an almost trivial generalization of the $D = 5$ case).

[2] See for instance [1].

In order to solve this difficulty the model has to be generalized by dropping the original Kaluza–Klein idea that a physical four-dimensional model with gravity and matter fields can be derived from a pure gravity model in $D > 4$. We have to include non-geometric fields even in $D > 4$, possible representing non-Abelian gauge fields and/or sources of the extra-dimensional curvature contributing to $\widetilde{R}_{mn} \neq 0$.

The advantage, as we shall see in the next subsection, is that appropriate higher-dimensional matter fields can automatically trigger the splitting of $\mathcal{M}_D$ into the product of two maximally symmetric manifolds (one of which is compact), thus implementing the so-called mechanism of "spontaneous compactification".

B.1.2 Spontaneous Compactification

Among the various mechanisms of spontaneous compactification (based on antisymmetric tensor fields, Yang–Mills fields, quantum fluctuations, monopoles, instantons, generalized higher-curvature actions, ...), we will concentrate here on the case of the antisymmetric tensor fields, which has been inspired by the dimensional reduction of the supergravity theory formulated in $D = 11$ dimensions (and which also finds applications in the context of ten-dimensional superstring theory).

Let us start by considering the general D-dimensional action for gravity with matter sources,

$$S = -\frac{1}{2}\int d^D x \sqrt{|\gamma|}\, R(\gamma) + S_m, \tag{B.43}$$

where we have set to one the gravitational coupling, working in units where $8\pi G_D = M_D^{2-D} = 1$. The corresponding gravitational equations are

$$R_{AB} - \frac{1}{2}\gamma_{AB} R = T_{AB}, \tag{B.44}$$

where T_{AB} represents the contribution of S_m.

Let us look for background solutions in which the geometry of the D-dimensional space–time manifold can be factorized as the product of two maximally symmetric spaces, $\mathcal{M}_D = \mathcal{M}_4 \otimes \mathcal{M}_{D-4}$, with metric

$$\gamma_{\mu\nu} = g_{\mu\nu}(x), \qquad \gamma_{mn} = g_{mn}(y), \qquad \gamma_{\mu m} = 0, \tag{B.45}$$

and with the corresponding Ricci tensors satisfying the conditions

$$R_{\mu\nu} = -\Lambda_x g_{\mu\nu}, \qquad R_{mn} = -\Lambda_y g_{mn}, \qquad R_{\mu m} = 0, \tag{B.46}$$

where Λ_x and Λ_y are constant parameters (see e.g. Eq. (6.44)). This gives, for the D-dimensional scalar curvature,

$$R(\gamma) = g^{\mu\nu} R_{\mu\nu} + g^{mn} R_{mn} = -4\Lambda_x + (4 - D)\Lambda_y. \tag{B.47}$$

Note that (like in the previous sections) we are splitting the D-dimensional coordinates x^A into 4 coordinates x^μ, with Greek indices running from 0 to 3, and $D-4$ coordinates y^m, with Latin indices running from 4 to $D-1$.

The above form of background geometry is clearly compatible with the Einstein equations (B.44) provided the sources satisfy the conditions

$$T_{\mu\nu} = T_x g_{\mu\nu}, \qquad T_{mn} = T_y g_{mn}, \qquad T_{\mu m} = 0, \tag{B.48}$$

where T_x and T_y are constant parameters. Let us see that such conditions can be satisfied by the energy-momentum of an antisymmetric tensor field of appropriate rank.

Consider the following action for the matter sources:

$$S_m = -k \int d^D x \sqrt{|\gamma|}\, F_{M_1\cdots M_r} F^{M_1\cdots M_r}, \tag{B.49}$$

where k is a model-dependent numerical coefficient (irrelevant for our discussion), and F is the field strength of a totally antisymmetric tensor A of rank $r-1$, namely:

$$F_{M_1\cdots M_r} = \partial_{[M_1} A_{M_2\cdots M_r]}. \tag{B.50}$$

The corresponding energy-momentum tensor, defined by the standard variational procedure (see Eq. (7.27)) referred to the metric γ^{AB}, is then given by

$$T_{AB} = -2kr\left(F_{AM_2\cdots M_r} F_B^{M_2\cdots M_r} - \frac{1}{2r}\gamma_{AB} F^2\right). \tag{B.51}$$

The variation of S_m with respect to A also provides the equation of motion of the tensor field,

$$\partial_N\left(\sqrt{|\gamma|}F^{NM_2\cdots M_r}\right) = 0, \tag{B.52}$$

to be satisfied together with the Einstein equations (B.44).

Let us now observe that, for our maximally symmetric background, $\sqrt{|\gamma|} = |\det g_{\mu\nu}|^{1/2} |\det g_{mn}|^{1/2}$. We also note that the constraints (B.48) imply, for the energy-momentum tensor (B.51), the following conditions:

$$\begin{aligned} -2kr\, F_{\mu M_2\cdots M_r} F_\nu^{M_2\cdots M_r} &= F_x g_{\mu\nu}, \\ -2kr\, F_{mM_2\cdots M_r} F_n^{M_2\cdots M_r} &= F_y g_{mn}, \end{aligned} \tag{B.53}$$

where F_x and F_y are constant parameters. This gives, in particular,

$$\begin{aligned} T_x &= \left(1-\frac{2}{r}\right)F_x - \frac{D-4}{2r}F_y, \\ T_y &= -\frac{2}{r}F_x + \left(1-\frac{D-4}{2r}\right)F_y. \end{aligned} \tag{B.54}$$

As discovered [16] in the context of $D = 11$ supergravity, a particular simultaneous solution of the conditions (B.53) and of the equations of motion (B.52), consistent with the assumed dimensionality split into 4 and $D - 4$ dimensions, is provided by the following (almost trivial) configurations: (i) $r = 4$ and

$$F^{\mu\nu\alpha\beta}(x) = c_x \, \eta^{\mu\nu\alpha\beta} = \frac{c_x}{\sqrt{|\det g_{\mu\nu}|}} \epsilon^{\mu\nu\alpha\beta}, \tag{B.55}$$

where c_x is a constant (and $F = 0$ for the components with one or more Latin indices); and (ii) $r = D - 4$ and

$$F^{m_4 \cdots m_{D-1}}(y) = c_y \, \eta^{m_4 \cdots m_{D-1}} = \frac{c_y}{\sqrt{|\det g_{mn}|}} \epsilon^{m_4 \cdots m_{D-1}}, \tag{B.56}$$

where c_y is a constant (and $F = 0$ for the components with one or more Greek indices). We have denoted with η the totally antisymmetric tensors of the two maximally symmetric spaces (see Sect. 3.2 for their definitions and properties).

Thanks to the presence of antisymmetric tensors of appropriate rank it is thus possible to find solutions with the required structure $\mathcal{M}_D = \mathcal{M}_4 \otimes \mathcal{M}_{D-4}$. But let us see now if we can also obtain, in this "spontaneous" way, a configuration in which the extra-dimensional manifold $\mathcal{M}_{D-4}$ is compact and characterized by $\Lambda_y > 0$, in such a way to have a finite volume and to admit non-Abelian isometries.

We can consider, for this purpose, both possibilities (i) and (ii) which, according to Eq. (B.53), are characterized, respectively, by (i) $r = 4$, $F_y = 0$, and (ii) $r = D - 4$, $F_x = 0$. In both cases we obtain, from Eq. (B.54), the condition $T_x + T_y = 0$, and this immediately gives an important relation between the curvature scales Λ_x, Λ_y of the two spaces.

In fact, by inserting the explicit configurations for the metric and the matter fields, Eqs. (B.46), (B.48), into the Einstein equations (B.44), together with the constraint (B.47), we obtain the relations:

$$\Lambda_x + \frac{D-4}{2}\Lambda_y = T_x, \qquad 2\Lambda_x + \frac{D-6}{2}\Lambda_y = T_y. \tag{B.57}$$

Hence, by imposing $T_x + T_y = 0$, we immediately obtain

$$\Lambda_x = -\frac{D-5}{3}\Lambda_y. \tag{B.58}$$

This shows that, in a model in which $D > 5$ and $\Lambda_y > 0$ (which admits a compact extra-dimensional space with a non-Abelian isometry group), we must necessarily accept a four-dimensional maximally symmetric space with a negative cosmological constant, $\Lambda_x < 0$, namely with an anti-de Sitter (AdS) geometry.

A background configuration $AdS_4 \otimes \mathcal{M}_{D-4}$ does not look very realistic, because of the huge cosmological constant ($|\Lambda_x| \sim \Lambda_y$) and also because of other phenomenological problems (such as the absence of four-dimensional "chiral" fermions, namely of fermions states of different helicity transforming as different

representations of the gauge group). All the phenomenological problems are basically related to the nonvanishing (and negative) value of the cosmological constant of $\mathcal{M}_4$, which forbids a four-dimensional Minkowski geometry.

In order to recover the Minkowski solution even for $D > 5$ the simplest possibility is probably that of accepting a Ricci-flat extra-dimensional space, setting $\Lambda_y = 0$ and giving up non-Abelian isometries. In that case the Yang–Mills gauge fields must be already present in the higher-dimensional action, where indeed they can themselves trigger the mechanism of spontaneous compactification (on a Ricci-flat manifold). This is what happens, for instance, in the so-called "heterotic" string model (see e.g. the books [22, 41] of the bibliography), where the chiral fermion problem is indeed solved in this way.

Another possibility is that of adding a suitable cosmological constant Λ_D to the D-dimensional action (B.43), in such a way as to exactly cancel the contribution of Λ_x (hence allowing $D = 4$ Minkowski solutions), while keeping a positive constant in the compact space $\mathcal{M}_{D-4}$ (to guarantee the presence of non-Abelian isometries). This, however, would require a high degree of "fine tuning" to exactly match the various contributions. In addition, the ad hoc introduction of Λ_D would explicitly break the supersymmetry of the higher-dimensional supergravity action.

An alternative mechanism, which relaxes the need for fine tuning—still providing a Ricci-flat four-dimensional geometry, $R_{\mu\nu} = 0$, together with a *non* Ricci-flat compact space, $R_{mn} \neq 0$—is based on the presence of a non-minimally coupled scalar field ϕ in the higher-dimensional action. Such a configuration is typical of the bosonic sector of superstring models, and we will present here a simple example based on the following D-dimensional action:

$$S = -\int d^D x \sqrt{|\gamma|}\left\{\frac{e^{-\phi}}{2}\left[R(\gamma) + \partial_M \phi \partial^M \phi\right] + V(\phi) + k F_{M_1 \cdots M_r} F^{M_1 \cdots M_r}\right\}, \tag{B.59}$$

where ϕ is the so-called "dilaton" field. By varying the action with respect to γ and ϕ we obtain, respectively, the gravitational equation

$$\begin{aligned} R_{AB} - \frac{1}{2}\gamma_{AB} R + \nabla_A(\partial_B \phi) + \frac{1}{2}\gamma_{AB}\partial_M \phi \partial^M \phi - \gamma_{AB}\nabla_M\left(\partial^M \phi\right) \\ = e^{\phi}(T_{AB} + \gamma_{AB} V), \end{aligned} \tag{B.60}$$

and the dilaton equation

$$R(\gamma) + \nabla_M\left(\partial^M \phi\right) - \partial_M \phi \partial^M \phi = 2e^{\phi} V' \tag{B.61}$$

(see e.g. the book [19] of the bibliography). Here $V' = \partial V/\partial\phi$, and T_{AB} is the energy-momentum tensor of Eq. (B.51). The variation with respect to A leads then to the equation of motion (B.52) for the antisymmetric tensor, exactly as before.

Let us look again for factorized solutions with the structure $\mathcal{M}_D = \mathcal{M}_4 \otimes \mathcal{M}_{D-4}$, where the metric satisfies the conditions (B.45), (B.46), the antisymmetric

tensor the condition (B.48), and, in addition, the scalar field is a constant, $\phi = \phi_0$. Inserting this ansatz into the gravitational equations we obtain

$$\begin{aligned} -\Lambda_x - \frac{R(\gamma)}{2} &= e^{\phi_0}(T_x + V_0), \\ -\Lambda_y - \frac{R(\gamma)}{2} &= e^{\phi_0}(T_y + V_0), \end{aligned} \tag{B.62}$$

while the dilaton equation (B.61) gives

$$R(\gamma) = 2e^{\phi_0} V_0', \tag{B.63}$$

where $V_0 = V(\phi_0)$ and $V_0' = (\partial V/\partial\phi)_{\phi=\phi_0}$.

We now use for the antisymmetric tensor field the Freund–Rubin solutions (B.55), (B.56), both characterized by the condition $T_x + T_y = 0$, which now implies (from Eqs. (B.62)):

$$\Lambda_x + \Lambda_y + R(\gamma) = -2e^{\phi_0} V_0. \tag{B.64}$$

We are interested, in particular, in a Ricci-flat four-dimensional space–time, characterized by

$$\Lambda_x = 0, \qquad \Lambda_y = -\frac{R(\gamma)}{D-4} \tag{B.65}$$

(we have used the condition (B.47) for the D-dimensional scalar curvature). This choice can simultaneously satisfy Eqs. (B.63) and (B.64) provided

$$\left(\frac{V'}{V}\right)_{\phi_0} = -\frac{D-4}{D-5}. \tag{B.66}$$

We can thus obtain the sought geometrical structure without fine adjustment of free dimensional parameters, at the price of imposing a simple differential condition on the functional form of the potential (satisfied, in this particular case, by an exponential potential $V \sim \exp[-\phi(D-4)/(D-5)]$).

This model of spontaneous compactification can be easily generalized to (more realistic) cases in which the dilaton coupling to the Einstein action is described by an arbitrary function $f(\phi)$ replacing $\exp(-\phi)$. In that case [17] Eq. (B.66) is replaced by a condition relating $(V'/V)_0$ to $(f'/f)_0$.

B.2 Brane-World Gravity

Another approach to the problem of the dimensional reduction, not necessarily alternative to the Kaluza–Klein scenario, is based on the assumption that the charges sourcing the gauge interactions are confined on 3-dimensional hypersurfaces called

"Dirichlet branes" (or D_3-branes), and that the associated gauge fields can propagate only on the "world-volume" swept by the time evolution of such branes. It follows that the gauge interactions are insensitive to the spatial dimensions orthogonal to the brane, even in the limiting case in which such dimensions are infinitely extended. According to such a "brane-world" scenario—suggested by superstring model of unified interactions—we are thus living on a four-dimensional "slice" of a D-dimensional space–time (also called "bulk" manifold).

Gravity, however, can propagate along all spatial directions, so that the gravitational theory must be formulated in D dimensions, and the geometry of the D-dimensional bulk space–time may be characterized by an arbitrary metric and curvature. We have thus to face the problems already met in the context of the Kaluza–Klein scenario: how to obtain (at least as a ground state solution) a flat Minkowski geometry in the four-dimensional space–time of the brane? and how to explain why we have not found (so far) any gravitational evidence of the extra dimensions? are they compactified on very small distance scales like in the Kaluza–Klein scenario?

In the following sections it will be shown that the compactification is *a possibility*, but *not a necessity* as in the Kaluza–Klein context. In this section we will first introduce a simple model illustrating the possibility of exact solutions with a flat four-dimensional space–time associated to a brane embedded in a curved bulk manifold.

Let us start with the general action for a D-dimensional bulk manifold $\mathcal{M}_D$,

$$S = \int d^D x \sqrt{|g_D|}\left(-\frac{M_D^{D-2}}{2} R_D + \mathcal{L}_D^{\text{bulk}}\right) + S_{p\text{-brane}}, \tag{B.67}$$

where we have included the Lagrangian density $\mathcal{L}_D^{\text{bulk}}$, generically representing the gravitational contributions of the bulk fields (and of their quantum fluctuations) to the geometry described by the D-dimensional metric g_{AB}. We have also included the action of a p-dimensional brane (p-brane, for short) embedded in $\mathcal{M}_D$, with $p+1 < D$, since it contributes to the bulk gravitational field with its own energy density, and with the energy-momentum density of the fields possibly living on it (namely, the fields confined on the $(p+1)$-dimensional hypersurface Σ_{p+1} swept by the brane evolution).

The action of the p-brane is proportional to the "world-volume" of the hypersurface Σ_{p+1}, just like the action of a point particle is proportional to the length of the "world-line" described by the particle evolution. Let us call $\xi^\mu = (\xi^0, \xi^1, \ldots, \xi^p)$ the coordinates on Σ_{p+1}, $x^A = (x^0, x^1, \ldots, x^{D-1})$ the coordinates on $\mathcal{M}_D$, and let us denote with

$$x^A = X^A(\xi^\mu), \quad A = 0, 1, \ldots, D-1, \ \mu = 0, 1, \ldots, p, \tag{B.68}$$

the parametric equations describing the embedding of Σ_{p+1} into $\mathcal{M}_D$. The so-called "induced metric" on the hypersurface Σ_{P+1} is then given by

$$h_{\mu\nu} = \frac{\partial X^A}{\partial \xi^\mu} \frac{\partial X^B}{\partial \xi^\nu} g_{AB}, \tag{B.69}$$

and the action of an "empty" p-brane can be written (in Nambu–Goto form) as follows:

$$S_{p\text{-brane}} = T_p \int d^{p+1}\xi \, \sqrt{|h|}. \tag{B.70}$$

Here $h = \det h_{\mu\nu}$, and T_p—the so-called "tension"—is a constant representing the vacuum energy density, i.e. the vacuum energy per unit of proper p-dimensional volume of the brane. If the brane contains matter fields then the "cosmological" constant T_p has to be replaced by the Lagrangian density $\mathcal{L}_p$ describing the gravitational sources living on the brane.

The above brane action can also be rewritten in an equivalent form which avoids the explicit presence of the square root—and is thus more convenient for variational computations—at the price of introducing an auxiliary tensor field $\gamma^{\mu\nu}$, acting as a Lagrange multiplier, and representing the "intrinsic" Riemannian metric of the manifold Σ_{p+1}. Such an equivalent form is the so-called Polyakov action,

$$S_{p\text{-brane}} = \frac{T_p}{2} \int d^{p+1}\xi \, \sqrt{|\gamma|} \left[\gamma^{\mu\nu} \frac{\partial X^A}{\partial \xi^\mu} \frac{\partial X^B}{\partial \xi^\nu} g_{AB} - (p-1) \right], \tag{B.71}$$

where $\gamma = \det \gamma_{\mu\nu}$. Its variation with respect to $\gamma^{\mu\nu}$ gives the constraint

$$h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\alpha\beta}h_{\alpha\beta} - \frac{1}{2}\gamma_{\mu\nu}(p-1) = 0, \tag{B.72}$$

which is identically solved by $\gamma_{\mu\nu} = h_{\mu\nu}$, where $h_{\mu\nu}$ is defined by Eq. (B.69). Using this result to eliminate $\gamma^{\mu\nu}$, and using the identity $h^{\mu\nu}h_{\mu\nu} = \delta^\mu_\mu = p+1$, one then finds that the Polyakov action exactly reduces to the Nambu–Goto form of Eq. (B.70).

It is finally convenient, for our purpose, to take into account that the brane contribution to the total action (B.67) is localized exactly *at the brane position* specified by the embedding equations (B.68), and it is vanishing for $x^A \neq X^A(\xi)$. We can thus express $S_{p\text{-brane}}$, in analogy with the bulk action, as an integral over a D-dimensional delta-function distribution,

$$S_{p\text{-brane}} = \int d^D x \, \sqrt{|g_D|} \, \mathcal{L}_D^{\text{brane}}, \tag{B.73}$$

where

$$\mathcal{L}_D^{\text{brane}} = \frac{T_p}{2\sqrt{|g_D|}} \int_{\Sigma_{p+1}} d^{p+1}\xi \, \sqrt{|\gamma|} \times \left[\gamma^{\mu\nu} \frac{\partial X^A}{\partial \xi^\mu} \frac{\partial X^B}{\partial \xi^\nu} g_{AB} - (p-1) \right] \delta^D\big(x - X(\xi)\big). \tag{B.74}$$

The total action (B.67) then becomes

$$S = \int d^D x \, \sqrt{|g_D|} \left(-\frac{M_D^{D-2}}{2} R_D + \mathcal{L}_D^{\text{bulk}} + \mathcal{L}_D^{\text{brane}} \right), \tag{B.75}$$

and can be easily varied with respect to our independent fields g_{AB}, X^A, $\gamma^{\mu\nu}$.

The variation with respect to g_{AB} gives the bulk Einstein equations,

$$R_{AB} - \frac{1}{2} g_{AB} R = M_D^{2-D} \big(T_{AB}^{\text{bulk}} + T_{AB}^{\text{brane}} \big), \tag{B.76}$$

where the energy-momentum tensor of the sources is provided by the standard variational definition (7.26), (7.27) (performed with respect to g^{AB}). For the brane, in particular, we have

$$T_{AB}^{\text{brane}} = \frac{T_p}{\sqrt{|g_D|}} \int_{\Sigma_{p+1}} d^{p+1}\xi \, \sqrt{|\gamma|} \gamma^{\mu\nu} \partial_\mu X_A \partial_\nu X_B \delta^D \big(x - X(\xi) \big), \tag{B.77}$$

where $\partial_\mu X^A = \partial X^A / \partial \xi^\mu$. The variation with respect to X^A gives the brane equation of motion,

$$\begin{aligned} &\partial_\mu \big[\sqrt{|\gamma|} \gamma^{\mu\nu} \partial_\nu X^B g_{AB}(x) \big]_{x = X(\xi)} \\ &\quad = \frac{1}{2} \big[\sqrt{|\gamma|} \gamma^{\mu\nu} \partial_\mu X^M \partial_\nu X^N \partial_A g_{MN}(x) \big]_{x = X(\xi)}. \end{aligned} \tag{B.78}$$

Finally, the variation with respect to γ^{muv} gives the constraint (B.72), which leads to identify $\gamma_{\mu\nu}$ with the induced metric $h_{\mu\nu}$.

Let us now consider the particular case $p = 3$, where the brane space–time Σ_4 has the appropriate number of dimensions to represent a possible model of our Universe. Also, let us assume that the bulk space–time has only one additional dimension, so that $D = 5$ (like in the original Kaluza–Klein proposal). Finally, let us concentrate on a very simple example where the only nonvanishing gravitational contribution of the bulk comes from the vacuum energy density (like the brane contribution), and has the form of a cosmological constant Λ. We set, in particular, $\mathcal{L}^{\text{bulk}} = -M^{D-2} \Lambda$, so that

$$M^{2-D} T_{AB}^{\text{bulk}} = \Lambda g_{AB}. \tag{B.79}$$

In this context we will look for particular solutions of Eqs. (B.76), (B.78) describing a flat (Minkowski) hypersurface Σ_4 embedded in a generally curved bulk manifold $\mathcal{M}_5$.

Let us call $x^A = (x^\mu, y)$ the bulk coordinates, and suppose that the hypersurface Σ_4 is rigidly fixed at $y = 0$, described by the trivial embedding:

$$\begin{aligned} &x^A = X^A(\xi) = \delta^A_\mu \xi^\mu, \quad A = 0, 1, 2, 3, \\ &x^4 \equiv y = 0. \end{aligned} \tag{B.80}$$

Also, suppose that Σ_4 has a globally flat geometry described by the Minkowski metric $\eta_{\mu\nu}$, and that the bulk metric is conformally flat, $g_{AB} = f^2(y) \eta_{AB}$, with a conformal factor f^2 which depends only on the y coordinate parametrizing the spatial direction normal to the brane. Since our configuration is symmetric under

$y \to -y$ reflections we thus look for a "warped" five-dimensional geometric structure described by the following line-element:

$$ds^2 = f^2(|y|)\left(\eta_{\mu\nu}\, dx^\mu\, dx^\nu - dy^2\right). \tag{B.81}$$

We can easily check that, for this type of background, the induced metric (B.69) reduces to $h_{\mu\nu} = f^2\eta_{\mu\nu} = \gamma_{\mu\nu}$, and that the brane equation (B.78) is identically satisfied thanks to the reflection symmetry, which implies $(\partial f/\partial y)_{y=0} = 0$ (see below). Let us then consider the Einstein equations (B.76).

For the energy-momentum of the sources we easily get, from Eq. (B.79),

$$M_5^{-3}\left(T_A{}^B\right)^{\text{bulk}} = \Lambda\delta_A^B, \tag{B.82}$$

and, from Eq. (B.77),

$$\begin{aligned} \left(T_4{}^4\right)^{\text{brane}} &= 0 \\ \left(T_\mu{}^\nu\right)^{\text{brane}} &= f^{-1}T_3\delta_\mu^\nu\delta(y). \end{aligned} \tag{B.83}$$

The five-dimensional Christoffel connection associated to the metric (B.81) , on the other hand, has the following nonvanishing components:

$$\Gamma_{44}{}^4 = \frac{f'}{f}, \qquad \Gamma_{\mu\nu}{}^4 = \frac{f'}{f}\eta_{\mu\nu}, \qquad \Gamma_{4\mu}{}^\nu = \frac{f'}{f}\delta_\mu^\nu \tag{B.84}$$

(a prime denotes differentiation with respect to y). Defining $F = f'/f$ we then obtain, from the components of the Einstein tensor,

$$\begin{aligned} G_4{}^4 &= R_4{}^4 - \frac{1}{2}R = -6f^{-2}F^2, \\ G_\mu{}^\nu &= R_\mu{}^\nu - \frac{1}{2}\delta_\mu^\nu R = -f^{-2}\left(3F' + 3F^2\right)\delta_\mu^\nu. \end{aligned} \tag{B.85}$$

Our Einstein equations, decomposed into the directions normal and tangential to the brane space–time Σ_4, thus reduce, respectively to:

$$6F^2 = -\Lambda f^2, \tag{B.86}$$

$$3F' + 3F^2 = -\Lambda f^2 - M_5^{-3}T_3 f\delta(y). \tag{B.87}$$

Note that f depends on the modulus of y, so that the second derivative of f (present into F') contains the derivative of the sign function, which generates a delta-function contribution to the left-hand side of Eq. (B.87). We have to match separately the finite parts of the equation and the coefficients of the singular contributions at $y = 0$.

In order to solve the above system of equations it is convenient to adopt the explicit representation

$$|y| = y\epsilon(y), \quad \epsilon(y) = \theta(y) - \theta(-y), \tag{B.88}$$

where $\theta(y)$ is the Heaviside step function and $\epsilon(y)$ the sign function, satisfying the properties:

$$\epsilon^2 = 1, \qquad \epsilon' = 2\delta(y). \tag{B.89}$$

We can thus set

$$f' = \frac{\partial f}{\partial |y|}\epsilon(y), \tag{B.90}$$

and Eq. (B.86) becomes

$$\left(\frac{\partial f}{\partial |y|}\right)^2 = -\frac{\Lambda}{6} f^4, \tag{B.91}$$

which admits real solutions provided $\Lambda < 0$. Assuming that the bulk cosmological constant is negative, and integrating, we then obtain the particular exact solution

$$f\big(|y|\big) = \big(1 + k|y|\big)^{-1}, \quad k = \left(-\frac{\Lambda}{6}\right)^{1/2}, \tag{B.92}$$

which inserted into the metric (B.81) describes an anti-de Sitter (AdS) bulk geometry, written in the conformally flat parametrization.

We have still to solve the second Einstein equation (B.87), which contains the explicit contributions of the brane. Using Eqs. (B.88)–(B.90) we can recast our equation in the form:

$$\frac{3}{f}\frac{\partial^2 f}{\partial |y|^2} + \frac{6}{f}\frac{\partial f}{\partial |y|}\delta(y) = -\Lambda f^2 - M_5^{-3} T_3 f \delta(y). \tag{B.93}$$

The finite part of this equation is identically satisfied by the solution given in Eq. (B.92). By equating the coefficients of the delta-function terms we are led to a condition between the tension of the brane and the curvature scale of the AdS bulk geometry:

$$T_3 = 6k M_5^3 = M_5^3 (-6\Lambda)^{1/2}. \tag{B.94}$$

If this condition is satisfied we obtain the so-called Randall–Sundrum model [43], in which the vacuum energy density of the brane (represented by its tension T_3) is exactly canceled by an opposite contribution generated by the bulk sources, and the geometry of the brane-world Σ_4 is allowed to be of the flat Minkowski type, as required.

B.2.1 Gravity Confinement

If we take seriously the possibility that the world explored by fundamental (strong and electroweak) interactions is the four-dimensional space–time of a 3-brane, embedded in a higher-dimensional manifold, we have to face the problem of why we

have not yet detected the extra dimensions by means of gravitational experiments. Indeed gravity, unlike the other gauge interactions, is expected to propagate along all spatial directions.

A possibility is that the dimensions external to Σ_4 have a very small, compact size, not accessible to presently available experimental sensitivities (as also assumed in the context of the Kaluza–Klein scenario). In the brane-world scenario, however, there is a second possibility based on an effect of "gravity confinement": an appropriate curvature of the bulk geometry can force the long-range component of tensor interactions to be strictly localized on Σ_4, just like the vector gauge interactions. In that case only a residual, short-range tail of the gravitational interaction (mediated by massive tensor particles) may propagate in the directions orthogonal to Σ_4, and make the extra dimensions detectable by experiments probing small enough corrections to long-range gravitational forces.

This interesting possibility can be illustrated considering the simple, five-dimensional Randall–Sundrum model introduced in the previous section, and by expanding to first order the fluctuations of the bulk metric tensor, $g_{AB} \to g_{AB} + \delta g_{AB}$, at *fixed* brane position, $\delta X^A = 0$, around the background solution (B.81). Let us call the fluctuations $\delta g_{AB} = h_{AB}$, and let us compute the perturbed action up to terms quadratic in h_{AB}.

We are interested, in particular, in the transverse and traceless part of the fluctuations of the four-dimensional geometry, $\delta g_{\mu\nu} = h_{\mu\nu}$, which describes the propagation of gravitational waves (see Chap. 9) in the brane space–time Σ_4. In the linear approximation they are decoupled from other (scalar and extra-dimensional) components of δg_{AB}. We shall thus assume that our perturbed geometric configuration is characterized by

$$h_{\mu 4} = 0, \qquad h_{\mu\nu} = h_{\mu\nu}\big(x^\alpha, y\big), \qquad g^{\mu\nu} h_{\mu\nu} = 0 = \partial^\nu h_{\mu\nu}. \tag{B.95}$$

For the computation of the perturbed, quadratic action we will follow the straightforward procedure introduced in Sect. 9.2 (which leads to the result (9.48)), taking into account, however, that we are now expanding around the non-trivial five-dimensional geometry (B.81). After using the unperturbed background equations we obtain

$$\begin{aligned} \delta S &= -\frac{M_5^3}{8} \int d^5x \sqrt{|g_5|}\, h_\mu{}^\nu \nabla_A \nabla^A h_\nu{}^\mu \\ &= -\frac{M_5^3}{8} \int d^5x \sqrt{|g_5|}\, f^3 \big[h_\mu{}^\nu \Box h_\nu{}^\mu - h_\mu{}^\nu h_\nu{}''{}^\mu - 3F h_\mu{}^\nu h_\nu{}'{}^\mu \big], \end{aligned} \tag{B.96}$$

where the covariant derivative ∇_A is referred to the unperturbed metric g_{AB}, and where $\Box = \partial_t^2 - \partial_i^2$ is the usual d'Alembert operator in four-dimensional Minkowski space. Integrating by parts to eliminate h'', decomposing $h_\mu{}^\nu$ into the two independent polarization modes (see Eq. (9.15)), and tracing over the polarization tensors,

the action for each polarization mode $h = h(t, x^i, y)$ can then be written as

$$\delta S = \frac{M_5^3}{4} \int dy\, f^3 \int d^4x \left(\dot{h}^2 + h\nabla h - h'^2\right), \tag{B.97}$$

where the dot denotes differentiation with respect to $t = x^0$, the prime with respect to y, and $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplace operator of 3-dimensional Euclidean space. The variation with respect to h finally gives the vacuum propagation equation for the linear fluctuations of the four-dimensional geometry:

$$\Box h - h'' - 3Fh' = 0. \tag{B.98}$$

It differs from the d'Alembert wave equation because the fluctuations are coupled to the gradients of the bulk geometry through their intrinsic dependence on the fifth coordinate y.

In order to solve the above equation we now separate the bulk and brane coordinates by setting

$$h\left(x^\mu, y\right) = \sum_m v_m(x)\psi_m(y), \tag{B.99}$$

and we find that the new variables v, ψ satisfy the following (decoupled) eigenvalue equations:

$$\begin{aligned} \Box v_m &= -m^2 v_m, \\ \psi_m'' + 3F\psi_m' &\equiv f^{-3}\left(f^3\psi_n'\right)' = -m^2\psi_m. \end{aligned} \tag{B.100}$$

If the spectrum is continuous, the sum of Eq. (B.99) is clearly replaced by integration over m. It is also convenient to rewrite the equation for ψ in canonical (Schrodinger-like) form by introducing the rescaled variable $\widehat{\psi}_m$, such that

$$\psi_m = \left(f^3 M_5\right)^{-1/2}\widehat{\psi}_m \tag{B.101}$$

(the dimensional factor $M_5^{-1/2}$ has been inserted for later convenience). The equation for ψ then becomes

$$\widehat{\psi}_m'' + \left[m^2 - V(y)\right]\widehat{\psi}_m = 0, \tag{B.102}$$

where

$$V(y) = \frac{3}{2}\frac{f''}{f} + \frac{3}{4}\left(\frac{f'}{f}\right)^2, \tag{B.103}$$

or, using the explicit background solution (B.92),

$$V(y) = \frac{15}{4}\frac{k^2}{(1+k|y|)^2} - \frac{3k\delta(y)}{1+k|y|}. \tag{B.104}$$

This is a so-called "volcano-like" potential, as the first term of $V(y)$ is peaked at $y = 0$, but the peak is in correspondence of a negative delta-function singularity, which looks like the crater of a volcano.

It is well known, from one-dimensional quantum mechanics, that the Schrodinger equation with an attractive delta-function potential admits one bound state only, associated with a square-integrable wave function which is localized around the position of the potential. In our case such a configuration corresponds to the eigenvalue $m = 0$, and to the reflection-symmetric solution of Eq. (B.102) given by

$$\widehat{\psi}_0 = c_0 f^{3/2}, \tag{B.105}$$

where c_0 is a constant to be determined by the normalization condition. It is important to stress that $\widehat{\psi}_0$, defined as in Eq. (B.101) (with ψ_0 dimensionless), has the correct canonical normalization to belong to the L_2 space of square integrable functions with measure dy (as in conventional quantum mechanics), and turns out to be normalizable even for an *infinite* extension of the dimension normal to the brane. In fact:

$$1 = \int dy\, |\widehat{\psi}_0|^2 = \int_{-\infty}^{+\infty} dy\, \frac{c_0^2}{(1 + k|y|)^3} = \frac{c_0^2}{k}. \tag{B.106}$$

We can express the same result in terms of the non-canonical variable ψ but, in that case, we must use inner products with dimensionless measure $dy\, M_5 f^3$.

This example clearly show how the *massless* components of the metric fluctuations (corresponding to long-range gravitational interactions) can be localized on the brane at $y = 0$ not because the fifth dimension is compactified on a very small length scale, but because the massless modes are "trapped" in a bound state generated by the bulk curvature. In this case, in particular, it is the AdS geometry which forces massless fluctuations to be peaked around the brane position.

Let us now take into account the massive part of the fluctuation spectrum, considering the Schrodinger equation (B.102) with $m \neq 0$. Even in that case there are exact solutions, with a continuous spectrum of positive values of m which extends up to infinity. However, as we shall see, these solutions are not bound states of the potential, and are not localized on the brane space–time Σ_4.

To obtain such solutions we can follow the standard quantum-mechanical treatment of a delta-function potential. Looking for reflection-symmetric functions $\widehat{\psi}_m(|y|)$ we first rewrite Eq. (B.102) as

$$\frac{d^2\widehat{\psi}_m}{d|y|^2} + 2\delta(y)\frac{d\widehat{\psi}_m}{d|y|} + \big(m^2 - V\big)\widehat{\psi}_m = 0, \tag{B.107}$$

where V is given by Eq. (B.104). Outside the origin ($y \neq 0$) this reduces to a Bessel equation, whose general solution can be written as a combination of Bessel functions J_ν and Y_ν of index $\nu = 2$ and argument $\alpha = m/(kf)$:

$$\widehat{\psi}_m = f^{-1/2}\big[A_m J_2(\alpha) + B_m Y_2(\alpha)\big]. \tag{B.108}$$

Imposing on this expression to satisfy Eq. (B.107) also at $y = 0$, and equating the coefficients of the delta-function terms, we obtain an additional condition which relates the two integration constants A_m and B_m:

$$B_m = -A_m \frac{J_1(m/k)}{Y_1(m/k)}. \tag{B.109}$$

The general solution can thus be rewritten as

$$\widehat{\psi}_m = c_m f^{-1/2} \left[Y_1\left(\frac{m}{k}\right) J_2(\alpha) - J_1\left(\frac{m}{k}\right) Y_2(\alpha) \right], \tag{B.110}$$

where c_m is an overall constant factor, to be determined by the normalization condition

$$\int dy\, \widehat{\psi}_m^* \widehat{\psi}_n \equiv \int dy\, M_5 f^3 \psi_m^* \psi_n = \delta(m,n). \tag{B.111}$$

Here $\delta(m,n)$ corresponds to the Kronecker symbol for a discrete spectrum, and to the Dirac delta function for a continuous spectrum. The normalization condition gives

$$c_m = \left(\frac{m}{2k}\right)^{1/2} \left[J_1^2\left(\frac{m}{k}\right) + Y_1^2\left(\frac{m}{k}\right) \right]^{-1/2}, \tag{B.112}$$

which completely fixes the continuous spectrum of the massive fluctuations.

Using the asymptotic behavior of the Bessel functions $J_2(\alpha)$, $Y_2(\alpha)$, with $\alpha = m/(kf) = m(1+k|y|)/k$, we see that the above solutions, instead of being damped, are asymptotically oscillating for $y \to \pm\infty$: hence, they cannot be localized on the brane. We may thus expect from these massive modes new (and genuinely higher-dimensional) effects: in particular, short-range corrections which are sensitive to the extra dimensions and which bear the direct imprint of the bulk geometry. This possibility will be discussed in the next subsection.

B.2.2 Short-Range Corrections

For a quantitative estimate of the gravitational corrections induced by the massive fluctuations of the brane-world geometry we need to compute, first of all, the effective coupling strengths of the massive modes. Such couplings can be obtained from the canonical form of the effective action (B.97), dimensionally reduced by integrating out the y dependence of the ψ_m wave functions.

We insert, for this purpose, the expansion (B.99) into the action (B.97), and note that the term h'^2 is proportional (modulo a total derivative) to the mass term of the mode ψ_m. In fact:

$$\int dy\, f^3 h'^2 = \sum_{m.n} v_m v_n \int dy\, f^3 \psi_m' \psi_n'$$

$$= \sum_{m.n} v_m v_n \int dy \left[\frac{d}{dy}\left(f^3 \psi_m \psi_n'\right) - \psi_m \left(f^3 \psi_n'\right)' \right]$$

$$= \sum_{m.n} v_m v_n \int dy\, f^3 m^2 \psi_m \psi_n. \tag{B.113}$$

In the last step we have neglected a total derivative and used Eq. (B.100) for ψ_m. Integrating over y, and taking into account the orthonormality condition (B.111), we get a dimensionally reduced action which contains only the components $v_m(x)$ of the metric fluctuations:

$$\delta S \equiv \sum_m \delta S_m = \sum_m \frac{M_5^2}{4} \int d^4x \left(\dot{v}_m^2 + v_m \nabla^2 v_m - m^2 v_m^2\right). \tag{B.114}$$

The summation symbol used here synthetically denotes that the contribution of the massless mode $m = 0$ has to be summed to the integral (from 0 to ∞) performed over all the continuous spectrum of massive modes.

Let us finally introduce the variable $\overline{h}_m$, representing the effective fluctuations of the four-dimensional Minkowski metric on the hypersurface Σ_4, namely:

$$\overline{h}_m(x) = \left[h_m(x, y)\right]_{y=0} \equiv v_m(x)\psi_m(0). \tag{B.115}$$

The action (B.114) becomes

$$\delta S = \sum_m \frac{M_5^2}{4\psi_m(0)} \int d^4x \left(\dot{\overline{h}}_m^2 + \overline{h}_m \nabla^2 \overline{h}_m - m^2 \overline{h}_m^2\right). \tag{B.116}$$

A comparison with the canonical form of the action for the tensor fluctuations of the Minkowski geometry (see Eq. (9.48), traced over the two polarization modes) immediately lead us to conclude that the effective coupling constant for the mode $\overline{h}_m$ is given by

$$8\pi G(m) \equiv M_{\rm P}(m) = M_5^{-2} \psi_m^2(0). \tag{B.117}$$

Note that this effective coupling depends not only on the bulk gravitational scale M_5, but also on the position of the brane on the bulk manifold (since the bulk is curved, and its geometry is not translational invariant).

For the massless fluctuations we have, from Eqs. (B.101), (B.106), $\psi_0 = (k/M_5)^{1/2}$; the corresponding coupling, that we may identity with the usual Newton constant G, is then given by

$$8\pi G(0) \equiv 8\pi G = \frac{k}{M_5^3}. \tag{B.118}$$

For the massive fluctuations, instead, the coupling is mass dependent: using the definitions $\psi_m(0) = M_5^{-1/2}\widehat{\psi}_m(0)$ and the solutions (B.110), (B.112), we obtain

$$8\pi G(m) = \frac{\alpha_0}{2M_5^3}\frac{[Y_1(\alpha_0)J_2(\alpha_0) - J_1(\alpha_0)Y_2(\alpha_0)]^2}{J_1^2(\alpha_0) + Y_1^2(\alpha_0)}, \tag{B.119}$$

where $\alpha_0 = m/k$. Note that $G(m)$ is referred to a continuous spectrum of values of m, hence it represents the effective coupling in the infinitesimal mass interval between m and $m + dm$.

We are now in the position of estimating the effective gravitational interactions on the four-dimensional brane space–time Σ_4, including the contribution of all (massless and massive) modes.

Let us consider, as a simple but instructive example, the static gravitational field produced by a point-like source of mass M localized on the brane. The linearized propagation equation for tensor metric fluctuations on the Minkowski space–time of the brane, including the sources, is given by Eq. (8.10). Including a possible mass term, and using the effective coupling (B.119), we obtain for a generic mode $\overline{h}_m^{\mu\nu}$:

$$\left(\Box + m^2\right)\overline{h}_m^{\mu\nu} = -16\pi G(m)\left(\tau^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\tau\right). \tag{B.120}$$

In the static limit we have $\Box \to -\nabla^2$, $\tau^{ij} \to 0$, $\tau = \eta^{\mu\nu}\tau_{\mu\nu} \to \tau_0^0 = \rho$, and $\overline{h}_m^{00} \to 2\overline{\phi}_m$, where $\overline{\phi}_m$ is the effective gravitational potential associated to the fluctuations of mass m. From the $(0,0)$ component of the above equation we then obtain

$$\left(-\nabla^2 + m^2\right)\overline{\phi}_m(x) = -4\pi G(m)\rho(x), \tag{B.121}$$

which represents a generalized Poisson equation controlling the massive mode contributions to the total static potential.

The general solution for $\overline{\phi}_m$ can be expressed using the standard method of the Green function, i.e. by setting

$$\overline{\phi}_m(x) = -\frac{1}{4\pi}\int d^3x'\,\mathcal{G}_m\left(x, x'\right)4\pi G(m)\rho\left(x'\right), \tag{B.122}$$

where $\mathcal{G}_m(x, x')$ satisfies

$$\left(-\nabla^2 + m^2\right)\mathcal{G}_m\left(x, x'\right) = 4\pi\delta\left(x - x'\right). \tag{B.123}$$

Hence, by Fourier transforming,

$$\mathcal{G}_m\left(x, x'\right) = 4\pi\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{p^2 + m^2}. \tag{B.124}$$

For the massless mode, in particular, we obtain

$$\mathcal{G}_0\left(x, x'\right) = \frac{2}{\pi}\int_0^\infty dp\frac{\sin(p|\boldsymbol{x} - \boldsymbol{x}'|)}{p|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{B.125}$$

and Eq. (B.122) gives, for a point-like source with $\rho(x') = M\delta^3(x')$,

$$\overline{\phi}_m(0) = -\frac{GM}{r}, \tag{B.126}$$

where $r = |\boldsymbol{x}|$ (we have used the coupling (B.118)). For a massive mode the Green function is given by

$$\mathcal{G}_m(x, x') = \frac{2}{\pi}\int_0^\infty dp\, \frac{p^2}{p^2+m^2}\frac{\sin(p|\boldsymbol{x}-\boldsymbol{x}'|)}{p|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{e^{-m|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{B.127}$$

and we obtain

$$\overline{\phi}_m(0) = -\frac{G(m)M}{r}e^{-mr}, \tag{B.128}$$

where $G(m)$ is defined by Eq. (B.119). The total static potential produced by the point-like source is finally given by the sum of all massless and massive contributions, namely by

$$\begin{aligned}\overline{\phi} &= \sum_m \overline{\phi}_m = \overline{\phi}_0 + \int_0^\infty dm\, \overline{\phi}_m \\ &= -\frac{GM}{r}\left[1 + \frac{1}{G}\int_0^\infty dm\, G(m)e^{-mr}\right].\end{aligned} \tag{B.129}$$

In the limit of weak fields, at large distances from the source, we see that the contribution of the massive fluctuations is exponentially suppressed, so that the dominant contribution to the above integral comes from the small-mass regime. For weak fields, we can then obtain an approximate estimate of the short-range corrections by using the small argument limit ($m \to 0$) of the Bessel function appearing in the definition of $G(m)$. In this limit we obtain

$$8\pi G(m) \underset{m\to 0}{\longrightarrow} \frac{m}{2kM_5^3} = \frac{m}{2k^2}\,8\pi G \tag{B.130}$$

(we have used Eq. (B.118)). The effective potential thus becomes, in the weak field limit,

$$\begin{aligned}\overline{\phi} &= -\frac{GM}{r}\left(1 + \frac{1}{2k^2}\int_0^\infty dm\, m e^{-mr}\right) \\ &= -\frac{GM}{r}\left(1 + \frac{1}{2k^2r^2}\right).\end{aligned} \tag{B.131}$$

It follows that the higher-dimensional corrections become important only at distance scales which are sufficiently small with respect to the bulk curvature scale: this means, in the particular care we are considering, at distances $r \lesssim k^{-1}$, where k^{-1} is

the curvature radius of the bulk AdS geometry (see Eq. (B.92)). At larger scales of distance the gravitational interaction experienced on the brane becomes effectively four-dimensional, *quite irrespectively* of the compactification and size of the extra dimensions. This result can be extended to space–times where the brane geometry is described by Ricci-flat metrics different from the Minkowski metric, and where the total number of dimensions is $D > 5$.

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